BABEŞ-BOLYAI UNIVERSITY
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INVESTIGATION IN CYCLIC MORSE THEORY AND APPLICATIONS

PhD thesis abstract

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Contents

Introduction

At the beginning of the last century (in 1925), Marston Morse [51], [52], [53] has initiated the study of critical points, a starting point for the Morse theory, a fundamental technique for investigating the topology of a smooth manifold, ([35], [44]). The development of the Morse theory over the years has led to circle-valued Morse functions, that is, C^∞ functions with values in S^1 having only non-degenerate critical points ([45], [56]).

The study of such functions was initiated by S.P. Novikov in 1980 ([54], [55]). Many other have developed the circle-valued Morse theory, a new branch of the Morse theory; we refer to M. Farber [32], [33], A. Ranicki [61], A. Pajitnov [56].

This paper is naturally divided into five chapters which will be described in the lines to come.

First chapter contains four sections and has a monographic character. The main purpose of this chapter is to introduce the basic notions and results that will be used in the paper. We follow here the excellent books of Y. Matsumoto [48], G. Cicortaş [27], J. Milnor [49], D. Andrica [3], R. Bott [23], [24], M. Agoston [1], J. Lee [44].

Chapter 2 is divided into four sections and is devoted to a brief presentation of circle-valued Morse theory. This chapter is based on the following papers of A. Pajitnov [56], M. Farber [32], M. Hutchings [40], S. Maksymenko [45], D. Andrica [3], R. Miron [50], M. Hirsch [39].

Chapter 3 aims to present our results regarding the circular φ-category of a manifold, [15]. Also, we present some results concerning the Morse-Smale characteristic for real Morse functions, following the papers of D. Andrica [3], [4], G. Rassias [62], B. Doubrovine [30], D. Andrica, D. Mangra, C. Pintea [15].

In Chapter 4 we present our results in the study of the Morse-Smale characteristic for circle-valued Morse functions, following the papers of D. Andrica, D. Mangra [12], [13] and D. Andrica, D. Mangra, C. Pintea [14].

Chapter 5 has a monographic character and it is devoted to the presentation of the Morse-Novikov inequalities for circle-valued functions, following papers [32], [56], [61].

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Chapter 1

Basic Aspects of Morse Theory

This first chapter aims to present the basic notions that are useful throughout the paper and it is naturally divided into four sections. First section presents Morse theory elements on finite-dimensional manifolds; we refer only at smooth manifolds ([49]). In this context, we define the notion of critical point, critical value, degenerate critical point, non-degenerate critical point, Morse function, ([23], [24]). Moreover, we present some examples of Morse functions and their properties, ([2], [27]). Section 1.2 is an introduction to Morse theory on surfaces and handle decomposition of a manifold. The last two sections are devoted to a brief presentation of a few results concerning the existence of Morse functions and Morse inequalities.

1.1 Morse theory for finite-dimensional manifolds

Let M and N be two smooth manifolds of dimension m respectively n and let $f : M \to N$ be a smooth function. Recall that for a point $x \in M$ the rank of f at x is

$$rank_x f = rank\ d(\psi \circ f \circ \varphi^{-1})(\varphi(x)) = rank J(\psi \circ f \circ \varphi^{-1})(\varphi(x)) \leq \min\{m, n\},$$

where (U, φ) is a local chart at x on M and (V, ψ) is a local chart at $f(x)$ on N.

If $(U, \varphi) = (U, x^1, \ldots, x^m)$ and $f = (f^1, \ldots, f^n)$ we can write

$$rank_x f = rank \left(\frac{\partial f^i}{\partial x^j}(\varphi(x)) \right)_{i=\overline{1,n}, j=\overline{1,m}} = \dim \mathrm{Im}(df)_x.$$

Definition 1.1.1 A point $x \in M$ with the property that $rank_x f = \min\{m, n\}$ is called a **regular point** of f. Otherwise, the point x is a **critical point** of f.

If M is an m-dimensional manifold without boundary and $f : M \to \mathbb{R}$ is a smooth function, a point $p_0 \in M$ is called a critical point of f if

$$\frac{\partial f}{\partial x_1}(p_0) = 0,\ \frac{\partial f}{\partial x_2}(p_0) = 0, \ldots, \frac{\partial f}{\partial x_m}(p_0) = 0$$

where $(x_1, x_2, \ldots, x_m)$ is a local coordinate system about p_0.

A real number c, such that $f(p_0) = c$, is called a **critical value** of f for a critical point p_0 of f.

We denote by $C(f) = \{p \in M : p$ is a critical point$\}$ the set of all critical points of f and by $B(f) = f(C(f))$, the bifurcation set of the smooth function f.

If $p \notin C(f)$ then p is a regular point of f. A number $c \in B(f)$ is called critical value of f. If $c \notin B(f)$, then c is called regular value of f.

Remark 1.1.1 1. The critical set $C(f)$ is closed in M.
2. The bifurcation set $B(f)$ is closed in $\mathbb{R}$.

Theorem 1.1.1 (Sard's theorem) *The set of critical values $B(f)$ of a smooth function $f : M \to \mathbb{R}^m$ has measure zero in $\mathbb{R}^m$.*

Proof. See [27] or [48].

Proposition 1.1.1 ([48]) *The definition of a critical point does not depend on the choice of the local coordinate system.*

The degenerate and non-degenerate critical points of a function defined on an m-dimensional manifold are defined using the Hessian matrix.

If $det H_f(p_0) \neq 0$, then p_0 is called a **non-degenerate critical point**.

If $det H_f(p_0) = 0$, then p_0 is a **degenerate critical point**.

Definition 1.1.2 A smooth function $f : M \to \mathbb{R}$ is a **Morse function** if all its critical points are non-degenerate.

Proposition 1.1.2 *Let M and N be two closed manifolds and let $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ be Morse functions. Define a function $F : M \times N \to \mathbb{R}$ by $F = (A + f)(B + g)$, where A and B are positive real numbers. The function F is a Morse function on $M \times N$.*

Proof. See [48].

Theorem 1.1.2 (The Morse lemma for finite-dimensional manifolds) *Let p_0 be a non-degenerate critical point of a function $f : M \to \mathbb{R}$. Then there exists a local coordinate system $(X_1, \ldots, X_m)$ about p_0 such that the local representation of f has the following form:*

$$f = -X_1^2 - X_2^2 - \ldots - X_\lambda^2 + X_{\lambda+1}^2 + \ldots + X_m^2 + f(p_0).$$

Proof. See [48].

Definition 1.1.3 The number λ which appears in the above theorem is called **the Morse index** of the non-degenerate critical point p_0.

The Morse index verifies the inequality $0 \leq \lambda \leq m$.

The non-degenerate critical points with index 0 are local minimum points and the non-degenerate critical points with index m are local maximum points.

Some important results are obtained from Morse lemma.

Proposition 1.1.3 *There exists a Morse function on any compact manifold.*

Proposition 1.1.4 *A non-degenerate critical point of a function $f : M \to \mathbb{R}$ is isolated in the critical set $C(f)$.*

Proposition 1.1.5 *Let M be a compact manifold and let $f : M \to \mathbb{R}$ be a Morse function. Then the critical set $C(f)$ is finite.*

Example 1.1.1 Let $\mathbb{T}^2 \subset \mathbb{R}^3$ be the 2-dimensional torus and let $f : \mathbb{T}^2 \to \mathbb{R}$, $f(x, y, z) = z$ be a smooth function (the height function on $\mathbb{T}^2$).

Then f is a Morse function with 4 critical points, all non-degenerate.

1.2 Morse functions on surfaces and handle decomposition

Theorem 1.2.1 *If M is a closed surface and $f : M \to \mathbb{R}$ is a Morse function with exactly two non-degenerate critical points then M and the sphere S^2 are diffeomorphic.*

For the proof of this result we refer to the excellent book of J. Milnor [49].

The Morse function $f : M \to \mathbb{R}$ takes a maximum value at the non-degenerate critical point p_0 in M and a minimum value at the non-degenerate critical point q_0 in M. The index of p_0 is 2 and the index of q_0 is 0.

Taking into account the Morse lemma, we can express the function f in a standard form with a suitable coordinate system (x, y) about p_0 and (X, Y) about q_0:

$$f = \begin{cases} -x^2 - y^2 + C & (\text{near } p_0) \\ X^2 + Y^2 + c & (\text{near } q_0) \end{cases}$$

where C and c are the maximum and minimum values of f.

For a small enough real number ε, we denote by $D(p_0)$ the set of points in a neighborhood of p_0 such that $C - \varepsilon \leq f(p) \leq C$ and by $D(q_0)$ the set of point in a neighborhood of q_0 such that $c \leq f(p) \leq c + \varepsilon$.

Removing the interior of the sets $D(p_0)$ and $D(q_0)$ from M one obtains a smooth surface with boundary denoted by M_0. The boundary of M_0 is denoted by ∂M_0. We obtain:

$$\partial M_0 = C(p_0) \cup C(q_0) \text{ and } int(M_0) = M_0 - \partial M_0,$$

where $C(p_0)$ and $C(q_0)$ are the boundary circles of $D(p_0)$ and $D(q_0)$.

Proposition 1.2.1 ([48]) *The surface M_0 is diffeomorphic to the direct product of one of the boundary circles and the unit interval $[0, 1]$, thus we have $M_0 \cong C(q_0) \times [0, 1]$. Since the boundary circle $C(q_0)$ and the unit circle S^1 are diffeomorphic, one obtains that $M_0 \cong S^1 \times [0, 1]$.*

Proposition 1.2.2 ([48]) *Consider two disks D_0 and D_1. If $k : \partial D_0 \to \partial D_1$ is a diffeomorphism then k can be extended to a diffeomorphism $K : D_0 \to D_1$.*

Proposition 1.2.3 ([48]) *Consider two disks D_1 and D_2 and a diffeomophism $h : \partial D_1 \to \partial D_2$. Then, by pasting D_1 and D_2 along their boundaries by the diffeomorphism h, we obtain a closed surface diffeomorphic to the two-dimensional sphere S^2.*

Lemma 1.2.1 ([48]) *A Morse function $f : M \to \mathbb{R}$ on a closed surface M has only a finite number of critical points.*

Let M be a closed surface and let $f : M \to \mathbb{R}$ be a Morse function.

We denote by $M_t = \{p \in M \mid f(p) \leq t\}$, $t \in \mathbb{R}$, the set of level t of M, consisting in all points at which f takes values less or equal than t.

Lemma 1.2.2 *Let $f : M \to [a, b]$ be a Morse function, where $a, b \in \mathbb{R}$ and $a < b$. If there is no critical value of f in the interval $[a, b]$ then M_a and M_b are diffeomorphic.*

Proof. See [3], [23].

Let $f : M \to \mathbb{R}$ be a Morse function such that $C(f)$ is finite

$$f(p_i) = c_i.$$

Assume that $f(p_1) < f(p_2) < \ldots < f(p_n) \Rightarrow c_1 < c_2 < \ldots < c_n$.

Consider t an arbitrary parameter such that if $t < c_1$ then $M_t = \emptyset$.

If $t > c_1$, the index of p_0 is 0, c_1 is the minimum value of f, thus $M_t = D^2$. This disk which corresponds to the critical point of index 0 is called 0-handle. The process continues and every time t passes a critical value c_i, a new 0, 1 or 2-handle is attached, depending on the index of the corresponding critical point. The last critical value c_n corresponds to a 2-handle.

Theorem 1.2.2 *Let M be a closed surface and let $f : M \to \mathbb{R}$ be a Morse function. The surface M can be described as a finite union of 0, 1 or 2-handles.*

Proof. See [48].

1.3 The existence of Morse functions

We use the notion of closed manifold, meaning a compact manifold without boundary.

Lemma 1.3.1 *If $\mathbb{R}^m = \{(x_1, \ldots, x_m)\}$ is the Euclidean space of dimension m, U is an open subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^m$ is a smooth function on U then*

$$g(x_1, \ldots, x_m) = f(x_1, \ldots x_m) - (a_1 x_1 + \ldots + a_m x_m)$$

is a Morse function, for $a_1, \ldots, a_m \in \mathbb{R}$.

The proof of this result can be found in paper [48].

For $\varepsilon > 0$ we say that f is a (C^2, ε)-**approximation** of g on $K \subseteq \mathbb{R}^m$ if for any point p in K, the following relations hold:

1. $|f(p) - g(p)| < \varepsilon$
2. $\left| \dfrac{\partial f}{\partial x_i}(p) - \dfrac{\partial g}{\partial x_i}(p) \right| < \varepsilon$, $i = 1, 2, \ldots, m$
3. $\left| \dfrac{\partial^2 f}{\partial x_i \partial x_j}(p) - \dfrac{\partial^2 g}{\partial x_i \partial x_j}(p) \right| < \varepsilon$, $i, j = 1, 2, \ldots, m$.

We cover the manifold M with a finite number of coordinate neighboorhoods $U_1, \ldots, U_k$ and for every $i = 1, \ldots, s$, we choose a compact set K_i in U_i such that

$$M = K_1 \cup \ldots \cup K_s.$$

The compact sets K_i cover M.

Definition 1.3.1 We say that a function $f : M \to \mathbb{R}$ is a (C^2, ε)-approximation of a function $g : M \to \mathbb{R}$ if f is a (C^2, ε)-approximation of g on K_i, for $i = 1, \dots, s$.

The next results were proved by Y. Matsumoto în [48].

Lemma 1.3.2 *If K is a compact subset of an m-dimensional manifold M and $g : M \to \mathbb{R}$ is a function with no degenerate critical point in K, then for a small enough number $\varepsilon > 0$, any (C^2, ε)-approximation f of g has no degenerate critical point in K.*

Theorem 1.3.1 (The existence of Morse functions) *Let M be a closed m-dimensional manifold and let $g : M \to \mathbb{R}$ be a smooth real function defined on M. Then there exists a Morse function $f : M \to \mathbb{R}$ which is an approximation of g.*

1.4 Morse inequalities

We will follow paper [27].

Theorem 1.4.1 *Let $f : M \to \mathbb{R}$ be a Morse function such that the compactness conditions are satisfied. Consider $B[f] \cap (a, b) = \{c\}$ and $C_c[f] = \{p_1, p_2, \dots, p_r\}$, the critical point p_i having the index k_i and the coindex l_i, $i = \overline{1, r}$. Then M_b is obtained from M_a by disjoint attaching of handles of the type $(k_1, l_1), \dots, (k_r, l_r)$.*

There is a very important connection between the topology of a manifold M and the critical points of a function $f : M \to \mathbb{R}$. This connection can be described in terms of some inequalities (Morse inequalities).

Denote by $H_*(X, A) = H_*(X, A; F)$ the relative homology with coefficients in F.

Definition 1.4.1 Let Y be a closed subspace of a space X and let $G : D^k \to X$ be a continuous function, $G(D^k) = e^k$. We denote $X = Y \cup_g e^k$ and we say that X is obtained from Y by attaching a k-cell with the attaching map $g = G|_{S^{k-1}}$, if:
i. $X = Y \cup e^k$
ii. $G|_{intD^k}$ is an homeomorphism on $e^k \setminus Y$
iii. g applies S^{k-1} on $\partial e^k = e^k \cap Y$.
G is called the characteristic map of the attaching.

Proposition 1.4.1 *If X is obtained from Y by attaching a k-cell then*

$$H_l(X, Y) = H_l(e^k, \partial e^k) = H_l(D^k, S^{k-1}) = \begin{cases} F, & \text{if } l = k \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See [27].

Let K be a convex subset of $\mathbb{R}^n$ and let A be a closed subset of K. If r is a retract of K on A, then $\rho(x, t) = (1 - t)x + tr(x)$ is a strong deformation retract of k on A.

It is obvious that the set $(0 \times D^k) \cup (D^l \times S^{k-1})$ is a strong deformation retract of $D^l \times D^k$. Indeed, $D^l \times D^k$ being a convex of $\mathbb{R}^l \times \mathbb{R}^K$, it is a sufficient to define a retract $r : D^l \times D^k \to (0 \times D^k) \cup (D^l \times S^{k-1})$. We define $r(0, y) = (0, y)$, and for $x \neq 0$ we consider

$$r(x, y) = \begin{cases} \left(0, \frac{2\|y\|}{2-\|x\|}\right), & \text{if } \| y \| \leq 1 - \frac{\|x\|}{2} \\ \left((\| x \| + 2 \| y \| - 2)\frac{x}{\|x\|}, \frac{y}{\|y\|}\right), & \text{otherwise.} \end{cases}$$

Theorem 1.4.2 *Let N and P be two smooth manifolds with boundary. If N is obtained from P by attaching a handle of the form (k,l), then $P \cup_g e^k$ is a strong deformation retract of N.*

In particular,

$$H_l(N,P) = \begin{cases} F, & \text{if } l = k \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See [27].

In the case of the disjoint attaching of $(k_1,l_1),\ldots,(k_r,l_r)$ handles one obtains the disjoint attaching of $e^{k_1},\ldots,e^{k_r}$ cells.

Consider a family X_i, $i = \overline{0,n}$ of closed subspaces of X such that

$$A = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq X$$

and X_{i+1} is homeomorphic to $X_i \cup_{g_i} e^{k_i}$. The pair (X,A) is called a relative spheric complex and the family of the attaching maps is called a cell decomposition of the pair (X,A).

If X_{i+1} is homotopic equivalent with $X_i \cup_{g_i} e^{k_i}$, then the pair (X,A) is called an homotopic spheric complex (homotopic cell decomposition).

We denote by ν_i the number of $e^{k_0},\ldots,e^{k_n-1}$ cells, with $k_j = i$. Thus ν_i is the number of the cells of dimension i that is attached to A in order to obtain X.

Let $f : M \to \mathbb{R}$ be a Morse function. For $0 \le k \le m$ we define the Morse numbers

$$\mu_k(f) = \text{the number of critical points of index } k.$$

For $a < b$ we define

$$\mu_k(f,a,b) = \text{the number of critical points of index } k \text{ in } f^{-1}(a,b).$$

Theorem 1.4.3 *Let M be a complete riemannian manifold, let $f : M \to \mathbb{R}$ be a Morse function which satisfies the Palais-Smale condition on M and let $a < b$ be regular values of f.*

Then (M_b, M_a) is a homotopic spheric complex. Actually, (M_b, M_a) admits an homotopic cell decomposition for which the number ν_k of the cells of dimension k is $\mu_k(f,a,b)$.

Corollary 1.4.1 *Any smooth compact manifold M is an homotopic spheric complex. For any Morse function $f : M \to \mathbb{R}$ there exists a homotopic cell decomposition of M such that $\nu_k = \mu_k(f)$.*

Consider admissible pairs of topological spaces (X,A), in other words a homotopic spheric complex (X,A).

For a pair (X,A) and a commutative field F we define the Betti numbers

$$b_k(X,A) = \dim\ H_k(X,A;F)$$

and the Euler-Poincaré characteristic of the pair (X,A),

$$\chi(X,A) = \sum_k (-1)^k b_k(X,A).$$

Let

$$S_k(X,A) = \sum_{m=0}^{k} (-1)^{k-m} b_m(X,A).$$

The following equalities hold:

$$\begin{aligned}
&S_0 = b_0 \\
&S_1 = b_1 - b_0 = b_1 - S_0 \\
&\dots \\
&S_k = b_k - b_{k-1} + \dots \pm b_0 = b_k - S_{k-1} \\
&\chi = b_0 - b_1 + b_2 - \dots
\end{aligned}$$

Proposition 1.4.2 *The Euler-Poincaré characteristic is additive and S_k is subadditive. If $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n$ and any pair (X_i, X_{i-1}) is admissible, then we have*

$$S_k(X_n, X_0) \le \sum_{i=1}^{n} S_k(X_i, X_{i-1})$$

$$\chi(X_n, X_0) = \sum_{i=1}^{n} \chi(X_i, X_{i-1}).$$

Proof. See [27].

Theorem 1.4.4 *Let (X, A) be an homotopic spheric complex which admits a homotopic cell decomposition with ν_k cells of dimension k. Then the following inequalities hold:*

$$\begin{aligned}
&b_0 \le \nu_0 \\
&b_1 - b_0 \le \nu_1 - \nu_0 \\
&\dots \\
&b_k - b_{k-1} + \dots \pm b_0 \le \nu_k - \nu_{k-1} + \dots \pm \nu_0.
\end{aligned}$$

Moreover, we have

$$\chi(X,A) = \sum_k (-1)^k b_k = \sum_k (-1)^k \nu_k.$$

Proof. See [27].

Corollary 1.4.2 *Let M be a complete riemannian manifold, let $f : M \to \mathbb{R}$ be a Morse function which satisfies the Palais-Smale condition on M and let $a < b$ be regular values of f. Let $\mu_k = \mu_k(f, a, b)$ and $b_k = b_k(M_b, M_a)$. Then the following relations hold:*

1. *More inequalities:*

$$\begin{aligned}
&b_0 \le \mu_0 \\
&b_1 - b_0 \le \mu_1 - \mu_0 \\
&\dots \\
&b_k - b_{k-1} + \dots \pm b_0 \le \mu_k - \mu_{k-1} + \dots \pm \mu_0
\end{aligned}$$

2. *Euler formula:*

$$\chi(X, A) = \sum_k (-1)^k b_k = \sum_k (-1)^k \nu_k$$

3. *Weak Morse inequalities:* $b_k \leq \mu_k$.

Chapter 2

Circle-valued Morse Theory

This chapter contains basic notions of circle-valued Morse theory. Most of the notions that were introduced in Chapter 1 are kept for circle-valued functions, ([40], [56]).

2.1 Circle-valued Morse functions

Let M^m be a manifold without boundary and let $f : M \to S^1$ be a smooth function. The circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is a 1-dimensional submanifold of $\mathbb{R}^2$ and it is endowed with the corresponding smooth structure.

For a point $x \in M$ we choose a neighborhood V of $f(x)$ in S^1 diffeomorphic to an open interval of $\mathbb{R}$. Denote $U = f^{-1}(V)$. The function $f|_U$ identifies with a smooth function from U to $\mathbb{R}$.

Definition 2.1.1 A smooth circle-valued function $f : M \to S^1$ is a **Morse function** if every critical point of f is non-degenerate.

Denote by $C(f)$ the set of all critical points of f and by $C_k(f)$ the set of all critical points of index k of f, $k = 0, \dots, m$.

Consider M a compact manifold. Then the critical set $C(f)$ is finite and we denote by $\mu(f)$ the cardinality of $C(f)$ and by $\mu_k(f)$ the cardinality of $C_k(f)$, $k = 0, \dots, m$. It is obvious that

$$\mu(f) = \mu_0(f) + ... + \mu_m(f).$$

It is well-known that the quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1 by the homeomorphism $f : \mathbb{R}/\mathbb{Z} \to S^1$, where $f(\widehat{x}) = e^{2\pi ix}$. Thus we can identify the circle S^1 with the quotient $\mathbb{R}/\mathbb{Z}$ and so circle-valued functions can be considered as multivalued real functions, ([40], [56]).

We will consider a covering space of the domain of definition such that the function becomes single-valued on the covering.

Consider the universal covering of the cirle

$$\exp : \mathbb{R} \to S^1, \quad \exp(t) = e^{2\pi it}.$$

The exponential map $\exp : \mathbb{R} \to S^1$, defined by $\exp(t) = e^{2\pi i t}$ is a covering projection, thus a local homeomorphism, ([50]).

The space $\mathbb{R}$ together with the exponential map exp is the covering space of the circle S^1.

The structure group of the covering is the subgroup $\mathbb{Z} \subset \mathbb{R}$ acting on $\mathbb{R}$ by translations.

We use the multiplicative notation for $\mathbb{Z}$ and denote by t the generator which corresponds to -1 in the additive notation.

For a circle-valued Morse function $f : M \to S^1$ and $\overline{M}$ a covering space of M, let $\pi : \overline{M} \to M$ be the infinite cyclic covering induced from the universal covering of the circle $\exp : \mathbb{R} \to S^1$, by function f.

By the definition of the induced covering we have a function $\overline{f} : \overline{M} \to \mathbb{R}$, such that the next diagram commutes.

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\overline{f}} & \mathbb{R} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \exp} \\ M & \xrightarrow{f} & S^1 = \mathbb{R}/\mathbb{Z} \end{array}$$

In some cases $\overline{M}$ can be considered as a subset of $M \times \mathbb{R}$. Function f lifts to a $\mathbb{Z}$-equivariant Morse function $\overline{f} : \overline{M} = f^*\mathbb{R} \to \mathbb{R}$ on the infinite cyclic covering, ([40]).

The function f is a Morse function if and only if $\overline{f}$ is a Morse function. The function $\overline{f}$ is equivariant with respect to the action of $\mathbb{Z}$ on $\overline{M}$ and $\mathbb{R}$, thus we have the relation $\overline{f}(tx) = \overline{f}(x) - 1$.

The classical Morse theory can not be applied here because the domain of definition of $\overline{f}$ is not compact and generally the number of critical points of $\overline{f}$ is not finite.

The solution to correct this is to consider the restriction of $\overline{f}$ to the fundamental domain of $\overline{M}$ with respect to the action of the group $\mathbb{Z}$.

For a regular value $a \in \mathbb{R}$ of $\overline{f}$ we denote the set $W = \overline{f}^{-1}([a-1, a])$ a compact cobordism, such that

$$\partial_1 W = \overline{f}^{-1}(a) \quad \text{and} \quad \partial_0 W = \overline{f}^{-1}(a-1).$$

The set W is called the fundamental cobordism.

For the Morse function $\overline{f}|_W : W \to [a-1, a]$, the cobordism W can be described as follows: consider $\alpha = \exp(a) \in S^1$, thus α is a regular value of f and $V = f^{-1}(\alpha)$ is a smooth submanifold of M. If M is cut along V, one obtains the cobordism W with both components of its boundary diffeomorphic to V.

For more details regarding circle-valued functions we refer to paper [56].

2.2 Morse forms

Definition 2.2.1 If M is a closed smooth manifold, then a 1-form ω on M

$$\omega : M \to T^*(M),$$

is a smooth section of the cotangent bundle $\pi^* : T^*(M) \to M$ such that

$$\omega \circ \pi^* = 1_M.$$

Denote by $\Omega^1(M)$ the set of 1-forms on M. In local coordinates $x_1, x_2, \ldots, x_m$, in an open subset $U \subset M$, any 1-form ω is given by

$$\omega = a_1 dx_1 + a_2 dx_2 + \ldots + a_m dx_m,$$

where $a_1, \ldots, a_m \in C^\infty(U)$.

The exterior derivative d is given by

$$d : \Omega^0(M) \to \Omega^1(M),\ f \to df,$$

where $\Omega^0(M) = C^\infty(M)$ and

$$\begin{aligned} d\omega &= \sum_{i=1}^{m} da_i \wedge dx_i = \sum_{i,j=1}^{m} \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_i \\ &= \sum_{i<j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right) \cdot dx_i \wedge dx_j. \end{aligned}$$

A 1-form ω is called **closed** if $d\omega = 0$.

Remark 2.2.1 The condition that ω is a closed form can be written as

$$\frac{\partial a_j}{\partial x_i} = \frac{\partial a_i}{\partial x_j}, \text{ for any } i, j = 1, \ldots, m.$$

Definition 2.2.2 A point $p \in M$ such that $\omega_p = 0$ is called **a zero** of the 1-form ω.

We define the set of zeros of ω by

$$Z(\omega) = \{p \in M : \omega_p = 0\}.$$

Remark 2.2.2 If ω is a closed 1-form ($d\omega = 0$), then there exists a smooth function $f_U : U \to \mathbb{R}$ such that $\omega|_U$ is an exact 1-form, $\omega|_U = df_U$, for any simply connected domain $U \subset M$. The zeros of ω in U are the critical points of f_U, ([32]).

Definition 2.2.3 A point $p \in Z(\omega) \cap U$ is a **non-degenerate zero** of the form ω if p is a non-degenerate critical point of any function $f_U : U \to \mathbb{R}$, which satisfies the relation $df_U = \omega_U$ ([39]).

Definition 2.2.4 Let $\omega \in \Omega^1(M)$:

1. The closed 1-form ω is called **Morse form** if every zero of ω is non-degenerate.

2. If ω is a Morse form, $p \in Z(\omega)$, we say that p has **the Morse index** k, $(0 \leq k \leq n)$ if p is a critical point of index k of f_U.

Example 2.2.1 Consider $M = \mathbb{R}^2 \setminus \{(0,0)\}$, $\omega \in \Omega^1(M)$, where ω is the angular 1-form described by

$$\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy.$$

Denote by $\omega|_{S^1} = d\theta \in \Omega^1(S^1)$. Note that $d\theta$ is local exact on M but is not exact, thus we have

$$Z(d\theta) = \emptyset.$$

Let $f : M \to S^1$ be a circle-valued function. Then f induces the function $f^* : \Omega^1(S^1) \to \Omega^1(M)$, where $\eta = f^*(d\theta)$ is a closed 1-form in $\Omega^1(M)$.

We present now some general notions and results regarding closed 1-forms.

The de Rham cohomology group, denoted by $H^1_{deRham}(M)$, is by definition the quotient of the space of all 1-forms by the subspace of the exact forms.

One obtains the isomorphism:

$$H^1(M, \mathbb{R}) \approx H^1_{deRham}(M)$$

between the de Rham cohomology group and the singular cohomology group with real coefficients. An important characteristic of closed 1-forms is the de Rham cohomology class.

For a closed 1-form, its image in the group $H^1(M, \mathbb{R}) = \dfrac{Kerd^1}{Imd^0}$,

$$\Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \Omega^2(M)$$

is called the de Rham cohomology class of ω and it is denoted by $[\omega]$.

It vanishes, $[\omega] = 0$, if and only if ω is exact ($\omega \in Imd^0$).

Circle-valued functions provides a lot of examples of closed 1-forms.

The 1-form dx on $\mathbb{R}^1$ is invariant with respect to the action of $\mathbb{Z}$ on $\mathbb{R}$ and defined a 1-form on $S^1 = \mathbb{R}/\mathbb{Z}$, denoted also by dx. The de Rham cohomology class of dx is the image of the generator $t \in H^1(S^1, \mathbb{Z})$ with respect to the inclusion map $H^1(S^1, \mathbb{Z}) \hookrightarrow H^1(S^1, \mathbb{R})$.

If $f : M \to S^1$ is a circle-valued smooth function, denote $df = f^*(dx)$.

Then df is a closed 1-form on M, called the differential of f.

The form df is a Morse form if and only if f is a circle-valued Morse function.

Proposition 2.2.1 ([56]) *The homology class of a C^∞ circle-valued Morse function $f : M \to S^1$ is determined by the de Rham cohomology class $[df]$ of its differential.*

Lemma 2.2.1 ([56]) *If $\omega \in \Omega^1(M)$ is a closed 1-form, then the following relations are equivalent:*

i. $\omega = df$ where $f : M \to S^1$ is a C^∞ function;

ii. $[\omega]$ is integral.

Let M be a manifold and $\xi \in H^1(M, \mathbb{R})$. Denote by L_ξ the set of all closed 1-forms ω such that $[\omega] = \xi$. A closed 1-form ω on M is regular on $U \subset M$ (or U-regular) if every zero $p \in U$ of ω is non-degenerate.

Theorem 2.2.1 ([56]) *If M is a manifold, $\xi \in H^1(M, \mathbb{R})$ and $U \subset M$ is a compact subset of M then the set of all regular 1-forms on U is open and dense in L_ξ.*

Theorem 2.2.2 ([56]) *If M is a closed manifold then the set of all Morse functions $f : M \to \mathbb{R}$ is open and dense in the set of all C^∞ functions.*

2.3 Gradients of Morse functions

Let M be a smooth manifold and let v be a vector field on M. Any C^1 function $\gamma : I \to M$ defined on an open interval $I \subset \mathbb{R}$ such that

$$\gamma'(t) = v(\gamma(t)) \text{ for any } t \in I$$

is called an integral curve of v.

Let M be a manifold without boundary and let $f : M \to \mathbb{R}$ be a Morse function. Consider v a C^∞ vector field on M such that

$$f'(x)(v(x)) > 0, \text{ for any } x \notin C(f). \tag{2.3.1}$$

The function $\phi(x) = f'(x)(v(x))$ vanishes on $C(f)$ and it is strictly positive on $M \setminus C(f)$. Every point $p \in C(f)$ is a point of local minimum of ϕ and $\phi'(p) = 0$.

Definition 2.3.1 A vector field v is called ***f*-gradient** if the condition (2.3.1) holds and every point $p \in C(f)$ is a non-degenerate minimum of

$$\phi(x) = f'(x)(v(x)),$$

thus the second derivative $\phi''(p)$ is a non-degenerate bilinear form on $T_p(M)$.

Denote by $G(f)$ the set of all f-gradients.

Lemma 2.3.1 ([56]) *If v is an f-gradient and $p \in C(f)$ then $v(p) = 0$.*

Definition 2.3.2 Let M be a manifold without boundary and let $f : M \to \mathbb{R}$ be a Morse function. A C^∞ vector field v is called a **gradient-like vector field** for f if the following conditions are satisfied:

1. for every critical point p of f we have the relation $f'(x)(v(x)) > 0$;
2. for every critical point p of f, with index k, there exists a Morse chart $\psi : U \to V \subset \mathbb{R}^m$ for f at p such that

$$\psi_*(v)(x_1, \ldots, x_m) = (-x_1, \ldots, -x_k, x_{k+1}, \ldots, x_m).$$

Lemma 2.3.2 ([56]) *A gradient-like vector field for f is an f-gradient.*

Definition 2.3.3 The **riemannian gradient** of a function f with respect to a riemannian metric $\langle \cdot, \cdot \rangle$ on M is defined by

$$\langle grad\ f(x), h \rangle = f'(x)(h),$$

where $x \in M$ and $h \in T_x(M)$.

The following results were proved by A. Pajitnov in [56].

Proposition 2.3.1 *Any riemannian gradient is an f-gradient.*

Lemma 2.3.3 *If p is a critical point of f and v is a riemannian gradient of f with respect to a riemannian metric on M, then*

$$\langle v'(p)h, k \rangle = f''(p)(h, k)$$

where $h, k \in T_p(M)$.

Definition 2.3.4 We say that a vector field v is a **weak gradient** for f if for any $x \notin C(f)$, we have $f'(x)(v(x)) > 0$.

Lemma 2.3.4 *If v is a weak gradient for f and $p \in C(f)$ then $v(p) = 0$.*

2.4 Gradients of Morse forms

In this section we present some properties for gradients associated with closed 1-forms, ([32], [56]).

Let M be a closed smooth manifold and let $\langle \cdot, \cdot \rangle$ be a riemannian metric on M. Any 1-form ω on M determines the gradient vector field $grad(\omega)$ on M, defined by

$$\langle grad(\omega)_p, h \rangle = \omega(h),$$

for any vector $h \in T_p(M)$, called the riemannian gradient of ω.

In local coordinates we have:

$$\omega = \sum_{i=1}^{n} a_i(x) dx_i$$

and the riemannian metric is given by the coefficients

$$g_{ij}(x) = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle.$$

We can write:

$$grad(\omega) = \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j}, \text{ where } b_j(x) = \sum_{i=1}^{n} a_i(x) g^{ij}(x).$$

Thus $grad(\omega)$ is a smooth vector field and the zeros of $grad(\omega)$ coincide with the zeros of the form ω.

Definition 2.4.1 A smooth vector field X on M is called a **gradient-like vector field** for a closed 1-form ω if the following conditions hold:

1. the function $\omega(X) > 0$ outside the set of zeros of ω;
2. for any zero $p \in M$ of ω, there exists a neigborhood $U \subset M$ that contains p such that $\omega|_U = df$, where $f : U \to \mathbb{R}$ is a smooth function and the field $X|_U$ coincides with the gradient field $grad(f)$ with respect to some riemannian metric on U.

Let ω be a Morse form. Using the Morse lemma, one can construct a gradient-like vector field X for ω such that for any zero $p \in M$ of ω, there exists the local coordinate $x_1, \ldots, x_n$ in a neighborhood U of p such that $x_j(p) = 0$ and for any $j = \overline{1, n}$, the form $\omega|_U$ is equal with

$$\omega = -\sum_{i=1}^{r} x_i dx_i + \sum_{i=r+1}^{n} x_i dx_i.$$

The field $X|_U$ can be written as:

$$X = -\sum_{i=1}^{r} x_i \frac{\partial}{\partial x_i} + \sum_{i=r+1}^{n} x_i \frac{\partial}{\partial x_i}.$$

Note that r is the Morse index for the zero p, ([32], [49]).

A systematic presentation of closed 1-forms can be found in paper [32].

Chapter 3

The $\varphi_{\mathcal{F}}$-category of a pair of differentiable manifolds. Some important examples of $\varphi_{\mathcal{F}}$-category

We present in this chapter the $\varphi_{\mathcal{F}}$-category of a pair of differentiable manifolds, our own results regarding the circular φ-category of a differentiable manifold and the Morse-Smale characteristic of a differentiable manifold, following the papers of D. Andrica [3], [4], G. Rassias [62], B. Doubrovine [30], D. Andrica, D. Mangra, C. Pintea [15].

3.1 The $\varphi_{\mathcal{F}}$-category of a pair of manifolds

Let M be a smooth manifold without boundary.

For a smooth real function $f \in C^\infty(M)$, we denote by $\mu(f)$ the number of critical point of f. Clearly, we have $0 \leq \mu(f) \leq +\infty$.

Definition 3.1.1 The number $\varphi(M)$ defined by

$$\varphi(M) = \min\{\mu(f) : f \in C^\infty(M)\} \tag{3.1.1}$$

is called **the φ-category** of the manifold M.

This number was intensively studied by F. Takens [66] for the classes of closed manifolds (compact and without boundary).

In this case the following inequalities hold

$$cat(M) \leq \varphi(M) \leq m + 1, \tag{3.1.2}$$

where $cat(M)$ is the Lusternik-Schnirelmann category of M.

For two diffeomorphic manifolds M and N, $\varphi(M) = \varphi(N)$, thus $\varphi(M)$ is a differential invariant of the manifold.

If M and N are two smooth manifolds without boundary then we have the relation

$$\varphi(M \times N) \leq \min\{\dim(M) + \dim(N) + 1, \varphi(M)\varphi(N)\}. \tag{3.1.3}$$

In [4] (see also [3], page 144), D. Andrica presents a generalization of this notion.

Let M^m, N^n be two smooth manifolds without boundary. If $f \in C^\infty(M,N)$, we denote by $\mu(f) = |C(f)|$ the cardinal number of the critical set of f.

Let $\mathcal{F} \subseteq C^\infty(M,N)$ be a family of smooth functions $M \to N$.

Definition 3.1.2 The $\varphi_{\mathcal{F}}$**-category** of the pair of manifolds (M,N) is

$$\varphi_{\mathcal{F}}(M,N) = \min\{\mu(f) \ : \ f \in \mathcal{F}\}.$$

It is obvious that $0 \leq \varphi_{\mathcal{F}}(M,N) \leq +\infty$ and $\varphi_{\mathcal{F}}(M,N) = 0$ if and only if the family $\mathcal{F}$ contains immersions, submersions or local diffeomorphisms, according to $m < n$, $m > n$ or $m = n$, respectively.

If $\mathcal{F} = C^\infty(M,N)$, then $\varphi_{\mathcal{F}}(M,N)$ represents the φ-category of pair (M,N), simply denoted by $\varphi(M,N)$, and it was studied by D. Andrica and L. Funar in [10] and [11], C. Pintea in [57], [58], and D. Andrica and C. Pintea in [16], [17].

Definition 3.1.3 Let (M,N) and (M',N') be two diffeomorphic pairs of manifolds. The families $\mathcal{F} \subseteq C^\infty(M,N)$ and $\mathcal{F}' \subseteq C^\infty(M',N')$ are **related** (by diffeomorphisms) if

$$\mathrm{Dif}\, f(N,N')\ \mathcal{F}\ \mathrm{Dif}\, f(M',M) = \mathcal{F}',$$

where $\mathrm{Dif}\, f(N,N')$ and $\mathrm{Dif}\, f(M',M)$ represents the sets of all diffeomorphisms from N to N' and from M' to M respectively.

More precisely, this definition shows that if $\lambda \in \mathrm{Dif}\, f(M,M')$, $\psi \in \mathrm{Dif}, f(N,N')$, $f \in C^\infty(M,N)$, $f' \in C^\infty(M',N')$, satisfy $f' = \psi \circ f \circ \lambda^{-1}$, in other words the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\psi} \\ M' & \xrightarrow{\ f'\ } & N' \end{array}$$

then $f \in \mathcal{F}$ if and only if $f' \in \mathcal{F}'$.

Proposition 3.1.1 *If the pairs (M,N) and (M',N') are diffeomorphic and families $\mathcal{F} \subseteq C^\infty(M,N)$ and $\mathcal{F}' \subseteq C^\infty(M',N')$ are related, then*

$$\varphi_{\mathcal{F}}(M,N) = \varphi_{\mathcal{F}'}(M',N').$$

Proof. See [3].

The above result shows that if the hypothesis of the Proposition are satisfied, then $\varphi_{\mathcal{F}}(M,N)$ is a differential invariant of pair (M,N).

We present now some important particular cases for the family $\mathcal{F}$.

1. Consider the case when $N = \mathbb{R}$, the real line, and the family $\mathcal{F}$ is given by $\mathcal{F}(M) = C^\infty(M,\mathbb{R})$, the algebra of all smooth real functions defined on M. In this situation $\varphi_{\mathcal{F}}(M,\mathbb{R})$ represents the φ-category of M and it is denoted by $\varphi(M)$. As we have mentioned before, the invariant $\varphi(M)$ was first investigated by F. Takens. The effective

computation of $\varphi(M)$ is a difficult problem, ([4]). It is interesting to remark that we have not an example of closed manifold M^m such that $cat(M) < \varphi(M)$ and also the equality $cat(M) = \varphi(M)$ is proved only for some isolated classes of manifolds. To understand the difficulty of the problem if $cat(M) = \varphi(M)$ for every closed manifold let us look only to the following particular situation: $cat(M) = \varphi(M) = 2$. From $cat(M) = 2$ one obtains that M is a homotopic sphere. Taking into account the well-known Reeb's result, from the equality $\varphi(M) = 2$ is follows that M is a topological m-sphere. Therefore, the equality $cat(M) = \varphi(M) = 2$ is equivalent to the Poincaré conjecture. According to the fact that Poincaré conjecture was proved to be true, it follows that for any closed manifold with $cat(M) = 2$ he have $\varphi(M) = 2$.

2. Let G be a compact Lie group which acts freely on the manifolds M^m and N^n. Recall that the function $f : M \to N$ is invariant (G-equivariant) if for any $g \in G$ and $p \in M$ we have $f(gp) = f(p)(f(gp) = gf(p))$. Consider $\mathcal{F} = C^\infty_{G,I}(M, N)$ the family of all smooth G-invariant functions, and we obtain $\varphi_{\mathcal{F}}(M, N) = \varphi_{G,I}(M, N)$ the G-invariant φ-category of pair (M, N). In an analogous way we can define the G-equivariant φ-category of the pair (M, N), denoted by $\varphi_{G,E}(M, N)$.

3.2 The circular φ-category of a differentiable manifold

We define, following paper D. Andrica, D. Mangra, C. Pintea [15], the circular φ-category of a manifold M by

(3.2.1) $$\varphi_{S^1}(M) = \min\{\mu(f) : f \in C^\infty(M, S^1)\},$$

where S^1 is the unity circle. Notice that we have the inequality $\varphi_{S^1}(M) \leq \varphi(M)$.

Indeed, considering a function $f \in C^\infty(M)$ with $\mu(f) = \varphi(M)$, then the function $\widetilde{f} = \exp \circ f$, where $\exp : \mathbb{R} \to S^1$ is the universal covering of the circle S^1, satisfies $C(\widetilde{f}) = C(f)$. Therefore, we obtain

$$\varphi_{S^1}(M) \leq \mu(\widetilde{f}) = \mu(f) = \varphi(M).$$

According to this inequality and using the relation (3.1.2), it follows

(3.2.2) $$\varphi_{S^1}(M) \leq \varphi(M) \leq m + 1.$$

The main purpose of this section is to point out some classes of closed manifolds such that $\varphi_{S^1}(M) \leq \varphi(M)$.

3.2.1 Circle-valued functions and their converings

In this section we relay on the lifting properties of the covering maps to obtain information on the size of critical sets of circular functions. The properties of covering maps $p : E \to M$, we have in mind, are:

1. The homomorhism $p_* : \pi(E) \to \pi(M)$, induced by p at the level of fundamental groups, is one-to-one.

2 The cardinality of the inverse images $p^{-1}(y)$ is independent of $y \in M$ whenever E is connected and it is equal to the index $[\pi(M) : \mathrm{Im}(p_*)]$, where p_* is the group homomorphism $p_* : \pi(E) \to \pi(M)$ induced by the projection p at the level of fundamental groups.

3. For every subgroup H of $\pi(M)$, there exists a covering map $q : E_H \to M$ such that $q_*(\pi(E_H)) = H$.

4. A necessary and sufficient condition for a continuous map $f : X \to M$ to be lifted to a map $\overline{f} : X \to E$ is the inclusion $f_*(\pi(X)) \subseteq \overline{f}_*(\pi(X))$. In other words, there exists a map $\overline{f} : X \to E$ such that $p \circ \overline{f} = f$ if and only if the following relation holds $f_*(\pi(X)) \subseteq \overline{f_*}(\pi(X))$.

Recall that the circular functions on a compact manifold whose fundamental group is a torsion group are rather real valued functions as they all can be lifted to the real line through the exponential covering map $\exp : \mathbb{R} \to S^1$, due to the triviality of $\mathrm{Hom}(\pi(M), \mathbb{Z}) = 0$.

More precisely, we have:

Remark 3.2.1 Let M be a connected differential manifold. If $\mathrm{Hom}(\pi(M), \mathbb{Z}) = 0$, then every circular map $f : M \to S^1$ can be lifted to a map $\widetilde{f} : M \to \mathbb{R}$ through the exponential covering map $\exp : \mathbb{R} \to S^1$. Indeed, since $f_* = 0$ and $\exp_* = 0$, the existence of a lifting $\widetilde{f} : M \to \mathbb{R}$ which factors as $f = \exp \circ \widetilde{f}$ follows from property (4) in the above list. A class of manifolds for which $\mathrm{Hom}(\pi_1(M), \mathbb{Z}) = 0$ consists in those manifolds whose fundamental group is a torsion group.

The following results were proved in paper D. Andrica, D. Mangra, C. Pintea [15].

Corollary 3.2.1 *If $m, n \geq 2$ are natural numbers, then*

$$\varphi_{S^1}(S^n) = \varphi(S^n) = 2 \quad \text{and} \quad \varphi_{S^1}(\mathbb{RP}^n) = \varphi(\mathbb{RP}^n) = n + 1.$$

Besides the manifolds having torsion fundamental groups, the group homomorphisms of connected sums of such manifolds are still trivial whenever the terms of connected sums, alongside the connected sums themselves, have dimension three or higher.

Proposition 3.2.1 *If $(G_1, \cdot), \ldots, (G_r, \cdot), (H, \cdot)$ are group and*

$$f : G_1 * \ldots * G_r \to H$$

is a given group homomorphism, then

$$\mathrm{Im}(f) \subseteq \langle \mathrm{Im}(f \circ i_1) \cup \ldots \cup \mathrm{Im}(f \circ i_r) \rangle,$$

where $i_k : G_k \to G_1 * \ldots * G_r$, $k = 1, \ldots, r$ are natural embeddings.

In particular, $\mathrm{Hom}(G_1 * \ldots * G_r, H) = 0$ whenever $G_1, \ldots, G_r$ are torsion groups and H is torsion free.

Corollary 3.2.2 *If $(G_1, \cdot), \ldots, (G_r, \cdot)$ are groups and $f : G_1 * \ldots * G_r \to \mathbb{Z}$ is a given group homomorphism, then $\mathrm{Im}(f) = \gcd(m_{i_1}, \ldots, m_{i_s})\mathbb{Z}$, where $\mathrm{Im}(f \circ i_j) = m_j\mathbb{Z}$, for $j = 1, \ldots, r$ and $m_{i_1}, \ldots, _{i_s} \neq 0$.*

If $m_1 = \ldots = m_r = 0$, $f \circ i_1 = \ldots = f \circ i_r = 0$, then $\mathrm{Im}(f) = 0$.

In particular, $\mathrm{Hom}(G_1 * \ldots * G_r, \mathbb{Z}) = 0$ whenever $G_1, \ldots, G_r$ are torsion groups.

Corollary 3.2.3 *If $M_1^n, \ldots, M_r^n$, $n \geq 3$ are connected manifolds such that $\pi(M_1), \ldots, \pi(M_r)$ are torsion groups, then $\mathrm{Hom}(\pi(M_1 \# \ldots \# M_r), \mathbb{Z}) = 0$.*

In particular, if each of $M_1^n, \ldots, M_r^n$, $n \geq 3$ is either a real projective space or a lens space, then $Hom(\pi(M_1 \# \ldots \# M_r), \mathbb{Z}) = 0$.

Remark 3.2.2 The requirement $n \geq 3$ in Corollary 3.2.3 is essential as the fundamental groups of the compact orientable surfaces $\Sigma_g = T^2 \# \ldots \# T^2$ admit, in contrast with the situation of torsion fundamental groups for the terms of the connected sums in Proposition 3.2.1, circular functions inducing onto group homomorphisms at the level of fundamental groups.

Theorem 3.2.1 *Let M be a compact differential manifold with abelian fundamental group. Every continuous circular function $f : M \to S^1$ which cannot be lifted to any real valued function via the exponential covering $\exp : \mathbb{R} \to S^1$, can be covered by a circular function $\overline{f} : \overline{M} \to S^1$ such that $\pi(\overline{M})$ is torsion free and the induced group homomorphism $\overline{f_*} : \pi(\overline{M}) \to \pi(S^1) = \mathbb{Z}$ is onto.*

More precisely, there are some covering maps $\overline{p} : \overline{M} \to M$ and $q : S^1 \to S^1$ which make the following diagrams commutative.

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\overline{f}} & S^1 \\ \Big\downarrow \overline{p} & & \Big\downarrow q \\ M & \xrightarrow{f} & S^1 \end{array} \qquad\qquad \begin{array}{ccc} \pi(\overline{M}) & \xrightarrow{\overline{f_*}} & \pi(S^1) = \mathbb{Z} \\ \Big\downarrow \overline{p_*} & & \Big\downarrow q_* \\ \pi(M) & \xrightarrow{f_*} & \pi(S^1) = \mathbb{Z} \end{array}$$

3.2.2 Manifolds satisfying $\varphi_{S^1}(M) = \varphi(M)$

From relations (3.2.2) we have the relation $\varphi_{S^1}(M) \leq \varphi(M)$, and from the following example we can see that the inequality could be strict.

Consider the m-dimensional torus $T^m = \underbrace{S^1 \times \ldots \times S^1}_{m \text{ times}}$ and according to [3] (Example 3.6.16) we have $\varphi(T^m) = m + 1$. On the other hand, the projection $T^m \to S^1$ is a trivial differentiable fibration, hence it has no critical points, implying $\varphi_{S^1}(T^m) = 0$.

This example can be incorporated in the following general observation.

Remark 3.2.3 For a closed manifold M we have $\varphi_{S^1}(M) = 0$ if and only if there is a differentiable fibration $M \to S^1$. The existence of a differentiable fibration $M \to S^1$ ensures the equality $\varphi_{S^1}(M) = 0$, as the fibration itself has no critical points at all. Conversely, the equality $\varphi_{S^1}(M) = 0$ ensures the existence of a submersion $M \to S^1$, which is also proper, as its inverse images of the compact sets in S^1 are obviously compact. Thus, by the Ehresmann's fibration theorem ([31] or [29] page 15) one can conclude that our submersion is actually a locally trivial fibration.

Proposition 3.2.2 ([15]) *Let M be a connected differential manifold. If $\mathrm{Hom}(\pi(M), \mathbb{Z}) = 0$, then $\varphi_{S^1}(M) = \varphi(M)$. In particular $\varphi_{S^1}(M) = \varphi(M)$ whenever M is connected and simply-connected.*

Corollary 3.2.4 ([15]) *If $m \geq 2$, then $\varphi_{S^1}(S^m) = \varphi(S^m) = 2$ and $\varphi_{S^1}(\mathbb{RP}^m) = \varphi(\mathbb{RP}^m) = m+1$, where S^m denotes the m-dimensional sphere and $\mathbb{RP}^m$ the m-dimensional real projective space.*

Corollary 3.2.5 ([15]) *If* $M_1^n, \dots, M_r^n$*,* $m \geq 3$*, are connected manifolds such that* $\pi(M_1), \dots, \pi(M_r)$ *are torsion groups, then*

$$\varphi_{S^1}(M_1 \# \dots \# M_r) = \varphi(M_1 \# \dots \# M_r).$$

In particular $\varphi_{S^1}(r\mathbb{RP}^n) = \varphi(r\mathbb{RP}^n)$*, where* $r\mathbb{RP}^n$ *stands for the connected sum* $\mathbb{RP}^n \# \dots \# \mathbb{RP}^n$ *of* r *copies of* $\mathbb{RP}^n$*.*

Corollary 3.2.6 ([15]) *If* $k, l, m_1, \dots, m_k \geq 2$ *are integers, then the following relations hold:*

1. $\varphi_{S^1}(S^{m_1} \times \dots \times S^{m_k}) = \varphi(S^{m_1} \times \dots \times S^{m_k}) = k + 1$.

2. $\varphi_{S^1}(\mathbb{RP}^{m_1} \times \dots \times \mathbb{RP}^{m_k}) = \varphi(\mathbb{RP}^{m_1} \times \dots \times \mathbb{RP}^{m_k}) \leq m_1 + m_2 + \dots + m_k + 1$.

3. $\varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = 5$*, where* $L(r,s)$ *stands for the lens space of dimensions 3 of type* (r,s)*.*

4. $\varphi_{S^1}(\mathbb{RP}^k \times S^l) = \varphi(\mathbb{RP}^k \times S^l) \leq k + 2$.

The following result is mentioned in the monograph [28] at page 221.

Lemma 3.2.1 *If* M *and* N *are closed manifolds, then the following inequality holds*

$$\varphi(M \# N) \leq \max\{\varphi(M), \varphi(N)\}.$$

In particular $\varphi(X \# X) \leq \varphi(X)$ *for every manifold* X*.*

Corollary 3.2.7 ([15]) *Let* $\Sigma_g = \underbrace{T^2 \# T^2 \# \dots \# T^2}_{g \text{ times}}$ *be the closed orientable surface of genus* g *and let* $P_g = \underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{g+1 \text{ times}}$ *the closed non-orientable surface of genus* g*. Then*

1. $\varphi_{S^1}(\Sigma_g) \leq \varphi(\Sigma_g) = 3,\ g \geq 1$.

2. $\varphi_{S^1}(P_g) \leq \varphi(P_g) = 3,\ g \geq 0$.

Corollary 3.2.8 ([15]) *If* $k, l \geq 2$ *are integers, then*

$$\varphi_{S^1}((S^k \times S^l) \# \dots \# (S^k \times S^l)) = \varphi((S^k \times S^l) \# \dots \# (S^k \times S^l)) = 3.$$

Open problem. Characterize all closed manifolds M^m with the property $\varphi_{S^1}(M) = 1$.

3.3 The real Morse-Smale characteristic of a differentiable manifold

Consider $N = \mathbb{R}$ and $\mathcal{F} = \mathcal{F}_m(M) \subset C^\infty(M, \mathbb{R})$ the set of all Morse functions defined on M. In this case one obtains $\varphi_{\mathcal{F}}(M, \mathbb{R}) = \gamma(M)$, the Morse-Smale characteristic of manifold M, an important invariant of M, intensively studied by many authors [62], [4].

An important case when the Morse-Smale characteristic can be computed in terms of the topology of M is given by the situation when the manifold M is simply-connected of dimension greater than five. This property was proved in the celebrated paper of S. Smale ([65]). Efforts have been made to generalize Smale's result to the case when M is not simply-connected. For example, V.V. Sharko proved that still it is possible to compute the Morse-Smale characteristic of manifold M when $\pi_1(M) = \mathbb{Z}$, ([64]). But a complete answer for general M is not known.

We define the numbers $\gamma_i(M)$, $i = 0, \ldots, m$, by

$$\gamma_i(M) = \min\{\mu_i(f) \ : \ f \in \mathcal{F}_m(M)\}.$$

Taking into account the monograph of B. Doubrovine [30] it follws

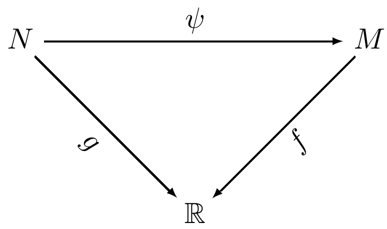
$$\gamma_0(M) = \gamma_m(M) = 1.$$

It is easy to see that

$$\gamma(M) \geq \sum_{i=0}^{m} \gamma_i(M)$$

The numbers $\gamma(M)$ and $\gamma_i(M)$ are differential invariants of M.

Let N be a smooth manifold, let $\psi : M \to N$ be a diffeormorphism and let $f : M \to \mathbb{R}$, $g : N \to \mathbb{R}$ be two smooth functions such that the following diagram is commutative



thus $g = f \circ \psi$.

Lemma 3.3.1 *The following relation holds:* $C(f) = \psi(C(g))$.

Lemma 3.3.2 *With the above notations, if* $f \in \mathcal{F}_m(M)$*, then* $g \in \mathcal{F}_m(N)$ *and the corresponding critical points via the diffeomorphism* ψ *have the same Morse index.*

Theorem 3.3.1 *If the manifolds* M *and* N *are diffeomorphic, then*

$$\gamma(M) = \gamma(N), \ \ \gamma_i(M) = \gamma_i(N), \ \ i = 0, \ldots, m,$$

in other words the numbers $\gamma(M)$ *and* $\gamma_i(M)$ *are differential invariants of the manifold.*

Proof. See [3].

Theorem 3.3.2 *Let* M^m *and* N^n *be two differentiable manifolds without boundary. The following relations hold:*

(i) $\gamma_i(M) = \gamma_{m-i}(M)$*,* $i = 0, \ldots, m$ *(symmetry);*

(ii) $\gamma(M \times N) \leq \gamma(M)\gamma(N)$ *(submultiplicity);*

(iii) $\gamma_i(M \times N) \leq \sum_{j+k=i} \gamma_j(M)\gamma_k(N), \ \ i = 0, \ldots, m+n.$

Proof. See [3].

In the paper of G. Rassias [63] one shows that $\gamma(M) = 0$ for any open smooth manifold.

3.4 The computation of the real Morse-Smale characteristic

Let M^m be a smooth compact m-dimensional manifold without boundary, $\partial M = 0$. It is well-known that $\mathcal{F}_m(M) \neq \emptyset$, that is there exists a Morse function defined on M.

Let $H_k(M;F)$, $k = \overline{0,m}$, be the singular homology groups with the coefficients in the field F and $\beta_k(M;F) = rank H_k(M;F) = \dim_F H_k(M;F)$, $k = \overline{0,m}$, the Betti numbers with respect to F. If $f \in \mathcal{F}_m(M)$, then the following relations hold $\mu_k(f) \geq \beta_k(M;F)$, $k = \overline{0,m}$ (weak Morse inequalities).

Definition 3.4.1 The Morse function $f \in \mathcal{F}_m(M)$ is **exact** (or minimal) if

$$\mu_k(f) = \gamma_k(M),\ \ k = \overline{0,m}.$$

Definition 3.4.2 The Morse function $f \in \mathcal{F}_m(M)$ is F**-perfect** if

$$\mu_k(f) = \beta_k(M;F),\ \ k = \overline{0,m}.$$

Taking into account the weak Morse inequalities and the definition of the Morse-Smale characteristic, one obtains that for any Morse function f on M and for any field F the following relations hold:

$$\mu_k(f) \geq \gamma_k(M) \geq \beta_k(M;F),\ \ k = \overline{0,m}.$$

It follows that any F-perfect Morse function on M is exact.

Theorem 3.4.1 ([3]) *The manifold M has F-perfect Morse functions if and only if*

$$\gamma(M) = \beta(M;F),$$

where $\beta(M;F) = \sum_{k=0}^{m} \beta_k(M;F)$ *is the total Betti number of the manifold M with respect to F.*

Because the manifold M^m is compact it follows that M has the topology type of a finite CW-complex ([25]), thus the singular homology groups $H_k(M;\mathbb{Z})$, $k = \overline{0,m}$, are finitely generated ([34]). For $k \in \mathbb{Z}$, one obtains the relation:

$$H_k(M;\mathbb{Z}) \simeq (\underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\beta_k\ times}) \oplus (\mathbb{Z}_{n_{k1}} \oplus \ldots \oplus \mathbb{Z}_{n_{kb(k)}})$$

where $\beta_k = \beta_k(M;\mathbb{Z})$, $k = \overline{0,m}$, are the Betti numbers of the manifold M with respect to the group $(\mathbb{Z},+)$, thus $\beta_k(M;\mathbb{Z}) = rank H_k(M;\mathbb{Z})$. It is well-known that $H_0(M;\mathbb{Z}) \simeq \mathbb{Z}$ and

$$H_m(M;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} \text{ if } M \text{ is orientable} \\ \{0\} \text{ otherwise} \end{cases}$$

thus $b(0) = b(m) = 0$.

Theorem 3.4.2 ([3]) *If M^m is a simply-connected compact manifold without boundary ($\partial M = \emptyset$) and $m \geq 6$, then the following relations hold:*

(i) $\gamma_k(M) = \beta_k(M;\mathbb{Z}) + b(k) + b(k-1),\ k \in \mathbb{Z}$;

(ii) $\gamma(M) = \beta(M;\mathbb{Z}) + 2\sum_{k=1}^{m-1} b(k)$,

where $\beta(M;\mathbb{Z}) = \sum_{k=0}^{m} \beta_k(M;\mathbb{Z})$ *is the total Betti number of the manifold M with respect to the group* $(\mathbb{Z},+)$.

Corollary 3.4.1 *Let M^m be a compact simply-connected manifold, without boundary with $m \geq 6$. Then M has $\mathbb{Q}$-perfect Morse functions if and only if $H_k(M;\mathbb{Z})$ has no torsion,* $k = \overline{0,m}$.

Example 3.4.1 1) The sphere S^m is a simply-connected compact manifold for $m \geq 2$. The singular homology of S^m is given by

$$H_k(S^m;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, m \\ \{0\} & \text{otherwise} \end{cases}$$

Taking into account Theorem 3.4.2, it follows that

$$\gamma_0(S^m) = \gamma_m(S^m) = 1,\ \gamma_k(S^m) = 0,\ 1 \leq k \leq m-1 \text{ and } \gamma(S^m) = 2 \text{ for } m \geq 6.$$

In fact these results are true for $m \geq 1$, since it is easy to see that the function $(x^1,\ldots,x^{m+1}) \longmapsto x^{m+1}$ is a $\mathbb{Q}$-perfect Morse function on S^m.

2) The complex projective space $P\mathbb{C}^m$ is a simply-connected compact manifold. The singular homology of $P\mathbb{C}^m$ is given by

$$H_k(P\mathbb{C}^m;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, 4, \ldots, 2m \\ \{0\} & \text{otherwise} \end{cases}$$

For $m \geq 6$ we have the relations $\gamma_{2i}(P\mathbb{C}^m) = 1$, $i = \overline{0,m}$, $\gamma_{2j-1}(P\mathbb{C}^m) = 0$, $j = \overline{1,m}$ and $\gamma(P\mathbb{C}^m) = m+1$. Moreover, using a direct method, N.H. Kuiper showed that $\gamma(P\mathbb{C}^m) = m+1$ for $m \geq 1$. In G.M. Rassias [62] it is mentioned that for m even the number $\gamma(P\mathbb{C}^m)$ is odd but it is given the value of $\gamma(P\mathbb{C}^m)$.

3) The singular homology of the quaternionic projective space PH^m is given by

$$H_k(PH^m;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, 4, 8, \ldots, 4m \\ \{0\} & \text{otherwise} \end{cases}$$

For $m \geq 6$ one obtains $\gamma_{4i}(PH^m) = 1$, $i = \overline{0,m}$, $\gamma_j(PH^m) = 0$ if $j \not\equiv 0 \pmod 4$ and $\gamma(PH^m) = m+1$.

An important problem which appears naturally is to get the manifolds M and N that satisfy the equalities in Theorem 3.3.2 (i), (ii) ([41], [62]). A sufficient condition is given by D. Andrica in [7].

Theorem 3.4.3 *If M^m and N^n are two compact manifolds without boundary ($\partial M = \partial N = \emptyset$), which have F-perfect Morse functions then*

$$\gamma(M \times N) = \gamma(M)\gamma(N) \text{ and } \gamma_i(M \times N) = \sum_{j+k=i} \gamma_j(M)\gamma_k(N),\ i = \overline{0, m+n}.$$

Corollary 3.4.2 *Let M^m and N^n be two simply-connected compact manifolds without boundary ($\partial M = \partial N = \emptyset$), $m, n \geq 6$. If the singular homology groups $H_k(M;\mathbb{Z})$, $H_j(N;\mathbb{Z})$, $k = \overline{0,m}$, $j = \overline{0,n}$ are torsion-free then*

$$\gamma(M \times N) = \gamma(M)\gamma(N) \text{ and } \gamma_i(M \times N) = \sum_{j+k=i} \gamma_j(M)\gamma_k(N),\ i = \overline{0, m+n}.$$

Example 3.4.2 Taking into account the equality $\gamma(S^m) = 2$, we have

$$\gamma(S^{m_1} \times \ldots \times S^{m_k}) = 2^k.$$

If $T^k = \underbrace{S^1 \times \ldots \times S^1}_{k\ times}$ is the k-dimensional, then $\gamma(T^k) = 2^k$.

Using the second equality from Corollary 3.4.2 we have

$$\gamma_i(T^k) = \binom{k}{i},\ i = \overline{0,m}.$$

We can obtain an extension of Theorem 3.4.2 for compact manifolds, not necessary simply-connected.

Let M^m be a compact manifold without boundary, $m \geq 6$ and let $p : \widetilde{M} \to M$ be an universal covering manifold of M. V.V. Sharko ([64]), showed that if $\pi_1(M) \simeq \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ (s times), $s \geq 0$, then there exists an exact Morse function f, defined on M such that

$$\mu_k(f) = \sum_{j=0}^{s} \binom{s}{j} \beta_{k+j-s}(\widetilde{M};\mathbb{Z}) + \sum_{i=0}^{s+1} \binom{s+1}{i} b(b_i - s - 1)$$

for $k \in \mathbb{Z}$.

Using these relations, one obtains the following result of D. Andrica, ([5], [6]).

Theorem 3.4.4 *If M^m is a compact manifold without boundary with $m \geq 6$ and*

$$\pi_1(M) \simeq \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}\ (s\ times),\ s \geq 0,$$

then:

(i) $\gamma_k(M) = \sum_{j=0}^{s} \binom{s}{j} \beta_{k+j-s}(\widetilde{M};\mathbb{Z}) + \sum_{i=0}^{s+1} \binom{s+1}{i} b(k+i-s-1),\ k = \overline{0,m}$

(ii) $\gamma(M) = \sum_{k=0}^{m} \left(\sum_{j=0}^{s} \binom{s}{j} \beta_{k+j-s}(\widetilde{M};\mathbb{Z}) \right) + \sum_{k=0}^{m} \left(\sum_{i=0}^{s+1} \binom{s+1}{i} b(k_i - s - 1) \right)$

where $p : \widetilde{M} \to M$ is any universal covering of M.

We will present another result obtained by D. Andrica ([5], [6]), an upper bound of the Lusternik-Schnirelmann category of M in terms of $\beta_k(\widetilde{M};\mathbb{Z})$, $b(k)$, $k = \overline{0,m}$.

Corollary 3.4.3 *Let M^m be a compact manifold without boundary with $m \geq 6$ and*

$$\pi_1(M) \simeq \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \ (s \ times), \ \ s \geq 0.$$

Then

$$cat(M) \leq \sum_{k=0}^{m} \left(\sum_{j=0}^{s} \binom{s}{j} \beta_{k+j-s}(\widetilde{M};\mathbb{Z}) \right) + \sum_{k=0}^{m} \left(\sum_{i=0}^{s+1} \binom{s+1}{i} b(k+i-s-1) \right),$$

where $p : \widetilde{M} \to M$ is any universal covering of M.

Let $p \geq 2$ be a prime number. Taking into account the relation

$$H_k(M;\mathbb{Z}) \simeq (\underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\beta_k \text{ times}}) \oplus (\mathbb{Z}_{n_{k1}} \oplus \ldots \oplus \mathbb{Z}_{n_{kb(k)}}),$$

we denote

$$d(M;p) = \text{card}\{n_{kj}, \ j = \overline{1, b(k)}, \ k = \overline{0, m} : \ p|_{n_{k_j}}\}.$$

It is obvious that $d(M;p) \leq \sum_{k=1}^{m-1} b(k)$.

The following result represents a necessary and sufficient condition in terms of $\gamma(M)$, $\beta(M;\mathbb{Z})$ and $d(M;p)$, for the existence of $\mathbb{Z}_p$-perfect Morse functions on the manifold M and it was obtained by D. Andrica in [8], [19].

Theorem 3.4.5 *The manifold M admits $\mathbb{Z}_p$-perfect Morse functions if and only if the following equality holds*

$$\gamma(M) = \beta(M;\mathbb{Z}) + 2d(M;p).$$

Corollary 3.4.4 *Let $p, q \geq 2$ be prime numbers. The manifold M has simultaneously $\mathbb{Z}_p$ and $\mathbb{Z}_p$-perfect Morse functions if and only if the following relations hold*

$$\gamma(M) = \beta(M;\mathbb{Z}) + 2d(M;p) \ \ and \ d(M;p) = d(M;q).$$

Corollary 3.4.5 *If M^m is a simply-connected compact manifold without boundary with $m \geq 6$ and the homology groups $H_k(M;\mathbb{Z})$, $k = \overline{0, m}$ are without torsion then the manifold M has $\mathbb{Z}_p$-perfect Morse functions for any prime number $p \geq 2$.*

Remark 3.4.1 The results of Corollary 3.4.5 and Corollary 3.4.1 can be extended. If we replace the condition that the manifold M is simply-connected with the condition

$$\pi_1(M) \simeq \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \ (s \text{ times}),$$

where $s \geq 0$ is an arbitrary integer and $\pi_1(M)$ represents the fundamental group of M. In this case we use the result given in V.V. Sharko ([64]) and the explicit formula for the Morse-Smale characteristic obtained in Theorem 3.4.4. (ii).

Theorem 3.4.6 *Let M^m be a compact manifold without boundary. If the integers m and $\beta(M;\mathbb{Z})$ are odd, then M has $\mathbb{Z}_p$-perfect Morse functions, for any prime number $p \geq 2$.*

Theorem 3.4.7 *(i) The m-dimensional sphere S^m admits $\mathbb{Q}$-perfect Morse functions.*
(ii) For any prime number $p \geq 2$, the sphere S^m has $\mathbb{Z}_p$-perfect Morse functions.

Let $P\mathbb{R}^m$ be the m-dimensional real projective space. It is well-known that $P\mathbb{R}^m$ is a compact differentiable smooth manifold without boundary and the homology of $P\mathbb{R}^m$ is given by

$$H_k(P\mathbb{R}^m;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \mathbb{Z}_2 & \text{if k is odd and } 0<k<m \\ \mathbb{Z} & \text{if k is odd and } k=m \\ \{0\} & \text{otherwise} \end{cases}$$

From this relation one obtains $\beta(P\mathbb{R}^m;\mathbb{Z}) = 1$ if m is even and $\beta(P\mathbb{R}^m;\mathbb{Z}) = 2$ if m is odd.

For a prime number $p \geq 2$ it follows

$$d(P\mathbb{R}^m;p) = \begin{cases} m/2 & \text{if } p=2 \text{ and m is even} \\ (m-1)/2 & \text{if } p=2 \text{ and m is odd} \\ 0 & \text{if } p \geq 3 \end{cases}$$

It is known that the Morse-Smale characteristic of $P\mathbb{R}^m$ is $\gamma(P\mathbb{R}^m) = m+1$ ([41]).

Theorem 3.4.8 *(i) $P\mathbb{R}^m$ has not $\mathbb{Q}$-perfect Morse functions.*
(ii) $P\mathbb{R}^m$ has $\mathbb{Z}_2$-perfect Morse functions.
(iii) For any prime $p \geq 3$, $P\mathbb{R}^m$ has not $\mathbb{Z}_p$-perfect Morse functions.

Notice that in the paper of N. H. Kuiper [41], is constructed a $\mathbb{Z}_2$-perfect Morse function on $P\mathbb{R}^m$.

Let T^2 be the 2-dimensional torus. We define the smooth, compact, connected, orientable surface of genus $g \geq 0$, by

$$T_g = \underbrace{T^2\#\ldots\#T^2}_{g \text{ times}},$$

thus T_g is the connected sum of g copies of T^2. If $g=0$ then $T_g = S^2$, the 2-dimensional sphere.

Consider P_g the smooth, compact, connected, non-orientable surface of genus $g \geq 0$, defined by

$$P_g = \underbrace{\mathbb{RP}^2 \,\#\, \mathbb{RP}^2 \,\#\, \ldots \,\#\, \mathbb{RP}^2}_{g+1 \text{ times}},$$

where $\mathbb{RP}^2$ is the real projective plane.

It is well-known that if M is a smooth, compact, connected surface without boundary, then M is diffeomorphic to T_g if it is orientable and M is diffeomorphic to P_g if it is non-orientable, for some values of g, ([37]).

The following result is a consequence of the exact Mayer-Vietoris sequence in the de Rham cohomology, ([36]):

$$\chi(T_g) = 2-2g,\ \chi(P_g) = 1-g.$$

N. H. Kuiper [42] proved the following relation between the Morse-Smale characteristic and the Euler-Poincaré characteristic of a smooth, connected, compact surface M, without boundary:

$$\gamma(M) = 4 - \chi(M).$$

Using the above relations one obtains

$$\gamma(T_g) = 2 + 2g, \ \gamma(P_g) = 3 + g.$$

Theorem 3.4.9 *(i) T_g has $\mathbb{Q}$-perfect Morse functions.*
(ii) For any prime number $p \geq 2$, T_g has $\mathbb{Z}_p$-perfect Morse functions.

Theorem 3.4.10 *(i) P_g has not $\mathbb{Q}$-perfect Morse functions.*
(ii) For any prime number $p \geq 3$, P_g has not $\mathbb{Z}_p$-perfect Morse functions.
(iii) P_g has $\mathbb{Z}_2$-perfect Morse functions.

In paper [26] are presented the following inequalities:

$$cat(M) \leq C(M) \leq \beta(M) \leq \gamma(M) \leq m + 1,$$

where $cat(M)$ is the Lusternik-Schnirelmann category of M (the minimal number of closed contractible sets which cover M), $C(M)$ is the minimal number of open disks necessary to cover M and

$$\beta(M) = \beta(M; \mathbb{Z}) = \sum_{k=0}^{m} H_k(M; \mathbb{Z}).$$

The purpose of the following result obtained by D. Andrica and M. Todea in [20], is to show that the inequality $\gamma(M) \leq m + 1$ from the above relation is not valid for every closed manifold M.

Let $\mathcal{M}_m$ be the set of all smooth closed m-dimensional manifolds.

Theorem 3.4.11 *The relation*

$$\sup\{\gamma(M) : \ M \in \mathcal{M}_m\} = \infty$$

holds for $m \geq 2$.

Remark 3.4.2 1) The relation $\sup\{\gamma(M) : \ M \in \mathcal{M}_m\} = \infty$ is not valid for $m = 1$. Taking into account the classification theorem of 1-dimensional closed manifolds it follows that a such manifold is diffeomorphic with S^1, thus $\gamma(M) = 2$.

2) If for a closed m-dimensional manifold $M \in \mathcal{M}_m$, one defines the number

$$\varphi(M) = \min\{\mu(f) : \ f \in C^{\infty}(M)\}$$

one obtain the φ-category of M. We have the inequality:

$$cat(M) \leq \varphi(M) \leq m + 1,$$

thus the result contained in the above theorem remains not true if one replace $\gamma(m)$ by $\varphi(M)$. In this case we have

$$\sup\{\varphi(M) : \ M \in \mathcal{M}_m\} = m + 1,$$

since for instance $\varphi(P\mathbb{R}^m) = m + 1$.

3) The relation $\sup\{\gamma(M) : M \in \mathcal{M}_m\} = \infty$ shows that there is not a positive constant $c_m > 0$ such that $\gamma(M) \leq c_m$, for any closed manifold $M \in \mathcal{M}_m$.

On the other hand, the result given by M. Gromov in [38], asserts the existence of a positive constant $c_m > 0$, such that $\beta(M; F) \leq c_m$, for any compact m-dimensional manifold M^m of positive curvature, where

$$\beta(M; F) = \sum_{i=0}^{m} \beta_i(M; F)$$

is the sum of the Betti numbers with respect to a field F.

Let M^m be a smooth, closed, differentiable manifold. We consider $\pi : \widetilde{M} \to M$ a k-covering of M, where $k \geq 2$. If $f \in \mathcal{F}_m(M)$ is a Morse function on M, let $h : \widetilde{M} \to \mathbb{R}$, be a function defined by $h = f \circ \pi$. Since π is a local diffeomorphism, it follows that

$$h \in \mathcal{F}_m(\widetilde{M}),\ C(f) = \pi(C(h)),$$

thus $\mu(h) = k\mu(f)$. Then for any Morse function $f \in \mathcal{F}_m(M)$ the inequality $\gamma(\widetilde{M}) \leq k\mu(f)$ holds. Taking into account the definition of the Morse-Smale characteristic, it results that

$$\gamma(\widetilde{M}) \leq k\gamma(M).$$

Theorem 3.4.12 ([3]) *Let M^m be a smooth, closed manifold and let $\pi : \widetilde{M} \to M$ be a k-covering of M, $k \geq 2$. Then the following inequality holds:*

$$\gamma(\widetilde{M}) \leq k\gamma(M) - 4(k - 1).$$

M. Gromov posed the following question: Let $\widetilde{M_k}$, $k \in \mathbb{N}$ be a sequence of manifolds, such that each $\widetilde{M_k}$ is an a_k-fold cover of M, where $a_k \to \infty$ as $k \to \infty$. What are the asymptotic properties of the sequence $\gamma(\widetilde{M_k})$ as $k \to \infty$?

Using the relation $\gamma(\widetilde{M}) \leq k\gamma(M) - 4(k - 1)$, one obtain

$$\gamma(\widetilde{M_k}) \leq a_k\gamma(M) - 4(a_k - 1).$$

It follows immediately, after a simple computation, a partial asymptotic estimation for the above question:

$$\lim_{k\to\infty} \sup \frac{\gamma(\widetilde{M_k})}{a_k} \leq \gamma(M) - 4.$$

Generally, the inequality proved in Theorem 3.4.12 is strict. Let $m \geq 3$ and the 2-covering $\pi : S^m \to P^m(\mathbb{R})$, where S^m is the m-dimensional sphere and $P^m(\mathbb{R})$ is the real projective space of dimension m. It is easy to show that $\gamma(S^m) = 2$ and $\gamma(P^m(\mathbb{R})) = m+1$. The inequality from Theorem 3.4.12 becomes strict, $2 < 2(m+1)-4$, because we considered $m \geq 3$.

Theorem 3.4.13 ([9]) *If M^2 is a smooth closed surface, orientable or not, then the relation*

$$\gamma(\widetilde{M}) = k\gamma(M) - 4(k - 1)$$

holds.

Corollary 3.4.6 *Let M^m be a smooth closed manifold and let G be a finite group which acts freely on M. Then:*

(i) $\gamma(M/G) \geq \dfrac{1}{|G|}(\gamma(M) + 4(|G| - 1))$

(ii) If M^2 is a closed smooth surface then

$$\gamma(M/G) = \frac{1}{|G|}(\gamma(M) + 4(|G| - 1)).$$

Chapter 4

The circular Morse-Smale characteristic of a differentiable manifold

4.1 Definition and some general properties

We present in this section our results regarding the Morse-Smale characteristic for circle-valued Morse functions, notion that was first introduced and studied by D. Andrica and D. Mangra in papers [12], [13].

Our goal was to investigate in which conditions some properties of the Morse-Smale characteristic are kept for circle-valued functions.

We define the Morse-Smale characteristic of a manifold M for circle-valued Morse functions.

Consider $N = S^1$ and the family of circle-valued Morse functions defined on M, $\mathcal{F} = \mathcal{F}_m(M, S^1) \subseteq C^\infty(M, S^1)$, ([18]).

In this case we denote $\varphi_{\mathcal{F}}(M, S^1)$ by $\gamma_{S^1}(M)$ and we call it **the circular Morse-Smale characteristic** of the manifold M.

This notion was introduced by D. Andrica and D. Mangra in paper [12].

From the definition it follows that

$$\gamma_{S^1}(M) = \min\{\mu(f) \ : \ f \in \mathcal{F}_m(M, S^1)\}.$$

We can define, in an analogous way the numbers $\gamma_{S^1}^{(i)}(M)$, for $i = 0, \ldots, m$, by

$$\gamma_{S^1}^{(i)}(M) = \min\{\mu_i(f) \ : \ f \in \mathcal{F}_m(M, S^1)\}.$$

From the relation $\mu(f) = \mu_0(f) + \ldots + \mu_m(f)$, it follows that for any $f \in \mathcal{F}_m(M, S^1)$ we have

$$\mu_(f) \geq \gamma_{S^1}^{(0)}(M) + \ldots + \gamma_{S^1}^{(m)}(M),$$

thus the following inequality holds:

$$\gamma_{S^1}(M) \geq \sum_{i=0}^{m} \gamma_{S^1}^{(i)}(M).$$

We will show next that the numbers $\gamma_{S^1}(M)$ and $\gamma_{S^1}^{(i)}(M)$ are differential invariants of the manifold M, $i = 0, \ldots, m$.

Let N be a smooth manifold, let $\varphi : M \to N$ be a diffeomorphism and let $f : M \to S^1$, $g : N \to S^1$ be two circle-valued smooth functions such that $g = f \circ \varphi$.

Clearly, we have the relation $C(f) = \varphi(C(g))$.

Proposition 4.1.1 *If $f \in \mathcal{F}_m(M, S^1)$, $\varphi : M \to N$ is a diffeomorphism, then*

$$g \in \mathcal{F}_m(N, S^1)$$

and the critical points $p \in C(g)$ and $\varphi(p) \in C(f)$ have the same Morse index.

The following results were proved in our paper [12].

Theorem 4.1.1 *If the manifolds M and N are diffeomorphic, then*

$$\gamma_{S^1}(M) = \gamma_{S^1}(N) \text{ and } \gamma_{S^1}^{(i)}(M) = \gamma_{S^1}^{(i)}(N),$$

for $i = 1, \ldots, m$. That is, these numbers are differential invariants of the manifolds.

Theorem 4.1.2 *The following relations hold:*

(i) (Symmetry) For any $i = 0, \ldots, m$, we have:

$$\gamma_{S^1}^{(i)}(M) = \gamma_{S^1}^{(m-i)}(M)$$

(ii) (Submultiplicity) For any two manifolds M and N we have

$$\gamma_{S^1}(M \times N) \leq \gamma_{S^1}(M) \times \gamma_{S^1}(N)$$

(iii) For any $i = 0, \ldots, m + n$, we have:

$$\gamma_{S^1}^{(i)}(M \times N) \leq \sum_{j+k=i} \gamma_{S^1}^{(j)}(M) \cdot \gamma_{S^1}^{(k)}(N).$$

We present next a general result from our paper [13].

Theorem 4.1.3 *(1) The following relation holds: $\gamma_{S^1}(M) \leq \gamma(M)$, where $\gamma(M)$ is the Morse-Smale characteristic of the manifold M.*

(2) If M is a simply-connected manifold, $\pi_1(M) = \{0\}$, where $\pi_1(M)$ is the fundamental group of M, then $\gamma_{S^1}(M) = \gamma(M)$.

As an example, consider the m-dimensional sphere S^m, where $m \geq 2$.

It is well-known that in this case the sphere S^m is simply-connected. Taking into account the second result of the above theorem we get $\gamma_{S^1}(S^m) \geq \gamma(S^m)$.

On the other hand, it is known that $\gamma(S^m) = 2$, hence $\gamma_{S^1}(S^m) = 2$.

4.2 The generalization of Theorem 4.1.3 and some applications

In this section, following paper [19], we will generalize the result from Theorem 4.1.3(2) for a class of manifolds that are not simply-connected.

Theorem 4.2.1 *If* $\operatorname{Hom}(\pi_1(M), \mathbb{Z}) = 0$ *for some differential connected manifold* M*, then* $\gamma_{S^1}(M) = \gamma(M)$*. In particular,* $\gamma_{S^1}(M) = \gamma(M)$ *whenever* M *is connected and simply-connected.*

Corollary 4.2.1 *If* $n \geq 2$ *is a natural number, then*

$$\gamma_{S^1}(S^n) = \gamma(S^n) = 2 \quad \textit{and} \quad \gamma_{S^1}(\mathbb{RP}^n) = \gamma(\mathbb{RP}^n) = n + 1.$$

Corollary 4.2.2 *If* $m_1, \ldots, m_k \geq 2$ *are natural numbers, then*

$$\gamma_{S^1}(S^{m_1} \times \ldots \times S^{m_k}) = \gamma(S^{m_1} \times \ldots \times S^{m_k}) = 2^k,$$

$$\gamma_{S^1}(\mathbb{RP}^{m_1} \times \ldots \times \mathbb{RP}^{m_k}) = \gamma(\mathbb{RP}^{m_1} \times \ldots \times \mathbb{RP}^{m_k}) = (m_1 + 1) \ldots (m_k + 1).$$

Another property relating the circular Morse-Smale characteristic of the total and base spaces of a finite-fold covering map is provided by the following result.

Proposition 4.2.1 *If* $\widetilde{M}$ *is a* k*-fold cover of* M*, then*

$$\gamma_{S^1}(\widetilde{M}) \leq k \cdot \gamma_{S^1}(M).$$

4.3 The circular Morse-Smale characteristic of the compact surfaces

The minimum number of critical points of all Morse functions on a manifold M, equally called the Morse-Smale characteristic of M is a lower bound for the total curvature of M with respect to its embeddings in Euclidean spaces.

In this section, following paper D. Andrica, D. Mangra, C. Pintea [14], we first compute the circular Morse-Smale characteristic of all closed surfaces. We also observe that the critical points of the real valued height functions alongside those of some S^1 valued functions on a surface $\Sigma_g \subset \mathbb{R}^3$, are the tangency points with respect to some involutive distributions.

We finally study the size of the tangency set of the compact orientable surface of genus g embedded in a certain way in the first Heisenberg group with respect to its highly noninvolutive horizontal distribution.

It is an interesting and challenging problem to compute the circular Morse-Smale characteristic for closed manifolds M for which $\operatorname{Hom}(\pi_1(M), \mathbb{Z}) \neq 0$. The main purpose of this section is to do this computation for smooth, compact, connected, orientable surfaces of genus $g \geq 1$.

Recall that such a surface Σ_g is defined by

$$\Sigma_g = T^2 \# T^2 \# \ldots \# T^2,$$

where the number of copies of the 2-dimensional torus $T^2 = S^1 \times S^1$ in the connected sum is equal to g. We can extend the definition for $g = 0$ by considering $\Sigma_0 = S^2$, the 2-dimensional sphere. From the classification theorem of surfaces, it follows that every smooth, compact, orientable, connected surface, without boundary, is diffeomorphic to Σ_g, for some value of $g \geq 0$.

Recall that the Morse-Smale characteristic was completely determined by N.H. Kuiper in [43], who proved the formula $\gamma(S) + \chi(S) = 4$ for every compact connected surface S. In this section, following paper D. Andrica, D. Mangra, C. Pintea [14], we will prove that for every closed surface S, except for the sphere S^2 and the projective plane $\mathbb{RP}^2$, one has $\gamma_{S^1}(S) + \chi(S) = 0$.

Producing a suitable embedding of the surface Σ_g in $\mathbb{R}^3 \setminus O_z$, where O_z stands for the z-axis $\{(x, 0, 0) : x \in \mathbb{R}\}$, and a submersion $f : \mathbb{R}^3 \setminus O_z \to S^1$, whose restriction $f|\Sigma_g$ is a circular Morse function with exactly $2(g-1)$ critical points, is a part of the strategy to compute the circular Morse-Smale characteristic of the surface Σ_g. In this respect we need to characterize somehow the critical points of such a restriction. In fact, the suitable submersion we are looking for is

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0). \tag{4.3.1}$$

Proposition 4.3.1 *Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface and $f : \mathbb{R}^3 \to N$, be a submersion, where N is either the real line or the circle S^1. The point $p = (x_0, y_0, z_0) \in \Sigma$ is critical for the restriction $f|_{\Sigma_g}$ if and only if the tangent plane of Σ at p is the tangent plane at p to the fiber $\mathcal{F}_p := f^{-1}(f(p))$ of the submersion (4.3.1) through p.*

Proposition 4.3.1 follows from the following more general statement:

Theorem 4.3.1 *Let M^m, N^n, P^p, $m \geq n > p$, be differential manifolds, let $f : M \to N$ be a differential map and $g : N \to P$ be a submersion. Then $x \in M$ is a regular point of $g \circ f$ is and only if $f \pitchfork_x \mathcal{F}_x$, where $\mathcal{F}_x$ stands for the fiber $g^{-1}(g(x))$ of g through x.*

The above result was mentioned in [2] and [59].

4.3.1 The case of orientable surface of genus g

According to the results of the previous section, we have

$$\gamma_{S^1}(\Sigma_0) = \gamma_{S^1}(S^2) = \gamma(S^2) = 2,$$

since the 2-dimensional sphere S^2 is simply-connected.

Also,

$$\gamma_{S^1}(\Sigma_1) = \gamma_{S^1}(T^2) = 0,$$

as the projection $T^2 = S^1 \times S^1 \to S^1$ is a submersion and it has no critical points.

More generally, we shall prove the following:

Theorem 4.3.2 *The Morse-Smale characteristic of closed surfaces is given by*

$$\gamma_{S^1}(\Sigma) = \begin{cases} |\chi(\Sigma)| & \text{if } \Sigma \neq \mathbb{RP}^2 \\ 3 & \text{if } \Sigma = \mathbb{RP}^2 \end{cases} \tag{4.3.2}$$

We only need to prove Theorem 4.3.2 for Σ_g - the compact orientable surface of genus $g \geq 1$, as it has been already done for $\Sigma = S^2$ within Corollary 4.2.1. In this respect we need:

1. to show that

$$\mu(F) := \mu_0(F) + \mu_1(F) + \mu_2(F) \geq 2(g-1),$$

for every circular Morse function $F : \Sigma_g \to S^1$, where $\mu_j(F)$ is the number of critical points of index j of F and $\mu(F)$ is the total number card $(C(F))$ of critical points of F;

2. to produce a circular Morse function on Σ_g with exact $2(g-1)$ critical points.

In order to do so, we first observe that

$$2 - 2g = \mu_0(F) - \mu_1(F) + \mu_2(F). \tag{4.3.3}$$

Indeed, by using the Poincaré-Hopf Theorem, one obtains

$$2 - 2g = \chi(\Sigma_g) = \sum_{p \in C(F)} ind_p(\nabla F),$$

where ∇F is the gradient vector field of F with respect to some riemannian metric on Σ_g. To finish the proof of relation (4.3.3.), we just need to observe that the index of ∇F at a critical point of index one is -1 and the index of ∇F at the critical points of index zero and two is 1. Indeed, the local behavior of F around the critical points of index one is $F = x^2 - y^2$ and its gradient behaves locally around such a point like the vector field $(x, -y)$. The degree of its normalized restriction to the circle S^1 is -1 as the normalized restriction is a diffeomorphism which reverses the orientation. Similarly, the index of ∇F at a critical point of index zero or two is one as the local behavior of F around such a critical point is $F = x^2 + y^2$ or $F = -x^2 - y^2$, and its gradient behaves locally around such a point like the vector field (x, y) or $(-x, -y)$ respectively. The normalized restrictions of these vector fields to the circle S^1 are diffeomorphisms preserving the orientation and their degree is therefore one. Thus the relation (4.3.3) is now completely proved via the Poincaré-Hopf Theorem.

For the second item of the above observation we prove the following.

Lemma 4.3.1 *The surface Σ_g can be suitably embedded into the three dimensional space $\mathbb{R}^3 \setminus O_z$, such that the restriction $f|_{\Sigma_g} : \Sigma_g \to S^1$ is a circular Morse function with exactly $2(g-1)$ critical points, where $f : \mathbb{R} \setminus O_z \to S^1$ is the submersion given by*

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0).$$

4.3.2 The embedding of Σ_g into $\mathbb{R}^3 \setminus O_z$

Recall that $\Sigma_1 = T^2 = S^1 \times S^1$ is being usually identified with the surface of revolution in $\mathbb{R}^3$ obtained by rotating a circle in the plane xOz centered at a point on the x-axis around the z-axis. The radius of the circle is supposed to be strictly smaller than the distance from the origin to its center. A certain embedding of the surface Σ_g in $\mathbb{R}^3$, obtained from the one of Σ_1 on which we perform some surgery will be useful in our approach. However the above mentioned embedding of Σ_1 in $\mathbb{R}^3$ has one circle on the top and one circle on the bottom where the Gauss curvature vanishes. The two circles form the critical set of the height function $f_{\overrightarrow{k}}$ in the z-axis direction, restricted to the embedded copy of T^2 in $\mathbb{R}^3$. Thus, this restricted height function is not a Morse function. In order to construct our suitable embedding of Σ_g we rather need to rotate around the z-axis a closed convex curve with a unique center of symmetry, on the x-axis, which lies in the plane xOz and has no overlaps with the z-axis. This curve is also required to contain two segments mutually symmetric with respect to the x-axis, one on the top and the other on its bottom. These two segments form the critical set of the height function $f_{\overrightarrow{k}}$ restricted to the curve itself.

Consider the embedding of Σ_1 obtained by rotating such a closed convex curve, instead a circle within the plane xOz, within the same plane. The obtained copy of Σ_1 is flat on the two annuli $\mathcal{A}$ and $\mathcal{A}'$ which lie in two horizontal parallel planes.

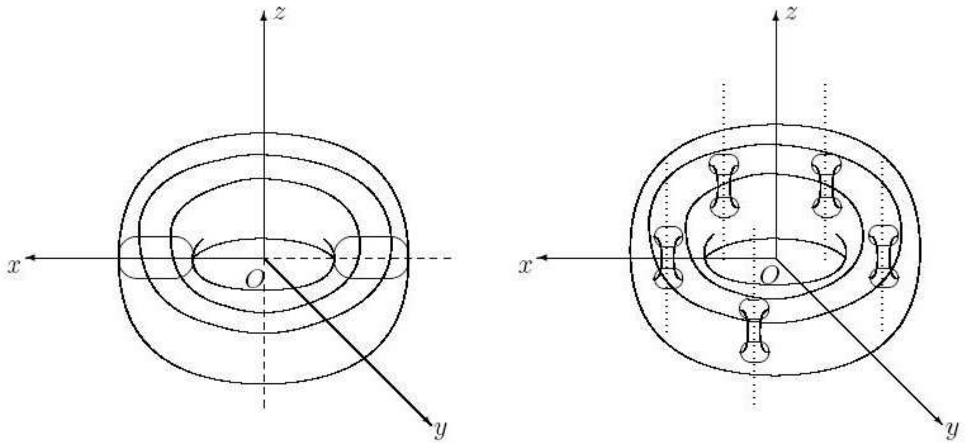


Figure 4.3.1. An embedded copy of Σ_g constructed out of an embedded copy of Σ_1

Consider the points $p_1, \ldots, p_{g-1} \in \mathcal{A}$ and $q_1, \ldots, q_{g-1} \in \mathcal{A}'$, such that the lines p_iq_i, $i = 1, \ldots, g-1$ are vertical, so parallel to the z-axis. In order to obtain a topological copy of the surface Σ_g we next remove some small open disks $D_1, \ldots, D_{g-1} \subseteq \mathcal{A}$ centered at $p_1, \ldots, p_{g-1}$ and $D'_1, \ldots, D'_{g-1} \subseteq \mathcal{A}'$ centered at $q_1, \ldots, q_{g-1}$, respectively. The radii of the disks D_i and D'_i are supposed to be the same. We next consider suitable planar curves $\gamma_i : [0,1] \to cl(\mathcal{B}) \cap \pi_i$, $i = 1, \ldots, g-1$ such that $\gamma_i(0) \in \partial D_i$ and $\gamma_i(1) \in \partial D'_i$, where $p_iq_i \cap xOy = \{(x_i, y_i, 0)\}$, π_i is the plane parallel to xOz through the point $(x_i, y_i, 0)$ ($\pi_i : y = y_i$) and $\mathcal{B}$ is the bounded component of the complement of the embedded copy of Σ_1. The curves γ_i are chosen in such a way to complete, by their rotation around the axes p_iq_i, the embedded copy of $\Sigma_1 \setminus [D_1 \cup \ldots \cup D_{g-1} \cup D'_1 \cup \ldots \cup D'_{g-1}]$ up to a smooth embedded copy of Σ_g.

4.3.3 The nondegeneracy of the critical point of $f|_{\Sigma_g}$ and the cardinality of its critical set

Since our embedded copy of Σ_g is constructed out of several surfaces of revolutions, we are going to investigate the critical set of the restriction of the submersion (4.3.1) to such a surface, by using the geometric interpretation coming from Proposition 4.3.1.

Proposition 4.3.2 *The following relation holds:* $\mathrm{card}(C(f|_{\Sigma_g})) = 2(g-1)$.

Proof. See [14].

Proposition 4.3.3 ([14]) *The restriction $f|_{\Sigma_g}$ is a circular Morse function (its critical points are non-degenerate). Moreover, the critical points of $f|_{\Sigma_g}$ have all index* 1.

Remark 4.3.1 No real valued Morse function defined on a compact manifold M^m ($m \geq 2$) can have just critical points of index one, as the global minimum of such a function has index zero and its global maximum has index $n = \dim(M)$. Thus the restriction $f|_{\Sigma_g}$ cannot be lifted to any map $\widetilde{f} : \Sigma_g \to \mathbb{R}$, $\exp \circ \widetilde{f} = f$, and the induced group homomorphism $f_* : \pi(\Sigma_g) \to \mathbb{Z} = \pi(S^1)$ is nontrivial therefore.

4.4 The case of non-orientable surfaces

Consider the compact orientable surfaces of genus $2g+1$ embedded in $\mathbb{R}^3$ as described before. Since the genus is odd, we may impose the extra-requirement on the embedded image of Σ_{2g+1} to be symmetric with respect to the origin, thus invariant with respect to the antipodal action of $\mathbb{Z}_2$ on $\mathbb{R}^3 \setminus \{0\}$. Because the restriction of this action to Σ_{2g+1} is orientation reversing, it follows that the quotient $\Sigma_{2g+1}/\mathbb{Z}_2$ is a compact non-orientable surface. Obviously the projection

$$\pi : \Sigma_{2g+1} \to \Sigma_{2g+1}/\mathbb{Z}_2$$

is the orientable double cover of $\Sigma_{2g+1}/\mathbb{Z}_2$. The reversing orientation property of the antipodal involution a follows from the reversing orientation property of the reflections σ_{xy}, σ_{xz} and σ_{yz} with respect to the coordinate planes xOy, xOz, yOz, respectively and the decomposition $a = \sigma_{xy} \circ \sigma_{xz} \circ \sigma_{yz}$.

Note that the three reflections commute with each other. The reversing orientation property of the reflection σ_{xy}, for example, follows by looking at the orientation behavior at a fixed point $p \in \mathrm{Fix}(\sigma_{xy}) = xOy \cap \Sigma_{2g+1}$. Since the tangent map of σ_{xy} at p reverses the orientation of the tangent space $T_p(\Sigma_{2g+1})$, it follows that σ_{xy}, and by similar arguments each of the reflections σ_{xz} and σ_{yz}, reverses the orientation of Σ_{2g+1}. Consequently, the antipodal map $a = \sigma_{xy} \circ \sigma_{xz} \circ \sigma_{yz}$ reverses, indeed, the orientation as well.

One can easily check that the non-orientable genus of $\Sigma_{2g+1}/\mathbb{Z}_2$ is $2g+2$, so $\Sigma_{2g+1}/\mathbb{Z}_2$ is diffeomorphic to $(2g+2)\mathbb{RP}^2$, where $k\mathbb{RP}^2$ stands for the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2$ of k copies of the projective plane.

Proof of Theorem 4.3.2 in the non-orientable case. We first observe that

$$f : \mathbb{RP}^2 \to \mathbb{R}, \quad f([x_1, x_2, x_3]) = \frac{x_1^2 + 2x_2^2 + 3x_3^2}{x_1^2 + x_2^2 + x_3^2}$$

is a perfect Morse function with three critical points of indices 0, 1, 2: a minimum point p, a maximum point q and a saddle point s. If $\varepsilon > 0$ is small enough, then the inverse images $D := f^{-1}(-\infty, f_2(p)+\varepsilon)$ and $D' := f^{-1}(f(q)-\varepsilon, \infty)$ are open disks and the inverse image

$$f^{-1}[f(p)+\varepsilon, f(q)-\varepsilon] = \mathbb{RP}^2 \setminus (D_1 \cup D_2)$$

is a compact surface with two circular boundary components $f^{-1}(f(p)+\varepsilon)$ and $f^{-1}(f(q)-\varepsilon)$ and observe that the restriction

$$f|_{\mathbb{RP}^2 \setminus (D \cup D')} : \mathbb{RP}^2 \setminus (D_1 \cup D_2) \to [f(p)+\varepsilon, f(q)-\varepsilon]$$

has one critical point of index one, so is the saddle point s. We next glue successively g copies of $\mathbb{RP}^2 \setminus (D \cup D')$, say

$$M_1 := \mathbb{RP}^2 \setminus (D_1 \cup D_1'), \ldots, M_g := \mathbb{RP}^2 \setminus (D_g \cup D_g'),$$

along the circular boundaries

$$\partial D_i' = f_i^{-1}(f_i(q) - \varepsilon) \subset M_i \quad \text{and} \quad \partial D_{i+1} := f_{i+1}^{-1}(f_{i+1}(p) + \varepsilon) \subset M_{i+1}$$

of $D_i' := f_i^{-1}(f_i(q) - \varepsilon, \infty)$ and $D_{i+1} := f_{i+1}^{-1}(-\infty, f_{i+1}(p) + \varepsilon)$ for $i = 1, \ldots, g-1$, where $f_i := f + iL : \mathbb{RP}^2 \to \mathbb{R}$ and $L := \text{length}([f(p) + \varepsilon, f(q) - \varepsilon]) = f(q) - f(p) - 2\varepsilon$. The obtained surface is $g\mathbb{RP}^2 \setminus (D_1 \cup D_g')$.

Note that f_i is a Morse function with one saddle point which is constant on each of the circular boundaries $\partial D_i = f_i^{-1}(f_i(p) + \varepsilon)$ and $\partial D_i' = f_i^{-1}(f_i(q) - \varepsilon)$ of M_i.

Moreover the equalities $f_i|_{\partial D_i'} = f_{i+1}|_{\partial D_{i+1}}$ hold for every $i = 1, \ldots, g-1$, which shows that the function

$$F : g\mathbb{RP}^2 \setminus (D_1 \cup D_g') \to \mathbb{R}, \quad F|_{M_i} := f_i$$

is well defined. In fact, F is a Morse function with g saddle points which is constant on the circle boundaries

$$\partial D_1 = f_1^{-1}(f_1(p) + \varepsilon) \subset M_1 \quad \text{and} \quad \partial D_g' = f_g^{-1}(f_g(q) - \varepsilon) \subset M_g.$$

Identifying the circle boundaries ∂D_1 and $\partial D_g'$ of $g\mathbb{RP}^2 \setminus (D_1 \cup D_g')$, via a suitable diffeomorphism $\varphi : \partial D_1 \to \partial D_g'$, we get the non-orientable surface $(g+2)\mathbb{RP}^2$.

Identifying $\min F$ with $\max F$ in $\text{Im}(F)$ we get the circle S^1. Also the Morse function

$$g\mathbb{RP}^2 \setminus (D_1 \cup D_g') \to \text{Im}(F), \quad x \mapsto F(x)$$

descends to a circular Morse function

$$f_0 : (g+2)\mathbb{RP}^2 = g\mathbb{RP}^2 \setminus (D_1 \cup D_g')/\{x = \varphi(x)\} \to S^1 = \text{Im}(F)/\{\min F = \max F\},$$

with g saddle points. This shows that $\gamma_{S^1}((g+2)\mathbb{RP}^2) \leq g$ for all $g \geq 1$.

For the opposite inequality we split the proof into two cases:

We first treat the case of closed non-orientable surfaces of even non-orientable genus, $(2g+2)\mathbb{RP}^2$, $g \geq 0$.

According to Proposition 4.2.1 one obtains

$$\gamma_{S^1}((2g+2)\mathbb{RP}^2) = \gamma_{S^1}(\Sigma_{2g+1}/\mathbb{Z}_2) \geq \frac{1}{2}\gamma_{S^1}(\Sigma_{2g+1}) = 2g = -\chi((2g+2)\mathbb{RP}^2).$$

For the case of closed non-orientable surfaces of odd non-orientable genus $2g+3$, $g \geq 0$, consider the oriented double cover of $(2g+3)\mathbb{RP}^2$

$$\Sigma_{2g+2} \to (2g+3)\mathbb{RP}^2.$$

According to Proposition 4.2.1 and the case of closed non-orientable surfaces of even non-orientable genus one obtains

$$\begin{aligned} \gamma_{S^1}(\Sigma_{2g+2}) \leq 2\gamma_{S^1}((2g+3)\mathbb{RP}^2) &\Leftrightarrow 2\gamma_{S^1}((2g+3)\mathbb{RP}^2) \geq 2(2g+2-1) \\ &\Leftrightarrow \gamma_{S^1}((2g+3)\mathbb{RP}^2) \geq 2g+1 \\ &\Leftrightarrow \mathbb{RP}^2((2g+3)\mathbb{RP}^2) \geq -\chi((2g+3)\mathbb{RP}^2). \end{aligned}$$

We therefore proved Theorem 4.3.2 in the non-orientable cases $g\mathbb{RP}^2$ with $g \geq 3$.

On the other hand $\gamma_{S^1}(\mathbb{RP}^2) = 3$ and $2\mathbb{RP}^2$ is the Klein Bottle which is a fibration over S^1 with fiber S^1, namely $\gamma_{S^1}(2\mathbb{RP}^2) = 0 = -\chi(2\mathbb{RP}^2)$. □

Remark 4.4.1 For the inequality $\gamma_{S^1}((2g+2)\mathbb{RP}^2) \leq 2g$ we can produce a particular circle-valued Morse function

$$f_0 : (2g+2)\mathbb{RP}^2 = \Sigma_{2g+1}/\mathbb{Z}_2 \to S^1$$

with exactly $2g$ critical points in the following different way. Pick the function $g_0 := f|_{\Sigma_{2g+1}} : \Sigma_{2g+1} \to S^1$ considered for the proof of Theorem 4.3.2 and recall that g_0 has precisely $4g$ critical points and $4g$ critical values, $\mathrm{card}(g_0(C(g_0)))$ is also $4g$. Indeed, the restriction $g_0|_{c(g_0)}$ is obviously one-to-one. Due to the way we embedded Σ_{2g+1}, the critical values of g_0, alongside its critical points, are pairwise antipodal in S^1 and in Σ_{2g+1}, respectively. By considering now the covering projection $p : S^1 \to P^1(\mathbb{R})$, $p(x) = [x] := \{-x, x\}$ one actually obtain a cyclic covering of order two $2p : S^1 \to S^1$, as $P^1(\mathbb{R})$ is diffeomorphic to S^1. The composed function $p \circ g_0$ is a circular Morse function with $2g$ critical values, each of whose inverse image consists of two critical points. Thus $\mathrm{card}(C(p \circ g_0)) = 4g$.

In fact $\pi^{-1}(\pi(x)) = \{-x, x\} \subseteq (p \circ g_0)^{-1}(x)$, for every $x \in \Sigma_{2g+1}$.

This shows that the restriction g_0 factors to a Morse function $f_0 : \Sigma_{2g+1}/\mathbb{Z}_2 \to S^1$ such that $p \circ g_0 = f_0 \circ \pi$.

Let us now observe that $\pi^{-1}(C(f_0)) = C(p \circ g_0)$ and therefore $\mathrm{card}(C(p \circ g_0)) = 2\mathrm{card}(C(f_0))$, thus $\mathrm{card}(C(f_0)) = \dfrac{1}{2}\mathrm{card}(C(p \circ g_0)) = 2g$.

4.5 On the number of tangency points of embedded surfaces into the first Heisenberg group

If $\Sigma \subset \mathbb{R}^3$ is a surface and $f : \mathbb{R}^3 \to N$ is a submersion, where N is either the real line or the circle S^1, then the critical points of the restriction $f|_\Sigma$ are the tangency points, as defined by Balogh [21], of the surface Σ with respect to the involutive distribution of the tangent planes to the fibers of f.

This observation bring us to study the minimal number of tangency points of a surface with respect to a highly noninvolutive distributions, for all embeddings of the surface in the support of the distribution. In fact this section is devoted to this subject.

For instance, the critical points of a height function are the tangency points of the embedded manifold in the environmental Euclidean space with respect to the involutive distribution of parallel hyperplanes which are perpendicular to the direction of the height function.

Indeed, this is in fact the distribution of fibers of the height function and the regular points of its restriction to the embedded manifold are, according to Theorem 4.3.1, precisely those points of the embedded manifold at which the tangent plane intersects transversally the fiber of the height function through that point, so the two planes are different.

Thus, the critical points of the restriction to the embedded manifold of the height function are precisely those at which the two planes are equal, the tangency points. Also the critical points of the restriction $f|_{\Sigma_g}$ are the tangency points of Σ_g with respect to the involutive distribution of half planes $f^{-1}(q)$, as q runs over the circle S^1. In other words the tangency point looks like an extended concept for critical point of real or S^1 valued functions to a more general context. This is the reason for us to evaluate, following paper D. Andrica, D. Mangra, C. Pintea [14], the size of a tangency set with respect to a highly non-involutive distribution, namely the horizontal distribution

$$\mathcal{H}_n = \mathrm{Span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$$

of the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$, where

$$X_i = \partial_{x_i} + 2y_i\partial_t \quad \text{and} \quad Y_i = \partial_{y_i} - 2x_i\partial_t$$

for $i = 1, \ldots, n$.

Some special attention will be payed to the minimum tangency number of the compact orientable surface of genus g, embedded into the first Heisenberg group $\mathbb{H}^1$.

Recall that a C^r, $r \in \mathbb{N}$, smooth distribution $\mathcal{D}$ of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ is a C^r smooth assignment to each point $z \in U$ of a linear n-dimensional subspace $\mathcal{D}(z) \preceq T_z(\mathbb{R}^{n+m})$. $\mathcal{D}$ is usually described either by n pointwise linearly independent C^r smooth vector fields

$$\{X_1, \ldots, X_n\} \tag{4.5.1}$$

such that $\{X_1(z), \ldots, X_n(z)\}$ forms a basis of $\mathcal{D}(z)$ for all $z \in U$ or as the intersection of the kernels of m linearly independent one-forms $\{\vartheta^1, \ldots, \vartheta^m\}$ with C^r smooth coefficients on U, thus

$$\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m). \tag{4.5.2}$$

Definition 4.5.1 Let $\mathcal{D}$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ and $S \subseteq U$ a C^1 smooth n-dimensional manifold. We call a point $z \in S$ a **tangency point** of S with respect to $\mathcal{D}$ if and only if $T_z(S) = \mathcal{D}(z)$. The set of such points is called the tangency set, or, in short, the tangency, of S with respect to $\mathcal{D}$ and denoted by

$$\tau(S, \mathcal{D}) := \{z \in S : T_z(S) = \mathcal{D}(z)\}. \tag{4.5.3}$$

Definition 4.5.2 If M^m is a differential manifold, then the **minimum tangency number** of M relative to the distribution $\mathcal{D}$ on $\mathbb{R}^{n+m}$ is defined by

$$\mu\tau\nu(M, \mathcal{D}) := \min\{\mathrm{card}(\tau(f(M), \mathcal{D})) : f \in \mathrm{Embed}(M, \mathbb{R}^{n+m})\},$$

where $\mathrm{Embed}(M, \mathbb{R}^{n+m})$ is the set of all embeddings of M into $\mathbb{R}^{n+m}$.

Remark 4.5.1 If M is a compact orientable $2n$-manifold of non-zero Euler-Poincaré characteristic, then according to [22, Example 8.9],

$$\mu\tau\nu(M, \mathcal{H}_n) \geq 2.$$

In fact $\mu\tau\nu(S^{2n}, \mathcal{H}_n) = 2$, as the Euler-Poincaré characteristic of the sphere S^{2n} is two and it admits an embedding into $\mathbb{H}^n$ with exactly two tangency points. The image of this embedding is the well-known Korányi sphere. On the other hand the standard torus $T^{2n} \subset \mathbb{H}^n$ has no tangency points at all [67], thus

$$\mu\tau\nu(T^{2n}, \mathcal{H}_n) = 0.$$

Theorem 4.5.1 *If $g \geq 2$, then*

$$2 \leq \mu\tau\nu(\Sigma_g, \mathcal{H}_1) \leq 4g - 4.$$

The inequality $\mu\tau\nu(\Sigma_g, \mathcal{H}_1) \geq 2$ is obvious, as

$$\chi(\Sigma_g) = 2 - 2g < 0.$$

For the opposite inequality we need to construct an embedding of Σ_g with $4g-4$, $\mathcal{H}_1$-tangency points. In this respect we use the possibility of Σ_1 to be embedded in $\mathbb{H}^1$ as a revolution surface and construct a suitable embedding of Σ_g out of Σ_1 by performing some suitable surgery on Σ_1. The handles we plan to glue will be surfaces of revolution as well. Therefore we are going to pay some special attention to the size of the tangency sets of revolution surfaces which lie inside $\mathbb{H}^1$ with respect to its horizontal distribution $\mathcal{H}_1$.

Problem 4.5.1 Is the upper estimate given by Theorem 4.5.1 on the number of horizontal points with respect to embeddings of the compact orientable surface of genus g in the first Heisenberg group sharp?

Note that the upper estimate $\mu\tau\nu(\Sigma_g, \mathcal{H}_1) \leq 4g - 4$ can be written in terms of the Euler-Poincaré characteristic of Σ_g, a strong invariant of the surface which determine its topological type, as $\mu\tau\nu(\Sigma_g, \mathcal{H}_1) + 2\chi(\Sigma_g) \leq 0$.

Problem 4.5.2 Who are the relevant invariants of a compact $2n$-dimensional manifold which can be embedded into the n^{th} Heisenberg group $\mathbb{H}^n$ for sharp estimates on the number of horizontal points of such embedded hypersurfaces with respect to all of its embeddings in the Heisenberg group $\mathbb{H}^n$?

4.5.1 Revolution surfaces in $\mathbb{H}^1$ with low number of horizontal points

Every revolution surface S obtained by rotating a plane curve

$$x = f(v), \quad z = v,$$

with $f > 0$, around the vertical line

$$x = x_0, \quad y = y_0,$$

admits a local parametrization of type

$$\begin{array}{l} x = x_0 + f(v)\cos u \\ y = y_0 + f(v)\sin u \\ z = g(v) \end{array} , \quad u \in I,\ v \in J,$$

where I is an open interval of length 2π and J will be symmetric with respect to the origin, so $J = (-a, a)$. The function f is subject to the following requirements:

$$f \text{ is bounded },\ f'' > 0 \text{ and } \lim_{v \to \pm a} f'(v) = \pm\infty. \tag{4.5.4}$$

The vector equation of our revolution surface is

$$\overrightarrow{r} = (x_0 + f(v)\cos u)\partial_x + (x_0 + f(v)\sin u)\partial_y + v\partial_t$$

and

$$\overrightarrow{r}_u = -(f(v)\sin u)\partial_x + (f(v)\cos u)\partial_y$$
$$\overrightarrow{r}_v = (f'(v)\cos u)\partial_x + (f'(v)\sin u)\partial_y + \partial_t$$
$$\overrightarrow{r}_u \wedge \overrightarrow{r}_v = (f(v)\cos u)\partial_x + (f(v)\sin u)\partial_y - f(v)f'(v)\partial_t.$$

On the other hand, the horizontal vector fields of the distribution $\mathcal{H}_1$ are

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t$$

and their vector product is

$$X \wedge Y = -2y\partial_x + 2x\partial_y + \partial_t.$$

Thus, the point $r(u,v) := (x(u,v), y(u,v), z(u,v)) \in S$ is a horizontal point if and only if the vectors $\overrightarrow{r}_u \wedge \overrightarrow{r}_v$, $X \wedge Y$ are linearly dependent at $r(u,v)$, thus we have

$$\sin u + f(v)f'(v)\cos u = -x_0 f'(v)$$

$$f(v)f'(v)\sin u - \cos u = -y_0 f'(v).$$

Thus

$$\sin u = -f'(v)\frac{x_0 + y_0 f(v)f'(v)}{1 + f^2(v)(f'(v))^2}$$
(4.5.5)
$$\cos u = -f'(v)\frac{x_0 f(v)f'(v) - y_0}{1 + f^2(v)(f'(v))^2}.$$

Remark 4.5.2 No revolution surface around the z-axis has $\mathcal{H}_1$-tangency points, as the equations (4.5.5) have no solutions at all for $x_0 = y_0 = 0$.

The identity $\sin^2 u + \cos^2 u = 1$ leads us to the equation

$$(4.5.6) \qquad \left(f'(v)\right)^2 = \frac{1}{\|(x_0, y_0)\|^2 - f^2(v)},$$

which has at least two solutions on the interval $J = (-a, a)$, as the right hand side of (4.5.6) is bounded and $(f')^2$ covers the positive real half line $[0, \infty)$ twice, once on the interval $(-a, 0]$ and once on the interval $[0, a)$. For suitable choices of the function f, the equation (4.5.6) has precisely two solutions. Such a choice is

$$(4.5.7) \qquad f(v) = 2 - \sqrt{\frac{2 - v^2}{2}}$$

for $a = \sqrt{2}$ and $\|(x_0, y_0)\| = 3$. Indeed, the equation (4.5.6), for the choice (4.5.7) of the function f, becomes:

$$2v^2\sqrt{2(2 - v^2)} = -2v^4 - 3v^2 + 4,$$

which has, indeed, precisely two solutions, as can be easily checked.

Proof of Theorem 4.5.1. The closed convex curve in the plane xOz described after the statement of Theorem 4.5.1 is supposed to have its unique center at the point $(3, 0, 0)$. The coordinates of the points p_i and q_i have the forms (x_i, y_i, z_i) and $(x_i, y_i, -z_i)$, respectively, for $i = 1, \ldots, g-1$. Moreover $\|(x_i, y_i)\|^2 := x_i^2 + y_i^2 = 3$, for all $i = 1, \ldots, g-1$. The handles we use within our surgery process are revolution surfaces around the vertical lines $x = x_i$ and $y = y_i$ of parametrized equations

$$\begin{cases} x = x_i + f(v)\cos u \\ y = y_i + f(v)\sin u \qquad u \in I,\ v \in J. \\ z = v \end{cases}$$

We denote by v_i and v_i' the roots of the equations

$$(f'(v))^2 = \frac{1}{\|(x_i, y_i)\|^2 - f^2(v)}, \tag{4.5.8}$$

with the choice (4.5.7) for the function f. The equations which corresponds to (4.5.5) are

$$\begin{cases} \sin u = -f'(v_i)\dfrac{x_i + y_i f(v_i) f'(v_i)}{1 + f^2(v_i)(f'(v_i))^2} \\ \cos u = -f'(v_i)\dfrac{x_i f(v_i) f'(v_i) - y_i}{1 + f^2(v_i)(f'(v_i))^2}, \end{cases} \tag{4.5.9}$$

$$\begin{cases} \sin u = -f'(v_i')\dfrac{x_i + y_i f(v_i') f'(v_i')}{1 + f^2(v_i')(f'(v_i'))^2} \\ \cos u = -f'(v_i')\dfrac{x_i f(v_i') f'(v_i') - y_i}{1 + f^2(v_i')(f'(v_i'))^2}. \end{cases} \tag{4.5.10}$$

Since the graphs of the sine and cosine functions on each interval of length 2π are intersected at most twice by any straight line parallel to the u-axis, it follows the equations (4.5.9) and (4.5.10) have at most two roots for each $i = 1, \ldots, g-1$. On the other hand the surface Σ_g embedded in $\mathbb{H}^1$, the way described right after Theorem 4.5.1 has no other $\mathcal{H}_1$-tangency points since on the two annuli $\mathcal{A}$ and $\mathcal{A}'$ its tangent planes are parallel to the xOy plane, a parallelism relation which happens for the planes of distribution $\mathcal{H}_1$ just along the z-axis and the two annuli have no common points with the z-axis. The remaining part of our embedded Σ_g is completely contained in Σ_1 which is, in its turn, a revolution surface around the z-axis which has no $\mathcal{H}_1$-tangency points, as we saw in Remark 4.5.2. Thus, our embedded surface Σ_g has at most $4(g-1)$ $\mathcal{H}_1$-tangency points.

A minimum tangency number, relative to a certain distribution $\mathcal{D}$ on $\mathbb{R}^{n+m}$, can be defined for a manifold M^n which is just immersible into $\mathbb{R}^{n+m}$ by

$$mtn(M, \mathcal{D}) := \min\{\text{card}\,(\tau(f, \mathcal{D})) : f \in \text{Imm}(M, \mathbb{R}^{n+m})\},$$

where $\text{Imm}(M, \mathbb{R}^{n+m})$ stands for the set of all immersions of M into $\mathbb{R}^{n+m}$ and $\tau(f, \mathcal{D}) := \{p \in M : \text{Im}(df)_p = \mathcal{D}(f(p))\}$. If M can be embedded into $\mathbb{R}^{n+m}$, then obviously $mtn(M, \mathcal{D}) \leq \mu\tau\nu(M, \mathcal{D})$.

Problem 4.5.3 We wonder whether $mtn(M, \mathcal{D}) = \mu\tau\nu(M, \mathcal{D})$.

Chapter 5

The Morse-Novikov inequalities for circle-valued functions

In this chapter we present the Morse-Novikov inequalities for circle-valued functions and some results regarding this subject, ([60], [61]).

The last section contains our results for the estimation of the number of critical points for circle-valued functions, following papers D. Andrica [4] and D. Mangra [46], [47].

5.1 The Morse-Smale complex

Definition 5.1.1 ([61]) **The Morse-Smale complex** $C^{MS}(M, f, v)$, defined on a Morse function $f : M \to \mathbb{R}$, a gradient like vector field $v \in \mathcal{G}(f)$, and a regular cover $\widetilde{M}$ of M with group of covering translations π, is a free $\mathbb{Z}[\pi]$-module chain complex with

$$d_i : C^{MS}(M, f, v)_i = \mathbb{Z}[\pi]^{c_i(f)} \to C^{MS}(M, f, v)_{i-1} = \mathbb{Z}[\pi]^{c_{i-1}(f)}$$

$$\widetilde{p} \to \sum_{\widetilde{q}} n(\widetilde{p}, \widetilde{q})\widetilde{q},$$

where $n(\widetilde{p}, \widetilde{q}) \in \mathbb{Z}$ is the finite signed number of $\widetilde{v}$-gradient flow lines $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ which start at a critical point $\widetilde{p} \in \widetilde{M}$ of $\widetilde{f} : \widetilde{M} \to \mathbb{R}$, with index i and terminate at a critical point $\widetilde{q} \in \widetilde{M}$ of index $i - 1$.

If we chose an arbitrary lift of each critical point $p \in M$ to a critical point $\widetilde{p} \in \widetilde{M}$ of $\widetilde{f}$, one obtains a basis for $C^{MS}(M, f, v)$.

The Morse-Smale complex is the cellular chain complex

$$C^{MS}(M, f, v) = C(\widetilde{M})$$

of the CW structure of $\widetilde{M}$ in which the i-cells are the lifts of the i-handles h^i.

The homology of the Morse-Smale complex is isomorphic to the ordinary homology of M:

$$H_*(C^{MS}(M, f, v)) \cong H_*(M).$$

If $\widetilde{M} = M$ then $C^{MS}(M, f, v) = C(M)$. This relation holds when the manifold M is simply-connected.

Proposition 5.1.1 *The Morse inequalities are:*

$$c_i(f) \geq b_i(M) + q_i(M) + q_{i-1}(M),$$

where $b_i(M)$ are the Betti numbers of the manifold M,

$$b_i(M) = \dim_{\mathbb{Z}}(H_i(M)/T_i(M)),$$

and $q_i(M)$ are defined by the minimum number of generators of $T_i(M)$, $i = 0, 1, \dots, m$.

Here $T_i(M) = \{x \in H_i(M) : nx = 0$ for any $n \neq 0 \in \mathbb{Z}\}$ is the torsion subgroup of $H_i(M)$.

In Morse theory, the Betti numbers represent lower bounds of the number of critical points of a Morse function.

If the manifold M is simply-connected, $\pi_1(M) = \{0\}$, and $m \geq 6$, then there exists a Morse function such that

$$c_i(f) = b_i(M) + q_i(M) + q_{i-1}(M),$$

for $i = 0, 1, \dots, m$.

This result was proved by S. Smale and implies Poincaré conjecture for dimension $m \geq 6$ ([32]).

5.2 The Novikov complex

The construction of the Novikov complex is similar with the construction of the Morse-Smale complex and is presented by A. Ranicki in paper [61].

Definition 5.2.1 Let $f : M \to S^1$ be a circle-valued Morse function and let $v \in \mathcal{G}(f)$ be a gradient like vector field of f.

The Novikov complex $C^{Nov}(M, f, v)$ is a free $\widehat{\mathbb{Z}[\Pi]}$-module chain complex with

$$d_i : C^{Nov}(M, f, v)_i = \mathbb{Z}[\pi]_\lambda((z))^{c_i(f)} \to C^{Nov}(M, f, v)_{i-1} = \mathbb{Z}[\pi]_\lambda((z))^{c_{i-1}(f)}.$$

$$\widetilde{p} \to \sum_{i=-\infty}^{\infty} \sum_{\widetilde{q}} n(\widetilde{p}, z^i \widetilde{q}) z^i \widetilde{q},$$

where $n(\widetilde{p}, \widetilde{q}) \in \mathbb{Z}$ is the finite signed number of $\widetilde{v}$-gradient flow lines $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ which start at a critical point $\widetilde{p} \in \widetilde{M}$ of $\widetilde{f} : \widetilde{M} \to \mathbb{R}$ with index i and terminate at a critical point $\widetilde{q} \in \widetilde{M}$ of index $i - 1$.

5.3 The Morse-Novikov inequalities

Definition 5.3.1 ([61]) The Novikov numbers of any CW-complex M and $f \in H^1(M)$ are $b_i^{Nov}(M, f)$ and $q_i^{Nov}(M, f)$, where

$$b_i^{Nov}(M, f) = \dim_{\mathbb{Z}((z))}(H_i^{Nov}(M, f)/T_i^{Nov}(M, f))$$

are the Betti numbers of the Novikov homology and $q_i^{Nov}(M, f)$ is the minimum number of generators of $T_i^{Nov}(M, f)$, with

$$T_i^{Nov}(M, f) = \{x \in H_i^{Nov}(M, f) : ax = 0, a \neq 0 \in \mathbb{Z}((z))\}$$

the torsion $\mathbb{Z}((z))$-submodule of $H_i^{Nov}(M, f)$.

Theorem 5.3.1 (Morse-Novikov inequalities) *For an m-dimensional compact manifold M and a circle-valued Morse function $f : M \to S^1$, the Morse-Novikov inequalities are, ([54]):*

$$c_i(f) \geq b_i^{Nov}(M, f) + q_i^{Nov}(M, f) + q_{i-1}^{Nov}(M, f), \ i = 0, 1, \dots, m.$$

The following result was proved by M. Farber in [33].

Theorem 5.3.2 ([33]) *Consider $\pi_1(M) = \mathbb{Z}$ and $m \geq 6$ and let $f : M \to S^1$ be a circle-valued Morse function, $1 \in [M, S^1] = H^1(M)$ with a minimum number of critical points.*

Then, for any $i = 0, 1, \dots, m$ the following relation holds:

$$c_i(f) = b_i^{Nov}(M, f) + q_i^{Nov}(M, f) + q_{i-1}^{Nov}(M, f).$$

5.4 Estimation of the number of critical points of circle-valued functions

We will use the Morse-Novikov inequalities to determine lower bounds for $\gamma_{S^1}(M)$, following papers D. Mangra [46], [47].

Let $f : M \to S^1$ be a circle-valued Morse function and let $f^* : H^1(S^1) \to H^1(M)$ be the induced homomorphism in cohomology. Denote

$$F^1(M) = \{f^*(1) : f \in \mathcal{F}(M, S^1)\} \subseteq H^1(M).$$

Theorem 5.4.1 *The following relation holds:*

$$\gamma_{S^1}(M) \geq \min\{b^{Nov}(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi) : \xi \in F^1(M)\},$$

where $b^{Nov}(\xi) = \sum_{i=0}^{m} b_i^{Nov}(\xi)$ is the total Betti number of the manifold M with respect to the cohomology class $\xi \in H^1(M)$.

Theorem 5.4.2 *If $\pi_1(M) = \mathbb{Z}$ and $m \geq 6$, the following relation holds:*

$$\gamma_{S^1}(M) = \min\{b^{Nov}(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi) : \xi \in F^1(M)\}.$$

Regarding the set $F^1(M)$ which appears in Theorem 5.4.1 and 5.4.2, we formulate the following problem:

Conjecture 4.4.1 For any compact manifold M, the equality

$$F^1(M) = H^1(M)$$

holds.

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