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Dual Representations and Subdifferential Formulae for Convex Risk Functions. Contributions to Extension Theorems for Set-Valued Maps

Ph.D. Thesis - Summary

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Contents

Introduction **1**

1 Preliminary results and notational conventions **7**

1.1 Preliminaries on sets 7

1.2 Preliminaries on functions 7

2 Dual representations for monotone and cash invariant hulls of risk functions **8**

2.1 Short summary on Lagrange duality 8

2.2 Risk Functions. Definitions and economic interpretations. 8

2.2.1 L^P spaces-short overview 8

2.2.2 Definitions and economic interpretations 9

2.3 Dual representations of monotone and cash invariant hulls 10

2.4 The situation of missing lower semicontinuity 14

3 Conjugate and subdifferential formulae for convex risk functions **18**

3.1 Conjugate functions and subdifferentiability-general approach 18

3.1.1 Conjugate functions 18

3.1.2 Subdifferentiability 19

3.1.3 Conjugate and subdifferentiability of sublinear functions 19

3.2 Conjugate and subdifferential formulae for convex risk functions via an utility model . . . 20

3.2.1 Entropic risk measure 22

3.2.2 The worst-case risk measure 23

3.3 Conjugate and subdifferential formulae for convex risk functions via duality theory 23

3.3.1 The generalized mean deviation of order p 25

3.3.2 The generalized mean upper/lower semideviations of order p from a target 27

3.4 An application - Conditional Value at Risk (CVaR) 29

3.4.1 Definition and economical signification 29

3.4.2 Conjugate and subdifferentiability of CVaR via duality theory 30

3.4.3 Conjugate and subdifferentiability of CVaR via an utility function 31

4 Extension theorems for convex set-valued maps **33**

4.1 Motivation 33

4.2 Background in notation and definitions for partially ordered spaces 34
4.3 Preliminaries on set-valued analysis 34
4.4 Extension theorems for linear continuous operators dominated by convex set-valued maps 35
4.5 Subgradients of Set-valued maps 37
4.5.1 General notions, definitions and remarks 37
4.5.2 Existence results 38
4.6 Extension theorems for closed convex processes 39

Bibliography **41**

Introduction

The optimization theory (alternatively called optimization or mathematical programming) is one of the most important branches of mathematics. It encompasses many areas of research with a wide range of applications in domains such as statistics, empirical sciences, computer science, engineering, economics, finance and even risk management. The origin of optimization theory may be traced in time around the 1940s, when Leonid Kantorovich developed the linear programming. Later, together with the apparition of the simplex method of George Dantzig, and the development of the duality theory [54], mathematics started to know a rapid progress. The ideas coming from linear programming inspired the development of the theory of convex functions, therefore, the works of Fenchel [48], Moreau [69] and Rockafeller [79] can be considered the cornerstones of convex analysis.

The study of classical optimization problems made as a result another step forward through the development of convex duality. Duality under all its three shapes, Lagrance, Fechel and Fechel-Lagrange, is an extensively studied method in all the areas of applied mathematics. As we shall see during this work we will also often employ this method in order to provide answers for the study of mathematical models in economy, insurance and risk management.

Economics is closely enough linked to optimization. In recent years there has been a growing attempt to use mathematical methods borrowed form economics and engineering for providing interpretations of the diversity of life. Conversely, different economical processes involving risk management can not be solved without employing different mathematical technics. Lately, significant progress has been made in developing the concept of a risk measure from both a theoretical and a numerical point of view. The first axiomatic way of defining risk measures has been given by Artzner, Delbaen, Eber and Heath in [2] and refers to *coherent risk measures*. Nevertheless, from the pioneer paper of Artzner et all., the literature on risk measures has known a rapid growth so it has become a standard in modern risk management to assess the riskiness of a portfolio by means of *convex risk measures*. The latter have been introduced by Föllmer and Schied in [52]. Rockafellar et.all. [81, 83, 84], being stipulated as an alternative class of risk functions, called *deviation measures* or *general deviation measures,* which are not translation invariant due to the fact that they are based on the difference $X - \mathbb{E}(X)$. The reader can find examples of coherent and convex risk measures or deviation measures in [31, 40, 41, 49, 50, 52, 75, 81, 82, 83, 84, 88, 93], some of them being objects of the investigations we make in the present thesis.

The present thesis is developed towards two main areas of research. In the first part we provide dual representation for monotone and invariant hulls of convex risk functions as well as subdifferential, and conjugate formulas for the most common risk measures presented in literature by means of two distinct techniques: the duality approach and by means of an utility model. The results provided in this direction have a wide range of applications in the risk management, portfolio optimization and in the financial field.

The other direction in this work is dedicated to a an area of research situated at the confluence of set-valued and nonsmooth analysis. Working with set-valued maps instead of functions became a necessity in modern analysis. Starting from the pioneering papers of Hahn and Banach [13],[57] which entirely revolutionized the functional analysis, there has been a good amount of interest in providing extension results not only for functionals and vector-valued functions but also for set-valued maps. Therefore through chapter 4, we aim to fill in this gap in the field and we provide some extension results for linear continuous operators dominated by convex set-valued maps under generalized interiority type conditions. We present also some applications under the form of existent results for weak and strong subgradients of set-valued maps.

One of the most widely used set of subgradients (subdifferential), appropriate for applications to optimization, is the one that first appeared in the context of convex analysis. From this model on, several types of subdifferentials appeared. Apart from the classical concept of Frèchet subdifferentiability, we mention also Mordukhovich, Iofee, Clark and the so called Dini Hadamard subdifferential. For the Dini-Hadamard-like ε-subdifferential we have recently provided in [10] calculus rules for the difference of important classes of nonsmooth functions. For more details on subgradients (subdifferential) of vector functions see the recent books of Mordukhovich [67, 68].

This thesis consists of four chapters, which are briefly presented in the following, underlining our most important results.

The **first Chapter**, as the title suggests, is dedicated to preliminary notions, conventional notations and a brief overview of the most important definitions and results from convex and functional analysis. The background for those notions is due to the monographs [24, 80] for finite dimensional spaces, while for the infinite dimensional case we mention [1, 27, 46, 86, 97].

In **Chapter 2**, we provide dual representations for monotone and cash invariant hulls of risk functions. In order to make this manuscript self contained in the first two subsections we simply draw the outlines for Lagrange duality and we fix the terminology for risk functions. Therefore in Section 2.1 we remind the most important generalized interiority notions which interfere in the expression of regularity conditions, $QC_1 - QC_4$. We use the above mentioned regularity conditions in order to ensure strong duality between the couple of problems (P) and (D). The Section 2.2 entitled, "Risk functions. Definitions and economic interpretations", consists of two subsections: the first one is dedicated to definitions and properties of L^p spaces (for $1 \leq p \leq +\infty$) while the last one is developed around the definition of risk function. The cornerstone of the investigations made in Chapter 2 and Chapter 3 is embodied in the notion of risk function. Definition 2.2.6 is an improved, refined adaptation of what the literature proposed in the field of risk analysis. Risk functions do not only have a crucial role in optimization under uncertainty but also a wide range of applications especially when dealing with the losses that may occure in finance and insurance industry. This is the reason why we analyze here the properties of risk functions from both a mathematical and an economical point of view.

The results of Section 2.3 are motivated by the paper of Filipović and Kupper [49], where for a convex risk function the so-called *monotone cash-invariant hull* has been introduced which is actually the greatest monotone and cash-invariant function majorized by the risk function in discussion. This function has been formulated in Definition 3, by making use of the *infimal convolution*. In other words, the monotone cash-invariant hull at a given point is nothing else than the optimal objective value of a convex optimization problem. Having as a starting point this observation, we give a dual representation of the monotone and cash-invariant hull by employing the Lagrange duality theory along with a qualification condition, under the hypothesis that the risk function is *lower semicontinuous*. This guarantees the vanishing of the duality gap and, implicitly, the validity of the dual representation. The examples considered in [49] are discussed from this new point of view. Furthermore, different to the approach in [49, Subsection 5.3], the use of the strong duality theory allows us to guarantee the attainment of the supremum in all the Examples 2.3.9 - 2.3.14.

In the last section of this chapter we deal with the same problem as in Section 2.3, but by considering this time a convex risk function which does not fulfill the lower semicontinuity assumption. For this function we can easily establish the *monotone hull* and we can also give a dual representation for it by making use of the quasi-relative interiority-type qualification condition (QC_4). We also refer to the limitations of this approach in the context of the determination of the *monotone cash-invariant hull* for the function in discussion.

The author's achievements within this chapter were published in [34] and are embodied in Theorems: 2.3.4, 2.3.7, Corollaries: 2.3.5, 2.3.6, Remarks: 2.3.8, 2.3.15, 2.4.1 and Examples: 2.3.9, 2.3.10, 2.3.11, 2.3.12, 2.3.13, 2.3.14. Also Section 2.4 consists entirely of original results but since this part furnished a limitation of the approach in the context of the determination of the monotone cash-invariant hull of a risk function, which is no longer lower semicontinuous, we can not punctually indicate a simple mathematical result which embodies the whole rationality. Nevertheless, the discussion provided during this section is based on viable mathematical results and remarks.

The **third Chapter** starts also with an introductory section. This time we aim to provide the reader with all the necessary tools to follow the conjugate and the subdifferential calculus developed further, under the context of risk measures. Therefore, subsection 3.1.1 collects the most important properties of conjugate functions since the conjugates play a central role in the development of the dual representations of risk functions. Section 3.1.2 is dedicated to the (convex)subdifferentia notion, since in economical applications having manageable subdifferential formulae for risk functions is vital for solving some classes of portfolio optimization problems. Finally, the last part of Section 3.1 joins both the conjugate and the subdifferential properties for sublinear functions, since the positively homogeneous property of convex functions often makes the difference between the most important classes of risk function, that is the convex and the coherent ones.

One of the most challenging topics in convex analysis is the formulation of optimality conditions for portfolio optimization problems with a convex risk measure as objective function. Since for this class of functions differentiability is not necessarily guaranteed, one will be forced to make use of the convex subdifferential when characterizing optimality (see for instance [32]). This is why being in the possession of easily handleable formulae for the subdifferential of the risk measures, is important to be taken in consideration in this context. Among the most relevant literature on this topic one has to mention [75, 81, 83, 84, 88].

We propose further two distinct ways of providing subdifferential formulae for convex risk function, by means of an utility function on one hand and combining classical results of convex analysis and duality theory on the other hand. We discuss and determine during this chapter the conjugate and subdifferential formulae for the case of usual risk functions and also for their natural extensions, which can easily interfere in some standard optimization problems.

In Section 3.2 we consider a generalized convex risk measure defined via a so-called *utility function* and associated with the Optimized Certainty Equivalent (OCE), a notion introduced and explored in [15, 16]. This convex risk measure is expressed as an infimal value function, thus we provide first of all a weak sufficient condition for the attainment of the infimum in its definition. Further, we give formulae for its conjugate function (Theorem 3.2.6) and its subdifferential (Theorem 3.2.8). The generalized convex risk measure we consider has the advantage that, for some particular choices of the utility function, it leads to some well-known convex risk measures. Consequently, we are able to derive, the conjugate and subdifferential formulae for the *entropic risk measure*, in Subsection 3.2.1 and the *worst-case risk measure* in Subsection 3.2.2.

Unfortunately a lot of risk functions, widely spread in practical applications, can be neither described by means of utility functions, nor inscribed in the general background of coherent or expectation bounded risk measures. For all those classes of functions the only hope for providing conjugate and subdifferential formulae lies in classical computation, using methods based on standard results of convex analysis, and this is what we do during our third section. What we present there is in fact a pattern which can be successfully applied even for risk functions which are not characterized by very good mathematical properties like positive homogeneity, monotonicity and cash invariance. We provide under this context dual representations and handleable formulas for the subdifferential of important generalized risk functions:

- *the generalized mean deviation*

$$\rho(X) = \|X - \mathbb{E}(X)\|_p^a - \mathbb{E}(X),\ \forall X \in L^p, p \in [1, \infty] \text{ and } a \geq 1,$$

as introduced in [31] and

- *the generalized mean upper/lower deviations of order p from a target,*

$$\rho_{\tau_\pm}(X) = \|(X - \tau)_\pm\|_p^a - \mathbb{E}(X),\ \forall X \in L^p, p \in [1, \infty], a \geq 1 \text{ and } \tau \in \mathbb{R}.$$

Moreover in order to exemplify the developments made during this Chapter we discuss in Section 3.4, the case of *the Conditional Value at Risk* (*CVaR*) from both of the perspectives presented above, namely by means of an utility model, on one hand and by means of the duality approach, on the other hand. In Section 3.4.2 we first derive the conjugate of $CVaR$, starting from the conjugate of a more general risk function -the so called *Generalized Conditional Value at Risk* (*GCVaR*) (see [65]). Having this conjugate, one can easily obtain, in terms of relation (3.2), the corresponding subdifferential formula. In the last section we use the method developed in Section 3.2, and we deliver, by means of an appropriate utility function, both the conjugate and the subdifferential formulae of the Conditional Value-at-risk.

The author's achievements within this area of research are synthesized as follows: Theorems: 3.2.6, 3.2.7, 3.2.8, 3.3.3, 3.3.5, 3.3.18, 3.3.19, 3.3.21, 3.3.22, 3.3.23, 3.3.24, 3.4.4, 3.4.7, Corollaries: 3.4.5, 3.4.6, Remarks: 3.2.1, 3.2.3, 3.3.4, 3.3.6, 3.3.7, 3.3.13, Propositions: 3.2.2, 3.2.5, Lemma 3.3.1 and the examples provided in the Subsections 3.2.1, 3.2.2 and 3.4.3 respectively. The results of this chapter are partially included into 5 papers, see [6], [7], [9], [11], [34].

In **Chapter 4** we aim on one hand to give new extension theorems for convex set-valued maps and on the other hand we intend to emphasize their applicability in the field of nonsmooth analysis. The Sections 4.1, 4.2 and 4.3 are introductory ones. We provide here our motivation for choosing this subject as well as the general background for the study. Section 4.3 briefly summarizes the most important definitions, notions and properties for set-valued maps. For a comprehensive study of set-valued analysis we refer to the books of Aubin and Frankowska [3] and Aubin and Ekeland [4].

Section 4.4 consists entirely of original results. This part of the work is motivated by a series of mistakes done in the pioneering papers concerning extension results for set-valued maps see [38, 70, 71]) which were recently underlined by Zălinescu in [96]. The Hahn-Banach theorem, in its geometrical (separation theorems) or analytical form, is a powerful tool which resonates through important fields of mathematics such as: functional analysis, convex analysis and optimization theory. Generalizations and variants of the extension theorems of dominated maps and implicitly generalized Hahn-Banach theorem, were developed in different directions in the past. Unfortunately most of those extensions were made only in the context of linear spaces, the case of topological spaces being superficially discussed. During this section we give new extension theorems for convex set-valued maps in partially ordered topological spaces, Lagrange multiplier theorems and sandwich theorems under weaker topological interiority assumptions, namely the *strong quasi relative interior* (*sqri*) ones .

Section 4.5 is devoted to applications, namely existence results for strong subgradients of set-valued maps. After a brief summary concerning the different notions of subgradients appeared in the field, we give existence results for two different approaches of the concept of subgradients of set-valued maps the one imposed by Borwein in [19] on one hand, and the recent ones introduces by E. Hernadez and L.R Marin in [58] on the other hand. We also discuss the connections between them.

And finally, the last section provides some norm preserving extension results for real valued closed convex processes. More precisely, Theorem 61 and Theorem 62 extend Hahn's extension theorems for continuous linear functionals to the general framework of set-valued analysis. As a direct consequence of those results, we characterize in Theorem 63 the elements of best approximation in normed linear spaces by elements of closed convex cones using closed convex processes.

The author's original contributions within this area of research are included in Theorems: 4.4.3, 4.4.8, 4.4.11, 4.5.7, 4.5.8, 4.5.9, 4.6.1, 4.6.2, 4.6.3, Corollaries: 4.4.10, 4.5.10, Remarks: 4.4.4, 4.4.5, 4.4.9, 4.5.3, 4.5.6, 4.5.11 and Examples: 4.4.6, 4.4.7, 4.4.12. The above results can be partially found in [8], [10] and [12].

Keywords: convex analysis, set-valued analysis, conjugate functions, (convex) subdifferential, Lagrange duality, monotone and cash-invariant hulls, qualification conditions, convex risk measures, utility function, optimized certainty equivalent, Conditional Value-at-Risk, Value-at-Risk, convex set-valued map, closed convex process, Hahn-Banch extension theorem, Lagrange multiplier theorem, sandwich theorem, strong subgradient.

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Chapter 1

Preliminary results and notational conventions

In this chapter we collect some standard definitions, results and notational conventions from convex and functional analysis, which will be frequently used throughout this work. For the background on functional analysis we refer to [1, 47, 85, 86] while for convex analysis we consider [24, 25, 27, 45, 46, 80, 97].

1.1 Preliminaries on sets

In this section we recall the definitions and main properties of set. We reminde the notions of convex and conical hulls and we present briefly the main classes of cones which interfere in convex analysis (the normal cone, Bouligand tangent cone and the asymptotic cone).

1.2 Preliminaries on functions

Let $\mathcal{X}$ be a separated locally convex space and $\mathcal{X}^*$ its topological dual space then $\langle x^*, x\rangle$ denotes the value of the linear continuous functional $x^* \in \mathcal{X}^*$ at $x \in \mathcal{X}$. For a given set $C \subseteq \mathcal{X}$, we consider its indicator function
$\delta_C : \mathcal{X} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, and its support function $\sigma_C : \mathcal{X}^* \to \overline{\mathbb{R}}$.

For a given function $f : \mathcal{X} \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom} f$ its effective domain and by $\operatorname{epi} f$ its epigraph, respectively. We recall further the definitions and main properties of proper, convex (concave), positively homogeneous, lower semicontinuous (upper semicontinuous) and asymptotic functions. Having $f_i : \mathcal{X} \to \overline{\mathbb{R}}, i = 1, \dots, n$, given proper functions we denote by $f_1 \square ... \square f_n : \mathcal{X} \to \overline{\mathbb{R}}$, $f_1 \square ... \square f_n(x) := \inf \left\{\sum_{i=1}^n f_i(x_i) : \sum_{i=1}^n x_i = x\right\}$, forall $x \in \mathcal{X}$, their infimal convolution. The main properties of infimal convolution are also stated.

Chapter 2

Dual representations for monotone and cash invariant hulls of risk functions

2.1 Short summary on Lagrange duality

First we turn our attention to the *Lagrange duality* for an optimization problem with geometric and cone constraints, as this is the key element for our development during the last two sections of this chapter.

The intention for this section is to create, as the title suggests, a short summary on Lagrange duality, starting with classical results of convex analysis (see [26, 27, 46, 80, 86, 97]) and ending with the most recent results regarding regularity conditions (see [30, 28, 29, 37, 55]).

We present in this section the definitions of the main generalized interiority notions, i.e. algebraic interior, strong quasi-relative interior, quasi-relative interior and quasi interior. We give also some generalized interiority type qualification conditions, which will be used several times in this work in order to guarantee strong duality between different classes of problems.

2.2 Risk Functions. Definitions and economic interpretations.

Risk functions have not only a crucial role in optimization under uncertainty but they also have a wide range of applications especially when dealing with the losses that my be incurred in finance and insurance industry. Since the entire theory of risk and uncertainty is developed in probability spaces, and lately it become a standard to work whit L^p spaces, $1 \leq p \leq +\infty$, we start this section with a small overview on L^p spaces for $p > 0$. We use for this background the books of Folland [51] and Aliprantis [1].

2.2.1 L^P spaces-short overview

Throughout this work, we assume that the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, is *atomless* (i.e. is rich enough to support a random variable with a continuous distribution). Here Ω denotes the space of future states ω, $\mathfrak{F}$ is a *σ-algebra* on Ω and $\mathbb{P}$ is a *probability measure* on $(\Omega, \mathfrak{F})$.

For a measurable random variable $X : \Omega \to \mathbb{R} \cup \{+\infty\}$ the *expectation value* with respect to $\mathbb{P}$ is defined by $\mathbb{E}(X) := \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$. Whenever X takes the value $+\infty$ on a subset of positive measure we have $\mathbb{E}(X) = +\infty$. The *essential supremum* of X, which represents the smallest essential upper bound of the random variable, is denoted $\operatorname{essup} X$. Similarly the *essential infimum* is defined by $\operatorname{esinf} X := -\operatorname{essup}(-X)$. The *characteristic function* of a set $G \in \mathfrak{F}$ is $\mathbf{1}_G : \Omega \to \mathbb{R}$.

For a measurable random variable $X : \Omega \to \mathbb{R}$ we consider for $0 < p < +\infty$, the norm $\|X\|_p = \left(\int_\Omega |X(\omega)|^p d\mathbb{P}\right)^{\frac{1}{p}} = (\mathbb{E}(|X|^p))^{\frac{1}{p}}$ and we define the spaces

$$L^p(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) := \left\{ X : \Omega \to \mathbb{R} : X \text{ is measurable}, \int_\Omega |X(\omega)|^p d\mathbb{P}(\omega) < +\infty \right\}.$$

To complete the picture of L^p spaces, we introduce the space of essentially bounded random variables, corresponding to the limiting value $p = \infty$, namely

$$L^\infty(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) := \{X : \Omega \to \mathbb{R} : X \text{ is measurable}, \ \operatorname{essup} |X| < +\infty\},$$

which is being equipped with the norm $\|X\|_\infty = \operatorname{essup} |X|$.

Theorem 2.2.2 [1, 51] For $1 \leq p \leq \infty$, the space L^p equipped with the norm $\|\cdot\|_p$ is a Banach space.

We denote the topological dual space of L^p by $(L^p)^*$ and for $p \in [1, \infty)$ one has that $(L^p)^* = L^q$, where $q \in (1, \infty]$ fulfills $q = p/(p-1)$ (with the convention $1/0 = \infty$). In what concerns $(L^\infty)^*$, the topological dual space of L^∞, can be identified with *ba,* the space of all bounded finitely additive measures on $(\Omega, \mathfrak{F})$ which are absolutely continuous with respect to $\mathbb{P}$. This is usually much bigger than L^1, i.e. $L^1 \subset (L^\infty)^*$. But endowing L^∞ with the weak topology $\sigma_\infty(L^\infty, L^1)$ and L^1 with the weak topology $\sigma_1(L^1, L^\infty)$ one obtains the dual pairing $(L^\infty, \sigma_\infty)^* = (L^1, \sigma_1)$. In the present thesis we will use this identification whenever we develop duality reasonings.

Each random variable $X : \Omega \to \mathbb{R}$ can be represented as $X = X_+ - X_-$, where $X_+, X_- : \Omega \to \mathbb{R}$ are the random variables defined by

$$X_+(\omega) = \max\{0, X(\omega)\} \quad \text{and} \quad X_-(\omega) = \max\{0, -X(\omega)\}, \ \forall \omega \in \Omega.$$

The equalities and inequalities between random variables are to be seen in an almost everywhere way (a.e.).

2.2.2 Definitions and economic interpretations

In this section we give a formal definition of risk functions and we discuss their properties from both the mathematical and the economical viewpoints.

The notion we introduce next is in fact an improved, refined definition of what the literature proposed in the field of risk analysis.

Definition 2.2.6 We call *risk function* a proper function $\rho : L^p \to \overline{\mathbb{R}}, p \in [1, \infty]$. The risk function ρ is said to be

(*i*) *convex*, if: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \forall \lambda \in [0,1], \forall X, Y \in L^p$;

(*ii*) *positively homogeneous*, if: $\rho(0) = 0$ and $\rho(\lambda X) = \lambda\rho(X), \forall \lambda > 0, \forall X \in L^p$;

(*iii*) *monotone*, if: $X \geq Y \Rightarrow \rho(X) \leq \rho(Y),\ \forall X, Y \in L^p$;

(*iv*) *expectation-bounded*, if: $\rho(X) \geq -\mathbb{E}(X),\ \forall X \in L^p$;

(*v*) *cash-invariant*, if: $\rho(X + a) = \rho(X) - a,\ \forall X \in L^p,\ \forall a \in \mathbb{R}$;

(*vi*) a *convex risk measure (cf. [52])*, if: ρ is convex, monotone and cash-invariant;

(*vii*) a *coherent risk measure (cf. [2])*, if: ρ is a positively homogeneous convex risk measure;

(*viii*) *expectation-bounded risk measure (cf. [83, 84])*, if: ρ is a positively homogeneous, convex, cash-invariant and expectation bounded risk function.

In order to avoid confusions we will call *convex risk function* each risk function which additionally satisfies only the hypothesis of convexity, as Ruszczynski and Shapiro proposed in [88, 89].

Judging from an economical point of view the elements of $L^p, p \in [1, \infty]$ can be seen as describing future net worths, while the value $\rho(X)$ can be understood as a capital requirement for X. Since the applicability of risk functions in risk management for insurance companies and financial markets is crucial, we present briefly, the economical significance of all the mathematical properties mentioned in Definition 2.2.6. Consequently, a convex risk measures guarantees that the capital requirement of the convex combination of two positions does not exceed the convex combination of the capital requirements of the positions taken separately. The monotonicity property says that if one has the certitude that Y will be smaller than X in (almost) every state of the world, than the capital requirement for Y should be greater than for X. Cash-invariance means that adding a constant amount of money a to X should reduce the capital requirement for X by a. For the economic interpretation of the other notions given in Definition 2.2.6 we refer to [2, 43, 52, 53, 65, 75, 77].

Strictly connected with expectation bounded risk measures are Rockafeller's deviation measures, which are also defined in this section.

2.3 Dual representations of monotone and cash invariant hulls

Throughout the economical literature one finds a vast variety of risk functions, along the coherent and convex ones some very irregular ones, which are neither monotone nor cash-invariant, being also present. In order to overcome the lack of monotonicity or cash-invariance and to provide better tools for quantifying risk, Filipović and Kupper have proposed in [49] the notions of *monotone* and *cash-invariant hulls*, which are the greatest monotone and, respectively, cash-invariant functions majorized by the risk function in discussion. For the majority of the examples treated in [49] these hulls are not given in their initial formulation, but tacitly some dual representations of them are used.

In this section we show that these dual representations are nothing else than the dual problems of the primal optimization problems hidden in the definition of the monotone and cash-invariant hulls and we also formulate sufficient qualification conditions for the existence of strong duality. This is the premise for making the dual representations viable. Finally, we discuss the examples from [49] and show that for those particular situations the qualification conditions are automatically fulfilled, fact which permits the formulation of *refined* dual representations.

For the beginning we work in the general setting of a separated locally convex vector space $\mathcal{X}$ with $\mathcal{X}^*$ its topological dual space. Further, let $\mathcal{P}$ be a nonempty convex closed cone in $\mathcal{X}$, $\Pi \in \mathcal{X} \setminus \{0\}$ and $f : \mathcal{X} \to \overline{\mathbb{R}}$ a proper function. The following notions have been introduced in [49] having as a starting point the corresponding ones in the definition of a convex risk measure.

Definition 2.3.1 The function f is called:

(*i*) $\mathcal{P}$-*monotone*, if: $x \geq_{\mathcal{P}} y \Rightarrow f(x) \leq f(y),\ \forall x, y \in \mathcal{X}$;

(ii) Π-*invariant*, if: $f(x + a\Pi) = f(x) - a,\ \forall x \in \mathcal{X},\ \forall a \in \mathbb{R}$.

If $\mathcal{X} = L^p$, $\mathcal{P} = L^p_+$ and $\Pi = 1$, then one rediscovers in the definition above the monotonicity and cash-invariance, respectively, as introduced in Definition 2.

Before introducing the following notions we consider the set $\mathcal{D} := \{x^* \in \mathcal{X}^* : \langle x^*, \Pi\rangle = -1\}$ and the conjugate of its indicator function, $\delta^*_{\mathcal{D}}$.

Definition 2.3.3 For the given function f we call

(*i*) $\mathcal{P}$-*monotone hull* of f the function $f_{\mathcal{P}} : \mathcal{X} \to \overline{\mathbb{R}}$ defined as

$$f_{\mathcal{P}}(x) := f \Box \delta_{\mathcal{P}}(x) = \inf\{f(y) : y \in \mathcal{X}, x \geq_{\mathcal{P}} y\};$$

(*ii*) Π-*invariant hull* of f the function $f_{\Pi} : \mathcal{X} \to \overline{\mathbb{R}}$ defined as

$$f_{\Pi}(x) := f \Box \delta^*_{\mathcal{D}}(x) = \inf_{a\in\mathbb{R}}\{f(x - a\Pi) - a\};$$

(*iii*) $\mathcal{P}$-*monotone* Π-*invariant hull* of f the function $f_{\mathcal{P},\Pi} : \mathcal{X} \to \overline{\mathbb{R}}$ defined as

$$f_{\mathcal{P},\Pi}(x) := f \Box \delta_{\mathcal{P}} \Box \delta^*_{\mathcal{D}}(x) = \inf\{f(y) - a : y \in \mathcal{X}, a \in \mathbb{R}, x \geq_{\mathcal{P}} y + a\Pi\}.$$

Obviously,

$$\operatorname{dom} f_{\mathcal{P}} = \operatorname{dom} f + \mathcal{P}, \quad \operatorname{dom} f_{\Pi} = \operatorname{dom} f + \mathbb{R}\Pi$$

and

$$\operatorname{dom} f_{\mathcal{P},\Pi} = \operatorname{dom} f + \mathcal{P} + \mathbb{R}\Pi.$$

Moreover, f is $\mathcal{P}$-monotone if and only if $f = f_{\mathcal{P}}$, while f is Π-invariant if and only if $f = f_{\Pi}$.

In the following we assume that f is a proper and convex function and we provide a dual representation for $f_{\mathcal{P},\Pi}$, by making use of the convex duality theory. This approach is based on the observation that the value of the $\mathcal{P}$-monotone Π-invariant hull at a given point is nothing else than the optimal objective value of a convex optimization problem.

Theorem 2.3.4 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a proper and convex function and $x \in \operatorname{dom} f + \mathcal{P} + \mathbb{R}\Pi$. If one of the following qualification conditions

$$\exists\, (y', a') \in \operatorname{dom} f \times \mathbb{R} \text{ such that } y' + a'\Pi - x \in -\operatorname{int} \mathcal{P} \tag{2.1}$$

and

$$\mathcal{X} \text{ is a Fréchet space}, f \text{ is lower semicontinuous and } x \in \operatorname{sqri}(\operatorname{dom} f + \mathbb{R}\Pi + \mathcal{P}) \tag{2.2}$$

is fulfilled, then one has

$$f_{\mathcal{P},\Pi}(x) = \max_{\substack{x^* \in -\mathcal{P}^* \\ \langle x^*, \Pi \rangle = -1}} \{\langle x^*, x\rangle - f^*(x^*)\}, \tag{2.3}$$

where by the use of max instead of sup we signalize the fact that the supremum is attained.

The case when the objective functions, f is either $\mathcal{P}$-monotone or Π-invariant is discussed in the following corollaries.

Corollary 2.3.5 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a proper, convex and $\mathcal{P}$-monotone function and $x \in \operatorname{dom} f + \mathcal{P} + \mathbb{R}\Pi$. If one of the qualification conditions (2.1) or (2.2) is fulfilled then one has

$$f_{\mathcal{P},\Pi}(x) = f_{\Pi}(x) = \max_{\langle x^*, \Pi \rangle = -1} \{\langle x^*, x\rangle - f^*(x^*)\}. \tag{2.4}$$

Corollary 2.3.6 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a proper, convex and f Π-invariant function and $x \in \operatorname{dom} f + \mathcal{P} + \mathbb{R}\Pi$. If one of the qualification conditions (2.1) or (2.2) is fulfilled then one has

$$f_{\mathcal{P},\Pi}(x) = f_{\mathcal{P}}(x) = \max_{x^* \in -\mathcal{P}^*} \{\langle x^*, x\rangle - f^*(x^*)\}. \tag{2.5}$$

In the following we investigate the verifiability of the qualification conditions in the context of risk measure theory, namely by assuming that $\mathcal{X} = L^p$ and $\mathcal{P} = L^p_+$ for $p \in [1, \infty]$. To this end we assume that f *is lower semicontinuous*. A situation when f fails to have this topological property will be adressed during the next section.

Theorem 2.3.7 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) For $p \in [1, \infty]$ let $f : L^p \to \overline{\mathbb{R}}$ be a convex risk function. If one of the following conditions

• when $p \in [1, \infty]$

$$\text{f is lower semicontinuous and } -L^p_+ \subseteq \operatorname{dom} f; \tag{2.6}$$

• when $p = \infty$

$$\operatorname{esinf} \Pi \cdot \operatorname{essup} \Pi > 0; \tag{2.7}$$

is fulfilled, then one has for all $X \in L^p$ that

$$f_{\mathcal{P},\Pi}(X) = \max_{\substack{X^* \in -(L^p_+)^* \\ \mathbb{E}(X^*\Pi) = -1}} \{\mathbb{E}(X^*X) - f^*(X^*)\},$$

where by the use of max instead of sup we signalize the fact that the supremum is attained.

Remark 2.3.8 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) One can notice that for $p = \infty$ the condition (2.7) in the theorem above is fulfilled when $\Pi \in L^\infty$ is a *constant numeraire.*

In the last part of this section we discuss the examples treated in [49] from the new perspective given by the duality theory. We investigate the fulfillment of the conditions (2.6) and (2.7) and we also provide some refined dual representations for the risk functions in discussion. The notions of *monotone* and *cash-invariant* will be used instead of L^p_+*-monotone* and 1*-invariant*, respectively. The same applies when we speak about the corresponding hulls. For this summary we mention only a few examples.

Example 2.3.9 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) For $p \in [1, \infty)$ and $c > 0$ consider the L^p *deviation risk measure* $f : L^p \to \mathbb{R}$ defined by

$$f(X) = c\|X - \mathbb{E}(X)\|_p - \mathbb{E}(X).$$

This is a convex, continuous and cash-invariant ($\Pi = 1$) risk function, but not monotone in general. For the conjugate formula of the L^p deviation risk measure we refer to [31]. This is for $X^* \in L^q$ given by

$$f^*(X^*) = \begin{cases} 0, & \text{if } \exists\, Y^* \in L^q \text{ such that } c(Y^* - \mathbb{E}(Y^*)) - 1 = X^*,\ \|Y^*\|_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

As $\operatorname{dom} f = L^p$, (2.6) is valid and thus the *monotone hull* of f looks for all $X \in L^p$ like (see also Remark 6)

$$f_{L^p_+,1}(X) = f_{L^p_+}(X) = \max_{\substack{\|Y^*\|_q \leq 1 \\ c(Y^* - \mathbb{E}(Y^*)) \leq 1}} c[\mathbb{E}(Y^*)\mathbb{E}(X) - \mathbb{E}(Y^*X)] - \mathbb{E}(X).$$

In this way we rediscover the formula given in [49, Subsection 5.1].

Example 2.3.11 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) For $p \in [1, \infty)$ and $c > 0$ consider the *mean-L^p risk measure* $f : L^p \to \mathbb{R}$ defined as

$$f(X) = c/p\mathbb{E}(|X|^p) - \mathbb{E}(X) = c/p\|X\|_p^p - \mathbb{E}(X),$$

which is a convex and continuous risk function but neither monotone nor cash-invariant ($\Pi = 1$). Its conjugate function can be easily derived from [31] and for $X^* \in L^q$ it looks like

$$f^*(X^*) = \frac{p-1}{pc^{\frac{1}{p-1}}}\mathbb{E}(|X^*+1|^q).$$

Again, $\operatorname{dom} f = L^p$, which means that the *monotone cash-invariant hull* of f has for all $X \in L^p$ the following formulation

$$f_{L^p_+,1}(X) = \max_{\substack{X^* \in -L^q_+ \\ \mathbb{E}(X^*)=-1}} \mathbb{E}\left[X^*X - \frac{1}{c^{q-1}q}|X^*+1|^q\right].$$

Different to the approach in [49, Subsection 5.3], the use of the strong duality theory allows us to guarantee the attainment of the supremum in the formula above.

Example 2.3.14 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) For $p=\infty$ the so-called *logarithmic risk measure* $f : L^\infty \to \overline{\mathbb{R}}$,

$$f(X) = \begin{cases} \mathbb{E}(-\ln(X)) - 1, & \text{if } X > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

is a convex, lower semicontinuous and monotone risk function which fails to be cash-invariant ($\Pi = 1$). Its conjugate function is given for $X^* \in (L^\infty)^*$ by

$$f^*(X^*) = \sup_{X>0}\{\langle X^*, X\rangle + \mathbb{E}(\ln(X)+1)\}$$

and can be further calculated by using [85, Theorem 14.60]. Indeed, one has

$$f^*(X^*) = \mathbb{E}\left\{\sup_{x>0}\{X^*x + \ln(x) + 1\}\right\} = \begin{cases} -\mathbb{E}(\ln(-X^*)), & \text{if } X^* < 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Before giving a dual representation for the cash-invariant hull of the logarithmic risk measure, one should notice that we are now in a situation where (2.6) fails, but (2.7) is valid. Consequently, the *cash-invariant hull* of f can be for all $X \in L^\infty$ given by

$$f_{L^\infty_+,1}(X) = f_1(X) = \max_{\substack{X^* \in (L^\infty)^*, X^*>0 \\ \mathbb{E}(X^*)=1}} \mathbb{E}[-X^*X + \ln(X^*)].$$

The next section points out also an interesting situation where no qualification condition of generalized interiority type can be applied.

2.4 The situation of missing lower semicontinuity

In the following we deal with the same problem of furnishing dual representations for the monotone and cash-invariant hull of a convex risk function by using the duality approach developed in Section 2.3, treating the particular case of a risk function which *fails to be lower semicontinuous*. We also discuss the difficulties which can arise when this topological assumption is missing. The section consists entirely in original results embodied in [34].

For $p \in [1, \infty]$ consider $f : L^p \to \overline{\mathbb{R}}$ defined by

$$f(X) = \begin{cases} \|X - \mathbb{E}(X)\|_p, & \text{if } X_- \in L^\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

This risk function is convex and fails to be lower semicontinuous for $p \in [1, \infty)$. One can easily verify that $\operatorname{dom} f = L^\infty + L^p_+$.

Like in the previous section we take as ordering cone L^p_+, but work with a *not necessarily constant numeraire* $\Pi \in L^p \setminus \{0\}$. Our goal is to furnish a dual representation for the *monotone* Π-*invariant hull* of f. To this end we will make use of the conjugate formula of $Y \mapsto \|Y - \mathbb{E}(Y)\|_p$, $p \in [1, \infty]$, which looks for $X^* \in (L^p)^*$ like (see [31, Fact 4.3])

$$(\|\cdot - \mathbb{E}(\cdot)\|_p)^*(X^*) = \begin{cases} 0, & \text{if } \exists\, Y^* \in (L^p)^*, \|Y^*\|_{(L^p)^*} \leq 1, \text{ s.t. } X^* = Y^* - \mathbb{E}(Y^*) \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.8}$$

The case $p = \infty$. In this situation $\operatorname{dom} f = L^\infty$, f is a convex and *continuous* function and one can, consequently, use the qualification condition (2.6), which is obviously fulfilled. Thus for the monotone Π-invariant hull of f one can employ again formula (2.3). This means that, by taking into consideration (2.8), the *monotone* Π-*invariant hull* of f looks for all $X \in L^\infty$ like

$$f_{L^\infty_+,\Pi}(X) = \max_{\substack{\|Y^*\|_{(L^\infty)^*} \leq 1, \mathbb{E}(Y^*) - Y^* \in (L^\infty_+)^* \\ \mathbb{E}(Y^*\Pi) - \mathbb{E}(Y^*)\mathbb{E}(\Pi) + 1 = 0}} \mathbb{E}(Y^*X) - \mathbb{E}(Y^*)\mathbb{E}(X). \tag{2.9}$$

One can easily notice that if Π is a *constant numeraire*, then $f_{L^\infty_+,\Pi} \equiv -\infty$.

The case $p \in [1, \infty)$. In this second case we proceed as follows: we first establish the monotone hull of f, along with a dual representation for it, then we discuss which are the difficulties that appear when trying to determine the dual representation of $f_{L^p_+,\Pi}$. Recall that $f_{L^p_+,\Pi}(X) = (f_{L^p_+})_\Pi(X)$ for all $X \in L^p$.

As $\operatorname{dom} f_{L^p_+} = \operatorname{dom} f + L^p_+ = L^\infty + L^p_+$, for every X outside this set one has $f_{L^p_+}(X) = +\infty$. For $X \in L^\infty + L^p_+$ we have

$$f_{L^p_+}(X) = \inf_{\substack{Y \in L^\infty + L^p_+ \\ Y - X \in -L^p_+}} \|Y - \mathbb{E}(Y)\|_p \tag{2.10}$$

and, obviously, $f_{L^p_+}(X) \geq 0$. On the other hand, since $X = Z + Y$ for $Z \in L^\infty$ and $Y \in L^P_+$, it holds $X \geq \operatorname{esinf} Z$, thus $\operatorname{esinf} Z$ is feasible for the optimization problem in the right-hand side of (2.10), which means that $f_{L^p_+}(X) = 0$. Consequently, $f_{L^p_+} = \delta_{L^\infty + L^p_+}$.

Before furnishing the monotone Π-invariant hull of f, let us shortly investigate how one could give dual representation for $f_{L^p_+}$. For $X \in L^\infty + L^p_+$ fixed one has to consider the convex optimization problem

$$\inf_{\substack{Y\in L^\infty+L^p_+\\ Y-X\in -L^p_+}} \|Y-\mathbb{E}(Y)\|_p \tag{2.11}$$

and its *Lagrange dual problem* (notice that L^∞ is dense in L^p)

$$\sup_{X^*\in -L^q_+}\{\langle X^*,X\rangle-(\|\cdot-\mathbb{E}(\cdot)\|_p)^*(X^*)\} = \sup_{\substack{\|Y^*\|_q\leq 1,\\ \mathbb{E}(Y^*)-Y^*\in L^q_+}} \mathbb{E}(Y^*X)-\mathbb{E}(Y^*)\mathbb{E}(X). \tag{2.12}$$

In order to show that for the primal-dual pair (2.11)-(2.12) strong duality holds, we verify a quasi-relative interior-type condition. Consequently, it follows that

$$f_{L^p_+}(X) = \max_{\substack{\|Y^*\|_q\leq 1,\\ \mathbb{E}(Y^*)-Y^*\in L^q_+}} \mathbb{E}(Y^*X)-\mathbb{E}(Y^*)\mathbb{E}(X)$$

and so one obtains for the monotone hull of f for all $X\in L^p$ the following dual representation

$$f_{L^p_+}(X) = \begin{cases} \max\limits_{\substack{\|Y^*\|_q\leq 1,\\ \mathbb{E}(Y^*)-Y^*\in L^q_+}} \mathbb{E}(Y^*X)-\mathbb{E}(Y^*)\mathbb{E}(X), & \text{if } X\in L^\infty+L^p_+,\\ +\infty, & \text{otherwise.}\end{cases}$$

The *monotone* Π-*invariant hull* of f is the Π-invariant hull of $f_{L^p_+}$ and for its derivation we use the direct formulation of the latter, $f_{L^p_+}=\delta_{L^\infty+L^p_+}$, as it is easier to handle with. For all $X\in L^P$ the monotone Π-invariant hull of f is

$$f_{L^p_+,\Pi}(X) = \inf_{a\in\mathbb{R}}\{f_{L^p_+}(X-a\Pi)-a\} = \inf_{\substack{(Y,a)\in(L^\infty+L^p_+)\times\mathbb{R}\\ Y+a\Pi-X=0}} -a.$$

Since $f_{L^p_+,\Pi}$ is the optimal objective value of a convex optimization problem, it is natural to ask if a dual formulation for it, via the duality theory, can be provided. Unfortunately, *we are not always able to answer this question.* What we can say is, that for $X\in L^\infty+L^p_+ +\mathbb{R}\Pi = \operatorname{dom} f_{L^p_+,\Pi}$ it holds $f_{L^p_+,\Pi}(X)=+\infty$. For $X\notin L^\infty+L^p_+ +\mathbb{R}\Pi$ one get as *Lagrange dual problem* to

$$\inf_{\substack{(Y,a)\in(L^\infty+L^p_+)\times\mathbb{R}\\ Y+a\Pi-X=0}} -a \tag{2.13}$$

the following optimization problem

$$\sup_{X^*\in L^q}\ \inf_{(Y,a)\in(L^\infty+L^p_+)\times\mathbb{R}}[-a+\langle X^*,Y+a\Pi-X\rangle],$$

which, since L^∞ is dense in L^p, is nothing else than

$$\sup_{X^* \in L^q} \left[-\langle X^*, X \rangle + \inf_{a \in \mathbb{R}} a(\langle X^*, \Pi \rangle - 1) + \inf_{Y \in L^p} \langle X^*, Y \rangle \right] = -\infty \tag{2.14}$$

Nevertheless, we cannot be sure that this is the value which $f_{L^p_+, \Pi}(X)$ takes, since no known qualification condition can be verified for (2.13)-(2.14). This applies as well as for the classical generalized interior ones ($L^\infty + L^p_+$ is not closed) as for the one of quasi-relative interior-type. This emphasizes the fact that one can have exceptional situations for which the approach we use is, unfortunately, not suitable.

Let us also mention that whenever $\Pi \in L^\infty$ (which includes the situation when Π is a *constant numeraire*), then for all $a \in \mathbb{R}$ there exists $Y \in L^\infty + L^p_+$ such that $X = a\Pi + Y$ and so $f_{L^p_+, \Pi}(X) = -\infty$. In this case we have for all $X \in L^p$

$$f_{L^p_+, \Pi}(X) = \begin{cases} -\infty, & \text{if } X \in L^\infty + L^p_+ + \mathbb{R}\Pi, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 2.4.1 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) The fact that $L^\infty + L^p_+$ is not closed does not make the applicability of the other main class of qualification conditions, the *closedness-type* ones, for the convex optimization problem in (2.13) possible, too.

Chapter 3

Conjugate and subdifferential formulae for convex risk functions

3.1 Conjugate functions and subdifferentiability-general approach

In order to make the chapter self sufficient we dedicate this first section to the notion of conjugate function and (convex) subdifferential. The general approach is due to the books [27, 46, 97].

3.1.1 Conjugate functions

Let $\mathcal{X}$ be a Hausdorff locally convex space and $\mathcal{X}^*$ its topological dual space endowed with weak* topology and let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a given function.

Definition 3.1.1 The function $f^* : \mathcal{X}^* \to \overline{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x \in \mathcal{X}} \{\langle x^*, x\rangle - f(x)\} \tag{3.1}$$

is said to be the *(Fenchel) conjugate function of* f.

To the function $f : \mathcal{X} \to \overline{\mathbb{R}}$ we can attach the so-called biconjugate function of f, which is defined as the conjugate function of the conjugate f^*, i.e.

$$f^{**} : \mathcal{X} \to \overline{\mathbb{R}},\ f^{**}(x) = (f^*)^*(x) = \sup_{x^* \in \mathcal{X}^*} \{\langle x^*, x\rangle - f^*(x^*)\}.$$

The main properties and results concerning conjugate functions are collected and proved in this section. For this summery we mention only the Fenchel-Moreau theorem.

Theorem 3.1.10 3.3.1 Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a proper function. Then $f = f^{**}$ if and only if f is convex and lower semicontinuous.

3.1.2 Subdifferentiability

Subdifferentiability is also an important notion in analysis and optimization. It allows, for instance to describe mathematical objects and models for practical problems and is of a tremendous importance in practical applications like game theory, economy and risk management. In economics and insurance for instance having handleable subdifferential formulae for risk functions it is vital for solving some classes of portfolio optimization problems. All these things will be emphasized in the general framework of risk analysis during our further developments.

Definition 3.1.16 Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a given function and take an arbitrary $x \in \mathcal{X}$ such that $f(x) \in \mathbb{R}$. The set

$$\partial f(x) = \{x^* \in \mathcal{X}^* : f(y) - f(x) \geq \langle x^*, y - x\rangle, \forall y \in \mathcal{X}\}$$

is said to be the *(convex) subdifferential* of f at x. Its elements are called *subgradients* of f at x. We say that the function f is subdifferentiable at x if $\partial f(x) \neq \emptyset$. If $f(x) \notin \mathbb{R}$ we consider by convention $\partial f(x) = \emptyset$.

Definition 3.1.17 Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be an arbitrary function and $\varepsilon \geq 0$ then if $f(x) \in \mathbb{R}$ the ε-*subdifferential* of f at x is the set

$$\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle - \varepsilon, \forall y \in X\},$$

while if $f(x) = \pm\infty$ we take by convention, $\partial_\varepsilon f(x) = \emptyset$.

Remark 3.1.18 For $\varepsilon = 0$ the ε-subdifferential coincides with the classical convex subdifferential, i.e. $\partial f(x) = \partial_0 f(x)$.

The connection between the conjugate function of f and its convex subdifferential will be used several times in the sequel.

Theorem 3.1.19 Let the function $f : \mathcal{X} \to \overline{\mathbb{R}}$ be given and $x \in \mathcal{X}$. Then

$$x^* \in \partial f(x) \Longleftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle, \forall x \in \mathcal{X}. \tag{3.2}$$

Next we have collected, some standard properties of the convex subdifferential as well as results concerning exact formulae for computing the subdifferential of the sum of two convex functions and the subdifferential of the composite convex function.

3.1.3 Conjugate and subdifferentiability of sublinear functions

Theorem 3.1.27 [17] Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a positively homogeneous convex function such that $f(0) = 0$. Then the following holds.

(a) The following properties are equivalent:

$$\begin{array}{rl} (i) & f \text{ is subdifferentiable at } 0, \\ (ii) & f \text{ is bounded from below on a neighborhood of } 0 \in \mathcal{X}, \\ (iii) & f \text{ is lower semicontinuous at } 0. \end{array}$$

(b) If one of the above conditions (i), (ii) or (iii) is satisfied then

$$\sigma_{\partial f(0)}(\cdot) = \overline{f(\cdot)}.$$

(c) Let $x \in \mathcal{X}$ be such that $f(x)$ is finite. Then

$$\partial f(x) = \{x^* \in \partial f(0) : \langle x^*, x \rangle = f(x)\} \tag{3.3}$$

The above relations for the sublinear functions are in fact the staring key for computing the subdifferential formulae of the coherent risk measures. Therefore this result is meaningful for the approaches presented in most of the recent papers [74, 88, 83, 84, 89].

3.2 Conjugate and subdifferential formulae for convex risk functions via an utility model

One of the most challenging topics in convex analysis is the formulation of optimality conditions for portfolio optimization problems with a convex risk measure as objective function. Since for this class of functions differentiability is not necessarily guaranteed, one will be forced to make use of the convex subdifferential when characterizing optimality (see for instance [32]). This is why it is important to be in the possession of easily handleable formulae for the subdifferential of the risk measures which could come into consideration with this respect. Among the most relevant literature on this topic one has to mention [75, 81, 83, 84, 88].

We propose further two distinct ways of providing subdifferential formulae for convex risk function, by means of an utility function on one hand and combining classical results of convex analysis and duality theory on the other hand.

In this section we furnish first formulae for both the conjugate and the subdifferential of a generalized convex risk measure, associated with the Optimized Certainty Equivalent (OCE). The Optimized Certainty Equivalent was introduced by Ben-Tal and Teboulle in [15] by making use of a *concave utility function*. For the investigations made in this section we adapt the definition of the Optimized Certainty Equivalent and the setting in which this has been introduced, by considering a *convex utility function*, as this better suits in the general framework of convex duality. We close the section by particularizing the general results to some convex risk measures widely used in the literature.

Assumption 21 *Let $u : \mathbb{R} \to \overline{\mathbb{R}}$ be a proper, convex, lower semicontinuous and nonincreasing function such that $u(0) = 0$ and $-1 \in \partial u(0)$.*

Consequently, we define for $p \in [1, \infty]$ the following *generalized convex risk function* $\rho_u : L^p \to \mathbb{R} \cup \{+\infty\}$

$$\rho_u(X) = \inf_{\lambda \in \mathbb{R}} \{\lambda + \mathbb{E}(u(X + \lambda))\}. \tag{3.4}$$

The main properties of the above risk function are synthesized in the following proposition.

Proposition 3.2.2 (**A.R. Baias**, D. Duca, [9]) The generalized risk function ρ_u described by relation (3.4) is a convex risk measure (i.e it is convex, monotone and cash invariant risk function) bounded by the expectancy.

Proposition 3.2.5 (**A.R. Baias**, D. Duca, [9]) The generalized risk function ρ_u described by relation (3.4) is monotonic with respect to the second order stochastic dominance.

In the sequel we obtain the conjugate and the subdifferential formulae of ρ_u.

Theorem 3.2.6 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) The conjugate function of ρ_u is the function $\rho_u^* : (L^p)^* \to \overline{\mathbb{R}}$, given by

$$\rho_u^*(X^*) = \begin{cases} \mathbb{E}(u^*(X^*)), & \text{if } \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.5}$$

Before providing a subdifferential formula for ρ_u, we deliver via Lagrange duality a sufficient condition the utility function u has to fulfill in order to guarantee the attainment of the infimum in the definition of $\rho_u(X)$ for all $X \in L^p$. According to [15, 16], for those $X \in L^p$ having as support a bounded and closed interval, the infimum in (3.4) is attained. But what we provide here, is a condition which ensures this fact independently from the choice of the random variable.

Let $X \in L^p$ be fixed. Consider the following primal optimization problem

$$\inf_{\substack{\Xi \in L^q \\ \mathbb{E}(\Xi) = -1}} \Big[\mathbb{E}(u^*(\Xi)) - \langle X, \Xi \rangle \Big], \tag{3.6}$$

where $q := \frac{p}{p-1}$, if $p \in [1, \infty)$, and $q := 1$, if $p = \infty$. The *Lagrange dual* optimization problem to (3.6) is given by

$$\sup_{\lambda \in \mathbb{R}} \Big[-\lambda - \mathbb{E}(u(X + \lambda)) \Big]. \tag{3.7}$$

Let us notice that the optimal objective value of the dual problem (3.7) is equal to $-\rho_u(X)$.

Theorem 3.2.7 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) Assume that the recession function of the utility function u fulfills the following condition

$$\{d \in \mathbb{R} : u_\infty(d) = -d\} = \{0\}. \tag{3.8}$$

Then for all $X \in L^p$ there exists $\bar{\lambda}(X) \in \mathbb{R}$ such that

$$\rho_u(X) = \bar{\lambda}(X) + \mathbb{E}(u(X + \bar{\lambda}(X))).$$

Next we provide a formula for the subdifferential of the general convex risk measure ρ_u.

Theorem 3.2.8 (R.I. Boţ, **A.R. Frătean (Baias)**, [34]) Assume that condition (3.8) is fulfilled. Let $X \in L^p$ and $\bar{\lambda}(X) \in \mathbb{R}$ be the element where the infimum in the definition of $\rho_u(X)$ is attained. Then it holds

$$\partial\rho_u(X) = \{X^* \in (L^p)^* : X^*(\omega) \in \partial u(X(\omega) + \bar{\lambda}(X)) \text{ for a.e. } \omega \in \Omega, \mathbb{E}(X^*) = -1\}. \tag{3.9}$$

The above theorem can be proved in an alternative manner, by means of the *infimal value function*, which is meaningful for the duality approach. Due to the beauty of the method we have presented here also its alternative proof.

In the sequel we rediscover for particular choices of the utility function u several well-known convex risk measures and we provide formulae for their conjugates and subdifferentials.

3.2.1 Entropic risk measure

Consider the utility function $u_1 : \mathbb{R} \to \mathbb{R}$, $u_1(t) = \exp(-t) - 1$, which obviously fulfills the hypotheses in the Assumption. The convex risk measure we define via u_1 is $\rho_{u_1} : L^p \to \mathbb{R}$,

$$\rho_{u_1}(X) = \inf_{\lambda \in \mathbb{R}} \{\lambda + \mathbb{E}(\exp(-X - \lambda) - 1)\}.$$

With the convention $0 \ln(0) = 0$ we have for all $t^* \in \mathbb{R}$ that

$$u_1^*(t^*) = \begin{cases} -t^* \ln(-t^*) + t^* + 1, & \text{if } t^* \leq 0, \\ +\infty, & \text{if } t^* > 0, \end{cases}$$

and, so, from Theorem 3.2.6 it follows that for all $X^* \in (L^p)^*$ one has

$$\rho_{u_1}^*(X^*) = \begin{cases} -\mathbb{E}(X^* \ln(-X^*)), & \text{if } X^* < 0,\ \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $(u_1)_\infty = \delta_{[0,+\infty)}$, condition (3.8) is fulfilled and for all $X \in L^p$ there exists $\bar{\lambda}(X) \in \mathbb{R}$ such that the infimum in the definition of $\rho_{u_1}(X)$ is attained at this point. But in this special case one can easily see that $\bar{\lambda}(X) = \ln(\mathbb{E}(\exp(-X))$ and therefore the risk measure can be equivalently written as $\rho_{u_1}(X) = \ln(\mathbb{E}(\exp(-X)))$. This is the so-called *entropic risk measure* introduced and investigated in [14].

Noticing that $\partial u_1(t) = \{\nabla u_1(t)\} = \{-\exp(-t)\}$ for all $t \in \mathbb{R}$, the subdifferential of the entropic risk measure at $X \in L^p$ is

$$\partial \rho_{u_1}(X) = \{\nabla \rho_{u_1}(X)\} = \left\{ \frac{-1}{\mathbb{E}(\exp(-X))} \exp(-X) \right\}.$$

3.2.2 The worst-case risk measure

By taking as utility function $u_2 = \delta_{[0,+\infty)}$, one rediscovers under $\rho_{u_2} : L^p \to \mathbb{R} \cup \{+\infty\}$,

$$\rho_{u_2}(X) = \inf_{\substack{\lambda \in \mathbb{R} \\ X+\lambda \geq 0}} \lambda = -\operatorname{esinf} X, \tag{3.10}$$

the so-called *worst-case risk measure*. As $u_2^* = \delta_{(-\infty,0]}$, we have for all $X^* \in (L^p)^*$ that

$$\rho_{u_2}^*(X^*) = \begin{cases} 0, & \text{if } X^* \leq 0,\ \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Noticing that $(u_2)_\infty = \delta_{[0,+\infty)}$, one can easily see that (3.8) is fulfilled, which means that for all $X \in L^p$ there exists $\bar{\lambda}(X) \in \mathbb{R}$ at which the infimum in (3.10) is attained. If $\operatorname{esinf} X = -\infty$, then one can take $\bar{\lambda}(X)$ arbitrarily in $\mathbb{R}$, while, when $\operatorname{esinf} X \in \mathbb{R}$, $\bar{\lambda}(X) = -\operatorname{esinf} X$. Since

$$\partial u_2(t) = \begin{cases} \emptyset, & \text{if } t < 0, \\ (-\infty, 0], & \text{if } t = 0, \\ \{0\}, & \text{if } t > 0, \end{cases}$$

we can provide via Theorem 3.2.8 the formula for the subdifferential of the worst-case risk measure. Indeed, for $X \in L^p$ with $\operatorname{esinf} X = -\infty$ one has $\partial \rho_{u_2}(X) = \emptyset$, while, if $\operatorname{esinf} X \in \mathbb{R}$, it holds

$$\partial \rho_{u_2}(X) = \left\{ X^* \in (L^p)^* : \mathbb{E}(X^*) = -1, \begin{array}{ll} X^*(\omega) \in (-\infty, 0], & \text{if } X(\omega) = \operatorname{esinf} X \\ X^*(\omega) = 0, & \text{if } X(\omega) > \operatorname{esinf} X \end{array} \right\}.$$

3.3 Conjugate and subdifferential formulae for convex risk functions via duality theory

Throughout the economical and financial literature one finds a vast variety of risk functions, along the coherent (see [2]), the convex (see [52]) and the expectation bounded (see [83]) ones some very irregular ones, which are neither positively homogeneous nor monotone or cash invariant, being also present. Of course that because of the remarkable mathematical properties it is easier to work with expectation-bounded or coherent risk measures, but what shall we do with the irregular ones? A first answer, at least in what concerns the subdifferentiability and the conjugates of risk functions has been given in the previous section, where conjugate and subdifferential formulae were obtained for risk functions which can be successfully described by utilities.

In this section we present the conjugate and subdifferential approach from a classical point of view, that of convex analysis, and we will provide conjugate and subdifferential formulae in the general context of L^p, $p \in [1, \infty]$ (with the convention $(L^\infty, \sigma_\infty)^* = (L^1, \sigma_1)$) for the most common risk measures, the mean absolute deviation, the lower and the upper semideviation, and the generalized mean deviation of order p from a target. What we present here is in fact a pattern which can be successfully applied for risk functions which are not characterized by very good mathematical properties like: positive homogeneity, monotonicity and cash invariance. Furthermore a lot of risk functions, widely spread in practical applications, can not be neither described by means of utility functions, nor inscribed in the general background of coherent or expectation bounded risk measures. For all those classes of functions the only hope for providing subdifferential formulae stays in classical computation, using methods based on standard results of convex analysis and duality theory.

The following lemma is of great importance for our further results.

Lemma 3.3.1 (**A.R. Baias**, D.M. Nechita, [11]) Consider the functions $g : L^p \to \mathbb{R}$, $g(X) = X_-$ and $h : L^p \to \mathbb{R}$, $h(X) = X_+$ respectively. The following assertions hold:

(a) $h(X) = g(-X)$ and $g(X) = h(-X), \quad \forall X \in L^p$;

(b) $h^*(X^*) = g^*(-X^*) \quad \forall X \in (L^p)^*$;

(c) $\partial h(X) = -\partial g(-X) \quad \forall X \in L^p$.

Since the most important risk measures and risk deviations used lately in the literature can not be described by utility functions we dedicate further a special attention to the subdifferentiability of L^p norm and lower and upper deviations of L^p norm respectively. We will show during this section that those norms dominate mainly the field of irregular risk functions and consequently we will provide subdifferential formulae for several risk function using the following proposition.

Proposition 3.3.2 Let $f_-, f_+ : L^p \to \mathbb{R}$ be the functions defined by $f_- = \|X_-\|_p$, and $f_+ = \|X_+\|_p$, $p \in [1, \infty]$. The following formulae for the conjugate and the subdifferential of f_- and f_+ hold:

$$(a) \quad f_+^*(X^*) = \begin{cases} 0, & \text{if } \|X^*\|_q \leq 1, X^* \geq 0, \\ +\infty, & \text{otherwise;} \end{cases}$$

$$(b) \quad f_-^*(X^*) = \begin{cases} 0, & \text{if } \|X^*\|_q \leq 1, X^* \leq 0, \\ +\infty, & \text{otherwise;} \end{cases}$$

$$(c) \ \partial f_+(x) = \begin{cases} \{X^* \in L^q : \|X^*\|_q \leq 1, X^* \geq 0\}, & \text{if } X = 0, \\ \{X^* \in L^q : \|X^*\|_q \leq 1, X^* \geq 0, \langle X^*, X\rangle = \|X_+\|_p\}, & \text{if } X \neq 0; \end{cases}$$

$$(d) \ \partial f_-(x) = \begin{cases} \{X^* \in L^q : \|X^*\|_q \leq 1, X^* \leq 0\}, & \text{if } X = 0, \\ \{X^* \in L^q : \|X^*\|_q \leq 1, X^* \leq 0, \langle X^*, X\rangle = \|X_-\|_p\}, & \text{if } X \neq 0. \end{cases}$$

3.3.1 The generalized mean deviation of order p

To start with we consider the generalized convex risk function $\rho : L^p \to \overline{\mathbb{R}}$, as introduced in [31],

$$\rho(X) = \|X - \mathbb{E}(X)\|_p^a - \mathbb{E}(X),\ \forall X \in L^p, \tag{3.11}$$

where $p \in [1, \infty]$ and $a \geq 1$.

This convex risk function is one of the most famous risk measures in economical and insurance literature. Different particular cases have been subject of analysis in books and papers like [31, 77, 83, 84, 88, 89]. Due to its importance in practical and computational applications we dedicate to this risk function a special attention.

Taking into account the lower and upper semideviations we obtain similarly the following two risk measures $\rho_- : L^p \to \mathbb{R}$ described by

$$\rho_-(X) = \|(X - \mathbb{E}(X))_-\|_p^a - \mathbb{E}(X),\ \forall X \in L^p, a \geq 1 \text{ and } p \in [1, \infty] \tag{3.12}$$

and $\rho_+ : L^p \to \mathbb{R}$ defined as

$$\rho_+(X) = \|(X - \mathbb{E}(X))_+\|_p^a - \mathbb{E}(X),\ \forall X \in L^p, a \geq 1 \text{ and } p \in [1, \infty], \tag{3.13}$$

respectively.

For the case $a = p = 1$ one rediscovers in the above formulae the so called lower and upper semideviations, while for the case $p = 2$ and $a = 1$ one gets the standard lower and standard upper semideviations, respectively.

ρ_- takes into account only the negative deviations from the mean and it may be considered as a measure of investors risk in a portfolio return. Consequently, downside risk has more attraction and its study become a problem of major interest for both of the mathematical and the economical approaches. However, the downside risk pays no attention to the right-hand side of the distribution of portfolio return. Responsible of this part is in fact, the risk measure described by ρ_+. Since from a mathematical point of view it is important to have a general picture of the whole distribution we will discuss here both of the situations. For the sake of generality and due to the particular importance of the case $a = 1$ for the above mentioned risk functions we treat distinctly the cases $a = 1$ and $a > 1$.

For this subsection our main results are the following:

Theorem 3.3.3 (**A.R. Baias**, D.M. Nechita, [11]) Let $\rho_1 : L^p \to \overline{\mathbb{R}}$ be defined by

$$\rho_1(X) = \|X - \mathbb{E}(X)\|_p - \mathbb{E}(X),\ \forall X \in L^p \text{ where } p \in [1, \infty]. \tag{3.14}$$

The subdifferential of ρ_1 is

$$\partial\rho_1(X) = \begin{cases} \{-1 + X^* - \mathbb{E}(X^*) : X^* \in L^q, \|X^*\|_q \leq 1)\}, \text{ if } X - \mathbb{E}(X) = 0, \\ \left\{-1 + \frac{1}{\|X - \mathbb{E}(X)\|_p^{\frac{p}{q}}}\left[(X - \mathbb{E}(X))^{\frac{p}{q}} - \mathbb{E}((X - \mathbb{E}(X))^{\frac{p}{q}})\right]\right\}, \text{otherwise.} \end{cases} \tag{3.15}$$

Remark 3.3.4 (**A.R. Baias**, D.M. Nechita, [11]) Those formulae were also obtained by Rockafellar et all. in [83] but only for the case of the Hilbert space L^2. Also the form of the subdifferential in the origin is given by A. Ruszczynski and A. Shapiro, by means of different approaches in [88, 89].

For positively homogeneous risk functions Rockafellar et all. [83, 84] have shown the existence of dual representation and characterized the subgradients set calling it *risk envelope*. This will be denoted further as in [83], by $\mathcal{Q}$. Anyway Rockafellar's risk envelope is linked with the subdifferential of the risk function ρ_1, by the relation $\mathcal{Q} = -\partial\rho_1(0)$.

Theorem 3.3.5 (**A.R. Baias**, D.M. Nechita, [11]) Let $\rho_{1_-}, \rho_{1_+} : L^p \to \overline{\mathbb{R}}$ be the risk functions defined by

$$\rho_{1_+}(X) = \|(X - \mathbb{E}(X))_+\|_p - \mathbb{E}(X),\ \forall X \in L^p,\ p \in [1, \infty] \tag{3.16}$$

and

$$\rho_{1_-}(X) = \|(X - \mathbb{E}(X))_-\|_p - \mathbb{E}(X),\ \forall X \in L^p,\ p \in [1, \infty], \tag{3.17}$$

respectively. Then:

(i) the subdifferential of ρ_{1_+} is

$$\partial\rho_{1_+}(X) = \begin{cases} \{-1 + X^* - \mathbb{E}(X^*) : X^* \in L^q, \|X^*\|_q \leq 1, X^* \geq 0\}, & \text{if } X - \mathbb{E}(X) = 0, \\ \left\{-1 + \frac{1}{\|(X-\mathbb{E}(X))_+\|_p^{\frac{p}{q}}} \left[[(X - \mathbb{E}(X))_+]^{\frac{p}{q}} - \mathbb{E}[(X - \mathbb{E}(X))_+]^{\frac{p}{q}}\right]\right\}, & \text{otherwise.} \end{cases} \tag{3.18}$$

(ii) the subdifferential of ρ_{1_-} is

$$\partial\rho_{1_-}(X) = \begin{cases} \{-1 + X^* - \mathbb{E}(X^*) : X^* \in L^q, \|X^*\|_q \leq 1, X^* \leq 0\}, & \text{if } X - \mathbb{E}(X) = 0, \\ \left\{-1 + \frac{1}{\|(X-\mathbb{E}(X))_-\|_p^{\frac{p}{q}}} \left[\mathbb{E}[(X - \mathbb{E}(X))_-]^{\frac{p}{q}} - [(X - \mathbb{E}(X))_-]^{\frac{p}{q}}\right]\right\}, & \text{otherwise.} \end{cases} \tag{3.19}$$

Remark 3.3.7 (**A.R. Baias**, D.M. Nechita, [11]) Notice that the relations between our subdifferential formulas and the risk envelopes for the negative and the positive semideviations (proposed by Rockafellar in [83]) are given by $\mathcal{Q}_+ = -\partial\rho_+(0)$ and $\mathcal{Q}_- = -\partial\rho_-(0)$, respectively. Therefore with Theorem 31, we extend in fact the characterization of the subdifferential of lower and upper mean semideviation to the context of L^p, $p \geq 1$. The results proposed by Rockafellar in [83, 84] are just particular cases of our results for $p = q = 2$.

For the case $a > 1$ we mention only the following observation.

Remark 3.3.13 (**A.R. Baias**, D.M. Nechita, [11]) The risk envelope for the risk functions ρ, ρ_- and ρ_+ consists only in the set $\{0\}$, which is uninteresting. Also the subdifferentiall of $\|X\|_p^a$ in the case $X \neq 0$ can be represented as $\left\{X^* \in L^q : \langle X^*, X\rangle = \|X\|_p^a + (a-1)\|\frac{1}{a}X^*\|_q^{\frac{a}{a-1}}\right\}$, for all $X \in L^p$.

3.3.2 The generalized mean upper/lower semideviations of order p from a target

To start with we define a generalized risk measure with a wide range of applications in the management or risk and portfolio optimization, the so called generalized mean upper semideviation of order p from a target.

Let then $\rho_{\tau_+} : L^p \to \overline{\mathbb{R}}$ defined by

$$\rho_{\tau_+}(X) = \|(X-\tau)_+\|_p^a - \mathbb{E}(X), \tag{3.20}$$

where $\tau \in \mathbb{R}$ is a fixed target and $a \geq 1$. The mean lower semideviation,
$\rho_{\tau_-} : L^p \to \overline{\mathbb{R}}$ can be symmetrically defines as

$$\rho_{\tau_-}(X) = \|(X-\tau)_-\|_p^a - \mathbb{E}(X), \ \forall a \geq 1. \tag{3.21}$$

For the particular situation of $a = 1$ we rediscover the classical mean upper/lower semideviations of order p for a target, as introduced in [88, 89]. As we have already argued in the case of classical mean semideviations we consider that is in important to treat both the negative and the positive outcomes.

In order to do this Lemma 27 will be used several times in the sequel, since it gives the connection between both the conjugates and the subdifferentials of the positive and the negative outcomes. Since the generalized mean semideviation can not be inscribed in the general framework or coherent risk measure, because of the lack of positive homogeneity we can not characterize its conjugate and subdifferentials by means of Theorem 3.1.27. Furthermore the generalized mean semideviations can not be described by any utility function therefore the approach presented in Section 3.2 fails too. Consequently the only possible approach stays in deriving the formulae by means of the duality theory. Also here, we treat distinctly the cases $a = 1$ an $a > 1$ respectively.

Theorem 3.3.18 (**A.R. Baias,** [7]) Let $\rho_{\tau 1_+} : L^p \to \overline{\mathbb{R}}$ be the risk function defined by

$$\rho_{\tau 1+}(X) = \|(X-\tau)_+\|_p - \mathbb{E}(X), \ \forall X \in L^p,$$

where $p \in [1, \infty]$. Then the conjugate function of $\rho_{\tau 1_+}$ is the function $\rho^*_{\tau 1_+} : L^q \to \overline{\mathbb{R}}$ given by

$$\rho^*_{\tau 1_+}(X^*) = \begin{cases} \tau\mathbb{E}(X^*+1), & \text{if } \ \|X^*+1\|_q \leq 1, X^* \geq -1, \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.22}$$

Theorem 3.3.19 (**A.R. Baias,** [7]) Let $\rho_{\tau 1_-} : L^p \to \overline{\mathbb{R}}$ be the risk function defined by

$$\rho_{\tau 1-}(X) = \|(X-\tau)_-\|_p - \mathbb{E}(X), \ \forall X \in L^p,$$

where $p \in [1, \infty]$. Then the conjugate function of $\rho_{\tau 1_-}$ is the function $\rho^*_{\tau 1_-} : L^q \to \overline{\mathbb{R}}$ is given by

$$\rho^*_{\tau 1_-}(X^*) = \begin{cases} -\tau \mathbb{E}(X^* + 1), & \text{if } \|X^* + 1\|_q \le 1, X^* \le -1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.23}$$

The conjugate functions of the generalized mean upper/lower deviation of order p from a target τ to the case $a > 1$ is given by the following results.

Theorem 3.3.21 (**A.R. Baias,** [7]) Let $\rho_{\tau a_+} : L^p \to \overline{\mathbb{R}}$ be defined by

$$\rho_{\tau a+}(X) = \|(X - \tau)_+\|_p^a - \mathbb{E}(X), \ \forall X \in L^p$$

where $p \in [1, \infty]$ and $a > 1$. Then the conjugate function of $\rho_{\tau a_+}$ is the function $\rho^*_{\tau a_+} : L^q \to \overline{\mathbb{R}}$ given by

$$\rho^*_{\tau a_+}(X^*) = \begin{cases} (a-1)\|\frac{1}{a}(X^* + 1)\|_q^{\frac{a}{a-1}} + \tau \mathbb{E}(X^* + 1), & \text{if } X^* \ge -1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.24}$$

Theorem 3.3.22 (**A.R. Baias,** [7]) Let $\rho_{\tau a_-} : L^p \to \overline{\mathbb{R}}$ be defined by

$$\rho_{\tau a-}(X) = \|(X - \tau)_-\|_p^a - \mathbb{E}(X), \ \forall X \in L^p$$

where $p \in [1, \infty]$ and $a > 1$. Then the conjugate function of $\rho_{\tau a_-}$ is the function $\rho^*_{\tau a_-} : L^q \to \overline{\mathbb{R}}$ given by

$$\rho^*_{\tau a_-}(X^*) = \begin{cases} (a-1)\|\frac{1}{a}(X^* + 1)\|_q^{\frac{a}{a-1}} - \tau \mathbb{E}(X^* + 1), & \text{if } X^* \le -1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.25}$$

Since the recent literature gives an important place to the dual characterizations of risk measures, we have presented also in this section the dual representations for the mean upper/lower deviations of order p from the target τ. Our development goes naturally since both the mean upper deviation $\rho_{\tau+}$ and the lower deviation $\rho_{\tau-}$ are convex, proper and lower semicontinuous functions and therefore the Fenchel-Moreau Theorem 3.1.10 applies.

For the subdifferential of mean lower/upper semideviations of order p form a target τ we have the following result.

Theorem 3.3.23 (**A.R. Baias,** [7]) Let $\rho_{\tau 1_+}, \rho_{\tau 1_-} : L^p \to \overline{\mathbb{R}}$ be the risk functions defined by

$$\rho_{\tau 1+}(X) = \|(X - \tau)_+\|_p - \mathbb{E}(X), \quad \forall X \in L^p, \, p \in [1, \infty], \tag{3.26}$$

and

$$\rho_{\tau 1-}(X) = \|(X - \tau)_-\|_p - \mathbb{E}(X), \ \forall X \in L^p, p \in [1, \infty], \tag{3.27}$$

respectively. Then:

(i) the subdifferential of $\rho_{\tau_{1+}}$ is

$$\partial\rho_{\tau_{1+}}(X) = \begin{cases} \{X^* - 1 : X^* \in L^q, \|X^*\|_q \leq 1, X^* \geq 0\}, & \text{if } X = \tau, \\ \left\{ \frac{((X-\tau)_+)^{\frac{p}{q}}}{\|(X-\tau)_+\|_p^{\frac{p}{q}}} - 1 \right\}, & \text{if } X \neq \tau; \end{cases} \tag{3.28}$$

(ii) the subdifferential of $\rho_{\tau_{1-}}$ is

$$\partial\rho_{\tau_{1-}}(X) = \begin{cases} \{X^* - 1 : X^* \in L^q, \|X^*\|_q \leq 1, X^* \leq 0\}, & \text{if } X = \tau, \\ \left\{ \frac{((X-\tau)_-)^{\frac{p}{q}}}{\|(X-\tau)_-\|_p^{\frac{p}{q}}} - 1 \right\}, & \text{if } X \neq \tau. \end{cases} \tag{3.29}$$

3.4 An application - Conditional Value at Risk (CVaR)

3.4.1 Definition and economical signification

Two of the most popular risk measures presented in the literature are the Value-at-Risk (VaR) and the Conditional Value-at-Risk (also known as expected shortfall, tail-VaR or Average Value-at-Risk).

Definition 3.4.1 *[84]* The Value-at-Risk of the loss associated with a decision X at the level $\beta \in (0, 1)$ is the value

$$\mathrm{VaR}_\beta(X) = -\inf\{\alpha : \mathbb{P}(X \leq \alpha) > \beta\}.$$

In other words VaR is defined as the minimum level of loss, at a given, sufficiently high, confidence level for a predefined time horizon. A very serious shortcoming of VaR is the lack of convexity which actually makes it undesirable from the mathematical point of view.

An alternative risk measure which quantifies also the losses which are situated in the tail of the loss distribution is the Conditional Value-at-Risk.

Definition 3.4.2 *[84]* For any $\beta \in (0, 1)$ the functional

$$\mathrm{CVaR}_\beta(X) = -\,[\text{expectation of the lower tail distribution of the variable } X \text{ at level } \beta].$$

Apart for its remarkable mathematical properties CVaR presents an amazing computational advantage, due to its minimization formula (see [81]).

$$\mathrm{CVaR}_\beta(X) = \inf_{\eta\in\mathbb{R}} \left\{ \eta + \frac{1}{\beta}\mathbb{E}[(X+\eta)_-] \right\}, \tag{3.30}$$

which can be successfully incorporated into optimization problems with respect to the random variable X. The CVaR has very good mathematical properties, being coherent in the way of Artzner et all. [2].

Of course that in time, CVaR it was extended and studied in various directions, but Lüthi and Doege [65] generalized the CVaR to a convex risk measure called Generalized Conditional Value-at-Risk

(GCVaR) which keeps both the computational advantage and faces the extreme risky outcomes, i.e.

$$\mathrm{GCVaR}_{\beta,l}(X) = \inf_{\substack{\eta\in\mathbb{R} \\ \mathbb{E}[(X+\eta)_-]\leq l}} \left\{\eta + \frac{1}{\beta}\mathbb{E}[(X+\eta)_-]\right\},$$

where $l \geq 0$ is a fixed parameter and $\beta \in (0,1)$ is as usually the confidence level.

Remark 3.4.3 It is obviously that for $l \geq \mathbb{E}[(X + \eta_{VaR})_-]$ the Generalized Conditional Value-at-Risk collapses into the classical notion of Conditional Value-at-Risk while for $l = 0$ one actually gets the worst-case risk measure, i.e. $\mathrm{GCVaR}_{\beta,0} = -\operatorname{esinf} X = \text{Max Loss}$.

In order to exemplify our recent developments we deduce further, the conjugate and the subdifferential formulae of the Conditional value-at-risk from both of the perspectives presented during this chapter, namely by means of an utility model, one one hand and by means of the duality approach, on the other hand.

3.4.2 Conjugate and subdifferentiability of CVaR via duality theory

In this section we aim to provide conjugate and subdifferential formulae for the conditional value-at-risk by means of the duality theory.

For the sake of generality, we first calculate the Lüthi and Doege's Generalized Conditional value-at-risk. As a consequence, we derive afterwards, the conjugates of the worst-case risk measure, denoted by *Max Loss*, and of the conditional value-at-risk (CVaR). Having this conjugates, in view of relation (3.2), one can easily obtain the subdifferential formula of CVaR.

Theorem 3.4.4 (**A.R. Baias,** [6]) Let $l \geq 0$ be a fixed parameter, $\beta \in (0,1)$, and let $GCVaR_{\beta,l} : L^p \to \overline{\mathbb{R}},\ p \geq 1$ be the risk function defined by

$$\mathrm{GCVaR}_{\beta,l}(X) = \inf_{\substack{\eta\in\mathbb{R} \\ \mathbb{E}[(X+\eta)_-]\leq l}} \left\{\eta + \frac{1}{\beta}\mathbb{E}[(X+\eta)_-]\right\}. \tag{3.31}$$

Then the conjugate function of GCVaR, is the function $\mathrm{GCVaR}^*_{\beta,l} : L^q \to \overline{\mathbb{R}}$, given by

$$\mathrm{GCVaR}^*_{\beta,l}(X^*) = \begin{cases} -l\min\{0, \operatorname{esinf}(X^* + \frac{1}{\beta})\}, & \text{if } \mathbb{E}(X^*) = -1, X^* \leq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.32}$$

In view of Remark 41 for $l = 0$ we rediscover in the above theorem the conjugate of the worst-case risk measure, while for $(X^* + \frac{1}{\beta}) \geq 0$ we actually get the conjugate of the classical Conditional Value-at-Risk, as the below corollaries shows.

Corollary 3.4.5 (**A.R. Baias,** [6]) Let $\text{Max Loss} : L^p \to \overline{\mathbb{R}},\ p \geq 1$ be the risk function defined by

$$\text{Max Loss} := -\operatorname{esinf} X.$$

Then the conjugate function of Max Loss is the function Max Loss$^* : L^q \to \overline{\mathbb{R}}$, given by

$$\text{Max Loss}^*(X^*) = \begin{cases} 0, & \text{if } \mathbb{E}(X^*) = -1,\ X^* \leq 0, \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.33}$$

Corollary 3.4.6 (**A.R. Baias,** [6]) Let $\beta \in (0,1)$ and let $\text{CVaR}_\beta : L^p \to \overline{\mathbb{R}}$, $p \geq 1$ be the risk function defined by relation (3.30). Then the conjugate function of CVaR is the function $\text{CVaR}^*_\beta : L^q \to \overline{\mathbb{R}}$, given by

$$\text{CVaR}^*_\beta(X^*) = \begin{cases} 0, & \text{if } \mathbb{E}(X^*) = -1,\ X^* \leq 0, (X^* + \frac{1}{\beta}) \geq 0, \\ +\infty, & \text{otherwise .} \end{cases} \tag{3.34}$$

Theorem 3.4.7 (**A.R. Baias,** [6]) Let $\beta \in (0,1)$ and let $\text{CVaR}_\beta : L^p \to \overline{\mathbb{R}}$, $p \geq 1$ be the risk function defined by relation (3.30). Then it holds

$$\partial \text{CVaR}_\beta(X) = \left\{ X^* \in L^q : \mathbb{E}(X^*) = -1, \begin{array}{ll} X^*(\omega) = -1/\beta, & \text{if } X(\omega) < -\text{VaR}_\beta(X) \\ X^*(\omega) \in [-1/\beta, 0], & \text{if } X(\omega) = -\text{VaR}_\beta(X) \\ X^*(\omega) = 0, & \text{if } X(\omega) > -\text{VaR}_\beta(X) \end{array} \right\}. \tag{3.35}$$

3.4.3 Conjugate and subdifferentiability of CVaR via an utility function

In the following we deal with the same problem of furnishing the conjugate and the subdifferential formula of the Conditional Value-at-risk, but this time by using the utility approach developed in Section 3.2.

Let therefore $\gamma_2 < -1 < \gamma_1 \leq 0$ and let $u : \mathbb{R} \to \mathbb{R}$ be the utility function defined by

$$u(t) = \begin{cases} \gamma_2 t, & \text{if } t \leq 0, \\ \gamma_1 t, & \text{if } t > 0, \end{cases}$$

Notice that it satisfies all the requirements in the Assumption 21, i.e. it is a proper, convex, lower semicontinuous and nonincreasing function, which additionally fulfills the normalization conditions. This gives rise to the following convex risk measure $\rho_u : L^p \to \mathbb{R}$,

$$\rho_u(X) = \inf_{\lambda \in \mathbb{R}} \{\lambda + \gamma_1 \mathbb{E}(X+\lambda)_+ - \gamma_2 \mathbb{E}(X+\lambda)_-\}.$$

For $\gamma_1 = 0$ and $\gamma_2 = -1/\beta$, where $\beta \in (0,1)$ is the confidence level, we rediscover in the above formula the classical conditional value-at-risk (see 3.30).

Since $u^* = \delta_{[\gamma_2, \gamma_1]}$, via Theorem 3.2.6 one gets for $\rho^*_u : L^q \to \overline{\mathbb{R}}$ the following expression

$$\rho^*_u(X^*) = \begin{cases} 0, & \text{if } \gamma_2 \leq X^* \leq \gamma_1, \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, for all $X^* \in L^q$ the conjugate of CVaR is the function $\mathrm{CVaR}_\beta^* : L^q \to \overline{\mathbb{R}}$ which looks like

$$\mathrm{CVaR}_\beta^*(X^*) = \begin{cases} 0, & \text{if } -\frac{1}{\beta} \leq X^* \leq 0, \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Noticing that for all $d \in \mathbb{R}$,

$$(u)_\infty(d) = \begin{cases} \gamma_2 d, & \text{if } d < 0, \\ 0, & \text{if } d = 0, \\ \gamma_1 d, & \text{if } d > 0, \end{cases}$$

one can easily see that condition (3.8) is satisfied. Thus for all $X \in L^p$ there exists $\bar{\lambda}(X) \in \mathbb{R}$ such that $\rho_u(X) = \bar{\lambda}(X) + \gamma_1 \mathbb{E}(X + \bar{\lambda}(X))_+ - \gamma_2 \mathbb{E}(X + \bar{\lambda}(X))_-$. Further, according to Theorem 3.2.8, we will make use of $\bar{\lambda}(X)$ when giving the formula for the subdifferential of ρ_u at X. Since

$$\partial u(t) = \begin{cases} \{\gamma_2\}, & \text{if } t < 0, \\ [\gamma_2, \gamma_1], & \text{if } t = 0, \\ \{\gamma_1\}, & \text{if } t > 0, \end{cases}$$

we obtain for all $X \in L^p$ the following formula

$$\partial \rho_u(X) = \left\{ X^* \in (L^p)^* : \mathbb{E}(X^*) = -1, \begin{array}{ll} X^*(\omega) = \gamma_2, & \text{if } X(\omega) < -\bar{\lambda}(X) \\ X^*(\omega) \in [\gamma_2, \gamma_1], & \text{if } X(\omega) = -\bar{\lambda}(X) \\ X^*(\omega) = \gamma_1, & \text{if } X(\omega) > -\bar{\lambda}(X) \end{array} \right\}.$$

As we have already seen, in the previous section, for all $X \in L^p$ the element where the infimum in the definition of $\mathrm{CVaR}_\beta(X)$ is attained, is the so-called *value-at-risk of* X *at level* β. Therefore we get for the subdifferential of CVaR the expression given by relation (3.35).

Chapter 4

Extension theorems for convex set-valued maps

4.1 Motivation

In this chapter we aim on one hand to give new extension theorems for convex set-valued maps under rather weak topological assumptions and, on the other hand, we intend to emphasize their applicability in the field of nonsmooth analysis.

The famous Hahn-Banach theorem, first stated in [57] and [13] is a powerful tool which resonates through important fields of mathematics such as: functional analysis, convex analysis and optimization theory. Among the numerous consequences of this result, we merely mention here Hahn's extension theorems for continuous linear functionals. Generalizations and variants of those theorems were developed in different directions in the past. We remind here only a few remarkable results for vector functions, in partially ordered spaces with least upper bound property, the so called Hahn Banach Kantorovich theorem, [38, 42, 63, 91, 100, 101], while for set-valued maps we recall [70, 71, 94, 99].

Although most of those results are mistaken, as Zălinescu showed in [98], they open the gates for further research in the domain. Most of those generalizations refer only to the case of linear spaces, where no topology is involved. As far as we know, only a few topological versions were developed, see for instance [19], [44] and the references therein. In all the above results the conditions for the existence of a linear extension are expressed by means of the classical topological interior or by means of the algebraic interior (core). The only exception is the paper of Zălinescu [98] in which a weaker version of algebraic interior was used (the intrinsic core), but only for the extension of pure algebraic Hahn-Banach-Kantorovich theorems.

Our goal for this chapter is to provide extension theorems for both linear continuous operators dominated by convex set-valued maps, and real valued closed convex processes. With this work we aim to fill in the gaps in the domain and to emphasize a new way of proving continuity for linear extensions in topological spaces. In order to justify the efficiency of our results we provide suggestive examples and a wide range of applications under the form of existence results for subgradients of set-valued maps.

4.2 Background in notation and definitions for partially ordered spaces

Lately, has become a standard to work in partially ordered spaces with the least upper bound property, when dealing with set-valued extension theorems. This is the reason why this section is dedicated to a short summary on the partially ordered spaces.

The general framework we work under in the present chapter is described below. All the assumptions made on the spaces $\mathcal{X}$ and $\mathcal{Y}$ and on the cone K will be valid for all the forthcoming sections if not otherwise specified.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Fréchet spaces and let $K \subset \mathcal{Y}$ be a pointed, closed, convex cone. The cone K induces a partial ordering on $\mathcal{Y}$, called *the strong ordering* and defined by

$$y_2 \geq y_1 \text{ if } y_2 - y_1 \in K, \ \forall y_1, y_2 \in \mathcal{Y}.$$

If the cone K is solid (i.e. $\operatorname{int} K \neq \emptyset$) we may also talk about *the weak ordering* on $\mathcal{Y}$, defined as

$$y_2 > y_1 \text{ if } y_2 - y_1 \in \operatorname{int} K, \ \forall y_1, y_2 \in \mathcal{Y}.$$

Further, we consider the cone K, solid, so that the whole theory developed in this chapter will suit both the weak and the strong ordering. For simplicity we assume also that K is normal for the topology on $\mathcal{Y}$. This is to say that there exists a basis of neighbourhoods V of zero in $\mathcal{Y}$ with

$$(V - K) \cap (K - V) = V.$$

Definition 4.2.1 (cf. [42]) A topological vector space $\mathcal{Y}$, partially ordered by a convex cone K, has *the least upper bound property* if every nonempty subset C which has an upper bound c in terms of the ordering (i.e. $\forall y \in C$, $c - y \in K$) has a least upper bound, called supremum of C (i.e. there exists $\tilde{c} \in \mathcal{Y}$ such that $\tilde{c}$ is an upper bound to C, and each upper bound c to C satisfies $c - \tilde{c} \in K$).

This section also contains examples and counter examples of infinite dimensional spaces whit least upper bound property.

4.3 Preliminaries on set-valued analysis

Our goal for this section is to draw the background of notations and definitions for set-valued maps. For prerequisite material on set-valued maps the reader is referred to [3, 21, 76, 78, 92]

For set-valued maps $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ we denote by $\operatorname{Gr}(\Lambda)$ its *graph*, by $\operatorname{Dom} \Lambda$ the *domain* of the set-valued map Λ, by $\operatorname{Epi}(\Lambda)$, its *epigraph* and by $\operatorname{Im} \Lambda$ the *image* or the *range* of the set-valued map.

We shall say that the set-valued map $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ is:

- *strict* if the images $\Lambda(x)$ are nonempty for all $x \in \mathcal{X}$;

- *convex* (or convex relation) if its graph is convex;
- *closed* if its graph is closed in $\mathcal{X} \times \mathcal{Y}$.
- *K-convex* if its epigraph is convex;
- *a process* (or positively homogeneous) if its graph is a cone;
- *lower semicontinuous* if for every $x_0 \in \mathrm{Dom}(\Lambda)$ and any open set $U \subseteq \mathcal{Y}$, with $\Lambda(x_0) \cap U \neq \emptyset$ there exist a neighbourhood V of x_0 such that $\Lambda(x) \cap U \neq \emptyset$ for all $x \in V$.

Similarly to the norm of a continuous linear functional we can define the norm of a closed convex process Λ as

$$\|\Lambda\| := \sup_{x \in \mathrm{Dom}\,\Lambda} \frac{d(0, \Lambda(x))}{\|x\|} = \sup_{x \in \mathrm{Dom}\,\Lambda} \inf_{u \in \Lambda(x)} \frac{\|u\|}{\|x\|}.$$

4.4 Extension theorems for linear continuous operators dominated by convex set-valued maps

As we have already mentioned one of our goals for this section is to provide extension theorems for linear continuous operators dominated by convex set-valued maps under rather weak interiority conditions. For all the results of the forthcoming section $\mathcal{X}$ and $\mathcal{Y}$ are Fréchet spaces, the latter with the least upper bound property, partially ordered by the closed, convex, pointed cone K. Furthermore K is normal and solid. Our main result for this section is the following.

Theorem 4.4.3 (**A.R. Baias**, [8]) Let $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ be a convex set-valued map. Suppose that

$$0 \in \mathrm{sqri}(\mathrm{Dom}(\Lambda)) \text{ and} \tag{4.1}$$

$$\Lambda(0) \geq 0 \text{ (i.e. } \forall y \in \Lambda(0) \text{ we have } y \geq 0). \tag{4.2}$$

Then there exist a continuous, linear operator $T : \mathcal{X} \to \mathcal{Y}$ such that

$$T(x) \leq \Lambda(x), \ \ \forall x \in \mathrm{Dom}\,\Lambda. \tag{4.3}$$

Remark 4.4.5 (**A.R. Baias**, [8]) If $\mathcal{X}$ is endowed with the finest convex topology for a convex set-valued map $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$, the following conditions are equivalent:

(a) Λ is lower semicontinuous at 0;

(b) $0 \in \mathrm{core}(\mathrm{Dom}(\Lambda))$;

(c) $0 \in \mathrm{sqri}(\mathrm{Dom}(\Lambda)) \cap \mathrm{qi}(\mathrm{Dom}(\Lambda))$.

The equivalence between condition (a) and (b) is due to [19, Proposition 2.1, (d)] while the equivalence between (b) and (c) is a simple consequence of the definitions of quasi relative interior and strong quasi relative interior respectively. Note that under this context all the above conditions imply condition (47).

The equivalence between (b) and (c) states for convex set-valued maps, under a more general framework, namely in topological vector spaces without any further assumptions.

In order to illustrate the applicability of our result we consider the following example.

Example 4.4.7 (**A.R. Baias**, [8])

Consider now the Fréchet space $\ell^2(\mathbb{N})$ and its closed linear subspace

$$\bar{\mathcal{X}} = \{(x_n)_n \in \mathbb{N} \in \ell^2 : x_{2n-1} + x_{2n} = 0, \forall n \in \mathbb{N}\}.$$

Define the map $\Lambda : \ell^2(\mathbb{N} \rightrightarrows \bar{\mathbb{R}}$ by

$$\Lambda(x) = \begin{cases} \{0\}, & \text{if } x \in \bar{\mathcal{X}}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

One can see that Λ is convex and $\operatorname{Dom}\Lambda = \bar{\mathcal{X}}$. Now since $\operatorname{cone}(\operatorname{Dom}\Lambda) = \bar{\mathcal{X}} \neq \ell^2$ we conclude that $0 \in \operatorname{sqri}(\operatorname{Dom}\Lambda)$ and our result applies. Furthermore $0 \notin \operatorname{core}(\operatorname{Dom}\Lambda)$ thus other similar results expressed by means of algebraic interior are not suitable for this problem.

Theorem 4.4.8 (Sandwich Theorem)(**A.R. Baias**, [8]) Let $\Lambda_1, \Lambda_2 : \mathcal{X} \rightrightarrows \mathcal{Y}$ be convex set-valued maps, such that

$$0 \in \operatorname{sqri}(\operatorname{Dom}\Lambda_1 - \operatorname{Dom}\Lambda_2).$$

Suppose that

$$\Lambda_2(x) \leq \Lambda_1(x), \quad \forall x \in \mathcal{X}. \tag{4.4}$$

Then there exists a continuous linear operator $T : \mathcal{X} \to \mathcal{Y}$ and $y_0 \in \mathcal{Y}$ with

$$\Lambda_2(x) \leq T(x) - y_0 \leq \Lambda_1(x), \quad \forall x \in \mathcal{X}.$$

Remark 4.4.9 (**A.R. Baias**, [8]) Theorem 50 contains Theorem 47 as a special case, for $\operatorname{Gr}(\Lambda_1) = \operatorname{Gr}(\Lambda)$ and $\operatorname{Gr}(\Lambda_2) = 0$.

We consider further the case of vector-valued maps and we provide the following sandwich result.

Corollary 4.4.10 (**A.R. Baias**, [8]) Let $\lambda_1, \lambda_2 : \mathcal{X} \to \mathcal{Y}$ be a K-convex and respectively a K-concave vector function, such that

$$0 \in \operatorname{sqri}(\operatorname{dom}\lambda_1 - \operatorname{dom}\lambda_2).$$

Suppose that

$$\lambda_2(x) \leq \lambda_1(x), \quad \forall x \in \operatorname{dom}\lambda_1 \cap \operatorname{dom}\lambda_2. \tag{4.5}$$

Then there exists a continuous linear operator $T : \mathcal{X} \to \mathcal{Y}$ and $y_0 \in \mathcal{Y}$ with

$$\lambda_2(x) \leq T(x) - y_0 \leq \lambda_1(x), \quad \forall x \in \mathcal{X}.$$

In the following we shall establish a new generalized Lagrange Multiplier Theorem.

Assumption 53 *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Fréchet spaces. $\mathcal{Y}$ has the least upper bound property with respect to the ordering induced by the normal, closed, convex, pointed cone K. Let $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ and $\Gamma : \mathcal{X} \rightrightarrows \mathcal{Z}$ be set-valued maps.*

We consider the following minimization problem

$$(P_\Lambda) \quad \inf_{0 \in \Gamma(x)} \Lambda(x).$$

If μ is a solution of problem (P_Λ), then we call μ a minimizer of (P_Λ).

Theorem 4.4.11 (Lagrange Multipliers)(**A.R. Baias**, [8]) Let the problem (P_Λ) be defined as above. Suppose that the set-valued map $\Phi : \mathcal{Z} \rightrightarrows \mathcal{Y}$, defined as

$$\Phi(z) = (\Lambda \circ \Gamma^{-1})(z) = \{\Lambda(x) : z \in \Gamma(x)\}$$

is convex and $0 \in \text{sqri}(\text{Dom}(\Phi))$. Then for any μ, minimum of (P_Λ), there exist a linear, continuous operator $T : \mathcal{Z} \to \mathcal{Y}$ such that

$$\mu \leq \Lambda(x) + (T \circ \Gamma)(x), \quad \forall x \in \mathcal{X}. \tag{4.6}$$

Problem (P_Λ) plays a central role in vector theory and optimization since the whole perturbation duality theory can be viewed as a particular case of the problem (P_Λ).

4.5 Subgradients of Set-valued maps

One of the most important research directions in nonsmooth analysis is represented by the existence results for subgradients of vector-valued or set-valued map. We refer in this section to the latter direction, which had grown amazingly, in the last few years. For more references and discussions on subgradients (subdifferentials) of vector-valued functions see the recent books of Mordukhovich [67, 68], which may be considered the cornerstone of this domain.

4.5.1 General notions, definitions and remarks

To start with we recall first the approaches regarding strong subgradients of set-valued maps.

Definition 4.5.1 [19] Let $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ be a set-valued map and let $x_0 \in \text{Dom}(\Lambda)$ and $y_0 \in \mathcal{Y}$ such that $y_0 \leq \Lambda(x_0)$. A linear and continuous operator $T : \mathcal{X} \to \mathcal{Y}$ is called *B*orwein-strong subgradient of Λ at (x_0, y_0) if

$$T(x - x_0) \leq \Lambda(x) - y_0, \quad \forall x \in \text{Dom}(\Lambda).$$

The set of Borwein-strong subgradients at (x_0, y_0) is denoted by $\partial_{y_0}^{B-S}\Lambda(x_0)$.

Definition 4.5.2 [58] Let $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ be a set-valued map and let $x_0 \in \mathrm{Dom}(\Lambda)$. A linear and continuous operator $T : \mathcal{X} \to \mathcal{Y}$ is called then *s*trong subgradient of Λ at x_0 if

$$T(x - x_0) \leq \Lambda(x) - \Lambda(x_0), \quad \forall x \in \mathrm{Dom}(\Lambda) \setminus \{x_0\}.$$

We denote by $\partial^S\Lambda(x_0)$ the set of strong subgradients of Λ at x_0.

Further the chapter present the "weak" concepts associated with the above. We compare and we discuss the connections between the above notions and other similar notions of weak and strong subgradients of set-valued maps. Although most of them have been introduced in the context of linear spaces where no topology is involved, we adapt the initial definition to the case of linear topological spaces .

4.5.2 Existence results

Due to the rapidly growth of the field of nonsmooth analysis, a lot of recent papers discussed and proved the existence of subgradients for set-valued or vector-valued maps under separation arguments. We mention here only a few of them as [19, 38, 39, 58, 71, 94].

Using our extension Theorem 4.4.3 we emphasize the existence of both strong and Borwein-strong subgradients.

Theorem 4.5.7 (**A.R. Baias**, [8]) Let $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ be a convex set-valued map and consider $x_0 \in \mathrm{sqri}(\mathrm{Dom}(\Lambda))$. If there exists $y_0 \in \mathcal{Y}$ such that $y_0 \leq \Lambda(x_0)$ then $\partial_{y_0}^{B-S}\Lambda(x_0)$ is non-empty.

Theorem 4.5.8 (**A.R. Baias**, [8]) Let $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ be a convex set-valued map and let $x_0 \in \mathcal{X}$ such that $x_0 \in \mathrm{sqri}(\mathrm{Dom}(\Lambda))$. Then $\partial^S\Lambda(x_0)$ is non-empty.

The convexity assumption of the above theorem can be easily relaxed if we assume it on an additional set-valued map, instead of the initial function Λ, as we shall see below.

Theorem 4.5.9 (**A.R. Baias**, [8]) Let $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$ be a set-valued map and $x_0 \in \mathrm{Dom}(\Lambda)$. Suppose that there exists a convex map $\Gamma : \mathcal{X} \rightrightarrows \mathcal{Y}$ such that

$$0 \in \mathrm{sqri}(\mathrm{Dom}(\Gamma)) \text{ with } \Gamma(0) \geq 0 \text{ and} \tag{4.7}$$

$$\Lambda(x) - \Lambda(x_0) \subset \Gamma(x - x_0) \text{ for all } x \in \mathcal{X} \setminus \{x_0\} \tag{4.8}$$

Then $\partial^S\Lambda(x_0)$ is non-empty.

In the above theorem the result still holds if we use instead of a convex arbitrary set-valued map Γ the contingent derivative of Λ.

Remark 4.5.11 (**A.R. Baias**, [8]) In [58, Theorem 4.2] existence theorems for strong subgradients were established by using a similar approach but under different assumptions. Since under the context of Remark 48 the condition (4.7) is weaker then the lower semicontinuity of Γ in the origin, Theorem 59 improves [58, Theorem 4.2].

When dealing with subdifferentials another direction intensively studied lately, was the one involving calculus rules for different classes of functions. With this respect we refer to the reader to [67, 68, 60, 33, 10].

4.6 Extension theorems for closed convex processes

In this section we work under a more general framework, namely in normed linear spaces, and we provide some norm preserving extension results for real valued closed convex processes. As a consequence, we characterize, the elements of best approximation in normed linear spaces by elements of closed convex cones using closed convex processes.

Hahn's extension theorems for continuous linear functionals, are extended to the general framework of set-valued analysis through the following two results.

Theorem 4.6.1 (**A.R. Baias**, T. Trif, [12]) Let $\mathcal{X}$ be a real normed linear space, let $\mathcal{X}_0$ be a linear subspace of $\mathcal{X}$, and let $\Gamma_0 : \mathcal{X}_0 \rightrightarrows \mathbb{R}$ be a closed convex process such that

$$\operatorname{Dom}\Gamma_0 = \mathcal{X}_0 \quad \text{and} \quad \|\Gamma_0\| < \infty. \tag{4.9}$$

Then there exists a closed convex process $\Gamma : \mathcal{X} \rightrightarrows \mathbb{R}$ such that

(*i*) $\operatorname{Dom}\Gamma = \mathcal{X}$ and $\|\Gamma\| = 1$;

(*ii*) $\Gamma(x) = \Gamma_0(x)$ for all $x \in \mathcal{X}_0$;

(*iii*) $\|\Gamma\| = \|\Gamma_0\|$.

Theorem 4.6.2 (**A.R. Baias**, T. Trif, [12]) Let $\mathcal{X}$ be a real normed linear space, let K_0 be a closed convex cone in $\mathcal{X}$, let $x_0 \in \mathcal{X} \setminus K_0$, and let $d_0 := d(x_0, K_0) = \inf_{x\in K_0} \|x - x_0\|$. Then there exists a closed convex process $\Gamma : \mathcal{X} \rightrightarrows \mathbb{R}$, satisfying the following conditions:

(*i*) $\operatorname{Dom}\Gamma = \mathcal{X}$ and $\|\Gamma\| = 1$;

(*ii*) $\min \Gamma(x) = 0$ for all $x \in K_0$;

(*iii*) $\min \Gamma(x_0) = d_0$.

As a direct consequence of Theorem 62 we obtain the characterization of elements of best approximation in normed linear spaces by elements of closed convex cones using closed convex processes.

Theorem 4.6.3 (**A.R. Baias**, T. Trif, [12]) Let $\mathcal{X}$ be a real normed linear space, let K_0 be a closed convex cone in $\mathcal{X}$, let $x_0 \in \mathcal{X} \setminus K_0$, and let $y_0 \in K_0$. Then $y_0 \in \mathrm{pr}_{K_0}(x_0)$ if and only if there exists a closed convex process $\Gamma : \mathcal{X} \rightrightarrows \mathbb{R}$, with the following properties:

(*i*) $\mathrm{Dom}\,\Gamma = \mathcal{X}$ and $\|\Gamma\| = 1$;

(*ii*) $\min \Gamma(x) = 0$ for all $x \in K_0$;

(*iii*) $\min \Gamma(x_0) = \|x_0 - y_0\|$.

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