Babeş-Bolyai University of Cluj-Napoca
Faculty of Mathematics and Computer Science

Contributions to the study of some algebraic constructions. Categorial aspects and applications to fuzzy arithmetics.

Ph.D. Thesis Summary

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Introduction

This thesis presents some algebraic constructions (group rings, trivial extensions and Dorroh extensions) treated by categorial and topological point of view, in the first part, respectively, some algebraic constructions on the set of fuzzy numbers, in the second part.

Chapter 1. Ring extensions. In this chapter we presented some algebraic and categorial properties of some ring extensions and it is structured as follows:

1.1. Group rings. In this section we presented some basic notions and results of the theory of group rings and some new results. Thus, here we introduced the category $\mathfrak{RngGrp}$ (which has as objects triples of the form (R, G, σ), where R is a ring with identity, G is a group and $\sigma : G \to \operatorname{Aut} R$ is a group homomorphism), the covariant functor $\mathrm{F} : \mathfrak{RngGrp} \to \mathfrak{Rng}$, which associate to (R, G, σ) the skew group ring $R *_\sigma G$ and we proved that this functor has a right adjoint (Theorem 1.1.4). Here, we also proved that the bifunctor $\mathrm{H_c} : \mathfrak{Rng}_\mathbf{c} \times \mathfrak{Ab} \to \mathfrak{Rng}_\mathbf{c}$ (which associate to a commutative ring with identity R and a commutative group G, the group ring $R[G]$) has a right adjoint (Theorem 1.1.7).

1.2. Commutative ring extensions. In this section, we gave a categorial presentation of the trivial extensions. Thus, here we presented the universal property of the trivial extension $R \ltimes M$ (Theorem 1.2.1) and some consequences of this theorem (Corollary 1.2.2, Proposition 1.2.4), results which facilitate the categorial constructions presented in this section. Here, we also characterized the group of units of the semidirect product $R * M$ (Proposition 1.2.9).

1.3. Generalized semidirect products. As a generalization of those presented in the previous section, we introduced the ring $R \ltimes_\alpha^\beta M$ (called the (α, β)–semidirect product of a ring R and an R-module M) and we studied some algebraic properties and categorial properties of this construction. Thus, in this section, we characterized the group of the units of the ring $R \ltimes_\alpha^\beta M$, we gave the universal property and we made some categorial constructions. Here, we also studied some topological properties, namely, the extensions of the norms on R and on M to the ring $R \ltimes_{\alpha}^{\beta} M$.

Chapter 2. **Dorroh extensions.** In this chapter of the thesis, we presented some basic properties of the Dorroh extensions and some original contributions related to this construction. Thus, we introduced two notions (to simplify the presentation), namely, the Dorroh pairs and the $\mathcal{D}$-homomorphism, the universal property of this ring (Theorem 2.1.6) and its consequence (Corollary 2.1.8), concepts and results that are useful for the following categorial constructions. We also described those rings that can be expressed as a certain Dorroh extension (Theorem 2.1.10 and Theorem 2.1.11), we characterized the group of units of the ring $R \bowtie M$ (Theorem 2.3.2) and we constructed the "Dorroh extension" functor ($\mathbf{D} : \mathfrak{D} \rightarrow \mathfrak{Rng}$) and we showed that it has a right adjoint (Theorem 2.2.1) and commutes with direct products and inverse limits (Proposition 2.2.2 and Proposition 2.2.3).

Chapter 3. **Fuzzy numbers. Generalities.** In this chapter we presented the definition, some basic properties, and some representations of fuzzy numbers. Thus, besides the well-known LU representation, we introduced some new representations of fuzzy numbers: the multivalued representation, the CE representation (core ecart) and the MCE representation (middle-core ecart). The CE and the MCE representations facilitates the construction of new operations with fuzzy numbers, operations presented in the following chapters.

Chapter 4. **Dorroh-type products on the set of fuzzy numbers.** As an application of the Dorroh extensions, we introduced a new algebraic structure on the set of fuzzy numbers and we studied some of its properties. By using the CE representation of the fuzzy numbers, we introduced a new product (denoted by "$\circledast$") on the set of fuzzy numbers, product which is based on the Dorroh extension of a semiring by a semimodule. Thus, $(\mathfrak{F}_+, +, \circledast)$ is a semiring (Theorem 4.2.1), where $\mathfrak{F}_+$ is the set of all fuzzy numbers with positive core. As a particularization of this general construction, there we obtained a new product, called "Dorroh-product ". We also constructed an equivalence relation compatible with the addition and the Dorroh product (Proposition 4.3.2 and Theorem 4.3.7).

Chapter 5. **Completely distributive products on the set of fuzzy numbers.** In this chapter, by using the MCE representation of the fuzzy numbers, we introduced two new products on the set of fuzzy numbers, products that are completely distributive over addition. Thus, Theorem 5.1.1, shown that $(\mathfrak{F}_c, +, \boxdot)$ is a commutative semiring with identity and $(\mathfrak{F}_c, +, \boxtimes)$ is a commutative semiring. Here we also introduced a new scalar multiplication (which, for a positive scalar coincides with the usual scalar multiplication) and which, in addition to the common properties, has a new property (Proposition 5.1.8.5). To define the topological structure of the set $\mathfrak{F}_c$, we introduced four types of norms and a new metric on the set $\mathfrak{F}_c$. Their properties are given in Proposition 5.2.1, Theorem 5.2.2 and Proposition 5.2.3. In the last paragraph of this chapter we presented some elementary functions, defined on the set of fuzzy numbers, their construction it being possible (in this form) due to using the MCE representation and the product $\boxdot$.

Chapter 6. Topological group structures on quotient sets of fuzzy numbers. A.M. Bica has constructed in [11] two isomorphic Abelian groups, defined on quotient set of the set of those unimodal fuzzy numbers which has strictly monotone and continuous sides. In this chapter, we extended the results of [11] to a larger class of fuzzy numbers and adding to it a topological structure. Here, we also characterized the constructed quotient groups, by using the set $\mathrm{BVC}\,[0,1]$ of the continuous functions with bounded variation, defined on $[0,1]$.

Finally, I would like to thank my scientific advisor, Professor Ioan Purdea, for his support, advice and supervision, while elaborating this thesis. I also want to thank to the members of the Chair of Algebra of "Babeş–Bolyai" University of Cluj Napoca.

KEYWORDS: group ring; skew group ring; near-ring; semiring; semimodule; semigroup with involution; trivial extension; Dorroh extension; (group) semidirect product; category; covariant functor; adjoint functors; pseudo-normed ring; normed ring; metrizable topological group; fuzzy number; fuzzy arithmetics; function with bounded variation;

Part I

Algebraic constructions - categorial aspects

Chapter 1

Ring extensions

1.1 Group rings

Throughout this section, by a *ring* we mean an associative ring with identity and by a ring homomorphism we mean an unitary ring homomorphism.

1.1.1 Skew group rings

Let R be a ring, G a group and $\sigma : G \to \operatorname{Aut} R$ be a group homomorphism. Denote $\sigma(g)(r)$ by r^g for all $g \in G$ and $r \in R$.

The *skew group ring* $R *_\sigma G$ (see, e.g. [75],[67]) is defined to be the free left R-module with G as a free generating set. The multiplication on $R *_\sigma G$ is defined distributively by using the following rule:

$$(r_1 g_1) \cdot (r_2 g_2) = r_1 r_2^{g_1} g_1 g_2 \,,$$

for all $r_1, r_2 \in R$ and $g_1, g_2 \in G$.

Theorem 1.1.1 ([75], [67]) *Let R be a ring, G be a group and $\sigma : G \to \operatorname{Aut} R$ be a group homomorphism. For any ring A, any ring homomorphism $\varphi : R \to A$ and any group homomorphism $f : G \to U(A)$, for which*

$$\varphi(r^g) = f(g) \cdot \varphi(r) \cdot (f(g))^{-1} \,,$$

for all $r \in R$ and $g \in G$, there exists a unique ring homomorphism $\Phi : R *_\sigma G \to A$, such that $\Phi(r) = \varphi(r)$, for all $r \in R$ and $\Phi(g) = f(g)$ for all $g \in G$.

Corollary 1.1.2 [38] *Let $f : G \to G'$ be a group homomorphism and $\varphi : R \to R'$ be a ring homomorphism. If $\sigma : G \to \operatorname{Aut} R$ and $\sigma' : G' \to \operatorname{Aut} R'$ are two group homomorphisms, such that the following diagram*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ {\scriptstyle \sigma(g)}\big\downarrow & & \big\downarrow{\scriptstyle (\sigma' \circ f)(g)} \\ R & \xrightarrow{\varphi} & R' \end{array}$$

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ {\scriptstyle \sigma(g)}\downarrow & & \downarrow{\scriptstyle (\sigma' \circ f)(g)} \\ R & \xrightarrow{\varphi} & R' \end{array} \tag{1.1}$$

is commutative $\left(i.e.,\ (\sigma' \circ f)(g) \circ \varphi = \varphi \circ \sigma(g)\right)$, *for all $g \in G$, then the mapping*

$$\begin{array}{rcccl} \Phi = \overline{(\varphi, f)} & : & R *_\sigma G & \longrightarrow & R' *_{\sigma'} G' \\ & & \sum_{i=1}^{n} r_i\, g_i & \longmapsto & \sum_{i=1}^{n} \varphi(r_i)\ f(g_i) \end{array}$$

is a ring homomorphism, which extends f and φ.

Corollary 1.1.3 (1) [38] *If R is a ring, G and G' are two groups and $f : G \to G'$, $\sigma : G \to \operatorname{Aut} R$ and $\sigma' : G' \to \operatorname{Aut} R$ are group homomorphisms, such that*

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ & {\scriptstyle \sigma}\searrow \quad \swarrow{\scriptstyle \sigma'} & \\ & \operatorname{Aut} R & \end{array} \qquad \sigma' \circ f = \sigma,$$

$\big($*or equivalently,* $r^{f(g)} = r^g$, *for all $g \in G$ and $r \in R$*$\big)$, *then the mapping*

$$\begin{array}{rcccl} \overline{f} & : & R *_\sigma G & \longrightarrow & R *_{\sigma'} G' \\ & & \sum_{i=1}^{n} r_i\, g_i & \longmapsto & \sum_{i=1}^{n} r_i\, f(g_i) \end{array}$$

is a ring homomorphism, which extends f.

(2) [38] *If G is a group R and R' are two rings, $\varphi : R \to R'$ is a ring homomorphism and $\sigma : G \to \operatorname{Aut} R$ and $\sigma' : G \to \operatorname{Aut} R'$ are two group homomorphisms such that*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ {\scriptstyle \sigma(g)}\big\downarrow & & \big\downarrow{\scriptstyle \sigma'(g)} \\ R & \xrightarrow{\varphi} & R' \end{array} \qquad \sigma'(g) \circ \varphi = \varphi \circ \sigma(g),\ \textit{for all } g \in G$$

$(or\ equivalently,\ \varphi(r^g) = \varphi(r)^g,\ for\ all\ g \in G\ and\ r \in R)$, *then the mapping*

$$\begin{array}{rcccl} \overline{\varphi} & : & R *_\sigma G & \longrightarrow & R' *_{\sigma'} G \\ & & \sum\limits_{i=1}^{n} r_i\, g_i & \longmapsto & \sum\limits_{i=1}^{n} \varphi(r_i)\ g_i \end{array}$$

is a ring homomorphism, which extends φ.

We can consider now, the following categories:.

1. If R is a fixed ring, consider the category $\mathfrak{Grp}_R$ for which the objects are pairs (G, σ), where G is a group and $\sigma : G \to \operatorname{Aut} R$ is a group homomorphism and

$$\operatorname{Hom}_{\mathfrak{Grp}_R}((G, \sigma), (G', \sigma')) = \{f \in \operatorname{Hom}_{\mathfrak{Grp}}(G, G') : \sigma' \circ f = \sigma\}.$$

2. If G is a fixed group, consider the category $\mathfrak{Rng}_G$, whose objects are pairs (R, σ), where R is a ring and $\sigma : G \to \operatorname{Aut} R$ is a group homomorphism and the set of morphisms from (R, σ) to (R', σ'), $\operatorname{Hom}_{\mathfrak{Rng}_G}((R, \sigma), (R', \sigma'))$ is

$$\{\varphi \in \operatorname{Hom}_{\mathfrak{Rng}}(R, R') : \sigma'(g) \circ \varphi = \varphi \circ \sigma(g),\ \forall\, g \in G\}.$$

3. We also consider the category $\mathfrak{RngGrp}$ constructed as follows:

   - the class of objects are the triplets (R, G, σ), where R is a ring, G is a group and $\sigma : G \to \operatorname{Aut} R$ is a group homomorphism;
   - the set of morphisms $\operatorname{Hom}_{\mathfrak{RngGrp}}((R, G, \sigma), (R', G', \sigma'))$, consist of all pairs (φ, f), where $\varphi : R \to R'$ is a ring homomorphism and $f : G \to G'$ is a group homomorphism, for which $(\sigma' \circ f)(g) \circ \varphi = \varphi \circ \sigma(g)$, for all $g \in G$.
   - if

$$\begin{array}{rcl} (\varphi, f) & \in & \operatorname{Hom}_{\mathfrak{RngGrp}}((R, G, \sigma), (R', G', \sigma')) \\ (\varphi', f') & \in & \operatorname{Hom}_{\mathfrak{RngGrp}}((R', G', \sigma'), (R'', G'', \sigma'')) \end{array}$$

   then $(\varphi', f') \circ (\varphi, f) = (\varphi' \circ \varphi, f' \circ f) \in \operatorname{Hom}_{\mathfrak{RngGrp}}((R, G, \sigma), (R'', G'', \sigma''))$.

Consider also, the following covariant functors:

1. If R is a fixed ring, we define the functor $\mathrm{I}_R : \mathfrak{Grp}_R \to \mathfrak{RngGrp}$ by

$$\begin{array}{ccc} (G, \sigma) & \longmapsto & \mathrm{I}_R(G, \sigma) = (R, G, \sigma) \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathrm{I}_R(f) = (\mathrm{id}_R, f)} \\ (G', \sigma) & \longmapsto & \mathrm{I}_R(G', \sigma) = (R, G', \sigma) \end{array}$$

2. If G is a fixed group, we define the functor $\mathrm{I}_G : \mathfrak{Rng}_G \rightarrow \mathfrak{RngGrp}$ by

$$\begin{array}{ccc} (R,\sigma) & \longmapsto & \mathrm{I}_G(R,\sigma) = (R,G,\sigma) \\ \downarrow \varphi & & \downarrow \mathrm{I}_G(\varphi)=(\varphi,\mathrm{id}_G) \\ (R',\sigma) & \longmapsto & \mathrm{I}_G(R',\sigma) = (R',G,\sigma) \end{array}$$

3. By Corollary 1.1.2, we can consider the functor $\mathrm{F} : \mathfrak{RngGrp} \rightarrow \mathfrak{Rng}$ defined by

$$\begin{array}{ccc} (R,G,\sigma) & \longmapsto & \mathrm{F}(R,G,\sigma) = R *_\sigma G \\ \downarrow (\varphi,f) & & \downarrow \mathrm{F}(\varphi,f) = \Phi \\ (R',G',\sigma') & \longmapsto & \mathrm{F}(R',G',\sigma') = R' *_{\sigma'} G' \end{array}$$

4. If R is a ring, then the mapping $\sigma_R : \mathrm{U}(R) \rightarrow \mathrm{Aut}\, R$, $r_0 \mapsto \sigma_{r_0}$,where $\sigma_{r_0}(x) = r_0\, x\, r_0^{-1}$, for all $x \in R$, is a group homomorphism. So, we can define the functor $\widetilde{\mathrm{U}} : \mathfrak{Rng} \rightarrow \mathfrak{RngGrp}$ by

$$\begin{array}{ccc} A & \longmapsto & \widetilde{\mathrm{U}}(A) = (A, \mathrm{U}(A), \sigma_A) \\ \downarrow \varphi & & \downarrow \widetilde{\mathrm{U}}(\varphi)=(\varphi,\mathrm{U}(\varphi)) \\ B & \longmapsto & \widetilde{\mathrm{U}}(B) = (B, \mathrm{U}(B), \sigma_B) \end{array}$$

where $\mathrm{U}(R)$ denotes the group of units of the ring R and $\mathrm{U}(\varphi) : \mathrm{U}(A) \rightarrow \mathrm{U}(B)$ is the group homomorphism induced by the ring homomorphism $\varphi : A \rightarrow B$.

Theorem 1.1.4 [38] *The functor* F *is left adjoint to* $\widetilde{\mathrm{U}}$.

1.1.2 Group rings

If $\sigma(g) = \mathrm{id}_R$, for all $g \in G$, then the skew group ring $R *_\sigma G$ coincides with the group ring $R[G]$.

If in Theorem 1.1.1, consider that $\sigma(g) = \mathrm{id}_R$, for all $g \in G$, we obtain:

Theorem 1.1.5 *Let R be a ring and G be a group. For any ring A, any ring homomorphism $\varphi : R \rightarrow A$ and any group homomorphism $f : G \rightarrow U(A)$, for which*

$$\varphi(r) \cdot f(g) = f(g) \cdot \varphi(r),$$

for all $r \in R$ and $g \in G$, there exists a unique ring homomorphism $\Phi : R[G] \rightarrow A$, such that $\Phi(r) = \varphi(r)$, for all $r \in R$ and $\Phi(g) = f(g)$, for all $g \in G$.

Corollary 1.1.6 *For any ring A, any ring homomorphism $\varphi : R \to R'$ and any group homomorphism $f : G \to G'$, there exists a unique ring homomorphism $\Phi : R\,[G] \to R'\,[G']$, such that $\Phi\,(r) = \varphi\,(r)$, for all $r \in R$ and $\Phi\,(g) = f\,(g)$, for all $g \in G$.*

So, by Corollary 1.1.6, we can consider the covariant functor $\mathrm{H} : \mathfrak{Rng}\times\mathfrak{Grp} \to \mathfrak{Rng}$, defined by:

$$\begin{array}{ccc} (R,G) & \longmapsto & \mathrm{H}(R,G) = R[G] \\ \Big\downarrow{\scriptstyle (\varphi,f)} & & \Big\downarrow{\scriptstyle \mathrm{H}(\varphi,f)=\Phi} \\ (R',G') & \longmapsto & \mathrm{H}(R',G') = R'[G'] \end{array}$$

Analogously, for the commutative case, consider the functor $\mathrm{H_c} : \mathfrak{Rng}_{\mathfrak{c}}\times\mathfrak{Ab} \to \mathfrak{Rng}_{\mathfrak{c}}$.

We also consider the functor $\widehat{\mathrm{U}} : \mathfrak{Rng}_{\mathfrak{c}} \to \mathfrak{Rng}_{\mathfrak{c}}\times\mathfrak{Ab}$, defined by:

$$\begin{array}{ccc} A & \longmapsto & \widehat{\mathrm{U}}(A) = (A, \mathrm{U}(A)) \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \widehat{\mathrm{U}}(\varphi)=(\varphi,\mathrm{U}(\varphi))} \\ B & \longmapsto & \widehat{\mathrm{U}}(B) = (B, \mathrm{U}(B)) \end{array}$$

Theorem 1.1.7 *The functor* $\mathrm{H_c}$ *is left adjoint to* $\widehat{\mathrm{U}}$.

1.2 Commutative ring extensions

Throughout this section, by a ring we mean an associative ring.

We consider the ring of endomorphisms $(\mathrm{End}\, M, +, \circ)$ of an Abelian group $(M,+)$, a commutative ring with identity $(R,+,\cdot)$ and $\delta : (R,+,\cdot) \to (\mathrm{End}\, M, +, \circ)$ a unitary ring homomorphism. If for all $a \in R$ and $x \in G$ we denote $\delta\,(a)\,(x) = ax$, we obtain that M is a left R-module. Conversely, if M is a left R-module, then the mapping $a \mapsto \delta_a$, where

$$\delta_a : G \to G, \quad x \mapsto ax$$

is a unitary ring homomorphism of $(R,+,\cdot)$ in $(\mathrm{End}\, G, +, \circ)$.

We also consider a multiplicative isomorphic copy $\overline{M}$ of the group G, i.e., $\overline{M} = \{\overline{x} : x \in M\}$, and

$$\overline{x} \cdot \overline{y} = \overline{x+y}, \text{ for all } x, y \in M.$$

1.2.1 Trivial extensions

Let $(R,+,\cdot)$ be a commutative ring with identity and M a left R-module. On the direct product $(R\times M,+)$ of the Abelian groups $(R,+)$ and $(M,+)$, we consider the multiplication

$$(a,x)\bullet(b,y)=(ab,bx+ay).$$

$(R\times M,+,\bullet)$ becomes a commutative ring with identity, called the trivial extension of R by M (or the idealization of M) and it is denoted by $R\ltimes M$ ([44], [54]). Moreover, $R\ltimes M$ is an $R-$algebra with the operation

$$R\times(R\ltimes M)\longrightarrow R\ltimes M,\quad (\alpha,(a,x))\longmapsto(\alpha a,\alpha x).$$

We consider the following mappings:

$$\begin{aligned}
i_{\overline{M}} &: \overline{M}\to R\times M,\ \ \overline{x}\mapsto(1,x);\\
i_R &: R\to R\times M,\ \ a\mapsto(a,0);\\
i_M &: M\to R\times M,\ \ x\mapsto(0,x)\\
\pi_R &: R\times M\to R,\ \ (a,x)\mapsto a;\\
\pi_{\overline{M}} &: U(R\ltimes M)\to\overline{M},\ \ (a,x)\mapsto\overline{a^{-1}x}.
\end{aligned}$$

These applications verifies the following properties:

1. $i_{\overline{M}}$ is an embedding of the group $\overline{M}$ in the group $U(R\ltimes M)$;

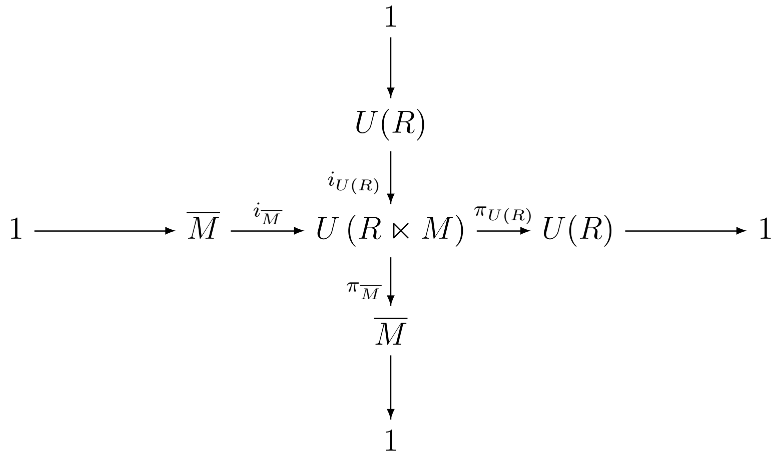
2. i_R is an embedding of the ring R in the ring $R\ltimes M$ and so its restriction $i_R\big|_{U(R)}=i_{U(R)}$ is an embedding of the group $U(R)$ in the group $U(R\ltimes M)$;

3. i_M is an embedding of the group $(M,+)$ in the additive group of $R\ltimes M$. If we identify the element $x\in M$ with $(0,x)\in R\times M$, we can consider that M is a subring of $R\times M$ with the multiplication $x_1\bullet x_2=0$.

4. π_R is a surjective homomorphism of the ring $R\ltimes M$ onto the ring R and so its restriction $\pi_R\big|_{U(R)}=\pi_{U(R)}$ is a group homomorphism of $U(R\ltimes M)$ onto $U(R)$,

5. $\pi_{\overline{M}}$ is a surjective group homomorphism;

6. the following sequences



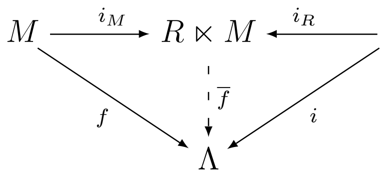
are exacts and $\pi_{\overline{M}} \circ i_{\overline{M}} = \mathrm{id}_{\overline{M}}$ and $\pi_{U(R)} \circ i_{U(R)} = \mathrm{id}_{U(R)}$. Therefore, $U(R \ltimes M) \cong U(R) \times \overline{M} \cong U(R) \times M$. The isomorphism is given by

$$\begin{array}{ccc} U(R) \times \overline{M} & \longrightarrow & U(R \ltimes M) \\ (a, \overline{x}) & \longmapsto & (a, ax) \end{array}$$

Theorem 1.2.1 [39] *Let $(R, +, \cdot)$ a commutative ring with identity and M a R-module. Then for every R-algebra Λ and every R-linear map $f : M \to \Lambda$, with the property*

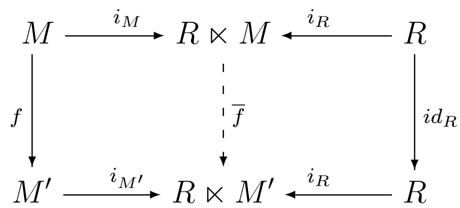
$$f(x) \cdot f(y) = 0, \quad \textit{for all } x, y \in M,$$

there exists an unique R-algebras homomorphism $\overline{f} : R \ltimes M \to \Lambda$, such that



$$\overline{f} \circ i_M = f \quad \textit{and} \quad \overline{f} \circ i_R = i.$$

Corollary 1.2.2 [39] *If M and M' are two R-modules and $f : M \to M'$ is a linear map, then there exists a unique $R-$algebras homomorphism $\overline{f} : R \ltimes M \to R \ltimes M'$ extending f, i.e., the following diagram*



is commutative.

Remark 1.2.3 *By Corollary* 1.2.2*, we can construct a covariant functor* $F : \mathrm{Mod}_R \to \mathrm{Alg}_R$*, as follows:*

$$\begin{array}{ccc} M & \longmapsto & \mathrm{F}(M) = R \ltimes M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{F}(f)=\overline{f}} \\ M' & \longmapsto & \mathrm{F}(M') = R \ltimes M' \end{array}$$

Proposition 1.2.4 [39] *Let* R_1 *and* R_2 *two unitary commutative rings,* $(M,+)$ *an Abelian group,* $\delta_1 : R_1 \to \mathrm{End}\, M$ *and* $\delta_2 : R_2 \to \mathrm{End}\, M$ *two unitary ring homomorphisms. If* $f : R_1 \to R_2$ *is a unitary ring homomorphism, such that*

$$\begin{array}{ccc} R_1 & \xrightarrow{\;f\;} & R_2 \\ & {\scriptstyle \delta_1}\searrow \quad \swarrow{\scriptstyle \delta_2} & \\ & \mathrm{End}\, M & \end{array} \qquad \delta_1 = \delta_2 \circ f,$$

then the mapping

$$\begin{array}{rccc} \overline{f} & : & R_1 \ltimes M & \longrightarrow & R_2 \ltimes M \\ & & (r_1, x) & \longmapsto & (f(r_1), x) \end{array}$$

is a unitary ring homomorphism which extend f*.*

Remark 1.2.5 *If* $(M,+)$ *is an Abelian group, we can consider the category* $\mathfrak{Rng}_M$*, whose objects are pairs of the form* (R,δ)*, where* R *is a unitary commutative ring and* $\delta : (R,+,\cdot) \to (\mathrm{End}\, M, +, \circ)$ *is a unitary ring homomorphism and*

$$\mathrm{Hom}_{\mathfrak{Rng}_M}((R_1,\delta_1),(R_2,\delta_2)) = \{f \in \mathrm{Hom}_{\mathfrak{Rng}}(R_1,R_2) : \delta_1 = \delta_2 \circ f\}.$$

By Proposition 1.2.4*, we can construct the covariant functor* $H : \mathfrak{Rng}_M \to \mathfrak{Rng}$ *, defined by*

$$\begin{array}{ccc} (R_1,\delta_1) & \longmapsto & \mathrm{H}(R_1,\delta_1) = R_1 \ltimes M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{H}(f)=\overline{f}} \\ (R_2,\delta_2) & \longmapsto & \mathrm{H}(R_2,\delta_2) = R_2 \ltimes M \end{array}$$

1.2.2 The semidirect product of a ring R with an R-module

Near-rings are generalized rings. They might generally be described as rings $(A,+,\cdot)$ where the addition is not necessarily abelian and only one distributive law holds:

Definition 1.2.6 [76] *A right (left) near-ring is a non-empty set* A*, together with two binary operations "* $+$ *" and "* $\cdot$ *", which satisfy the following conditions:*

1. *$(A, +)$ is a group (not necessarily abelian);*

2. *$(A, \cdot)$ is a semigroup;*

3. *the right (left) distributivity law is satisfied.*

Further, by a near-ring we mean a right near-ring.

On the direct product $(R \times M, +)$ of the Abelian groups $(R, +)$ and $(M, +)$ we also consider the multiplication

$$(a, x) \cdot (b, y) = (ab, x + ay)$$

Proposition 1.2.7 *$(R \times M, +, \cdot)$ is a right near-ring with identity.*

Definition 1.2.8 *The near-ring $(R \times M, +, \cdot)$ is called the semidirect product of the ring R with M and it is denoted by $R * M$.*

We consider the mappings:

$$\begin{aligned} i_{\overline{M}} &: \overline{M} \to R \times M, \ \ \overline{x} \mapsto (1, x)\,; \\ i_R &: R \to R \times M, \ \ a \mapsto (a, 0)\,; \\ \pi_R &: R \times M \to R, \ \ (a, x) \mapsto a. \end{aligned}$$

Then:

1. $i_{\overline{M}}$ is an embedding of the group $\overline{M}$ in the group $U(R * M)$;

2. i_R is an embedding of the ring R in the near-ring $R * M$ and so, its restriction $i_R\big|_{U(R)} = i_{U(R)}$, is an embedding of the group $U(R)$ in the group $U(R * M)$;

3. π_R is a surjective homomorphism of the near-ring $R * M$ onto the ring R and so, its restriction $\pi_R\big|_{U(R)} = \pi_{U(R)}$ is a group homomorphism of $U(R * M)$ onto $U(R)$.

Proposition 1.2.9 [39] *The group of units $U(R * M)$ of the near-ring $R * M$ is isomorphic to the semidirect product $\overline{M} \times_{U(\delta)} U(R)$ of the groups $\overline{M}$ and $U(R)$, where $U(\delta) : U(R) \to \mathrm{Aut}\,\overline{M}$ is the group homomorphism induced by the ring homomorphism $\delta : R \to \mathrm{End}\, M$.*

1.3 Generalized semidirect products

1.3.1 The construction of the generalized semidirect product

We consider an Abelian group $(M,+)$, a commutative ring with identity $(R,+,\cdot)$, an unitary ring homomorphism $\delta:(R,+,\cdot)\to(\operatorname{End}M,+,\circ)$ and two functions $\alpha,\,\beta:R\to R$. If $a\in R$ and $x\in G$, we denote $\delta(a)(x)=a\cdot x$.

On the direct product $(R\times G,+)$ of the additive groups $(R,+)$ and $(G,+)$, we consider the multiplication

$$(a,x)\cdot(b,y):=(ab,\alpha(b)\cdot x+\beta(a)\cdot y). \tag{1.2}$$

Proposition 1.3.1 [37] *As above, we have that:*

1. *If* $\alpha(a)\cdot\beta(b)=\beta(b)\cdot\alpha(a)$, *for all* $a,b\in R$, $\alpha\in\operatorname{End}^*(R,\cdot)$ [1] *and* $\beta\in\operatorname{End}(R,\cdot)$, *then* $(R\times M,\cdot)$ *is a semigroup;*
2. *if* R *is a ring with identity and* $\alpha(1)=1$, *then* $(1,0)$ *is a right unit of the multiplication defined by* (1.2)*;*
3. *if* R *is a ring with identity and* $\beta(1)=1$, *then* $(1,0)$ *is a left unit of the multiplication defined by* (1.2)*;*
4. *if* $\alpha\in\operatorname{End}(R,+)$, *then the multiplication* (1.2) *distributes over addition on the left;*
5. *if* $\beta\in\operatorname{End}(R,+)$, *then the multiplication* (1.2) *distributes over addition on the right.*

We consider an Abelian group $(M,+)$, a commutative ring with identity $(R,+,\cdot)$, an unitary ring homomorphism $\delta:(R,+,\cdot)\to(\operatorname{End}M,+,\circ)$ and two functions $\alpha,\,\beta:R\to R$. If $a\in R$ and $x\in G$, we denote $\delta(a)(x)=a\cdot x$.

Corollary 1.3.2 [37] *Let* R *be a ring,* $(M,+)$ *an Abelian group and* $\delta:(R,+,\cdot)\to(\operatorname{End}M,+,\circ)$ *a ring homomorphism. If* $\alpha\in\operatorname{End}^*(R,+,\cdot)$ *and* $\beta\in\operatorname{End}(R,+,\cdot)$ *verifies the property that*

$$\alpha(a)\cdot\beta(b)=\beta(b)\cdot\alpha(a)\,,\ \textit{for all}\ \ a,b\in R, \tag{1.3}$$

[1] i.e., α is an anti-endomorphism

then $(R \times M, +, \cdot)$ *is a ring. If in addition,* R *is with identity and* α, β, δ *are unitary homomorphisms, then* $(R \times M, +, \cdot)$ *is a ring with identity.*

Definition 1.3.3 [37] *The ring* $(R \times M, +, \cdot)$ *(see, Corollary* 1.3.2*) is called the* (α, β)*-the* ***semidirect product*** *of* R *with* M *and it is denoted by* $R \ltimes_{\alpha}^{\beta} M$*.*
If R *is commutative and* $\alpha = \beta$*, this ring is denoted by* $R \ltimes_{\alpha} M$*.*

Denote by:

- Ω the class of all systems $(R, M, \delta, \alpha, \beta)$, where $(R, +, \cdot)$ is a ring, $(M, +)$ is an Abelian group, $\delta : (R, +, \cdot) \to (\operatorname{End} M, +, \circ)$ is a ring homomorphism and $\alpha \in \operatorname{End}^* (R, +, \cdot)$, $\beta \in \operatorname{End}(R, +, \cdot)$ which satisfies the condition (1.3).
- Ω_c the class of all systems $(R, M, \delta, \alpha, \beta) \in \Omega$, where $(R, +, \cdot)$ is a commutative ring;
- Ω_1 the class of all systems $(R, M, \delta, \alpha, \beta) \in \Omega$, where $(R, +, \cdot)$ is a ring with identity, δ is a unitary ring homomorphism and $\alpha \in \operatorname{End}^* (R, +, \cdot, 1)$, $\beta \in \operatorname{End}(R, +, \cdot, 1)$;
- $\Omega_{c,1} = \Omega_c \cap \Omega_1$.

Remark 1.3.4 *Thus:*

1. $(R, M, \delta, \alpha, \beta) \in \Omega \implies R \ltimes_{\alpha}^{\beta} M$ *is a ring;*
2. $(R, M, \delta, \alpha, \beta) \in \Omega_1 \implies R \ltimes_{\alpha}^{\beta} M$ *is a ring with identity;*
3. $(R, M, \delta, \alpha, \alpha) \in \Omega_{c,1} \implies R \ltimes_{\alpha} M$ *is a commutative ring with identity.*

1.3.2 The group of units of the ring $R \ltimes_{\alpha}^{\beta} M$

We consider that $(R, M, \delta, \alpha, \beta) \in \Omega_1$. We also consider a multiplicative isomorphic copy $\overline{M}$ of the group G, i.e., $\overline{M} = \{\overline{x} : x \in M\}$, and

$$\overline{x} \cdot \overline{y} = \overline{x + y}, \text{ for all } x, y \in M.$$

Proposition 1.3.5 *If* $(a, x) \in R \ltimes_{\alpha}^{\beta} M$*, then* $(a, x) \in \mathrm{U}\left(R \ltimes_{\alpha}^{\beta} M, +, \cdot\right)$ *if and only if* $a \in \mathrm{U}(R, +, \cdot)$*. In this case,*

$$(a, x)^{-1} = \left(a^{-1}, -\alpha\left(a^{-1}\right) \cdot \beta\left(a^{-1}\right) \cdot x\right).$$

We consider the following functions:

$$\begin{aligned} i_{\overline{M}} &: \overline{M} \to R \ltimes_{\alpha}^{\beta} M, \ \ \overline{x} \mapsto (1, x)\,; \\ i_R &: R \to R \ltimes_{\alpha}^{\beta} M, \ \ a \mapsto (a, 0)\,; \\ \pi_R &: R \ltimes_{\alpha}^{\beta} M \to R, \ \ (a, x) \mapsto a; \\ \pi_{\overline{M}} &: U\left(R \ltimes_{\alpha}^{\alpha} M\right) \to \overline{M}, \ \ (a, x) \mapsto \overline{\alpha\left(a^{-1}\right) \cdot x}. \end{aligned}$$

It is easy to see that:

1. $i_{\overline{M}}$ is an embedding of the group $\overline{M}$ in the group $U\left(R \ltimes_{\alpha}^{\beta} M\right)$;

2. i_R is an embedding of the ring R in the ring $R \ltimes_{\alpha}^{\beta} M$, and so its restriction to $U(R)$, $i_R\big|_{U(R)} = i_{U(R)}$, is an embedding of the group $U(R)$ in the group $U\left(R \ltimes_{\alpha}^{\beta} M\right)$;

3. π_R is a surjective ring homomorphism and its restriction to $U(R)$, $\pi_R\big|_{U(R)} = \pi_{U(R)}$, is a surjective group homomorphism of $U\left(R \ltimes_{\alpha}^{\beta} M\right)$ onto $U(R)$,

4. $\pi_{\overline{M}}$ is a surjective group homomorphism.

Since the following sequences

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & U(R) & & & & \\ & & & & \downarrow{\scriptstyle i_{U(R)}} & & & & \\ 1 & \longrightarrow & \overline{M} & \xrightarrow{i_{\overline{M}}} & U(R \ltimes_{\alpha}^{\beta} M) & \xrightarrow{\pi_{U(R)}} & U(R) & \longrightarrow & 1 \end{array}$$

are exacts and $\pi_{U(R)} \circ i_{U(R)} = \mathrm{id}_{U(R)}$, we have that

$$U\left(R \ltimes_{\alpha}^{\beta} M\right) \cong U(R) \times_{\phi} \overline{M},$$

where $\phi : (U(R), \cdot) \to \mathrm{Aut}\left(\left(\overline{M}, \cdot\right), \circ\right)$ is defined by

$$\phi(a)(\overline{x}) = \overline{\alpha\left(a^{-1}\right) \cdot \beta(a) \cdot x}, \qquad \forall\, a \in U(R), \ \ \forall\, \overline{x} \in \overline{M}.$$

The multiplication of the (group) semidirect product $U(R) \times_{\phi} \overline{M}$ is defined by

$$(a, \overline{x}) \cdot (b, \overline{y}) = (ab, \overline{x} \cdot \phi(a)(\overline{y})) = \left(ab, \overline{x + \alpha\left(a^{-1}\right) \cdot \beta(a) \cdot y}\right),$$

and the isomorphism between $U(R) \times_{\phi} \overline{M}$ and $U\left(R \ltimes_{\alpha}^{\beta} M\right)$ is given by

$$\begin{array}{ccc} U(R) \times_{\phi} \overline{M} & \longrightarrow & U\left(R \ltimes_{\alpha}^{\beta} M\right) \\ (a, \overline{x}) & \longmapsto & (a,\ \alpha(a) \cdot x) \end{array}$$

Proposition 1.3.6 *The groups* $U(R) \times_{\phi} \overline{M}$ *and* $U\left(R \ltimes_{\alpha}^{\beta} M\right)$ *are isomorphic.*

Remark 1.3.7 *If* $(R, M, \delta, \alpha, \alpha) \in \Omega_{c,1}$, *then* $U(R \ltimes_{\alpha} M) \cong U(R) \times \overline{M}$.

1.3.3 Categorial aspects

If $(R, M, \delta, \alpha, \beta) \in \Omega$, then the function

$$\sigma_{M} : M \to R \ltimes_{\alpha}^{\beta} M, \quad x \mapsto (0, x)$$

is an embedding of the group $(M, +)$ in the additive group $\left(R \ltimes_{\alpha}^{\beta} M, +\right)$. If we identify the elements $x \in M$ with $(0, x) \in R \ltimes_{\alpha}^{\beta} M$, we can consider that M is a subring of the ring $R \ltimes_{\alpha}^{\beta} M$, the product of M being the null multiplication, i.e.,

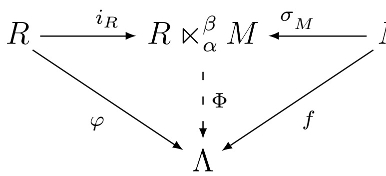
$$x_1 \cdot x_2 = 0, \quad \forall x_1, x_2 \in M.$$

Moreover, M is an ideal of $R \ltimes_{\alpha}^{\beta} M$.

Theorem 1.3.8 (The universal property) *Let* $(R, M, \delta, \alpha, \beta) \in \Omega$. *For every ring* Λ, *for every ring homomorphism* $\varphi : R \to \Lambda$ *and for every group homomorphism* $f : (M, +) \to (\Lambda, +)$, *which satisfies the properties:*

1. $f(\alpha(r) \cdot x) = f(x) \cdot \varphi(r), \ \forall r \in R, \ \forall x \in M$;
2. $f(\beta(r) \cdot x) = \varphi(r) \cdot f(x), \ \forall r \in R, \ \forall x \in M$;
3. $f(x) \cdot f(y) = 0, \ \forall x, y \in M$;

there exists an unique ring homomorphism $\Phi : R \ltimes_{\alpha}^{\beta} M \to \Lambda$, *which extend* f *and* φ, *i.e.,*



$$\Phi \circ \sigma_{M} = f \quad \text{and} \quad \Phi \circ i_R = \varphi.$$

If $(R, M, \delta, \alpha, \beta) \in \Omega_1$, Λ *is a ring with identity and* φ *is an unitary homomorphism, then* Φ *is an unitary homomorphism.*

Corollary 1.3.9 *If* $(R, M, \delta, \alpha, \beta), (R, M', \delta', \alpha, \beta) \in \Omega$ *and* $f : (M, +) \to (M', +)$ *is a group homomorphism such that:*

1. $f(\alpha(r)\cdot x)=\alpha(r)\cdot f(x)\,,\ \forall\, r\in R,\ \forall\, x\in M;$

2. $f(\beta(r)\cdot x)=\beta(r)\cdot f(x)\,,\ \forall\, r\in R,\ \forall\, x\in M;$

then there exists an unique ring homomorphism $\overline{f}: R\ltimes_{\alpha}^{\beta} M \to R\ltimes_{\alpha}^{\beta} M'$ *which extend* f*, i.e., the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & M' \\ \Big\downarrow{\scriptstyle \sigma_M} & & \Big\downarrow{\scriptstyle \sigma_{M'}} \\ R\ltimes_{\alpha}^{\beta} M & \overset{\overline{f}}{\dashrightarrow} & R\ltimes_{\alpha}^{\beta} M' \end{array}$$

is commutative and $\overline{f}\big|_R=\mathrm{id}_R\,.$

Corollary 1.3.10 *If* $(R,M,\delta,\alpha,\beta)\,,\ (R',M,\delta',\alpha',\beta')\in\Omega$ *and* $\varphi: R\to R'$ *is a ring homomorphism such that:*

1. $\alpha(r)\cdot x=\alpha'(\varphi(r))\cdot x,\quad \forall\, r\in R,\quad \forall\, x\in M;$

2. $\beta(r)\cdot x=\beta'(\varphi(r))\,\cdot x,\quad \forall\, r\in R,\quad \forall\, x\in M;$

then there exists an unique ring homomorphism $\overline{\varphi}: R\ltimes_{\alpha}^{\beta} M \to R'\ltimes_{\alpha'}^{\beta'} M$ *which extend* φ*, i.e., the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\ \varphi\ } & R' \\ \Big\downarrow{\scriptstyle i_R} & & \Big\downarrow{\scriptstyle i_{R'}} \\ R\ltimes_{\alpha}^{\beta} M & \overset{\overline{\varphi}}{\dashrightarrow} & R'\ltimes_{\alpha'}^{\beta'} M \end{array}$$

is commutative and $\overline{f}\big|_M=\mathrm{id}_M\,.$

Corollary 1.3.11 *If* $(R,M,\delta,\alpha,\beta)\,,(R',M',\delta',\alpha',\beta')\in\Omega,\ \varphi: R\to R'$ *is a ring homomorphism and* $f: M\to M'$ *is a group homomorphism such that:*

1. $f(\alpha(r)\cdot x)=\alpha'(\varphi(r))\cdot f(x)\,,\quad \forall\, r\in R,\quad \forall\, x\in M;$

2. $f(\beta(r)\cdot x)=\beta'(\varphi(r))\cdot f(x)\,,\quad \forall\, r\in R,\quad \forall\, x\in M;$

then there exists an unique ring homomorphism $\Phi : R \ltimes_{\alpha}^{\beta} M \to R' \ltimes_{\alpha'}^{\beta'} M'$, *which extend* φ *and* f, *i.e.,*

$$\begin{array}{ccccc} R & \xrightarrow{i_R} & R \ltimes_{\alpha}^{\beta} M & \xleftarrow{\sigma_M} & M \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle f} \\ R' & \xrightarrow{i_{R'}} & R' \ltimes_{\alpha'}^{\beta'} M' & \xleftarrow{\sigma_{M'}} & M' \end{array}$$

$\Phi \circ i_R = i_{R'} \circ \varphi$ *and* $\Phi \circ \sigma_M = \sigma_{M'} \circ f$.

Now, we consider the category $\mathfrak{C}$ defined by:

1. $\operatorname{Ob}\mathfrak{C} = \Omega$;

2. If $(R, M, \delta, \alpha, \beta), (R', M', \delta', \alpha', \beta') \in \Omega$, then $\operatorname{Hom}_{\mathfrak{C}}((R, M, \delta, \alpha, \beta), (R', M', \delta', \alpha', \beta'))$ is the set of all pairs (φ, f), where $\varphi : R \to R'$ is a ring homomorphism and $f : M \to M'$ is a group homomorphism which verifies the conditions of Corollary 1.3.11.

3. If

$$\begin{array}{rcl} (\varphi, f) & \in & \operatorname{Hom}_{\mathfrak{C}}((R, M, \delta, \alpha, \beta), (R', M', \delta', \alpha', \beta')) \\ (\varphi', f') & \in & \operatorname{Hom}_{\mathfrak{C}}((R', M', \delta', \alpha', \beta'), (R'', M'', \delta'', \alpha'', \beta'')), \end{array}$$

   then, we define $(\varphi', f') \circ (\varphi, f) = (\varphi' \circ \varphi,\ f' \circ f)$.

We can consider now, the covariant functor $F : \mathfrak{C} \to \mathfrak{Rng}$ defined by:

$$\begin{array}{ccc} (R, M, \delta, \alpha, \beta) & \longmapsto & \mathrm{F}(R, M, \delta, \alpha, \beta) = R \ltimes_{\alpha}^{\beta} M \\ {\scriptstyle(\varphi, f)}\downarrow & & \downarrow{\scriptstyle \mathrm{F}(\varphi, f) = \Phi} \\ (R', G', \delta', \alpha', \beta') & \longmapsto & \mathrm{F}(R', M', \delta', \alpha', \beta') = R' \ltimes_{\alpha'}^{\beta'} M' \end{array}$$

1.3.4 Norm extensions

Definition 1.3.12 *The function* $\|\cdot\| : A \to \mathbb{R}_+$ *is called a norm on the Abelian group* $(A, +)$, *if:*

1. $\|a\| = 0$ *if and only if* $a = 0$;

2. $\|a - b\| \le \|a\| + \|b\|, \ \forall\, a, b \in A$;

If $\|a+b\| \leq \max(\|a\|, \|b\|)$, *for all* $a, b \in A$, *the norm is called non-Archimedean.*

Definition 1.3.13 *The function* $\|\cdot\| : R \to \mathbb{R}_+$ *is called a pseudo-norm* (*norm*) *on the ring* R, *if:*

1. $\|a\| = 0$ *if and only if* $a = 0$;

2. $\|a - b\| \leq \|a\| + \|b\|$, $\forall\, a, b \in R$;

3. $\|a \cdot b\| \leq \|a\| \cdot \|b\| \quad (\|a \cdot b\| = \|a\| \cdot \|b\|)$, $\forall\, a, b \in R$.

4. $\|1\| = 1 \quad$ (*if* R *is with identity*).

Definition 1.3.14 *Let* R *be a pseudo-normed* (*normed*) *ring with identity and* M *be a left* R*-module. The function* $\|\cdot\| : M \to \mathbb{R}_+$ *is called a pseudo-norm* (*norm*) *on* M, *if:*

1. $\|x\| = 0$ *if and only if* $x = 0$;

2. $\|x - y\| \leq \|x\| + \|y\|$, $\forall\, x, y \in M$;

3. $\|a \cdot x\| \leq \|a\| \cdot \|x\| \quad (\|a \cdot x\| = \|a\| \cdot \|x\|)$, $\forall\, a \in R$, $\forall\, x \in M$.

We consider now $(R, M, \delta, \alpha, \beta) \in \Omega_1$, such that M is a pseudo-normed R-module and we assume that

$$\|\alpha(r)\| \leq \|r\| \text{ and } \|\beta(r)\| \leq \|r\|,$$

for all $r \in R$.

For each natural numbers k, we define the applications $\|\cdot\|_k : R \ltimes_\alpha^\beta M \to \mathbb{R}_+$ as follows:

$$\begin{aligned} \|(a,x)\|_0 &= \max(\|a\|, \|x\|) \\ \|(a,x)\|_1 &= \|a\| + \|x\| \\ \|(a,x)\|_k &= \sqrt[k]{\|a\|^k + \|x\|^k}, \qquad \text{if } k \geq 2. \end{aligned}$$

Theorem 1.3.15 [37] $\|\cdot\|_1$ *is a pseudo-norm on the ring* $R \ltimes_\alpha^\beta M$ *and if the pseudo-norm of* M *is non-Archimedean, then* $\|\cdot\|_0$ *and* $\|\cdot\|_k$ (*for* $k > 1$) *are pseudo-norms on the ring* $R \ltimes_\alpha^\beta M$. *Moreover, the pseudo-norms* $\|\cdot\|_k$ *extends the pseudo-norms of* R *and* M, *for all* k.

Chapter 2

Dorroh extensions

Throughout this chapter, by a ring we mean an associative ring.

2.1 Dorroh extensions

To simplify the presentation, we give the following definition:

Definition 2.1.1 *A pair (R, M) of (associative) rings, is called a* ***Dorroh-pair*** *if M is also an (R, R)-bimodule and for all $a \in R$ and $x, y \in M$, are satisfied the following compatibility conditions:*

$$(ax)\ y = a\ (xy)\,, \quad (xy)\ a = x\ (ya)\,, \quad (xa)\ y = x\ (ay)\,.$$

We denote further with $\mathcal{D}$, the class of all Dorroh-pairs.

If $(R, M) \in \mathcal{D}$, on the (Abelian groups) direct sum $R \oplus M$, we introduce the multiplication

$$(a, x) \cdot (b, y) = (ab,\ xb + ay + xy)\,.$$

$(R \oplus M, +, \cdot)$ is a ring, it is denoted by $R \bowtie M$ and it is called the **Dorroh extension**. Moreover, $R \bowtie M$ is an (R, R)-bimodule under the scalar multiplications defined by

$$\lambda\,(a, x) = (\lambda a, \lambda x)\,, \qquad (a, x)\,\lambda = (a\lambda, x\lambda)\,,$$

and $(R, R \bowtie M)$ is also a Dorroh-pair.

If R has the unit 1, then $(1, 0)$ is an unit of the ring $R \bowtie M$.

Remark 2.1.2 *Dorroh first used this construction (see [28]), with $R = \mathbb{Z}$, as a means of embedding a ring without identity into a ring with identity.*

Remark 2.1.3 *If M is a zero ring, the Dorroh extension $R \bowtie M$ coincides with the trivial extension $R \ltimes M$.*

Example 2.1.4 *If R is a ring, then $(R, R), (R, M_n(R)) \in \mathcal{D}$.*

Since the applications

$$\begin{aligned} i_R &: R \hookrightarrow R \bowtie M, \quad a \mapsto (a, 0) \\ i_{_M} &: M \hookrightarrow R \bowtie M, \quad x \mapsto (0, x) \end{aligned}$$

are injective and both ring homomorphisms and (R, R) linear maps, we can identify further the element $a \in R$ with $(a, 0) \in R \bowtie M$ and $x \in M$ with $(0, x) \in R \bowtie M$. The application

$$\pi_R : R \bowtie M \to R, \qquad (a, x) \mapsto a$$

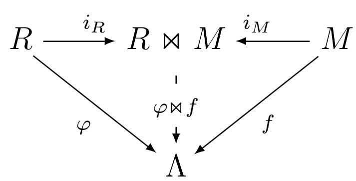
is a surjective ring homomorphism, which is also (R, R) linear.

Consequently, R is a subring of $R \bowtie M$ and M is an ideal of the ring $R \bowtie M$, with $(R \bowtie M)/M \simeq R$.

Definition 2.1.5 *Let (R, M) and (R', M') two Dorroh-pairs. By a $\mathcal{D}$-homomorphism of (R, M) to (R', M') we mean a pair (φ, f), where $\varphi : R \to R'$ and $f : M \to M'$ are ring homomorphisms for which, for all $\alpha \in R$ and $x \in M$ we have that*

$$f(\alpha \cdot x) = \varphi(\alpha) \cdot f(x) \qquad \text{and} \quad f(x \cdot \alpha) = f(x) \cdot \varphi(\alpha).$$

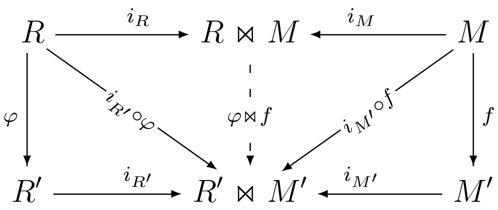
Theorem 2.1.6 [35] *If (R, M) is a Dorroh-pair, then for any ring Λ and any $\mathcal{D}$-homomorphism $(\varphi, f) : (R, M) \to (\Lambda, \Lambda)$, there exists an unique ring homomorphism $\varphi \bowtie f : R \bowtie M \to \Lambda$ such that*



$(\varphi \bowtie f) \circ i_{_M} = f$ and $(\varphi \bowtie f) \circ i_R = \varphi$.

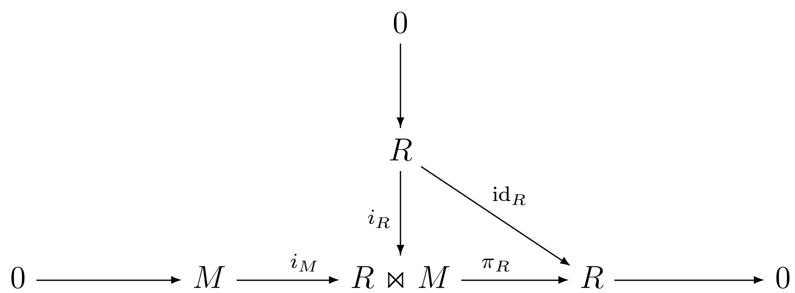
Remark 2.1.7 1. *$\varphi \bowtie f$ is injective if and only if φ and f are injective and $\operatorname{Im} \varphi \cap \operatorname{Im} f = \{0\}$.*
2. *$\varphi \bowtie f$ is surjective if and only if $\operatorname{Im} \varphi + \operatorname{Im} f = \Lambda$.*

Corollary 2.1.8 [35] *If* (R, M) *and* (R', M') *are two Dorroh-pairs, and* $(\varphi, f) : (R, M) \to (R', M')$ *is a* $\mathcal{D}$*-homomorphism, there exists an unique ring homomorphism* $\varphi \bowtie f : R \bowtie M \to \Lambda$ *such that*



$(\varphi \bowtie f) \circ i_R = i_{R'} \circ \varphi$ *and* $(\varphi \bowtie f) \circ i_M = i_{M'} \circ f$.

Remark 2.1.9 *If* (R, M) *is a Dorroh-pair, the sequences*



(*as Abelian groups sequences*) *are exacts and* $\pi_R \circ i_R = \mathrm{id}_R$. *Moreover, all homomorphisms are ring homomorphisms,* i_R *and* π_R *are unitary* (*if* R *is with identity*) *but* i_M *is not unitary* (*if* M *is with identity*).

Theorem 2.1.10 *Let* T *a ring with identity,* M *an ideal of* T *and* R *a subring of* T. *If* $R \cap M = \{0\}$ *and* $T = R + M$, *then the rings* T *and* $R \bowtie M$ *are isomorphic.*

Theorem 2.1.11 *Let* M *a ring,* R *and* T *two rings with identity,* $\alpha : M \to T$ *a ring homomorphism and* $\beta : T \to R$ *an unitary ring homomorphism. If the sequence*

$$0 \longrightarrow M \xrightarrow{\alpha} T \xrightarrow{\beta} R \longrightarrow 0$$

(*as Abelian groups sequence*) *is exact and* $s : R \to T$ *is an unitary ring homomorphism such that* $\beta \circ s = \mathrm{id}_R$, *then:*

(*i*) M *is a* $(R, R)-$*bimodule with the scalar multiplications defined by:*

$$a \cdot x \;:\; = \alpha_0^{-1} (s(a) \cdot \alpha_0(x))$$
$$x \cdot a \;:\; = \alpha_0^{-1} (\alpha_0(x) \cdot s(a))$$

($a \in R$, $x \in M$, *and* $\alpha_0 : M \to \operatorname{Im} \alpha$ *is the isomorphism induced by the injective homomorphism* α) *and* (R, M) *is a Dorroh-pair;*

(ii) the rings T and $R \bowtie M$ are isomorphic.

2.2 Categorial aspects

We consider now, the category $\mathfrak{D}$ whose objects are the class $\mathcal{D}$ of the Dorroh-pairs and the homomorphisms between two objects are the Dorroh-pairs homomorphisms, respectively, the category $\mathfrak{Rng}$ of the associative rings.

By Corollary 2.1.8, we can consider the covariant functor $\mathbf{D} : \mathfrak{D} \to \mathfrak{Rng}$, defined as follows: if $(R, M) \in \mathcal{D}$, then $\mathbf{D}(R, M) = R \bowtie M$, and if $(\varphi, f) : (R, M) \to (R', M')$ is a $\mathcal{D}$-homomorphism, then $\mathbf{D}(\varphi, f) = \varphi \bowtie f$.

We also consider the functor $\mathbf{B} : \mathfrak{Rng} \to \mathfrak{D}$, defined as follows: if A is a ring, then $\mathbf{B}(A) = (A, A)$ and if $h : A \to B$ is a ring homomorphism, $\mathbf{B}(h) = (h, h)$.

Theorem 2.2.1 [35] *The functor $\mathbf{D}$ is left adjoint of $\mathbf{B}$.*

Proposition 2.2.2 [35] *We consider $\{(R_i, M_i) : i \in I\}$ a family of Dorroh-pairs and the direct products $\prod_{i\in I} R_i$ and $\prod_{i\in I} M_i$ (with the canonical projections p_i and π_i, respectively, the canonical embeddings q_i and σ_i). Then $\left(\prod_{i\in I} R_i, \prod_{i\in I} M_i\right)$ is also a Dorroh-pair, for all $i \in I$, (p_i, π_i) and (q_i, σ_i) are $\mathcal{D}$-homomorphisms and*

$$\left(\prod_{i\in I} R_i\right) \bowtie \left(\prod_{i\in I} M_i\right) \cong \prod_{i\in I} (R_i \bowtie M_i).$$

Proposition 2.2.3 [35] *Let I a directed set and $\left\{(R_i, M_i)_{i\in I}\ ; (\varphi_{ij}, f_{ij})_{i,j\in I}\right\}$ an inverse system of Dorroh-pairs. Then $\left\{(R_i \bowtie M_i)_{i\in I}, (\varphi_{ij} \bowtie f_{ij})_{i,j\in I}\right\}$ is an inverse system of rings and*

$$\varprojlim (R_i \bowtie M_i) \cong \left(\varprojlim R_i\right) \bowtie \left(\varprojlim M_i\right).$$

2.3 The group of units of the ring $R \bowtie M$

Let (R, M) a Dorroh-pair, where R is a ring with identity and we consider the Dorroh extension $R \bowtie M$.

In this section we will describe the group of units of the ring $R \bowtie M$. Firstly, we observe that, if $(a, x) \in \mathbf{U}(R \bowtie M)$, then $a \in \mathbf{U}(R)$.

The set of all elements of M forms a monoid under the circle composition on M, $x \circ y = x + y + xy$, 0 being the neutral element. The group of units of this monoid we will denoted by M°.

Remark 2.3.1 *It is easy to see that the function* $\delta : \mathbf{U}(R) \to \operatorname{Aut} M^{\circ}$, $a \longmapsto \delta_a$ *where,*

$$\delta_a : M^{\circ} \to M^{\circ}, \qquad x \mapsto axa^{-1}.$$

is a group homomorphism.

Theorem 2.3.2 [35] *The group of units* $\mathbf{U}(R \bowtie M)$ *of the Dorroh extension* $R \bowtie M$ *is isomorphic with the semidirect product* $\mathbf{U}(R) \times_{\delta} M^{\circ}$ *of the groups* $\mathbf{U}(R)$ *and* M°.

Remark 2.3.3 *If* M *is a ring with identity, the correspondence* $x \mapsto x - 1$ *establishes an isomorphism between the groups* $\mathbf{U}(M)$ *and* M°, *and therefore the group* $\mathbf{U}(R \bowtie M)$ *is isomorphic with a semidirect product of the groups* $\mathbf{U}(R)$ *and* $\mathbf{U}(M)$.

Part II

Algebraic structures on the set of fuzzy numbers

Chapter 3

Fuzzy numbers. Generalities

3.1 The definition of a fuzzy number

Definition 3.1.1 [5] *A fuzzy number is a function* $A : \mathbb{R} \to [0,1]$ *which satisfies the following properties:*

1. *A is normal (i.e., there exists* $x_0 \in \mathbb{R}$*, such that* $A(x_0) = 1$*);*
2. *A is convex (i.e.,* $A(\lambda x + (1-\lambda) y) \geq \min\{A(x), A(y)\}$*, for all* $x, y \in \mathbb{R}$ *and* $\lambda \in [0,1]$*);*
3. *A is upper semicontinuous on* $\mathbb{R}$ *(i.e., for all* $x_0 \in \mathbb{R}$ *and for all* $\varepsilon > 0$ *there exists a neighborhood* V_0 *of* x_0 *such that* $A(x) - A(x_0) \leq \varepsilon$*, for all* $x \in V_0$*);*
4. *A has compact support (i.e., the closure of the set* $\{x \in \mathbb{R} : A(x) > 0\}$ *is a compact interval of* $\mathbb{R}$*).*

Denote the set of fuzzy numbers by $\mathfrak{F}$.
As usual, if $A : \mathbb{R} \to [0,1]$ is a fuzzy number, then

$$\operatorname{supp} A = \overline{\{x \in \mathbb{R} : A(x) > 0\}}$$

is called the support of A, respectively,

$$\operatorname{core} A = \{x \in \mathbb{R} : A(x) = 1\}$$

is called the core of A.

Remark 3.1.2 *By Definition* 3.1.1*,* $\operatorname{supp} A$ *and* $\operatorname{core} A$ *are compact intervals.*

Definition 3.1.3 [5] *In the case that* $\operatorname{core} A$ *is an one point set, we say that* A *is unimodal, respectively, if* $\operatorname{core} A$ *is a nontrivial compact interval, we say that the fuzzy number* A *is flat.*

Remark 3.1.4 [5] *By Definition 3.1.1, we conclude that the function* $A : \mathbb{R} \to [0,1]$ *is a fuzzy number if and only if there exists* $\alpha_1, a_1, a_2, \alpha_2 \in \mathbb{R}$, *with* $\alpha_1 \leq a_1 \leq a_2 \leq \alpha_2$ *such that:*

1. *the restriction* $A_1 = A|_{[\alpha_1, a_1]} : [\alpha_1, a_1] \to [0,1]$ (*called the left side of* A) *is upper semicontinuous and increasing function;*

2. *the restriction* $A_2 = A|_{[a_2, \alpha_2]} : [a_2, \alpha_2] \to [0,1]$ (*called the right side of* A) *is upper semicontinuous and decreasing function;*

3. $A(x) = 1$, *for all* $x \in [a_1, a_2]$;

4. $A(x) = 0$, *if* $x \notin [\alpha_1, \alpha_2]$.

With these notations, $\operatorname{supp} A = [\alpha_1, \alpha_2]$ *and* $\operatorname{core} A = [a_1, a_2]$.

3.2 Representations of fuzzy numbers

3.2.1 The LU representation of a fuzzy number

If $A : \mathbb{R} \to [0,1]$ is a fuzzy number, the $t-$level sets $[A]_t$ of A, defined by

$$[A]_t = \begin{cases} \overline{\{x \in \mathbb{R} : A(x) > 0\}}, & \text{if} \quad t = 0 \\ \{x \in \mathbb{R} : A(x) \geq t\}, & \text{if} \quad 0 < t \leq 1 \end{cases}$$

are compact intervals for each $t \in [0,1]$.

Remark 3.2.1 ([45],[5]) *If* $[A]_t = \left[x_A^-(t), x_A^+(t)\right]$, *for each* $t \in [0,1]$, *then the functions* $x_A^-, x_A^+ : [0,1] \to \mathbb{R}$ (*defining the endpoints of the* $t-$*level sets*) *satisfies the following properties:*

1. x_A^- *and* x_A^+ *are bounded;*

2. x_A^- *and* x_A^+ *are left-continuous in* $(0,1]$ *and continuous at* 0;

3. x_A^- *is increasing and* x_A^+ *is decreasing;*

4. $x_A^-(t) \leq x_A^+(t)$, *for all* $t \in [0,1]$.

Moreover, Goetschel and Woxmann proves that, a fuzzy number A is completely determined by a pair $x_A = (x_A^-, x_A^+)$ of functions $x_A^-, x_A^+ : [0,1] \to \mathbb{R}$ satisfying the above conditions.

This representation of a fuzzy number as a pair of functions that satisfy these conditions, is called the LU representation.

3.2.2 The CE-representation of a fuzzy number

If the fuzzy number $A : \mathbb{R} \to [0,1]$ has the $t-$level sets $[A]_t = [x_A^-(t), x_A^+(t)]$, the functions $\delta_A^-, \delta_A^+ : [0,1] \to \mathbb{R}_+$ (where $\mathbb{R}_+ = [0,+\infty)$), called the left, respectively, the right deviation of the fuzzy number A, defined by

$$\begin{cases} \delta_A^-(t) = a_1 - x_A^-(t) \\ \delta_A^+(t) = x_A^+(t) - a_2 \end{cases}, \quad \text{for each} \quad t \in [0,1]$$

are bounded, decreasing, left-continuous in $(0,1]$, continuous at 0 and $\delta_A^-(1) = \delta_A^+(1) = 0$.

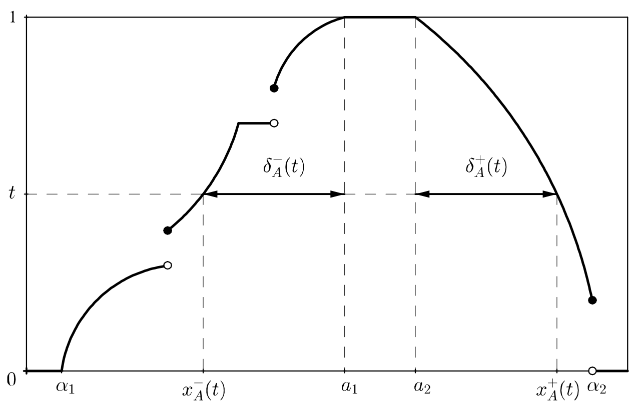
In consequence, a fuzzy number $A \in \mathfrak{F}$ can be also represented as a system

$$A = ((a_1, \delta_A^-), (a_2, \delta_A^+))$$

where

1. $a_1, a_2 \in \mathbb{R}$, with $a_1 \leq a_2$;
2. $\delta_A^-, \delta_A^+ : [0,1] \to [0,+\infty)$ are two bounded, decreasing, left-continuous in $(0;1]$ and continuous at 0 functions, with the property that $\delta_A^-(1) = \delta_A^+(1) = 0$.

Pointwise, we can represent the fuzzy number A, by $A = ((a_1, \delta_A^-(t)), (a_2, \delta_A^+(t)))_{t \in [0,1]}$



Remark 3.2.2 [36] *If Ω is the set of all functions f of $[0,1]$ in $\mathbb{R}_+$ which are bounded, decreasing, left-continuous in $(0,1]$ and continuous at 0, with the property that $f(1)=0$, then for every fuzzy number $A=\left(\left(a_1,\delta_A^-\right),\left(a_2,\delta_A^+\right)\right)\in\mathfrak{F}$, the pairs $\left(a_1,\delta_A^-\right)$ and $\left(a_2,\delta_A^+\right)$ are elements of the Cartesian product $\mathbb{R}\times\Omega$, and so, we can identify the set $\mathfrak{F}$ of all fuzzy numbers with a subset of $(\mathbb{R}\times\Omega)^2$ i.e.,*

$$\mathfrak{F}=\left\{A=\left(\left(a_1,\delta_A^-\right),\left(a_2,\delta_A^+\right)\right):\left(a_1,\delta_A^-\right),\left(a_2,\delta_A^+\right)\in\mathbb{R}\times\Omega,\ a_1\le a_2\right\}.$$

Definition 3.2.3 *This representation of a fuzzy number is called the CE-representation (core-ecart representation).*

Remark 3.2.4 *If $\theta\in\Omega$ is the null function, then every real number (crisp number) $a\in\mathbb{R}$ can be represented as $((a,\theta),(a,\theta))\in\mathfrak{F}$, respectively, every compact interval (crisp interval) $[a_1,a_2]\subset\mathbb{R}$ can be represented as $((a_1,\theta),(a_2,\theta))\in\mathfrak{F}$.*

Definition 3.2.5 *The fuzzy number $A=\left(\left(a_1,\delta_A^-\right),\left(a_2,\delta_A^+\right)\right)\in\mathfrak{F}$ is said to be with:*

1. *positive core* $(\operatorname{core} A\ge 0)$, *if* $a_1\ge 0$;
2. *negative core* $(\operatorname{core} A\le 0)$, *if* $a_2\le 0$;
3. *strictly positive core* $(\operatorname{core} A> 0)$, *if* $a_1> 0$;
4. *strictly negative core* $(\operatorname{core} A< 0)$, *if* $a_2< 0$.

Notations:

$$\begin{aligned}\mathfrak{F}_+&=\{A\in\mathfrak{F}:\operatorname{core} A\ge 0\}\\ \mathfrak{F}_-&=\{A\in\mathfrak{F}:\operatorname{core} A\le 0\}\\ \mathfrak{F}_+^*&=\{A\in\mathfrak{F}:\operatorname{core} A> 0\}\\ \mathfrak{F}_-^*&=\{A\in\mathfrak{F}:\operatorname{core} A< 0\}\end{aligned}$$

3.2.3 The MCE-representation of a fuzzy number

For a fuzzy number $A:\mathbb{R}\to[0,1]$ having the $t-$level sets $[A]_t=\left[x_A^-(t),x_A^+(t)\right]$, the following functions $\Theta_A^-,\Theta_A^+,\Delta_A:[0,1]\to\mathbb{R}_+$ (where $\mathbb{R}_+=[0,+\infty)$), defined by

$$\begin{cases}\Theta_A^-(t)=a-x_A^-(t)\\ \Theta_A^+(t)=x_A^+(t)-a\end{cases},\quad \text{for each}\quad t\in[0,1]$$

respectively,

$$\Delta_A=x_A^+-x_A^-=\Theta_A^-+\Theta_A^+$$

where

$$a = \frac{1}{2}\left(x_A^-(1) + x_A^+(1)\right)$$

is the middle point of the core A, are bounded, decreasing, left-continuous on $(0,1]$ and continuous in 0 and $\Theta_A^-(1) = \Theta_A^+(1)$.

Definition 3.2.6 *We call Θ_A^-, Θ_A^+ by the left and the right deviation, relatively to the middle point of the core of A, and Δ_A by the width of the fuzzy number A.*

In consequence, a fuzzy number $A \in \mathfrak{F}$ can be also represented as a system $A = \left(a; \Theta_A^-, \Theta_A^+\right)$ where $a \in \mathbb{R}$, and $\Theta_A^-, \Theta_A^+ : [0,1] \to [0,+\infty)$ are bounded, decreasing, left-continuous on $(0,1]$ and continuous in 0, functions with the property that $\Theta_A^-(1) = \Theta_A^+(1)$.

Definition 3.2.7 [33] *We call this representation the middle core ecart-representation of a fuzzy number (MCE-representation).*

Pointwise, we can represent a fuzzy number A, by $A = \left(a; \Theta_A^-(t), \Theta_A^+(t)\right)_{t\in[0,1]}$

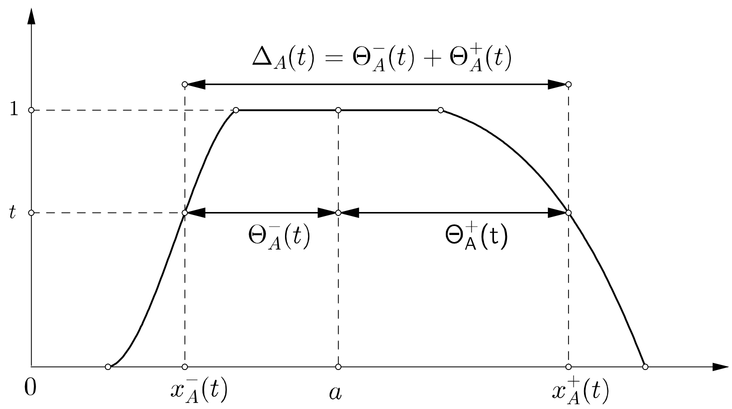


Figure 1.

We consider the sets

$$\mathfrak{F}_c = \left\{A \in \mathfrak{F} : x_A^-, x_A^+ \in C[0,1]\right\}$$

and

$$\Xi = \{(f_1, f_2) \in C_{DP}[0,1] \times C_{DP}[0,1] : f_1(1) = f_2(1)\}$$

where $C_{DP}[0,1]$ is the set of all positive valued, continuous and decreasing functions.

With these notations, we can identify the set $\mathfrak{F}_{\mathrm{c}}$ with the Cartesian product $\mathbb{R} \times \Xi$.

Obviously, $(\mathrm{C}_{\mathrm{DP}}[0,1], +, \cdot)$ and $(\Xi, +, \cdot)$ are commutative semi-rings with identity.

Definition 3.2.8 [33] *If $A = \left(a; \Theta_A^-, \Theta_A^+\right)$ and $B = \left(a; \Theta_B^-, \Theta_B^+\right)$ are two fuzzy numbers, we define the order " $\preccurlyeq$ " on $\mathfrak{F}$ by*

$$A \preccurlyeq B \Longleftrightarrow \begin{cases} a \leq b \\ \Theta_A^-(t) \leq \Theta_B^-(t), & \forall\ t \in [0,1] \\ \Theta_A^+(t) \leq \Theta_B^+(t), & \forall\ t \in [0,1] \end{cases}$$

It is obvious that, it is a partial order on the set $\mathfrak{F}$ of all fuzzy numbers.

3.2.4 The multivalued representation of a fuzzy number

Further, in order to simplify the presentation, we will introduce the following notations

$$\mathcal{P}_c[0,1] = \{[\alpha, \beta] : 0 \leq \alpha \leq \beta \leq 1\}$$
$$\mathcal{P}_c^*[0,1] = \{[\alpha, \beta] : 0 \leq \alpha < \beta \leq 1\}$$

for the set of all compact subintervals of $[0,1]$, respectively, for the set of all compact and nontrivial subintervals of $[0,1]$. More generally, we can consider the sets

$$\begin{aligned} \mathcal{P}_c(I) &= \{[\alpha, \beta] \subseteq I : \alpha \leq \beta\} \\ \mathcal{P}_c^*(I) &= \{[\alpha, \beta] \subseteq I : \alpha < \beta\} \end{aligned}$$

of all compact subintervals of a real interval I. We will identify the "interval" $[\alpha, \alpha] \in \mathcal{P}_c(I)$ with $\alpha \in I$.

We also consider , the functions

$$\begin{aligned} \mathrm{L} &: \mathcal{P}_c(\mathbb{R}) \to \mathbb{R}, \quad [\alpha, \beta] \mapsto \alpha \\ \mathrm{U} &: \mathcal{P}_c(\mathbb{R}) \to \mathbb{R}, \quad [\alpha, \beta] \mapsto \beta \end{aligned}$$

which gives the lower and upper endpoints of a compact interval.

Also, if $f : I \to \mathbb{R}$ is a function, where $I \subseteq \mathbb{R}$ is an interval, denote by $\mathrm{D}(f)$ the set of all discontinuity points of the function f.

Remark 3.2.9 *It is known that, if $f : I \to \mathbb{R}$ is a monotone function, then all the points of discontinuity of f are either removable or jump discontinuities and hence, of the first kind (see, [83]). Moreover, by Froda's theorem (see, [41]), the set $\mathrm{D}(f)$ of all discontinuities of the function f is at most countable. In the case that $\mathrm{D}(f)$ is a finite set, obviously, the elements of the set $\mathrm{D}(f)$ are isolated points of $\mathbb{R}$, but, when the set $\mathrm{D}(f)$ is an infinite set, the elements of $\mathrm{D}(f)$ are not necessarily isolated.*

Remark 3.2.10 *Let $f : [a, b] \to [0, 1]$ be an upper semicontinuous and monotone function. If the set $\mathrm{D}(f)$, consists just of isolated points and the set $\{x \in [a, b] : f(x) = 1\}$, has only one element, we can consider the multivalued function $\widehat{f} : [a, b] \to \mathcal{P}_c[0, 1]$, defined as follows:*

$$\widehat{f}(x) = \begin{cases} [f(x-0), f(x)], & \text{if } \ a < x \leq b \\ [0, f(a)], & \text{if } \ x = a \end{cases}$$

if f is increasing, respectively,

$$\widehat{f}(x) = \begin{cases} [f(x+0), f(x)], & \text{if } \ a \leq x < b \\ [0, f(b)], & \text{if } \ x = b \end{cases}$$

if f is decreasing ($f(x-0)$ and $f(x+0)$ denotes the left, respectively the right limit of the function f in x).

Proposition 3.2.11 *The multivalued function $\widehat{f}$ introduced in Remark 3.2.10 has the following properties:*

1. $\widehat{f}(x) = f(x)$, *for all* $x \in (a, b) \setminus \mathrm{D}(f)$;
2. $\left\{x \in [a, b] : \widehat{f}(x) \in \mathcal{P}_c^*[0, 1]\right\}$ *is a discrete set;*
3. $\Gamma\left(\widehat{f}\right) = \left\{(x, y) \in [a, b] \times [0, 1] : y \in \widehat{f}(x)\right\}$ *is a continuous plane curve;*
4. *if f is increasing, then $(a, 0), (b, 1) \in \Gamma\left(\widehat{f}\right)$, respectively, if f is decreasing, then $(a, 1), (b, 0) \in \Gamma\left(\widehat{f}\right)$;*
5. *if f is increasing (decreasing), then $\widehat{f}$ is increasing (decreasing), that is for all $x_1, x_2 \in [a, b]$ with $a \leq x_1 < x_2 \leq b$ we have that*

$$t_1 \leq t_2 \quad (t_2 \leq t_1), \text{ whenever } t_1 \in f(x_1) \quad \text{and} \quad t_2 \in f(x_2);$$

6. *the image of* $\widehat{f}$ *is* $[0,1]$, *i.e.,* $\operatorname{Im}\widehat{f} = \bigcup_{x\in[a,b]} \widehat{f}(x) = [0,1]$.

Remark 3.2.12 *We consider now, a fuzzy number* $A:\mathbb{R}\to[0,1]$, *represented as in Remark 3.1.4. If A has only isolated discontinuities, then* $\mathrm{D}(A_1)$ *and* $\mathrm{D}(A_2)$ *are discrete sets. Therefore, we can construct the multivalued functions*

$$\widehat{A}_1 : [\alpha_1, a_1] \to \mathcal{P}_c[0,1] \quad \text{and} \quad \widehat{A}_2 : [a_2, \alpha_2] \to \mathcal{P}_c[0,1]$$

by

$$\widehat{A}_1(x) = \begin{cases} [A_1(x-0), A_1(x)], & \text{if} \quad \alpha_1 < x \le a_1 \\ [0, A_1(\alpha_1)], & \text{if} \quad x = \alpha_1 \end{cases}$$

and

$$\widehat{A}_2(x) = \begin{cases} [A_2(x+0), A_2(x)], & \text{if} \quad a_2 \le x < \alpha_2 \\ [0, A_2(\alpha_2)], & \text{if} \quad x = \alpha_2 \end{cases}.$$

Proposition 3.2.13 *If* $A:\mathbb{R}\to[0,1]$ *is a fuzzy number, then the multivalued functions* $\widehat{A}_1 : [\alpha_1, a_1] \to \mathcal{P}_c[0,1]$ *and* $\widehat{A}_2 : [a_2, \alpha_2] \to \mathcal{P}_c[0,1]$ *constructed in Remark 3.2.12 have the following properties:*

1. $\left\{x \in [\alpha_1, a_1] : \widehat{A}_1(x) \in \mathcal{P}_c^*[0,1]\right\}$ *and* $\left\{x \in [a_2, \alpha_2] : \widehat{A}_2(x) \in \mathcal{P}_c^*[0,1]\right\}$ *are discrete sets;*
2. $(\alpha_1, 0), (a_1, 1) \in \Gamma\left(\widehat{A}_1\right)$ *and* $(a_2, 1), (\alpha_2, 0) \in \Gamma\left(\widehat{A}_2\right)$;
3. $\operatorname{Im}\widehat{A}_1 = \operatorname{Im}\widehat{A}_2 = [0,1]$;
4. $\widehat{A}_1$ *is increasing and* $\widehat{A}_2$ *is decreasing;*
5. $\Gamma\left(\widehat{A}_1\right)$ *and* $\Gamma\left(\widehat{A}_2\right)$ *are continuous plane curve.*

Conversely, if $\alpha_1 \le a_1 \le a_2 \le \alpha_2$ *and*

$$\widehat{A}_1 : [\alpha_1, a_1] \to \mathcal{P}_c[0,1] \quad \text{and} \quad \widehat{A}_2 : [a_2, \alpha_2] \to \mathcal{P}_c[0,1]$$

are two multivalued functions, which satisfy the above properties $(1)-(5)$, *then the functions*

$$A_1 : [\alpha_1, a_1] \to [0,1] \quad \text{and} \quad A_2 : [a_2, \alpha_2] \to [0,1],$$

defined by

$$A_i(x) = \mathrm{U}\left(\widehat{A}_i(x)\right), \quad i \in \{1,2\},$$

can be considered as a left and right parts of a fuzzy number A.

Therefore, if A is a fuzzy number with discrete set of discontinuities, A is uniquely determined by a pair $\left(\widehat{A}_1, \widehat{A}_2\right)$ of multivalued functions (constructed as in Remark 3.2.12).

Definition 3.2.14 *The above representation of a fuzzy number A, as a pair of multivalued functions, is called the multivalued representation of this fuzzy number.*

Remark 3.2.15 *If $a_1 \neq a_2$ or if $a_1 = a_2 = a$ and $A_1(a-0) = A_2(a+0)$, then the multivalued function*

$$\widehat{A} : \mathbb{R} \to \mathcal{P}_c[0,1], \qquad \widehat{A}(x) = \begin{cases} \widehat{A}_1(x), & if\ x \in [\alpha_1, a_1] \\ 1, & if\ x \in (a_1, a_2) \\ \widehat{A}_2(x), & if\ x \in [a_2, \alpha_2] \\ 0, & otherwise \end{cases}$$

has the following properties:

1. $\left\{x \in \mathbb{R} : \widehat{A}(x) \in \mathcal{P}_c^*[0,1]\right\}$ *is a discrete set;*
2. *there exists $x_0 \in \mathbb{R}$, such that $1 \in \widehat{A}(x_0)$;*
3. $\mathrm{U}\left(\widehat{A}(\lambda x + (1-\lambda) y)\right) \geq \min\left\{\mathrm{U}\left(\widehat{A}(x)\right), \mathrm{U}\left(\widehat{A}(y)\right)\right\}$, *for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$;*
4. *for all $x_0 \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists a neighborhood V_0 of x_0, such that $\mathrm{U}\left(\widehat{A}(x)\right) - \mathrm{U}\left(\widehat{A}(x_0)\right) \leq \varepsilon$, for all $x \in V_0$;*
5. *the closure of the set $\left\{x \in \mathbb{R} : 0 \notin \widehat{A}(x)\right\}$, is a compact interval of $\mathbb{R}$;*

Conversely, if $\widehat{A} : \mathbb{R} \to \mathcal{P}_c[0,1]$ is a multivalued function, which satisfy the above properties $(1)-(5)$, then the function $A : \mathbb{R} \to [0,1]$, defined by

$$A(x) = \begin{cases} \widehat{A}(x), & if\ \ \widehat{A}(x) \in [0,1] \\ \mathrm{U}\left(\widehat{A}(x)\right), & if\ \ \widehat{A}(x) \in \mathcal{P}_c^*[0,1] \end{cases}$$

is a fuzzy number and $\mathrm{D}(A) = \left\{x \in \mathbb{R} : \widehat{A}(x) \in \mathcal{P}_c^*[0,1]\right\}$. ‘

Example 3.2.16 *In Figure* 2*, in the left side is given a fuzzy number A and in the right side is its multivalued representation.*

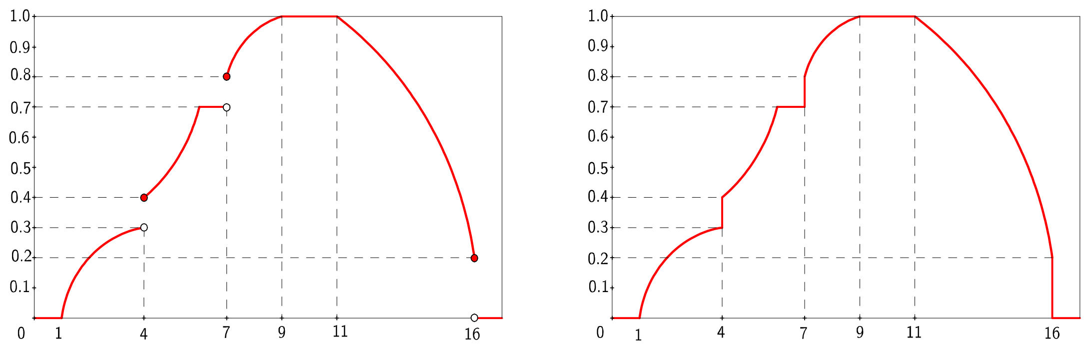


Figura 2*.*

We observe that $\mathrm{D}(A) = \{4, 7, 16\}$ *and*

$$\widehat{A}_1(4) = [0.3, 0.4], \quad \widehat{A}_1(7) = [0.7, 0.8], \quad \widehat{A}_2(16) = [0, 0.2]$$

Chapter 4

Dorroh-type products on the set of fuzzy numbers

4.1 Algebraic preliminaries

Definition 4.1.1 *A (commutative) semiring is an algebraic structure* $(S, +, \cdot, 0)$ *such that:*

1. $(S, +, 0)$ *is a commutative monoid;*
2. $(S, \cdot)$ *is a (commutative) semigroup;*
3. *the distributivity law is fulfilled;*
4. $0 \cdot a = 0 = a \cdot 0$, *for all* $a \in S$.

If $(S, \cdot, 1)$ *is a monoid, the semiring is said to be with identity.*

Definition 4.1.2 *Let* S *be a commutative semiring with identity. A (left)* S*-semimodule is a commutative monoid* $(M, +, 0)$ *with an external operation with coefficients in* S, $(a, x) \mapsto a \cdot x$, *called scalar multiplication, such that the following conditions hold for all* $a, b \in S$ *and* $x, y \in M$:

1. $(ab) \cdot x = a \cdot (b \cdot x)$;
2. $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
3. $(a + b) \cdot x = (a \cdot x) + (b \cdot x)$;

4. $0_S \cdot x = 0_M = a \cdot 0_M$;

5. $1 \cdot x = x$;

Remark 4.1.3 *If in the definition of the trivial extension and the Dorroh - extension, we replace all rings with semirings and the module structure, with a semimodule structure, then we obtain that* $R \ltimes M$ *and* $R \bowtie M$ *are commutative semirings.*

4.2 The Dorroh-product

Recall that Ω, denote the set of all functions $f : [0,1] \to \mathbb{R}_+$, which are bounded, decreasing, left-continuous in $(0,1]$ and continuous at 0, with the property that $f(1) = 0$ and we consider the subset Ω_0 of Ω, which contain all the continuous functions of Ω.

Obviously, the set $\mathbb{R}_+$ of the positive real numbers together with the usual addition and multiplication is a commutative semiring with identity and Ω is a $\mathbb{R}_+$-semimodule together with the pointwise addition $(f, g) \longmapsto f + g$ and the pointwise scalar multiplication $(a, f) \longmapsto a \cdot f$. We consider now, a semiring structure $(\Omega, +, *)$, such that

$$(a \cdot f) * g = a \cdot (f * g), \quad \text{for all } a \in \mathbb{R}_+ \text{ and } f, g \in \Omega$$

and the Dorroh extension $(\mathbb{R}_+ \bowtie \Omega, +, \bullet)$.

If $A = \left(\left(a_1, \delta_A^-\right), \left(a_2, \delta_A^+\right)\right) \in \mathfrak{F}$ and $B = \left(\left(b_1, \delta_B^-\right), \left(b_2, \delta_B^+\right)\right) \in \mathfrak{F}$ are two fuzzy numbers, we define their sum by

$$A + B = \left(\left(a_1 + b_1, \delta_A^- + \delta_B^-\right), \left(a_2 + b_2, \delta_A^+ + \delta_B^+\right)\right) \in \mathfrak{F}$$

respectively, if $A, B \in \mathfrak{F}_+$, we define their product by

$$A \circledast B = \left(\left(a_1, \delta_A^-\right) \bullet \left(b_1, \delta_B^-\right), \left(a_2, \delta_A^+\right) \bullet \left(b_2, \delta_B^+\right) \right) = \left(\left(a_1 b_1, \delta_{A \circledast B}^-\right), \left(a_2 b_2, \delta_{A \circledast B}^+\right) \right),$$

where,

$$\begin{cases} \delta_{A \circledast B}^- = a_1 \delta_B^- + b_1 \delta_A^- + \delta_A^- * \delta_B^- \\ \delta_{A \circledast B}^+ = a_2 \delta_B^+ + b_2 \delta_A^+ + \delta_A^+ * \delta_B^+ \end{cases}.$$

It is obvious that, if $A, B \in \mathfrak{F}_+$, then $A + B, A \circledast B \in \mathfrak{F}_+$.

Theorem 4.2.1 [36] $(\mathfrak{F}_+, +, \circledast)$ *is a commutative semiring with identity.*

If we consider that the product of the semiring $(\Omega, +, *)$ is the usual pointwise product of Ω, i.e.

$$(f * g)(t) = (f \cdot g)(t) = f(t) \cdot g(t), \text{ for all } t \in [0,1]$$

for all $f, g \in \Omega$, we denote the above defined multiplication of $\mathfrak{F}_+$ by "$\odot$". Therefore, in this case, $A \odot B = \left(\left(a_1 b_1, \delta^-_{A\odot B}\right), \left(a_2 b_2, \delta^+_{A\odot B}\right)\right)$, where

$$\begin{cases} \delta^-_{A\odot B}(t) = a_1 \delta^-_B(t) + b_1 \delta^-_A(t) + \delta^-_A(t) \cdot \delta^-_B(t) \\ \delta^+_{A\odot B}(t) = a_2 \delta^+_B(t) + b_2 \delta^+_A(t) + \delta^+_A(t) \cdot \delta^+_B(t) \end{cases}, \text{ for all } t \in [0,1].$$

Definition 4.2.2 [36] *The multiplication "$\odot$" of $\mathfrak{F}_+$ (defined above), is called the Dorroh-product.*

Remark 4.2.3 *In [3], A.I. Ban and B. Bede have introduced and studied the main properties of the cross product of fuzzy numbers. If $A = \left[x^-_A(t), x^+_A(t)\right]_{t\in[0,1]}$ and $B = \left[x^-_B(t), x^+_B(t)\right]_{t\in[0,1]}$ are two fuzzy numbers with positive core, the cross product is defined by*

$$A \circ B = \left[x^-_{A\circ B}(t), x^+_{A\circ B}(t)\right]_{t\in[0,1]}$$

where,

$$\begin{cases} x^-_{A\circ B}(t) = x^-_A(t) \cdot x^-_B(1) + x^-_A(1) \cdot x^-_B(t) - x^-_A(1) \cdot x^-_B(1) \\ x^+_{A\circ B}(t) = x^+_A(t) \cdot x^+_B(1) + x^+_A(1) \cdot x^+_B(t) - x^+_A(1) \cdot x^+_B(1) \end{cases}$$

for each $t \in [0,1]$.

If we consider now, that $A = \left(\left(a_1, \delta^-_A\right), \left(a_2, \delta^+_A\right)\right)$ and $B = \left(\left(b_1, \delta^-_B\right), \left(b_2, \delta^+_B\right)\right)$, then

$$A \circ B = \left(\left(a_1 \cdot b_1, \delta^-_{A\circ B}\right), \left(a_2 \cdot b_2, \delta^+_{A\circ B}\right)\right)$$

where,

$$\begin{cases} \delta^-_{A\circ B}(t) = a_1 \cdot b_1 - x^-_{A\circ B}(t) = \delta^-_A(t) \cdot b_1 + a_1 \cdot \delta^-_B(t) \\ \delta^+_{A\circ B}(t) = x^+_{A\circ B}(t) - a_2 \cdot b_2 = \delta^+_A(t) \cdot b_2 + a_2 \cdot \delta^+_B(t) \end{cases}$$

and so, the cross product defined on the set of fuzzy numbers is a particular case of the product introduced above on $\mathfrak{F}_+$, which is obtained for the null product of Ω.

If $A = \left(\left(a_1, \delta^-_A\right), \left(a_2, \delta^+_A\right)\right) \in \mathfrak{F}$, define its opposite $-A$, by $-A = \left(\left(-a_2, \delta^+_A\right), \left(-a_1, \delta^-_A\right)\right)$.

Proposition 4.2.4 [36] *The Dorroh-product defined on* $\mathfrak{F}_+$ *can be extended to* $\mathfrak{F}_+ \cup \mathfrak{F}_-$*, as follows:*

$$A \odot B = \begin{cases} -\left((-A) \odot B\right), & \text{if } A \in \mathfrak{F}_- \text{ and } B \in \mathfrak{F}_+ \\ -\left(A \odot (-B)\right), & \text{if } A \in \mathfrak{F}_+ \text{ and } B \in \mathfrak{F}_- \\ (-A) \odot (-B), & \text{if } A \in \mathfrak{F}_- \text{ and } B \in \mathfrak{F}_- \end{cases}$$

and this has the following properties:

1. $A \odot B = B \odot A$, *for all* $A, B \in \mathfrak{F}_+ \cup \mathfrak{F}_-$;
2. $(A \odot B) \odot C = A \odot (B \odot C)$, *for all* $A, B, C \in \mathfrak{F}_+ \cup \mathfrak{F}_-$;
3. $A \odot (B + C) = A \odot B + A \odot C$, *if* $(B, C \in \mathfrak{F}_+)$ *or* $(B, C \in \mathfrak{F}_-)$ *or* $(A \in \mathbb{R})$;

Example 4.2.5 [36] *If* $A, B \in \mathfrak{F}_+$ *where*

$$\begin{aligned} A &= [t+2, 5-t]_{t\in[0,1]} = ((3, 1-t), (4, 1-t))_{t\in[0,1]} \\ B &= [2t+3, 7-t]_{t\in[0,1]} = ((5, 2(1-t)), (6, 1-t))_{t\in[0,1]} \end{aligned}$$

then their products are:

1. *the usual product:*

$$A \cdot B = [(t+2)(2t+3), (5-t)(7-t)] = \left(\left(15, -2t^2 - 7t + 9\right), \left(24, t^2 - 12t + 11\right)\right)$$

2. *the cross product:*

$$A \circ B = [11t + 4, 34 - 10t] = ((15, 11(1-t)), (24, 10(1-t)))$$

3. *the Dorroh-product:*

$$A \odot B = \left[-2t^2 + 15t + 2, t^2 - 12t + 35\right] = \left(\left(15, 2t^2 - 15t + 13\right), \left(24, t^2 - 12t + 11\right)\right)$$

These are represented in Figure 4.

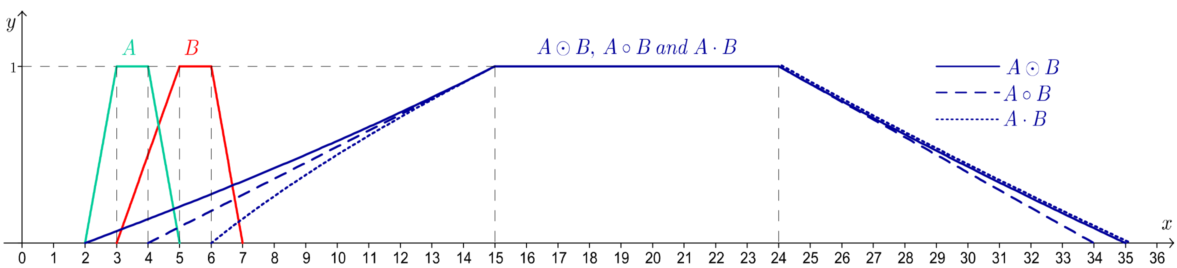


Figure 4.

4.3 A congruence relation on the set of fuzzy numbers

If $A \in \mathfrak{F}$ is a fuzzy number, then its left and right parts are strictly monotone (i.e., A_1 is strictly increasing and A_2 is strictly decreasing) if and only if the functions δ_A^- and δ_A^+ are continuous.

We consider now $\mathfrak{F}_0$, the set of all fuzzy numbers with discrete set of discontinuities and with strictly monotone left and right parts. If Ω_0 is the set of continuous and decreasing functions f of $[0,1]$ in $\mathbb{R}_+$ with the property that $f(1) = 0$, then

$$\mathfrak{F}_0 = \left\{ A = \left(\left(a_1, \delta_A^-\right), \left(a_2, \delta_A^+\right) \right) \in \mathfrak{F} : \delta_A^-, \delta_A^+ \in \Omega_0 \right\}.$$

Obviously, $(\mathfrak{F}_0, +)$ is a submonoid of the monoid $(\mathfrak{F}, +)$ and $(\mathfrak{F}_0 \cap \mathfrak{F}_+, +, \odot)$ is a subsemiring of the semiring $(\mathfrak{F}_+, +, \odot)$.

Remark 4.3.1 *If $f \in \Omega_0$ and $f^{-1}(x)$ is the inverse image of an element $x \in \mathbb{R}_+$ under the function f (i.e., $f^{-1}(x) = \{t \in [0,1] : f(t) = x\}$), then $f^{-1}(x)$ is either a set consisting of a single element, or is the empty set, or it is in $\mathcal{P}_c^*[0,1]$. Moreover, for each $x, x' \in \mathbb{R}_+$ with $x \neq x'$, we have that $f^{-1}(x) \cap f^{-1}(x') = \emptyset$.*

If $f \in \Omega_0$, we define

$$V(f) = \bigcup_{x \geq 0} \left\{ f^{-1}(x) : f^{-1}(x) \in \mathcal{P}_c^*[0,1] \right\},$$

respectively, if $A = \left(\left(a_1, \delta_A^-\right), \left(a_2, \delta_A^+\right) \right) \in \mathfrak{F}_0$, we define $V_1(A) = V\left(\delta_A^-\right)$ and $V_2(A) = V\left(\delta_A^+\right)$. Equivalently, if $\left(\widehat{A}_1, \widehat{A}_2\right)$ is the multivalued representation of the fuzzy number $A \in \mathfrak{F}$, where $\widehat{A}_1 : [\alpha_1, a_1] \to \mathcal{P}_c[0,1]$ and $\widehat{A}_2 : [a_2, \alpha_2] \to \mathcal{P}_c[0,1]$, then

$$V_1(A) = \bigcup_{\alpha_1 \leq x \leq a_1} \left\{ \widehat{A}_1(x) : \widehat{A}_1(x) \in \mathcal{P}_c^*[0,1] \right\}$$
$$V_2(A) = \bigcup_{a_2 \leq x \leq \alpha_2} \left\{ \widehat{A}_2(x) : \widehat{A}_2(x) \in \mathcal{P}_c^*[0,1] \right\}.$$

and so, $V_1(A)$ and $V_2(A)$ are the left, respectively, the right vertical parts of the multivalued representation of A. This vertical parts appears only in the discontinuity points of A, and so, the fuzzy number A is continuous if and only if $V_1(A) = V_2(A) = \emptyset$.

Proposition 4.3.2 [36] *If* $A = \left(\left(a_1, \delta_A^-\right), \left(a_2, \delta_A^+\right)\right) \in \mathfrak{F}_0$ *and* $B = \left(\left(b_1, \delta_B^-\right), \left(b_2, \delta_B^+\right)\right) \in \mathfrak{F}_0$ *are two fuzzy numbers, then the relation*

$$A \sim B \Longleftrightarrow V_1(A) = V_1(B) \quad \text{and} \quad V_2(A) = V_2(B),$$

is an equivalence relation on $\mathfrak{F}_0$.

Remark 4.3.3 *An equivalence class* $[A]_\sim$ *relatively to the relation "* $\sim$ *" consists of all fuzzy numbers with the same (left and right) "vertical" parts.*

Lemma 4.3.4 [36] *Let* $f_1, f_2 \in \Omega_0$ *and let* $x \in \mathbb{R}_+$. *Then* $(f_1 + f_2)^{-1}(x) \in \mathcal{P}_c^*[0,1]$, *if and only if there exists* $x_1, x_2 \in \mathbb{R}_+$, *uniquely determined, such that* $x = x_1 + x_2$ *and* $f_1^{-1}(x_1) \cap f_2^{-1}(x_2) \in \mathcal{P}_c^*[0,1]$.
Moreover, in this case, we have that

$$(f_1 + f_2)^{-1}(x) = f_1^{-1}(x_1) \cap f_2^{-1}(x_2).$$

Lemma 4.3.5 [36] *Let* $f_1, f_2 \in \Omega_0$ *and let* $y \in \mathbb{R}_+$. *Then* $(f_1 \cdot f_2)^{-1}(y) \in \mathcal{P}_c^*[0,1]$, *if and only if there exists* $y_1, y_2 \in \mathbb{R}_+$, *uniquely determined, such that* $y = y_1 \cdot y_2$ *and* $f_1^{-1}(y_1) \cap f_2^{-1}(y_2) \in \mathcal{P}_c^*[0,1]$.
Moreover, in this case, we have that

$$(f_1 \cdot f_2)^{-1}(y) = f_1^{-1}(y_1) \cap f_2^{-1}(y_2).$$

Let $C = \cup\{C_i : i \in I\}$ and $C' = \cup\left\{C'_j : j \in J\right\}$ where $\{C_i : i \in I\}$ and $\left\{C'_j : j \in J\right\}$ are two families of pairwise disjoint elements of $\mathcal{P}_c^*[0,1]$ (i.e., $C_{i_1} \cap C_{i_2} = \emptyset$ and $C'_{j_1} \cap C'_{j_2} = \emptyset$, whenever $i_1 \neq i_2$ and $j_1 \neq j_2$). We define $C \sqcap C'$ by

$$C \sqcap C' = \cup\left\{C_i \cap C'_j : i \in I,\ j \in J,\ \ C_i \cap C'_j \in \mathcal{P}_c^*[0,1]\right\}.$$

Proposition 4.3.6 [36] *If* $A, B \in \mathfrak{F}_0$, *then*

$$V_i(A + B) = V_i(A) \sqcap V_i(B), \quad \text{for each } i \in \{1, 2\}$$

and if A *and* B *are with strictly positive core, then*

$$V_i(A \odot B) = V_i(A) \sqcap V_i(B), \quad \text{for each } i \in \{1, 2\}.$$

Theorem 4.3.7 [36] *The relation "* $\sim$ *" is a congruence of the monoid* $(\mathfrak{F}_0, +)$ *and the restriction of the relation "* $\sim$ *" to* $\mathfrak{F}_0 \cap \mathfrak{F}_+^*$ *is a congruence of the monoid* $\left(\mathfrak{F}_0 \cap \mathfrak{F}_+^*, \odot\right)$.

Remark 4.3.8 *The subset* $\mathfrak{F}_c$ *of the set* $\mathfrak{F}_0$ *which contain the continuous fuzzy numbers, respectively, the set* I *of crisp numbers together with crisp intervals, are two important equivalence classes of the factor set* $\mathfrak{F}_0/_\sim$. *Their importance lies in:*

1. I *is the neutral element of the factor monoids* $(\mathfrak{F}_0/_\sim, +)$ *and* $\left(\left(\mathfrak{F}_0 \cap \mathfrak{F}_+^*\right)/_\sim, \odot\right)$;
2. $\mathfrak{F}_c$ *is an ideal of the monoid* $(\mathfrak{F}_0, +)$, *respectively,* $\mathfrak{F}_c \cap \mathfrak{F}_+^*$ *is an ideal of the monoid* $\left(\mathfrak{F}_0 \cap \mathfrak{F}_+^*, \odot\right)$, *that is, if* $A \in \mathfrak{F}_c$ *and* $B \in \mathfrak{F}_0$, *then* $A + B \in \mathfrak{F}_c$, *respectively, if* A *and* B *are with strictly positive core, then* $A \odot B \in \mathfrak{F}_c$.

Proposition 4.3.9 [36] *If* $A \in \mathfrak{F}_c$ *and* $B \in \mathfrak{F}_0$, *then* $A + B \in \mathfrak{F}_c$, *respectively, if* A *and* B *are with strictly positive core, then* $A \odot B \in \mathfrak{F}_c$.

Chapter 5

Completely distributive products on the set of fuzzy numbers

In this chapter, consider the set

$$\mathfrak{F}_{\mathrm{c}} = \left\{ A \in \mathfrak{F} : x_A^-, x_A^+ \in \mathrm{C}\,[0,1] \right\}.$$

By using the MCE-representation, the set $\mathfrak{F}_{\mathrm{c}}$ is identified with the Cartesian product $\mathbb{R} \times \Xi$, where

$$\Xi = \left\{ (f_1, f_2) \in \mathrm{C}_{\mathrm{DP}}\,[0,1] \times \mathrm{C}_{\mathrm{DP}}\,[0,1] : f_1\,(1) = f_2\,(1) \right\}.$$

5.1 Semiring structures on the set $\mathfrak{F}_{\mathrm{c}}$

We consider now, two fuzzy numbers $A = \left(a; \Theta_A^-, \Theta_A^+\right)$ and $B = \left(b; \Theta_B^-, \Theta_B^+\right)$ and we define the following operations:

$$\begin{aligned} A + B &= \left(a + b;\ \Theta_A^- + \Theta_B^-,\ \Theta_A^+ + \Theta_B^+\right) \\ A \boxdot B &= \left(a \cdot b;\ \Theta_A^- \cdot \Theta_B^-,\ \Theta_A^+ \cdot \Theta_B^+\right) \\ A \boxtimes B &= \left(a \cdot b;\ \Theta_A^- \cdot \Theta_B^- + \Theta_A^+ \cdot \Theta_B^+,\ \Theta_A^- \cdot \Theta_B^+ + \Theta_A^+ \cdot \Theta_B^-\right) \end{aligned}$$

Since $\Theta_A^-, \Theta_A^+, \Theta_B^-$ and Θ_B^+ are positive valued decreasing functions, then so are the functions $\Theta_{A\boxdot B}^-, \Theta_{A\boxdot B}^+, \Theta_{A\boxtimes B}^-$ and $\Theta_{A\boxtimes B}^+$. Also, since $\Theta_A^-(1) = \Theta_A^+(1)$ and $\Theta_B^-(1) = \Theta_B^+(1)$, then $\Theta_{A\boxdot B}^-(1) = \Theta_{A\boxdot B}^+(1)$ and $\Theta_{A\boxtimes B}^-(1) = \Theta_{A\boxtimes B}^+(1)$. Therefore the above introduced products are well defined.

Theorem 5.1.1 [33] $(\mathfrak{F}_{\mathrm{c}}, +, \boxdot)$ *is a commutative semiring with identity and* $(\mathfrak{F}_{\mathrm{c}}, +, \boxtimes)$ *is a commutative semiring.*

Remark 5.1.2 [33] *If* $A, B \in \mathfrak{F}_c$, *then* $\Delta_{A+B} = \Delta_A + \Delta_B$ *and* $\Delta_{A \boxtimes B} = \Delta_A \cdot \Delta_B$.

Remark 5.1.3 *If* $a \in \mathbb{R}$, *the crisp number* $\widetilde{a}$ *has the ecart-representation* $(a; \theta, \theta)$. *Since* $\widetilde{a} + \widetilde{b} = \widetilde{a+b}$ *and* $\widetilde{a} \boxdot \widetilde{b} = \widetilde{a} \boxtimes \widetilde{b} = \widetilde{a+b}$, *for each* $a, b \in \mathbb{R}$, *we conclude that the field of real numbers is embedded in both semirings* $(\mathfrak{F}_c, +, \boxdot)$ *and* $(\mathfrak{F}_c, +, \boxtimes)$ *as a subsemiring, but the unit of* $\mathbb{R}$ *differs from the unit of the semiring* $(\mathfrak{F}_c, +, \boxdot)$.

Also, the group of units of the semiring $(\mathfrak{F}_c, +, \boxdot)$ *consist of the non-trivial intervals of the form* $[a - x, a + x] = (a; x, x)$, *with* $a \in \mathbb{R} - \{0\}$ *and* $x > 0$. *Obviously, the inverse of* $(a; x, x)$ *is* $\left(\frac{1}{a}; \frac{1}{x}, \frac{1}{x}\right) = \left[\frac{1}{a} - \frac{1}{x}, \frac{1}{a} + \frac{1}{x}\right]$.

Example 5.1.4 *If*

$$\begin{aligned} A &= [t+2,\ 7-2t] = (4;\ 2-t,\ 3-2t) \\ B &= [3t+3,\ 9-t] = (7;\ 4-3t,\ 2-t) \end{aligned}$$

are two fuzzy numbers, then

$$\begin{aligned} A \cdot B &= [(t+2)(3t+3), (7-2t)(9-t)] = \left(29;\ -3t^2 - 9t + 23, 2t^2 - 25t + 34\right) \\ A \boxdot B &= (28;\ (2-t)(4-3t), (3-2t)(2-t)) = \left[-3t^2 + 10t + 20, 2t^2 - 7t + 34\right] \\ A \boxtimes B &= \left(28;\ (2-t)(4-3t) + (3-2t)(2-t), (2-t)^2 + (3-2t)(4-3t)\right) \\ &= \left[-5t^2 + 17t + 14, 7t^2 - 21t + 44\right] \end{aligned}$$

where $A \cdot B$ *is the usual product (based on the Zadeh's extension principle, defined by* $A \cdot B = \left[x_A^- \cdot x_B^-,\ x_A^+ \cdot x_B^+\right]$) *and* $A \boxdot B$ *and* $A \boxtimes B$ *are the two above introduced products. These are represented in Figure 7.*

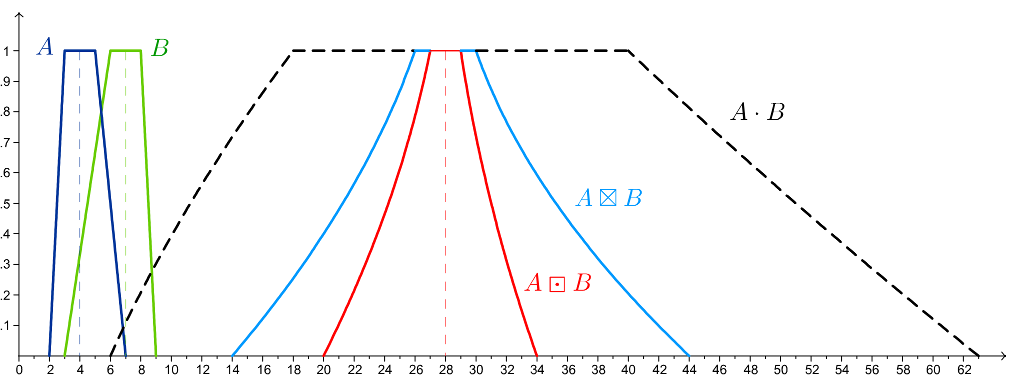


Figure 7.

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Remark 5.1.5 *If* $A = \left[x_A^-, x_A^+\right]$ *and* $B = \left[x_B^-, x_B^+\right]$, *then*

1. $\begin{cases} x_{A+B}^- = x_A^- + x_B^- \\ x_{A+B}^+ = x_A^+ + x_B^+ \end{cases}$;

2. $\begin{cases} x_{A\boxdot B}^- = ab - \Theta_{A\boxdot B}^- = a \cdot x_B^- + b \cdot x_A^- - x_A^- \cdot x_B^- \\ x_{A\boxdot B}^+ = ab + \Theta_{A\boxdot B}^+ = 2ab - a \cdot x_B^+ - b \cdot x_A^+ + x_A^+ \cdot x_B^+ \end{cases}$;

3. $\begin{cases} x_{A\boxtimes B}^- = ab - \Theta_{A\boxtimes B}^- = a\left(x_B^- + x_B^+\right) + b\left(x_A^- + x_A^+\right) - x_A^- \cdot x_B^- - x_A^+ \cdot x_B^+ - ab \\ x_{A\boxtimes B}^+ = ab + \Theta_{A\boxtimes B}^+ = a\left(x_B^- + x_B^+\right) + b\left(x_A^- + x_A^+\right) - x_A^- \cdot x_B^+ - x_A^+ \cdot x_B^- - ab \end{cases}$.

Definition 5.1.6 [33] *If* $\lambda \in \mathbb{R}$ *and* $A \in \mathfrak{F}_{\mathrm{c}}$, *we define the scalar multiplication by*

$$\lambda A = \left(\lambda \cdot a; |\lambda| \cdot \Theta_A^-, |\lambda| \cdot \Theta_A^+\right)$$

Remark 5.1.7 *Since*

$$\begin{cases} x_{\lambda A}^- = \lambda a - |\lambda| \cdot \Theta_A^- = (\lambda - |\lambda|)\, a + |\lambda| \cdot x_A^- \\ x_{\lambda A}^+ = \lambda a + |\lambda| \cdot \Theta_A^+ = (\lambda - |\lambda|)\, a + |\lambda| \cdot x_A^+ \end{cases}$$

we infer that in the case that $\lambda \geq 0$, *the above scalar multiplication coincides with the classic scalar multiplication, i.e.*

$$\lambda \cdot \left[x_A^-, x_A^+\right] = \left[\lambda \cdot x_A^-, \lambda \cdot x_A^+\right].$$

Proposition 5.1.8 [33] *The scalar multiplication has the following properties:*

1. $\lambda (A + B) = \lambda A + \lambda B$, *for all* $\lambda \in \mathbb{R}$ *and* $A, B \in \mathfrak{F}_{\mathrm{c}}$;
2. $\lambda (A \boxdot B) = (\lambda A) \boxdot B = A \boxdot (\lambda B)$, *for all* $\lambda \in \mathbb{R}$ *and* $A, B \in \mathfrak{F}_{\mathrm{c}}$;
3. $\lambda (A \boxtimes B) = (\lambda A) \boxtimes B = A \boxtimes (\lambda B)$, *for all* $\lambda \in \mathbb{R}$ *and* $A, B \in \mathfrak{F}_{\mathrm{c}}$;
4. $1 \cdot A = A;\ 0 \cdot A = \overline{0}$;
5. $(\alpha + \beta) A \preccurlyeq \alpha A + \beta A$, *for all* $\alpha, \beta \in \mathbb{R}$ *and* $A \in \mathfrak{F}_{\mathrm{c}}$;
6. $(\alpha + \beta) A = \alpha A + \beta A \Leftrightarrow \alpha \cdot \beta \geq 0$.

5.2 The topological structure of the set $\mathfrak{F}_{\mathrm{c}}$

For each fuzzy number $A = \left(a; \Theta_A^-, \Theta_A^+\right) \in \mathfrak{F}_{\mathrm{c}}$ define $\langle A \rangle$ by

$$\langle A \rangle = \sup_{t\in[0,1]} \max\left(\Theta_A^-(t), \Theta_A^+(t)\right)$$

Since Θ_A^- and Θ_A^+ are positive valued and decreasing functions, it follows that $\langle A \rangle = \max\left(\Theta_A^-(0), \Theta_A^+(0)\right) \geq 0$.

We also define, for each $n \in \{1,2,3,4\}$, the functions $\|\cdot\|_n : \mathfrak{F}_c \to [0,+\infty)$ by:

$$\begin{aligned}\|A\|_1 &= \max(|a|, \langle A\rangle)\\ \|A\|_2 &= |a| + \langle A\rangle\\ \|A\|_3 &= \max(|a|, 2\langle A\rangle)\\ \|A\|_4 &= |a| + 2\langle A\rangle\end{aligned}$$

Proposition 5.2.1 [33] *For each $A, B \in \mathfrak{F}_c$ and $\lambda \in \mathbb{R}$, the functions $\|\cdot\|_n$ satisfy the following properties:*

1. $\|A\|_n = 0 \Leftrightarrow A = \overline{0}$ *for* $n \in \{1,2,3,4\}$;
2. $\|A+B\|_n \leq \|A\|_n + \|B\|_n$ *for* $n \in \{1,2,3,4\}$;
3. $\|A \boxdot B\|_n \leq \|A\|_n \cdot \|B\|_n$ *for* $n \in \{1,2\}$;
4. $\|A \boxtimes B\|_n \leq \|A\|_n \cdot \|B\|_n$ *for* $n \in \{3,4\}$;
5. $\|\lambda A\|_n = |\lambda| \cdot \|A\|_n$ *for* $n \in \{1,2,3,4\}$;

Theorem 5.2.2 [33] *The function* $\mathrm{d} : \mathfrak{F}_c \times \mathfrak{F}_c \to [0,+\infty)$, *defined by*

$$\mathrm{d}(A,B) = |a-b| + \sup_{t\in[0,1]} \max\left(\left|\Theta_A^-(t) - \Theta_B^-(t)\right|, \left|\Theta_A^+(t) - \Theta_B^+(t)\right|\right)$$

is a (complete) metric on $\mathfrak{F}_c$.

Proposition 5.2.3 [33] *The metric* d *on $\mathfrak{F}_c$ satisfies the following properties:*

1. $\mathrm{d}(A+C, B+C) = \mathrm{d}(A,B)$;
2. $\mathrm{d}(A+C, B+D) \leq \mathrm{d}(A,B) + \mathrm{d}(C,D)$;
3. $\mathrm{d}(A \boxdot C, B \boxdot C) \leq \|C\|_1 \cdot \mathrm{d}(A,B) \leq \|C\|_2 \cdot \mathrm{d}(A,B)$;
4. $\mathrm{d}(A \boxtimes C, B \boxtimes C) \leq \|C\|_3 \cdot \mathrm{d}(A,B) \leq \|C\|_4 \cdot \mathrm{d}(A,B)$;
5. $\mathrm{d}(\lambda A, \lambda B) = |\lambda| \cdot \mathrm{d}(A,B)$;

for all $A, B, C, D \in \mathfrak{F}_c$ and $\lambda \in \mathbb{R}$.

Definition 5.2.4 *We say that the sequence $(A_n)_{n\geq 1} \subset \mathfrak{F}_c$ converges to $A \in \mathfrak{F}_c$ if $\lim\limits_{n\to\infty} \mathrm{d}(A_n, A) = 0$ and we will use, in this case, the notation $\lim\limits_{n\to\infty} A_n = A$.*

Remark 5.2.5 *If $A_n = \left(a_n; \Theta_{A_n}^-, \Theta_{A_n}^+\right)$ and $A = \left(a; \Theta_A^-, \Theta_A^+\right)$, then $\lim\limits_{n\to\infty} A_n = A$, if and only if*

$$\begin{cases} \lim\limits_{n\to\infty} a_n = a \\ \lim\limits_{n\to\infty} \Theta_{A_n}^-(t) = \Theta_A^-(t), & \textit{for all } t \in [0,1] \\ \lim\limits_{n\to\infty} \Theta_{A_n}^+(t) = \Theta_A^+(t), & \textit{for all } t \in [0,1] \end{cases}.$$

5.3 Some elementary functions defined on $\mathfrak{F}_{\rm c}$

Let us denote by $\mathfrak{F}_{\rm c}^*$ the set

$$\left\{A=\left(a;\Theta_A^-,\Theta_A^+\right)\in\mathfrak{F}_{\rm c}: a>0,\text{ and }\Theta_A^-(t),\Theta_A^+(t)\geq 1,\ \forall\, t\in[0,1]\right\}.$$

Since $A,B\in\mathfrak{F}_{\rm c}^*$ implies that $A\boxdot B\in\mathfrak{F}_{\rm c}^*$, it follows that $(\mathfrak{F}_{\rm c}^*,\boxdot)$ is a submonoid of $(\mathfrak{F}_{\rm c},\boxdot)$.

We define the exponential function $\exp:\mathfrak{F}_{\rm c}\to\mathfrak{F}_{\rm c}^*$, by

$$A\longmapsto e^A=\left(e^a;\ e^{\Theta_A^-},e^{\Theta_A^+}\right)$$

and the logarithmic function $\ln:\mathfrak{F}_{\rm c}^*\to\mathfrak{F}_{\rm c}$, by

$$A\longmapsto\ln A=\left(\ln a;\ \ln\circ\Theta_A^-,\ln\circ\Theta_A^+\right)$$

where $A=\left(a;\Theta_A^-,\Theta_A^+\right)$ and " $\circ$ " denotes the composition of functions.

Proposition 5.3.1 [33] *The functions* exp *and* ln, *defined above establish the isomorphism between the monoids* $(\mathfrak{F}_{\rm c},+)$ *and* $(\mathfrak{F}_{\rm c}^*,\cdot)$, *and* $\ln=\exp^{-1}$.

Remark 5.3.2 [33] *If* A^k *denotes* $A\boxdot...\boxdot A$ $(k-$ *times*), *then*

$$\lim_{n\to\infty}\left(\overline{1}+\frac{A}{1!}+\frac{A^2}{2!}+...+\frac{A^n}{n!}\right)=e^A.$$

Definition 5.3.3 [33] *If* $A=\left(a;\Theta_A^-,\Theta_A^+\right)$ *and* $B=\left(b;\Theta_B^-,\Theta_B^+\right)$ *are two fuzzy numbers such that:*

1. a^b *is defined* (*in* $\mathbb{R}$);
2. $\left(\Theta_A^-(t)\right)^{\Theta_B^-(t)}$ *and* $\left(\Theta_A^+(t)\right)^{\Theta_B^+(t)}$ *are defined for each* $t\in[0,1]$;
3. *the functions* $\left(\Theta_A^-\right)^{\Theta_B^-}$ *and* $\left(\Theta_A^+\right)^{\Theta_B^+}$ *are decreasing;*

then we define the $B-$*power of* A *by* $A^B=\left(a^b;\left(\Theta_A^-\right)^{\Theta_B^-},\left(\Theta_A^+\right)^{\Theta_B^+}\right)$..

Remark 5.3.4 *For instance, if* $A\in\mathfrak{F}_{\rm c}^*$, *then* A^B *can be constructed for any* $B\in\mathfrak{F}_{\rm c}$.

Proposition 5.3.5 [33] *If* $A\in\mathfrak{F}_{\rm c}^*$ *and* $B\in\mathfrak{F}_{\rm c}$, *then:*

1. $A^{\overline{0}}=\overline{1}$ *and* $A^{\overline{1}}=A$;

2. $A^{B+C} = A^B \boxdot A^C$ *and* $A^{B \boxdot C} = \left(A^B\right)^C$.

If $A = \left(a; \Theta_A^-, \Theta_A^+\right) \in \mathfrak{F}_c$, we can define, for a positive integer n,

$$A^n = \underbrace{A \boxdot ... \boxdot A}_{n-times} = \left(a^n; \left(\Theta_A^-\right)^n, \left(\Theta_A^+\right)^n\right),$$

and,

$$\sqrt[n]{A} = \left(\sqrt[n]{a}; \sqrt[n]{\Theta_A^-}, \sqrt[n]{\Theta_A^+}\right)$$

(where for even n it is supposed that $a \geq 0$).

Remark 5.3.6 [33] *If for* $A \in \mathfrak{F}_c$ *we have* $\operatorname{supp} A \subset (0,1)$, *that is* $\left[x_A^-(t), x_A^+(t)\right] \subset (0,1)$, $\forall t \in [0,1]$, *then it is easy to see that* $\operatorname{supp}(A \boxdot A) \subset (0,1)$ *and consequently,* $\operatorname{supp}(A^n) \subset (0,1)$ *for all* $n \in \mathbb{N}^*$. *So, for* $A \in \mathfrak{F}_c$ *with* $\operatorname{supp} A \subset (0,1)$ *we have*

$$\begin{aligned} \lim_{n\to\infty}\left(\overline{1} + A + ... + A^n\right) &= \lim_{n\to\infty}\left(\frac{1-a^{n+1}}{1-a}, \frac{1-\left(\Theta_A^-\right)^{n+1}}{1-\Theta_A^-}, \frac{1-\left(\Theta_A^+\right)^{n+1}}{1-\Theta_A^+}\right) \\ &= \left(\frac{1}{1-a}, \frac{1}{1-\Theta_A^-}, \frac{1}{1-\Theta_A^+}\right) \overset{not.}{=} \frac{\overline{1}}{\overline{1}-A} \end{aligned}$$

Similarly, for $A \in \mathfrak{F}_c$ *with* $\operatorname{supp} A \subset (0,1)$ *we obtain*

$$\lim_{n\to\infty}\left(A + \frac{1}{2}\cdot A^2 + ... + \frac{1}{n}\cdot A^n\right) = \ln\left(\frac{\overline{1}}{\overline{1}-A}\right).$$

We mention that $\frac{\overline{1}}{\overline{1}-A}$ *is just a notation and it not represent the inverse of* $\overline{1}-A$, *respectively* $\overline{1}-A$ *is not a "subtraction".*

Chapter 6

Topological group structures on quotient sets of fuzzy numbers

6.1 Preliminaries

In this chapter we consider only those fuzzy numbers for which the functions x_A^- and x_A^+ are continuous and we denote by $\mathfrak{F}$, the set of all these fuzzy numbers.

Thus, the set $\mathfrak{F}$ can be represented as the set of elements of the type $A = \left[x_A^-, x_A^+\right]$ where $x_A^-, x_A^+ \in \mathrm{C}\,[0,1]\,,\ x_A^-$ is increasing, x_A^+ is decreasing and $x_A^-(t) \leq x_A^+(t)\,,$ for all $t \in [0,1]\,.$

We also consider the set $\mathfrak{F}_+$ of all positive fuzzy numbers $A \in \mathfrak{F}$ (i.e., $x_A^-(t) > 0$, for $t \in [0,1]$).

We consider the sets:

- $\mathrm{C}\,[a,b] -$ the set of real-valued and continuous functions on $[a,b]\,;$
- $\mathrm{C}_+\,[a,b] -$ the subset of $\mathrm{C}\,[a,b]$ of strictly positive-valued functions
- $\mathrm{BV}\,[a,b] -$ the set of real-valued functions with bounded variation on $[a,b]\,.$
- $\mathrm{BVC}\,[a,b] = \mathrm{C}\,[a,b] \cap \mathrm{BV}\,[a,b]\,;$
- $\mathrm{BVC}_+\,[a,b] = \mathrm{C}_+\,[a,b] \cap \mathrm{BV}\,[a,b]\,.$

In the theory of the functions with bounded variation it is well known that:

Theorem 6.1.1 [69] *If* $f, g \in \mathrm{BV}\,[a,b]$ *and* $\lambda \in \mathbb{R}$, *then* $f \pm g,\ \lambda f,\ f \cdot g \in \mathrm{BV}\,[a,b]\,,$ *and if* $\dfrac{1}{g}$ *is bounded, then* $\dfrac{f}{g} \in \mathrm{BV}\,[a,b]\,.$

Theorem 6.1.2 [69] *A function $f \in \mathrm{C}[a,b]$ is with bounded variation on $[a,b]$ if and only if there exist two increasing functions f_1 and f_2, such that $f = f_1 - f_2$.*

Theorem 6.1.3 [53] *If $[a,b] \xrightarrow{f} [c,d] \xrightarrow{g} \mathbb{R}$ where $f \in \mathrm{BV}[a,b]$, then $g \circ f \in \mathrm{BV}[a,b]$ if and only if g satisfies the Lipschitz condition on $[c,d]$.*

Proposition 6.1.4 *A continuous function $f \in \mathrm{C}_+[a,b]$ is of bounded variation on $[a,b]$ if and only if there exist two increasing functions $\alpha, \beta \in \mathrm{C}_+[a,b]$, such that $f = \dfrac{\alpha}{\beta}$.*

Remark 6.1.5 *If $f \in \mathrm{BVC}[a,b]$, then we can choose an increasing function $u \in \mathrm{C}[a,b]$ and a decreasing function $v \in \mathrm{C}[a,b]$ such that $f = \dfrac{u+v}{2}$ and $u(t) < v(t)$, for all $t \in [a,b]$. Also, if $f \in \mathrm{BVC}_+[a,b]$ then we can choose an increasing function $u \in \mathrm{C}_+[a,b]$ and a decreasing function $v \in \mathrm{C}_+[a,b]$ such that $f = \sqrt{u \cdot v}$ and $u(t) < v(t)$, for all $t \in [a,b]$.*

It is known that $(\mathrm{BVC}[a,b], +)$ and $(\mathrm{BVC}_+[a,b], \cdot)$ are topological groups with the topology induced by the distance defined by

$$\mathrm{D}(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|.$$

Moreover, the correspondence $f \mapsto e^f$ establishes a topological isomorphism between the topological groups $\mathrm{BVC}[a,b]$ and $\mathrm{BVC}_+[a,b]$.

6.2 Monoids with involution - algebraic and topological overviews

Let $(M, \cdot)$ be a semigroup. An involution in M is a unary operation $x \mapsto x^*$ on M, such that $(x \cdot y)^* = y^* \cdot x^*$ and $x^{**} = x$, for all $x, y \in M$. An element $x \in M$ is called Hermitian if and only if $x^* = x$.

We consider now, the class $\mathfrak{M}$ of all systems $(M, \cdot, e, ^*)$, where $(M, \cdot, e)$ is a cancelative and commutative monoid and * is an involution in M. If $(M_1, \cdot, e_1, ^*)$ and $(M_2, \bullet, e_2, ^\star)$ are in $\mathfrak{M}$, a function $f : M_1 \to M_2$ is called a $\mathfrak{M}$ -homomorphism, if f is a monoid homomorphism and $f(x^*) = (f(x))^\star$, for all $x \in M_1$.

Remark 6.2.1 *If $(G, \cdot)$ is an Abelian group, then $(G, \cdot, 1, \cdot^{-1}) \in \mathfrak{M}$ and every group homomorphism between two Abelian groups is a $\mathfrak{M}$-homomorphism.*

Remark 6.2.2 *If* $(M, \cdot, e,^{*}) \in \mathfrak{M}$*, then the set*

$$S(M) = \{x \in M : x^{*} = x\}$$

of all Hermitian elements of M*, is a submonoid of* M *and its elements have the following properties:*

1. $x \in S(M) \Leftrightarrow x^{*} \in S(M)$;
2. $x \cdot x^{*} \in S(M)\,,\ \forall\ x \in M$;
3. *if* $x, x \cdot y \in S(M)$ *then* $y \in S(M)$;
4. *if* $x, y \in M$*, then* $x \cdot y^{*} \in S(M) \Leftrightarrow x \cdot y^{*} = x^{*} \cdot y$.

Proposition 6.2.3 *If* $(M, \cdot, e,^{*}) \in \mathfrak{M}$*, the relation "* $\sim_{*}$ *" on* M*, defined by*

$$x \sim_{*} y \Longleftrightarrow x \cdot y^{*} \in S(M)$$

is a congruence relation on $(M, \cdot, e,^{*})$.

We consider now the quotient set

$$M/_{\sim_{*}} = \widehat{M} = \{[x] : x \in M\}\,,$$

where

$$[x] = \{y \in M : x \cdot y^{*} = x^{*} \cdot y\}$$

is the equivalence class of $x \in M$ and we consider the induced operation on $\widehat{M}$,

$$[x] \odot [y] = [x \cdot y]$$

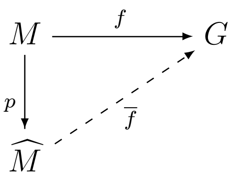
and the canonical homomorphism $p : M \to \widehat{M}$, defined by $x \mapsto [x]$.

Proposition 6.2.4 *If* $(M, \cdot, e,^{*}) \in \mathfrak{M}$*, then* $\left(\widehat{M}, \odot\right)$ *is an abelian group, where* $[e] = S(M)$ *is the neutral element and the inverse of* $[x] \in \widehat{M}$ *is* $[x^{*}] \in \widehat{M}$.

Remark 6.2.5 *We consider* $(M, \cdot, e,^{*}) \in \mathfrak{M}$*. If there exist an Abelian group* $(G, \bullet)$ *and a surjective* $\mathfrak{M}-$*homomorphism* $f : (M, \cdot, e,^{*}) \to (G, \bullet, 1, \cdot^{-1})$*, such that*

$$x \sim_{*} y \Leftrightarrow f(x) = f(y) \tag{6.1}$$

for all $x, y \in M$, *then* (*by the first isomorphism theorem*), *the function* $\overline{f} : \widehat{M} \to G$, $[x] \mapsto f(x)$ *is a group isomorphism and*



$$\overline{f} \circ p = f.$$

Remark 6.2.6 *As above, if*

$$\ker f = \{(x, y) \in M \times M : f(x) = f(y)\}$$

is the kernel of f *as a function, the condition* (6.1) *is equivalent with* $\sim_* = \ker f$. *Also, if*

$$\operatorname{Ker} f = \{x \in M : f(x) = 1\}$$

is the kernel of f *as a monoid homomorphism, the condition* (6.1) *is equivalent with* $\operatorname{Ker} f = S(M)$, *too.*

Theorem 6.2.7 [34] *If* (M, d_1) *and* (G, d_2) *are metric spaces such that*

1. $(M, \cdot, e, ^*, \tau_{\mathrm{d}_1})$ *is a topological monoid with continuous involution;*

2. $(G, \bullet, \tau_{\mathrm{d}_2})$ *is a topological Abelian group;*

3. $f : M \to G$ *is a continuous* $\mathfrak{M}$ *- homomorphism,*

then $\left(\widehat{M}, \widehat{\mathrm{d}}\right)$ *is a metric space, where* $\widehat{\mathrm{d}} : \widehat{M} \times \widehat{M} \to \mathbb{R}$ *is defined by*

$$\widehat{\mathrm{d}}([x], [y]) = \mathrm{d}_2(f(x), f(y)), \quad \text{for all } [x], [y] \in \widehat{M}.$$

Moreover, the canonical homomorphism $p : M \to \widehat{M}$ *is continuous and* $\left(\widehat{M}, \odot, \tau_{\widehat{\mathrm{d}}}\right)$ *is a topological Abelian group* (*with the induced topology*) *which is topologically isomorphic with* $(G, \bullet, \tau_{\mathrm{d}_2})$.

6.3 Topological group structures on quotient sets of $\mathfrak{F}$

Recall that, if $A = \left[x_A^-, x_A^+\right] \in \mathfrak{F}$ and $B = \left[x_B^-, x_B^+\right] \in \mathfrak{F}$, then their (usual) sum is defined by

$$A + B = \left[x_A^- + x_B^-, x_A^+ + x_B^+\right]$$

and $-A$ is defined by $-A = \left[-x_A^+, -x_A^-\right]$. Also, if $A, B \in \mathfrak{F}_+$, then their (usual) product is defined by

$$A \cdot B = \left[x_A^- \cdot x_B^-, x_A^+ \cdot x_B^+\right]$$

and $A^{-1} = \dfrac{1}{A}$ is defined by $\dfrac{1}{A} = \left[\dfrac{1}{x_A^+}, \dfrac{1}{x_A^-}\right]$. Denote $\overline{0} = [0,0]$ and $\overline{1} = [1,1]$.

The Hausdorff distance $\mathrm{d} : \mathfrak{F} \times \mathfrak{F} \to [0, +\infty)$ on the set of fuzzy numbers is defined by

$$\mathrm{d}(A, B) = \sup_{t \in [0,1]} \left(\left|x_A^-(t) - x_B^-(t)\right| + \left|x_A^+(t) - x_B^+(t)\right| \right).$$

Proposition 6.3.1 [34] *$\left(\mathfrak{F}, +, \overline{0}, -\right)$ and $\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}\right)$ are elements of $\mathfrak{M}$ and they are topological monoids with continuous involutions, relatively to the distance* d.

If $S_0 = \mathrm{S}\left(\mathfrak{F}, +, \overline{0}, -\right)$ and $S_1 = \mathrm{S}\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}\right)$, then

$$S_0 = \{A \in \mathfrak{F} : A = -A\} = \left\{A \in \mathfrak{F} : x_A^- + x_A^+ = 0\right\}$$
$$S_1 = \left\{A \in \mathfrak{F}_+ : A = A^{-1}\right\} = \left\{A \in \mathfrak{F} : x_A^- \cdot x_A^+ = 1\right\}$$

and the induced congruence relations on $\left(\mathfrak{F}, +, \overline{0}, -\right)$ and $\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}\right)$ are defined by

$$A \sim B \Leftrightarrow A + (-B) \in S_0 \Leftrightarrow x_A^- + x_A^+ = x_B^- + x_B^+$$

if $A, B \in \mathfrak{F}$, respectively,

$$A \approx B \Leftrightarrow A \cdot B^{-1} \in S_1 \Leftrightarrow x_A^- \cdot x_A^+ = x_B^- \cdot x_B^+$$

if $A, B \in \mathfrak{F}_+$.

The corresponding equivalence classes are

$$[A] = \{B \in \mathfrak{F} : A \sim B\}$$

if $A \in \mathfrak{F}$, respectively

$$\langle A \rangle = \{B \in \mathfrak{F}_+ : A \approx B\}$$

if $A \in \mathfrak{F}_+$.

We denote by $\widehat{\mathfrak{F}}$ and by $\widetilde{\mathfrak{F}}_+$ the corresponding quotient sets $\mathfrak{F}/_\sim$ and $\mathfrak{F}_+/_\approx$ respectively, and so $\widehat{\mathfrak{F}} = \{[A] : A \in \mathfrak{F}\}$ and $\widetilde{\mathfrak{F}}_+ = \{\langle A \rangle : A \in \mathfrak{F}_+\}$.

By Proposition 6.2.4, we have that:

- $\left(\widehat{\mathfrak{F}}, \oplus\right)$ is an Abelian group with the operation defined by $[A] \oplus [B] = [A + B]$. The neutral element is $\left[\overline{0}\right] = S_0$ and the additive inverse of $[A] \in \widehat{\mathfrak{F}}$ is $-[A] = [-A]$;

- $\left(\widetilde{\mathfrak{F}}_+, \odot\right)$ is an Abelian group with the operation defined by $\langle A \rangle \odot \langle B \rangle = \langle A \cdot B \rangle$. The neutral element is $\langle \overline{1} \rangle = S_1$ and the multiplicative inverse of $\langle A \rangle \in \widetilde{\mathfrak{F}}_+$ is $\langle A \rangle^{-1} = \langle A^{-1} \rangle$.

Theorem 6.3.2 [34] $\left(\widehat{\mathfrak{F}}, \oplus\right)$ *is a metrizable topological group which is topologically isomorphic with* (BVC $[0,1]$, +).

Theorem 6.3.3 [34] $\left(\widetilde{\mathfrak{F}}_+, \odot\right)$ *is a metrizable topological group which is topologically isomorphic with* (BVC$_+$ $[0,1]$, ·).

Theorem 6.3.4 [34] $\left(\widehat{\mathfrak{F}}, \oplus\right) \cong_{\text{top}} \left(\widetilde{\mathfrak{F}}_+, \odot\right)$.

Remark 6.3.5 *The equivalence class* $[A] \in \widehat{\mathfrak{F}}$ *of a fuzzy number* $A \in \mathfrak{F}$ *is defined by the arithmetic mean of* A*, respectively the equivalence class* $\langle A \rangle \in \widetilde{\mathfrak{F}}_+$ *of a fuzzy number* $A \in \mathfrak{F}_+$ *is defined by the the geometric mean of* A*. These are illustrated (for a positive fuzzy number) in Figure* 8.

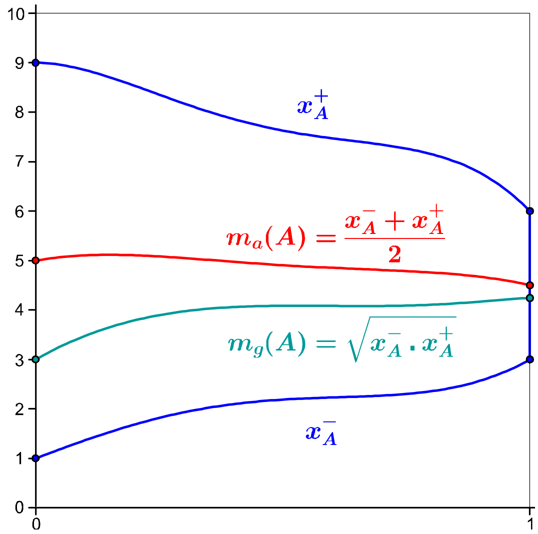


Figure 8.

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