„BABEŞ-BOLYAI" UNIVERSITY CLUJ-NAPOCA

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Integral transformations for certain classes of univalent functions

DOCTORAL THESIS RESUME

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Published articles

1) O. M. Pârva, D. Breaz, *Univalence properties of an integral operator*, Afrika Matematika, https://doi.org/10.1007/s13370-022-00975-0, Vol. 33, No. 37, (2022) - ISI journal.

2) O. M. Pârva, D. Breaz, S. Owa, *Properties of the coefficients of an integral operator*, General Mathematics, Vol. 31, No. 1 (2023) - ZMath journal.

3) O. M. Pârva, D. Breaz, *Univalence conditions for analytic functions on the exterior unit disk*, Journal of Advanced Mathematical Studies, Vol. 16, no. 2 (2023), pp. 125 -133 - ZMath journal, ISC journal, EBSCO journal.

4) O. M. Pârva, D. Breaz, *Univalence conditions of an integral operator on the exterior unit disk*, Studia Universitatis Babeș-Bolyai Mathematica, Vol 69, No. 4 (2024), pp. 749 -758 - ISI journal.

Index of notations

$\Re(\alpha)$ - the real part of the complex number α
$Im(\alpha)$ - the imaginary part of the complex number α

Sets:
$\mathbb{C}$ - the complex plane
$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ - the extended complex plane
$\mathbb{C}^*$ - the set of non-zero complex numbers
$\mathbb{R}$ - the set of real numbers
$\mathbb{N}_1^*$ - the set of non-zero natural numbers except 1
C_r - the image of the circle $\{z \in \mathbb{C} : |z| = r, 0 < r < 1\}$ under an holomorphic function f
U - the interior of the unit disk where $U := \{z \in \mathbb{C} : |z| < 1\}$
U_R - the disk of radius R
U^* - the interior of the unit disk with a hole where $U^* = U - \{0\}$
$\dot{U}(z_0; R) = U(z_0, R) - \{z_0\}$ - the punctured disk centered at z_0 with radius $R > 0$
$\mathbb{H}(U)$ - the set of holomorphic functions in the unit disk
$\mathbb{H}_u(U)$ - the set of univalent (holomorphic and injective) functions in the unit disk
$E(q)$ - the exceptional set, $E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty, q'(\zeta) \neq 0, \zeta \in \partial U \setminus E(q)\}$
$\overline{U}$ - the closed unit disk
$\overline{U}(a, r) = \{z \in \mathbb{C} : |z - a| \leq r, r > 0\}$ - the closed unit disk centered at a with radius r
Q - the set of functions that are holomorphic and injective on $\overline{U} \setminus E(q)$
W - the exterior of the unit disk, $W = \{z \in \mathbb{C} : 1 < |z| < \infty\}$
U^- - the exterior of the open unit disk, $U^- = \{z \in \mathbb{C}_\infty : |z| > 1\}$
W_R - the exterior of the unit disk of radius R, $W_R = \{z \in \mathbb{C} : |z| > R\}$
Simple pole - an isolated singularity z_0 of a function $f \in \mathbb{H}(U)$ where $\exists \lim_{x\to z_0} f(z) = \infty$ and f can be extended at z_0 by defining $f(z_0) = \infty$

Classes and subclasses:
A - the class of analytic functions defined on U, normalized with the conditions: $f(0) = 0 = f'(0) - 1$ - p. 12
S - subclass of class A, containing univalent functions from U^* - p. 12
O - the class of analytic functions defined on the exterior of the unit disk - p. 12
Σ - subclass of O containing univalent functions - p. 12
O_1 - subclass of O containing meromorphic and injective functions - p. 12
O_j - subclass of O - p. 12

Operators:

Theorems, Corollaries, Lemmas:

Introduction

Complex analysis, also known as the analysis of functions of complex variables, has a history spanning several centuries and is closely related to the development of mathematics.

The concept of complex numbers began to emerge in the 16th century, primarily due to the works of G. Cardano, who addressed the solutions to cubic equations, where he encountered square roots of negative numbers.

Between the 16th and 17th centuries, mathematicians such as R. Bombelli began to accept imaginary numbers (i, where i^2 = -1) and use them in calculations. The definition of complex numbers as ordered pairs of real numbers was introduced in 1836 by W. Hamilton. As stated by academician S. Marcus, "Complex numbers were not introduced simply out of a desire to extend the concept of numbers, but because mathematics, mechanics, and physics needed these numbers."

In the 18th and 19th centuries, complex analysis started to take shape as an essential part of mathematics, with significant contributions from L. Euler and A. L. Cauchy.

Cauchy formulated the fundamental theorem of complex analysis, which established connections between analytic functions and their integrals.

Complex analysis is one of the disciplines where the Romanian school of mathematics made important contributions, and it is also one of the classical branches of mathematics with wide applications in various fields of science and technology. Two important directions of complex analysis are the theory of conformal representations and the geometric theory of analytic functions.

The theory of functions of complex variables evolved in the 19th century, with the development of concepts such as holomorphic functions and the theory of residues. These notions are essential for evaluating complex integrals and applying complex analysis in other fields such as physics.

At the beginning of the 20th century, mathematicians such as H. Poincaré and K. Weierstrass extended the theories of complex analysis, and their applications spread into theoretical physics, engineering, and even communication theory.

The geometric theory of functions of a complex variable took shape as a distinct branch of complex analysis in the 20th century, thanks to important works in this field by mathematicians such as P. Köebe (1907), T. H. Gronwall (1914), J. W. Alexander (1915), and L. Bieberbach (1916).

Today, numerous treatises and monographs are dedicated to the study of univalent functions, including works by P. Montel, Z. Nehari, L. V. Ahlfors, Ch. Pommerenke, A. W. Goodman, P. L. Duren, D. J. Hallenbeck, T. H. Mac Gregor, S. S.

Miller, and P. T. Mocanu.

The problem of extending results from the geometric theory of functions of one complex variable to multiple complex variables was first formulated by H. Cartan in the appendix of P. Montel's book published in 1933. The extension of geometric properties for biolomorphic applications was initiated between 1960 and 1980 by Japanese mathematicians I. Ono, T. Higuchi, K. Kikuchi, and was revisited in the last two decades by J. A. Pfaltzgraff, T. J. Suridge, C. FitzGerald, S. Gong, I. Graham, G. Kohr, H. Hamada, P. Liczberski, P. Curt, T. Bulboacă, D. Breaz, M. Acu, Gr. St. Sălăgean, R. Szasz, L. I. Cotîrlă, D. Răducanu, B. Arpad, N. N. Pascu, and others.

In the work [8] (1916), L. Bieberbach formulated his famous conjecture, which bears his name, in which he demonstrated the exact estimate of the coefficient a_2 for functions in the class S, normalized and univalent functions in the unit disk U of the complex plane, that is, $|a_2| \leq 2$.

Bieberbach's conjecture, namely: if $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in S, z \in U$, then $|a_n| \leq n$, with equality if and only if f is a rotation of Köebe's function, was proven only in 1984 by Louis de Branges [11], during which research in the field of univalent functions was significantly propelled, one of the directions being the exact estimation of the coefficients for various subclasses of univalent functions.

At the same time, new research methods appeared and were developed, such as: L. Lowner's parametric method, the variational methods initiated by M. Schiffer [70], G. M. Goluzin [25], K. Sakaguchi [69], methods based on H. Grunsky's inequalities [29] and G. M. Goluzin's [26], the extreme functions method by L. Brickman [18], [19] and T. H. MacGregor [37], etc.

Romanian mathematicians also played an important role in the development of this field of mathematics.

G. Călugăreanu is the creator of the Romanian school of univalent function theory, which obtained the first necessary and sufficient conditions for univalence expressed with the help of coefficients, and P. T. Mocanu introduced the concept of α-convexity, addressed the problem of injectivity for non-analytic functions, and developed, together with S. S. Miller, the well-known method of studying certain classes of univalent functions, called the "method of admissible functions," "the method of differential subordinations," and more recently, "the theory of superdifferential subordinations."

The theory of univalent functions is important due to its numerous applications in various branches of natural sciences, such as theoretical physics (especially fluid mechanics, electricity, and heat theory) and engineering, as well as in many branches of mathematics, such as algebra, analytic number theory, differential equations, etc.

Integral operators were studied starting with the 20th century by several mathematicians, including J. W. Alexander, R. Libera, S. Bernardi, S. S. Miller, and more recently, P. T. Mocanu, M. O. Reade, R. Singh, R. Sijuk, E. Deniz, M. Caglar, H. Orhan, G. Murugusundaramoorthy, L. I. Cotîrlă, A. K. Wanas, and others.

The study of integral operators has seen continuous development, yielding many remarkable results over time.

In the work "Geometric Theory of Univalent Functions" [45], the authors T.

Bulboacă, P. T. Mocanu, and Gr. St. Sălăgean mention important results obtained from the class of univalent functions outside the unit disk, such as: "Area Theorem," "Coefficient Delimitation Theorem for Functions in the Class Σ," and the definitions of the classes of star-like and convex functions outside the unit disk.

These represented the starting point for the present work "Integral Transformations for Certain Classes of Univalent Functions," in which two chapters entirely cover the results related to the study of properties of univalent functions inside the unit disk and properties of meromorphic functions defined outside the unit disk.

In this work, new results were obtained regarding some subclasses of analytic functions. Properties of univalence, stellarity, and convexity are studied, both for known integral operators and for new integral operators.

The work includes an index of notations, an introduction, three chapters, conclusions, and a bibliography.

The first chapter, entitled "Basic Concepts and Preliminary Results," contains 6 paragraphs where basic concepts regarding functions of a complex variable and integral operators are introduced. These concepts will later be used in the proofs of the results in this work.

Thus, the notions of: holomorphic function, analytic function, univalent function, as well as the General Schwarz Lemma, which plays a key role in proving the main results, are defined. Furthermore, the main notions related to subordinations, the method of differential subordinations, and the class of admissible functions are described.

Next, several special classes of univalent functions are presented, including the class of star-like functions, the class of convex functions, the class of star-like functions of a certain order, and the class of convex functions of a certain order.

In the last two paragraphs of this chapter, some criteria for univalence and known integral operators in the literature are presented, concepts that will be used in proving the results of the following chapters.

Chapters two and three are dedicated to the contributions brought by the author in the field of Geometric Theory of Functions, some of the results being published, while others have been submitted or accepted for publication.

Chapter titled "Properties of Certain Univalent Integral Operators" contains, in the first subsection, the study of certain particular cases of the integral operator $F_\beta(f,g)(z)$ obtained in the paper "Properties of a New Integral Operator" by R. Bucur, L. Andrei, and D. Breaz. The author of this thesis also contributes by improving the conditions for univalence and membership of the operator $F_\beta(f,g)(z)$ in the class S.

In section 2.2, the univalence conditions of the integral operator $F_{n,\beta}(z)$ are discussed.

By applying N. N. Pascu's Criterion and Schwarz's General Lemma, new properties of this operator were found, which was introduced in the paper *Mapping properties of a new Integral Operator* by P. Dicu, R. Bucur, and D. Breaz.

Several univalence conditions of a new integral operator $G_{\beta,\gamma}(f,g)(z)$ are presented in section 2.3, the proofs of which were carried out with the help of N. N. Pascu's Criterion and Schwarz's Lemma.

A univalence criterion for the operator $G_{n,\beta}(z)$ is given in section 2.4, where it is defined as a generalization of the function of the given operator from section 2.2 of this work.

The following chapter, titled *"Properties of certain classes of meromorphic functions defined on the exterior unit disk"*, contains seven sections, in which properties and univalence conditions of meromorphic functions, respectively integral operators defined on the exterior unit disk, are described. The author has found sufficient conditions for univalence, convexity, and stellarity, conditions on the coefficients of certain classes of univalent meromorphic functions defined on the exterior of the unit disk, and for various subclasses of analytic functions, with examples provided.

In section 3.1, the properties of functions from the class of injective meromorphic functions of order γ, $O_1^*(\gamma)$, and functions from the class of convex meromorphic functions of order γ, $O_k(\gamma)$, are studied. Univalence conditions of certain integral operators formed from functions defined on the exterior unit disk are presented in sections 3.2, 3.3, and 3.6. The operators mentioned in the previous sections were formed starting from the operator $F_{\alpha_i,\beta}(z)$ introduced by N. Seenivasagan and D. Breaz in the paper [71], and the obtained results were published in journals such as Afrika Matematika [55], Journal of Advanced Mathematical Studies [57], Studia Universitatis Babeş-Bolyai Mathematica [58].

In section 3.5, certain values of the coefficients of the integral operator $E(z)$ were obtained, which represent a particular case of the operator $G_{\alpha_i,\beta}(z)$, proving the membership of the operator $E(z)$ in the class of meromorphic star-like functions of order 0, $O_1^*(0)$. These results were published in the journal General Mathematics [56].

The thesis concludes with a bibliography containing 77 titles, of which ten works are authored by the author, four of them being published, and six submitted or accepted for publication in prestigious journals in the field of Geometric Function Theory, both in the country and abroad, with results also presented at conferences. The paper "Univalence properties of an integral operator" was presented at the scientific event: "13th Joint Conference on Mathematics and Informatics, ELTE, Hungary, 1-3 October, 2020."

In conclusion, I would like to thank Prof. Univ. Dr. Valer Daniel Breaz for his guidance, support, and encouragement throughout the development and writing of this thesis.

I would also like to thank my colleagues with whom I collaborated in the study and development of the subject discussed.

Chapter 1

Notions and preliminary results

In this chapter, basic notions and results used in the development and proof of the found results are presented.

1.1 Definitions and notations

Let $\mathbb{C}$ be the set of complex numbers. The interior of the unit disk is denoted by $U = z \in \mathbb{C} : |z| < 1$, and the punctured interior of the unit disk is denoted by $U^* = z \in \mathbb{C} : 0 < |z| < 1 = U - 0$. The punctured disk centered at z_0 with radius $R > 0$ is denoted by $\dot{U}(z_0; R) = U(z_0, R) - z_0$.

The set of holomorphic functions in the unit disk U is denoted by $\mathbb{H}(U)$, and the set of univalent (holomorphic and injective) functions in the unit disk U is denoted by $\mathbb{H}_u(U)$.

Definition 1.1.1. *[45] Let $D \subset \mathbb{C}$ be an open set. A complex function $f : D \to \mathbb{C}$ is called holomorphic on D if f is differentiable at every point z_0 in D.*

The set of all holomorphic functions on D is denoted by $\mathbb{H}(D)$.

Definition 1.1.2. *[45] Let $M \subset \mathbb{C}$ be any set. A function $f : M \to \mathbb{C}$ is called holomorphic on the set $M \subset \mathbb{C}$ if there exists an open set D that contains M such that f is holomorphic on D.*

A holomorphic function on $\mathbb{C}$ is called an entire function.

Definition 1.1.3. *[30] Let f be a holomorphic function on the open set $G \subset \mathbb{C}$. A point $a \in G$ is called a zero of f if $f(a) = 0$. If there exists an $n \in \mathbb{N}^*$ such that $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$, then a is called a zero of order n of the function f. If $n = 1$, a is called a simple zero.*

Definition 1.1.4. *[30] Let f be a holomorphic function on the open set $G \subset \mathbb{C}$. A point $z_0 \in \mathbb{C}$ is called an isolated singular point of the function f if $z_0 \notin G$, but there exists a punctured neighborhood of z_0 included in G, i.e., there exists an $R > 0$ such that $\dot{U}(z_0; R) \subset G$.*

Definition 1.1.5. *[30] An isolated singular point z_0 of the function $f \in \mathbb{H}(G)$ is called a pole if $\lim_{z\to z_0} f(z) = \infty$; it is called an essential singular point if f does not have a limit at z_0.*

If z_0 is a pole of the function, then f can be extended to z_0 by defining $f(z_0) = \infty$. In this way, the function becomes continuous at z_0 in the topology of $\mathbb{C}_\infty$.

Theorem 1.1.1. *[30] If z_0 is an isolated singular point of the function $f \in \mathbb{H}(G)$, then the following statements are equivalent:*

a) z_0 is a pole.

b) z_0 is a regular point and specifically a zero of $\frac{1}{f}$.

c) There exists a unique $n \in \mathbb{N}^*$ such that in a punctured disk centered at z_0, the expansion

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n, a_{-n} \neq 0, \tag{1.1}$$

holds.

d) There exists a unique $n \in \mathbb{N}^*$ and a unique function $g \in \mathbb{H}(G \cup z_0)$ such that $g(z_0) \neq 0$ and

$$f(z) = (z-z_0)^{-n} g(z), \forall z \in G. \tag{1.2}$$

Let A be the class of analytic functions defined on the interior of the unit disk, normalized with the conditions: $f(0) = 0$ and $f'(0) = 1$.

The exterior of the unit disk is denoted by $W = z \in \mathbb{C}_\infty : |z| > 1$.

Let O be the class of analytic functions g defined on the exterior of the unit disk.

The subclass of O that contains univalent functions in W is denoted by Σ.

Let O_1 be the subclass of O containing meromorphic, normalized, and injective functions with a unique simple pole at $z = \infty$ in W, which have a Laurent series expansion of the form

$$g(z) = z + \sum_{k=3}^{\infty} \frac{b_k}{z^k}, 1 < |z| < \infty, \tag{1.3}$$

with $g(\infty) = \infty$ and $g'(\infty) = 1$.

The subclass of O that contains functions of the form

$$g(z) = z + \sum_{k=j+1}^{\infty} \frac{b_k}{z^k}, j \in \mathbb{N}1^* = \mathbb{N} - 0, 1, \tag{1.4}$$

is denoted by Oj.

Let S be the subclass of class A, which contains univalent functions on the unit disk satisfying the conditions: $f(0) = 0$ and $f'(0) = 1$.

We will denote by $S = f \in A : f$ is univalent in U.

Property 1.1.1. *[45] Every function $f \in A$ admits a power series expansion of the form*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in U. \tag{1.5}$$

Definition 1.1.6. *[45] A holomorphic and injective function on a domain D in $\mathbb{C}$ is called a univalent function on D.*

The set of univalent functions on D is denoted by $\mathbb{H}_u(D)$.

Definition 1.1.7. *[45] Let $f : D \to \mathbb{C}, z_0 \in D$. We say that the function f is analytic at the point z_0 or can be expanded in a Taylor series at z_0 if there exists a disk,*

$U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\} \subset D$ such that f is the sum of a Taylor series, i.e.:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, z \in U(z_0, r).$$

We say that f is analytic on the domain D if it is analytic at every point in D.

Theorem 1.1.2. *[45] (Theorem of the analyticity of holomorphic functions) A function $f : D \to \mathbb{C}$ is holomorphic on D if and only if f is analytic on D.*

Theorem 1.1.3. *[45] If $f \in \mathbb{H}_u(D)$, then $f'(z) \neq 0, \forall z \in D$.*

The converse of this theorem is not generally valid, as can be seen from the example of the function $z \mapsto e^z$, which is not univalent on $\mathbb{C}$ although its derivative does not vanish at any point in $\mathbb{C}$.

Note that for real differentiable functions, the non-vanishing of the derivative on an interval is a sufficient but not necessary condition for injectivity, as shown by the example $x \mapsto x^3 (x \in \mathbb{R})$.

This essential difference between the complex case and the real case is explained by the fact that for complex functions, the mean value theorem of Lagrange does not hold.

Theorem 1.1.4. *[45] If the function f is holomorphic in U and $|f(z)| < 1$ in U, then for any $\xi \in U$ and $z \in U$ the following inequalities hold:*

$$\left| \frac{f(\xi) - f(z)}{1 - \overline{f(z)} f(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \bar{z}\xi} \right|, \tag{1.6}$$

and:

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}. \tag{1.7}$$

Equality holds in the case where the function is:

$$f(z) = \frac{\varepsilon(z + t)}{1 + \bar{t}z},$$

where $|\varepsilon| = 1$ and $|t| < 1$.

Observation 1.1.1. *[45] For $z = 0$, the inequalities from Theorem 1.1.4 become:*

$$\left| \frac{f(\xi) - f(0)}{1 - \overline{f(0)} f(\xi)} \right| \leq |\xi|, \tag{1.8}$$

and:

$$|f(\xi)| \leq \frac{|\xi| + |f(0)|}{1 + |f(0)| \cdot |\xi|}. \tag{1.9}$$

Considering $f(0) = a$ *and* $\xi = z$, *we get:*

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a| \cdot |z|}, \forall z \in U. \tag{1.10}$$

An essential result from the theory of univalent functions is the Area Theorem, obtained by L. Bieberbach [9], [8], and later by T. Gronwall [27].

Theorem 1.1.5. *[45] (Area Theorem) If* $g(z) = z + \sum_{k=0}^{\infty} \frac{b_k}{z^k}$ *is a function from the class* O_1, *then the area is:*

$$E(g) = \pi \left(1 - \sum_{k=1}^{\infty} k|b_k|^2\right) \geq 0, \tag{1.11}$$

where $U^- = \{z \in \mathbb{C}_\infty : |z| > 1\}$ *and* $E(g) = \mathbb{C} - g(U^-)$.
Therefore, $\sum_{k=1}^{\infty} k|b_k|^2 \leq 1$, *and the area is understood in the sense of the two-dimensional Lebesgue measure.*

Using the area theorem, the following bound on the coefficients of functions from the class Σ is deduced.

Corollary 1.1.1. *[45] (Theorem of the coefficient bounds for functions in* Σ*) If* $g(z) = z + b_0 + \frac{b_1}{z} + \cdots \in \Sigma$, *then* $|b_1| \leq 1$, *and equality* $|b_1| = 1$ *holds if and only if* $g(z) = z + b_0 + \frac{e^{i\tau}}{z}$, *for* $z \in U^-, \tau \in \mathbb{R}$.

Theorem 1.1.6. *[45] (Bieberbach's Theorem on the coefficient* a_2*) If* $f \in S$, $f(z) = z + a_2 z^2 + \ldots$, *then* $|a_2| \leq 2$.

Equality $|a_2| = 2$ *holds if and only if* f *is of the form*

$$K_\tau(z) = \frac{z}{(1 + e^{i\tau} z)^2}, \tag{1.12}$$

where K_τ *is the Koebe function.*

Lemma 1.1.1. *[45] ([49]) (General Schwarz Lemma) Let* f *be a holomorphic function in the disk:*

$$U_R = \{z \in \mathbb{C} : |z| < R\},$$

with the property that:

$$|f(z)| < M, \text{ for fixed } M. \tag{1.13}$$

If f *has a zero of multiplicity greater than* m *at* $z = 0$, *then:*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, z \in U_R. \tag{1.14}$$

Equality in relation 1.14 holds for $z \neq 0$ *if and only if:*

$$f(z) = e^{i\tau} \frac{M}{R^m} z^m, z \in U_R, \tag{1.15}$$

where τ *is a constant.*

Lemma 1.1.2. *(Schwarz's Lemma) ([48], [49], [13]) If f is a holomorphic function on the unit disk $U = U(0,1)$ that satisfies the conditions $f(0) = 0$ and $|f(z)| < 1$ for all $z \in U$, then:*

$$|f(z)| \leq |z|, \forall z \in U,$$

and:

$$|f'(0)| \leq 1.$$

If $|f(z_0)| = |z_0|$, for $z_0 \in U$, or if $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for all $z \in U$.

It is known that there are the following relations between the class S and the class Σ:

Proposition 1.1.1. *[33]*

i) Let $f \in S$ and $g(\zeta) = \frac{1}{f\left(\frac{1}{\zeta}\right)}$, then $g \in \Sigma$ and $g(\zeta) \neq 0$, for $\zeta \in W$.

ii) If $g \in \Sigma$ and $g(\zeta) \neq 0$, for $\zeta \in W$, then $f \in S$, with $f(z) = \frac{1}{g\left(\frac{1}{z}\right)}$, for $z \in U$.

1.2 Method of differential subordination

The method of differential subordination represents a synthesis between functional analysis and complex geometry, serving as a powerful tool for investigating analytic functions. This method has significant applications in the study of univalent functions, conformal mappings, and the geometric theory of functions, allowing for the characterization and constraint of analytic functions through differential relations and the concept of subordination, which emerged initially in the 20th century.

In 1923, K. Loewner introduced a differential equation, the Loewner equation, to study univalent functions. This opened the path for the application of differential methods in the analysis of subordinations.

Loewner demonstrated that univalent functions can be characterized as solutions to a differential equation depending on a real parameter.

In the second half of the 20th century, the method was extended and formalized by mathematicians such as J. D. Miller, W. T. Scott, and B. Pommerenke. They combined the concept of subordination with differential equations to characterize large classes of analytic and univalent functions.

The method has become an essential tool in the study of classes of analytic functions, such as star-like and convex-univalent functions and other associated classes.

In modern theory, the method is used in complex geometry, holomorphic dynamics, and the analysis of flow models. Its historical development from the concepts of subordination and Loewner's equations to modern applications has demonstrated the versatility of this method in understanding and classifying complex functions.

Definition 1.2.1. *[45] Let $f, g \in \mathbb{H}(U)$. We say that function f is subordinated to function g (or g is superordinated to function f) and we denote:*

$$f \prec g \text{ or } f(z) \prec g(z),$$

if there exists a function $w \in \mathbb{H}(U)$*, with* $w(0) = 0$ *and* $|w(z)| < 1$*,* $z \in U$*, such that:*

$$f(z) = g[w(z)],\ z \in U.$$

Proposition 1.2.1. *[45]*
1) If $f \prec g$*, then* $f(0) = g(0)$ *and* $f(U) \subseteq g(U)$*.*

2) If $f \prec g$*, then* $f(\overline{U_r}) \subseteq g(\overline{U_r})$*,* $r < 1$*, equality holds if and only if* $f(z) = g(\lambda z), |\lambda| = 1$*.*

3) If $f \prec g$*, then* $\max\{|f(z)| : |z| \leq r\} \leq \max\{|g(z)| : |z| \leq r\}, r < 1$*, equality holds if and only if* $f(z) = g(\lambda z)$*,* $|\lambda| = 1$*.*

4) If $f \prec g$*, then* $|f'(0)| \leq |g'(0)|$*, equality holds if and only if* $f(z) = g(\lambda z), |\lambda| = 1$*.*

If the function g is univalent, we have the following theorem that characterizes the subordination relation.

Theorem 1.2.1. *[45] [66] Let* $f, g \in \mathbb{H}(U)$ *and assume that* g *is univalent in* U*. Then* $f \prec g$ *if and only if* $f(0) = g(0)$ *and* $f(U) \subseteq g(U)$*.*

Corollary 1.2.1. *[35] [66] (Lindelöf's Subordination Principle)*
Let the functions $f, g \in \mathbb{H}(U)$ *be such that* g *is univalent in* U *and* $f(0) = g(0)$*.*

1) *If* $f(U) \subseteq g(U)$*, then* $f(\overline{U_r}) \subseteq g(\overline{U_r})$*,* $0 < r < 1$*.*
2) *The equality* $f(\overline{U_r}) = g(\overline{U_r})$ *for some* $r < 1$ *holds if and only if* $f(U) = g(U)$*, or* $f(z) = g(\lambda z)$*,* $|\lambda| = 1$*.*

This corollary is a consequence of the previous theorem and the properties above, and represents a generalization of Schwarz's Lemma, with multiple applications in the geometric theory of analytic functions.

Proposition 1.2.2. *[45]*
Let $f, g \in \mathbb{H}_u(U)$*. If* $f \prec g$*, we have:*

1) $|g^{-1}(w)| \leq |f^{-1}(w)|$*, for any* $w \in f(U)$*.*
   Equality holds if and only if $f(z) = g(\lambda z)$*,* $|\lambda| = 1$*.*
2) *If in addition, there exists a* $z_0 \in U$ *with* $|z_0| = r$ *such that* $f(z_0) \in \partial g(U_r)$*, then* $f(U) = g(U)$*, or* $f(z) = g(\lambda z)$*,* $|\lambda| = 1$*.*

In the works [43], [42], S.S. Miller and P.T. Mocanu inaugurated the theory of differential subordination, which was later developed in many other works.

The method of *differential subordinations* (or the *method of admissible functions*) is one of the newest methods used in the geometric theory of analytic functions, having a significant merit both in simplifying the demonstration of already known results and in obtaining many new results. This method is a technique used in complex analysis and the theory of analytic functions to approximate solutions to various problems, based on the selection of a class of analytic functions that satisfy certain conditions imposed by the problem, and using these functions to construct approximate solutions.

The method of admissible functions has three stages:

- *Defining a class of functions* – a set of analytic functions is chosen that are compatible with the restrictions imposed by the problem (e.g., boundary conditions, regularity restrictions).
- *Constraints* – the functions must satisfy certain conditions, such as being analytic in a given domain and meeting the specific requirements of the problem.
- *Optimization* – a function from this class is sought that minimizes or maximizes a certain functional associated with the problem.

The *method of differential subordinations* is presented in general form below.

Let $\Omega, \Delta \subset \mathbb{C}, p \in \mathbb{H}(U)$ with $p(0) = a$, $a \in \mathbb{C}$, and $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. The problem is to study implications of the form:

$$\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\} \subset \Omega \Rightarrow p(U) \subset \Delta. \tag{1.16}$$

We note that the function ψ can also be considered with values in $\mathbb{C}_\infty$, i.e., $\psi : \mathbb{C}^3 \times U \to \mathbb{C}_\infty$, and the theory presented here remains valid for such a function ψ.

In relation to the implication (1.16), three types of problems can be formulated:

Problem 1) Given the sets Ω and Δ, find conditions on the function ψ such that the implication (1.16) holds. Such a function ψ is called an admissible function.

Problem 2) Given the function ψ and the set Ω, find the set Δ such that the implication (1.16) holds. Additionally, find the "smallest" set Δ with this property.

Problem 3) Given the function ψ and the set Δ, find the set Ω such that the relation (1.16) holds. Additionally, find the "largest" set Ω with this property.

If Ω and Δ are simply connected domains in $\mathbb{C}$, different from $\mathbb{C}$, the implication (1.16) can be rewritten in terms of subordination.

It is known that if Ω and Δ are simply connected domains in $\mathbb{C}$, different from $\mathbb{C}$, and $a \in \Delta$, then there exist conformal transformations:

$$q : U \to \Delta,\ q(U) = \Delta,\ q(0) = a,$$

and

$$h : U \to \Omega,\ h(U) = \Omega,\ h(0) = \psi(a, 0, 0; 0).$$

If, additionally, ψ is holomorphic in U, then (1.16) becomes:

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \Rightarrow p(z) \prec q(z). \tag{1.17}$$

Definition 1.2.2. *[45]*

1) *Let* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, *and the function* h *be univalent in* U. *If the function* $p \in \mathbb{H}[a,n]$ *satisfies the differential subordination:*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z),\ z \in U, \tag{1.18}$$

then the function p *is called an* (a,n) *solution of the differential subordination (1.18), or simply a solution of the differential subordination (1.18).*

2) *The subordination (1.18) is called a second-order differential subordination, and the univalent function* q *in* U *is called the* (a,n) *dominant of the solutions of the differential subordination (1.18), or more simply, the dominant of the differential subordination (1.18), if* $p(z) \prec q(z)$ *for any function* p *satisfying the relation (1.18).*

3) *A dominant* $\tilde{q}$ *such that* $\tilde{q}(z) \prec q(z)$ *for any dominant* q *of (1.18) is called the best* (a,n) *dominant, or simply the best dominant of the differential subordination (1.18).*

Remark 1.2.1. *[45]*

1) *The best dominant is unique, abstracting from a rotation in* U, *since if* $q_1 \prec q_2$ *and* $q_2 \prec q_1$, *then* $q_1(z) = q_2(e^{i\theta}z), \theta \in \mathbb{R}$.

2) *Let* Ω *be a set in* $\mathbb{C}$, *and assume that (1.18) is replaced with the relation:*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,\ z \in U.$$

Although this is a "differential inclusion" and $\psi(p(z), zp'(z), z^2p''(z); z)$ *may not be analytic in* U, *it is still called a second-order differential subordination.*

If Ω and Δ in the relation (1.16) are simply connected domains different from $\mathbb{C}$, Problems 1), 2), 3) can be reformulated as follows:

Problem 1') Given the univalent functions h and q, determine a class of admissible functions $\Psi[h,q]$ such that (1.17) holds.

Problem 2') Given the differential subordination (1.18), find a dominant q of it. Additionally, find the best dominant of it.

Problem 3') Given ψ and a dominant q, determine the largest class of univalent functions h such that (1.17) holds.

In 1962, K. Sakaguchi proved in the work [68] that if the function $p \in \mathbb{H}(U)$, $\Re(p(0)) > 0$ and $\alpha \in \mathbb{R}$, then:

$$\Re\left(p(z) + \alpha\frac{zp'(z)}{p(z)}\right) > 0,\ z \in U \Rightarrow \Re(p(z)) > 0,\ z \in U.$$

Next, we present an example mentioned in the work [45], where the choice of the function ψ with values in $\mathbb{C}_\infty$ is justified.

Example 1.2.1. *Let* $\Omega = \Delta = \{w \in \mathbb{C} : \Re(w) > 0\}$ *and consider* $\psi(r, s, t; z) = r + \alpha\frac{s}{r}$, *then the implication becomes:*

$$\left\{p(z) + \alpha\frac{zp'(z)}{p(z)} : z \in U\right\} \subset \Omega \Rightarrow p(U) \subset \Delta,$$

and this implication is of the form (1.16).

Thus, we observe that $\psi : \mathbb{C}^3 \times U \to \mathbb{C}_\infty$.

1.3 Fundamental theorems of certain classes of admissible functions

The fundamental theorems associated with the class of admissible functions represent a cornerstone of complex analysis. They allow for the precise characterization of analytic functions and provide powerful tools for understanding their geometric and analytic properties.

Their importance extends from fundamental theoretical problems to practical applications, such as conformal mapping and optimization of complex functions.

Definition 1.3.1. *[45] We denote by* Q *the set of functions* q *that are holomorphic and injective on* $\overline{U} \setminus E(q)$, *where:*

$$E(q) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\right\}, q'(\zeta) \neq 0, \zeta \in \partial U \setminus E(q). \tag{1.19}$$

The set $E(q)$ is called the exceptional set.

Remark 1.3.1. *[45] If* $q \in Q$, *then the domain* $\Delta = q(U)$ *is simply connected, and its boundary consists either of a single closed analytic curve or a union, possibly infinite, of disjoint simple analytic curves tending to infinity in both directions.*

Definition 1.3.2. *[42], [41], [45] Let* $\Omega \subset \mathbb{C}$, *and let the function* $q \in Q, n \in \mathbb{N}, n \geq 1$. *We say that* $\Psi_n[\Omega, q]$ *is the class of functions* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ *that satisfy the condition:*

$$\psi(r, s, t; z) \notin \Omega, \tag{1.20}$$

when:

$$r = q(\zeta), s = m\zeta q'(\zeta), \Re\left[\frac{t}{s} + 1\right] \geq m\Re\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right], \tag{1.21}$$

where $z \in U$, $\zeta \in \partial U \setminus E(q), m \geq n$.

The set $\Psi_n[\Omega, q]$ is called the class of admissible functions, and the condition in equation (1.20) is called the admissibility condition.

Remark 1.3.2. *[45] Let* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}_\infty$

1) If $\Omega \subset \tilde{\Omega}$, *then* $\Psi_n[\tilde{\Omega}, q] \subset \Psi_n[\Omega, q]$.

2) $\Psi_n[\Omega, q] \subset \Psi_{n+1}[\Omega, q]$.

3) In the particular case $\Psi : \mathbb{C}^2 \times U \to \mathbb{C}$, *the admissibility condition becomes:*

$$(A')\ \psi(r, s; z) \notin \Omega,$$

when:

$$r = q(\zeta), s = m\zeta q'(\zeta),$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$, $m \geq n$.

4) In the particular case $\psi : \mathbb{C} \times U \to \mathbb{C}$, *the admissibility condition becomes:*

$$(A'')\ \psi(r; z) \notin \Omega,$$

when:

$$r = q(\zeta),$$

where $z \in U, \zeta \in \partial U \setminus E(q)$.

5) We denote $\Psi_1[\Omega, q]$ *by* $\Psi[\Omega, q]$.

Theorem 1.3.1. *[44], [45] Let* $\psi \in \Psi_n[\Omega, q]$ *where* $q(0) = a$. *If the function* $p \in \mathbb{H}[a, n]$ *satisfies the condition:*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,\ z \in U,$$

then $p(z) \prec q(z)$.

In the special case when $\Omega \subset \mathbb{C}, \Omega \neq \mathbb{C}$ is a simply connected domain, and $h \in \mathbb{H}_u(U)$, $h(U) = \Omega$, denoting $\Psi_n[h(U), q]$ by $\Psi_n[h, q]$ we obtain:

Theorem 1.3.2. *[44], [45] Let* $h \in \mathbb{H}_u(U)$, $\psi \in \Psi_n[h, q]$ *where* $q(0) = a$. *If the function* $p \in \mathbb{H}[a, n]$ *and the function* $\psi(p(z), zp'(z), z^2p''(z); z) \in \mathbb{H}(U)$, *then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \Rightarrow p(z) \prec q(z).$$

Theorem 1.3.1 is used to show that the solutions of certain second-order differential equations take values in a specific domain, as we can observe in the following corollary:

Corollary 1.3.1. *[45] Let the function* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If the function* $f \in \mathbb{H}(U)$ *satisfies* $f(U) \subset \Omega$ *and if the differential equation*

$$\psi(p(z), zp'(z), z^2p''(z); z) = f(z),$$

has a solution $p \in \mathbb{H}[a, n]$, *then* $p(z) \prec q(z)$.

From Theorem 1.3.1 we obtain the following result:

Theorem 1.3.3. *[45] Let the function* $p \in \mathbb{H}[a, n], \Re(a) > 0$.

(i) If $\psi \in \Psi_n\{\Omega, a\}$, *then:*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,\ z \in U \Rightarrow \Re(p(z)) > 0,\ z \in U.$$

(ii) If $\psi \in \Psi_n\{a\}$, *then:*

$$\Re(\psi(p(z), zp'(z), z^2p''(z); z)) > 0,\ z \in U \Rightarrow \Re(p(z)) > 0,\ z \in U.$$

Next, we consider a particular case of Theorem 1.3.3.
Suppose that the function ψ is defined as $\psi : \mathbb{C}^2 \times U \to \mathbb{C}_\infty$. From Observation 1.3.2 we have the admissibility condition (A') and, furthermore, assuming that $\psi \in \Psi_n\{\Omega, 1\}$, we obtain from (A_0'') the following admissibility condition:

$$(A_0''')\ \psi(\rho i, \sigma; z) \notin \Omega,$$

when

$$\rho, \sigma \in \mathbb{R},\ \sigma \leq -\frac{n}{2}(1+\rho^2), z \in U, n \geq 1.$$

From Theorem 1.3.3 (point (i)) we obtain the following result:

Theorem 1.3.4. *[45] Let* $n \in \mathbb{N}^*$, $\Omega \subset \mathbb{C}$, $p \in \mathbb{H}[1, n]$ *and* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}_\infty$. *If the admissibility condition* (A_0''') *is satisfied, then:*

$$\psi(p(z), zp'(z); z) \in \Omega,\ z \in U \Rightarrow \Re(p(z)) > 0,\ z \in U.$$

Next, we will present some immediate applications of the admissible function method.

Theorem 1.3.5. *[45] Let* $p \in \mathbb{H}[a, n]$ *with* $\Re(a) > 0$ *and let* $P : U \to \mathbb{C}$ *be a function with* $\Re(P(z)) > 0,\ z \in U$. *If:*

$$\Re[p(z) + P(z)zp'(z)] > 0,\ z \in U,$$

then $\Re(p(z)) > 0,\ z \in U$.

The next theorem is a generalization of the previous result.

Theorem 1.3.6. *[45] Let* h *be a convex function in* U *and let the function* $P : U \to \mathbb{C}$ *with* $\Re(P(z)) > 0,\ z \in U$. *If* $p \in \mathbb{H}[h(0), 1]$, *then:*

$$p(z) + P(z)zp'(z) \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 1.3.1. *[45] Let* $p \in \mathbb{H}[a, n]$ *with* $\Re(a) > 0$ *and let* $\alpha : U \to \mathbb{R}$. *If:*

$$\Re\left[p(z) + \alpha(z)\frac{zp'(z)}{p(z)}\right] > 0,\ z \in U,$$

then $\Re(p(z)) > 0,\ z \in U$.

Lemma 1.3.2. *[42] Suppose the function* $\Psi :\ \mathbb{C}^2 \longrightarrow \mathbb{C}$ *satisfies the condition:*

$$\Re\{\Psi(is, t)\} \leq 0, \quad s, t \in \mathbb{R}; t \leq \frac{1+s^2}{2}.$$

If the function $p(z) = p(1) + \frac{p_1}{z} + ...$ *is analytic in* W *and*

$$\Re\{\Psi(z^2p(z) + 2, z^2\left(zp'(z) + 1\right))\} > 0, z \in W$$

then,

$$\Re(p(z)) > 0, z \in W.$$

1.4 Classes of functions

Let T denote the subclass of univalent functions that satisfy the condition:

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1, \quad z \in U;$$

T_2 is the subclass of univalent functions from the class T for which $f''(0) = 0$.

Let $T_{2,\mu}$ be the subclass of univalent functions in the class T_2 that satisfy the condition:

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq \mu, \quad z \in U,$$

for $0 < \mu \leq 1$, and we have that $T_{2,1} \equiv T_2$;

For a real number p with $0 < p \leq 2$, the subclass $S(p)$ of class A is defined, containing all functions that satisfy the condition:

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p, \quad z \in U.$$

In the work [73], S. Singh proved that if $f(z) \in S(p)$, then $f(z)$ satisfies the condition:

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p, z \in U. \tag{1.22}$$

Let V denote the subclass of univalent functions of O for which

$$\left|\frac{g'(z)}{z^2} + 1\right| > 1, \quad z \in W,\ g(z) \in V.$$

V_j is the subclass of V for which $g^{(k)}(\infty) = 0, k = 2, 3, \ldots, j$;

$V_{j,\mu}$ is the subclass of V_j containing functions of the form (1.3) that satisfy the relation:

$$\left|\frac{g'(z)}{z^2} + 1\right| > \mu, z \in W, \mu > 1. \tag{1.23}$$

We denote $V_{j,1} \equiv V_j$.

Let $p \in \mathbb{R}$ with $1 < p \leq 2$, and $\sum_j(p)$ be the subclass of O containing all functions $g \in O_j$ for which

$$\left|\left(\frac{g(z)}{z}\right)''\right| > p, \quad z \in W, \tag{1.24}$$

$$\left|\frac{g'(z)}{z^2} + 1\right| \geq \frac{p}{|z|^j}, \quad j \in \mathbb{N}_1^*. \tag{1.25}$$

We denote $\sum_2(p) \equiv \sum(p)$.

1.4.1 The class of star-like functions

The class of stellar functions is one of the most studied subclasses of univalent functions. They represent a starting point for exploring other classes of analytic functions and provide a solid foundation for the development of the theory of extreme coefficients, used in Bieberbach theory. This class was first studied by Alexander in the work [4].

At the beginning of the 20th century, through the works of L. Bieberbach and P. Koebe, stellar functions were identified as an important subclass of univalent functions. Bieberbach demonstrated significant connections between the properties of stellar functions and their coefficients, laying the groundwork for the Bieberbach conjecture, one of the most important conjectures in complex analysis.

In the second half of the 20th century, mathematicians like B. Pommerenke expanded the theory of stellar functions, exploring the connections with other classes of analytic functions, such as convex-univalent and spiral-like functions. Their research solidified the position of stellar functions in modern complex analysis, with applications in complex geometry and functional analysis.

Let f be a holomorphic function in U, satisfying the conditions $f(0) = 0$ and $f(z) \neq 0$, for $z \neq 0$. We denote by C_r the image of the circle $\{z \in \mathbb{C} : |z| = r, 0 < r < 1\}$ under the function f.

Definition 1.4.1. *[45] We say that C_r is a stellar curve with respect to the origin, or simply stellar, if the angle $\varphi = \varphi(r, \tau) = \arg f(re^{i\tau})$, which the radius vector of the point $f(z), z = re^{i\tau}$, makes with the positive real axis, is an increasing function of τ, as τ increases from 0 to 2π, i.e.:*

$$\frac{\partial \varphi}{\partial \tau} = \frac{\partial}{\partial \tau} \arg f(z) > 0, \quad z = re^{i\tau}, \tau \in (0, 2\pi). \tag{1.26}$$

We will say that f is stellar on the circle $\{z \in \mathbb{C} : |z| = r\}$ if C_r is a stellar curve.

Since $f(z) \neq 0$, for $z \neq 0$, we can write:

$$\text{Log} f(z) = \log |f(z)| + i \arg f(z), \quad z = re^{i\tau}.$$

Differentiating with respect to τ, and noting that:

$$\frac{\partial z}{\partial \tau} = \frac{\partial re^{i\tau}}{\partial \tau} = rie^{i\tau} = iz,$$

we obtain:

$$\frac{izf'(z)}{f(z)} = \frac{\partial}{\partial \tau} \log |f(z)| + i \frac{\partial}{\partial \tau} \arg f(z).$$

From this equality, we deduce that:

$$\frac{\partial}{\partial \tau} \arg f(z) = \text{Re} \left(\frac{zf'(z)}{f(z)}, \, z = re^{i\tau} \right). \tag{1.27}$$

Thus, condition (1.26) can be rewritten as:

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad |z| = r, \tag{1.28}$$

which expresses the necessary and sufficient condition for f to be stellar on the circle $\{z \in \mathbb{C} : |z| = r\}$.

Since the function $\frac{zf'(z)}{f(z)}$ is harmonic, it follows that if this inequality holds for $|z| = r$, it will also hold for $|z| \leq r$. From this, we deduce that if f is stellar on the circle $\{z \in \mathbb{C} : |z| = r\}$, it will also be stellar on any circle $\{z \in \mathbb{C} : |z| = r'\}$, where $0 < r' < r$.

Definition 1.4.2. *[45] The radius of stellarity of the function f is the number $r^*(f)$ defined by:*

$$r^*(f) = \sup\left\{r;\ Re\left(\frac{zf'(z)}{f(z)}\right) > 0, |z| \leq r\right\}. \tag{1.29}$$

If $r^*(f) \geq 1$, we say that the function f is stellar in the unit disk U, or simply stellar.

Observation 1.4.1. *[45]*

1) *The equality $Re\left(\frac{zf'(z)}{f(z)}\right) = 0$ for a point $z \in U$ cannot hold, because in this case, the function f would reduce to a constant, which would contradict the conditions imposed on the function f.*

2) *If f satisfies $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$, $|z| < 1$, then necessarily $f(z) \neq 0$ for $0 < |z| < 1$.*

3) *From the definition, it follows that f is stellar in U if and only if it is stellar on every circle $\{z \in \mathbb{C} : |z| = r, 0 < r < 1\}$.*

4) *The condition of stellarity $Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U$, does not ensure the univalence of the function f in the unit disk, so the problem arises of finding an additional condition that will guarantee the univalence of the function.*

   If we additionally assume the condition $f'(0) \neq 0$, then the condition $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ implies the univalence of the function f, as well as the fact that $f(U)$ is a domain stellar with respect to the origin, i.e., the segment joining any point in $f(U)$ with the origin is contained in $f(U)$.

Theorem 1.4.1. *[45] Let f be a holomorphic function in U with $f(0) = 0$. Then f is univalent and $f(U)$ is a domain stellar with respect to the origin if and only if $f'(0) \neq 0$ and*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \forall z \in U. \tag{1.30}$$

We denote by S^* the class of holomorphic functions in U with $f(0) = 0, f'(0) = 1$ and which are stellar with respect to the origin in U.

Thus, $S^* = \left\{f \in A,\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U\right\}$.

Observation 1.4.2. *[45] Using the definition of subordination, the class S^* can be defined as follows:*
If

$$f(z) = z + a_2 z^2 + \dots, \quad z \in U, \tag{1.31}$$

then $f \in S^*$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in U. \tag{1.32}$$

The name of deformation theorems relative to univalent functions comes from the fact that a conformal transformation can be viewed as a "deformation" of one domain into another.

Since the Koebe function $K_\tau(z) = \frac{z}{(1+e^{i\tau}z)^2}$, $\tau \in \mathbb{R}$ for a suitable choice of τ, is stellar, it follows that the deformation theorem for the class S also holds for the class S^*.

Theorem 1.4.2. *[45] (Deformation Theorem) If $f \in S^*$, then the following exact bounds hold:*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, \tag{1.33}$$

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}, \tag{1.34}$$

$$\frac{1-r}{1+r} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+r}{1-r}, \tag{1.35}$$

where $z \in U$, $|z| = r$, and the extremal function is the function of Köebe $f = K_\tau$, for a convenient choice of τ.

Note:

$$M[a,b] = \left\{ \mu : [a,b] \to \mathbb{R}_+, \mu \text{ increasing on } [a,b], \int_a^b d\mu(t) = \mu(b) - \mu(a) = 1 \right\}. \tag{1.36}$$

Theorem 1.4.3. *[45] The function $f(z) = z + a_2 z^2 + \dots$, $z \in U$ belongs to the class S^* if and only if there exists a function $\mu \in M[0, 2\pi]$ such that:*

$$f(z) = z \exp\left\{ -2 \int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t) \right\}, \ z \in U. \tag{1.37}$$

1.4.2 The class of convex functions

Convex functions were first studied by E. Study in the paper [75], followed by significant results in the Geometric Theory of Functions obtained by K. Löwner [36], T. H. Gronwall [27], and J. W. Alexander [4] in 1915.

Definition 1.4.3. *[45] The curve C_r is called convex if the angle*

$$\psi(r,\tau) = \frac{\pi}{2} + \arg z f'(z), \;\; z = re^{i\tau},$$

formed by the tangent to the curve C_r at the point $f(z)$ with the positive real axis is an increasing function of τ on $[0, 2\pi]$.

Definition 1.4.4. *[45] The function f is said to be convex on the circle $\{z \in \mathbb{C} : |z| = r\}$ if C_r is a convex curve.*

It is shown that f is convex on the circle $\{z \in \mathbb{C} : |z| = r\}$ if and only if:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \;\; |z| = r. \tag{1.38}$$

From this definition, we deduce that if f is convex on the circle $\{z \in \mathbb{C} : |z| = r\}$, then it will be convex on any circle $\{z \in \mathbb{C} : |z| = r'\}$, where $0 < r' < r$.

Definition 1.4.5. *[45] The radius of convexity of the function f is defined as the number:*

$$r^c(f) = \sup\left\{r; \Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, |z| \le r\right\}. \tag{1.39}$$

Observation 1.4.3. *[45] If $r^c(f) \ge 1$, we will say that the function f is convex in the unit disk U or, more simply, convex.*

This means that f satisfies the condition:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \;\; |z| < 1. \tag{1.40}$$

Observation 1.4.4. *[45] The relation (1.40) implies that $f'(z) \ne 0$, for any $0 < |z| < 1$.*

Observation 1.4.5. *[45] The condition:*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \;\; z \in U,$$

does not ensure the univalence of the function f in the unit disk as shown by the example:

$$f(z) = z^2.$$

Next, we will present a sufficient condition for univalence:

Theorem 1.4.4. *[45] A function f holomorphic in U is univalent and $f(U)$ is a convex domain if and only if $f'(0) \ne 0$ and*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \;\; z \in U. \tag{1.41}$$

Definition 1.4.6. *[45] We say that S^c is the class of holomorphic functions f in U, with the property that $f(0) = 0,\ f'(0) = 1$ and are convex in U.*

We will denote

$$S^c = \left\{ f \in A, \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U \right\}$$

and $S^c \subset S$.

The relationship between the classes S^* and S^c is given by the following theorem.

Theorem 1.4.5. *[45] (Alexander's Duality Theorem)*

Let $f \in A$ and $g(z) = zf'(z)$. Then $f \in S^c$ if and only if $g \in S^*$.

The integral operator $I_A : A \to A,\ f = I_A(g),\ g \in A$, where:

$$f(z) = \int_0^z \frac{g(t)}{t} dt,\ z \in U,$$

is called the Alexander operator.

Using this operator, we can reformulate Theorem 3.7.3 as $S^c = I_A(S^*)$, and I_A establishes a bijection between S^* and S^c.

Other relationships between the classes S^* and S^c can also be established, such as the one in the following theorem.

Theorem 1.4.6. *[38], [45], [74] (Theorem of A. Marx and E. Strohhäcker)*

If $f \in A$, then the following implications hold:

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,\ z \in U \Rightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2},\ z \in U \Rightarrow \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2},\ z \in U,$$

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,\ z \in U \Rightarrow \Re\left(\sqrt{f'(z)}\right) > \frac{1}{2},\ z \in U \Rightarrow \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2},\ z \in U.$$

Observation 1.4.6. *[45] The function $f(z) = \frac{z}{1-z}$ shows that all these implications are exact.*

Therefore, we have $S^c \subset S^*(1/2)$.

Regarding the coefficients of functions in the class S^c, the following theorem holds.

Theorem 1.4.7. *[45] If $f(z) = z + a_2z^2 + a_3z^3 + ...$ belongs to the class S^c, then $|a_n| \leq n$, for any $n \geq 2$. Equality holds if and only if the function f has the form:*

$$f(z) = \frac{z}{1 + e^{i\tau}z}, \tau \in \mathbb{R},\ z \in U.$$

The following deformation theorem holds.

Theorem 1.4.8. *[45] If $f \in S^c$, then the following exact bounds hold:*

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}, \tag{1.42}$$

$$\frac{1}{(1+r)^2} \leq |f'(z)| \leq \frac{1}{(1-r)^2}, z \in U, |z| = r < 1. \tag{1.43}$$

Equality holds for the function $f(z) = \frac{z}{1+e^{i\tau}z}, \tau \in \mathbb{R}, z \in U$, for a convenient choice of τ.

From relation (1.42) it follows that S^c is compact.

Observation 1.4.7. *[45] Applying $r \to 1$ in (1.42), we deduce the constant of Köebe for the class S^c, which is 1/2.*

1.4.3 The class of star-like functions and the class of convex functions of a certain order

Among the subfamilies of the class S^*, we mention the class of star-shaped functions of order α, $0 \leq \alpha < 1$, denoted by $S^*(\alpha)$, and the class of strongly star-shaped functions of order α, $0 < \alpha \leq 1$, denoted by $S^*(\alpha)$.

Definition 1.4.7. *[45] A function $f \in A$ is called star-shaped of order $\alpha, 0 \leq \alpha < 1$, if it satisfies the inequality:*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in U. \tag{1.44}$$

We denote by $S^*(\alpha)$ the class of these functions.

Definition 1.4.8. *[45] A function $f \in A$ is called strongly star-shaped of order $\alpha, 0 < \alpha \leq 1$, if it satisfies the inequality:*

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2}, \ z \in U. \tag{1.45}$$

It is observed that $S^*(0) = S^*$ and $S^*(1) = S^*$.

Definition 1.4.9. *[45] A function $f \in A$ is convex of order α, $0 \leq \alpha < 1$, if it satisfies the inequality:*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in U. \tag{1.46}$$

We denote by $S^c(\alpha)$ the class of convex functions of order $\alpha, 0 \leq \alpha < 1$, where

$$S^c(\alpha) = \left\{f \in A : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in U\right\}. \tag{1.47}$$

It is observed that for $\alpha = 0$, we have $S^c(0) = S^c$.

Theorem 1.4.9. *[45] (Duality Theorem Between the Classes $S^*(\alpha)$ and S^*)*
Let α be a real number with $0 \leq \alpha < 1$.
1) We have the inclusions $S^*(\alpha) \subset S^*$, $S^c(\alpha) \subset S^c$.
2) A function $f \in S^*(\alpha)$ if and only if the function $g \in S^*$, where

$$g(z) = z\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\alpha}},$$

where by $\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\alpha}}$ we understand the holomorphic branch for which $\left.\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\alpha}}\right|_{z=0} = 1$.

Observation 1.4.8. *[45] For $0 \leq \alpha < 1$, it is easily verified that a function $f \in S^c(\alpha)$ if and only if the function $g(z) = zf'(z) \in S^*(\alpha)$, and applying the above theorem, we deduce the following duality result between the classes $S^c(\alpha)$ and S^*.*

Corollary 1.4.1. *[45] If $0 \leq \alpha < 1$, then a function $f \in S^c(\alpha)$ if and only if the function $g \in S^*$, where*

$$g(z) = z\,[f'(z)]^{\frac{1}{1-\alpha}}, z \in U.$$

Theorem 1.4.10. *[45] (Deformation theorem for the class $S^c(\alpha)$)*
If the function $f \in S^c(\alpha)$, $0 \leq \alpha < 1$, and $|z| = r < 1$, then the following exact bounds hold:

$$\frac{1}{(1+r)^{2(1-\alpha)}} \leq |f'(z)| \leq \frac{1}{(1-r)^{2(1-\alpha)}},$$

$$\left.\begin{matrix} \alpha \neq \frac{1}{2}, & \frac{(1+r)^{2\alpha-1}-1}{2\alpha-1} \\ \alpha = \frac{1}{2}, & \log(1+r) \end{matrix}\right\} \leq |f(z)| \leq \left\{\begin{matrix} \frac{1-(1-r)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq \frac{1}{2} \\ -\log(1-r), & \alpha = \frac{1}{2}. \end{matrix}\right.$$

The extremal function is:

$$f_{\alpha(z)} = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq \frac{1}{2} \\ -\log(1-z), & \alpha = \frac{1}{2}. \end{cases}$$

Theorem 1.4.11. *[45] (Deformation theorem for the class $S^*(\alpha)$) If the function $f \in S^*(\alpha)$, $0 \leq \alpha < 1$ and $|z| = r < 1$, then the following exact bounds hold:*

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leq |f(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}}.$$

The extremal function is $f_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$.

1.4.4 Star-like and convexity conditions for several classes of meromorphic functions

This paragraph includes concepts from the specialized literature that I used to obtain new foundational results related to meromorphic functions.

In various complex analysis problems, it is necessary to extend the set $\mathbb{C}$ of complex numbers by adding an improper number denoted by ∞, where by definition, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}, \infty \notin \mathbb{C}$.

The relationship between the numbers in $\mathbb{C}$ and the element ∞ is established by extending the operations with complex numbers to this element, such that $a + \infty = \infty + a = \infty$ and $a \cdot \infty = \infty \cdot a = \infty$ for $a \in \mathbb{C}_\infty \setminus \{0\}$.

By special convention regarding division, we write $a/0 = \infty$ for $a \in \mathbb{C}_\infty \setminus \{0\}$ and $a/\infty = 0$ for $a \in \mathbb{C}$.

The operations $\infty - \infty, 0 \cdot \infty, 0/0, \infty/\infty$ are not defined.

Thus, regarding the algebraic structure of $\mathbb{C}_\infty$, the algebraic operations from $\mathbb{C}$ can be extended without being defined everywhere.

By convention, $|\infty| = +\infty$ extends the modulus from $\mathbb{C}$ to $\mathbb{C}_\infty$.

To study a function f in a neighborhood of the point ∞, we will consider the function $g = f \circ k$, where $k(z) = \frac{1}{z}$. Since k transforms a neighborhood of 0 into a neighborhood of ∞, by examining the behavior of f at ∞, we understand the behavior of g at 0.

Definition 1.4.10. *[30] Let $\tilde{G}$ be an open set in $\mathbb{C}$ or $\mathbb{C}_\infty$. We say that f is a meromorphic function on $\tilde{G}$ if there exists a set $E \subset \tilde{G}$ such that $f \in H(\tilde{G} \setminus E)$, and E consists of removable singularities or poles for the function f.*

Denoting by G the set of regular points and by B the set of poles in $\tilde{G}$, we have $\tilde{G} = G \cup B$.

Remark 1.4.1. *[17] A meromorphic function is a uniformly analytic function in the complex plane $\mathbb{C}$ that has no singularities other than poles.*

Entire functions, on the one hand, and rational functions, on the other hand, are particular cases of meromorphic functions.

The point ∞ for a meromorphic function can be ordinary, a pole, or essential, isolated or an accumulation point of poles.

Since poles are isolated singular points, it follows that a meromorphic function cannot have more than countably many poles in $\mathbb{C}$, which must accumulate at infinity.

Remark 1.4.2. *[17] A meromorphic function in a domain is a uniformly analytic branch corresponding to that domain, which admits only poles as singularities within the domain. These can be a finite number or countably infinite, but in the latter case, they must accumulate on the boundary of the domain.*

We will denote by $M(\tilde{G})$ the set of meromorphic functions on $\tilde{G}$.

If $f \in M(\tilde{G})$, then f can be extended to any point $z_0 \in \tilde{G}$ by $\tilde{f}(z_0) = \lim_{z \to z_0} f(z)$.

The function $\tilde{f} : \tilde{G} \to \mathbb{C}_\infty$ is $\mathbb{C}_\infty$-continuous and $\tilde{f} \in \mathbb{H}(G)$. Sometimes, the extended function is still denoted by f.

Several examples are presented below.

Example 1.4.1. *[30] Any holomorphic function on $\tilde{G}$ is also meromorphic, i.e., $\mathbb{H}(\tilde{G}) \subset M(\tilde{G})$.*

In this case, $B = \emptyset$ and $\tilde{G} = G$.

Example 1.4.2. *[30] Any rational function is meromorphic on $\mathbb{C}_\infty$.*

Example 1.4.3. *[30] The function ctg is meromorphic on $\mathbb{C}$, with poles at $z_k = k\pi$, $k \in \mathbb{Z}$;*

The point ∞ is an accumulation point of poles, so the function *ctg* cannot be meromorphic on $\mathbb{C}_\infty$.

Example 1.4.4. *[30] The function $tg\frac{1}{z}$ is meromorphic on $\mathbb{C}_\infty \setminus \{0\}$, since ∞ is a regular point, and the points $z_k = \frac{2}{(2k+1)\pi}$ are poles, which accumulate at the origin.*

The study of meromorphic and univalent functions can be done in parallel with class S, considering the class Σ_u of meromorphic functions φ with the unique pole (simple) $z = \infty$ and univalent in $U^- = \{z \in \mathbb{C}_\infty : |z| > 1\}$, which have Laurent series expansion of the form:

$$g(z) = z + \alpha_0 + \frac{\alpha_1}{z} + \cdots + \frac{\alpha_n}{z^n} + \ldots, |z| > 1. \qquad (1.48)$$

Thus, functions $g \in \Sigma_u$ are normalized with the conditions $g(\infty) = \infty, g'(\infty) = 1$.

Denoting

$$E(g) = \mathbb{C} \setminus g(U^-),$$

this will be a continuum in $\mathbb{C}$, i.e., a compact and connected set containing more than one point.

The subclass of functions $g \in \Sigma_u$ that do not vanish outside the unit disk is denoted as Σ_0, i.e.,

$$\Sigma_0 = \{g \in \Sigma_u : g(z) \neq 0, z \in U^-\},$$

and thus the following property easily follows.

Property 1.4.1. *[45] There is a bijection between the classes S and Σ_0, so the class Σ_0 is "larger" than the class S.*

It is observed that if $g \in \Sigma_u$ and $c \in E(g)$, then the function:

$$f(z) = \frac{1}{g\left(\frac{1}{z}\right) - c} = z + (c - \alpha_0)z^2 + \ldots, z \in U, \qquad (1.49)$$

has the property that $f \in S$.

Definition 1.4.11. *[45] We say that a function g of the form (1.48) is stellar in U^- if g is univalent in U^- and the set $E(g)$ is stellar with respect to the origin.*

We denote by Σ^* the class of stellar functions outside the unit disk, i.e.,

$$\Sigma^* = \{g \in \Sigma_0 : g \text{ is stellar in } U^-\}.$$

The transformation T is a bijection, $T(S) = \Sigma_0$ and $T^{-1}(\Sigma_0) = S$.

It follows that

$$\frac{zg'(z)}{g(z)} = \frac{zf'(z)}{f(z)}, z = \frac{1}{z}, z \in U^-,$$

from which it will result that the function $g \in \Sigma^*$ if and only if $f \in S^*$.

Thus, we deduce that $g \in \Sigma^*$ if and only if:

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > 0, z \in U^-.$$

In conclusion, we have:

$$\Sigma^* = \{g \in \Sigma_0 : \Re\left(\frac{zg'(z)}{g(z)}\right) > 0, z \in U^-\}, \quad \Sigma^* = T(S^*).$$

From Definition 1.4.11, it follows that if g is stellar, then $E(g)$ is a stellar set with respect to the origin, meaning $0 \in E(g)$, i.e., $g \in \Sigma_0$ (the set of meromorphic functions normalized, univalent that do not vanish in U^-).

Definition 1.4.12. *[46] We say that a function $f \in S$ is meromorphic stellar of order α, with $0 \leq \alpha < 1$, and belongs to the class $S^*(\alpha)$, if it satisfies the inequality:*

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha.$$

Definition 1.4.13. *[45] Let the function $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n + \ldots, 0 < |z| < 1$, be a meromorphic function in U. We say that the function g is stellar in U if the function $g(z) = f\left(\frac{1}{z}\right)$, for $z \in U^-$, is stellar in U^-.*

Theorem 1.4.12. *[45] (The Analytic Characterization Theorem of the Starlikeness of Meromorphic Functions) Let $f(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + ...$, $0 < |z| < 1$, be a meromorphic function in U with $f(z) \neq 0$, $z \in U$. Then, f is starlike in U if and only if f is univalent in U and*

$$\Re\left(-\frac{zf'(z)}{f(z)}\right) > 0, z \in U.$$

Definition 1.4.14. *[45] We say that the function g of the form (1.48) is convex in U^- if g is univalent in U^- and the set $E(g)$ is convex.*

We mention that if g is convex in U^-, then it is not necessarily starlike, as g may vanish in U^-, i.e., $0 \notin E(g)$.

If $g \in \Sigma_0$ and g is a convex function, then it is evidently also starlike in U^-.

We denote by Σ^c the class of functions that are convex in the exterior of the unit disk and do not vanish in U^-, i.e.,

$$\Sigma^c = \{g \in \Sigma_0 : g \text{ is convex in } U^-\}.$$

It is evident that $\Sigma^c \subset \Sigma^*$.

Remark 1.4.3. *[45] It is known that a function $g \in \Sigma_0$ is convex in the exterior of the unit disk if and only if it satisfies the condition:*

$$\Re\left(\frac{zg''(z)}{g'(z)}+1\right) > 0, z \in U^-.$$

Thus,

$$\Sigma^c = \left\{g \in \Sigma_0 : \Re\left(\frac{zg''(z)}{g'(z)}+1\right) > 0, z \in U^-\right\}.$$

Definition 1.4.15. *[45] Let f be a meromorphic function in U,*

$$f(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + ... + \alpha_n z^n + ..., 0 < |z| < 1.$$

We say that f is convex in U if the function $g(z) = f\left(\frac{1}{z}\right), z \in U^-$ is convex in U^-.

Theorem 1.4.13. *[45] (Analytic characterization theorem of convexity for meromorphic functions) Let $f(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \ldots, 0 < |z| < 1$, a meromorphic function in U with $f(z) \neq 0, z \in U$. Then f is convex in U if and only if f is univalent in U and*

$$\Re\left\{-\left(\frac{zf''(z)}{f'(z)}+1\right)\right\} > 0,\ z \in \dot{U}.$$

Definition 1.4.16. *[46], [31] We say that a function $f \in S$ is convex, meromorphic of order $\alpha, 0 \leq \alpha < 1$, if it satisfies the inequality:*

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, z \in U.$$

We denote by $S_k(\alpha)$ the class of meromorphic, convex functions of order $\alpha, 0 \leq \alpha < 1$,

$$S_k(\alpha) = \left\{f \in S : -\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, z \in U\right\}. \tag{1.50}$$

1.5 Integral operators

Integral operators have played an essential role in the development of complex analysis, providing a powerful framework for solving problems related to analytic functions, conformal transformations, and differential equations. Starting with Cauchy's works in the 19th century, those who initiated the study of integral operators include: J. W. Alexander, R. Libera, S. Bernardi, S. D. Miller, P.T. Mocanu, R. Singh, M. O. Reade, and others.

The study of integral operators remains relevant, as evidenced by numerous works in recent years [12], [15], [16], [24], [77], etc., and the numerous citations of already existing works.

We say that an integral operator is univalent if it transforms univalent functions into univalent functions. The star/convex integral operator is one that transforms star functions into star functions/convex functions into convex functions.

A central problem in the theory of complex-variable functions is the study of integral operators defined on certain subclasses of these functions.

The first integral operator was introduced in 1915 by the mathematician J. W. Alexander in [4]. The Alexander integral operator I_A is defined in [4] as

$$I_A : A \to A, I_A(F) = f,$$

where:

$$I_A(F) = f(z) = \int_0^z \frac{F(t)}{t} dt, \tag{1.51}$$

For this integral operator, Alexander proved that $I_A(S^*) \subset S^*$.

In 1965, R. J. Libera defined in his work [34] the following integral operator:

$$L : A \to A, L_f(z) = \frac{2}{z} \int_0^z f(t)dt, \tag{1.52}$$

called the Libera operator, and he demonstrated that $L_A(S^*) \subset S^*$.

S. D. Bernardi in [7] introduced a generalization of the Libera operator,

$$I_a : A \to A, I_a(F) = f, a = 1, 2, 3, \ldots, \text{ where:}$$

$$f(z) = \frac{1+a}{z^a} \int_0^z F(t)t^{a-1}dt, \tag{1.53}$$

and this was called the Bernardi integral operator, demonstrating that $I_a(S^*) \subset S^*$.

A few years later, in 1963, W. M. Causey introduced the operator:

$$J_4(f)(z) = \int \left[\frac{f(t)}{t}\right]^\alpha dt. \tag{1.54}$$

The operator J_4 was studied by S. S. Miller, P. T. Mocanu, and M. O. Reade, who later provided a generalization of the operator in their work [40].

Numerous generalizations of the previous operators have been studied, among which the most general form using only one function under the integral sign is given by the operator L_a.

This integral operator L_a is defined in [52] as

$$f(z) = \frac{1+a}{z^a} \int_0^z F(t)t^{a-1}dt, \tag{1.55}$$

where $a \in \mathbb{C}, \Re(a) \geq 0$.

It was introduced in this general form where $a \in \mathbb{C}, \Re(a) \geq 0$, by N. N. Pascu in [52], and was named the Libera-Pascu integral operator by D. Blezu in his work [10].

The integral operator $I_{c+\delta}: A \to A$, where $0 < u \leq 1\,,\ 1 \leq \delta < \infty\,,\ 0 < c < \infty\,,$ is defined in [2] as:

$$f(z) = I_{c+\delta}(F)(z) = (c+\delta)\int_0^1 t^{c+\delta-2}F(tz)dt\,. \tag{1.56}$$

Observation 1.5.1. *[22] For $\delta = 1$ and $c = 1, 2, \ldots$ from the integral operator $I_{c+\delta}$, given by the relation (1.56), we obtain the Bernardi integral operator defined by the relation (1.53).*

Observation 1.5.2. *[22] Let $F(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$. From the relation (1.56) we obtain:*

$$f(z) = z + \sum_{j=2}^{\infty} \frac{c+\delta}{c+j+\delta-1} a_j z^j.$$

We observe that

$$0 < \frac{c+\delta}{c+j+\delta-1} < 1,$$

where $0 < c < \infty,\ j \geq 2,\ 1 \leq \delta < \infty$.

Observation 1.5.3. *[22] For $F \in T, f = I_{c+\delta}(F)$, we have $f \in T$, where $I_{c+\delta}$ is the integral operator defined by the relation (1.56).*

Definition 1.5.1. *[22] Let $F \in A$, $F(z) = z + b_2 z^2 + \cdots + b_n z^n + \ldots$, $b_j \geq 0, j \geq 2$ and $a \in \mathbb{R}^*$. We define the integral operator $L : A \to A$ by the relation:*

$$f(z) = L(F)(z) = \frac{1+a}{z^a}\int_0^z F(t)\cdot t^{a-1} + t^{a+1}dt. \tag{1.57}$$

In the work [23], P. Dicu introduces a new integral operator:

$$I_n(z) = \int_0^z \prod_{i=1}^{n} \left[\frac{e^{f_i(t)}}{g_i'(t)}\right]^{\alpha_i} dt, \tag{1.58}$$

where the parameters $\alpha_i \in \mathbb{C}$, $\Re(\alpha_i) > 0$ and the functions $f_i, g_i \in A, i \in \{1, 2, ..., n\}$ are restricted (constrained by appropriate restrictions).

The integral operator I_n generalizes the integral operator:

$$I_1(z) = \int_0^z \left[\frac{e^{f(t)}}{g'(t)}\right]^{\alpha} dt. \tag{1.59}$$

Let $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ be the integral operator studied by N. Seenivasagan and D. Breaz in the work [71]:

$$F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left[\frac{f_i(t)}{t}\right]^{\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}}, \tag{1.60}$$

with $f_i(t) \in T_2$, T_2 being a subclass of T.

If $\alpha_i = \alpha, \forall i = 1, 2, ...,$ then $F_{\alpha_i,\beta}(z)$ becomes the operator $F_{\alpha,\beta}(z)$ [14].

1.6 Univalence criteria

Univalence criteria are a cornerstone in complex analysis, providing essential tools for understanding and classifying analytic functions. They allow for the identification and classification of injective (univalent) functions in specific regions of the complex plane. The univalence property is crucial because it ensures that analytic functions preserve local geometric structure and are free of ambiguities in their representations. These criteria evolved from the classical theorems of P. Koebe (around 1907, achieving a major breakthrough with the formulation of the univalence theorem and the study of univalent functions defined on the unit disk) and Bieberbach to modern methods based on geometry and differential equation analysis.

The first criteria were developed to analyze injectivity through derivatives and other properties of analytic functions. Some of the most well-known univalence criteria include:

- Schwarz's criterion, which establishes the univalence of analytic functions using the Schwarzian derivative,

- Nehari's criterion (1949), which links univalence to conditions on the curvature of the images of the analytic function.

The study of univalence criteria led to the discovery of special functional spaces such as the Hardy space and the Bergman space. Univalent analytic functions are used to perform conformal transformations, which preserve angles and the local structure of the domain. These transformations are fundamental in complex geometry and in practical applications such as electrical network modeling and fluid flow analysis. In contemporary analysis, univalence criteria are applied to the study of complex dynamics, fractals, and spectral theories associated with analytic operators.

In 1972, S. Ozaki and M. Nunokawa in the work [50] demonstrated the following result:

Theorem 1.6.1. *([50]) If $f \in T$ satisfies the following condition:*

$$\left| \frac{z^2 \cdot f'(z)}{f^2(z)} - 1 \right| \le 1, \forall z \in U,$$

then the function f is univalent in U.

The following theorem demonstrates a univalence condition given by N. Pascu in the work [52].

Theorem 1.6.2. *[52] Let $\alpha, \beta \in \mathbb{C}$ and $\Re(\beta) \ge \Re(\alpha) \ge \frac{3}{|\alpha|}$. If $f \in T_2$ satisfies the condition:*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, |f(z)| \le 1; \forall z \in U,$$

then the integral operator $H_{\alpha,\beta}(z)$ *defined by*

$$H_{\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\frac{1}{\alpha}} dt\right]^{\frac{1}{\beta}}$$

belongs to the class S.

Regarding the class of analytic functions, Becker in [5] demonstrated in 1972, using the Löwner chain method, the following univalence criterion.

Theorem 1.6.3. *[5] If the function* f *is regular in the unit disk* U, *with the properties:*

$$f(z) = z + a_2z^2 + ...,$$

and

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \forall z \in U,$$

then f *is univalent in* U.

A year later, Ahlfors in [3] and J. Becker in [6] generalized Becker's criterion, given by the following theorem.

Theorem 1.6.4. *[3] [6] Let* c *be a complex number,* $|c| \leq 1$, $c \neq -1$. *If* $f(z) = z + a_2z^2 + ...$ *is a function regular in* U *and*

$$\left|c|z|^2 + (1-|z|^2)\frac{zf''(z)}{f'(z)}\right| \leq 1, \forall z \in U,$$

then the function f *is regular and univalent in* U.

V. Pescar in [64] found a new univalence criterion (a generalization of Ahlfors and Becker's univalence criterion given in Theorem 1.6.4), given by the following theorem.

Theorem 1.6.5. *[64] Let* α *and* c *be two complex numbers,* $\Re(\alpha) > 0, |c| \leq 1$, $c \neq -1$. *If* $f(z) = z + a_2z^2 + ...$ *is a function regular in* U *and*

$$\left|c|z|^{2\alpha} + (1-|z|^{2\alpha})\frac{zf''(z)}{\alpha f'(z)}\right| \leq 1, \forall z \in U,$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z t^{\alpha-1} f'(t)dt\right]^{\frac{1}{\alpha}} = z + a_2z^2 + ...,$$

is regular and univalent in U.

In [53] N. N. Pascu and I. Radomir obtained the following result.

Theorem 1.6.6. *[53] Let β and c be two complex numbers, $\Re(\beta) > 0, |c| \leq 1$, $c \neq -1$ and $f(z) = z + a_2z^2 + ...$ be a function regular in U. If:*

$$\left| ce^{-2t\beta} + (1 - e^{-2t\beta}) \frac{e^{-t}zf''(e^{-t}z)}{\beta f'(e^{-t}z)} \right| \leq 1,$$

is preserved for $z \in U$ and $t \geq 0$, then the function

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)dt \right]^{\frac{1}{\beta}} = z + a_2z^2 + ...$$

is regular and univalent in U.

Theorem 1.6.7. *[51] Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. If f is an analytic function in U with the property that*

$$\left| \frac{1 - e^{-2t\alpha}}{\alpha} \cdot \frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)} \right| \leq 1, \forall z \in U, t \geq 0,$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u)du \right]^{\frac{1}{\alpha}},$$

is regular and univalent in U.

In [52] [51], Pascu demonstrated the following theorem.

Theorem 1.6.8. *[52] [51] Let $\beta \in \mathbb{C}$, $\Re(\beta) \geq \gamma > 0$. If $f \in A$ satisfies the condition:*

$$\frac{1 - |z|^{2\gamma}}{\gamma} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,\ z \in U,$$

then the integral operator

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)dt \right]^{\frac{1}{\beta}} \in S.$$

Using Theorem 1.6.8 and Theorem 1.6.2, D. Breaz and N. Breaz in the paper [14] obtained the following theorem.

Theorem 1.6.9. *[14] Let $\alpha, \beta \in \mathbb{C}$ and $\Re(\beta) \geq \Re(\alpha) \geq \frac{3n}{|\alpha|}$, let $f_i \in T_2$ be defined as:*

$$f_i(z) = z + \sum_{k=3}^{\infty} a_k^i z^k,\ z \in U,\ \forall i = 1, 2, ..., n, \forall n \in \mathbb{N}^*.$$

If $|f_i(z)| \leq 1, z \in U$, then the integral operator

$$F_{\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha}} dt \right]^{\frac{1}{\beta}},$$

belongs to the class S.

Theorem 1.6.10. *([64]) Let c and β be complex numbers such that $\Re(\beta) > 0, |c| \leq 1$, and $c \neq -1$, and let $f(z) = z + a_2 z^2 + \dots$ be a regular function in U. If:*

$$\left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zf''(z)}{\beta f'(z)} \right| \leq 1, \; \forall z \in U,$$

then the function

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta - 1} \cdot f'(t) dt \right\}^{\frac{1}{\beta}},$$

is regular and univalent in U.

Theorem 1.6.11. *[52] Let α be a complex number, $\Re(\alpha) > 0$, c be a complex number, $|c| \leq 1, c \neq 1$, and $f \in A$. If:*

$$\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - |c|, \; \forall z \in U, \tag{1.61}$$

then for any complex number β, $\Re(\beta) \geq \Re(\alpha)$, the function $F_\beta(z)$ defined by

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta - 1} f'(t) dt \right)^{\frac{1}{\beta}},$$

is in the class S.

Theorem 1.6.12. *[52] (N. N. Pascu's Univalence Criterion) Let $f \in A$ and $\beta \in \mathbb{C}$. If $\Re(\beta) > 0$ and*

$$\frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \; \forall z \in U,$$

then the function $F_\beta(z)$ defined by:

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta - 1} f'(t) dt \right)^{\frac{1}{\beta}},$$

is in the class S.

For $c = 0$ in Theorem 1.6.11, we obtain the univalence criterion obtained by N. N. Pascu in the paper [52].

Theorem 1.6.13. *[51] Let $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $f \in A$. If f satisfies:*

$$\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \left| \frac{z \cdot f''(z)}{f'(z)} \right| \leq 1, \; \forall z \in U,$$

then, for any complex number β with $\Re(\beta) \geq \Re(\alpha)$, the integral operator

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta - 1} \cdot f'(t) dt \right\}^{\frac{1}{\beta}},$$

belongs to the class S.

Equality holds if $f(z) = e^{i\tau} \cdot \frac{M}{R^m} \cdot z^m$, where τ is a constant.

In 2004, D. Răducanu, I. Radomir, M. E. Gageone, and N. R. Pascu in the paper [67] demonstrated one of the generalizations of the criterion of S. Ozaki and M. Nunokawa.

Theorem 1.6.14. *[67] Let $f \in A$ and $m > 0$ such that:*

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) - \frac{m-1}{2}|z|^{m+1}\right| \leq \frac{m+1}{2}|z|^{m+1}, \ \forall z \in U,$$

then the function f is analytic and univalent in U.

In the next theorem, we observe sufficient conditions for the univalence of the operator I_n using the univalence criterion of J. Becker.

Theorem 1.6.15. *[23] Let the functions $f_i \in A$ and $m_i > 0$ satisfy:*

$$\left|\left(\frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1\right) - \frac{m_i-1}{2}|z|^{m_i+1}\right| \leq \frac{m_i+1}{2}|z|^{m_i+1}, \ \forall z \in U, i \in \{1,2,...,n\}. \quad (1.62)$$

Additionally, assume that M_i, N_i are positive real numbers and the functions $g_i \in A$ are such that:

$$|f_i(z)| < M_i, \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq N_i, \ \forall z \in U, i \in \{1,2,...,n\}. \quad (1.63)$$

If:

$$\sum_{i=1}^{n} |\alpha_i| \left[(m_i+1)M_i^2 + N_i\right] \leq \frac{3\sqrt{3}}{2}, \ \forall \alpha_i \in \mathbb{C}, \Re(\alpha_i) > 0, i \in \{1,2,...,n\}, \quad (1.64)$$

then the integral operator I_n from relation (1.58) belongs to the class S.

For the particular case $m_i = 1, M_i = M, N_i = 1$, we obtain the following result.

Corollary 1.6.1. *[23] Let the functions $f_i, g_i \in A$ and M be a positive real number such that the inequalities:*

$$\left|\frac{z \cdot f_i'(z)}{(f_i(z))^2} - 1\right| \leq |z|^2, |f_i(z)| < M, \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1,$$

are satisfied for any $z \in U, i \in \{1,...,n\}$.

If

$$(2M^2+1)\sum_{i=1}^{n} |\alpha_i| \leq \frac{3\sqrt{3}}{2},$$

where $\alpha_i \in \mathbb{C}, \Re(\alpha_i) > 0, i \in \{1,2,...,n\}$, then the integral operator I_n belongs to the class S.

Next, setting $n = 1$ in Theorem 1.6.15, we obtain the following result.

Corollary 1.6.2. *[23] Let $m > 0$ and the function $f \in A$ satisfy the hypotheses of Theorem 1.6.14. Assume that $\alpha \in \mathbb{C}, \Re(\alpha) > 0$, and M, N are positive real numbers, and the function $g \in A$. If*

$$|f(z)| < M, \left|\frac{g^{''}(z)}{g^{'}(z)}\right| \leq N,$$

for any $z \in U$ and

$$|\alpha| \left[(m+1)M^2 + N\right] \leq \frac{3\sqrt{3}}{2},$$

then the integral operator I_1 defined by relation (1.59) belongs to the class S.

Chapter 2

Properties of some univalent integral operators

This chapter, structured into four sections, is dedicated to the study of sufficient conditions for univalence, convexity, and starlikeness for analytic functions defined in the interior of the unit disk. The original results were obtained based on the use of univalence criteria established by J. Becker, N. N. Pascu, V. Pescar, and others, with some of our own results being generalizations and improvements of those found in the work [23].

In the first subsection, the author of this work presents her own contributions regarding the conditions for the membership of the operator $F_\beta(f,g)(z)$ in the class S. Univalence conditions for the integral operator $F_{n,\beta}(z)$ are presented in Section 2.2, where, by applying the criterion of N. N. Pascu and the general Schwarz Lemma, new properties of this operator, introduced by P. Dicu, R. Bucur, and D. Breaz in [23], were discovered.

Section 2.3 contains several univalence conditions for a new integral operator $G_{\beta,\gamma}(f,g)(z)$, whose proofs were obtained using the criterion of N. N. Pascu and the Schwarz Lemma, while Section 2.4 illustrates a univalence criterion for the operator $G_{n,\beta}(z)$, which is defined as a generalization of an n-function, an operator introduced in Section 2.2 of this work.

2.1 Univalence conditions for the integral operator $F_\beta(f,g)(z)$

In this subsection, we will present sufficient conditions ensuring the univalence of the integral operator $F_\beta(f,g)$, defined below.

For functions $f,g \in A$, we introduce the integral operator $F_\beta(f,g)$ defined by:

$$F_\beta(f,g)(z) = \left\{\beta \int_0^z t^{\beta-1} \frac{e^{f(t)}}{g'(t)} dt\right\}^{\frac{1}{\beta}}, \beta \in \mathbb{C} \setminus \{0\}, z \in U. \tag{2.1}$$

Remark 2.1.1. *The operator* $F_\beta(f,g)$ *generalizes the operator*

$$I_\alpha(f,g)(z) = \int_0^z \left(\frac{e^{f(t)}}{g'(t)}\right)^\alpha dt, \Re(\alpha) \le 1,$$

which was introduced and studied in [20].

In the following theorem, a univalence condition for the integral operator $F_\beta(f,g)$ is presented using the univalence criterion of N. N. Pascu [52].

Theorem 2.1.1. *[60] Let* $f \in A$ *satisfy the condition:*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1, z \in U. \tag{2.2}$$

Assume that M, N *are positive real numbers and that* $g \in A$ *satisfies:*

$$|f(z)| < M, \left|\frac{g''(z)}{g'(z)}\right| \le N, z \in U. \tag{2.3}$$

If $\beta \in \mathbb{C}, \Re(\beta) = a > 0$ *and*

$$c(2M^2 + N) \le 1, \tag{2.4}$$

where

$$c = \frac{2}{1+2a}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}},$$

then the integral operator $F_\beta(f,g)$ *defined by* (2.1) *belongs to the class* S.

For $N = 1$, we obtain the following result.

Corollary 2.1.1. *[60] Let* $f \in A$ *satisfy the condition:*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1, z \in U. \tag{2.5}$$

Assume that M *is a positive real number and that* $g \in A$ *satisfies:*

$$|f(z)| < M, \left|\frac{g''(z)}{g'(z)}\right| \le 1, z \in U. \tag{2.6}$$

If $\beta \in \mathbb{C}, \Re(\beta) = a > 0$ *and*

$$c(2M^2 + 1) \le 1, \tag{2.7}$$

where $c = \frac{2}{1+2a} \cdot \left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$ *, then the integral operator* $F_\beta(f,g)$ *defined by* (2.1) *belongs to the class* S.

For $M = 1$ in Corollary 2.1.1, we obtain the following result.

Corollary 2.1.2. *[60] Assume that the functions $f, g \in A$ satisfy the conditions*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1, |f(z)| < 1, \tag{2.8}$$

and

$$\left|\frac{g''(z)}{g'(z)}\right| \leq 1, z \in U. \tag{2.9}$$

If $\beta \in \mathbb{C}, \Re(\beta) = a > 0$ and $c \leq \frac{1}{3}$, where $c = \frac{2}{1+2a}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$, then the operator $F_\beta(f,g)(z)$ defined by (2.1) *belongs to the class S.*

Remark 2.1.2. *[60] For $\beta = 1$ in Corollary 2.1.2, we obtain that the operator*

$$I(f,g)(z) = \int_0^z \frac{e^{f(t)}}{g'(t)} dt,$$

belongs to the class S and the constant c is exact.

These results improve upon those obtained in [20].

2.2 Univalence conditions for the integral operator $F_{n,\beta}(z)$

In this subsection, we consider a generalization of the result from Theorem 2.1.1, taking the integral operator as depending on n functions belonging to the class A.

For functions $f_i, g_i \in A,\ i \in \{1, 2, ..., n\}$, we introduce the integral operator $F_{n,\beta}$ defined by:

$$F_{n,\beta}(z) := \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \frac{e^{f_i(t)}}{g_i'(t)} dt\right\}^{\frac{1}{\beta}}, z \in U, \tag{2.10}$$

where $\beta \in \mathbb{C} \setminus \{0\}$, $\alpha_i \in \mathbb{C}, i \in \{1, 2, ..., n\}$.

Remark 2.2.1. *[61] The operator $F_{n,\beta}$ generalizes the operator $F_\beta(f,g)$ defined by equation (2.1).*

Using N. N. Pascu's criterion, we present the following theorem, which provides sufficient conditions for the univalence of the operator $F_{n,\beta}$ introduced and studied in [23].

Theorem 2.2.1. *[61] Let the functions $f_i \in A,\ i \in \{1, 2, ..., n\}$, satisfy the condition:*

$$\left|\frac{z^2 f_i'(z)}{\left(f_i(z)\right)^2} - 1\right| < 1, z \in U. \tag{2.11}$$

Assume that M_i, N_i are positive real numbers and that the functions $g_i \in A$ satisfy:

$$|f_i(z)| < M_i,\ \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq N_i, z \in U. \tag{2.12}$$

If $\beta \in \mathbb{C}, \Re(\beta) = a > 0$ *and*

$$c\sum_{i=1}^{n}(2M_i^2 + N_i) \leq 1, \tag{2.13}$$

where

$$c = \frac{2}{1+2a} \cdot \left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}, \tag{2.14}$$

then the operator $F_{n,\beta}$ *defined by equation (2.10) belongs to the class* S.

If we take $M_i = N_i = M,\ i \in \{1, 2, ..., n\}$, positive real numbers in Theorem 2.2.1, we obtain the following result.

Corollary 2.2.1. *[61] Let the functions* $f_i \in A,\ i \in \{1, 2, ..., n\}$ *satisfy the condition:*

$$\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| < 1, z \in U. \tag{2.15}$$

Assume that M *is a positive real number and that the functions* $g_i \in A,\ i \in \{1, 2, ..., n\}$ *satisfy:*

$$\begin{aligned} &|f_i(z)| < M, \\ &\left|\frac{g_i''(z)}{g_i'(z)}\right| \leq M, z \in U. \end{aligned} \tag{2.16}$$

If $\beta \in \mathbb{C}, \mathrm{Re}(\beta) = a > 0$ *and*

$$cM(2M+1)n \leq 1, \tag{2.17}$$

where $c = \frac{2}{1+2a}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$, *then the operator* $F_{n,\beta}$ *defined by equation (2.10) belongs to the class* S.

Taking $M = 1$ in Corollary 2.2.1, we obtain the following result.

Corollary 2.2.2. *[61] Let the functions* $f_i, g_i \in A,\ i \in \{1, 2, ..., n\}$ *satisfy the conditions*

$$\begin{aligned} &\left|\frac{z^2 f_i'(z)}{\left(f_i(z)\right)^2} - 1\right| < 1, \\ &|f_i(z)| < 1, \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1, z \in U. \end{aligned} \tag{2.18}$$

If $\beta \in \mathbb{C}, \Re(\beta) = a > 0$ *and*

$$c \leq \frac{1}{3n}, \tag{2.19}$$

where $c = \frac{2}{1+2a} \cdot \left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$, *then the operator* $F_{n,\beta}$ *defined by equation (2.10) belongs to the class* S.

Remark 2.2.2. *[61] For* $n = 1$ *in Corollary 2.2.2, we obtain Corollary 2.1.2.*

2.3 Univalence conditions for the integral operator $G_{\beta,\gamma}(f,g)(z)$

For the functions $f, g \in A$, we introduce a new integral operator defined by:

$$G_{\beta,\gamma}(f,g)(z) = \left\{\beta \int_0^z t^{\beta-1} \left(\frac{e^{f(t)}}{g'(t)}\right)^{\gamma} dt\right\}^{\frac{1}{\beta}}, \tag{2.20}$$

where $\beta \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{C}$, $z \in U$.

This operator generalizes the operators introduced in the paper [23] by P. Dicu, R. Bucur, and D. Breaz.

In this section, we present the univalence conditions for the operator $G_{\beta,\gamma}(f,g)$.

To demonstrate the univalence of the operator $G_{\beta,\gamma}(f,g)$, we use N. N. Pascu's criterion [52].

Theorem 2.3.1. *[62] Let the function $f \in A$ satisfy the condition:*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1, z \in U. \tag{2.21}$$

Let M, N be positive real numbers, and $g \in A$ such that:

$$\begin{aligned} &|f(z)| < M, \\ &\left|\frac{g''(z)}{g'(z)}\right| \le N, z \in U, \end{aligned} \tag{2.22}$$

$\beta, \gamma \in \mathbb{C}, \Re(\beta) = a > 0$ and we have that

$$c \cdot |\gamma| \cdot (2M^2 + N) \le 1, \tag{2.23}$$

where $c = \frac{2}{1+2a}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$, then the operator $G_{\beta,\gamma}(f,g)$ defined by relation (2.20) belongs to the class S.

If we take $N = 1$ in Theorem 2.3.1, we obtain the following result.

Corollary 2.3.1. *[62] Let the function $f \in A$ satisfy the condition:*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1, z \in U. \tag{2.24}$$

Let M be a positive real number, and the function $g \in A$ such that:

$$\begin{aligned} &|f(z)| < M, \\ &\left|\frac{g''(z)}{g'(z)}\right| \le 1, z \in U, \end{aligned} \tag{2.25}$$

$\beta, \gamma \in \mathbb{C}, \Re(\beta) = a > 0$ and we have that

$$c \cdot |\gamma| \cdot (2M^2 + 1) \le 1, \tag{2.26}$$

where $c = \frac{2}{1+2a} \cdot \left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$, then the operator $G_{\beta,\gamma}(f,g)$ defined by relation (2.20) belongs to the class S.

By setting $M = 1$ in Corollary 2.3.1, we obtain the following corollary.

Corollary 2.3.2. *[62] Let the function $f \in A$ satisfy the condition:*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1, z \in U. \tag{2.27}$$

Let M be a positive real number, and $g \in A$ such that:

$$\begin{aligned} &|f(z)| < 1, \\ &\left|\frac{g''(z)}{g'(z)}\right| \le 1, z \in U, \end{aligned} \tag{2.28}$$

$\beta, \gamma \in \mathbb{C}, \Re(\beta) = a > 0$ and we have that

$$c|\gamma| \le \frac{1}{3}, \tag{2.29}$$

where $c = \frac{2}{1+2a}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$. Then the operator $G_{\beta,\gamma}(f, g)$ defined by relation (2.20) belongs to the class S.

Remark 2.3.1. *[62] For $\beta = 1$ and $\gamma = \alpha$ the univalence condition of the operator $I_1(z) = \int_0^z \left(\frac{e^{f(t)}}{g'(t)}\right)^\alpha dt$ is obtained from relation (1.59).*

2.4 Univalence conditions for the integral operator $G_{n,\beta}(z)$

For the functions $f_i, g_i \in A$, $i \in \{1, 2, ..., n\}$, we introduce the integral operator $G_{n,\beta}(z)$ defined by:

$$G_{n,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{e^{f_i(t)}}{g_i'(t)}\right)^{\gamma_i} dt\right\}^{\frac{1}{\beta}}, \tag{2.30}$$

where $\beta, \gamma \in \mathbb{C}, \beta \neq 0, \alpha_i \in \mathbb{C}, i \in \{1, 2, ..., n\}, z \in U$.

In this section, we present a generalization of the operator from Theorem 2.3.1, considering the operator depending on n analytic functions.

We will demonstrate the univalence of this operator using the univalence criterion of N. N. Pascu.

Theorem 2.4.1. *[63] Let the functions $f_i \in A$, $i \in \{1, 2, ..., n\}$, satisfy the condition:*

$$\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| < 1, z \in U. \tag{2.31}$$

Assume that M_i, N_i are positive real numbers and $g_i \in A, i \in \{1, 2, ..., n\}$ satisfy:

$$|f_i(z)| < M_i, \left|\frac{g_i''(z)}{g_i'(z)}\right| \le N_i, z \in U. \tag{2.32}$$

If $\beta \in \mathbb{C}, \Re(\beta) = a > 0, \gamma_i \in \mathbb{C}$, *and:*

$$c\sum_{i=1}^{n} |\gamma_i|(M_i^2 + N_i) \leq 1, \tag{2.33}$$

where $c = \frac{2}{1+2a} \cdot \left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$ *, then the operator* $G_{n,\beta}(z)$ *defined by relation (2.30) belongs to the class* S.

For $M_i = N_i = M, i \in \{1, 2, ..., n\}$ in Theorem 2.4.1, we obtain:

Corollary 2.4.1. *[63] Let the functions* $f_i \in A, i \in \{1, 2, ..., n\}$ *satisfy:*

$$\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| < 1, z \in U. \tag{2.34}$$

Assume that M *is a positive real number and* $g_i \in A, i \in \{1, 2, ..., n\}$ *satisfy:*

$$\begin{aligned} &|f_i(z)| < M, \\ &\left|\frac{g_i''(z)}{g_i'(z)}\right| \leq M, z \in U. \end{aligned} \tag{2.35}$$

If $\beta \in \mathbb{C}, \Re(\beta) = a > 0,\ \gamma_i \in \mathbb{C}, i \in \{1, 2, ..., n\}$ *and:*

$$c \cdot M \cdot (M + 1) \sum_{i=1}^{n} |\gamma_i| \leq 1, \tag{2.36}$$

where $c = \frac{2}{1+2a} \cdot \left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$ *, then the operator* $G_{n,\beta}(z)$ *defined by relation (2.30) belongs to the class* S.

If we take $M = 1$ in Corollary 2.4.1, we obtain the following result:

Corollary 2.4.2. *[63] Let the functions* $f_i, g_i \in A,\ i \in \{1, 2, ..., n\}$, *satisfy:*

$$\begin{aligned} &\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| < 1, \\ &|f_i(z)| < 1, \\ &\left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1, z \in U. \end{aligned} \tag{2.37}$$

If $\beta \in \mathbb{C}, \Re(\beta) = a > 0,\ \gamma_i \in \mathbb{C}, i \in \{1, 2, ..., n\}$ *and*

$$2 \cdot c \cdot \sum_{i=1}^{n} |\gamma_i| \leq 1, \tag{2.38}$$

where $c = \frac{2}{1+2a} \cdot \left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}$ *, then the operator* $G_{n,\beta}$ *defined by relation (2.30) belongs to the class* S.

Remark 2.4.1. *[63] For* $\beta = 1$ *and* $\gamma_i = \alpha_i$, *we obtain another univalence condition for the operator*

$$I_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{e^{f_i(t)}}{g_i'(t)}\right)^{\alpha_i} dt$$

defined in [23], the first condition being given in Theorem 1.6.15 of the paper [23].

Chapter 3

Properties of certain classes of meromorphic functions defined on the exterior unit disk and new integral operators

In this chapter, we aim to study and find new sufficient conditions for univalence, convexity, and star-likeness, as well as conditions on the coefficients of certain classes of univalent functions, defined on the exterior unit disk for various subclasses of analytic functions. These meromorphic functions have a unique simple pole at $z = \infty$.

The results of this chapter, which consists of seven original sections.

Section 3.1 covers properties of functions from the class of injective meromorphic functions, star-like of order γ, $O_1^*(\gamma)$, and functions from the class of convex meromorphic functions of order γ, $O_k(\gamma)$.

In sections 3.2, 3.3, 3.4, 3.6 and 3.7, conditions for univalence of some integral operators formed from functions defined on the exterior of the unit disk are presented. These operators were formed starting from the operator $F_{\alpha_i,\beta}(z)$ introduced by N. Seenivasagan and D. Breaz in the paper [71], and the original results obtained by the author of this thesis have been published in journals such as Afrika Matematika [55], Journal of Advanced Mathematical Studies [57], Studia Universitatis Babeş-Bolyai Mathematica [58]. A particular case of the operator $G_{\alpha_i,\beta}(z)$ is represented by the integral operator $E(z)$, for which, in section 3.5, certain values of the coefficients were obtained, demonstrating that the operator belongs to the class of star-like meromorphic functions of order 0, $O_1^*(0)$, and these results are published in the journal General Mathematics [56].

3.1 Properties of some meromorphic functions from certain special subclasses

If, in Definition 1.4.12, we apply the transformation

$$\begin{aligned} z &\to \frac{1}{z} \\ dz &\to \frac{-1}{z^2}dz, \quad g(z) = \frac{1}{f(\frac{1}{z})}, \end{aligned} \tag{3.1}$$

we obtain:

$$\begin{aligned} -\Re\left(\frac{\frac{1}{z}f'(\frac{1}{z})}{f(\frac{1}{z})}\right) &= -\Re\left(\frac{f'(\frac{1}{z})}{z\cdot f(\frac{1}{z})}\right) = -\Re\left(\frac{g(z)\cdot\left(\frac{1}{g(z)}\right)'}{z}\right) \\ &= -\Re\left(\frac{-g'(z)}{z\cdot g(z)}\right) = \Re\left(\frac{g'(z)}{z\cdot g(z)}\right). \end{aligned} \tag{3.2}$$

To illustrate the relationship (3.2), we consider $b_3 = 1, \gamma = 0$ in the relationship (1.3).

Thus, we have $g(z) = z + \frac{1}{z^3}$ and

$$\Re\left(\frac{g'(z)}{z\cdot g(z)}\right) = \Re\left(\frac{(z+z^{-3})'}{z(z+z^{-3})}\right) = \Re\left(\frac{1-\frac{3}{z^4}}{z^2+\frac{1}{z^2}}\right) = \Re\left(\frac{z^4-3}{z^6+z^2}\right).$$

We will consider several particular cases to see if $\Re\left(\frac{z^4-3}{z^6+z^2}\right)$ is positive or negative. To simplify the calculations, we will use the Symbolab application [76].

z	$1+i$	$2-3i$	$2+3i$	$3-i$	$3+2i$	$-4-5i$	$-5-4i$	$-5+4i$
$\Re\left(\frac{z^4-3}{z^6+z^2}\right)$	$=0$	<0	<0	>0	>0	<0	>0	>0

Table 3.1: The values of $\Re\left(\frac{z^4-3}{z^6+z^2}\right)$ for a given z.

It is observed that it is necessary to add the condition $|\Re(z)| > |Im(z)|$, in order to formulate the following definition.

Definition 3.1.1. *[54] A meromorphic function $g \in O_1$, star-like of order γ, $0 \leq \gamma < 1$, belongs to the class $O_1^*(\gamma)$ if it satisfies the inequalities:*

$$\Re\left(\frac{g'(z)}{zg(z)}\right) > \gamma, \tag{3.3}$$

$$\left|\Re(z)\right| > \left|Im(z)\right|, z \in W.$$

If, in Definition 1.4.16, we apply the transformation (3.1) and use relation (3.3), we obtain:

$$-\Re\left(1+\frac{\frac{1}{z}f''\left(\frac{1}{z}\right)}{f'\left(\frac{1}{z}\right)}\right)=-\Re\left(1+\frac{\left(\frac{1}{g(z)}\right)''}{z\cdot\left(\frac{1}{g(z)}\right)'}\right),$$

$$-\Re\left(1+\frac{\frac{1}{z}f''\left(\frac{1}{z}\right)}{f'\left(\frac{1}{z}\right)}\right)=-\Re\left(1+\frac{g''(z)\cdot g^2(z)-g'(z)\cdot 2g(z)\cdot g'(z)}{-\frac{z\cdot g'(z)}{g^2(z)}\cdot g^4(z)}\right),$$

$$-\Re\left(1+\frac{\frac{1}{z}f''\left(\frac{1}{z}\right)}{f'\left(\frac{1}{z}\right)}\right)=-\Re\left(1+\frac{g''(z)}{z\cdot g'(z)}-2\cdot\frac{g'(z)}{z\cdot g(z)}\right),$$

$$-\Re\left(1+\frac{\frac{1}{z}f''\left(\frac{1}{z}\right)}{f'\left(\frac{1}{z}\right)}\right)=-\Re\left(1+\frac{g''(z)}{z\cdot g'(z)}\right)+2\cdot\Re\left(\frac{g'(z)}{z\cdot g(z)}\right)$$

$$-\Re\left(1+\frac{\frac{1}{z}f''\left(\frac{1}{z}\right)}{f'\left(\frac{1}{z}\right)}\right)>2\cdot\gamma-\gamma>\gamma.$$

We want to illustrate:

$$\Re\left(1+\frac{g''(z)}{z\cdot g'(z)}\right)>\gamma.$$

Thus, we will consider $b_3=1,\gamma=0$ in the function defined in relation (1.3).
To simplify the calculations, I used the Symbolab application ([76]).
Thus, we have $g(z)=z+\frac{1}{z^3}$. Then

$$\Re\left(1+\frac{g''(z)}{z\cdot g'(z)}\right)=\Re\left(1+\frac{[(z+z^{-3})']'}{z\cdot(z+z^{-3})'}\right)=\Re\left(1+\frac{12}{z^6-3z^2}\right).$$

We will consider particular cases of z to see if $\Re\left(1+\frac{12}{z^6-3z^2}\right)$ is positive or negative.

z	$1+i$	$1+2i$	$2-3i$	$3-i$	$3+2i$	$-4-5i$	$-5-4i$
$\Re\left(1+\frac{12}{z^6-3z^2}\right)$	>0	>0	>0	>0	>0	>0	>0

Table 3.2: The values of $\Re\left(1+\frac{12}{z^6-3z^2}\right)$ for a given z.

It is observed that in all the cases considered above, $\Re\left(1+\frac{12}{z^6-3z^2}\right)>0$.

We can thus formulate the following definition.

Definition 3.1.2. *[54] A meromorphic function $g\in O_1$, convex of order γ, $0\le\gamma<1$, belongs to the class $O_k(\gamma)$ if it satisfies the inequality:*

$$\Re\left(1+\frac{g''(z)}{z\cdot g'(z)}\right)>\gamma, z\in W.$$

Proposition 3.1.1. *[54] A function $g\in O_1$ is meromorphic, normalized, and injective if:*

$$\Re\left(\frac{z\cdot g'(z)}{g(z)}\right)<1, \tag{3.4}$$

$$\left|\Re(z)\right| > \left|Im(z)\right|, \tag{3.5}$$

and

$$\Re(z^4) > 0, \forall z \in W.$$

3.2 A univalence condition for the operator $K_{\alpha,\beta}(z)$

Starting from the operator $F_{\alpha_i,\beta}$ defined in equation (1.60), we can define a new operator $K_{\alpha,\beta}(z) = \left\{\beta \int_1^z t^{-1-\beta+\frac{1}{\alpha}} g(t)^{\frac{-1}{\alpha}} dt\right\}^{\frac{1}{\beta}}$, for which we will present univalence conditions in this subsection.

Theorem 3.2.1. *[55] Let $\alpha, \beta \in \mathbb{C}$, $z \in W$ and $\Re(\beta) \geq \Re(\alpha) \geq \frac{3}{|\alpha|}$. If $f \in T_2$ and $g \in V_2$, and the following conditions are satisfied:*

$$\begin{aligned} &\left|\frac{g'(z)}{z^2} + 1\right| > 1, \\ &\left|f\left(\frac{1}{z}\right)\right| = \left|\frac{1}{g(z)}\right| \geq 1, z \in W, \end{aligned} \tag{3.6}$$

then the integral operator, $K_{\alpha,\beta}(z)$, belongs to the class Σ.

3.3 Univalence conditions for the operators $G_{\alpha_i,\beta}(z)$ and $G_\beta(z)$

Starting from the integral operator $F_{\alpha_i,\beta}(z)$, defined in equation (1.60), we can define new operators denoted by $G_{\alpha_i,\beta}(z) = \left\{\beta \int_1^z t^{-1-\beta} \prod_{i=1}^n \left(\frac{t}{g_i(t)}\right)^{\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}}$ and $G_\beta(z) = \left\{\beta \int_1^z t^{-1-\beta} \frac{g'(t)}{g^2(t)} dt\right\}^{\frac{1}{\beta}}$. For these operators, we will further provide univalence conditions.

Let $g_i(t) = \frac{1}{f_i(\frac{1}{t})} \in O_1$, with $g_i(t) \neq 0, t \in O_1, (t \neq 0)$.

Since O_1 is a subclass of O that contains meromorphic and injective functions g, defined in equation (1.3), we can say that there is a bijection between T_2 and O_1.

We start from the operator $F_{\alpha_i,\beta}(z)$ and apply the following transformations:

$$\begin{aligned} &t \to \frac{1}{t}, \\ &dt \to \frac{-1}{t^2} dt, \\ &g_i(t) = \frac{1}{f_i(\frac{1}{t})} \in O_1. \end{aligned} \tag{3.7}$$

Note that we must also apply transformations to the integration limits, as follows:

- when $t = 0$, we will have $t = \frac{1}{0_+} = +\infty$, - when $t = z$, we will have $t = \frac{1}{z} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{|z|^2} > 1$.

Therefore, $\int_0^z$ becomes $\int_\infty^1$, but since z is outside the unit disk, i.e. $\int_z^1 = -\int_1^z$.

The integral operator is defined as:

$$G_{\alpha_i,\beta}(z) = \left\{ \beta \int_1^z t^{-1-\beta} \prod_{i=1}^{n} \left(\frac{t}{g_i(t)} \right)^{\frac{1}{\alpha_i}} dt \right\}^{\frac{1}{\beta}}. \tag{3.8}$$

If $\beta = 1$, then the integral operator $G_{\alpha_i,\beta}$ takes the form:

$$G_{\alpha_i,1}(z) = \int_1^z t^{-2} \prod_{i=1}^{n} \left(\frac{t}{g_i(t)} \right)^{\frac{1}{\alpha_i}} dt. \tag{3.9}$$

Theorem 3.3.1. *[55] Let $\beta \in \mathbb{C}$, $\Re\beta \geq \gamma > 0$. If $g \in O$ satisfies the condition*

$$\frac{|z|^{2\gamma} - 1}{\gamma |z|^{2\gamma}} \cdot \left| \frac{g''(z)}{zg'(z)} \right| > 1,$$

then the operator $G_\beta(z)$ belongs to the class Σ.

Theorem 3.3.2. *[55] Let $\alpha_i, \beta \in \mathbb{C}$ and $\Re(\beta) \geq \Re(\alpha_i) \geq \frac{3n}{|\alpha_i|}$. Let $g_i \in O_2$, where O_2 is a subclass of O_1, with:*

$$g_i(z) = z + \sum_{k=3}^{\infty} \frac{{b_k}^i}{z^k}, \forall i \in \{1, 2, ..., n\}, n \in \mathbb{N}^*.$$

If $|g_i(z)| > 1, z \in W$, then the integral operator $G_{\alpha_i,\beta}(z)$ belongs to O_1.

Theorem 3.3.3. *[55] Let $m > 1$, $g_i \in V_{2,\mu_i}$, (V_{2,μ_i} is a subclass defined in equation (1.23)), $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \gamma$ and*

$$\gamma = \sum_{i=3}^{n} \frac{(1 + \mu_i)m - 1}{|\alpha_i|}, \mu_i > 1, i \in \{1, 2, ..., n\}, n \in \mathbb{N}^*.$$

If

$$|g_i(z)| > m, z \in W, i \in \{1, 2, .., n\},$$

then the integral operator

$$G_{\alpha_i,\beta}(z) = \left\{ \beta \int_1^z t^{-1-\beta} \prod_{i=1}^{n} \left(\frac{t}{g_i(t)} \right)^{\frac{1}{\alpha_i}} dt \right\}^{\frac{1}{\beta}},$$

belongs to the class Σ.

Theorem 3.3.4. *[55] Let $m > 1, g_i \in S(p)$, (g_i defined in Theorem 3.3.2) and*

$$\gamma_1 = \sum_{i=3}^{n} \frac{(1+p)m-1}{|\alpha_i|}, i = 1, 2, ..., n, n \in \mathbb{N}^*,$$

and p with the properties from relations (1.24) and (1.25).
If

$$|g_i(z)| > m, z \in W, i = 1, 2, .., n,$$

then we obtain that the operator $G_{\alpha_i,\beta}(z)$ belongs to the class Σ.

Lemma 3.3.1. *[58] Let the analytic function g be regular on the exterior of the unit disk $W_R = \{z \in \mathbb{C} : |z| > R\}$ and let $g(\infty) = \infty, g'(\infty) = 1$.*
If $|g(z)| \geq 1$, then the following inequalities hold:

$$\left| f\left(\frac{1}{z}\right) \right| \leq \left| \frac{1}{z} \right|,$$

$$\frac{1}{|g(z)|} \leq \frac{1}{|z|}, z \in W.$$

Equality holds only if $|g(z)| = K \cdot z$ and $K = 1$.

In Lemma 1.1.1 [58], we apply the transformations from equation (3.7) and obtain the following lemma.

Lemma 3.3.2. *[57] Let the function g be regular on the exterior of the unit disk $W_R = \{z \in \mathbb{C} : |z| > R\}$, with $|f(z)| > M$, for fixed M.*
If the order of the multiplicity of the zeros is one more than m for $z = \infty$, then:

$$\left| f\left(\frac{1}{z}\right) \right| \leq \frac{M}{R^m} \cdot \left| \frac{1}{z} \right|^m,$$

$$\left| \frac{1}{g(z)} \right| \leq \frac{M}{R^m} \cdot \frac{1}{|z|^m}, z \in W.$$

Equality holds only if $f(z) = e^{i\tau} \cdot \frac{R^m}{M} \cdot z^m$, where τ is a constant.

Theorem 3.3.5. *[57] Let $g \in O_1$ such that:*

$$\left| \frac{g'(z)}{z^2} + 1 \right| \geq 1, \forall z \in W. \tag{3.10}$$

Then g is univalent in W.

Theorem 3.3.6. *[57] Let c and β be complex numbers such that $\Re\{\beta\} > 0, |c| \geq 1$, and $c \neq -1$. Let $k(z) = z + \frac{b_3}{z^3} + \frac{b_4}{z^4} + ...$ be a regular function in W. If*

$$\left| \frac{c}{|z|^{2\beta}} + \left(1 - \frac{1}{|z|^{2\beta}}\right) \cdot \frac{k''(z)}{\beta \cdot z \cdot k'(z)} \right| \geq 1, \forall z \in W,$$

then the operator $G_{\alpha,\beta}(z)$, defined by

$$G_{\alpha,\beta}(z) = \left\{ \beta \int_1^z t^{-\beta-1} \cdot k'(t) dt \right\}^{\frac{1}{\beta}}, z \in W,$$

is a regular and univalent function in W.

Theorem 3.3.7. *[57] Let $M \geq 1$ and the functions $g_i \in O_1, i \in \{1, 2, ..., n\}$, which satisfy condition (3.10), and let β be a real number, $\beta \leq \sum_{i=1}^{n} \frac{1}{M|\alpha_i|}$, where c and α_i are complex numbers, $\alpha_i \neq 0$. If*

$$\begin{aligned} &|c| \geq |z|^{2\beta} - \left(1 - |z|^{2\beta}\right) \frac{1}{\beta} \sum_{i=1}^{n} \frac{1}{M|\alpha_i|}, \\ &|g_i(z)| \geq M, \\ &|z| \geq M, \forall z \in W, \end{aligned} \tag{3.11}$$

then the operator $G_{\alpha_i,\beta}(z)$ defined in (3.8) belongs to the class Σ.

Theorem 3.3.8. *[57] Let $M \geq 1$ and the function $g_i \in O_1$ for $i \in \{1, 2, ..., n\}$, which satisfies relation (3.10), β a real number, $\beta \leq \frac{n}{M|\alpha|}$, and $c, \alpha \in \mathbb{C}$, $\alpha \neq 0$. If:*

$$|c| \geq |z|^{2\beta} - \left(1 - |z|^{2\beta}\right) \frac{1}{\beta} \cdot \frac{n}{M|\alpha|},$$
$$|g_i(z)| > M,$$
$$|z| > M, z \in W,$$

then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to the class Σ.

Corollary 3.3.1. *[57] Let the function $g_i \in O_1$ that satisfies (3.10) and β a real number, $\beta \leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|}$, where $c, \alpha \in \mathbb{C}$, $\alpha \neq 0$. If the relation*

$$|c| \geq |z|^{2\beta} - \left(1 - |z|^{2\beta}\right) \frac{1}{\beta} \cdot \sum_{i=1}^{n} \frac{1}{|\alpha_i|},$$
$$|g_i(z)| > 1, \forall z \in W,$$

then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to the class Σ.

Corollary 3.3.2. *[57] Let $M \geq 1$ and the function $g \in O_1$ that satisfies condition (3.10), $\beta \in \mathbb{R}$, $\beta \leq \frac{1}{M|\alpha|}$ and $c \in \mathbb{C}$. If:*

$$|c| \geq |z|^{2\beta} - \left(1 - |z|^{2\beta}\right) \frac{1}{\beta} \cdot \frac{1}{M|\alpha|},$$
$$|g(z)| > M,$$
$$|z| > M, \forall z \in W,$$

then the operator $G_{\alpha,\beta}(z), z \in W$, belongs to the class Σ.

Corollary 3.3.3. *[57] Let the function $g \in O_1$ that satisfies condition (3.10), $\beta \in \mathbb{R}$, $\beta \leq \frac{1}{|\alpha|}$ and $c, \alpha \in \mathbb{C}$, $\alpha \neq 0$. If:*

$$|c| \geq |z|^{2\beta} - \left(1 - |z|^{2\beta}\right) \frac{1}{\beta} \cdot \frac{1}{|\alpha|},$$
$$|g(z)| > 1, \forall z \in W,$$

then the operator $G_{\alpha,\beta}(z), z \in W$ belongs to the class Σ.

We will next give the conditions for the membership of the integral operator $G_\beta(z)$ in the class Σ.

Theorem 3.3.9. *[58] Let $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $k \in O$. If k satisfies the inequalities*

$$\frac{|z|^{2\Re(\alpha)} - 1}{\Re(\alpha) \cdot |z|^{2\Re(\alpha)}} \cdot \left| \frac{k''(z)}{z \cdot k'(z)} \right| > 1, \forall z \in W,$$

$$\left| \frac{k''(z)}{zk'(z)} \right| > \Re(\alpha) \cdot |z|, \forall z \in W, \tag{3.12}$$

then, for any complex number β with $\Re(\beta) \leq \Re(\alpha)$, the operator

$$G_\beta(z) = \left\{ \beta \int_1^z t^{-\beta-1} \cdot k'(t) dt \right\}^{\frac{1}{\beta}},$$

belongs to the class Σ.

Next, we will give the conditions for the univalence of the operator $G_{\alpha_i,\beta}(z)$ in the class Σ.

Theorem 3.3.10. *[58] Let g_i defined by*

$$g_i(z) = z + \sum_{k=j+1}^{\infty} \frac{b_k^i}{z^k}, |z| > 1, \tag{3.13}$$

from the class $V_j, i \in \{1, 2, ...n\}, n \in \mathbb{N}^*, j \in \mathbb{N}_1^*$.
If $|g_i(z)| \geq M_i, M_i \geq 1, z \in W$, then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) is in the class Σ,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{M_i |\alpha_i|}, \tag{3.14}$$

and $\Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}$.

Corollary 3.3.4. *[58] Let g_i defined in (3.13) from the class $V_j, i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) is in the class Σ, and

$$\Re(\alpha) \leq \frac{1}{M|\alpha|},$$

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{M|\alpha_i|}, \Re(\beta) \leq \Re(\alpha), \alpha_i, \beta \in \mathbb{C}.$$

Corollary 3.3.5. *[58] Let g_i defined in relation (3.13) from the class $V_j, i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{M|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Corollary 3.3.6. *[58] Let g_i defined in (3.13) from the class $V_2, i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{M|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Corollary 3.3.7. *[58] Let g_i defined in (3.13) from the class $V_2, i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$.*
If $|g_i(z)| \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Theorem 3.3.11. *[58] Let g_i defined in (3.13) from the class $V_{j,\mu_i}, i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M_i, M_i \geq 1, z \in W$, then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha_i) \leq \sum_{i=1}^{n} \frac{1}{(1+\mu_i)M_i|\alpha_i|}, \Re(\beta) \leq \Re(\alpha_i), \alpha_i, \beta \in \mathbb{C}.$$

Corollary 3.3.8. *[58] Let g_i defined in (3.13) from the class $V_{j,\mu_i}, i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha_i) \leq \sum_{i=1}^{n} \frac{1}{(1+\mu_i)M|\alpha_i|}, \Re(\beta) \leq \Re(\alpha_i), \alpha_i, \beta \in \mathbb{C}.$$

Corollary 3.3.9. *[58] Let g_i defined in (3.13) from the class $V_{j,\mu_i}, i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+\mu_i)M|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Corollary 3.3.10. *[58] Let g_i defined in (3.13) from the class $V_{j,\mu}$ for $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{(1+\mu)M|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Corollary 3.3.11. *[58] Let g_i defined in (3.13) from the class V_{2,μ_i} for $i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha_i) \leq \sum_{i=1}^{n} \frac{1}{(1+\mu_i)M|\alpha_i|}, \Re(\beta) \leq \Re(\alpha_i), \alpha_i, \beta \in \mathbb{C}.$$

Corollary 3.3.12. *[58] Let g_i defined in (3.13) from the class $V_{2,\mu}$ for $n \in \mathbb{N}^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{(1+\mu)M|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Corollary 3.3.13. *[58] Let g_i defined in (3.13) from the class $V_{2,\mu}$ for $n \in \mathbb{N}^*$.*
If $|g_i(z)| \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{(1+\mu)|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Theorem 3.3.12. *[58] Let g_i defined in (3.13) from the class $\Sigma_j(p_i), i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*, j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M_i, M_i \geq 1, z \in W$, then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha_i) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M_i|\alpha_i|}, \Re(\beta) \leq \Re(\alpha_i), \alpha_i, \beta \in \mathbb{C}.$$

Corollary 3.3.14. *[58] Let g_i defined in (3.13) from the class $\Sigma_j(p_i), i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*, j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha_i) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M|\alpha_i|}, \Re(\beta) \leq \Re(\alpha_i), \alpha_i, \beta \in \mathbb{C}.$$

Corollary 3.3.15. *[58] Let g_i defined in (3.13) from the class $\Sigma_j(p_i), i \in \{1, 2, ...n\}, n \in \mathbb{N}^*, j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Corollary 3.3.16. *[58] Let g_i defined in (3.13) from the class $\Sigma_j(p_i), i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*, j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha_i) \leq \sum_{i=1}^{n} \frac{1}{(1+p)M|\alpha_i|}, \Re(\beta) \leq \Re(\alpha_i), \alpha_i, \beta \in \mathbb{C}.$$

Corollary 3.3.17. *[58] Let g_i defined in (3.13) from the class $\Sigma_j(p), i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*, j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{(1+p)M|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Corollary 3.3.18. *[58] Let g_i defined in (3.13) from the class $\Sigma_2(p), i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*, j \in \mathbb{N}_1^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the integral operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha_i) \leq \sum_{i=1}^{n} \frac{1}{(1+p)M|\alpha_i|}, \Re(\beta) \leq \Re(\alpha_i), \alpha_i, \beta \in \mathbb{C}.$$

Corollary 3.3.19. *[58] Let g_i defined in (3.13) from the class $\Sigma_2(p), i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$.*
If $|g_i(z)| \geq M, M \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{(1+p)M|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

Corollary 3.3.20. *[58] Let g_i defined in (3.13) from the class $\Sigma_2(p), i \in \{1, 2, ...n\}$, $n \in \mathbb{N}^*$.*
If $|g_i(z)| \geq 1, z \in W$, then the operator $G_{\alpha,\beta}(z)$ defined in relation (3.8) belongs to class Σ, and

$$\Re(\alpha) \leq \frac{n}{(1+p)|\alpha|}, \Re(\beta) \leq \Re(\alpha), \alpha, \beta \in \mathbb{C}.$$

3.4 Stellarity and convexity of the operator $G_{\alpha_i,1}(z)$

We will define the operator $G_{\alpha_i,1}(z) = \int_1^z t^{-2} \prod_{i=1}^n \left(\frac{t}{g_i(t)}\right)^{\frac{1}{\alpha_i}} dt$. This operator is also a generalization of the operator $F_{\alpha_i,\beta}(z)$, defined in [71].

Theorem 3.4.1. *[54] Let $g_i \in O_1, \alpha_i \in \mathbb{C}, i \in \{1, ..., n\}$.*
If

$$\Re\left(\frac{zg_i'(z)}{g_i(z)}\right) < 1, \tag{3.15}$$

$$\left|\Re(z)\right| > \left|Im(z)\right|,$$

and

$$\Re(z^4) > 0, z \in W,$$

then $G_{\alpha_i,1}(z)$ belongs to the class $O_1^*(0)$.

To simplify the writing, we will denote $G(z)$ in place of $G_{\alpha_i,1}(z)$.

Theorem 3.4.2. *[54] Let $i \in \{1, 2, ..., n\}$, $\alpha_i \in \mathbb{C}$, and $g_i \in O(\gamma_i), 0 \leq \gamma_i < 1$.*
If $0 < \sum_{i=1}^{n} \frac{1}{\alpha_i}(1-\gamma_i) \leq 1, z \in W$, and:

$$|\Re(z)| > |Im(z)|,$$

$$\Re\left(-\frac{zg_i'(z)}{g_i(z)}\right) > -\gamma_i, z \in W, \tag{3.16}$$

then $G_{\alpha_i,1}(z)$ defined in relation (3.9) belongs to the class $O_1^*(\mu)$, where $\mu = \sum_{i=1}^{n} \frac{1}{\alpha_i}(1-\gamma_i)$.

If we take $\gamma_i = \gamma, i \in \{1, 2, ..., n\}$ in Theorem 3.7.2, we obtain the following corollary.

Corollary 3.4.1. *[54] Let $g_i \in O_1(\gamma), 0 \leq \gamma < 1, \alpha_i \in \mathbb{C}, i \in \{1, 2, ..., n\}$. If $0 < \sum_{i=1}^{n} \frac{1}{\alpha_i} \leq \frac{1}{1-\gamma}$, and*

$$|\Re(z)| > |Im(z)|$$
$$\Re\left(\frac{-z \cdot g_i'(z)}{g_i(z)}\right) > -\gamma, \forall z \in W,$$

then $G_{\alpha_i,1}(z)$ defined in relation (3.9) is stellar of order μ, where $\mu = (1-\gamma)\sum_{i=1}^{n} \frac{1}{\alpha_i}$.

Theorem 3.4.3. *[54] Let $g_i \in O_k(\gamma_i), 0 \leq \gamma_i < 1, i \in \{1, 2, ..., n\}, \alpha_i \in \mathbb{C}$. If $0 < \sum_{i=1}^{n} \frac{1}{\alpha_i} \cdot 2\gamma_i \leq 1$ and*

$$|\Re(z)| > |Im(z)|,$$

$$\Re(z^4) > 0, \forall z \in W,$$

then $G_{\alpha_i,1}(z)$ defined in relation (3.9) belongs to the class $O_k^*(\mu)$, where $\mu = \sum_{i=1}^{n} \frac{1}{\alpha_i} \cdot 2\gamma_i$.

Letting $\gamma_i = \gamma, i \in \{1, 2, ..., n\}$ in Theorem 3.7.3, we obtain the following corollary.

Corollary 3.4.2. *[54] Let $g_i \in O_k(\gamma), -1 \leq \gamma < 1, i \in \{1, 2, ..., n\}, \alpha_i \in \mathbb{C}$. If $0 < \sum_{i=1}^{n} \frac{1}{\alpha_i} \leq \frac{1}{2\cdot\gamma}$,*

$$|\Re(z)| > |Im(z)|,$$

and

$$\Re(z^4) > 0, \forall z \in W,$$

then $G_{\alpha_i,1}(z)$ given by relation (3.9) is stellar of order μ, where $\mu = 2\gamma \cdot \sum_{i=1}^{n} \frac{1}{\alpha_i}$.

3.5 Properties of the coefficients of the operator $E(z)$

For the integral operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8), we take the particular case: $\beta = 1, \alpha_i = 1$.

For simplicity, we will write $E(z)$ instead of $G_{1,1}(z)$, that is

$$E(z) = \int_1^z t^{-1-1} \left(\frac{t}{g(t)}\right) dt = \int_1^z \frac{1}{t \cdot g(t)} dt \tag{3.17}$$

Condition (3.4) can be rewritten as follows:

$$\Re\left(\frac{-z \cdot g'(z)}{g(z)}\right) > -1,$$

$$\Re\left(1 + \frac{z \cdot g'(z)}{g(z)}\right) < 2,$$

$$\frac{1}{\Re\left(1 + \frac{z \cdot g'(z)}{g(z)}\right)} > \frac{1}{2}.$$

Taking into account the condition from relation (1.3), that is $1 < |z| < \infty$, we obtain:

$$\Re(z^2) > 1.$$

We can easily conclude that if $0 \leq \gamma < 1$, then:

$$\frac{5-\gamma}{2} > 2 \Rightarrow \frac{1-\gamma}{2} > 0. \tag{3.18}$$

Theorem 3.5.1. *[56] Let $g \in O_{1(\gamma)}$. If*

$$-\Re\left\{\frac{z \cdot E'''(z)}{E''(z)}\right\} > 0,$$

then $E(z) \in O_1^*(0)$, where $E(z)$ is the integral operator given by relation (3.17).

Corollary 3.5.1. *[56] Let $g \in O_{1(\gamma)}$. If*

$$\Re\left\{-\frac{z \cdot E'''(z)}{E''(z)} - 2\right\} > \frac{1-\gamma}{2},$$

then $E(z) \in O_1^*(0)$, where $E(z)$ is the operator from relation (3.17).

We will consider a few examples.

Example 3.5.1. *[56]*

For meromorphic, normalized, and injective functions g, from relation (1.3):

$$g(z) = z + \sum_{k=3}^{\infty} \frac{b_k}{z^k}, \quad 1 < |z| < \infty,$$

let $b_k = 0$. Then we obtain

$$g(z) = z. \tag{3.19}$$

We want to check if the conditions of Theorem 3.5.1 are satisfied. We will find the new forms of $E(g(z) = z)$, $E'(g(z) = z)$, $E''(g(z) = z)$, and $E'''(g(z) = z)$.

Applying relation (3.19) in relation (3.17), we have

$$E(g(z) = z) = \int_1^z \frac{1}{t^2} dt = 1 - \frac{1}{z}. \tag{3.20}$$

After successive differentiation of $E(z)$ defined in relation (3.20), we obtain:

$$E'(g(z) = z) = \frac{1}{z^2},$$

$$E''(g(z) = z) = -\frac{2}{z^3}, \tag{3.21}$$

$$E'''(g(z) = z) = \frac{6}{z^4}. \tag{3.22}$$

We will multiply relation (3.22) by z and divide the result by relation (3.21):

$$\frac{z \cdot E'''(g(z) = z)}{E''(g(z) = z)} = -3.$$

Thus we obtain

$$-\Re\left\{\frac{z \cdot E'''(g(z) = z)}{E''(g(z) = z)}\right\} = 3 > 0.$$

Thus, $E(g(z) = z) \in O_1^*(0)$.

We check if the conditions from Corollary 3.5.1 hold. We have that

$$\frac{z \cdot E'''(g(z) = z)}{E''(g(z) = z)} + 2 = -3 + 2 = -1,$$

$$-\Re\left\{\frac{z \cdot E'''(g(z) = z)}{E''(g(z) = z)} + 2\right\} = 1 > 0,$$

thus $E(g(z) = z) \in O_1^*(0)$.

Example 3.5.2. *[56]*

For meromorphic, normalized, and injective functions g, from relation (1.3):

$$g(z) = z + \sum_{k=3}^{\infty} \frac{b_k}{z^k}, \quad 1 < |z| < \infty,$$

let $k = 3$ and $b_k = 1$. We thus find the function:

$$g(z) = z + \frac{1}{z^3}. \tag{3.23}$$

We want to check if Theorem 3.5.1 holds in this case, finding the new forms of $E(g(z))$, $E'(g(z))$, $E''(g(z))$, and $E'''(g(z))$.

We apply the function g defined above in relation (3.23) to the operator defined in relation (3.17) and thus we obtain:

$$E(z) = \int_1^z \frac{1}{t \cdot \left(t + \frac{1}{t^3}\right)} dt = \int_1^z \frac{t^2}{t^4 + 1} dt. \tag{3.24}$$

After successive differentiation of $E(z)$ defined in relation (3.24), we obtain:

$$E'(z) = \frac{z^2}{1 + z^4} - \frac{1}{2},$$

$$E''(z) = \frac{2z - 2z^5}{(1 + z^4)^2}, \tag{3.25}$$

$$E'''(z) = \frac{2(1 - 12z^4 + 3z^8)}{(1 + z^4)^3}. \tag{3.26}$$

We will multiply relation (3.26) by z and divide the result by relation (3.25), thus obtaining:

$$\begin{aligned} \frac{z \cdot E'''(z)}{E''(z)} &= \frac{2z(1 - 12z^4 + 3z^8)}{(1 + z^4)^3} \cdot \frac{(1 + z^4)^2}{2z(1 - z^4)}, \\ &= -3 + \frac{12}{1 + z^4} - \frac{8}{1 - z^8}. \end{aligned} \tag{3.27}$$

It is known that z is a complex number of the form $z = a + ib$, with $a, b \in \mathbb{R}, |z| > 1, a^2 + b^2 > 1$.

If we take the case where $z = 1 + i$, we have $|z| = \sqrt{2} > 1$,

$$z^4 = -4, z^8 = 16,$$

$$\frac{z \cdot E'''(z)}{E''(z)} = -3 + \frac{12}{1 - 4} - \frac{8}{1 - 16} = -\frac{97}{15} = -6.4(6),$$

$$-\Re\left\{\frac{z \cdot E'''(z)}{E''(z)}\right\} = +6.4(6) > 0.$$

We have obtained that: $-\Re\left\{\frac{z\cdot E'''(z)}{E''(z)}\right\} > 0$. Thus, $E(z) \in O_1^*(0)$.

We check if Corollary 3.5.1 holds in this case.

$$\frac{z\cdot E'''(z)}{E''(z)} + 2 = -6.4(6) + 2 = -4.4(6),$$

$$-\Re\left\{\frac{z\cdot E'''(z)}{E''(z)} + 2\right\} = +4.4(6) > 0.$$

We obtain:

$$-\Re\left\{\frac{z\cdot E'''(z)}{E''(z)} + 2\right\} > 0.$$

Therefore, we have that $E(z) \in O_1^*(0)$.

3.6 Univalence conditions for the operator $T_{\alpha_i,\beta}(z)$

We consider the operator $T_{\alpha_i,\beta}(z)$ defined by

$$T_{\alpha_i,\beta}(z) = \left\{\beta\int_1^z t^{-1-\beta}\prod_{i=1}^{n}\left(\frac{t}{g_i(t)\cdot e^{g_i(t)}}\right)^{\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}}.$$

This operator is also a generalization of the operator $F_{\alpha_i,\beta}(z)$ as well as the operator $G_{\alpha_i,\beta}(z)$ defined in relation (3.8).

Theorem 3.6.1. *[59] Let $m > 1$, $g_i \in V_{2,\mu_i}$,*

$$g_i(z) = z + \sum_{k=3}^{\infty}\frac{b_k{}^i}{z^k}, \forall i = 1, 2, ..., n, n \in \mathbb{N}^*$$

and $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \gamma$, where:

$$\gamma = \sum_{i=3}^{n}\frac{m - 2(\mu_i - 1)}{|\alpha_i|\cdot m}, \mu_i > 1, i = 1, 2, ..., n; n \in \mathbb{N}^*.$$

If:

$$|g_i(z)| > m, z \in W, i = 1, 2, .., n,$$

then we obtain that the integral operator

$$T_{\alpha_i,\beta}(z) = \left\{\beta\int_1^z t^{-1-\beta}\prod_{i=1}^{n}\left(\frac{t}{g_i(t)\cdot e^{g_i(t)}}\right)^{\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}},$$

belongs to the class Σ.

Theorem 3.6.2. *[59] Let $m > 1, g_i \in S(p), (g_i$ defined in Theorem 3.3.2) and:*

$$\gamma_1 = \sum_{i=3}^{n} \frac{m - 2p + 2}{m \cdot |\alpha_i|}, i = 1, 2, ..., n; n \in \mathbb{N}^*$$

and p with the properties from relations (1.24) and (1.25), that is

$$\left| \left(\frac{g(z)}{z} \right)'' \right| > p, z \in W, \left| \frac{g'(z)}{z^2} + 1 \right| \geq \frac{p}{|z|^j}, j \in \mathbb{N}_1^*.$$

If:

$$|g_i(z)| > m, z \in W, i = 1, 2, .., n,$$

then the operator

$$T_{\alpha_i,\beta}(z) = \left\{ \beta \int_1^z t^{-1-\beta} \prod_{i=1}^{n} \left(\frac{t}{g_i(t) \cdot e^{g_i(t)}} \right)^{\frac{1}{\alpha_i}} dt \right\}^{\frac{1}{\beta}}$$

belongs to the class Σ.

3.7 Stellarity and convexity of the operator $T_{\alpha_i,1}(z)$

In this section, we will introduce the integral operator:

$$T_{\alpha_i,\beta}(z) = \left\{ \beta \int_1^z t^{-1-\beta} \prod_{i=1}^{n} \left(\frac{t}{g_i(t) \cdot e^{g_i(t)}} \right)^{\frac{1}{\alpha_i}} dt \right\}^{\frac{1}{\beta}}, \tag{3.28}$$

where $g_i(t) \neq 0; g_i(t) \in O_1, \alpha_i \in \mathbb{C}^*, \forall i \in 1, 2, .., n$, which is a generalization of the operator $F_{\alpha_i,\beta}(z)$ defined in [71] and of the operator $G_{\alpha_i,\beta}(z)$ defined in (3.8).

For $\beta = 1$, the operator $T_{\alpha_i,\beta}(z)$ becomes

$$T_{\alpha_i,1}(z) = \int_1^z t^{-2} \prod_{i=1}^{n} \left(\frac{t}{g_i(t) \cdot e^{g_i(t)}} \right)^{\frac{1}{\alpha_i}} dt. \tag{3.29}$$

The following theorem provides conditions for the belonging of the introduced operator to the class $O_1^*(0)$.

Theorem 3.7.1. *[59] Let $g_i \in O_1, \alpha_i \in \mathbb{C}^*, i \in \{1, ..., n\}$.*
If

$$\Re \left(\frac{z g_i'(z)}{g_i(z)} \right) < 1, \Re(1 + g_i(z)) < 1, \tag{3.30}$$

$$\big|\Re(z)\big| > \big|Im(z)\big|,$$

and

$$\Re(z^4) > 0, z \in W,$$

then $T_{\alpha_i,1}(z)$ belongs to the class $O_1^*(0)$.

The following theorem gives us a condition for the belonging of the operator $T_{\alpha_i,1}(z)$ to the class $O_1^*(\mu)$.

Theorem 3.7.2. *[59] For $i \in \{1,2,...,n\}$, let $\alpha_i \in \mathbb{C}^*$ and $g_i \in O(\gamma_i), 0 \leq \gamma_i < 1$. If $0 < \sum_{i=1}^{n} \frac{1}{\alpha_i}(1-\gamma_i) \leq 1$*

$$|\Re(z)| > |Im(z)|,$$

$$\Re(1 + g_i(z)) > 1, \tag{3.31}$$

and

$$\Re\left(-\frac{zg_i'(z)}{g_i(z)}\right) > -\gamma_i, \forall z \in W,$$

then the operator $T_{\alpha_i,1}(z)$ given by relation (3.29) belongs to the class $O_1^*(\mu)$, where $\mu = \sum_{i=1}^{n} \frac{1}{\alpha_i}(1-\gamma_i)$.

If we take $\gamma_i = \gamma$, $i \in \{1,2,...,n\}$ in Theorem 3.7.2, we obtain the following corollary.

Corollary 3.7.1. *[59] For $i \in \{1,2,...,n\}$, let $\alpha_i \in \mathbb{C}^*$, $g_i \in O_1(\gamma), 0 \leq \gamma < 1$. If $0 < \sum_{i=1}^{n} \frac{1}{\alpha_i} \leq \frac{1}{1-\gamma}$,*

$$|\Re(z)| > |Im(z)|,$$

$$\Re(1 + g_i(z)) > 1,$$

$$\Re\left(\frac{-z \cdot g_i'(z)}{g_i(z)}\right) > -\gamma, \forall z \in W,$$

then the operator $T_{\alpha_i,1}(z)$ given by relation (3.29) is starlike of order μ, where $\mu = (1-\gamma)\sum_{i=1}^{n} \frac{1}{\alpha_i}$.

The following theorem gives us a condition for the belonging of the operator $T_{\alpha_i,1}(z)$ to the class $O_k^*(\mu)$.

Theorem 3.7.3. *[59] For $i \in \{1,2,...,n\}$, let $\alpha_i \in \mathbb{C}^*$ and $g_i \in O_k(\gamma_i), 0 \leq \gamma_i < 1$. If $0 < \sum_{i=1}^{n} \frac{1}{\alpha_i} \cdot (2-\gamma_i) \leq 1, T_{\alpha_i,1}(z)$*

$$|\Re(z)| > |Im(z)|,$$

$$\Re(g_i(z)) > 1,$$

$$\Re(z^4) > 0, \forall z \in W,$$

then the operator $T_{\alpha_i,1}(z)$ given by relation (3.29) belongs to the class $O_k^*(\mu)$, where $\mu = \sum_{i=1}^{n} \frac{1}{\alpha_i} \cdot (2-\gamma_i)$.

Taking $\gamma_i = \gamma, i \in \{1, 2, ..., n\}$ in Theorem 3.7.3, we obtain the following result:

Corollary 3.7.2. *[59] For $i \in \{1, 2, ..., n\}$, let $\alpha_i \in \mathbb{C}^*$ and $g_i \in O_k(\gamma), -1 \leq \gamma < 1$. If $0 < \sum_{i=1}^{n} \frac{1}{\alpha_i} \leq \frac{1}{2-\gamma}$,*

$$|\Re(z)| > |Im(z)|,$$

$$\Re(g_i(z)) > 1,$$

and

$$\Re(z^4) > 0, \forall z \in W,$$

then the operator $T_{\alpha_i,1}(z)$ given by relation (3.29) is starlike of order μ, where $\mu = (2-\gamma) \cdot \sum_{i=1}^{n} \frac{1}{\alpha_i}$.

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