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MODERN APPROACHES IN THE THEORY OF UNIVALENT MAPPINGS IN $\mathbb{C}$ AND $\mathbb{C}^n$

PhD Thesis – Summary

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Cluj-Napoca
2025

Contents

Introduction

This thesis explores both classical and novel results in the theory of univalent functions of complex variable in one and higher dimensions. The thesis follows the course of this domain over the last years, i.e. we start with notions related to univalent functions of one complex variable, then we discuss families of biholomorphic mappings in $\mathbb{C}^n$ and end our study with a part dedicated to biholomorphic mappings and extension operators in complex Banach spaces. The theory of univalent functions is a major topic in the geometric function theory being intensively studied by several authors who contributed to the development of this field.

The results included in this thesis continue, on a smaller scale, the excellent work of the great professors from Cluj-Napoca (we mention here Professor Petru T. Mocanu, Professor Grigore S. Sălăgean, Professor Gabriela Kohr and Professor Mirela Kohr) being inspired by their innovative ideas used over the years. We note here the special contribution of Professor Gabriela Kohr together with her collaborators, especially Professor Ian Graham, Professor Hidetaka Hamada and Professor Mirela Kohr (see e.g. [45]).

A very important outcome in the theory of univalent functions in $\mathbb{C}$ is the Riemann mapping theorem which affirms the conformally equivalence of simply connected domains in one dimensional case (see e.g. [45], [77]). Taking into account this result, the study of univalent functions of one variable can be reduced to the unit disc $\mathbb{U}$. In higher dimensions the Riemann's theorem does not hold (see e.g. [45]) and this is one of the major differences between $\mathbb{C}$ and $\mathbb{C}^n$, where $n \geq 2$, proved by Poincaré (see [112]).

Starting with Bieberbach (see e.g. [4]) who proved in 1916 the sharp second coefficient estimation for the class S of normalized univalent functions on $\mathbb{U}$, the theory of univalent functions suffered important developments (especially due to those who worked to prove the conjecture proposed by Bieberbach related to the coefficient bounds for functions in S). One of the important tools used in this context was the theory of Loewner chains. Based on these ideas L. de Branges proved the Bieberbach's conjecture and opened new directions for the study of univalent functions. Moreover, the Loewner theory was also useful to prove univalence criteria, analytical characterizations of geometric properties (starlikeness, convexity, spirallikeness), radii of starlikeness, convexity, univalence and other strong results related to univalent functions in $\mathbb{C}$. Another important step was made by Pommerenke (see [114]) who proved that any $f \in S$ admits parametric representation, i.e. there is a Loewner chain $f(z,t)$ such that $f = f(\cdot, 0)$ is the first element of the chain. Various aspects and applications of this theory can be found in the monographs of Duren [19], Graham and Kohr [45], Pommerenke [114] and also in Conway [12], Goluzin [23], Mocanu, Bulboacă and Sălăgean [102].

In higher dimensions, Cartan studied the class $S(\mathbb{B}^n)$ of normalized biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ (see e.g. [7], [45], [83]). He proved that $S(\mathbb{B}^n)$ is not compact, since it is not locally uniformly bounded. Taking into account this property, we obtain that $S(\mathbb{B}^n)$ does not admit a growth and distortion theorem (see e.g. [45]). This problem was solved by Graham, Hamada and Kohr who introduced the class $S^0(\mathbb{B}^n)$ of mappings which admit parametric representation on $\mathbb{B}^n$ (see e.g. [32]; see also [114]). For $n = 1$, we have that $S^0(\mathbb{B}^1) = S$ (see e.g. [114]). However, if $n \geq 2$, then $S^0(\mathbb{B}^n)$ is strictly included in $S(\mathbb{B}^n)$. Moreover, Graham, Kohr and Kohr (see [48]) proved that $S^0(\mathbb{B}^n)$ is compact (see e.g. [32], [45], [48]). This result is one of the results that presents the clear distinction between one and several complex variables cases. On the other hand, it opened new ways of studying biholomorphic mappings theory in higher dimensions.

Another important direction in the theory of biholomorphic mappings in higher dimensions is the investigation of geometric properties of biholomorphic mappings (e.g. starlikeness, convexity, spirallikeness).

Matsuno (see e.g. [99]) studied for the first time the notion of starlikeness on $\mathbb{B}^n$ in $\mathbb{C}^n$, while Suffridge (see e.g. [126]) dealt with similar results on the unit polydisc $\mathbb{U}^n$. In infinite dimensional case, Gurganus [57] and Suffridge [127] obtained characterizations of starlikeness. Other important properties of starlike mappings on $\mathbb{B}^n$ have been proved over the time by Curt [14], Gong [24], Graham, Hamada and Kohr [32], Graham and Kohr [45], Kikuchi [80], Kohr [83], Kubicka and Poreda [86] and others. The idea of starlikeness of order $\alpha \in [0, 1)$ in $\mathbb{C}^n$ was introduced by Kohr in [81] and the almost starlikeness of order $\alpha \in [0, 1)$ was defined by Feng in [22] in infinite dimensional case. The family of convex mappings was studied by Suffridge [126] on $\mathbb{U}^n$, respectively by Kikuchi and Gong [80] on $\mathbb{B}^n$. Later, Hamada and Kohr [68] obtained generalizations of Kikuchi's necessary and sufficient condition of convexity to the case of complex Hilbert spaces. The same problem was studied by Suffridge [127] on the unit ball in $\mathbb{C}^n$ with respect to different norms. Other important contributions in this field are due to Curt [14], Graham, Hamada and Kohr [32], Graham and Kohr [45], Liu [91], Roper and Suffridge [121]. In [57] Gurganus proposed the idea of spirallikeness with respect to a linear operator that is normal and has the property that its eigenvalues have positive real part. This concept was extended by Suffridge in infinite dimensional case in [128] (see also the generalizations considered by Liu and Liu [94]).

As in one dimensional case, the theory of Loewner chains remains an important instrument in the analysis of biholomorphic mappings of several complex variables. Pfaltzgraff [109] is the first contributor in this area, obtaining generalizations on the Euclidean unit ball in $\mathbb{C}^n$ of the results proved on the unit disc in $\mathbb{C}$. On the other hand, Poreda extended these results on $\mathbb{U}^n$ in $\mathbb{C}^n$ in the context of growth and distortion theorems (see e.g. [115], [116]). Various properties and results in the theory of Loewner chains in $\mathbb{C}^n$ were obtained by Duren, Graham, Hamada and Kohr [20], Graham, Hamada and Kohr [32], Graham, Kohr and Kohr [47], Graham, Kohr and Pfaltzgraff [49] and also by Arosio [2], Curt and Kohr [15], Cristea [16], Poreda [117], Vodă [130]. Among the most important results, we mention that Graham, Hamada and Kohr (see [32]) showed that in higher dimensions, there are normalized biholomorphic mappings that cannot be embedded as the initial elements of Loewner chains. Also, there exist mappings that do not have parametric representation on the unit ball of $\mathbb{C}^n$, where $n \geq 2$. Another significant distinction between one dimensional case and higher dimensions consists in the fact that in $\mathbb{C}$ the Loewner differential equation has a unique normalized univalent solution. In contrast, in higher dimensions this result does not hold (see e.g. [32], [114]). Duren, Graham, Hamada and Kohr studied the form of general solutions of the Loewner differential equation in [20].

A turning point in the evolution of the theory of biholomorphic mappings of several complex variables was the proving of the compactness of the Carathéodory family $\mathcal{M}$ by Graham, Hamada and Kohr in 2002 (see [44], [68]). This result revived the study of biholomorphic mappings in higher dimensions and opened new study opportunities in the geometric function theory. Other aspects and applications of the theory of Loewner chains, respectively the Carathéodory class in several complex variables may be found in [14], [16], [17], [32], [45], [48], [109], [117].

A problem that appeared in the study of biholomorphic mappings in higher dimensions was the construction of examples of convex mappings in $\mathbb{C}^n$. Those who took the first steps towards solving this problem were K. Roper and T.J. Suffridge (see [121]). They introduced the extension operator $\Phi_n : \mathcal{LS} \to \mathcal{LS}_n(\mathbb{B}^n)$ given by

$$\Phi_n(f)(z) = \left(f(z_1), \tilde{z}\sqrt{f'(z_1)}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where the branch of the square root function is taken such that $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$ as a tool that preserve the notion of convexity from $\mathbb{U}$ to $\mathbb{B}^n$. This property was later obtained also by Graham and Kohr in a different manner (see [43]). Moreover, they showed that the operator preserves also the notion of starlikeness. Different authors obtained strong extension results related to the preservation of starlikeness of order 1/2 (Hamada, Kohr and Kohr in [73]), starlikeness of order $0 < \alpha < 1$ (Liu in [92] and Chirilă in [11]), spirallikeness of type δ, where $\delta \in \mathbb{R}$ is such that $|\delta| < \frac{\pi}{2}$ (Graham, Kohr and Kohr in [48]). Taking into account the method of g-Loewner chains, Chirilă (see [11]) proved that Φ_n preserves the almost starlikeness of order α and type γ, where $\alpha, \gamma \in [0, 1)$, respectively the spirallikeness of type δ and order α, where $\delta \in \mathbb{R}$ with $|\delta| < \frac{\pi}{2}$ and $\alpha \in [0, 1)$. The preservation of spirallikeness of type δ and order α was obtained also by Liu and Liu (see [94]) using a different method. Other properties of the extension operator Φ_n can be found in [30], [45].

Starting with Roper and Suffridge, the theory of extension operators became an important topic in the study of biholomorphic mappings in higher dimensions. Other extension operators with strong properties were introduced by

- Graham and Kohr (see [43], [44]), i.e. the Graham-Kohr extension operator $\Psi_{n,\alpha}$ defined by

$$\Psi_{n,\alpha}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} \tilde{z}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

  for any function $f \in \mathcal{LS}$ with $f(z_1) \neq 0$ for $z_1 \in \mathbb{U} \setminus \{0\}$. The branch of the power function is taken such that $\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\big|_{z_1=0} = 1$.

- Graham, Hamada, Kohr and Suffridge (see [42]), i.e. the extension operator $\Phi_{n,\alpha,\beta}$ defined by

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} \left(f'(z_1)\right)^{\beta} \tilde{z}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

  where $\alpha, \beta \geq 0$ and $f \in \mathcal{LS}$ has the property that $f(z_1) \neq 0$ for $z_1 \in \mathbb{U} \setminus \{0\}$. Here, the branches of the power functions have the properties $\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\big|_{z_1=0} = 1$ and $\left(f'(z_1)\right)^{\beta}\big|_{z_1=0} = 1$.

- Pfaltzgraff and Suffridge (see [111]), i.e. the Pfaltzgraff-Suffridge extension operator $\Gamma_n : \mathcal{LS}_n(\mathbb{B}^n) \to \mathcal{LS}_{n+1}(\mathbb{B}^{n+1})$ defined by

$$\Gamma_n(f)(z) = \left(f(z'), z_{n+1}\left[J_f(z')\right]^{\frac{1}{n+1}}\right), \quad z = (z', z_{n+1}) \in \mathbb{B}^{n+1},$$

  where $J_f(z') = \det Df(z')$, for $z' \in \mathbb{B}^n$. We take the power function such that $\left[J_f(z')\right]^{\frac{1}{n+1}}\big|_{z'=0} = 1$.

All these extension operators preserve the notion of parametric representation (in particular, starlikeness, starlikeness of order $\alpha \in (0,1)$, spirallikeness). An important remark is that Graham, Hamada, Kohr and Suffridge proved that $\Phi_{n,\alpha,\beta}$ preserves the convexity only if $\alpha = 0$ and $\beta = \frac{1}{2}$. In particular, the Graham-Kohr extension operator $\Psi_{n,\alpha}$ does not preserve convexity for $n \geq 2$. The problem of preservation of convexity under the operator Γ_n was partially solved by Graham, Kohr and Pfaltzgraff (see [49]), respectively by Chirilă (see [10]) for a modified Pfaltzgraff-Suffridge operator. More details about generalizations of Roper-Suffridge type extension operators and their properties can be found in [25], [27], [43], [46], [84], [93], [103], [138].

In recent years, the study of biholomorphic mappings has focused on the infinite dimensional case. Part of the results from one variable, respectively several complex variables, can be extended in the case of complex Banach spaces (e.g. the theory of univalent functions, families of univalent functions with special geometric properties, the theory of Loewner chains, the theory of extension operators and others). However, many of the problems proposed in the infinite dimensional case are still open and of great interest to researchers. Among those who started the study of biholomorphic mappings in infinite dimensions we mention K. Gurganus [57], J. Mujica [106], T. Poreda [118] and T.J. Suffridge [127]. One of the important differences in this context is that the notions of univalence and biholomorphy are not equivalent, i.e. there exist univalent mappings which are not biholomoprhic (see e.g. [107], [119], [128]). This result is in contrast with the finite dimensional case (see [45]). General properties of the class $S(\mathbb{B}_X)$ of normalized biholomorphic mappings on the unit ball $\mathbb{B}_X$ of the complex Banach space X were obtained in the books of Graham and Kohr [45], Hille and Phillips [79], Mujica [106].

Related to the generalization of the Carathéodory class in infinite dimnesional case, important contributions have been made by Bracci, Elin, Shoikhet [6], Graham, Hamada, Honda, Kohr and Shon [31], Graham, Hamada, Kohr and Kohr (see e.g. [36], [38]). They improved some results obtained initially by Gurganus [57] and Pfaltzgraff [109]. Moreover, Hamada and Kohr (see [68]) obtained the final version of the analytical characterization of starlikeness studied by Suffridge [127] and Gurganus [57]. They also proved the analytical characterization of convexity in the case of Hilbert spaces (see e.g. [68]). Other families of biholomorphic mappings on $\mathbb{B}_X$ were studied by Gong and Liu [25], Graham and Kohr [45], Hamada, Kohr and Kohr [74], Wang and Wang [133].

Recently, the study of subordination chains in infinite dimensional spaces started by Poreda (see e.g. [117], [118]) was continued and improved by Graham, Hamada, Kohr and Kohr (see e.g. [34], [38], [39], [40], [41]), Hamada and Kohr (see e.g. [70], [72]), Arosio, Bracci, Hamada and Kohr (see e.g. [2], [3]) who obtained important results related to Loewner chains and Loewner PDE in infinite dimensional case. Another strong development in the theory of Loewner chain was introduced by Arosio, Bracci, Hamada and Kohr in [3]. They considered Loewner chains on complete hyperbolic complex manifolds and obtained a one-to-one correspondence between L^d-Loewner chains and L^d-evolution families. Moreover, they construct L^d-Loewner chains produced by the Roper-Suffridge extension operator. The theory of Loewner chains and its generalizations play a key role in the study of biholomorphic mappings and extension operators in complex Banach spaces.

Another domain that has been extended to the infinite dimensional case is that of extension operators. Among those who study extension operators in complex Banach spaces we mention Graham, Hamada, Kohr and Kohr (see e.g. [39], [40]), Muir Jr. (see e.g. [104], [105]), Wang and Zhang (see e.g. [131], [132], [134]), Zhang and Thang (see e.g. [139]). Recently, they obtained important results related to generalized Roper-Suffridge extension operator, respectively Muir-type extension operators, g-Loewner chains and generalized parametric representation in complex Banach spaces. Note that an important recent tool to generate extension operators is also the semigroup theory studied by Elin (see e.g. [21]).

Nowadays, the most recent approach to the Loewner theory was realized by Hamada and Kohr in [72]. They studied a new concept, namely the inverse Loewner chain, in infinite dimensional case. Their important work represents a new way of studying the results related to the Loewner chains and extension operators in complex Banach spaces.

The content of this thesis is structured in three parts organized in six chapters. We present here a brief description of each part, highlighting the main results we obtained in each chapter. The original results included in the thesis are mainly derived from the author's six articles presented in the bibliography (a short presentation of these results can be consulted in the conclusions part at the end of the thesis). Moreover, it is important to mention that throughout this thesis are addressed some conjectures and open questions that lead to the final chapter dedicated to further research directions.

Part I contains results related to univalent functions of one complex variable. It includes the first two chapters of the thesis that contain both classical and original results (the latter being obtained by the author in [50], [51] and [54]). The main sources cited in this section are [19], [29], [45], [77], [85], [87], [102], [114]

- In **Chapter 1** we include general results related to univalent functions of one complex variable in $\mathbb{C}$. We begin with basic notations, notions and preliminary results that will be used during the first part of the thesis.

  In Section 1.1 we shortly discuss the theory of holomorphic functions in $\mathbb{C}$ including the open mapping theorem and the minimum/maximum modulus theorem with its applications (e.g. the Schwarz's lemma or Schwarz-Pick's lemma). In the final part of the first section we revisit the concepts of normal families and locally uniformly bounded families of holomorphic functions in $\mathbb{C}$. We end this section with the equivalence of the previous two notions proved by Montel (see e.g. [77], [85], [87]), the characterization of compactness of closed families of holomorphic functions in terms of locally uniformly boundedness (see e.g. [85]) and one of the most important application of the Montel's theorem, namely Vitali's theorem (see e.g. [85], [87]).

  Section 1.2 contains some classical results regarding to the notion of subordination, respectively the Carathéodory class $\mathcal{P}$ in $\mathbb{C}$. The family of holomorphic functions with positive real part is an important tool in the characterization of univalent functions and in the theory of Loewner chains on $\mathbb{U}$. We include here the growth and distortion theorem for the class $\mathcal{P}$, coefficient estimations and the Herglotz representation formula that characterize the Carathéodory family $\mathcal{P}$ (see e.g. [29], [45], [102]).

In Section 1.3 we discuss general results related to univalent functions on $\mathbb{U}$. We present several properties of univalent functions, necessary and sufficient conditions of univalence and some examples that we will refer to during the thesis (see e.g. [19], [85]). We end this section with one of the most important results in the theory of univalent functions in $\mathbb{C}$, i.e. the Riemann mapping theorem which establishes the conformal equivalence of every simple connected domain $D \subsetneq \mathbb{C}$ with the unit disc $\mathbb{U}$.

In Section 1.4 we consider the class S of normalized univalent functions on $\mathbb{U}$ and some particular subclasses of S (starlike, respectively almost starlike of order α, convex of order α and spirallike). For these families of univalent functions on $\mathbb{U}$ we present the well-known results related to growth, distortion and coefficient bounds, as well as the analytical characterizations of the families $S^*(\alpha)$, $K(\alpha)$ and $\hat{S}_\delta$.

It is important to mention here that even if this chapter is an introductory one, it also contains original results regarding to general distortion theorems for starlike functions of order α (see Theorem 1.4.9), respectively convex functions of order α (see Theorem 1.4.19). The original outcomes were derived by the author in [51].

Section 1.5 of this chapter is focused on the study of the class $\mathcal{R}$ of normalized holomorphic functions whose derivative has positive real part. Here, we include the most important results related to the class $\mathcal{R}$ obtained in [89], [90], [96] or [129]. Together with the classical results we present also several original results obtained by the author in [50]. In §1.5.3 and §1.5.4 we define two new subclasses of functions, namely $\mathcal{R}_p$ and $\mathcal{R}_p(\alpha)$, and study some of their properties (see Theorems 1.5.3, 1.5.9). The class $\mathcal{R}_p(\alpha)$ was introduced in order to generalize the class $\mathcal{R}(\alpha)$ described in §1.5.2. The idea of considering a parameter $\alpha \in [0,1)$ is inspired from the extensions that Robertson made in [120] for starlike, respectively convex functions of order α. The relation with the Carathéodory family is an important tool that can be used in characterization of the new subclasses introduced by the author in [50].

In Section 1.6 we present important results regarding Loewner theory in $\mathbb{C}$. Since the Loewner chains are strongly connected to the Loewner differential equation, we present here some of the most important results in this theory that will be used in the study of univalent functions mentioned above. In the second part of this section we refer to the analytical characterization of some subclasses of S through Loewner chains and finally, we present the notion of parametric representation on $\mathbb{U}$ (see e.g. [45], [114]) The characterization of the geometric properties of univalent functions in terms of Loewner chains will play an important role in the Chapter 2, where we use the Loewner theory to describe some new subclasses of univalent functions on $\mathbb{U}$ (see §2.2.2).

- The main idea of **Chapter 2** consists in the study of a new differential operator and two new subclasses of univalent functions on the unit disc $\mathbb{U}$ defined with this operator. This chapter is made up entirely of original results obtained by the author in [54].

  In Section 2.1 we present the differential operator $\mathcal{G}_k$ defined on the family $\mathcal{H}_0(\mathbb{U})$ of normalized holomorphic functions on $\mathbb{U}$. Using the operator $\mathcal{G}_k$ we can construct some particular subclasses of univalent functions on $\mathbb{U}$ that are strongly related to the families S^*, respectively K, as we can see in §2.2. Several properties of the operator $\mathcal{G}_k$ are studied in this section, e.g the linearity of $\mathcal{G}_k$, convolution product and a sufficient condition of univalence for $\mathcal{G}_k$ (see Propositions 2.1.3 – 2.1.6). It is important to mention here that the differential operator $\mathcal{G}_k$ is different from the Sălăgean differential operator D^n (see Remark 2.2.6; see also [124]). Another important remark is that the operator $\mathcal{G}_k$ can be extended in the case of several complex variables (see Chapter 4; see also [53]).

  Using the differential operator $\mathcal{G}_k$ mentioned above, we can construct some particular subclasses of univalent functions on $\mathbb{U}$ in $\mathbb{C}$. These subclasses, denoted here by $E_k^*(\alpha)$, respectively $E_k(\alpha)$, where $\alpha \in [0,1)$, are related to the classes of starlike, respectively convex functions of order α on $\mathbb{U}$. An important remark is that for $k=0$ we obtain $E_0^*(\alpha) = S^*(\alpha)$ and $E_0(\alpha) = K(\alpha)$, so we can start our study of these new subclasses in terms of the well-known families $S^*(\alpha)$ and $K(\alpha)$ introduced by Robertson in [120]. On the other hand, we have that E_1 is strictly included in the family $K(1/2)$ of convex functions of order $1/2$ (see Proposition 2.2.25) and $E_1^*(\alpha) = K(\alpha)$. As we already mentioned above, the operator $\mathcal{G}_k$ and the subclasses introduced in this chapter can be extended also in the case of several complex variables (see e.g. [53]). However, in higher dimensions, some properties are different as can be seen in the results included by the author in Chapter 4.

Section 2.2 is dedicated to the study of subclasses $E_k(\alpha)$ and $E_k^*(\alpha)$ in $\mathbb{C}$, where $k \in \mathbb{N}$ and $\alpha \in [0, 1)$. Together with general properties of these subclasses (growth and distortion theorems, coefficient estimations, analytical characterization, connection with Loewner chains presented in Theorems 2.2.7–2.2.8, 2.2.10–2.2.18, 2.2.31–2.2.32 and others), we also study particular cases (e.g. $k = 1$ and $\alpha = 0$) that are of interest being in close connection with the classes of univalent functions mentioned in the first chapter (see e.g. Propositions 2.2.25 and 2.2.26 in §2.2.2). All the results in this chapter are original and were obtained by the author in [54].

Part II contains results related to biholomorphic mappings of several complex variables. It includes chapters 3 and 4 of the thesis that contain both classical and original results. This part is based on several important books (e.g. [45], [83], [107], [119], [123]) and papers (e.g. [32], [37], [44], [128]) and contains also original results obtained by the author in [52] and [53].

- In **Chapter 3** we present general results related to biholomorphic mappings of several complex variables in $\mathbb{C}^n$. We begin with basic notations, notions and preliminary results that will be used during the second part of the thesis.

  In Section 3.1 we refer to the theory of holomorphic functions, respectively holomorphic mappings in $\mathbb{C}^n$, including the open mapping theorem, the minimum/maximum modulus theorem and its applications (e.g. the Schwarz's lemma). We recall also the definition of a set of uniqueness (see e.g. [45], [83]) and two important results related to this notion, namely the Montel, respectively Vitali's theorem in $\mathbb{C}^n$ (see e.g. [83], [107], [119]). In the final part of this section we mention some general results about holomorphic mappings in $\mathbb{C}^n$ and the main results that will be used in this chapter (e.g. the Schwarz-Pick's lemma).

  Section 3.2 contains classical notions related to the generalization of the Carathéodory class to higher dimensions (i.e. the class of functions with positive real part). We refer here especially to growth and distortion theorems obtained by Graham, Hamada and Kohr (see [32]), Pfaltzgraff (see [109]) and Poreda (see [115]). One of the most important results that was proved by Graham, Hamada and Kohr in 2002 (see [32], [68]) is the compactness of the Carathéodory family $\mathcal{M}$. This result had a strong impact on the history of the geometric function theory in higher dimensions.

  Sections 3.3 and 3.4 are intended for the study of certain subclasses of biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$, respectively on the unit polydisc $\mathbb{U}^n$ in $\mathbb{C}^n$. For $n \geq 2$, we denote by $S(\mathbb{B}^n)$ the family of biholomorphic and normalized mappings on $\mathbb{B}^n$ (see e.g. [45], [83]). It is known that the set $S(\mathbb{B}^n)$ is not locally uniformly bounded and then it does not admit a growth and distortion theorem. As an important consequence of this property due to Cartan (see e.g. [7], [45]) we obtain that $S(\mathbb{B}^n)$ is not compact for $n \geq 2$. Among the most important subclasses of $S(\mathbb{B}^n)$, we mention the family of starlike, starlike of order α, convex and spirallike mappings on $\mathbb{B}^n$. For these mappings we recall analytical and geometric characterizations, growth and distortion results together with suggestive examples that are used throughout this chapter.

  Section 3.5 contains extensions of the notions presented in §1.6 related to Loewner chains, Loewner differential equation and parametric representation in higher dimensions. Pfaltzgraff (see e.g. [109]) was the first who obtained generalizations of the Loewner chains and Loewner differential equation on $\mathbb{B}^n$. The study was extended by Poreda in the case of the unit polydisc in $\mathbb{C}^n$ (see e.g. [115], [116]), respectively by Kubicka and Poreda (see e.g. [86]). Important results were obtained over time by Duren, Graham, Hamada and Kohr (see e.g. [20]), Graham, Hamada and Kohr (see e.g. [32]), Graham, Hamada, Kohr and Kohr (see e.g. [36], [37]) and others. One of the most important difference between the one dimensional case and the higher dimension is the compactness of the family of normalized biholomorphic mappings. It is known that $S(\mathbb{U})$ is a compact set (see Theorem 1.4.4) while the set $S(\mathbb{B}^n)$ is not compact for $n \geq 2$ (see e.g. [7], [45]). This problem was solved by Graham, Hamada and Kohr who introduced the class $S^0(\mathbb{B}^n)$ of mappings which admit parametric representation on $\mathbb{B}^n$ (see e.g. [32]; see also [114]). For $n = 1$, we have that $S^0(\mathbb{B}^1) = S$ (see e.g. [114]). However, if $n \geq 2$, then $S^0(\mathbb{B}^n)$ is strictly included in $S(\mathbb{B}^n)$. Moreover, Graham, Kohr and Kohr (see [48]) proved that $S^0(\mathbb{B}^n)$ is compact (see e.g. [32], [45], [48]). This result is one of the results that presents the clear distinction between one and several complex variables cases. On the other hand, it opened new ways of studying geometric function theory in higher dimensions. Another important problem that was solved by Graham, Hamada, Kohr and Kohr is the existence in $\mathbb{C}^n$ of mappings which cannot be embedded as the first elements of a Loewner chain. Using the family $S^0(\mathbb{B}^n)$ they succeed to prove the analogous of Pommerenke's theorem (see Theorem 1.6.3) in higher dimensions (see [48]; see also [32]). Moreover, the notion of parametric representation was extended to g-parametric representation by Graham, Hamada and Kohr (see e.g. [32]). More details about the class $S^0_g(\mathbb{B}^n)$ will be discussed in the last part of the thesis.

Section 3.6 is devoted to the study of convex combinations of biholomorphic mappings on $\mathbb{B}^n$. We consider mappings of the form $h_\lambda = (1-\lambda)f + \lambda g$, where $f, g \in S(\mathbb{B}^n)$ and $\lambda \in (0, 1)$. It is known that, in general, the convex combination of two normalized biholmorphic mappings is not biholomorphic on $\mathbb{B}^n$ (see e.g. [45], [83]). The phenomenon also occurs in the one dimensional case and was intensively studied by several authors (see e.g. [9], [58], [97], [100]). The main idea of this section is to obtain biholomorphic mappings h_λ on $\mathbb{B}^n$ (or even starlike mappings) as convex combinations of the form $h_\lambda = (1 - \lambda)f + \lambda g$, where $f, g \in S(\mathbb{B}^n)$ and $\lambda \in (0, 1)$. The results presented in this section are original and were obtained by Grigoriciuc in [52].

A powerful tool in the study of biholomorphic mappings in higher dimensions is the theory of extension operators. In Section 3.7 we present extension operators that preserve geometric and analytic properties on the unit ball in $\mathbb{C}^n$. We start our discussion with the Roper-Suffridge extension operator Φ_n (considered by K. Roper and T.J. Suffridge in [121]) and the Graham-Kohr extension operator $\Psi_{n,\alpha}$ (defined by I. Graham and G. Kohr in [44]; see also [43]). Then we will look at two generalizations of the Roper-Suffridge extension operator introduced by Graham, Hamada, Kohr, Kohr and Suffridge (see e.g. [42], [47]) that map a locally univalent function on $\mathbb{U}$ into a locally biholomorphic mapping on $\mathbb{B}^n$. In the final part of this section we present the extension operator introduced by Pfaltzgraff and Suffridge (see [111]) and a generalization of their operator (see e.g. [10]).

Section 3.8 concludes this chapter with an interesting study that combine the ideas presented above, namely extension operators and convex combinations of biholomorphic mappings in $\mathbb{C}^n$. Hence, we discuss about convex combinations of extension operators on $\mathbb{B}^n$. In particular, we consider a new extension operator obtained as a convex combination of two Graham-Kohr type extension operators (see e.g. [43], [44]). The results presented in this section are original.

- **Chapter 4** contains extensions of the main results presented in Chapter 2 related to a new differential operator, respectively new subclasses of biholomorphic mappings on $\mathbb{B}^n$ in $\mathbb{C}^n$.

  In Section 4.1 we discuss about the n-dimensional form of the operator $\mathcal{G}_k$, denoted here by $\mathcal{G}_{n,k}$, for every $n \in \mathbb{N}$ with $n \geq 2$ and $k \in \mathbb{N}$. The operator $\mathcal{G}_{n,k}$ will be used to extend the subclasses E_k and E_k^* from the unit disc $\mathbb{U}$ to the unit ball B^n in $\mathbb{C}^n$ with respect to an arbitrary norm. Even if these classes can be defined in a very general context, the case of the Euclidean unit ball $\mathbb{B}^n$ will be addressed in particular in our discussion, considering the properties that are preserved (or not) from the one dimensional case to higher dimensions.

  The main result that is highlighted in Section 4.2 shows that the family $E_1^*(\mathbb{B}^n)$ coincides with the class K of convex functions for $n = 1$ (see Theorem 4.2.1; see also Proposition 2.2.4). However, for $n \geq 2$, we obtain that $E_1^*(\mathbb{B}^n) \cap K(\mathbb{B}^n) \neq \emptyset$, but $E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n)$. Note that in the case of the subclass $E_1^*(\mathbb{B}^n)$ we obtain a major difference between the one dimensional case and the one of several complex variables, i.e. the family of convex mappings is not the same with the subclass $E_1^*(\mathbb{B}^n)$. Another result that is proved in this section (see Theorem 4.2.3) says something about the connection between $E_1(\mathbb{B}^n)$ and the family $K(\mathbb{B}^n; 1/2)$ of convex mappings of order $1/2$. The inclusion $E_1 \subset K(1/2)$ that holds in the one dimensional case can be partially extended in $\mathbb{C}^n$. Other properties and relevant examples are presented in this section in order to describe the new subclasses introduced by the author (e.g. a Marx-Strohhäcker type theorem for our subclasess).

In Section 4.3 we include a study of two particular cases of the Graham-Kohr extension operator $\Psi_{n,\alpha}$ (presented in §3.7) applied to the family of convex functions K. Although the operator $\Psi_{n,\alpha}$ does not preserve the notion of convexity (see e.g. [44]), we can prove an important property related to the subclass E_1^*. We know that $E_1^* = K$ in $\mathbb{C}$ and thus, in §4.3 we show that $\Psi_{n,\alpha}(K) \subseteq E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n)$ for $\alpha \in \{0, 1\}$. With this result, not only we managed to connect the results proved in Chapters 2 and 4 with the help of the Graham-Kohr extension operator, but we also obtained a new property of the operator $\Psi_{n,\alpha}$. Along with these results, we also propose some questions and open problems related to the Graham-Kohr extension operator and the subclass E_k^* in higher dimensions. All the original results presented here have been obtained by Grigoriciuc in [53].

Part III contains results related to biholomorphic mappings in complex Banach spaces. It includes the last two chapters of the thesis which contain both known results (based on the references [39], [40], [41], [45], [106], [117], [127]) and original results (obtained by the author in [55]).

- **Chapter 5** is dedicated to a short study on biholomorphic mappings and Extension operators in complex Banach spaces. We include here extensions of most of the results presented in previous chapters. Among those who have made important contributions in the geometric function theory of complex variables in the infinite dimensional case are J. Mujica, T. Poreda, T.J. Suffridge (see e.g. [106], [117], [118], [127]) and more recently F. Bracci, I. Graham, H. Hamada, G. Kohr and M. Kohr (see e.g. [3], [34], [38], [39], [40], [41]). We start our discussion from the very recent paper published by Graham, Hamada, Kohr and Kohr regarding biholomorphic mappings, Loewner chains and Extention operators in complex Banach spaces (see e.g. [39], [40], [41]). These papers constitute the basis of our study, containing some of the fundamental ideas in obtaining all the other results in this part.

  Section 5.1 contains basic results and properties of holomorphic functions and holomorphic mappings in infinite dimensions. We present the main notions and results that will be used during this chapter (e.g. the maximum modulus theorem, the Schwarz's lemma). For more details, one may consult [45], [78], [79], [106], [127], [128]. Moreover, we recall here the generalization of the Carathéodory family and the growth results obtained by Gurganus (see [57]), respectively by Bracci, Elin, Shoikhet (see [6]) and Graham, Hamada, Honda, Kohr and Shon (see [31]) in infinite dimensional case.

  Section 5.2 is dedicated to some particular families of biholomorphic mappings in complex Banach spaces. We present here the classes of starlike, convex, respectively ε-starlike mappings together with their analytical characterization. Important contributions were made by Suffridge (see [127]), Gurganus (see [57]), Hamada and Kohr (see e.g. [68], [74]), Gong and Liu (see [25], [26]).

  In Section 5.3 we discuss some general results related to the theory of Loewner chains in complex Banach spaces that will be used in our main results. The study of subordination chains in infinite dimensional spaces was started by Poreda (see e.g. [117], [118]). These ideas were continued and improved by Graham, Hamada, Kohr and Kohr (see e.g. [34], [38], [39], [40], [41]), Hamada and Kohr (see e.g. [70], [72]), Arosio, Bracci, Hamada and Kohr (see e.g. [2], [3]) who obtained important results related to Loewner chains and Loewner PDE in infinite dimensional spaces. The second part of this section contains results related to the notion of parametric representation in infinite dimensions. This notion is due to Graham, Hamada, Kohr and Kohr (see [38]) and represents the generalization of the parametric representation presented in Definition 3.5.11. Also, we discuss in this section about g-parametric representation, g-Loewner chain and particular families of biholomorphic mappings associated to g-Loewner chains. For details, one may consult [32], [34], [45], [60], [61], [62], [74].

- **Chapter 6** contains original results obtained based on the ideas presented by Graham, Hamada, Kohr and Kohr in [39] and [40]. Part of the original results have been obtained by Grigoriciuc in [55].

  In Section 6.1 we consider the Graham-Kohr extension operator Ψ_α on the domain $\Omega_{p,r} = \{(z_1, w) \in \mathcal{Y} = \mathbb{C} \times X : |z_1|^p + \|w\|_X^r < 1\}$, where X is a complex Banach space, $\alpha \in [0, 1]$ and $p, r \geq 1$. Based on the results proved by Graham, Hamada, Kohr and Kohr in [39] for $p = 2$ (see also [40]), we try to obtain extension properties for the general case $p \in [1, \infty)$.

Section 6.2 is dedicated to study of preservation of Loewner chains by the Pfaltzgraff-Suffridge extension operator from one dimension to infinite dimensional complex Banach spaces. Recently, Graham, Hamada, Kohr and Kohr (see e.g. [40]) proved that the Pfaltzgraff-Suffridge extension operator preserves the first elements of Loewner chains from the open unit ball $\mathbb{B}_X$ of an n-dimensional JB*-triple X into a domain $\mathbb{D}_\alpha \subseteq \mathbb{B}_X \times \mathbb{B}_Y$, where Y is a complex Banach space (for the complete results and their proofs, one may consult [33], [35] and [40]). Inspired by these ideas, we prove that the Pfaltzgraff-Suffridge type extension operator preserves the first elements of Loewner chains from the unit ball B^n of $\mathbb{C}^n$ (with respect to different norms, i.e. the Euclidean norm, the maximum norm) to the unit ball of $\mathcal{W} = \mathbb{C}^n \times Y$, where Y is a complex Banach space.

In the final part, we present a list of the main original results included in this thesis. We mention again that Chapters $1-4$ and 6 contains original results.

- **Chapter 1:** Theorem 1.4.9, Theorem 1.4.19, Theorem 1.5.3, Theorem 1.5.7, Proposition 1.5.8, Theorem 1.5.9
- **Chapter 2:** Proposition 2.1.3, Proposition 2.1.4, Proposition 2.1.5, Proposition 2.1.6, Proposition 2.2.4, Theorem 2.2.7, Theorem 2.2.8, Corollary 2.2.9, Theorem 2.2.10, Theorem 2.2.16, Theorem 2.2.18, Corollary 2.2.19, Theorem 2.2.21, Proposition 2.2.25, Proposition 2.2.26, Theorem 2.2.27, Corollary 2.2.28, Lemma 2.2.30, Theorem 2.2.31, Theorem 2.2.32
- **Chapter 3:** Lemma 3.6.4, Lemma 3.6.5, Proposition 3.6.6, Theorem 3.6.8, Proposition 3.8.2, Lemma 3.8.4, Proposition 3.8.5, Theorem 3.8.6, Theorem 3.8.7, Proposition 3.8.8, Theorem 3.8.9
- **Chapter 4:** Remark 4.1.4, Theorem 4.2.1, Theorem 4.2.3, Theorem 4.2.5, Corollary 4.2.6, Proposition 4.3.1, Lemma 4.3.2, Corollary 4.3.3, Theorem 4.3.4, Lemma 4.3.5, Corollary 4.3.6
- **Chapter 6:** Lemma 6.1.1, Theorem 6.1.4, Corollary 6.1.5, Corollary 6.1.6, Corollary 6.1.7, Theorem 6.1.8, Corollary 6.1.9, Theorem 6.1.11, Corollary 6.1.14, Corollary 6.1.15, Theorem 6.2.6, Corollary 6.2.7, Corollary 6.2.8, Corollary 6.2.9, Corollary 6.2.10, Theorem 6.2.11, Theorem 6.2.13, Corollary 6.2.14, Theorem 6.2.16, Corollary 6.2.17, Corollary 6.2.18

Part of the original results listed above are published (or under publication) in the following papers:

- **Grigoriciuc E.S.**, *On some classes of holomorphic functions whose derivatives have positive real part*, Stud. Univ. Babeş-Bolyai Math. **66**(3) (2021), 479–490. **WoS-ESCI**, **IF**(2022): 0.400
- **Grigoriciuc E.S.**, *Some general distortion results for $K(\alpha)$ and $S^*(\alpha)$*, Mathematica **64**(87) (2022), 222–232. (**Scopus**)
- **Grigoriciuc E.S.**, *On Some Convex Combinations of Biholomorphic Mappings in Several Complex Variables*, Filomat **36**(16) (2022), 5503–5519. **WoS-SCIE**, **IF**(2021): 0.988
- **Grigoriciuc E.S.**, *New Subclasses of Univalent Mappings in Several Complex Variables: Extension Operators and Applications*, Comput. Methods Funct. Theory **23**(3) (2023), 533–555. **WoS-SCIE**, **IF**(2022): 1.155
- **Grigoriciuc E.S.**, *New subclasses of univalent functions on the unit disc in $\mathbb{C}$*, Stud. Univ. Babeş-Bolyai Math. **69**(4) (2024), 769–787. **WoS-ESCI**, **IF**(2022): 0.400
- **Grigoriciuc E.S.**, *g-Loewner chains and the Graham-Kohr extension operator in complex Banach spaces*, Comput. Methods Funct. Theory, accepted

The original results included in this thesis were also exposed in more than thirty national or international conferences and research seminars. We mention here the participation in the

- **International Conference of Young Mathematicians**, Institute of Mathematics of National Academy of Sciences, Kyiv, Ukraine, online, 3 – 5 June 2021;
- **8th European Congress of Mathematics**, Portorož, Slovenia, online, 20 – 26 June 2021 (Talk in the Minisymposium *Current topics in Complex Analysis*);
- **The International Conference on Complex Analysis and Related Topics** (*dedicated to the 90-th anniversary of Anatolii Asirovich Goldberg*, 1930-2008), Ivan Franko National University of Lviv, Ukraine, online, 28 June – 1 July 2021;
- **16th International Symposium on Geometric Function Theory and Applications (GFTA 2021)** – *Dedicated to the memory of Professor Gabriela Kohr*, "Lucian Blaga" University of Sibiu, Romania, online, 15 – 18 October 2021;
- **The 14th Joint Conference on Mathematics and Computer Science (14th MaCS)**, "Babeş-Bolyai" University of Cluj-Napoca, Romania, 24 – 27 November 2022;
- **2nd Edition of The Workshop dedicated to the memory of Professor Gabriela Kohr** – *Geometric Function Theory in Several Complex Variables and Complex Banach Spaces*, "Babeş-Bolyai" University of Cluj-Napoca, Romania, 1 – 3 December 2022;
- **9th International Conference on Mathematics and Informatics**, Sapientia Hungarian University of Transylvania, Târgu Mureş, Romania, 7 – 8 September 2023;
- **3rd Edition of The Workshop dedicated to the memory of Professor Gabriela Kohr** – *Geometric Function Theory in Several Complex Variables and Complex Banach Spaces*, "Babeş-Bolyai" University of Cluj-Napoca, Romania, 1 – 3 December 2023;
- **Sixth Romanian Itinerant Seminar on Mathematical Analysis and its Applications (RISMAA)**, "Babeş-Bolyai" University of Cluj-Napoca, Romania, 30 – 31 May 2024.
- **4th Edition of The Workshop dedicated to the memory of Professor Gabriela Kohr** – *Geometric Function Theory in Several Complex Variables and Complex Banach Spaces*, "Babeş-Bolyai" University of Cluj-Napoca, Romania, 29 November – 1 December 2024;

MSC 2020: 32H02 (primary), 30C45 (secondary).

Keywords: univalent function, biholomorphic mapping, Carathéodory family, starlike mapping, convex mapping, convex combination, Loewner chain, g-Loewner chain, parametric representation, g-parametric representation, g-starlikeness, Graham-Kohr extension operator, Pfaltzgraff-Suffridge extension operator, Muir extension operator, complex Banach space.

Acknowledgments

Dedicated to Professor Gabriela Kohr

This thesis is the result of a special collaboration that I have had since 2016 when, for the first time, I had the privilege of meeting ***Professor Gabriela Kohr****. In the following years, I had the honor of working together with* ***Professor Gabriela Kohr*** *for both my bachelor's and master's theses. As one who had the enormous chance to meet* ***Professor Gabriela Kohr*** *and to start my PhD studies under her careful guidance, I want to dedicate this thesis to her.*

I would like to express my sincere thanks to my scientific advisors, **Professor Gabriela Kohr** and **Professor Mirela Kohr**, for their patience, guidance, motivation, continuous support and encouraging words. I am grateful for the opportunity to write this thesis under their supervision, for their effort, and for the help offered in such difficult moments. I especially want to thank **Professor Mirela Kohr** for her sacrifice and guidance in completing this thesis, thus continuing the work started with our dear **Professor Gabriela Kohr**. I am honored and grateful to be one of the students of **Professors Gabriela** and **Mirela Kohr**.

My sincere thanks also go to the members of the Complex Analysis Research Group "Prof. Dr. Gabriela Kohr" at Babeş-Bolyai University for their valuable suggestions and discussions during our Research Seminar.

Finally, I would like to express my gratitude to my parents, Anca and Sorin, for their unwavering understanding and patience over the years. Their sacrifice and continuous support have made all of this possible.

Soli Deo Gloria!

Part I

Contributions in the theory of univalent functions of one complex variable

Chapter 1

Univalent functions of one complex variable

In the first chapter we include classical and well-known results related to univalent functions of one complex variable. We begin with basic notations, notions and preliminary results that will be used during the first part of the thesis. We refer here to the topic of holomorphic functions in $\mathbb{C}$ including the open mapping theorem and the minimum/maximum modulus theorem with its applications (e.g. the Schwarz's lemma or Schwarz-Pick's lemma). In the final part of the first section we recall the concepts of locally uniformly bounded family and normal family of holomorphic functions in $\mathbb{C}$. We end this section with the equivalence of the previous two notions proved by Montel (see e.g. [77], [85], [87]), the characterization of compactness of closed families of holomorphic functions in terms of locally uniformly boundedness (see e.g. [85]) and one of the most important application of the Montel's theorem, namely Vitali's theorem (see e.g. [77], [85], [87]).

Next, we revisit the concept of subordination and we present classical results related to the Carathéodory class $\mathcal{P}$ in $\mathbb{C}$. The family of holomorphic functions with positive real part is an important tool in the characterization of univalent functions and in the theory of Loewner chains on the unit disc $\mathbb{U}$. We include here the distortion theorem for the class $\mathcal{P}$, coefficient estimations and the Herglotz representation formula that characterize the Carathéodory family $\mathcal{P}$.

The third section contains general results related to univalent functions on $\mathbb{U}$. We present several properties of univalent functions, necessary and sufficient conditions of univalence and some examples that we will refer to during the thesis. We end this section with the Riemann mapping theorem which claims the conformal equivalence of every simple connected domain $D \subsetneq \mathbb{C}$ with $\mathbb{U}$. This result is among the most important in the theory of univalent functions in $\mathbb{C}$, but which is not valid in $\mathbb{C}^n$, for $n \geq 2$.

Further, we consider the class S of normalized univalent functions on $\mathbb{U}$ and some particular subclasses of S (starlike, respectively almost starlike of order α, convex of order α and spirallike). For these families of univalent functions on $\mathbb{U}$ we present the well-known results related to growth, distortion and coefficient bounds. Also, we recall here the analytical characterizations of the families $S^*(\alpha)$, $K(\alpha)$ and $\hat{S}_\delta$. It is important to mention here that even if this chapter is an introductory one, it also contains original results obtained by the author in [51] regarding to general distortion theorems for starlike functions of order α (see Theorem 1.4.9), respectively convex functions of order α (see Theorem 1.4.19).

The fifth section of this chapter is dedicated to study the class $\mathcal{R}$ of normalized holomorphic functions whose derivative has positive real part. Here, we include the most important results related to the class $\mathcal{R}$ obtained in [89], [90], [96] or [129]. Together with the classical results we present also several original results obtained by the author in [50]. In §1.5.3 and §1.5.4 we define two new subclasses of functions, namely $\mathcal{R}_p$ and $\mathcal{R}_p(\alpha)$, and study some of their properties (see Theorems 1.5.3, 1.5.9). The class $\mathcal{R}_p(\alpha)$ was introduced in order to generalize the class $\mathcal{R}(\alpha)$ described in §1.5.2. The idea of considering a parameter $\alpha \in [0, 1)$ is inspired from the extensions that Robertson made in [120] for starlike, respectively convex functions of order α. The connection with the Carathéodory family is an important tool that can be used in characterization of these new subclasses.

The last section is focused on the Loewner theory in $\mathbb{C}$. Given the strong relation between Loewner chains and the Loewner differential equation, we present here some key results related to both, which will be utilized in the study of univalent functions mentioned above. In the second part of this section we refer to the analytical characterization of special subfamilies of S via Loewner chains and finally, we present the notion of parametric representation on $\mathbb{U}$ (see e.g. [45], [114]) The characterization of the geometric properties of univalent functions in terms of Loewner chains will play an important role in the Chapter 2, where we use the Loewner theory to describe some new subclasses of univalent functions on $\mathbb{U}$ (see §2.2.2).

The main bibliographic references used to compose this chapter are [19], [29], [45], [85], [77], [96], [102], [113], [114]. For details, one may consult also [8], [12], [23], [87].

1.1 General notions regarding holomorphy in $\mathbb{C}$

The first section contains basic notions, notations and well-known results about holomorphic functions of one complex variable. We include here the main results that are of interest in the theory of holomoprhic functions in $\mathbb{C}$ that will be used during this thesis. The main sources referenced here are [77], [85]. For details, one may consult also [8], [87].

1.1.1 Preliminaries

Let $\mathbb{C}$ be the complex plane, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. During this thesis we denote by

$$\mathcal{U}(w,r) = \big\{z \in \mathbb{C} \,:\, |z-w| < r\big\}$$

the open disc of center $w \in \mathbb{C}$ and radius $r > 0$, respectively by

$$\overline{\mathcal{U}}(w,r) = \big\{z \in \mathbb{C} \,:\, |z-w| \leq r\big\}$$

the closed disc of center $w \in \mathbb{C}$ and radius $r > 0$. In particular, we denote by $\mathbb{U} = \mathcal{U}(0,1)$ *the open unit disc* in $\mathbb{C}$ and by $\partial\mathbb{U}$ the unit circle in $\mathbb{C}$. Also, for simplicity, we use the notation $\mathcal{U}_r = \mathcal{U}(0,r)$ for the open disc of center zero and radius $r > 0$.

1.1.2 Holomorphic functions in $\mathbb{C}$

Let $D \subseteq \mathbb{C}$ be an open set. Then we denote by

$$\mathcal{H}(D) = \big\{f : D \to \mathbb{C} \,:\, f \textit{ is holomorphic on } D\big\}$$

the family of all holomorphic functions on D. In particular, the set $\mathcal{H}(\mathbb{C})$ contains the entire functions on $\mathbb{C}$ (holomorpic on the whole complex plane).

Remark 1.1.1. Let $D \subseteq \mathbb{C}$ be a domain such that $0 \in D$. Then $f \in \mathcal{H}(D)$ is normalized if $f(0) = 0$ and $f'(0) = 1$. For the sake of brevity, let us denote by $\mathcal{H}_0(D)$ *the family of normalized holomorphic functions* on D.

Next, we introduce some classical results related to holomorphic functions that will be used in the subsequent sections (see e.g. [77], [85]). The first result is commonly referred to in the literature as the *open mapping theorem* for holomorphic functions (see e.g [85]). It is important to mention here that this result can be generalized also in the case of holomorphic functions from domains in $\mathbb{C}^n$ into $\mathbb{C}$, respectively in the case of locally biholomoprhic mappings from domains in $\mathbb{C}^n$ into $\mathbb{C}^n$ for $n \geq 2$ (see e.g. [119]).

Theorem 1.1.2 (Open mapping theorem). *Let $f \in \mathcal{H}(D)$, where $D \subseteq \mathbb{C}$ is a domain and f is nonconstant. Then $f(D) \subseteq \mathbb{C}$ is a domain.*

Another important result related to holomorphic functions of one complex variable is *the maximum (minimum) modulus theorem* (see e.g. [77], [85]).

Theorem 1.1.3 (**Maximum/minimum modulus theorem**). *Let $D \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(D)$. If $\exists z_0 \in D$ such that*

$$|f(z_0)| = \max\left\{|f(z)| \,:\, z \in D\right\} \qquad \text{or} \qquad |f(z_0)| = \min\left\{|f(z)| \,:\, z \in D\right\},$$

then f is constant on D.

The first important application of Theorem 1.1.3 is commonly referred to in the literature as the *Schwarz's Lemma* (see e.g. [85]) which says that:

Lemma 1.1.4 (**Schwarz's lemma**). *Let $f \in \mathcal{H}(\mathbb{U})$ satisfy $f(0) = 0$ and $|f(z)| < 1$, for $z \in \mathbb{U}$. Then $|f(z)| \leq |z|$, for $z \in \mathbb{U}$ and $|f'(0)| \leq 1$. Moreover, if $\exists z_0 \in \mathbb{U} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then $\exists a \in \mathbb{C}$ with $|a| = 1$ such that $f(z) = az$, for $z \in \mathbb{U}$.*

1.2 The Carathéodory family in $\mathbb{C}$

The second section of this chapter is devoted to the Carathéodory family in $\mathbb{C}$. We describe here the concept of subordination in $\mathbb{C}$ together with some well-known results related to holomorphic functions with positive real part. The primary references used in this section are [102] and [114].

Definition 1.2.1. Let $f, g \in \mathcal{H}(\mathbb{U})$. Then f *is subordinate to* g and we write $f \prec g$ if $\exists v \in \mathcal{H}(\mathbb{U})$ with $v(0) = 0$ and $|v(z)| < 1$, $z \in \mathbb{U}$ (i.e. Schwarz function) such that $f = g \circ v$ on $\mathbb{U}$.

In order to describe the subordination relation between two holomorphic functions we can use the following characterization result (see e.g. [102], [114]):

Theorem 1.2.2. *Let $f, g \in \mathcal{H}(\mathbb{U})$ be such that g is injective on $\mathbb{U}$. Then $f \prec g$ is equivalent to the fact that $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.*

A well-known family of holomorphic functions (that plays an important role in the characterization of univalent functions and also in the theory of Loewner chains on $\mathbb{U}$) is *the Carathéodory family* $\mathcal{P}$ (see e.g. [45], [102], [114]). Recall that the Carathéodory class is defined by

$$\mathcal{P} = \left\{p \in \mathcal{H}(\mathbb{U}) \,:\, p(0) = 1, \mathfrak{Re}\, p(z) > 0, z \in \mathbb{U}\right\}$$

and contains all holomorphic functions with positive real part on $\mathbb{U}$.

Remark 1.2.3. A simple characterization of the Carathéodory class shows that $p \in \mathcal{P}$ if and only if $\exists \phi$ a Schwarz function with $p(z) = \frac{1+\phi(z)}{1-\phi(z)}$, for all $z \in \mathbb{U}$ (see e.g. [114]).

Another important characterization of the functions from class $\mathcal{P}$ is given by the *Herglotz representation formula*. This result consists in an integral representation of the Carathéodory family on $\mathbb{U}$. Based on this result, we obtain the *growth and distortion theorem* for the Carathéodory class (see e.g. [102]).

Theorem 1.2.4 (**Growth and distortion theorem**). *Let $p \in \mathcal{P}$. Then*

$$\frac{1-|z|}{1+|z|} \leq \mathfrak{Re}\, p(z) \leq |p(z)| \leq \frac{1+|z|}{1-|z|} \tag{1.2.1}$$

and

$$|p'(z)| \leq \frac{2\mathfrak{Re}\, p(z)}{1-|z|^2} \leq \frac{2}{(1-|z|)^2}, \quad z \in \mathbb{U}. \tag{1.2.2}$$

These estimates are sharp and the extremal function is $p : \mathbb{U} \to \mathbb{C}$ given by $p(z) = \frac{1+\lambda z}{1-\lambda z}$, for all $z \in \mathbb{U}$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

The Herglotz representation formula can be used in various ways in the study of univalent functions (see e.g. [19], [45], [102], [114]). One of these applications is to prove the sharp coefficients bounds for the class $\mathcal{P}$ (see e.g. [102]).

Proposition 1.2.5. *Let $p \in \mathcal{P}$ be such that $p(z) = 1 + p_1 z + p_2 z^2 + ... + p_n z^n + ...$, for $z \in \mathbb{U}$. Then*

$$|p_n| \leq 2, \quad n \geq 1. \tag{1.2.3}$$

This result is sharp and the equality holds for the extremal function $p(z) = \frac{1+\lambda z}{1-\lambda z}$, where $z \in \mathbb{U}$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

The final important property of the Carthéodory class that is presented here is related to its compactness as a subset of $\mathcal{H}(\mathbb{U})$ (see [45], [102]).

Theorem 1.2.6. *The Carathéodory family $\mathcal{P} \subseteq \mathcal{H}(\mathbb{U})$ is compact.*

1.3 General results regarding univalent functions in $\mathbb{C}$

Section 1.3 is focused on the idea of univalent function in $\mathbb{C}$. We include here well-known results in this area (univalence, locally univalence, conformal equivalence and other notions). In the final part of the section we include a very powerful result in this setting: the Riemann mapping theorem. The main references used in this section are [19], [29], [45], [85], [77], [114].

Definition 1.3.1. Let $D \subseteq \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$. Then f is *univalent on D* if f is holomorphic and injective on D. We denote by

$$\mathcal{H}_u(D) = \big\{ f : D \to \mathbb{C} \,:\, f \textit{ is univalent on } D \big\}$$

the *set of all univalent functions* on D.

Definition 1.3.2. Let $D \subseteq \mathbb{C}$ be a domain and let $f \in \mathcal{H}(D)$. Then f is called *locally univalent* on D if for each $z \in D$, there exists $r > 0$ such that $f\big|_{\mathcal{U}(z,r)}$ is univalent.

For more details about the notions of univalence and locally univalence presented in the previous two definitions, one may consult [85], [114].

A necessary condition of univalence is presented in the following result (see e.g. [77], [85], [114]).

Theorem 1.3.3. *Let $D \subseteq \mathbb{C}$ and $f \in \mathcal{H}_u(D)$. Then $f' \not\equiv 0$ on D.*

As we said above, the previous result does not ensure a sufficient condition of univalence. To solve this problem, Alexander [1], Noshiro [108], Warschawski [135] and Wolff [136] have obtained an improved version of the condition in Theorem 1.3.3 on particular domains (i.e. convex domains). Recall that a domain $D \subseteq \mathbb{C}$ is convex if for any two points $z_1, z_2 \in D$, the entire segment $[z_1, z_2]$ lies in D, i.e. $(1-t)z_1 + tz_2 \in D$, for all $t \in [0, 1]$.

Theorem 1.3.4. *Let $f \in \mathcal{H}(D)$, where D is a convex domain in $\mathbb{C}$. If $\mathfrak{Re} f'(z) > 0$, for all $z \in D$, then $f \in \mathcal{H}_u(D)$.*

Probably the most important example of univalent function on the unit disc $\mathbb{U}$ is *the Koebe function* presented below (see e.g. [19], [45]):

Example 1.3.5. Let $f : \mathbb{U} \to \mathbb{C}$ be defined by $f(z) = \frac{z}{(1-z)^2}$, for all $z \in \mathbb{U}$. Then $f(\mathbb{U}) = \mathbb{C} \setminus \big\{ w \in \mathbb{C} : \mathfrak{Re} w \leq -1/4, \mathfrak{Im} w = 0 \big\}$ and $f \in \mathcal{H}_u(\mathbb{U})$. The Koebe function plays an important role in the study of the extremal problems.

Univalent functions have also the property that they preserve the simply connected domains in $\mathbb{C}$ (see e.g. [77], [85]) as we can see in the following result:

Theorem 1.3.6. *Let $D \subseteq \mathbb{C}$ be a simply connected domain and $f \in \mathcal{H}_u(D)$. Then $f(D)$ is a simply connected domain in $\mathbb{C}$.*

In the final part of this section we revisit the concept of *conformal equivalence of domains in* $\mathbb{C}$ and the fundamental result in this context, the Riemann mapping theorem (see e.g. [19], [77], [85], [114]).

Definition 1.3.7. Let $D_1, D_2 \subseteq \mathbb{C}$ be two domains.

a) The domains D_1 and D_2 are *conformally equivalent* if $\exists f : D_1 \to D_2$ such that $f \in \mathcal{H}_u(D_1)$ and $f(D_1) = D_2$.

b) The function $f : D_1 \to D_2$ with the previous properties is called a *conformal mapping* between D_1 and D_2.

Finally, we have all the necessary notions and results to present the fundamental result of univalent functions in $\mathbb{C}$, namely the *Riemann mapping theorem* (see [45], [77]). It is very important to mention here that this result is not true in $\mathbb{C}^n$ for $n \geq 2$ (see e.g. [107], [119]).

Theorem 1.3.8 (Riemann mapping theorem). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain. Then D and $\mathbb{U}$ are conformlly equivalent. In addition, if $z_0 \in D$ is given, then $\exists f : D \to \mathbb{U}$ unique conformal mapping such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

1.4 Families of univalent functions on the unit disc $\mathbb{U}$

In this section we present some important families of univalent functions on the unit disc $\mathbb{U}$. First, we shortly describe the class S of normalized univalent functions on $\mathbb{U}$ and then we continue with subclasses of univalent functions that have special geometric properties: the family S^* of normalized starlike (with respect to zero) functions, respectively the family K of normalized convex functions on $\mathbb{U}$. We include also results about the families of starlike and convex functions of order $\alpha \in [0, 1)$, the class of almost starlike functions of order α and the class of spirallike functions of type $\delta \in \big(-\pi/2, \pi/2\big)$. We focus our attention on the analytical and geometric characterizations of the previous families of functions, growth and distortion theorems and coefficient estimations. Among the references used here, we mention [19], [45], [102], [114]. It is important to mention here that §1.4.3 and §1.4.6 contains new results obtained by the author in [51].

1.4.1 Normalized univalent functions

The first class of univalent function that is considered in this section is the class S of normalized univalent functions on $\mathbb{U}$. For this family of functions we present coefficient bounds, growth and distortion theorems. For more details about class S, one may consult [19], [45], [29], [85], [102], [114]. Recall that the family of all *normalized univalent functions* on the unit disc $\mathbb{U}$ is defined by

$$S = \big\{ f \in \mathcal{H}_u(\mathbb{U}) \, : \, f(0) = f'(0) - 1 = 0 \big\}.$$

It is well-known that every function $f \in S$ admits a Taylor series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ..., \tag{1.4.1}$$

for all $z \in \mathbb{U}$. It is clear that the coefficients $a_0 = 0$ and $a_1 = 1$ can be inserted in the previous sum without inducing any change.

Next we present two well-known, but important examples of functions that belongs to class S (see e.g. [19], [85]).

Example 1.4.1. The Koebe function presented in Example 1.3.5 belongs to the class S, since $f \in \mathcal{H}_u(\mathbb{U})$, $f(0) = 0$ and $f'(0) = 1$. Also, $f_\theta \in S$, where f_θ is the generalization of the Koebe function.

It is known that if we consider $f \in S$ of the form (1.4.1), then $|a_2| \leq 2$ and the equality is obtained if and only if f is a rotation of the Koebe function. This result is due to L. Bieberbach (see [4]). Starting from the above estimation, Bieberbach formulated in 1916 the following conjecture (see [4]):

Theorem 1.4.2 (**Bieberbach's conjecture**). *Let $f \in S$ be of the form (1.4.1). Then $|a_n| \leq n$, for all $n \geq 2$. These estimations are exact and the equality is obtained if and only if f is a rotation of the Koebe function.*

The Bieberbach's conjecture was solved by L. de Branges in 1985 (see [18]). Until this year a lot of partial results were obtained by different authors (for details, one may consult [12], [19], [23], [114]).

Another important result related to the class S is *the growth and distortion theorem* (see e.g. [4], [45], [85]).

Theorem 1.4.3. *Let $f \in S$. Then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \tag{1.4.2}$$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \tag{1.4.3}$$

and

$$\frac{1-|z|}{1+|z|} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|}, \tag{1.4.4}$$

for all $z \in \mathbb{U}$. These estimates are sharp and the equality is obtained if and only if f is a rotation of the Koebe function.

We end this first part of our section with a result related to the compactness of the class S, established through the upper bounds from the estimates (1.4.2). For details, one may consult [45], [102].

Theorem 1.4.4. *The class $S \subseteq \mathcal{H}(\mathbb{U})$ is compact.*

1.4.2 Starlike functions

Among the special subclasses of the class S is the family S^* of normalized starlike functions on $\mathbb{U}$. In Subsection 1.4.2 we present results related to the analytical characterization of starlikeness, coefficient estimates, distortion and growth results. Other properties of starlike functions on the unit disc may be found in [19], [29], [45], [102], [114].

First, let us remember the definition of a starlike domain with respect to a given point in $\mathbb{C}$, respectively of a starlike function on $\mathbb{U}$.

Definition 1.4.5. Let $D \subseteq \mathbb{C}$ be a domain and let $z_0 \in D$. Then D is *starlike with respect to z_0* if the closed line segment $[z_0, z]$ lies entirely in D, $\forall z \in D$.

Taking into account the previous definition, we recall the concept of starlike function on $\mathbb{U}$. This definition was presented by Alexander in [1].

Definition 1.4.6. Let $f \in \mathcal{H}(\mathbb{U})$ be such that $f(0) = 0$. Then f is *starlike* on the unit disc $\mathbb{U}$ if $f \in \mathcal{H}_u(\mathbb{U})$ and $f(\mathbb{U})$ is a starlike (with respect to 0) domain in $\mathbb{C}$. We denote by S^* the family of normalized starlike functions on $\mathbb{U}$.

The next important result in this context is the analytical characterization of starlikeness (see e.g. [19], [45], [102]). This result plays a very important role in the geometric function theory in $\mathbb{C}$.

Theorem 1.4.7. *Let $f \in \mathcal{H}(\mathbb{U})$ satisfy $f(0) = 0$. Then $f \in S^*$ if and only if $f'(0) \neq 0$ and*

$$\mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in \mathbb{U}.$$

It is clear that $S^* \subseteq S$ and $f_\theta \in S^*$, for all $\theta \in \mathbb{R}$. Hence, the growth and distortion result for the class S (see Theorem 1.4.3) remains true and sharp also for the class S^* (see e.g. [95], [102]). As a direct consequence we obtain that the family S^* is a compact subset of $\mathcal{H}(\mathbb{U})$ (see e.g. [45], [95]).

1.4.3 Starlike functions of order α

The next important subclass of S that is studied in this section is the family of normalized starlike functions of order $\alpha \in [0,1)$ on $\mathbb{U}$. The idea of starlikeness of order α was described by M.S. Robertson in [120]. For details, one may consult also [19], [29], [45], [102].

Definition 1.4.8. Let $0 \le \alpha < 1$ and $f \in \mathcal{H}(\mathbb{U})$. The function $f : \mathbb{U} \to \mathbb{C}$ is *starlike of order* α if $f(0) = 0$, $f'(0) \neq 0$ and $\mathfrak{Re}\big[\frac{zf'(z)}{f(z)}\big] > \alpha$, for all $z \in \mathbb{U}$. In this thesis, we denote by

$$S^*(\alpha) = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha, \quad z \in \mathbb{U} \right\}$$

the *family of normalized starlike functions of order* α on $\mathbb{U}$.

It is clear that $S^*(\alpha) \subseteq S$ for every $\alpha \in [0,1)$ and $S^*(0) = S^*$.

Based on the coefficient bounds for the class $S^*(\alpha)$, we can obtain a generalized distortion theorem. This result was obtained by Grigoriciuc in [51].

Theorem 1.4.9. *Let $\alpha \in [0,1)$ and $f \in S^*(\alpha)$. Then*

$$|f^{(k)}(z)| \le \frac{B(k,\alpha)\big[k + |z| \cdot (1-2\alpha)\big]}{(1-|z|)^{k+2-2\alpha}}, \quad z \in \mathbb{U}, \quad k \ge 1, \tag{1.4.5}$$

where

$$B(k,\alpha) = \begin{cases} \dfrac{1}{1-2\alpha} \displaystyle\prod_{m=1}^{k} (m-2\alpha), & \alpha \neq \dfrac{1}{2} \\ (k-1)!, & \alpha = \dfrac{1}{2}. \end{cases}$$

These bounds are sharp.

1.4.4 Almost starlike functions of order α

The notion of almost starlikeness of order α was defined by Feng (see [22]) in the case of complex Banach spaces. We present here the family of almost starlike functions of order $\alpha \in [0,1)$ on the unit disc $\mathbb{U}$ in $\mathbb{C}$.

Definition 1.4.10. Let $\alpha \in [0,1)$ and let $f \in \mathcal{H}_0(\mathbb{U})$. Then f is *almost starlike of order* α if

$$\mathfrak{Re}\left[\frac{f(z)}{zf'(z)}\right] > \alpha, \quad z \in \mathbb{U}. \tag{1.4.6}$$

1.4.5 Convex functions

In the following subsection we briefly describe the class K of normalized convex functions on $\mathbb{U}$. Starting with E. Study who introduced this notion in 1913, many other authors contributed to the study of family K (see e.g. T. Gronwall [56] and K. Loewner [95]). Here we present the analytical characterization of convexity on $\mathbb{U}$, coefficients bounds and growth and distortion results. Two other important results that establish the connection between classes K and S^* are presented here, namely the Alexander's duality theorem and the Marx-Strohhäcker theorem. For details an other results related to convex functions one may consult [19], [29], [45], [102], [114].

Definition 1.4.11. Let $D \subseteq \mathbb{C}$ be a domain and let $z_0 \in D$. Then D is *convex* if for all $z_1, z_2 \in D$, the segment $[z_1, z_2] \subseteq D$, i.e. $(1-t)z_1 + tz_2 \in D$, for every $t \in [0,1]$.

Definition 1.4.12. Let $f \in \mathcal{H}(\mathbb{U})$. Then f is *convex* on the unit disc $\mathbb{U}$ if $f \in \mathcal{H}_u(\mathbb{U})$ and $f(\mathbb{U})$ is a convex domain in $\mathbb{C}$. We denote by K the family of normalized convex functions on $\mathbb{U}$.

It is clear that K and S^* are both subsets of S. Moreover, we have that $K \subset S^* \subset S$ (see e.g. [19], [29], [45]).

In order to describe a function from class K we can use the following analytical characterization of convexity (see e.g. [19], [29], [45], [102]):

Theorem 1.4.13. *Let $f \in \mathcal{H}(\mathbb{U})$. Then $f \in K$ if and only if $f'(0) \neq 0$ and $\mathfrak{Re}\big[1 + \frac{zf''(z)}{f'(z)}\big] > 0$, for all $z \in \mathbb{U}$.*

The following growth and distortion result is true for the family K (see e.g. [45], [56], [95]):

Theorem 1.4.14. *Let $f \in K$. Then*

$$\frac{|z|}{1+|z|} \leq |f(z)| \leq \frac{|z|}{1-|z|} \tag{1.4.7}$$

and

$$\frac{1}{(1+|z|)^2} \leq |f'(z)| \leq \frac{1}{(1-|z|)^2}, \quad z \in \mathbb{U}. \tag{1.4.8}$$

These estimates are sharp and the equality is obtained for $f(z) = \frac{z}{1-\lambda z}$, where $z \in \mathbb{U}$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

According to the previous result we deduce that the family K is locally uniformly bounded. Hence, we obtain that K is compact, since the class is also closed (see e.g. [45], [102]). Another important result is given by the following coefficient estimations for the class K (see e.g. [95]).

Proposition 1.4.15. *Let $f \in K$ be of the form (1.4.1). Then $|a_n| \leq 1$ for all $n \geq 2$. The estimations are sharp and the equality is obtained for $f(z) = \frac{z}{1-\lambda z}$, where $z \in \mathbb{U}$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

The next result is due to Alexander [1] and describes the relation between the families S^* and K (see e.g. [102]). Note that in the case of normalized convex mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ this result is not true (see e.g. [45], [121], [128]).

Theorem 1.4.16 (Alexander's duality theorem). *Let $f \in \mathcal{H}(\mathbb{U})$ be a function with the property $f(0) = 0$. Then $f \in K$ if and only if $F \in S^*$, where $F(z) = zf'(z)$, for $z \in \mathbb{U}$.*

1.4.6 Convex functions of order α

Closely related to the class K is the family of normalized convex functions of order α, with $\alpha \in [0, 1)$. This notion was also introduced by M.S. Robertson in [120]. For details, one may consult also [19], [29], [45], [102].

Definition 1.4.17. Let $0 \leq \alpha < 1$ and $f \in \mathcal{H}(\mathbb{U})$. The function $f : \mathbb{U} \to \mathbb{C}$ is *convex of order* α if $f'(0) \neq 0$ and $\mathfrak{Re}\big[1 + \frac{zf''(z)}{f'(z)}\big] > \alpha$, for all $z \in \mathbb{U}$. In this thesis, we denote by

$$K(\alpha) = \Big\{f \in \mathcal{H}_0(\mathbb{U}) : \mathfrak{Re}\Big[1 + \frac{zf''(z)}{f'(z)}\Big] > \alpha, \quad z \in \mathbb{U}\Big\}$$

the *family of normalized convex functions of order* α on $\mathbb{U}$.

It is clear that $K(\alpha) \subseteq S$, for every $\alpha \in [0, 1)$ and $K(0) = K$. Moreover, an Alexander type theorem holds for the classes $K(\alpha)$ and $S^*(\alpha)$, for $0 \leq \alpha < 1$ (see [45]).

For every function that belong to the class $K(\alpha)$ the following coefficient bounds were obtained (see e.g. [29], [120]):

Proposition 1.4.18. *Let $\alpha \in [0, 1)$ and $f \in K(\alpha)$. Then*

$$|a_n| \leq \frac{1}{n!} \prod_{m=2}^{n} (m - 2\alpha), \quad n \geq 2. \tag{1.4.9}$$

These estimates are sharp.

In view of Proposition 1.4.18 we obtain a generalized distortion theorem for the class $K(\alpha)$, with $\alpha \in [0,1)$. This result is the analogue of Theorem 1.4.9 from the previous section and was obtained by Grigoriciuc in [51].

Theorem 1.4.19. *Let $\alpha \in [0,1)$ and $f \in K(\alpha)$. Then*

$$|f^{(k)}(z)| \leq \frac{B(k,\alpha)}{(1-|z|)^{k+1-2\alpha}}, \quad z \in \mathbb{U}, \quad k \geq 1,$$

where

$$B(k,\alpha) = \begin{cases} \dfrac{1}{1-2\alpha} \displaystyle\prod_{m=1}^{k} (m-2\alpha), & \alpha \neq \dfrac{1}{2} \\ (k-1)!, & \alpha = \dfrac{1}{2} \end{cases}$$

These bounds are sharp.

1.4.7 Spirallike functions

In the last part of this section we present some general results related to spirallikeness on $\mathbb{U}$. This notion was defined by L. Špaček in 1932 (see [125]) as a generalization of starlikeness on $\mathbb{U}$. For details, one may consult also [19], [45], [102].

Definition 1.4.20. Let $\delta \in \mathbb{R}$ be such that $|\delta| < \frac{\pi}{2}$.

a) A *logarithmic δ-spiral* (or *δ-spiral*) is a curve in $\mathbb{C}$ given by

$$s(t) = s_0 e^{-(\cos\delta - i\sin\delta)t}, \quad t \in \mathbb{R} \text{ and } s_0 \in \mathbb{C}^*.$$

b) A domain $D \subseteq \mathbb{C}$ with $0 \in D$ is called *δ-spirallike* if for every point $z_0 \in D \setminus \{0\}$, the arc of the δ-spiral between z_0 and the origin lies in D.

Definition 1.4.21. Let $\delta \in \mathbb{R}$ be such that $|\delta| < \frac{\pi}{2}$ and $f \in \mathcal{H}(\mathbb{U})$ with $f(0) = 0$. Then f is called

a) *spirallike of type* δ on the unit disc $\mathbb{U}$ if $f \in \mathcal{H}_u(\mathbb{U})$ and $f(\mathbb{U})$ is a δ-spirallike domain in $\mathbb{C}$;

b) *spirallike* if there exists $\delta \in \mathbb{R}$ with $|\delta| < \pi/2$ such that f is spirallike of type δ on the unit disc $\mathbb{U}$.

Recall that we denote by $\hat{S}_\delta$ the family of all normalized spirallike functions of type δ on $\mathbb{U}$. It is clear that $\hat{S}_\delta \subseteq S$ and $\hat{S}_0 = S^*$ (see e.g. [29], [102]).

Following this, we present the analytical characterization of spirallikeness of type δ on $\mathbb{U}$ due to Špaček (see [125]; see also [45], [102]):

Theorem 1.4.22. *Let $f \in \mathcal{H}(\mathbb{U})$ satisfy $f(0) = 0$ and $f'(0) \neq 0$. Also let $\delta \in (-\pi/2, \pi/2)$. Then $f \in \hat{S}_\delta$ if and only if*

$$\mathfrak{Re}\left[e^{i\delta}\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in \mathbb{U}. \tag{1.4.10}$$

1.5 Functions whose derivative has positive real part

Strongly related to the class S is the family $\mathcal{R}$ of normalized holomorphic functions whose derivative has positive real part. In view of the result proved by Alexander, Noshiro, Warschawski and Wolff (see Theorem 1.3.4; see also [19], [77]) it follows that that every function $f \in \mathcal{R}$ is also univalent on $\mathbb{U}$.

Hence, $\mathcal{R}$ is a subclass of the class S (see e.g. [19], [96] or [102]) and it is usually called the Noshiro-Warschawski class. We present in this section some classical results related to the class $\mathcal{R}$ and extensions of this class (see e.g. [89], [90], [96], [102]), as well as original results obtained by Grigoriciuc in [50].

1.5.1 General results related to the class $\mathcal{R}$

First, recall that we denote by

$$\mathcal{R} = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, \mathfrak{Re} f'(z) > 0, \quad z \in \mathbb{U} \right\}$$

the *family of normalized holomorphic functions whose derivative has positive real part* (see e.g. [29], [45] or [102]). As we already mention above, it is known that $\mathcal{R} \subseteq S$ (see e.g. [19], [96] or [102]).

Taking into account the definition of class $\mathcal{P}$ it easy to observe that $f \in \mathcal{R}$ if and only if $f' \in \mathcal{P}$ (see e.g. [102]). This is also equivalent with the property that $|\arg f'(z)| < \frac{\pi}{2}$, for every $z \in \mathbb{U}$.

Example 1.5.1. Let $f : \mathbb{U} \to \mathbb{C}$ be given by

$$f(z) = -z - \frac{2}{\lambda} \log(1 - \lambda z), \tag{1.5.1}$$

for all $z \in \mathbb{U}$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then $f \in \mathcal{R}$.

The first result presented here is regarding to the growth and distortion theorem and the coefficient estimations for the class $\mathcal{R}$ (see e.g. [96], [102] or [129]).

Proposition 1.5.2. *Let $f \in \mathcal{R}$. Then*

$$|a_n| \leq \frac{2}{n}, \quad n \geq 2, \tag{1.5.2}$$

$$\frac{1-|z|}{1+|z|} \leq \mathfrak{Re} f'(z) \leq |f'(z)| \leq \frac{1+|z|}{1-|z|} \tag{1.5.3}$$

and

$$-|z| + 2\log(1+|z|) \leq |f(z)| \leq -|z| - 2\log(1-|z|), \tag{1.5.4}$$

for all $z \in \mathbb{U}$. These estimates are exact and the equality is obtained for the function given by (1.5.1).

Starting from the previous result, we can prove a general distortion theorem for the class $\mathcal{R}$. This result was obtained by Grigoriciuc in [50].

Theorem 1.5.3. *If $f \in \mathcal{R}$, then*

$$|f^{(k)}(z)| \leq \frac{2(k-1)!}{(1-|z|)^k}, \quad z \in \mathbb{U}, \quad k \geq 2.$$

The estimates are sharp and the equality is obtained for the function given by (1.5.1).

1.5.2 The class $\mathcal{R}(\alpha)$

A first generalization of the class $\mathcal{R}$ was considered by Krishna, RamReddy and Venkateswarlu in [89] and [90]. For a real parameter $\alpha \in [0, 1)$, they denoted by

$$\mathcal{R}(\alpha) = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, \mathfrak{Re} f'(z) > \alpha, \quad z \in \mathbb{U} \right\}$$

the *class of normalized holomorphic functions whose derivative has positive real part of order α.* This family of functions was studied also by Grigoriciuc in [50].

Remark 1.5.4. It is easy to prove that f belongs to the class $\mathcal{R}(\alpha)$ if and only if $g \in \mathcal{P}$, where $g : \mathbb{U} \to \mathbb{C}$ is given by $g(z) = \frac{1}{1-\alpha}\big(f'(z) - \alpha\big)$, for all $z \in \mathbb{U}$.

Based on the previous equivalence between $\mathcal{R}(\alpha)$ and $\mathcal{P}$, we can obtain the following example:

Example 1.5.5. Let $\alpha \in [0, 1)$ and let $f : \mathbb{U} \to \mathbb{C}$ be given by

$$f(z) = \frac{1}{\lambda}\Big[(2\alpha - 1)\lambda z - 2(1-\alpha)\log(1-\lambda z)\Big], \tag{1.5.5}$$

where $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. Then $f \in \mathcal{R}(\alpha)$.

Next we present the coefficient bounds for the class $\mathcal{R}(\alpha)$ obtained by Grigoriciuc in [50] (see e.g. [90] for a different proof of this result).

Proposition 1.5.6. *Let $\alpha \in [0,1)$ and $f \in \mathcal{R}(\alpha)$. Then $|a_n| \leq \frac{2(1-\alpha)}{n}$, for all $n \geq 2$. These estimates are sharp and the equality is obtained for the function given by (1.5.5).*

For the class $\mathcal{R}(\alpha)$ we can obtain also a growth and distortion result. This result is original and was obtained by Grigoriciuc in [50].

Theorem 1.5.7. *Let $\alpha \in [0,1)$ and $f \in \mathcal{R}(\alpha)$. Then*

$$|f(z)| \leq (2\alpha - 1)|z| + 2(\alpha - 1)\log(1 - |z|) \tag{1.5.6}$$

and

$$|f(z)| \geq -|z| - 2(\alpha - 1)\log(1 + |z|), \quad z \in \mathbb{U}. \tag{1.5.7}$$

Moreover,

$$\frac{1 - 2\alpha - |z|}{1 + |z|} \leq |f'(z)| \leq \frac{1 + (1 - 2\alpha)|z|}{1 - |z|}, \quad z \in \mathbb{U}. \tag{1.5.8}$$

These estimates are sharp and the extremal function is given by (1.5.5).

1.5.3 The class $\mathcal{R}_p$

In the third part of this section we consider another extension of the class $\mathcal{R}$, namely the class

$$\mathcal{R}_p = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, f^{(p)}(0) = 1, \mathfrak{Re} f^{(p)}(z) > 0, \quad z \in \mathbb{U} \right\}, \quad p \geq 1,$$

of *normalized holomorphic functions whose p-th derivative has positive real part.* The original results presented in this part have been obtained in [50].

It is important to mention here that the connection with the Carathéodory class $\mathcal{P}$ is preserved. Indeed, if $p \in \mathbb{N}^* = \{1, 2, ...\}$ is arbitrary fixed, then $f \in \mathcal{R}_p$ if and only if $f^{(p)} \in \mathcal{P}$. Hence, we can study the properties of the class $\mathcal{R}_p$ in terms of the Carathéodory class. It is clear that $\mathcal{R}_1 = \mathcal{R}$.

The following statement due to Grigoriciuc (see [50]) presents the coefficient bounds for the class $\mathcal{R}_p$ and is a generalization of Proposition 1.5.6.

Proposition 1.5.8. *Let $p \in \mathbb{N}^*$ and $f \in \mathcal{R}_p$ be of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, for $z \in \mathbb{U}$. Then*

$$|a_n| \leq \frac{2(n-p)!}{n!}, \quad n \geq p + 1. \tag{1.5.9}$$

Next, we present a general distortion theorem obtained by Grigoriciuc in [50].

Theorem 1.5.9. *Let $p \in \mathbb{N}^*$ and $f \in \mathcal{R}_p$ be of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, for $z \in \mathbb{U}$. Then*

$$|f^{(k)}(z)| \leq \frac{2(k-p)!}{(1 - |z|)^{k-p+1}}, \quad z \in \mathbb{U}, \quad k \geq p. \tag{1.5.10}$$

1.5.4 The class $\mathcal{R}_p(\alpha)$

The last part of this section contains a generalization of the previous class (according to the model presented in [89] and [90]) introduced by Grigoriciuc in [50]. For $\alpha \in [0,1)$ and $p \in \mathbb{N}^*$ we denote by

$$\mathcal{R}_p(\alpha) = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, f^{(p)}(0) = 1, \mathfrak{Re} f^{(p)}(z) > \alpha, \quad z \in \mathbb{U} \right\}.$$

the *class of normalized holomorphic functions whose p-th derivative has positive real part of order α*. The class $\mathcal{R}_p(\alpha)$ was introduced in order to generalize the class $\mathcal{R}_p$ described in the previous subsection. The idea of considering a parameter $\alpha \in [0,1)$ is taken from the extensions that Robertson made in [120] for starlike, respectively convex functions.

In the light of the results presented previously, we have that $f \in \mathcal{R}_p(\alpha)$ if and only if $g \in \mathcal{P}$, where $g : \mathbb{U} \to \mathbb{C}$ is given by $g(z) = \frac{f^{(p)}(z) - \alpha}{1 - \alpha}$, for all $z \in \mathbb{U}$.

It is important that the general class $\mathcal{R}_p(\alpha)$ can be described in terms of Carathéodory functions, i.e. that belong the class $\mathcal{P}$ and hence, we can obtain coefficient bounds and a distortion result for the class $\mathcal{R}_p(\alpha)$.

1.6 The theory of Loewner chains in $\mathbb{C}$

In the sixth section we include a short introduction in the theory of Loewner chains in $\mathbb{C}$. We recall some well-known definitions (e.g. univalent subordination chain, Loewner chain) and some important results related to them. In the second part of this section we refer to the analytical characterization of some subclasses of S via Loewner chains and finally, we present the notion of parametric representation on $\mathbb{U}$. The main references used in this section are [45], [102], [113], [114]. More details about the theory of Loewner chains can be found in [12], [16], [17], [19], [23].

1.6.1 General results related to Loewner chains in $\mathbb{C}$

We start this last section by presenting preliminary notions and results related to the theory of Loewner chains in $\mathbb{U}$ (see e.g. [45], [102], [113], [114]).

Definition 1.6.1. Let $f = f(z,t) : \mathbb{U} \times [0,\infty) \to \mathbb{C}$ be a function.

- If the conditions $f(\cdot,t) \in \mathcal{H}_u(\mathbb{U})$, $f(0,t) = 0$, for $t \geq 0$ and $f(\cdot,s) \prec f(\cdot,t)$, for $0 \leq s \leq t < \infty$ are fulfilled, then $f(z,t)$ is a *univalent subordination chain* on $\mathbb{U}$.
- If $f(z,t)$ satisfies also the property that $f'(0,t) = e^t$, for all $t \geq 0$, then f is called a *Loewner chain* (*normalized univalent subordination chain*) on $\mathbb{U}$.

Note that we use the notation $f'(z,t)$ for the partial derivative $\frac{\partial f}{\partial z}(z,t)$.

Remark 1.6.2. Let $f(z,t)$ be a Loewner chain. Then $\exists v = v(z,s,t)$ a unique Schwarz function associated to $f(z,t)$ such that

$$f(z,s) = f\big(v(z,s,t),t\big), \quad \forall z \in \mathbb{U}, \quad 0 \leq s \leq t < \infty, \tag{1.6.1}$$

called *transition function* of f (see e.g. [45]).

Based on the normalization of $f(z,t)$, we easily obtain that $v'(0,s,t) = e^{s-t}$, for all $0 \leq s \leq t < \infty$. From (1.6.1), it follows that $v = v(z,s,t)$ satisfies the *semigroup property*

$$v(z,s,T) = v\big(v(z,s,t),t,T\big), \tag{1.6.2}$$

for all $z \in \mathbb{U}$ and $0 \leq s \leq t \leq T < \infty$. Moreover, the function $|v(z,s,t)|$ is decreasing with respect to $t \in [s,\infty)$, for all $z \in \mathbb{U}$ and $s \geq 0$.

Probably one of the key results in the theory of Loewner chains is related to the connection between normalized univalent functions and Loewner chains. This result is due to Pommerenke (see [114]) and says that

Theorem 1.6.3 (**Pommerenke's theorem**). *Let $f \in S$ and let $f(\cdot,t)$ be a Loewner chain on $\mathbb{U}$, for all $t \geq 0$. Then $f(\cdot,0) = f$.*

Next we present some classical, but very significant results related to the theory of Loewner chains and the Loewner differential equation on $\mathbb{U}$ (see e.g. [45], [114]). The first theorem was proved by Pommerenke (see [114]) and shows a method that generates Loewner chains (see e.g. [45]).

Theorem 1.6.4. *Assume that $p = p(z,t) : \mathbb{U} \times [0,\infty) \to \mathbb{C}$ satisfy the properties: $p(\cdot,t)$ belongs to the class $\mathcal{P}$, for all non-negative numbers t and for any $z \in \mathbb{U}$, $p(z,\cdot)$ belongs to the class of measurable functions on $[0,\infty)$. Let also*

$$\begin{cases} \dfrac{\partial v}{\partial t} = -vp(v,t), \quad a.e. \quad t \geq s \\ v(z,s,s) = z \end{cases} \tag{1.6.3}$$

be an initial value problem. Then $\forall z \in \mathbb{U}$ and $s \geq 0$, the problem (1.6.3) is uniquely solvable and its solution $v(z,s,\cdot)$ is locally absolutely continuous with $v'(0,s,t) = e^{s-t}$. Morevoer, if $s \geq 0$ and $z \in \mathbb{U}$, then $v(z,s,\cdot)$ belongs to the class of Lipschitz continuous functions on $[s,\infty)$ locally uniformly with respect to z

and $v(\cdot, s, t)$ belongs to the class of univalent Schwarz functions for every $t \geq s$. In addition, for all $s \geq 0$, the limit $\lim_{t\to\infty} e^t v(z, s, t) = f(z, s)$ exists locally uniformly on $\mathbb{U}$ and $f(z, s)$ is a Loewner chain which satisfies the equation

$$\frac{\partial f}{\partial t}(z,t) = zp(z,t)f'(z,t), \quad a.e. \quad t \geq 0, \quad \forall z \in \mathbb{U}. \tag{1.6.4}$$

Recall that the differential equation (1.6.4) is known as the *Loewner(-Kufarev) differential equation* (see e.g. [45], [114]).

The next result is a characterization of the Loewner chains and was obtained by Pommerenke in [113] (see also [45], [114]).

Theorem 1.6.5. *Let $f : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ be such that $f(0, t) = 0$ and $f'(0, t) = e^t$, for all $t \geq 0$. Then $f(z, t)$ is a Loewner chain if and only if*

a) $\exists \rho \in (0, 1)$ and $\alpha > 0$ such that $f(\cdot, t)$ belongs to the family $\mathcal{H}(\mathcal{U}_\rho)$ for every $t \geq 0$, $f(z, \cdot)$ belongs to the class of locally absolutely continuous functions on $[0, \infty)$ locally uniformly with respect to $z \in \mathcal{U}_\rho$ and $|f(z, t)| \leq \alpha e^t$, for all $z \in \mathcal{U}_\rho$ and $t \geq 0$.

b) $\exists p = p(z, t)$ such that $p(\cdot, t)$ belongs to the class $\mathcal{P}$ for each $t \geq 0$, $p(z, \cdot)$ belongs to the family of measurable functions on $[0, \infty)$ for each $z \in \mathbb{U}$ and for almost every $t \geq 0$, we have that

$$\frac{\partial f}{\partial t}(z,t) = zp(z,t)f'(z,t), \quad z \in \mathcal{U}_\rho.$$

1.6.2 Loewner chains and univalent functions in $\mathbb{C}$

We end this section by showing how the theory of Loewner chains can be used in the characterization of the geometric properties of the univalent functions (see e.g. [45], [114], [137]). First, we present the characterization of spirallikeness (in particular, starlikeness) via Loewner chains (see e.g. [45], [114]).

Theorem 1.6.6. *Let $f \in \mathcal{H}_0(\mathbb{U})$, $\delta \in \mathbb{R}$ with $|\delta| < \frac{\pi}{2}$ and $\alpha = \tan\delta$. Then $f \in \hat{S}_\delta$ if and only $f(z,t) = e^{(1-i\alpha)t} f(e^{i\alpha t} z)$ is a Loewner chain, for all $z \in \mathbb{U}$ and $t \geq 0$. In particular, for $\delta = 0$, we obtain that $f \in S^*$ if and only if $f(z, t) = e^t f(z)$, for all $z \in \mathbb{U}$ and $t \geq 0$.*

Next, we mention the characterization of almost starlikeness of order α via Loewner chains (see e.g. [137]).

Theorem 1.6.7. *Let $\alpha \in [0, 1)$ and $f \in \mathcal{H}_0(\mathbb{U})$. Then f is almost starlike of order α if and only if $f(z,t) = e^{\frac{t}{1-\alpha}} f\big(e^{\frac{\alpha t}{\alpha-1}} z\big)$ is a Loewner chain, for all $z \in \mathbb{U}$ and $t \geq 0$.*

Finally, we can also obtain a characterization of convexity through Loewner chains (see e.g. [45], [114]).

Theorem 1.6.8. *Let $f \in \mathcal{H}_0(\mathbb{U})$. Then $f \in K$ if and only if $f(z, t) = f(z) + (e^t - 1)zf'(z)$ is a Loewner chain, for all $z \in \mathbb{U}$ and $t \geq 0$.*

1.6.3 Parametric representation on $\mathbb{U}$

In this subsection we present the notion of parametric representation on $\mathbb{U}$ (see e.g. [45], [114]). Since every function in the class S can be seen as the first element of a Loewner chain (see Theorem 1.6.3; see also [45], [114]), it is clear that it has parametric representation.

Definition 1.6.9. A function $f \in \mathcal{H}_0(\mathbb{U})$ has *parametric representation on* $\mathbb{U}$ if $\exists p : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ such that $p(\cdot, t)$ belongs to the class $\mathcal{P}$, for $t \geq 0$, $p(z, \cdot)$ belongs to the family of measurable functions on $[0, \infty)$, for $z \in \mathbb{U}$ and the limit $\lim_{t\to\infty} e^t v(z, t) = f(z)$ is locally uniformly on $\mathbb{U}$, where $v(z, \cdot)$ is the solution of the problem (1.6.3) for $s = 0$ with the property that v is unique locally absolutely continuous on $[0, \infty)$.

If we denote by $S^0(\mathbb{U})$ the family of mappings which have parametric representation, then $S^0(\mathbb{U}) = S$ based on Theorem 1.6.3 (see e.g. [45], [114]). However, in higher dimensions, this result is not true (see e.g. [32]) as we shall see in Chapter 3.

Chapter 2

New subclasses of univalent functions on $\mathbb{U}$

The main idea of the second chapter consists in the study of a new differential operator and two new subclasses of univalent functions on the unit disc $\mathbb{U}$ defined with this operator. This chapter is made up entirely of original results obtained by the author in [54].

In the first section we present the differential operator $\mathcal{G}_k$ defined on the family $\mathcal{H}_0(\mathbb{U})$ of normalized holomorphic functions on $\mathbb{U}$. Using the operator $\mathcal{G}_k$ we can construct some particular subclasses of univalent functions on the unit disc $\mathbb{U}$ that are strongly related to the families S^*, respectively K, as we can see in §2.2. Several properties of the operator $\mathcal{G}_k$ are studied in this section, e.g the linearity of $\mathcal{G}_k$, convolution product and a sufficient condition of univalence for $\mathcal{G}_k$ (see Propositions 2.1.3 − 2.1.6). It is important to mention here that the differential operator $\mathcal{G}_k$ is different from the Sălăgean differential operator D^n (see Remark 2.2.6; see also [124]). Another important remark is that the operator $\mathcal{G}_k$ can be extended in $\mathbb{C}^n$ (see Chapter 4; see also [53]).

Using the differential operator $\mathcal{G}_k$ mentioned above, we can construct some particular families of univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$. These subclasses, denoted here by $E_k^*(\alpha)$, respectively $E_k(\alpha)$, where $\alpha \in [0,1)$, are related to the classes of starlike, respectively convex functions of order α on $\mathbb{U}$. An important remark is that for $k = 0$ we obtain $E_0^*(\alpha) = S^*(\alpha)$ and $E_0(\alpha) = K(\alpha)$, so we can start our study of these new subclasses in terms of the well-known families $S^*(\alpha)$ and $K(\alpha)$ introduced by Robertson in [120]. On the other hand, we have that E_1 is strictly included in the family $K(1/2)$ of convex functions of order $1/2$ (see Proposition 2.2.25) and $E_1^*(\alpha) = K(\alpha)$. As we already mentioned above, the operator $\mathcal{G}_k$ and the subclasses introduced in this chapter can be extended also in the case of several complex variables (see e.g. [53]). However, in higher dimensions, some properties are different as can be seen in Chapter 4.

The second section of this chapter is dedicated to the study of subclasses $E_k(\alpha)$ and $E_k^*(\alpha)$ in $\mathbb{C}$, where $k \in \mathbb{N}$ and $\alpha \in [0,1)$. Together with general properties of these subclasses (growth and distortion theorems, coefficient estimations, analytical characterization, connection with Loewner chains presented in Theorems 2.2.7 − 2.2.8, 2.2.10 − 2.2.18, 2.2.31 − 2.2.32 and others), we also study particular cases (e.g. $k = 1$ and $\alpha = 0$) that are of interest being in close connection with the classes of univalent functions mentioned in the first chapter (see e.g. Propositions 2.2.25 and 2.2.26 in §2.2.2). All the results in this chapter are original and were obtained by the author in [54]. Other important bibliographic sources used to prepare this chapter are [19], [29], [45], [85].

2.1 The differential operator $\mathcal{G}_k$

In this section we introduce the differential operator $\mathcal{G}_k$ defined on the family $\mathcal{H}_0(\mathbb{U})$ of normalized holomorphic functions on $\mathbb{U}$. Using the operator $\mathcal{G}_k$ we can construct some particular subclasses of univalent functions on the unit disc $\mathbb{U}$ that are strongly related to the families S^*, respectively K, as we can see in the next section.

It is important to mention here that the differential operator $\mathcal{G}_k$ is different from the Sălăgean differential operator D^n (see Remark 2.2.6; see also [124]). Another interesting remark is that the operator $\mathcal{G}_k$ can be extended in the case of several complex variables (see Chapter 4; see also [53]).

For the differential operator $\mathcal{G}_k$ in $\mathbb{C}$ we present some properties related to the linearity and univalence on the unit disc $\mathbb{U}$ and we discuss about how the convolution product is preserved under the action of the operator $\mathcal{G}_k$. Note that the original results discussed in this section were derived by the author in [54].

Definition 2.1.1. Let $k \in \mathbb{N} = \{0, 1, 2, ...\}$ and let $\mathcal{G}_k : \mathcal{H}_0(\mathbb{U}) \to \mathcal{H}(\mathbb{U})$ be the differential operator defined on the class of normalized holomorphic functions on $\mathbb{U}$, as follows

$$(\mathcal{G}_k f)(z) = \begin{cases} z^k f^{(k)}(z) + a_{k-1}z^{k-1} + ... + a_2 z^2 + a_1 z + a_0, & k \geq 1 \\ f(z) & k = 0, \end{cases} \tag{2.1.1}$$

for all $f \in \mathcal{H}_0(\mathbb{U})$ and $z \in \mathbb{U}$. Notice that, for $k \geq 1$, $a_0, ..., a_{k-1}$ are the first k coefficients from the Taylor series expansion of the function $f \in \mathcal{H}_0(\mathbb{U})$.

Remark 2.1.2. In view of the above definition, it is easy to see that the operator $\mathcal{G}_0$ (of order 0) is the identity operator, i.e. $\mathcal{G}_0 f = f$. Another particular form of the operator $\mathcal{G}_k$ is for $k = 1$ (of order 1). In this case, $(\mathcal{G}_1 f)(z) = zf'(z)$, for $z \in \mathbb{U}$.

The connection between two differential operators of consecutive orders $k-1$, respectively k, where $k \in \mathbb{N}$ with $k \geq 1$, is given in the following result (see [54])

Proposition 2.1.3. *Let $f \in \mathcal{H}_0(\mathbb{U})$. Then for any $k \in \mathbb{N}^* = \{1, 2, ...\}$ the following relation holds*

$$(\mathcal{G}_k f)(z) = z(\mathcal{G}_{k-1} f)'(z) - (k-1)(\mathcal{G}_{k-1} f)(z) + \sum_{n=0}^{k-1} (k-n) a_n z^n, \quad z \in \mathbb{U}. \tag{2.1.2}$$

Proposition 2.1.4. *Let $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathcal{H}_0(\mathbb{U})$. Then*

$$\mathcal{G}_k(\alpha f + \beta g) = \alpha \mathcal{G}_k f + \beta \mathcal{G}_k g. \tag{2.1.3}$$

Another property of the operator $\mathcal{G}_k$ is related to the Hadamard (convolution) product (for details, one may consult [19], [29], [45]). Let $f, g \in \mathcal{H}_0(\mathbb{U})$ be given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. We denote by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U} \tag{2.1.4}$$

the Hadamard (convolution) product of the functions f and g on $\mathbb{U}$ (see e.g. [19], [29], [45]). There is a nice connection between the convolution product of two different operators and the operator applied on a convolution product, as follows in the next result (see [54]).

Proposition 2.1.5. *Let $k \in \mathbb{N}$ and $f, g \in \mathcal{H}_0(\mathbb{U})$. Then*

1. *$\mathcal{G}_k(f * g) = (\mathcal{G}_k f) * g = f * (\mathcal{G}_k g)$;*
2. *$(\mathcal{G}_k f) * (\mathcal{G}_k g) = \mathcal{G}_k(\mathcal{G}_k(f * g))$.*

It is important that we can prove a sufficient condition of univalence for $\mathcal{G}_k$ (in terms of modulus of coefficients a_n), as follows

Proposition 2.1.6. *Let $k \in \mathbb{N}$ and $f \in \mathcal{H}_0(\mathbb{U})$. Also, let σ_k be defined by*

$$\sigma_k = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{n \cdot n!}{(n-k)!} |a_n|, & k \leq 2 \\ \displaystyle\sum_{n=2}^{k-1} n|a_n| + \sum_{n=k}^{\infty} \frac{n \cdot n!}{(n-k)!} |a_n|, & k \geq 3. \end{cases} \tag{2.1.5}$$

If $\sigma_k \leq 1$, then $\mathcal{G}_k f$ is univalent on $\mathbb{U}$. In particular, $\mathcal{G}_k f \in S$.

Remark 2.1.7. In particular, for $k = 0$, we obtain the well-known univalence condition for a holomorphic function on $\mathbb{U}$ (see e.g. [45, Exercise 1.1.4]): if $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then f is univalent on $\mathbb{U}$.

2.2 Subclasses of univalent functions

Using the differential operator $\mathcal{G}_k$ defined above, we can construct some particular subclasses of univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$. These subclasses, denoted here by $E_k^*(\alpha)$, respectively $E_k(\alpha)$, where $\alpha \in [0,1)$, are related to the classes of starlike, respectively convex functions of order α on $\mathbb{U}$. An important remark is that for $k=0$ we obtain $E_0^*(\alpha) = S^*(\alpha)$ and $E_0(\alpha) = K(\alpha)$, so we can start our study of these new subclasses in terms of the well-known families $S^*(\alpha)$ and $K(\alpha)$ introduced by Robertson in [120]. On the other hand, we have that E_1 is strictly included in the family $K(1/2)$ of convex functions of order $1/2$ (see Proposition 2.2.25). The original results included in this section can be found in [54].

2.2.1 The subclass $E_k^*(\alpha)$

First, we present some general results about the subclass $E_k^*(\alpha)$ and connections of this class with another important classes of univalent functions (for example, the class of starlike functions of order α or the class of univalent functions introduced by Sălăgean in [124]).

Definition 2.2.1. Let $\alpha \in [0,1)$ and $k \in \mathbb{N}$. Let $\mathcal{G}_k$ be the differential operator defined by formula (2.1.1). Then

$$E_k^*(\alpha) = \{f \in S \,:\, \mathcal{G}_k f \in S^*(\alpha)\}$$

is the family of normalized univalent functions f on the unit disc such that $\mathcal{G}_k f$ is starlike of order α. In particular, we denote by $E_k^* = E_k^*(0)$.

Remark 2.2.2. It is clear that $E_0^*(\alpha) = S^*(\alpha)$ is the family of normalized starlike functions of order α on $\mathbb{U}$.

Remark 2.2.3. Taking into account the definition of starlikeness of order α (see Definition 1.4.8), we deduce that

$$E_k^*(\alpha) = \Big\{f \in S \,:\, \mathfrak{Re}\Big[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\Big] > \alpha, \quad z \in \mathbb{U}\Big\}. \tag{2.2.1}$$

Indeed, if $f \in S$, then $\mathcal{G}_k f \in \mathcal{H}(\mathbb{U})$, $(\mathcal{G}_k f)(0) = 0$ and $(\mathcal{G}_k f)'(0) = 1$. Together with the condition $\mathfrak{Re}\big[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\big] > \alpha$, for all $z \in \mathbb{U}$, all the assumptions from the definition of starlikeness of order α are satisfied.

Proposition 2.2.4. *Let $\alpha \in [0,1)$. Then $E_1^*(\alpha) = K(\alpha)$.*

Remark 2.2.5. As a consequence of the previous two remarks, we obtain that $E_0^* = S^*$ and $E_1^* = K$. It is important to mention here that the second equality is no longer true in the case of several complex variables (see [53]).

Remark 2.2.6. It is very important to mention here that

$$E_0^*(\alpha) = S_0(\alpha) \quad \text{and} \quad E_1^*(\alpha) = S_1(\alpha),$$

where $S_0(\alpha)$ and $S_1(\alpha)$ are particular forms of the class $S_n(\alpha)$ introduced by Sălăgean in [124] for $\alpha \in [0,1)$. These equalities hold because

$$D^0 f(z) = f(z) = (\mathcal{G}_0 f)(z) \quad \text{and} \quad D^1 f(z) = zf'(z) = (\mathcal{G}_1 f)(z),$$

for all $z \in \mathbb{U}$, where D^n is the differential operator introduced by Sălăgean. However, for $n = k \geq 2$, we have that

$$E_k^*(\alpha) \neq S_n(\alpha),$$

since the Sălăgean differential operator $D^n f$ (see [124]) is different from the operator $\mathcal{G}_k f$, for every $n = k \geq 2$. For example, if $n = 2$, then

$$D^2 f(z) = D(Df(z)) = z^2 f''(z) + zf'(z) \neq z^2 f''(z) + z = (\mathcal{G}_2 f)(z),$$

for all $z \in \mathbb{U}$. Hence, the common results from this thesis and the ones obtained by Sălăgean in [124] are only for the particular cases $k = 0$ and $k = 1$ (which are already well-known, as reduces to the classes $S^*(\alpha)$, respectively $K(\alpha)$).

Using a similar argument as in Proposition 2.1.6, we can prove the following result. We mention here that this result is a general form of the theorem proved by Merkes, Robertson and Scott in [101].

Theorem 2.2.7. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in S$. Also, let $\sigma_{k,\alpha}$ be defined by*

$$\sigma_{k,\alpha} = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \le 2 \\ \displaystyle\sum_{n=2}^{k-1}(n-\alpha)|a_n| + \sum_{n=k}^{\infty} \frac{(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \ge 3. \end{cases} \tag{2.2.2}$$

If $\sigma_{k,\alpha} \le 1-\alpha$, then $f \in E_k^*(\alpha)$.

Next, we provide some results regarding to coefficient estimates and distortion theorems for the class $E_k^*(\alpha)$. For the proof of our first result, we use the coefficient estimates for the class $S^*(\alpha)$ given by Robertson in [120] (see [45]). Note that this result was obtained by Grigoriciuc in [54].

Theorem 2.2.8. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in E_k^*(\alpha)$. Then*

$$|a_n| \le \frac{(n-k)!}{(n-1)!\cdot n!} \prod_{m=2}^{n} (m-2\alpha), \quad n \ge k \ge 2. \tag{2.2.3}$$

Corollary 2.2.9. *Let $k \in \mathbb{N}$ and $f \in E_k^*$. Then*

$$|a_n| \le \frac{n}{n(n-1)(n-2)\cdot \ldots \cdot (n-k+1)} = \frac{n\cdot (n-k)!}{n!}, \quad n \ge k. \tag{2.2.4}$$

Following the idea presented by Duren in [19] and treated by Goodman in [29] (also by Grigoriciuc in [51]), we can prove a general distortion result for the class E_k^*. In fact, we obtain upper bounds for the m-th derivative of a function $f \in E_k^*$, where $m \in \mathbb{N}$ such that $m \ge k$ (see [54]).

Theorem 2.2.10. *Let $k \in \mathbb{N}$. If $f \in E_k^*$, then*

$$\left|f^{(m)}(z)\right| \le \frac{\big[m + (1-k)|z|\big]\cdot (m-k)!}{(1-|z|)^{m-k+2}}, \tag{2.2.5}$$

for all $m \ge k$ and $z \in \mathbb{U}$.

Remark 2.2.11. Obviously, for $k \in \{0,1\}$ we obtain the classical results proved by Goodman in [29].

Based on the previous theorem and the result proved in [51], we propose the following conjecture (already proved for the particular cases $k=0$, $\alpha=0$ and $\alpha=\frac{1}{2}$):

Conjecture 2.2.12. *Let $\alpha \in [0,1)$ and $m,k \in \mathbb{N}$. If $f \in E_k^*(\alpha)$, then*

$$\left|f^{(m)}(z)\right| \le \frac{\big[m + (1-k)(1-2\alpha)|z|\big]\cdot B(m-k,\alpha)}{(1-|z|)^{m-k+2-2\alpha}}, \tag{2.2.6}$$

for all $m \ge k+1$ and $z \in \mathbb{U}$, where

$$B(m-k,\alpha) = \begin{cases} \frac{1}{m}(m-k)!, & \alpha = \frac{1}{2} \\ \displaystyle\frac{1}{1-2\alpha}\prod_{j=1}^{m-k}(j-2\alpha), & \alpha \ne \frac{1}{2}. \end{cases} \tag{2.2.7}$$

Remark 2.2.13. It is clear that for $k=0$, Conjecture 2.2.12 reduces to Theorem 1.4.9 (see e.g. [51]). Moreover, for $\alpha = 0$, the previous Conjecture reduces to Theorem 2.2.10 proved in this section.

2.2.2 The subclass $E_k(\alpha)$

Similarly as in the previous section, we can use the operator $\mathcal{G}_k$ to define the class $E_k(\alpha)$ of holomorphic functions on $\mathbb{U}$ for which $\mathcal{G}_k f$ is a convex function of order α on $\mathbb{U}$. In the first part, we discuss some general results for the class $E_k(\alpha)$ related to coefficient estimates and general distortion results. The final part of this section is dedicated to the particular case $k = 1$.

In this subsection we introduce the subclass $E_k(\alpha)$ together with some general properties of it. The original results presented in this part have been obtained by Grigoriciuc in [54].

Definition 2.2.14. Let $\alpha \in [0, 1)$ and $k \in \mathbb{N}$. Let $\mathcal{G}_k$ be the differential operator defined by formula (2.1.1). Then

$$E_k(\alpha) = \{f \in S \,:\, \mathcal{G}_k f \in K(\alpha)\}$$

is the family of normalized univalent functions f on the unit disc such that $\mathcal{G}_k f$ is convex of order α. In particular, we denote by $E_k = E_k(0)$.

Remark 2.2.15. Taking into account the definition of convexity of order α (see Definition 1.4.17; see also [45], [120], [102]), we deduce that

$$E_k(\alpha) = \left\{f \in S \,:\, \mathfrak{Re}\left[1 + \frac{z(\mathcal{G}_k f)''(z)}{(\mathcal{G}_k f)'(z)}\right] > \alpha, \quad z \in \mathbb{U}\right\}. \tag{2.2.8}$$

It is clear that $E_0(\alpha) = K(\alpha)$ is the family of normalized convex functions of order α on $\mathbb{U}$.

Taking into account Theorem 2.2.7, we can prove a similar criteria for the family $E_k(\alpha)$, as follows

Theorem 2.2.16. *Let $\alpha \in [0, 1)$, $k \in \mathbb{N}$ and $f \in S$. Also, let $\sigma_{k,\alpha}$ be defined by*

$$\sigma_{k,\alpha} = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{n(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \le 2 \\ \displaystyle\sum_{n=2}^{k-1} n(n-\alpha)|a_n| + \sum_{n=k}^{\infty} \frac{n(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \ge 3. \end{cases} \tag{2.2.9}$$

If $\sigma_{k,\alpha} \le 1 - \alpha$, then $f \in E_k(\alpha)$.

Remark 2.2.17. If $k = 0$, then $E_0(\alpha) = K(\alpha)$ and we obtain the sufficient condition for convexity of order α (one may consult [45] or [101]).

Similar with Theorem 2.2.8, we can obtain some bounds for the coefficients of a function $f \in E_k(\alpha)$, as follows

Theorem 2.2.18. *Let $\alpha \in [0, 1)$, $k \in \mathbb{N}$ and $f \in E_k(\alpha)$. Then*

$$|a_n| \le \frac{(n-k)!}{n!\cdot n!} \prod_{m=2}^{n} (m - 2\alpha), \quad n \ge k \ge 2. \tag{2.2.10}$$

Corollary 2.2.19. *Let $k \in \mathbb{N}$ and $f \in E_k$. Then*

$$|a_n| \le \frac{1}{n(n-1)(n-2)\cdot \ldots \cdot (n-k+1)} = \frac{(n-k)!}{n!}, \quad n \ge k. \tag{2.2.11}$$

Remark 2.2.20. If $k = 0$, then $E_0 = K$ and we obtain the classical result related to the coefficient estimates for convex functions (see e.g. [19]).

Following the remarks presented before Theorem 2.2.10, we can prove the following general distortion result (see [54]):

Theorem 2.2.21. *Let $k \in \mathbb{N}$. If $f \in E_k$, then*

$$\left|f^{(m)}(z)\right| \leq \frac{(m-k)!}{(1-|z|)^{m-k+1}}, \tag{2.2.12}$$

for all $m \geq k$ and $z \in \mathbb{U}$.

Remark 2.2.22. It is clear that for $k = 0$ we obtain the result proved by Goodman in [29, Theorem 9, Chapter 8].

We end the first part of this subsection with the following characterization of functions from E_k in terms of Loewner chains. Based on the Alexander's duality theorem (see 1.4.16) and the characterization of starlikeness, respectively convexity with Loewner chains (see Theorems 1.6.6 and 1.6.8), we can construct two different Loewner chains starting from the same function $f \in E_k$, as follows

Theorem 2.2.23. *Let $k \in \mathbb{N}$ and $f \in \mathcal{H}_0(\mathbb{U})$. Then $f \in E_k$ if and only if*

$$f_1(z,t) = (\mathcal{G}_k f)(z) + (e^t - 1)z(\mathcal{G}_k f)'(z) \tag{2.2.13}$$

or

$$f_2(z,t) = e^t z(\mathcal{G}_k f)'(z) \tag{2.2.14}$$

is a Loewner chain, for all $z \in \mathbb{U}$ and $t \geq 0$. Moreover,

$$f_2(z,t) - f_1(z,t) = z(\mathcal{G}_k f)'(z) - (\mathcal{G}_k f)(z), \quad z \in \mathbb{U}, \quad t \geq 0.$$

The particular case $k = 1$ and $\alpha = 0$

The next section is dedicated to the study of a special form ($k = 1$ and $\alpha = 0$) of the class $E_k(\alpha)$. Because we consider such a particular case, we obtain some nice results and examples related to classical properties of univalent functions on $\mathbb{U}$. According to Definition 2.2.14, we have that E_1 is defined by

$$E_1 = \left\{f \in S \,:\, \mathcal{G}_1 f \in K\right\},$$

where $\mathcal{G}_1 f(z) = zf'(z)$, for all $z \in \mathbb{U}$.

Example 2.2.24. Let $f : \mathbb{U} \to \mathbb{C}$ be defined by $f(z) = -\log(1-z)$ for all $z \in \mathbb{U}$, where log denotes the principal branch of the complex logarithm. Then f belongs to the class E_1.

Next, we present an important result that establishes the connection between classes E_1 and $K(1/2)$. In particular, we obtain that every function from E_1 is also convex (see [54]). This proof of this result was given by the author and is based on the proof of [45, Theorem 2.3.2] given by Suffridge.

Proposition 2.2.25. *If f belongs to the class E_1, then f belongs to the class $K(1/2)$. This result is sharp.*

Proposition 2.2.26. *If f belongs to the class E_1, then f belongs to the class $\mathcal{R}(1/2)$, i.e. $\mathfrak{Re} f'(z) > 1/2$, for all $z \in \mathbb{U}$.*

Theorem 2.2.27. *Let $f \in E_1$. Then*

$$\log(1+|z|) \leq |f(z)| \leq -\log(1-|z|) \tag{2.2.15}$$

and

$$\frac{1}{1+|z|} \leq |f'(z)| \leq \frac{1}{1-|z|}, \tag{2.2.16}$$

for all $z \in \mathbb{U}$. All of these estimates are sharp.

Corollary 2.2.28. *If f belongs to the class E_1, then the open disc $\mathcal{U}_{\ln 2}$ is included in $f(\mathbb{U})$.*

The last result in this subsection is a particular form of Theorem 2.2.23 and presents the characterization of mappings in class E_1 via Loewner chains.

Theorem 2.2.29. *Let $f \in \mathcal{H}_0(\mathbb{U})$. Then $f \in E_1$ if and only if*

$$f_1(z,t) = e^t z f'(z) + (e^t - 1) z^2 f''(z) \tag{2.2.17}$$

or

$$f_2(z,t) = e^t z f'(z) + e^t z^2 f''(z) \tag{2.2.18}$$

is a Loewner chain, for all $z \in \mathbb{U}$ and $t \geq 0$. Moreover,

$$f_2(z,t) - f_1(z,t) = z^2 f''(z), \quad z \in \mathbb{U}, \quad t \geq 0.$$

2.2.3 Connections between E_k^* and E_k

Based on the Alexander's duality theorem between convex and starlike functions on $\mathbb{U}$ (see [1], [19], [102]), we prove in this section similar duality results for the subclasses E_k^* and E_k (see [54]).

Lemma 2.2.30. *Let $k \in \mathbb{N}$ and $f, g \in S$ be such that $g(z) = zf'(z)$, for all $z \in \mathbb{U}$. Then*

$$z(\mathcal{G}_k f)'(z) = (\mathcal{G}_k g)(z), \quad z \in \mathbb{U}. \tag{2.2.19}$$

Based on the previous lemma, we can obtain an Alexander type theorem for the families E_k^* and E_k. This result was proved by Grigoriciuc in [54].

Theorem 2.2.31. *Let $k \in \mathbb{N}$ and $f, g \in S$. Then $f \in E_k$ if and only if $g \in E_k^*$, where $g(z) = zf'(z)$, for all $z \in \mathbb{U}$.*

Theorem 2.2.32. *Let $k \in \mathbb{N}$. If $f \in E_k$, then $f \in E_k^*(1/2)$.*

Remark 2.2.33. It is clear that Theorem 2.2.32 is a generalization of Proposition 2.2.25 (where $k = 1$). On the other hand, if $k = 0$, then Theorem 2.2.32 reduces to [45, Theorem 2.3.2] due to Marx and Strohhäcker.

Finally, we end this section with some questions related to the subclasses E_k and E_k^* studied above. First question is a generalization of Proposition 2.2.25:

Question 2.2.34. *Is it true that $E_{k+1} \subset E_k$, for all $k \in \mathbb{N}$?*

Clearly, a similar question can be formulated also for the subclass E_k^*. Another important property of these subclasses is the compactness. Hence, one may ask

Question 2.2.35. *Is it true that the subclasses E_k and E_k^* are compact in $\mathcal{H}(\mathbb{U})$?*

Since E_k^* and E_k are subclasses of the class S, it would be interesting to study also other geometric and analytic properties of them.

2.2.4 The subclass $E_{\mathbb{N}}$

We end this chapter with some remarks on a particular class related to the one presented above. For this, let $k \in \mathbb{N}$ and $f \in \bigcap\limits_{k\in\mathbb{N}} E_k$. Then, for every $k \in \mathbb{N}$, we have that $f \in E_k$. Moreover, according to Corollary 2.2.19, it follows that for every $k \in \mathbb{N}$

$$|a_n| \leq \frac{(n-k)!}{n!}, \quad n \geq k.$$

In particular, for $n = k$ we obtain that $|a_k| \leq \frac{1}{k!}$, for every $k \in \mathbb{N}$. Let us denote by

$$E_{\mathbb{N}} = \left\{ f \in S \,:\, |a_n| \leq \frac{1}{n!},\, n \geq 2 \right\}. \tag{2.2.20}$$

Then, we obtain the following remark

Remark 2.2.36. Let $E_{\mathbb{N}}$ be the set defined by (2.2.20). Then $\bigcap_{k\in\mathbb{N}} E_k \subsetneqq E_{\mathbb{N}}$, i.e. the intersection of all subclasses E_k is included in $E_{\mathbb{N}}$, but it is not equal with $E_{\mathbb{N}}$.

Part II

Contributions in the theory of biholomorphic mappings of several complex variables

Chapter 3

Biholomorphic mappings and Extension operators in several complex variables

In this chapter we include general aspects related to biholomorphic mappings of several complex variables in $\mathbb{C}^n$. We begin with basic notations, notions and preliminary results that will be used during the second part of the thesis. We refer here to the theory of holomorphic functions, respectively holomorphic mappings in $\mathbb{C}^n$, including the open mapping theorem and the minimum/maximum modulus theorem with its applications (e.g. the Schwarz's lemma). We recall also the definition of a set of uniqueness (see e.g. [45], [83]) and two important results related to this notion, namely the Montel, respectively Vitali's theorem in $\mathbb{C}^n$ (see e.g. [83], [107], [119]). In the final part of this section we present some general results about holomorphic mappings in $\mathbb{C}^n$ and the main results that will be used in this chapter (e.g. the Schwarz-Pick's lemma).

The second section contains results about subordination in $\mathbb{C}^n$ and generalizations to higher dimensions of the notions related to the Carathéodory class of functions with positive real part on $\mathbb{U}$. We refer here especially to growth and distortion theorems obtained by Graham, Hamada and Kohr (see [32]), Pfaltzgraff (see [109]) and Poreda (see [115]). One of the most important results that was proved by Graham, Hamada and Kohr (see [32], [68]) is the compactness of the Carathéodory family $\mathcal{M}$. Their result had a strong impact on the evolution of the geometric function theory in $\mathbb{C}^n$.

The next two sections are dedicated to the presentation of special subclasses of biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$, respectively on the unit polydisc $\mathbb{U}^n$ in $\mathbb{C}^n$. For $n \geq 2$, we denote by $S(\mathbb{B}^n)$ the family of biholomorphic and normalized mappings on $\mathbb{B}^n$ (see e.g. [45], [83]). It is known that the set $S(\mathbb{B}^n)$ is not locally uniformly bounded and then it does not admit a growth and distortion theorem. As an important consequence of this property due to Cartan (see e.g. [7], [45]) we obtain that $S(\mathbb{B}^n)$ is not compact for $n \geq 2$. Among the most important subclasses of $S(\mathbb{B}^n)$ we mention the family of starlike, starlike of order α, convex and spirallike mappings on $\mathbb{B}^n$. For these mappings we recall analytical and geometric characterizations, growth and distortion results together with suggestive examples that are used throughout this chapter.

Section 3.5 contains generalizations of the notions presented in §1.6 related to Loewner chains, Loewner differential equation and parametric representation in higher dimensions. Pfaltzgraff (see e.g. [109]) was the first who obtained generalizations of the Loewner chains and Loewner differential equation on $\mathbb{B}^n$. The study was extended on $\mathbb{U}^n$ by Poreda (see e.g. [115], [116]), respectively by Kubicka and Poreda (see e.g. [86]). Important results were obtained over time by Duren, Graham, Hamada and Kohr (see e.g. [20]), Graham, Hamada and Kohr (see e.g. [32]), Graham, Hamada, Kohr and Kohr (see e.g. [36], [37]) and others. One of the most important distinction between the one dimensional case and the higher dimensions is the compactness of the family of normalized biholomorphic mappings. It is known that $S(\mathbb{U})$ is a compact set (see Theorem 1.4.4) while the set $S(\mathbb{B}^n)$ is not compact for $n \geq 2$ (see e.g. [7], [45]). This problem was solved by Graham, Hamada and Kohr who introduced the class $S^0(\mathbb{B}^n)$ of mappings which admit parametric representation on $\mathbb{B}^n$ (see e.g. [32]; see also [114]). For $n = 1$, we have that $S^0(\mathbb{B}^1) = S$ (see e.g. [114]). However, if $n \geq 2$, then $S^0(\mathbb{B}^n)$ is strictly included in $S(\mathbb{B}^n)$. Moreover, Graham, Kohr and Kohr (see [48]) proved that $S^0(\mathbb{B}^n)$ is compact (see e.g. [32], [45], [48]). This result is one of the results that presents the clear distinction between one and several complex variables cases. On the other hand, it opened new ways of studying geometric function theory in higher dimensions. Another important problem that was solved by Graham, Hamada, Kohr and Kohr is the existence in $\mathbb{C}^n$ of mappings which cannot be embedded as the first elements of a Loewner chain. Using the family $S^0(\mathbb{B}^n)$ they succeed to prove the analogous of Pommerenke's theorem (see Theorem 1.6.3) in higher dimensions (see [48]; see also [32]). Moreover, the notion of parametric representation was extended to g-parametric representation by Graham, Hamada and Kohr (see e.g. [32]). More details about the class $S_g^0(\mathbb{B}^n)$ will be discussed in the last part of the thesis.

In the next section we study convex combinations of the form $(1-\lambda)f + \lambda g$, where $f, g \in S(\mathbb{B}^n)$ and $\lambda \in (0,1)$. It is known that, in general, the convex combination of two normalized biholmorphic mappings is not biholomorphic on $\mathbb{B}^n$ (see e.g. [45], [83]). The phenomenon also occurs in the one dimensional case and was intensively studied by several authors (see e.g. [9], [58], [97], [100]). The main idea of this section is to obtain biholomorphic mappings on $\mathbb{B}^n$ (or even starlike mappings) as convex combinations of the form $(1-\lambda)f + \lambda g$, where $f, g \in S(\mathbb{B}^n)$ and $\lambda \in (0,1)$. The results presented in this section original and were obtained by Grigoriciuc in [52].

A powerful tool in the study of biholomorphic mappings in higher dimensions is the theory of extension operators. In §3.7 we present various extension operators that preserve geometric and analytic properties on the unit ball in $\mathbb{C}^n$. We start our discussion with the Roper-Suffridge extension operator Φ_n (defined by K. Roper and T.J. Suffridge in [121]) and the Graham-Kohr extension operator $\Psi_{n,\alpha}$ (defined by I. Graham and G. Kohr in [44]; see also [43]). Then we will look at two generalizations of the Roper-Suffridge extension operator introduced by Graham, Hamada, Kohr, Kohr and Suffridge (see e.g. [42], [47]) that map a locally univalent function on $\mathbb{U}$ into a locally biholomorphic mapping on $\mathbb{B}^n$. In the final part of this section we present the extension operator introduced by Pfaltzgraff and Suffridge (see [111]) and a generalization of their operator (see e.g. [10]).

We end this chapter with an short study that combine the ideas presented above, namely extension operators and convex combinations of biholomorphic mappings in $\mathbb{C}^n$. Hence, we discuss about convex combinations of extension operators on $\mathbb{B}^n$. In particular, we consider a new extension operator obtained as a convex combination of two Graham-Kohr type extension operators (see e.g. [43], [44]). The results presented in this section are original.

Finally, we mention that the main references used in this chapter are [14], [24], [25], [27], [32], [45], [46], [48], [83], [84], [88], [107], [109], [117], [128], [138].

3.1 General notions regarding holomorphy in $\mathbb{C}^n$

This section is focused on the study of the properties of holomorphic functions and holomorphic mappings in higher dimensions. We discuss here the main results that can be generalized from one dimension to the case of several complex variables. For details, one may consult [8], [45], [83], [88], [119].

3.1.1 Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, ..., z_n)$ equipped with the Euclidean inner product $\langle z, w\rangle = \sum_{k=1}^n z_k \overline{w}_k$ and the Euclidean norm $\|z\| = \sqrt{\langle z, z\rangle}$, where $z, w \in \mathbb{C}^n$. We denote by

$$\mathbb{B}^n(a, r) = \left\{z \in \mathbb{C}^n \,:\, \|z - a\| < r\right\}$$

the open ball of center $a \in \mathbb{C}^n$ and radius $r > 0$ with respect to the Euclidean norm. For simplicity, we use the notation $\mathbb{B}^n_r = \mathbb{B}^n(0, r)$ for the open ball of center zero and radius r. In particular, we denote by $\mathbb{B}^n = \mathbb{B}^n(0,1)$ *the (open) Euclidean unit ball in* $\mathbb{C}^n$. The open polydisc $U^n(a, R)$ of center a and (multi)radius R is defined by

$$U^n(a, R) = \mathcal{U}(a_1, r_1) \times ... \times \mathcal{U}(a_n, r_n),$$

where $a = (a_1, ..., a_n) \in \mathbb{C}^n$ and $R = (r_1, ..., r_n) \in \mathbb{R}^n_+$. If $r_j = r$, for all $j = \overline{1, n}$, then we denote this polydisc by $U^n(a, r)$. In particular, we denote by $\mathbb{U}^n = U^n(0, 1)$ *the (open) unit polydisc in* $\mathbb{C}^n$. Is it clear that $\mathbb{U}^n$ is the open unit ball in $\mathbb{C}^n$ with respect to the maximum norm $\|z\|_\infty = \max\{|z_j| : j = \overline{1,n}\}$, for all $z = (z_1, ..., z_n) \in \mathbb{C}^n$.

During this thesis we are working on different domains (especially, unit balls in $\mathbb{C}^n$ with respect to different norms), as follows:

- $\mathbb{B}^n$ – the Euclidean unit ball in $\mathbb{C}^n$ with respect to the Euclidean norm $\|z\| = \sqrt{\sum_{j=1}^n |z_j|^2}$, for all $z = (z_1, ..., z_n) \in \mathbb{C}^n$.

For the Euclidean case, we denote the Euclidean norm simply $\|\cdot\|$ without any index. Then, every time we use the notation $\|\cdot\|$, we automatically refer to the Euclidean norm.

- $\mathbb{U}^n$ – the unit polydisc in $\mathbb{C}^n$ with respect to the maximum norm $\|z\|_\infty = \max\{|z_j| : j = \overline{1,n}\}$, for all $z = (z_1, ..., z_n) \in \mathbb{C}^n$.

- B_p^n – the unit ball in $\mathbb{C}^n$ with respect to the p-norm $\|z\|_p = \left[\sum_{j=1}^n |z_j|^p\right]^{1/p}$, for all $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $p \in [1, \infty)$.

In $\mathbb{C}$ each of the sets $\mathbb{B}^1$, $\mathbb{U}^1$ and B_p^1 coincides with $\mathbb{U}$. Mention that when we work with an arbitrary norm, it will be denoted $\|\cdot\|_*$. But, when the domains are those described above, we use the particular notations for unit balls and norms presented for each case.

3.1.2 Holomorphic functions in $\mathbb{C}^n$

The first part of this section contains results regarding holomorphic functions in several complex variables (see e.g. [45], [88], [107])

Definition 3.1.1. Let $\Omega \subseteq \mathbb{C}^n$ be an open set. If $f : \Omega \to \mathbb{C}$ is holomorphic in each variable and continuous on Ω, then f is *holomorphic*. We denote by $\mathcal{H}(\Omega, \mathbb{C}) = \{f : \Omega \to \mathbb{C} \,:\, f \textit{ is holomorphic on } \Omega\}$ the family of all holomorphic functions from Ω to $\mathbb{C}$.

Note that, in view of the Hartogs result, the assumption of continuity from the previous definition can be neglected. Hence, it follows that every holomorphic function in each variable individually is, in fact, holomorphic (see e.g. [8], [88]).

Next we present some known properties of holomorphic functions in $\mathbb{C}^n$. These results are generalizations of the properties presented in the first chapter for the case of one complex variable. First, we state *the open mapping theorem* in $\mathbb{C}^n$ (see e.g. [83], [107], [119]).

Theorem 3.1.2 (Open mapping theorem). *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. If $f : \Omega \to \mathbb{C}$ is holomorphic and nonconstant, then $f(\Omega) \subseteq \mathbb{C}$ is a domain.*

An application of Theorem 3.1.2 is *the Schwarz lemma for holomorphic functions* in $\mathbb{C}^n$ (see e.g. [83], [107]):

Lemma 3.1.3 (Schwarz's lemma). *Let $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C})$ with $f(0) = 0$ and $|f(z)| < 1$, for all $z \in \mathbb{B}^n$. Then $|f(z)| \leq \|z\|$, for all $z \in \mathbb{B}^n$. Morevoer, if $\exists z_0 \in \mathbb{B}^n \setminus \{0\}$ such that $|f(z_0)| = \|z_0\|$, then $|f(az_0)| = \|az_0\|$, for all $a \in \mathbb{C}$ with $|a| \leq \frac{1}{\|z_0\|}$.*

3.1.3 Holomorphic mappings in $\mathbb{C}^n$

Next, we briefly describe the case of holomorphic mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$, where $m, n \in \mathbb{N}$ such that $m, n \geq 2$. Among the references used in this subsection, we mention [45], [83], [88], [107], [119].

Definition 3.1.4. Let $\Omega \subseteq \mathbb{C}^n$ be an open set and let $f : \Omega \to \mathbb{C}^m$. If $f_k \in \mathcal{H}(\Omega, \mathbb{C})$, for each $k = \overline{1,m}$, then $f = (f_1, ..., f_m)$ is *holomorphic* on Ω.

We denote by $\mathcal{H}(\Omega, \mathbb{C}^m)$ the family of all holomorphic mappings from the open set $\Omega \subseteq \mathbb{C}^n$ into $\mathbb{C}^m$. In particular, if $m = n$, then we use the simple notation $\mathcal{H}(\Omega)$.

If $\Omega \subseteq \mathbb{C}^n$ is a domain such that $0 \in \Omega$, then we say that $f \in \mathcal{H}(\Omega, \mathbb{C}^m)$ is normalized if $f(0) = 0$ and $Df(0) = I_m$, where the differential

$$Df(z) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \dots & \frac{\partial f_1}{\partial z_n} \\ \dots & \dots & \dots \\ \frac{\partial f_m}{\partial z_1} & \dots & \frac{\partial f_m}{\partial z_n} \end{pmatrix}$$

is a complex linear mapping from $\mathbb{C}^n$ into $\mathbb{C}^m$ at the point $z \in \Omega$ and I_m is the identity operator in $\mathbb{C}^m$. When $m = n$, we denote by $J_f(z) = \det Df(z)$ for $z \in \Omega$ and it is the complex Jacobian determinant of f at z. In this case, we denote by $\mathcal{H}_0(\Omega)$ the set of all normalized holomorphic mappings from Ω into $\mathbb{C}^n$.

The following result presents the Schwarz's lemma for holomorphic mappings (see e.g. [83], [107]). For this, let us consider an arbitrary norm $\|\cdot\|_*$ on $\mathbb{C}^n$ and $B^n \subseteq \mathbb{C}^n$ the unit ball.

Theorem 3.1.5 (Schwarz's lemma). *Let $f \in \mathcal{H}(B^n, \mathbb{C}^n)$ with $f(0) = 0$ and $\|f(z)\|_* < 1$, for all $z \in B^n$. Then $\|f(z)\|_* \le \|z\|_*$, for all $z \in B^n$ and $\|Df(0)\| \le 1$. Moreover, if $\exists z_0 \in B^n \setminus \{0\}$ with the property that $\|f(z_0)\|_* = \|z_0\|_*$, then $\|f(az_0)\|_* = \|az_0\|_*$, for all $a \in \mathbb{C}$ with $|a| \le 1/\|z_0\|_*$.*

We end this subsection with an important result, namely *the Schwarz-Pick Lemma* for holmorphic mapping on the Euclidean unit ball $\mathbb{B}^n$. This result will play a key role in the final part of this thesis (see e.g. [76], [123]).

Lemma 3.1.6. *Let $f \in \mathcal{H}(\mathbb{B}^n)$ be such that $f(\mathbb{B}^n) \subseteq \mathbb{B}^n$. Then*

$$|J_f(z)| \le \left[\frac{1 - \|f(z)\|^2}{1 - \|z\|^2}\right]^{\frac{n+1}{2}}, \quad z \in \mathbb{B}^n. \tag{3.1.1}$$

This inequality is sharp and equality at a point $z \in \mathbb{B}^n$ holds if and only if f is an automorphism of $\mathbb{B}^n$.

3.2 The Carathéodory family in $\mathbb{C}^n$

In Section 3.2 we focus our attention on the notion of subordination in $\mathbb{C}^n$ and the generalization of the Carathéodory class in $\mathbb{C}^n$ (see e.g. [32], [45], [109]). For this, let $\mathbb{B}^n$ be the Euclidean unit ball in $\mathbb{C}^n$.

Definition 3.2.1. Let $f, g, \phi \in \mathcal{H}(\mathbb{B}^n)$. Then

1. ϕ is a *Schwarz mapping* if $\|\phi(z)\| \le \|z\|$, for all $z \in \mathbb{B}^n$;
2. $f \prec g$ if there is a Schwarz mapping ϕ such that $f(z) = g(\phi(z))$, for all $z \in \mathbb{B}^n$ (we already know that this means that f *is subordinate to* g)

The family of normalized holomorphic mapping on $\mathbb{B}^n$ that extend the Carathéodory class in $\mathbb{C}^n$ (see e.g [109], [127], [128]; see also [45], [83]) is given by

$$\mathcal{M}(\mathbb{B}^n) = \Big\{ h \in \mathcal{H}_0(\mathbb{B}^n) \,:\, \mathfrak{Re}\langle h(z), z\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\} \Big\}.$$

When $\mathbb{C}^n$ is endowed with the Euclidean norm, we denote $\mathcal{M}(\mathbb{B}^n)$ simply by $\mathcal{M}$. The class $\mathcal{M}$ is very powerful in the theory of Loewner chains in $\mathbb{C}^n$, as well as in the characterizing different subclasses of univalent mappings in $\mathbb{C}^n$ (see e.g. [45], [83], [109]).

In the case of one complex variable it is not difficult to observe that $h \in \mathcal{M}$ if and only if $p \in \mathcal{P}$, where $h(\zeta) = \zeta p(\zeta)$, for all $\zeta \in \mathbb{U}$.

Next we list a few properties of the class $\mathcal{M}$ in the simplest case of the Euclidean norm in $\mathbb{C}^n$. Note that these properties are also true for an arbitrary norm. First, we state the growth result obtained by Pfaltzgraff in [109] (see also [57]).

Theorem 3.2.2. *If $h \in \mathcal{M}(\mathbb{B}^n)$, then*

$$\|z\|^2 \frac{1-\|z\|}{1+\|z\|} \leq \mathfrak{Re}\langle h(z), z\rangle \leq \|z\|^2 \frac{1+\|z\|}{1-\|z\|}, \quad z \in \mathbb{B}^n. \tag{3.2.1}$$

These inequalities are sharp.

Similar to the previous bounds, Graham, Hamada and Kohr obtained a stronger result (see [32]), as follows:

Theorem 3.2.3. *If $h \in \mathcal{M}(\mathbb{B}^n)$, then*

$$\|z\| \frac{1-\|z\|}{1+\|z\|} \leq \|h(z)\| \leq \frac{4\|z\|}{(1-\|z\|)^2}, \quad z \in \mathbb{B}^n. \tag{3.2.2}$$

Based on the results presented above, Graham, Hamada and Kohr obtained the compactness of the class $\mathcal{M}$ (see [32], [68]).

Corollary 3.2.4. *The family $\mathcal{M}(\mathbb{B}^n)$ is compact.*

It is important to mention that the results presented above may be extended to an arbitrary norm in $\mathbb{C}^n$. For more details, one may consult [45], [57], [127], [128].

3.3 General results regarding biholomorphic mappings in $\mathbb{C}^n$

In the last subsection we study the biholomorphic mappings in $\mathbb{C}^n$ and a few properties that are fulfilled by these mappings. Also, we present the notion of univalence in several complex variables (see e.g. [45], [83], [88], [107]).

Definition 3.3.1. Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Then $f : \Omega \to \mathbb{C}^n$ is

a) *univalent* on Ω if $f \in \mathcal{H}(\Omega)$ and f is injective on Ω;

b) *biholomorphic* on Ω if $f \in \mathcal{H}(\Omega)$ and $\exists f^{-1} \in \mathcal{H}(\Delta)$, where $\Delta = f(\Omega)$.

If $f \in \mathcal{H}(\Omega)$ is biholomorphic, then the domains Ω and Δ are *biholomorphically equivalent*. Moreover, if the domains coincide, then f is an *automorphism* of Ω.

Similar to the case of one complex variables, the notions of biholomorphy and univalence are equivalent (see e.g. [107], [119]). However, in the infinite dimensional case, this equivalence is no longer valid (see [128]).

Theorem 3.3.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Then $f : \Omega \to \mathbb{C}^n$ is univalent on Ω if and only if f is biholomorphic from Ω into $f(\Omega)$.*

One of the important results in $\mathbb{C}^n$ is the Poincaré theorem (see e.g. [112]) which shows that $\mathbb{B}^n$ and $\mathbb{U}^n$ are not biholomorphically equivalent. Hence, the Riemann's mapping theorem is not valid in higher dimensions (see e.g. [107], [119]).

Theorem 3.3.3 (Poincaré). *If $n \geq 2$, then the Euclidean unit ball $\mathbb{B}^n$ and the unit polydisc $\mathbb{U}^n$ are not biholomorphically equivalent.*

3.4 Families of biholomorphic mappings on the unit ball $\mathbb{B}^n$

In Section 3.4 we discuss general properties of some families of biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$. We recall here the classes of normalized starilke, starlike of order α, almost starlike of order α, convex and spirallike mappings on $\mathbb{B}^n$ together with the most important results. We include in this part the analytical characterizations of these classes, growth theorems and some examples. The most important references used in this section are [24], [45], [83], [128].

3.4.1 Normalized biholomoprhic mappings

Let $L(\mathbb{C}^n, \mathbb{C}^n)$ be the set of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the (standard) operator norm

$$\|A\| = \sup\big\{\|A(z)\| \,:\, \|z\| = 1\big\}.$$

Recall that if $\Omega \subseteq \mathbb{C}^n$ is a domain such that $0 \in \Omega$ and $f \in \mathcal{H}(\Omega)$, then f is normalized if $f(0) = 0$ and $Df(0) = I_n$, where I_n is the identity operator in $L(\mathbb{C}^n, \mathbb{C}^n)$ and $Df(z)$ is the Fréchet derivative of f at z.

In this thesis we denote by $S(\mathbb{B}^n)$ the set of normalized biholomorphic mappings on $\mathbb{B}^n$ in $\mathbb{C}^n$ and by $\mathcal{LS}_n(\mathbb{B}^n)$ the set of normalized locally biholomorphic mappings on $\mathbb{B}^n$ in $\mathbb{C}^n$. In particular, if $n = 1$, then $S(\mathbb{B}^1) = S$ is the well-known family of normalized univalent functions on $\mathbb{U}$, respectively $\mathcal{LS}_1(\mathbb{B}^1) = \mathcal{LS}$ is the family of locally univalent functions on $\mathbb{U}$ (for details, one may consult [24], [45], [83], [128]).

3.4.2 Starlike mappings

In this subsection we present the notion of starlikeness in $\mathbb{C}^n$ together with some results related to the class of starlike mappings on $\mathbb{B}^n$. For simplicity, we consider the Euclidean case, but all the results presented here are valid with respect to any norm in $\mathbb{C}^n$ (see e.g. [24], [45], [83]).

Recall that a domain $\Omega \subseteq \mathbb{C}^n$ is starlike (with respect to 0) if the closed segment $[0, z] \subseteq \Omega$, for all $z \in \Omega$ (see e.g. [83]). Next, we reiterate the definition of a starlike mapping on $\mathbb{B}^n$ (see e.g. [24], [45]).

Definition 3.4.1. Let $f \in \mathcal{H}(\mathbb{B}^n)$. We say that f is *starlike on* $\mathbb{B}^n$ if $f(0) = 0$, f is biholomorphic on $\mathbb{B}^n$ and $f(\mathbb{B}^n)$ is a starlike domain with respect to the origin.

We denote by $S^*(\mathbb{B}^n)$ the class of normalized starlike mappings on the Euclidean unit ball $\mathbb{B}^n$. Clearly, if $n = 1$, then $S^*(\mathbb{B}^1) = S^*$.

The first result stated in this section is the analytical characterization of starlikeness on $\mathbb{B}^n$ given by Matsuno (see e.g. [99]). Other authors obtained extensions of this result for the unit ball of a Banach space (Gurganus [57] and Suffridge [127]) and for the unit polydisc in $\mathbb{C}^n$ (Suffridge [126]).

Theorem 3.4.2. *Let $f \in \mathcal{LS}(\mathbb{B}^n)$ be such that $f(0) = 0$. Then $f \in S^*(\mathbb{B}^n)$ if and only if*

$$\mathfrak{Re}\langle [Df(z)]^{-1} f(z), z\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{3.4.1}$$

It is not difficult to observe that for $n = 1$, the previous result reduces to the analytical characterization of starlikeness on $\mathbb{U}$ (see Theorem 1.4.7).

3.4.3 Starlike mappings of order α

We continue this section by presenting the class of starlike mappings of order α on $\mathbb{B}^n$, where $\alpha \in [0, 1)$. This notion was first considered by Kohr (see [81]; see also [75]) and Curt (see [13]).

Definition 3.4.3. Let $\alpha \in [0, 1)$. We say that $f \in \mathcal{LS}(\mathbb{B}^n)$ is *starlike of order* α if

$$\mathfrak{Re}\left[\frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z\rangle}\right] > \alpha, \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{3.4.2}$$

If $n = 1$, then the previous definition reduces to the starlikeness of order α on $\mathbb{U}$.

We denote by $S^*_\alpha(\mathbb{B}^n)$ the family of all starlike mappings of order α on $\mathbb{B}^n$. Hence, $S^*_0(\mathbb{B}^n) = S^*(\mathbb{B}^n)$ and $S^*_\alpha(\mathbb{B}^n) \subseteq S^*(\mathbb{B}^n)$, for every $\alpha \in [0, 1)$.

3.4.4 Almost starlike mappings of order α

Another important notion that is considered in this section is the almost starlikeness of order α on $\mathbb{B}^n$. Feng defined this concept in the case of complex Banach spaces (see [22]).

Definition 3.4.4. Let $\alpha \in [0,1)$. Then $f \in \mathcal{LS}(\mathbb{B}^n)$ is *almost starlike of order α* if

$$\frac{1}{\|z\|^2}\mathfrak{Re}\langle [Df(z)]^{-1}f(z), z\rangle > \alpha, \quad z \in \mathbb{B}^n \setminus \{0\}.$$

We denote by $\mathcal{AS}^*_\alpha(\mathbb{B}^n)$ the family of almost starlike mappings of order α on $\mathbb{B}^n$.

Notice that for $n=1$ the previous definition reduces to Definition 1.4.10 and we can denote $\mathcal{AS}^*_\alpha(\mathbb{B}^1) = \mathcal{AS}^*_\alpha$.

3.4.5 Convex mappings

In the subsequent subsection, we describe the family of convex mappings on the polydisc, respectively on the Euclidean unit ball in $\mathbb{C}^n$. Since $\mathbb{U}^n$ and $\mathbb{B}^n$ are not biholomorphically equivalent for $n \geq 2$, it is not trivial to study both of these cases in $\mathbb{C}^n$. We present the analytical characterizations and growth theorems for normalized convex mappings on these two domains in $\mathbb{C}^n$. The key bibliographic sources utilized in this part include [24], [45], [83], [128].

Recall that a domain $\Omega \subseteq \mathbb{C}^n$ is convex if the closed segment $[z_1, z_2] \subseteq \Omega$, for all $z_1, z_2 \in \Omega$ (see e.g. [24], [83]). Next, we given the definition of a convex mapping on the unit ball B^n in $\mathbb{C}^n$ with respect to an arbitrary norm (see e.g. [83]).

Definition 3.4.5. Let $f \in \mathcal{H}(B^n)$. Then f is *convex on B^n* if f is biholomorphic on B^n and $f(B^n)$ is a convex domain.

We denote by $K(B^n)$ the class of normalized convex mappings on B^n. Clearly, if $n=1$, then $K(B^1) = K$.

Convexity on the unit polydisc $\mathbb{U}^n$

On the unit polydisc $\mathbb{U}^n$ in $\mathbb{C}^n$ we have the following analytical characterization of convexity due to Suffridge (see [126]).

Theorem 3.4.6. *Let $f \in \mathcal{LS}(\mathbb{U}^n)$. Then f belongs to the class $K(\mathbb{U}^n)$ if and only if $\exists \varphi_k \in K$, for every $k = \overline{1,n}$ such that $f(z) = \big(\varphi_1(z_1), ..., \varphi_n(z_n)\big)$, for all $z = (z_1, ..., z_n) \in \mathbb{U}^n$.*

Convexity on the Euclidean unit ball $\mathbb{B}^n$

In the case of Euclidean unit ball $\mathbb{B}^n$, Kikuchi obtained the following analytical characterization (see [80]). Gong, Wang and Yu obtained an equivalent characterization on $\mathbb{B}^n$ in [28].

Theorem 3.4.7. *Let $f \in \mathcal{LS}(\mathbb{B}^n)$. Then f is convex if and only if*

$$1 - \mathfrak{Re}\langle [Df(z)]^{-1}D^2f(z)(w,w), z\rangle > 0, \tag{3.4.3}$$

for every $z \in \mathbb{B}^n$ and $w \in \mathbb{C}^n$ with $\|w\| = 1$ and $\mathfrak{Re}\langle z, w\rangle = 0$.

It is clear that if $n = 1$, then the previous result reduces to the analytical characterization of convexity on $\mathbb{U}$ (see Theorem 1.4.13).

We end this subsection with the extension of Marx-Strohhäcker's theorem (see [45], [102]). In the case of several complex variables, this generalization was obtained by Kohr [81] and Curt [13].

Theorem 3.4.8. *If f belongs to the class $K(\mathbb{B}^n)$, then f belongs to the class $S^*_{1/2}(\mathbb{B}^n)$ and this result is sharp.*

Remark 3.4.9. An important remark regarding the convexity in higher dimensions is that the generalization of Alexander's duality theorem (see Theorem 1.4.16) does not hold on $\mathbb{B}^n$, when $n \geq 2$ (see e.g. [45], [83]). For details and examples, one may consult [81], [83], [128].

3.4.6 Spirallike mappings

The last part of this section explores the concept of spirallikeness on $\mathbb{B}^n$ in $\mathbb{C}^n$. The spirallikeness relative to a normal linear operator having eigenvalues with positive real part was introduced by Gurganus (see [57]). In the case of complex Banach spaces, this idea was extended by Suffridge (see [128]). For more details, one may consult also [37], [67], [94].

Definition 3.4.10. Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be characterized by $m(A) > 0$, where $m(A) = \min\big\{\mathfrak{Re}\langle A(z), z\rangle : \|z\| = 1\big\}$. Then $f \in S(\mathbb{B}^n)$ is *spirallike relative to* A if $e^{-tA}f(\mathbb{B}^n) \subseteq f(\mathbb{B}^n)$, for all $t \geq 0$, where

$$e^{-tA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k A^k.$$

Let us consider $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with the property $m(A) > 0$. We now present the analytical characterization given by Suffridge (see [128]; see also [57]) of the spiralikeness relative to A.

Theorem 3.4.11. *Let $f \in \mathcal{LS}(\mathbb{B}^n)$. Then f belongs to the class of spirallike mappings (relative to A) if and only if*

$$\mathfrak{Re}\langle [Df(z)]^{-1} A f(z), z\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{3.4.4}$$

In particular, if $A = e^{-i\delta} I_n$, where $\delta \in \mathbb{R}$ with $|\delta| < \frac{\pi}{2}$, then we obtain the class $\hat{S}_\delta(\mathbb{B}^n)$ of spirallike mappings of type δ. This class was introduced by Hamada and Kohr in [67].

3.5 The theory of Loewner chains in $\mathbb{C}^n$

The fifth section of this chapter contains generalizations of the notions presented in §1.6. Pfaltzgraff (see e.g. [109]) was the first who obtained generalizations of the Loewner chains and Loewner differential equation on the Euclidean unit ball. The study was extended by Poreda in the case of the unit polydisc in $\mathbb{C}^n$ (see e.g. [115], [116]), respectively by Kubicka and Poreda (see e.g. [86]). Important results were obtained over time by Duren, Graham, Hamada and Kohr (see e.g. [20]), Graham, Hamada and Kohr (see e.g. [32]), Graham, Hamada, Kohr and Kohr (see e.g. [36], [37], [47]) and others.

Probably the most important distinction between the one dimensional case and the higher dimension is the compactness of the family of normalized univalent mappings. It is known that $S(\mathbb{U})$ is a compact set (see Theorem 1.4.4) while the set $S(\mathbb{B}^n)$ is not compact for $n \geq 2$ (see e.g. [45]). This problem was solved by Graham, Hamada and Kohr who introduced the class $S^0(\mathbb{B}^n)$ of mappings which admit parametric representation on $\mathbb{B}^n$ (see e.g. [32]; see also [114]). For $n = 1$, we have that $S^0(\mathbb{B}^1) = S$ (see e.g. [114]). However, if $n \geq 2$, then $S^0(\mathbb{B}^n)$ is strictly included in $S(\mathbb{B}^n)$. Moreover, Graham, Kohr and Kohr (see [48]) proved that $S^0(\mathbb{B}^n)$ is compact (see e.g. [32], [45], [48]).

Another important problem that was solved by Graham, Hamada, Kohr and Kohr is the existence in $\mathbb{C}^n$ of mappings which cannot be embedded as the first elements of a Loewner chain. Using the family $S^0(\mathbb{B}^n)$ they succeed to prove the analogous of Pommerenke's theorem (see Theorem 1.6.3) in higher dimensions (see [48]; see also [32]). Other important results can be found in [14], [16], [17], [32], [45], [48].

3.5.1 General results related to Loewner chains in $\mathbb{C}^n$

This section opens with some preliminary notions and results connected to the theory of Loewner chains on $\mathbb{B}^n$ (see e.g. [14], [45], [48], [109], [117]).

Definition 3.5.1. Let $f = f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ be a mapping.

- Let f satisfies the properties: $f(\cdot,t)$ is biholomorphic on $\mathbb{B}^n$, $f(0,t) = 0$, for all $t \geq 0$ and $f(\cdot,s) \prec f(\cdot,t)$, for all $0 \leq s \leq t < \infty$. Then $f(z,t)$ is a *univalent subordination chain* on $\mathbb{B}^n$.
- If $f(z,t)$ satisfies also the property that $Df(0,t) = e^t I_n$, for all $t \geq 0$, then $f(z,t)$ is called a *Loewner chain* (*normalized univalent subordination chain*) on $\mathbb{B}^n$.

Remark 3.5.2. If $f(z,t)$ is a Loewner chain, then there is a unique biholomorphic Schwarz mapping $v = v(z,s,t)$ such that

$$f(z,s) = f\big(v(z,s,t),t\big), \quad z \in \mathbb{B}^n, \quad 0 \le s \le t < \infty. \tag{3.5.1}$$

The mapping $v(z,s,t)$ is called *the transition mapping* of $f(z,t)$ (see e.g. [45], [109]).

Based on the normalization of $f(z,t)$, we easily deduce that $Dv(0,s,t) = e^{s-t}I_n$, for all $0 \le s \le t < \infty$. From (3.5.1), it follows that $v = v(z,s,t)$ satisfies also the *semigroup property*

$$v(z,s,T) = v\big(v(z,s,t),t,T\big), \quad z \in \mathbb{B}^n, \quad 0 \le s \le t \le T < \infty. \tag{3.5.2}$$

The next result was obtained by Graham, Kohr and Kohr (see [48]), respectively Curt and Kohr (see [15]) and presents the strong connection between the Loewner chains and transition mappings (see also [45]).

Theorem 3.5.3. *Let $f(z,t)$ be a Loewner chain and let $v(z,s,t)$ be its transition mapping. Let also $(t_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}_+^*$ be such that $\lim_{k\to\infty} t_k = \infty$ and the limit*

$$\lim_{k\to\infty} e^{-t_k} f(z,t_k) = F(z) \in \mathcal{H}(\mathbb{B}^n)$$

is locally uniformly on $\mathbb{B}^n$. Then $\exists \lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$ locally uniformly on $\mathbb{B}^n$, for all $s \in [0,\infty)$.

The second important result in this section was given by Pfaltzgraff (see [109]). In the case of complex Banach spaces the result was studied by Poreda (see [117]). The theory of Loewner chains was also considered in the abstract setting of complex hyperbolic manifolds and was studied by Arosio, Bracci, Hamada and Kohr (see [3]). Other contributions can be found in [2], [5].

Theorem 3.5.4. *Let $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ be such that $h(\cdot,t)$ belongs to the class $\mathcal{M}$, for all $t \ge 0$ and $h(z,\cdot)$ belongs to the family of measurable functions on $[0,\infty)$, for all $z \in \mathbb{B}^n$. Let also*

$$\begin{cases} \dfrac{\partial v}{\partial t} = -h(v,t), \quad a.e. \quad t \ge s \\ v(s) = z \end{cases} \tag{3.5.3}$$

be an initial value problem. Then $\forall s \in [0,\infty)$ and $z \in \mathbb{B}^n$, the problem (3.5.3) is uniquely solvable and its solution $v(t) = v(z,s,t) = e^{s-t}z + ...$ is locally absolutely continuous. Moreover, $v(\cdot,s,t,)$ belongs to the family of univalent Schwarz mappings on $\mathbb{B}^n$ for a fixed $s \le t \in [0,\infty)$. For fixed $s \ge 0$ and $z \in \mathbb{B}^n$ we know that $v(z,s,\cdot)$ belongs to the family of Lipschitz functions locally uniformly with respect to z.

Remark 3.5.5. Recall that the application $h = h(z,t)$ presented in the previous result is known as a *Herglotz vector field* (see e.g. [45]). The differential equation (3.5.3) is called *Loewner (ordinary) differential equation* associated to h.

The subsequent result indicates that if the transition mapping solves the problem (3.5.3), then it generates a Loewner chain. This result is due to Poreda (see [117]), Hamada and Kohr (see [68]).

Theorem 3.5.6. *Let us consider $h(z,t)$ a Herglotz vector field and $v(z,s,t)$ the solution of the Cauchy problem (3.5.3). Then*

$$\forall s \ge 0, \quad \exists \lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$$

locally uniformly on $\mathbb{B}^n$. Moreover, $f(\cdot,s)$ belongs to the family of biholomorphic mappings on $\mathbb{B}^n$ and

$$f(z,s) = f\big(v(z,s,t),t\big), \quad z \in \mathbb{B}^n, 0 \le s \le t < \infty.$$

Consequently, $f(z,t)$ is a Loewner chain with the property that the family $\{e^{-t}f(\cdot,t)\}_{t\ge0}$ is normal on $\mathbb{B}^n$ and $f(z,\cdot)$ belongs to the family of locally Lipschitz functions on $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$. In this context, $f(z,t)$ satisfies also the equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad a.e. \quad t \ge 0, z \in \mathbb{B}^n. \tag{3.5.4}$$

The differential equation (3.5.4) is called *the (generalized) Loewner differential equation* associated to h (see e.g. [45]).

The main result in the theory of Loewner chains in higher dimensions is state in the following theorem. This result was obtained by Pfaltzgraff (see [109]) and was generalized by Poreda (see [117]) in the case of complex Banach spaces. Other notable contributions have been achieved by Hamada and Kohr(see [68]).

Theorem 3.5.7. *Let us consider $h(z,t)$ a Herglotz vector field and let $f = f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ satisfy the properties: $f(\cdot,t)$ belongs to the family $\mathcal{H}(\mathbb{B}^n)$, $f(0,t) = 0$, $Df(0,t) = e^t I_n$, for all $t \geq 0$ and $f(z,\cdot)$ belongs to the class of locally absolutely continuous functions on $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$. Also, suppose that the relation (3.5.4) holds. If $(t_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}_+^*$ is increasing such that $\lim\limits_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} e^{-t_k} f(z,t_k) = F(z)$ locally uniformly on $\mathbb{B}^n$, then $f(z,t)$ is a Loewner chain and for all $s \geq 0$, the limit $\lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$ holds locally uniformly on $\mathbb{B}^n$, where $v(z,s,t)$ solves the problem (3.5.3) for all $z \in \mathbb{B}^n$.*

The following result was achieved by Graham, Hamada, and Kohr (see [32]). See also the results obtained by Curt and Kohr in [15].

Theorem 3.5.8. *Assume that $f = f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is a Loewner chain. Then $\exists h = h(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ with the properties: $h(\cdot,t)$ belongs to the class $\mathcal{M}$, for all $t \geq 0$, $h(z,\cdot)$ belongs to the family of measurable functions on $[0,\infty)$, for $z \in \mathbb{B}^n$ and*

$$\frac{\partial f}{\partial t} = Df(z,t)h(z,t), \quad a.e. \quad t \geq 0, z \in \mathbb{B}^n.$$

Morevoer, if $\exists (t_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}_+^*$ with $\lim\limits_{k\to\infty} t_k = \infty$ and the limit $\lim_{k\to\infty} e^{-t_k} f(z,t_k) = F(z)$ is locally uniformly on $\mathbb{B}^n$, then for every $s \geq 0$, there exists the limit $\lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$ locally uniformly on $\mathbb{B}^n$, where $v(z,s,t)$ solves the problem (3.5.3) for all $z \in \mathbb{B}^n$.

We end this section with the growth theorem for Loewner chains that satisfy the property that the family $\left\{e^{-t} f(\cdot,t)\right\}_{t\geq 0}$ is normal on $\mathbb{B}^n$ (see e.g. [32]). Note that in $\mathbb{C}^n$ there are Loewner chains that do not meet this assumption (see [32], [45]).

Theorem 3.5.9. *Let $f(z,t)$ be a Loewner chain such that the family $\left\{e^{-t} f(\cdot,t)\right\}_{t\geq 0}$ is normal on $\mathbb{B}^n$. Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|e^{-t} f(z,t)\| \leq \frac{\|z\|}{(1-\|z\|)^2},$$

for all $z \in \mathbb{B}^n$ and $t \geq 0$.

3.5.2 Loewner chains and biholomorphic mappings in $\mathbb{C}^n$

This subsection outlines some characterizations of certain subclasses of $S(\mathbb{B}^n)$ via Loewner chains. Based on these results, one may easily construct examples of Loewner chains in $\mathbb{C}^n$ (see e.g. [45]).

The first result states the characterization of mappings from the class $\hat{S}_\delta(\mathbb{B}^n)$ and was obtained by Hamada and Kohr (see [67]). In particular, for $\delta = 0$, we obtain the characterization of starlikeness on $\mathbb{B}^n$ proved by Pfaltzgraff and Suffridge (see [110]).

Theorem 3.5.10. *Let $f \in \mathcal{LS}(\mathbb{B}^n)$, $\delta \in \mathbb{R}$ be such that $|\delta| < \frac{\pi}{2}$ and $a = \tan\delta$. Then $f \in \hat{S}_\delta(\mathbb{B}^n)$ if and only if $f(z,t) = e^{(1-ia)t} f(e^{iat}z)$ is a Loewner chain, for all $z \in \mathbb{B}^n$ and $t \geq 0$. For $\delta = 0$ we obtain the characterization of starlikeness on $\mathbb{B}^n$.*

3.5.3 Parametric representation on $\mathbb{B}^n$

The third part of this section is dedicated to study of mappings which admit parametric representation on $\mathbb{B}^n$ in $\mathbb{C}^n$. We present here some general notions, growth and distortion theorems, as well as the connection between these mappings and Loewner chains.

First, the univalent mappings that admit parametric representation were studied by Poreda on $\mathbb{U}^n$ (see [115], [116]) and by Kohr on $\mathbb{B}^n$ in $\mathbb{C}^n$ (see [82]). Graham, Hamada, and Kohr generalized these results to the case of arbitrary norm (see [32]; see also [48]).

Definition 3.5.11. Let $f \in \mathcal{H}_0(\mathbb{B}^n)$. Then f has *parametric representation* if $\exists h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ a Herglotz vector field such that $\lim_{t\to\infty} e^t v(z,t) = f(t)$ locally uniformly on $\mathbb{B}^n$, where $v(z,\cdot)$ uniquely solves the problem (3.5.3) on $[0,\infty)$ for $s = 0$. We denote by $S^0(\mathbb{B}^n)$ the class of mappings which admit parametric representation on $\mathbb{B}^n$.

The analogous of Pommerenke's theorem (see Theorem 1.6.3) in higher dimension was proved by Graham, Kohr and Kohr (see [48]; see also [32]).

Theorem 3.5.12. *Let $f \in \mathcal{H}_0(\mathbb{B}^n)$. Then $f \in S^0(\mathbb{B}^n)$ if and only if $\exists f = f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ a Loewner chain that satisfy the property that $\left\{e^{-t} f(z,t)\right\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$ and $f = f(\cdot,0)$.*

Recall that for $n = 1$, we have that $S^0(\mathbb{B}^1) = S$ (see e.g. [114]). However, if $n \geq 2$, then $S^0(\mathbb{B}^n)$ is strictly included in $S(\mathbb{B}^n)$. This inclusion and the property that many subsets of $S(\mathbb{B}^n)$ are also subsets of $S^0(\mathbb{B}^n)$ was proved by Graham, Hamada, Kohr and Kohr (see [32], [48], [115], [116]).

We present the growth theorem for mappings with parametric representation on $\mathbb{B}^n$. This result was obtained by Graham, Hamada and Kohr for the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm (see [32]).

Theorem 3.5.13. *If $f \in S^0(\mathbb{B}^n)$, then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in \mathbb{B}^n.$$

These inequalities are sharp. As a consequence, we obtain that $f(\mathbb{B}^n) \supseteq \mathbb{B}^n_{1/4}$.

Based on the previous result we obtain the compactness of the class $S^0(\mathbb{B}^n)$. This result was proved by Graham, Kohr and Kohr (see [48]) and it is contrast with the class $S(\mathbb{B}^n)$ which is not compact (see e.g. [32], [45], [48]).

Corollary 3.5.14. *The family $S^0(\mathbb{B}^n) \subseteq \mathcal{H}(\mathbb{B}^n)$ is compact.*

3.5.4 g-parametric representation on $\mathbb{B}^n$

Strongly related to the notion of parametric representation is the notion of g-parametric representation (see e.g. [32]). First, let us consider the following assumption that will be used during this thesis.

Assumption 3.5.15. Let $g \in \mathcal{H}_u(\mathbb{U})$ be such that $g(0) = 1$, $g(\overline{\zeta}) = \overline{g(\zeta)}$ and $\mathfrak{Re}\, g(\zeta) > 0$, for all $\zeta \in \mathbb{U}$. In addition, assume that for $\rho \in (0,1)$ we have that

$$\min_{|\zeta|=\rho} \mathfrak{Re}\, g(\zeta) = \min\left\{g(\rho), g(-\rho)\right\} \quad \text{and} \quad \max_{|\zeta|=\rho} \mathfrak{Re}\, g(\zeta) = \max\left\{g(\rho), g(-\rho)\right\}.$$

Considering the previous assumption, we introduce the class $\mathcal{M}_g(\mathbb{B}^n)$ that is the analogous of the Carathéodory family in $\mathbb{C}^n$ (see [32]).

Definition 3.5.16. Let $g : \mathbb{U} \to \mathbb{C}$ be such that Assumption 3.5.15 is satisfied. Then we denote by

$$\mathcal{M}_g(\mathbb{B}^n) = \left\{h \in \mathcal{H}_0(\mathbb{B}^n) \, : \, \left\langle h(z), \frac{z}{\|z\|^2}\right\rangle \in g(\mathbb{U}), \quad z \in \mathbb{B}^n\right\},$$

where we consider the particular case $\left\langle h(z), \frac{z}{\|z\|^2}\right\rangle\big|_{z=0} = 1$. When the space $\mathbb{C}^n$ is endowed with the Euclidean norm we denote $\mathcal{M}_g(\mathbb{B}^n)$ simply by $\mathcal{M}_g$. This class was introduced and intensively studied by Graham, Hamada and Kohr (see [32]).

Note that $\mathcal{M}_g \neq \emptyset$, since $id_{\mathbb{B}^n} \in \mathcal{M}_g$ and it is clear that $\mathcal{M}_g \subseteq \mathcal{M}$. In particular, if $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, for all $\zeta \in \mathbb{U}$, then $\mathcal{M}_g \equiv \mathcal{M}$.

Next, we revisit the definition of mappings that have g-parametric representation given by Graham, Hamada and Kohr in [32] (see also [82]).

Definition 3.5.17. Let $g : \mathbb{U} \to \mathbb{C}$ satisfy the Assumption 3.5.15. Then $f \in \mathcal{H}_0(\mathbb{B}^n)$ has *g-parametric representation* if $\exists h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ a Herglotz vector field that satisfy the properties: $h(\cdot,t)$ belongs to the class $\mathcal{M}_g$, for all $t \geq 0$ and

$$\lim_{t\to\infty} e^t v(z,t) = f(t)$$

locally uniformly on $\mathbb{B}^n$, where $v(z,\cdot)$ uniquely solves the problem (3.5.3) on $[0,\infty)$ for $s = 0$. We denote by $S_g^0(\mathbb{B}^n)$ the family of mappings which have g-parametric representation on $\mathbb{B}^n$.

Remark 3.5.18. Graham, Hamada and Kohr (see [32]) proved that

$$S_g^0(\mathbb{B}^n) \subseteq S^0(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n) \tag{3.5.5}$$

and the equality holds in the first inclusion if $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, for all $\zeta \in \mathbb{U}$.

Definition 3.5.19. Let $f : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ be a mapping. Then $f(z,t)$ is a *g-Loewner chain* if $f(z,t)$ is a Loewner chain such that $\left\{e^{-t} f(\cdot,t)\right\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$ and the mapping $h(z,t)$ from equation (3.5.4) has the property that $h(\cdot,t)$ belongs to the family $\mathcal{M}_g$ for almost every $t \geq 0$.

The characterization of the class $S_g^0(\mathbb{B}^n)$ via Loewner chains was given by Graham, Hamada and Kohr in [32].

Proposition 3.5.20. *A mapping $f \in \mathcal{H}_0(\mathbb{B}^n)$ belongs to the class $S_g^0(\mathbb{B}^n)$ if and only if $\exists f(z,t)$ a g-Loewner chain that satisfy the property $f = f(\cdot,0)$.*

3.6 New results on convex combinations of biholomorphic mappings on $\mathbb{B}^n$

In this section we study convex combinations of the form $(1-\lambda)f + \lambda g$, where $f, g \in S(\mathbb{B}^n)$ and $\lambda \in [0,1]$. It is known that, in general, the convex combination of two normalized biholmorphic mappings is not biholomorphic on $\mathbb{B}^n$ (see e.g. [45], [83]). The phenomenon also occurs in the case of one dimension and was intensively studied by several authors (see e.g. [9], [58], [97], [100]).

The main idea of this section is to obtain biholomorphic mappings on $\mathbb{B}^n$ (or even starlike mappings) as convex combinations of the form $(1-\lambda)f + \lambda g$, where $f, g \in S(\mathbb{B}^n)$ and $\lambda \in [0,1]$. The results presented in this section are due to Grigoriciuc (see [52]) and represent partial extensions of the results obtained by Chichra and Singh in [9].

3.6.1 Preliminaries

We start the first part of this section with some examples of convex combinations of univalent functions in $\mathbb{C}$. This classical examples show that the linear combination of two normalized univalent functions is not, in general, univalent on $\mathbb{U}$ in $\mathbb{C}$ (see e.g. [19], [85], [97], [102]).

Example 3.6.1. Let $f, g : \mathbb{U} \to \mathbb{C}$ be given by $f(\zeta) = \dfrac{\zeta}{(1-\zeta)^2}$ and $g(\zeta) = \dfrac{\zeta}{(1+\zeta)^2}$, for all $\zeta \in \mathbb{U}$. Then $f, g \in S$, but $h = \frac{f+g}{2}$ does not belong to S.

In Example 3.6.1, the functions f and g are not only normalized and univalent, they are even starlike on $\mathbb{U}$. However, the function h is not starlike on $\mathbb{B}$ (in fact, h is not even univalent on $\mathbb{U}$). On the other hand, MacGregor (see [97]) proved that the linear combination of two convex functions is not necessarily univalent on the unit disc.

Next, we can extend the statement of Example 3.6.1 to the case of several complex variables. For $n = 2$, we obtain the following example (see e.g. [45], [83]):

Example 3.6.2. Let $F, G : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$F(z) = \left(\frac{z_1}{(1-z_1)^2}, \frac{z_2}{(1-z_2)^2}\right) \quad \text{and} \quad G(z) = \left(\frac{z_1}{(1+z_1)^2}, \frac{z_2}{(1+z_2)^2}\right),$$

for all $z = (z_1, z_2) \in \mathbb{B}^2$. Then $H = \frac{1}{2}(F+G) \notin S^*(\mathbb{B}^2)$. In fact, $H \notin S(\mathbb{B}^2)$.

Although linear combinations of univalent functions are not always univalent (for more details about these results, one may consult [58], [97]), there exist subclasses of the class S that satisfy this condition (see e.g. [9], [100] in one dimension). Our purpose is to extend in the case of several complex variables a result proved by P.N. Chichra and R. Singh (see [9] for the case of one complex variable). In the following, we state their result for the case $n = 1$

Theorem 3.6.3. *Let $\lambda \in (0,1)$. If $f \in S^*$ and $\mathfrak{Re} f'(\zeta) > 0$, for all $\zeta \in \mathbb{U}$, then*

$$h_\lambda(\zeta) = (1-\lambda)\zeta + \lambda f(\zeta) \tag{3.6.1}$$

is starlike with respect to zero in $\mathbb{U}$ and $\mathfrak{Re} h'(\zeta) > 0$, for all $\zeta \in \mathbb{U}$.

3.6.2 Univalence of convex combinations in $\mathbb{C}^n$

In view of the results presented in the previous section we can prove some criteria for univalence of a convex combination of normalized holomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$. In fact, we can obtain a condition for a convex combination to be a mapping which has parametric representation on $\mathbb{B}^n$. The original results presented in this part have been obtained by Grigoriciuc in [52].

Lemma 3.6.4. *Let $f \in \mathcal{H}_0(\mathbb{B}^n)$ be such that $\mathfrak{Re}\langle Df(z)(u), u\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$. Also let $h_\lambda : \mathbb{B}^n \to \mathbb{C}^n$ be given by*

$$h_\lambda(z) = (1-\lambda)z + \lambda f(z), \tag{3.6.2}$$

for all $z \in \mathbb{B}^n$ and $\lambda \in [0,1]$. Then $h_\lambda \in S(\mathbb{B}^n)$.

Lemma 3.6.5. *Let $f \in \mathcal{H}_0(\mathbb{B}^n)$ be such that $\|Df(z) - I_n\| < 1$, for all $z \in \mathbb{B}^n$ and let h_λ be given by (3.6.2) for $\lambda \in [0,1]$. Then $h_\lambda \in S^0(\mathbb{B}^n)$. In particular, h_λ is univalent on $\mathbb{B}^n$.*

3.6.3 Starlikeness of convex combinations on $\mathbb{B}^n$

In this subsection we discussed about the starlikeness of a convex combination of biholomorphic mappings on $\mathbb{B}^n$. We present also some examples that illustrate how this property occurs in several particular cases.

Proposition 3.6.6. *Let $\lambda \in [0,1]$ and let $f_j \in S^*$ be such that $\mathfrak{Re} f_j'(\zeta) > 0$, for $j = \overline{1,n}$ and $\zeta \in \mathbb{U}$. Also, let $f(z) = \big(f_1(z_1), ..., f_n(z_n)\big)$, for $z \in \mathbb{B}^n$. Then $h_\lambda \in S^*(\mathbb{B}^n)$, where h_λ is given by (3.6.2). Moreover, $\mathfrak{Re}\langle Dh_\lambda(z)(u), u\rangle > 0$, for $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$.*

It is clear that the mapping used in the previous result has a very particular form (has on each component a starlike function of one complex variable). However, we can obtain similar results for arbitrary starlike mappings on $\mathbb{B}^n$, as in the following examples (see e.g. [47], [128]). Here, we use arbitrary starlike mappings to construct convex combinations that are starlike on $\mathbb{B}^n$.

Example 3.6.7. Let $f : \mathbb{B}^2 \to \mathbb{C}^2$ be defined by $f(z) = \big(z_1 + az_2^2, z_2\big)$, for all $z = (z_1, z_2) \in \mathbb{B}^2$ with $|a| \le \frac{3\sqrt{3}}{2}$. We know (see e.g. [24], [45], [83]) that $f \in S^*(\mathbb{B}^2)$. Moreover,

$$\begin{aligned} h_\lambda(z) = (1-\lambda)z + \lambda f(z) &= (1-\lambda)\big(z_1, z_2\big) + \lambda\big(z_1 + az_2^2, z_2\big) \\ &= \big((1-\lambda)z_1 + \lambda z_1 + \lambda a z_2^2, (1-\lambda)z_2 + \lambda z_2\big) \\ &= \big(z_1 + \lambda a z_2^2, z_2\big), \end{aligned}$$

for all $z \in \mathbb{B}^2$. Since $\lambda \in [0,1]$ and $|a| \le \frac{3\sqrt{3}}{2}$, it follows that $|\lambda a| \le \frac{3\sqrt{3}}{2}$ and then $h_\lambda \in S^*(\mathbb{B}^2)$.

Based on the ideas presented in the previous examples, we can prove a second version of Chichra-Singh's theorem (see Theorem 3.6.3) in $\mathbb{C}^n$, for $n \geq 2$. Note that this result was obtained by Grigoriciuc in [52].

Theorem 3.6.8. *Let $\lambda \in (0,1)$, $\mu = \lambda/(1-\lambda)$ and let $f \in \mathcal{LS}_n(\mathbb{B}^n)$ be such that*

$$\left\|Df(z) - I_n\right\| < \lambda^{-1} \tag{3.6.3}$$

and

$$\mathfrak{Re}\left\langle \left(I_n + \mu Df(z)\right)^{-1}\left(z + \mu f(z)\right), z\right\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}. \tag{3.6.4}$$

Then $h_\lambda \in S^*(\mathbb{B}^n)$, for all $\lambda \in (0,1)$, where h_λ is given by (3.6.2).

Inspired by the previous result, we can define a subclass $\mathcal{L}^*_\lambda(\mathbb{B}^n)$ of normalized locally biholomorphic mappings on $\mathbb{B}^n$ that satisfies conditions from Theorem 3.6.8.

Definition 3.6.9. Let us consider $\lambda \in (0,1)$ and $\mu = \lambda/(1-\lambda)$. We say that $f \in \mathcal{L}^*_\lambda(\mathbb{B}^n)$ if $f \in \mathcal{LS}_n(\mathbb{B}^n)$ such that (3.6.3) and (3.6.4) are satisfied. Note that $\mathcal{L}^*_\lambda(\mathbb{B}^n) \neq \emptyset$ since $I_n \in \mathcal{L}^*_\lambda(\mathbb{B}^n)$.

Next we offer a non-trivial example of a mapping $f \in \mathcal{L}^*_\lambda$ (see also [24], [45], [83], [128]).

Example 3.6.10. Let $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by $f(z) = \left(z_1 + az_2^2, z_2\right)$, for all $z \in \mathbb{B}^2$ with $|a| \leq \frac{1}{2}$. Then $f \in \mathcal{L}^*_\lambda(\mathbb{B}^2)$. Moreover, $h_\lambda \in S^*(\mathbb{B}^2)$, where h_λ is given by (3.6.2).

We end this section by proposing a conjecture which generalize the result proved by Chichra and Singh (see [9]). Starting from the result proved in $\mathbb{C}$, we consider the following conjecture on $\mathbb{B}^n$ in $\mathbb{C}^n$ (see [52]):

Conjecture 3.6.11. *Let $\lambda \in (0,1)$. If $f \in S^*(\mathbb{B}^n)$ and $\mathfrak{Re}\left\langle Df(z)(u), u\right\rangle > 0$, for all $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$, then $h_\lambda(z) = (1-\lambda)z + \lambda f(z)$ is a starlike mapping on $\mathbb{B}^n$. Moreover, $\mathfrak{Re}\left\langle Dh_\lambda(z)(u), u\right\rangle > 0$, for $z \in \mathbb{B}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$. In particular, h_λ is biholomorphic on $\mathbb{B}^n$.*

It is clear that in $\mathbb{C}$ the statement of Conjecture 3.6.11 proposed by Grigoriciuc in [52] is true, as it reduces to Theorem 3.6.3 obtained by Chichra and Singh in [9].

3.7 Extension operators in $\mathbb{C}^n$

In this section we present some extension operators that preserve geometric and analytic properties on the unit ball in $\mathbb{C}^n$. Our discussion starts with the Roper-Suffridge extension operator Φ_n (considered by K. Roper and T.J. Suffridge in [121]) and the Graham-Kohr extension operator $\Psi_{n,\alpha}$ (defined by I. Graham and G. Kohr in [44]; see also [43]). The third part of this section contains two generalizations of the Roper-Suffridge extension operator introduced by Graham, Hamada, Kohr, Kohr and Suffridge (see e.g. [42], [47]) that map a locally univalent function on $\mathbb{U}$ into a locally biholomorphic mapping on $\mathbb{B}^n$. In the final part of this section we briefly look at the extension operator introduced by Pfaltzgraff and Suffridge (see [111]) and a generalization of their operator (see e.g. [10]).

During the following sections, for $n \geq 2$, we use the variable $z = (z_1, \tilde{z}) \in \mathbb{C}^n$, where $\tilde{z} = (z_2, ..., z_n) \in \mathbb{C}^{n-1}$. Recall that $\mathcal{LS}_n(\mathbb{B}^n)$ is the family of normalized locally biholomorphic mappings on $\mathbb{B}^n$ and $\mathcal{LS}_1(\mathbb{B}^1) = \mathcal{LS}$.

3.7.1 The Roper-Suffridge extension operator Φ_n

The Roper-Suffridge extension operator $\Phi_n : \mathcal{LS} \to \mathcal{LS}_n(\mathbb{B}^n)$ is given by

$$\Phi_n(f)(z) = \left(f(z_1), \tilde{z}\sqrt{f'(z_1)}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n, \tag{3.7.1}$$

where the branch of the square root function has the property $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$.

The first important result related to the extension operator Φ_n is due to Roper and Suffridge. They proved that Φ_n preserves the notion of convexity from one dimensional case to higher dimensions (see e.g. [121]). The same result was proved by Graham and Kohr in a different manner (see [43]).

Theorem 3.7.1. *If $f \in K$, then $\Phi_n(f)$ belongs to the class $K(\mathbb{B}^n)$. Hence, $\Phi_n(K) \subseteq K(\mathbb{B}^n)$.*

Another important property of the operator Φ_n is related to the preservation of starlikeness of order $\alpha \in [0,1)$. During the time, several authors obtained strong extension results, as follows:

- Graham and Kohr (see [43]) proved that Φ_n preserves the starlikeness;
- Hamada, Kohr and Kohr (see [73]) obtained that Φ_n preserves the stralikeness of order $1/2$;
- Liu (see [92]) obtained that Φ_n preserves the starlikeness of order $\alpha \in (0,1)$;

The last result is due to Graham, Kohr and Kohr (see [48]) and states the relation between the Loewner theory and the Roper-Suffridge extension operator.

Theorem 3.7.2. *If f belongs to the class S, then $\Phi_n(f) \in S^0(\mathbb{B}^n)$. Hence, $\Phi_n(S)$ is a subset of $S^0(\mathbb{B}^n)$, i.e. Φ_n maps the functions that have parametric representation on $\mathbb{U}$ to mappings that have the same property on $\mathbb{B}^n$.*

3.7.2 The Graham-Kohr extension operator $\Psi_{n,\alpha}$

The second extension operator that is presented here was defined by I. Graham and G. Kohr (see [43], [44]). For $\alpha \in [0,1]$, let $\Psi_{n,\alpha}$ be defined by

$$\Psi_{n,\alpha}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} \tilde{z}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n \tag{3.7.2}$$

for any function $f \in \mathcal{LS}$ with $f(z_1) \neq 0$ for $z_1 \in \mathbb{U} \setminus \{0\}$. We take the branch of the power function with the property $\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\big|_{z_1=0} = 1$. The particular case $\Psi_{n,1}$ was studied by Pfaltzgraff and Suffridge in [111].

Graham and Kohr (see [44]) proved that the extension operator $\Psi_{n,\alpha}$ has the following important properties. It is important to mention here that the extension results remain true also for the norm $\|\cdot\|_p$, where $1 \leq p \leq \infty$ (see e.g. [44]).

Theorem 3.7.3. *Let $\alpha \in [0,1]$.*

a) If $f \in S$, then $\Psi_{n,\alpha}(f)$ belongs to the class $S^0(\mathbb{B}^n)$. Hence, $\Psi_{n,\alpha}(S)$ is a subset of $S^0(\mathbb{B}^n)$.

b) If $f \in S^*$, then $\Psi_{n,\alpha}(f)$ belongs to the class $S^*(\mathbb{B}^n)$. Hence, $\Psi_{n,\alpha}(S^*)$ is a subset of $S^*(\mathbb{B}^n)$.

c) The extension operator $\Psi_{n,\alpha}$ does not preserve convexity for $n \geq 2$.

3.7.3 Generalizations of the Roper-Suffridge extension operator

In the third part of this section we focus our attention on two generalizations of the Roper-Suffridge extension operator introduced by Graham, Kohr and Kohr (see [47]), respectively by Graham, Hamada, Kohr and Suffridge (see [42]). Other results related to the generalized Roper-Suffridge extension operator can be found in [25], [27], [43], [46], [84], [93], [103], [138].

The generalized Roper-Suffridge extension operator $\Phi_{n,\beta}$

A first general form of the operator Φ_n was considered by Graham, Kohr and Kohr in [47]. For $\beta \in [0,1/2]$, they defined the operator $\Phi_{n,\beta} : \mathcal{LS} \to \mathcal{LS}_n(\mathbb{B}^n)$ given by

$$\Phi_{n,\beta}(f)(z) = \left(f(z_1), \big(f'(z_1)\big)^{\beta} \tilde{z}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n, \tag{3.7.3}$$

where the branch of the power function has the property $\big(f'(z_1)\big)^{\beta}\big|_{z_1=0} = 1$. It is clear that $\Phi_{n,1/2} = \Phi_n$ is the Roper-Suffridge extension operator given by (3.7.1).

The following properties of the operator $\Phi_{n,\beta}$ were obtainted by Graham, Kohr and Kohr in [47].

Theorem 3.7.4. *Let $\beta \in [0,1/2]$ and $\delta \in \mathbb{R}$ be such that $|\delta| < \frac{\pi}{2}$.*

a) If $f \in S$, then $\Phi_{n,\beta}(f)$ belongs to the class $S^0(\mathbb{B}^n)$. Hence, $\Phi_{n,\beta}(S)$ is a subset of $S^0(\mathbb{B}^n)$.

b) If $f \in \hat{S}_\delta$, then $\Phi_{n,\beta}(f)$ belongs to the class $\hat{S}_\delta(\mathbb{B}^n)$. Hence $\Phi_{n,\beta}(\hat{S}_\delta)$ is a subset of $\hat{S}_\delta(\mathbb{B}^n)$. In particular, if $\delta = 0$, then $\Phi_{n,\beta}(S^*)$ is a subset of $S^*(\mathbb{B}^n)$.

It is important to mention here that the operator $\Phi_{n,\beta}$ preserves the convexity only if $\beta = \frac{1}{2}$ (i.e. the Roper-Suffridge extension operator). This result was obtained by Graham, Kohr and Kohr (see [47]).

The generalized Roper-Suffridge-Graham-Kohr extension operator $\Phi_{n,\alpha,\beta}$

The second generalization of the Roper-Suffridge, respectively of the Graham-Kohr extension operator was introduced by Graham, Hamada, Kohr and Suffridge in [42]. They considered the extension operator $\Phi_{n,\alpha,\beta}$ defined by

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} \big(f'(z_1)\big)^{\beta} \tilde{z}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where $\alpha, \beta \geq 0$ and $f \in \mathcal{LS}$ has the property that $f(z_1) \neq 0$ for $z_1 \in \mathbb{U}\backslash\{0\}$. Here, the branches of the power functions are taken such that $\big(\frac{f(z_1)}{z_1}\big)^{\alpha}\big|_{z_1=0} = 1$ and $\big(f'(z_1)\big)^{\beta}\big|_{z_1=0} = 1$. It is clear that $\Phi_{n,0,1/2} = \Phi_n$ is the Roper-Suffridge extension operator, $\Phi_{n,0,\beta} = \Phi_{n,\beta}$ is the generalized Roper-Suffridge extension operator and $\Phi_{n,\alpha,0} = \Psi_{n,\alpha}$ is the Graham-Kohr extension operator.

Graham, Hamada, Kohr and Suffridge (see [42]) studied the operator $\Phi_{n,\alpha,\beta}$ and obtained the following extension results:

Theorem 3.7.5. *Let $\alpha \in [0,1]$ and $\beta \in [0,1/2]$ be such that $\alpha + \beta \leq 1$.*

a) If $f \in S$, then $\Phi_{n,\alpha,\beta}(f)$ belongs to the class $S^0(\mathbb{B}^n)$. Hence, $\Phi_{n,\alpha,\beta}(S)$ is a subset of $S^0(\mathbb{B}^n)$.

b) If $f \in S^*$, then $\Phi_{n,\alpha,\beta}(f)$ belongs to the class $S^*(\mathbb{B}^n)$. Hence, $\Phi_{n,\alpha,\beta}(S^*)$ is a subset of $S^*(\mathbb{B}^n)$.

In addition, Graham, Hamada, Kohr and Suffridge (see [42]) proved that $\Phi_{n,\alpha,\beta}$ preserves the notion of convexity only if $\alpha = 0$ and $\beta = \frac{1}{2}$ (i.e. the Roper-Suffridge extension operator).

3.7.4 The Pfaltzgraff-Suffridge extension operator Γ_n

Pfaltzgraff and Suffridge considered a different extension of the Roper-Suffridge operator in [111]. They defined an operator that extend locally biholomorphic mappings on $\mathbb{B}^n$ in $\mathbb{C}^n$ to a local biholomorphic mapping on $\mathbb{B}^{n+1}$ in $\mathbb{C}^{n+1}$. Here, let $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$, where $z' = (z_1, ..., z_n) \in \mathbb{C}^n$.

For $n \geq 1$, the Pfaltzgraff-Suffridge extension operator $\Gamma_n : \mathcal{LS}_n(\mathbb{B}^n) \to \mathcal{LS}_{n+1}(\mathbb{B}^{n+1})$ is given by

$$\Gamma_n(f)(z) = \left(f(z'), z_{n+1}\big[J_f(z')\big]^{\frac{1}{n+1}}\right), \quad z = (z', z_{n+1}) \in \mathbb{B}^{n+1}, \tag{3.7.4}$$

where $J_f(z') = \det Df(z')$, for $z' \in \mathbb{B}^n$. We consider the power function such that $\big[J_f(z')\big]^{\frac{1}{n+1}}\big|_{z'=0} = 1$. Note that, if $n = 1$, then $\Gamma_1 = \Phi_2$ is a particular form of the Roper-Suffridge extension operator Φ_n.

Remark 3.7.6. It is clear that if $f \in \mathcal{LS}_n(\mathbb{B}^n)$, then $\Gamma_n(f) \in \mathcal{LS}_{n+1}(\mathbb{B}^{n+1})$. Moreover, $\Gamma_n\big(S(\mathbb{B}^n)\big) \subseteq S(\mathbb{B}^{n+1})$. For details, one may consult [111].

Graham, Kohr and Suffridge (see [49]) proved that the extension operator preserves the first element of a Loewner chain from $\mathbb{B}^n$ to $\mathbb{B}^{n+1}$. For $n \geq 2$, Graham, Hamada and Kohr (see [35]; see also [10], [33]) obtained the same property in the setting of bounded symmetric domains.

Theorem 3.7.7. *If $f \in S^0(\mathbb{B}^n)$, then $\Gamma_n(f)$ belongs to the class $S^0(\mathbb{B}^{n+1})$. Hence, $\Gamma_n\big(S^0(\mathbb{B}^n)\big)$ is a subset of $S^0(\mathbb{B}^{n+1})$.*

3.8 Convex combinations of Graham-Kohr type extension operators

In the last section of this chapter we combine the ideas presented above, namely extension operators and convex combinations of biholomorphic mappings in $\mathbb{C}^n$ (see e.g. [9], [19], [97] for convex combinations of univalent functions in $\mathbb{C}$; see e.g. [45], [52], [83] for convex combinations of biholomorphic mappings in $\mathbb{C}^n$; see also [42], [44], [111], [121] for extension operators). Hence, we discuss about convex combinations of extension operators on $\mathbb{B}^n$. In particular, we consider a new extension operator obtained as a convex combination of two Graham-Kohr type extension operators (see e.g. [43], [44]). The results presented in this section are original.

3.8.1 The extension operator $\mathcal{K}_{n,\lambda}^{\alpha,\beta}$

First, for $\lambda, \alpha, \beta \in [0,1]$, let us introduce the extension operator $\mathcal{K}_{n,\lambda}^{\alpha,\beta}$ in $\mathbb{C}^n$. This operator is obtained as a convex combination of two Graham-Kohr type extension operators (see e.g. [44]).

Definition 3.8.1. Let $\lambda, \alpha, \beta \in [0,1]$. We define the operator

$$\begin{aligned}\mathcal{K}_{n,\lambda}^{\alpha,\beta}(f,g)(z) &= (1-\lambda)\Psi_{n,\alpha}(g)(z) + \lambda\Psi_{n,\beta}(f)(z)\\ &= \left((1-\lambda)g(z_1) + \lambda f(z_1), \quad (1-\lambda)\tilde{z}\left[\frac{g(z_1)}{z_1}\right]^{\alpha} + \lambda\tilde{z}\left[\frac{f(z_1)}{z_1}\right]^{\beta}\right), \end{aligned}\tag{3.8.1}$$

for all $z = (z_1, \tilde{z}) \in \mathbb{B}^n$, where $f, g \in \mathcal{LS}$ such that $f(z_1) \neq 0$ and $g(z_1) \neq 0$, for all $z_1 \in \mathbb{U} \setminus \{0\}$ and $\Psi_{n,\alpha}$ and $\Psi_{n,\beta}$ are the Graham-Kohr extension operators defined by (3.7.2). We consider the branch of the power functions such that $\left(\frac{g(z_1)}{z_1}\right)^{\alpha}\big|_{z_1=0} = 1$ and $\left(\frac{f(z_1)}{z_1}\right)^{\beta}\big|_{z_1=0} = 1$.

Proposition 3.8.2. *Let $\lambda, \alpha, \beta \in [0,1]$ and let $\mathcal{K}_{n,\lambda}^{\alpha,\beta}$ be the operator defined by (3.8.1). Also let $f, g : \mathbb{U} \to \mathbb{C}$ be two functions with the properties from Definition 3.8.1. Then $\mathcal{K}_{n,\lambda}^{\alpha,\beta}(f,g) \in \mathcal{H}_0(\mathbb{B}^n)$.*

Taking into account Definition 3.8.1 it is natural to consider $\lambda \in (0,1)$, $\alpha, \beta \in [0,1]$ and the particular cases of the operator $\mathcal{K}_{n,\lambda}^{\alpha,\beta}$ presented in the next definition. We also impose some supplementary conditions on the function f in order to obtain a complete generalization of the Graham-Kohr extension operator $\Psi_{n,\alpha}$.

Definition 3.8.3. Let $\alpha, \beta, \gamma \in [0,1]$ be such that $\alpha \neq \beta$ and $\lambda \in (0,1)$. Let $f \in \mathcal{LS}$ be such that $f(z_1) \neq 0$, for all $z_1 \in \mathbb{U} \setminus \{0\}$ and $\left(\frac{f(z_1)}{z_1}\right)^{\gamma}\big|_{z_1=0} = 1$. Moreover, let us consider the assumptions

A) $f'(z_1) \neq \frac{\lambda-1}{\lambda}$, for all $z_1 \in \mathbb{U}$ and $\lambda \in (0,1)$

B) $\left[\frac{f(z_1)}{z_1}\right]^{\gamma} \neq \frac{\lambda-1}{\lambda}$, for all $z_1 \in \mathbb{U}$, $\lambda \in (0,1)$ and $\gamma \in [0,1]$.

Then we define

- the extension operator $\mathcal{K}_{\lambda}^{\beta}$ by

$$\begin{aligned}\mathcal{K}_{\lambda}^{\beta}(f)(z) = \mathcal{K}_{n,\lambda}^{\alpha,\beta}(f, id_{\mathbb{U}})(z) &= (1-\lambda)I_n(z) + \lambda\Psi_{n,\beta}(f)(z)\\ &= \left((1-\lambda)z_1 + \lambda f(z_1), \quad (1-\lambda)\tilde{z} + \lambda\tilde{z}\left[\frac{f(z_1}{z_1}\right]^{\beta}\right), \quad z \in \mathbb{B}^n, \end{aligned}\tag{3.8.2}$$

  where I_n is the identity operator in $\mathbb{C}^n$ and

- the extension operator $\mathcal{K}_{\lambda}^{\alpha,\beta}$ by

$$\begin{aligned}\mathcal{K}_{\lambda}^{\alpha,\beta}(f)(z) = \mathcal{K}_{n,\lambda}^{\alpha,\beta}(f,f)(z) &= (1-\lambda)\Psi_{n,\alpha}(f)(z) + \lambda\Psi_{n,\beta}(f)(z)\\ &= \left(f(z_1), \quad (1-\lambda)\tilde{z}\left[\frac{f(z_1)}{z_1}\right]^{\alpha} + \lambda\tilde{z}\left[\frac{f(z_1)}{z_1}\right]^{\beta}\right), \quad z \in \mathbb{B}^n. \end{aligned}\tag{3.8.3}$$

Note that $\Psi_{n,\alpha}$ and $\Psi_{n,\beta}$ are Graham-Kohr type extension operators defined by (3.7.2).

3.8.2 The preservation of biholomorphy through the operator $\mathcal{K}_\lambda^\beta$

In the second part of this section we study particular properties of the operator $\mathcal{K}_\lambda^\beta(f)$. We observe that some additional conditions on the normalized locally univalent function f can ensures the locally biholomorphy, respectively the univalence of the mapping $\mathcal{K}_\lambda^\beta(f)$ in $\mathbb{C}^n$. The original results presented here have been obtained by the author.

General properties

In view of relation (3.8.2) we deduce that if $f \in \mathcal{LS}$, then $\mathcal{K}_\lambda^\beta(f) \in \mathcal{H}_0(\mathbb{B}^n)$, for all $\lambda \in (0,1)$ and $\beta \in [0,1]$, where

$$D\mathcal{K}_\lambda^\beta(f)(z) = (1-\lambda)I_n + \lambda D\Psi_{n,\beta}(f)(z), \quad z \in \mathbb{B}^n. \tag{3.8.4}$$

Indeed, if $f \in \mathcal{LS}$, then $\Psi_{n,\beta}(f) \in \mathcal{LS}_n(\mathbb{B}^n)$, for all $\beta \in [0,1]$. In addition, under the assumptions considered in Definition 3.8.3, we obtain the locally biholomorphy of the mapping $\mathcal{K}_\lambda^\beta(f)$, as follows:

Lemma 3.8.4. *Let $f \in \mathcal{LS}$ be such that $f(z_1) \neq 0$, for all $z_1 \in \mathbb{U} \setminus \{0\}$ and $\left(\frac{f(z_1)}{z_1}\right)^\beta\Big|_{z_1=0} = 1$, for all $\beta \in [0,1]$. Also, let us consider that f satisfies assumptions A) and B) from Definition 3.8.3. Then $\mathcal{K}_\lambda^\beta(f) \in \mathcal{LS}_n(\mathbb{B}^n)$, for all $\lambda \in (0,1)$ and $\beta \in [0,1]$.*

In view of the previous result, we deduce the following statement:

Proposition 3.8.5. *Let $f \in \mathcal{LS}$ be with $f(z_1) \neq 0$, for all $z_1 \in \mathbb{U}\setminus\{0\}$ and $\left(\frac{f(z_1)}{z_1}\right)^\beta\Big|_{z_1=0} = 1$, for $\beta \in [0,1]$. Also, assume that $\mathfrak{Re} f'(z_1) > 0$, for all $z_1 \in \mathbb{U}$. Then $\mathcal{K}_\lambda^\beta(f) \in \mathcal{LS}_n(\mathbb{B}^n)$, for $\lambda \in (0,1)$ and $\beta \in [0,1]$.*

For some particular choices of β, we can obtain even a better result which ensures the univalence of the mapping $\mathcal{K}_\lambda^\beta(f)$. These particular forms of the operator $\mathcal{K}_\lambda^\beta$ are not trivial, as can be seen in the following result.

Theorem 3.8.6. *Let $\lambda \in (0,1)$ and $f \in S$ be such that $\mathfrak{Re} f'(z_1) > 0$, for all $z_1 \in \mathbb{U}$. Then $\mathcal{K}_\lambda^\beta(f) \in S^0(\mathbb{B}^n)$, for all $\lambda \in (0,1)$ and $\beta \in \{0,1\}$.*

3.8.3 The preservation of starlikeness through the operator $\mathcal{K}_\lambda^\beta$

Based on the result obtained above, we can prove that under similar hypothesis, the operator $\mathcal{K}_\lambda^\beta$ preserve the starlikeness from the unit disc $\mathbb{U}$ to the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. The following results are due to Grigoriciuc.

Theorem 3.8.7. *Let $\lambda \in (0,1)$ and $f \in S^*$ be such that $\mathfrak{Re} f'(z_1) > 0$, for all $z_1 \in \mathbb{U}$. Then $\mathcal{K}_\lambda^1(f) \in S^*(\mathbb{B}^n)$, for all $\lambda \in (0,1)$.*

By applying Theorem 3.8.6 and employing a similar argument as in the previous proof, we can derive the following result concerning the starlikeness of the mapping $\mathcal{K}_\lambda^0(f)$:

Proposition 3.8.8. *Let $\lambda \in (0,1)$ and $f \in S^*$ be such that $\mathfrak{Re} f'(z_1) > 0$, for all $z_1 \in \mathbb{U}$. Then $\mathcal{K}_\lambda^0(f) \in S^*(\mathbb{B}^n)$, for all $\lambda \in (0,1)$.*

Until this point, we proved that if $f \in S^*$ with $\mathfrak{Re} f'(z_1) > 0$, for all $z_1 \in \mathbb{U}$, then $\mathcal{K}_\lambda^\beta(f) \in S^*(\mathbb{B}^n)$, for all $\lambda \in (0,1)$ and $\beta \in \{0,1\}$. Even if these results hold, the case when $\beta \in (0,1)$ remains an open question. Hence, it is natural to consider the following question:

3.8.4 The preservation of local biholomorphy by the operator $\mathcal{K}_\lambda^{\alpha,\beta}$

We end this section by presenting two simple properties of the operator $\mathcal{K}_\lambda^{\alpha,\beta}$ given by (3.8.3). First, we observe that the operators $\mathcal{K}_\lambda^\beta$ and $\mathcal{K}_\lambda^{\alpha,\beta}$ are different for all $\alpha, \beta \in [0,1]$ with $\alpha \neq \beta$ and $\lambda \in (0,1)$, even in the simplest form of the second one. Then, it is not trivial to prove a local biholomorphy criteria for the operator $\mathcal{K}_\lambda^{\alpha,\beta}$. This subsection contains original results obtained by the author.

Theorem 3.8.9. *Let $f \in \mathcal{LS}$ be such that $\mathfrak{Re} f'(z_1) > 0$, for all $z_1 \in \mathbb{U}$ and let $0 \leq \alpha < \beta \leq 1$. Then $\mathcal{K}_\lambda^{\alpha,\beta}(f) \in \mathcal{LS}_n(\mathbb{B}^n)$, for all $\lambda \in (0,1)$ and $0 \leq \alpha < \beta \leq 1$.*

Chapter 4

New subclasses of biholomorphic mappings on $\mathbb{B}^n$

The fourth chapter of this thesis contains extensions of the main results presented in Chapter 2 related to a new differential operator, respectively new subclasses of biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$.

First, we discuss about the n-dimensional form of the operator $\mathcal{G}_k$, denoted here by $\mathcal{G}_{n,k}$, for every $n \in \mathbb{N}$ with $n \geq 2$ and $k \in \mathbb{N}$. The operator $\mathcal{G}_{n,k}$ will be used to extend the subclasses E_k and E_k^* from the unit disc $\mathbb{U}$ to the unit ball B^n in $\mathbb{C}^n$ with respect to an arbitrary norm. Even if these classes can be defined in a very general context, the case of the Euclidean unit ball $\mathbb{B}^n$ will be addressed in particular in our discussion, considering the properties that are preserved (or not) from the one dimensional case to higher dimensions.

The main result that is highlighted in §4.2 shows that the family $E_1^*(\mathbb{B}^n)$ coincides with the class K of convex functions for $n = 1$ (see Theorem 4.2.1; see also Proposition 2.2.4). However, for $n \geq 2$, we obtain that $E_1^*(\mathbb{B}^n) \cap K(\mathbb{B}^n) \neq \emptyset$, but $E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n)$. Note that in the case of the subclass $E_1^*(\mathbb{B}^n)$ we obtain a notable difference between the one dimensional case and the one of several complex variables, i.e. the family of convex mappings is not the same with the subclass $E_1^*(\mathbb{B}^n)$. Another result that is proved in this section (see Theorem 4.2.3) says something about the connection between $E_1(\mathbb{B}^n)$ and the family $K(\mathbb{B}^n; 1/2)$ of convex mappings of order $1/2$. The inclusion $E_1 \subset K(1/2)$ that holds in the one dimensional case can be partially extended in $\mathbb{C}^n$. Other properties and relevant examples are presented in this section in order to describe the new subclasses introduced by the author (e.g. a Marx-Strohhäcker type theorem for our subclasess).

We end this chapter with the study of two particular cases of the Graham-Kohr extension operator $\Psi_{n,\alpha}$ (presented in in §3.7) applied to the family of convex functions K. Although the operator $\Psi_{n,\alpha}$ does not preserve the notion of convexity (see e.g. [44]), we can prove an important property related to the subclass E_1^*. We know that $E_1^* = K$ in $\mathbb{C}$ and thus, in §4.3 we show that $\Psi_{n,\alpha}(K) \subseteq E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n)$ for $\alpha \in \{0, 1\}$. With this result, not only we managed to connect the results proved in Chapters 2 and 4 with the help of the Graham-Kohr extension operator, but we also obtained a new property for this operator. Along with these results, we also propose some questions and open problems related to the Graham-Kohr extension operator and the subclass E_k^* in higher dimensions.

Finally, let us mention that all original outcomes detailed in this chapter were attained by Grigoriciuc in [53]. Other important bibliographic sources used to prepare this chapter are [19], [44], [45], [71], [83], [122].

4.1 Preliminaries

In this section we introduce the n-dimensional version of the differential operator $\mathcal{G}_k$ defined in Chapter 2. We denote this operator in $\mathbb{C}^n$ by $\mathcal{G}_{n,k}$, for every $n \in \mathbb{N}$ with ≥ 2 and $k \in \mathbb{N}$. Using the differential operator $\mathcal{G}_{n,k}$ we can extend also the subclasses E_k^*, respectively E_k from the unit disc $\mathbb{U}$ to the unit ball B^n in $\mathbb{C}^n$ with respect to an arbitrary norm $\|\cdot\|_*$. We give the definitions of the mentioned subclasses in a general setting (on the unit ball B^n in $\mathbb{C}^n$ with respect to an arbitrary norm $\|\cdot\|_*$), but we will focus our attention on the particular case of the Euclidean unit ball $\mathbb{B}^n$. The original results from this section can be found in [53].

Definition 4.1.1. Let $k \in \mathbb{N} = \{0, 1, 2, ...\}$ and let $f \in \mathcal{H}_0(B^n)$ be of the form $f(z) = z + \sum_{m=2}^{\infty} P_m(z)$, where

$$P_m(z) = \frac{1}{m!} D^m f(0)(z^m), \quad z \in B^n, \quad m \geq 2. \tag{4.1.1}$$

Then we define the differential operator $\mathcal{G}_{n,k} : \mathcal{H}_0(B^n) \to \mathcal{H}(B^n)$ by

$$\big(\mathcal{G}_{n,k} f\big)(z) = \begin{cases} D^k f(z)(z^k) + z + \displaystyle\sum_{m=2}^{k-1} P_m(z), & k \geq 3 \\ D^2 f(z)(z^2) + z, & k = 2 \\ Df(z)(z), & k = 1 \\ f(z), & k = 0, \end{cases} \tag{4.1.2}$$

for all $z \in B^n$. It is clear that $\mathcal{G}_{1,k} = \mathcal{G}_k$ is the differential operator defined in Chapter 2 (see Definition 2.1.1) for every $k \in \mathbb{N}$. Moreover, $\mathcal{G}_{n,0} = I_n$ is the identity operator in $\mathbb{C}^n$ and, as in the case of one dimension, also here $\mathcal{G}_{n,k}(I_n) = I_n$, for all $k \in \mathbb{N}$.

Taking into account the definition of the operator $\mathcal{G}_{n,k}$, we can define the n-dimensional version of the subclasses E_k^*, respectively E_k presented in Chapter 2 (see Definitions 2.2.1 and 2.2.14). These subclasses were introduced by Grigoriciuc in [53].

Definition 4.1.2. Let $k \in \mathbb{N}$. Then we denote by

$$E_k^*(B^n) = \big\{f \in S(B^n) \,:\, \mathcal{G}_{n,k} f \in S^*(B^n)\big\} \tag{4.1.3}$$

the subclass of normalized univalent mappings on B^n for which $\mathcal{G}_{n,k} f$ is starlike on B^n, respectively by

$$E_k(B^n) = \big\{f \in S(B^n) \,:\, \mathcal{G}_{n,k} f \in K(B^n)\big\}. \tag{4.1.4}$$

the subclass of normalized univalent mappings on B^n for which $\mathcal{G}_{n,k} f$ is convex on B^n.

Remark 4.1.3. According to the previous definition, it is clear that

1. if $k = 0$, then $E_0^*(B^n) = S^*(B^n)$ and $E_0(B^n) = K(B^n)$;
2. if $k = 1$, then $E_1^*(B^n) = \big\{f \in S(B^n) \,:\, \mathcal{G}_{n,1} f \in S^*(B^n)\big\}$ and $E_1(B^n) = \big\{f \in S(B^n) \,:\, \mathcal{G}_{n,1} f \in K(B^n)\big\}$, where $\mathcal{G}_{n,1} f(z) = Df(z)(z)$, for $z \in B^n$.

An important remark on the previous subclasses of biholomorphic mappings in $\mathbb{C}^n$ is based on the fact that Alexander's duality theorem is no longer true in higher dimensions (see Remark 3.4.9; see e.g. [45], [83]). According to this, we obtain the following remarks:

Remark 4.1.4. If $n \geq 2$, then

$$K(B_1^n) \subsetneq E_1^*(B_1^n) \quad \text{and} \quad K(\mathbb{U}^n) \subsetneq E_1^*(\mathbb{U}^n), \tag{4.1.5}$$

where B_1^n is the unit ball in $\mathbb{C}^n$ with respect to the 1-norm (recall that the 1-norm is given by $\|z\|_1 = \sum_{j=1}^n |z_j|$, for all $z = (z_1, ..., z_n) \in \mathbb{C}^n$), respectively $\mathbb{U}^n$ is the unit polydisc in $\mathbb{C}^n$ (for details, one may consult §3.1.1).

Remark 4.1.5. In contrast to Remark 4.1.4, we can prove (see Theorem 4.2.1 presented in the next section) that

$$E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n), \quad n \geq 2, \tag{4.1.6}$$

where $\mathbb{B}^n$ is the Euclidean unit ball in $\mathbb{C}^n$. Hence, it is not trivial to define the subclasses E_k^* and E_k of univalent mappings in the case of several complex variables. Even the simplest case $k = 1$ is important for the Euclidean unit ball $\mathbb{B}^n$ in view of the difference provided by relation (4.1.6).

Next we present an example of mapping which belong to the class $E_1^*(B_p^n)$ for the general case of the unit ball B_p^n in $\mathbb{C}^n$ with respect to a p-norm (recall that B_p^n is defined in §3.1.1). This example was also considered in [45], [83], [71], [122]. We present here this example in order to show that the family $E_1^*(B^n)$ is nonempty.

Example 4.1.6. Let $f : B_p^2 \subset \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $f(z) = \big(z_1 + az_2^2, z_2\big)$, for all $z = (z_1, z_2) \in B_p^2$. Then $f \in E_1^*(B_p^2)$ if and only if $|a| \leq \frac{1}{2}\big(\frac{p^2-1}{4}\big)^{1/p}\big(\frac{p+1}{p-1}\big)$, for all $p > 1$.

4.2 General properties of the subclasses $E_k^*(\mathbb{B}^n)$ and $E_k(\mathbb{B}^n)$

The second section is dedicate to study of some general properties of the subclasses defined above. We will highlight the connection between subclasses E_1^* (respectively E_1) on $\mathbb{U}$ and the class of convex mappings $K(B_p^n)$ in $\mathbb{C}^n$. It is important to mention that results from the case $n = 1$ are not longer true in $\mathbb{C}^n$, for $n \geq 2$. The following results are novel and were achieved in [53].

Theorem 4.2.1. *Regarding to the class E_1^*, the following statements are true:*

1. *If $n = 1$, then $E_1^*(\mathbb{U}) = K(\mathbb{U}) = K$.*

2. *If $n \geq 2$, then $E_1^*(\mathbb{B}^n) \cap K(\mathbb{B}^n) \neq \emptyset$ and $E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n)$.*

Remark 4.2.2. According to Theorem 4.2.1, it is clear that if $n = 1$, then $K(\mathbb{U}) = E_1^*(\mathbb{U})$. However, if $n \geq 2$, then $K(B_1^n) \subsetneq E_1^*(B_1^n)$ and $K(\mathbb{U}^n) \subsetneq E_1^*(\mathbb{U}^n)$ in view of Remark 4.1.4 and $K(\mathbb{B}^n) \neq E_1^*(\mathbb{B}^n)$ in view of Theorem 4.2.1.

Theorem 4.2.3. *Regarding to the class E_1, the following statements are true:*

1. *If $n = 1$, then $E_1(\mathbb{U}) \subsetneq K(1/2)$.*

2. *If $n \geq 2$, then $E_1(\mathbb{B}^n) \cap K(\mathbb{B}^n; 1/2) \neq \emptyset$ and $K(\mathbb{B}^n; 1/2) \setminus E_1(\mathbb{B}^n) \neq \emptyset$, i.e. there exist also convex mappings of order $1/2$ on $\mathbb{B}^n$ which does not belong to class $E_1(\mathbb{B}^n)$.*

Definition 4.2.4. Let $k \in \mathbb{N}$ and $\alpha \in [0, 1)$. In view of Definition 4.1.2, we denote by

$$E_k^*(B^n; \alpha) = \Big\{f \in S(B^n) \;:\; \mathcal{G}_{n,k} f \in S_\alpha^*(B^n)\Big\}, \tag{4.2.1}$$

where $S_\alpha^*(B^n)$ is the family of starlike mappings of order α in $\mathbb{C}^n$ (see Definition 3.4.3; see also [13], [81]) and B^n is the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm $\|\cdot\|_*$. Clearly, for $\alpha = 0$, we have that $S_0^*(B^n) = S^*(B^n)$ and then $E_k^*(B^n; 0) = E_k^*(B^n)$.

Using the previous definition, we can obtain another form of the Marx-Strohhäcker theorem for the classes E_k and E_k^* (see Theorem 3.4.8 for the case of several complex variables; see also [13], [45]). This result was proved by the author in [53].

Theorem 4.2.5. *Let $k \in \mathbb{N}$. Then $E_k(B^n) \subseteq E_k^*(B^n; 1/2) \subseteq E_k^*(B^n)$, where B^n is the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm $\|\cdot\|_*$.*

We end this section with some consequences of the previous result related to the inclusions between subclasses studied in this part. In particular, we obtain some well-known results in one and higher dimensions (see e.g. [13], [29], [45], [83], [102]).

Corollary 4.2.6. *Let us consider $k \in \mathbb{N}$.*

1. *If $n = 1$, then $E_k(\mathbb{U}) \subseteq E_k^*(\mathbb{U}; 1/2) \subseteq E_k^*(\mathbb{U})$. In particular, for $k \in \{0, 1\}$ we obtain that*

$$K = E_0(\mathbb{U}) \subseteq E_0^*(\mathbb{U}; 1/2) = S^*(1/2) \subseteq S^* = E_0^*(\mathbb{U}) \tag{4.2.2}$$

and

$$E_1(\mathbb{U}) \subseteq E_1^*(\mathbb{U}; 1/2) = K(1/2) \subseteq K = E_1^*(\mathbb{U}). \tag{4.2.3}$$

2. *On the other hand, if $n \geq 2$ and $k \in \{0, 1\}$, then*

$$E_0(B_p^n) = K(B_p^n) \subseteq E_0^*(B_p^n; 1/2) = S^*(B_p^n; 1/2) \subseteq S^*(B_p^n) \tag{4.2.4}$$

and

$$E_1(B_p^n) \subseteq E_1^*(B_p^n; 1/2), \quad 1 \leq p < \infty, \tag{4.2.5}$$

where B_p^n is the unit ball in $\mathbb{C}^n$ with respect to the p-norm and

$$E_1(\mathbb{U}^n) \subseteq E_1^*(\mathbb{U}^n; 1/2), \tag{4.2.6}$$

where $\mathbb{U}^n$ is the unit polydisc in $\mathbb{C}^n$.

4.3 Geometric properties preserved by the Graham-Kohr extension operator

In the third part of this section we consider two particular cases of the Graham-Kohr extension operator $\Psi_{n,\alpha}$ for $\alpha \in \{0, 1\}$. Although the operator $\Psi_{n,\alpha}$ does not preserve the notion of convexity (see e.g. [44]), we can still observe an important property related to the subclass E_1^* (see Definition 2.2.1) in the particular cases mentioned above, i.e. the subclass E_1^* is preserved by the Graham-Kohr extension operator $\Psi_{n,\alpha}$. The original results discussed in this section were derived by Grigoriciuc in [53].

Proposition 4.3.1. *If f belongs to the family K, then $\Psi_{n,0}(f) \in E_1^*(\mathbb{B}^n)$. Hence,*

$$\Psi_{n,0}(K) = \Psi_{n,0}\big(E_1^*(\mathbb{U})\big) \subseteq E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n). \tag{4.3.1}$$

The following property was observed and proved by Grigoriciuc in [53].

Lemma 4.3.2. *Let $\alpha = 0$ and $k \in \{0, 1, 2\}$. Then $\Psi_{n,0}(\mathcal{G}_{n,k} f) = \mathcal{G}_{n,k}(\Psi_{n,0}(f))$.*

As a direct consequence of Lemma 6.1.1 we have that

Corollary 4.3.3. *Let $\alpha = 0$ and $k \in \{0, 1, 2\}$. Then $\Psi_{n,0}\big(E_k^*(\mathbb{U})\big) \subseteq E_k^*(\mathbb{B}^n)$.*

The second important result in this section is related to the operator $\Psi_{n,1}$ in the particular case $\alpha = 1$.

Theorem 4.3.4. *If $f \in K$, then $\Psi_{n,1} \in E_1^*(\mathbb{B}^n)$. Hence, $\Psi_{n,1}(K) = \Psi_{n,1}\big(E_1^*(\mathbb{U})\big) \subseteq E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n)$.*

Lemma 4.3.5. *Let $\alpha = 1$ and $k \in \{0, 1, 2\}$. Then $\Psi_{n,1}(\mathcal{G}_{n,k} f) = \mathcal{G}_{n,k}(\Psi_{n,1}(f))$.*

Based on Lemma 6.1.2, we obtain the following direct consequence:

Corollary 4.3.6. *Let $\alpha = 1$ and $k \in \{0, 1, 2\}$. Then $\Psi_{n,1}\big(E_k^*(\mathbb{U})\big) \subseteq E_k^*(\mathbb{B}^n)$.*

Recall that Graham and Kohr proved in [44] that the extension operator $\Psi_{n,\alpha}$ does not preserve convexity for $n \geq 2$, for all $\alpha \in [0, 1]$. However, in this section we have proved that $\Psi_{n,\alpha}\big(E_k^*(\mathbb{U})\big) \subseteq E_k^*(\mathbb{B}^n)$, for $\alpha \in \{0, 1\}$ and $k \in \{1, 2\}$. In particular, this leads to $\Psi_{n,\alpha}(K) \subseteq E_1^*(\mathbb{B}^n)$, for $\alpha \in \{0, 1\}$. However, these are just some particular cases of the extension operator, respectively of the class E_k^* so one might reasonably ask the following question:

Question 4.3.7. *Let $\alpha \in [0, 1]$ and $k \in \mathbb{N}$*

- *Is it true that $\Psi_{n,\alpha}\big(E_k^*(\mathbb{U})\big) \subseteq E_k^*(\mathbb{B}^n)$?*
- *In particular, is it true that $\Psi_{n,\alpha}(K) \subseteq E_1^*(\mathbb{B}^n)$, for all $\alpha \in [0, 1]$?*

On the other hand, Roper and Suffridge stated in [121] that their extension operator preserves convexity from $\mathbb{U}$ to the unit ball of a complex Hilbert space. Graham, Hamada, Kohr and Kohr proved in [36] that the extension operator $\Psi_{\alpha,\beta}$ preserves the notion of starlikeness from the unit disc $\mathbb{U}$ to the unit ball B_H of a complex Hilbert space H. In view of these results, we consider the following question:

Question 4.3.8. *Let $\alpha \in [0, 1]$ and $k \in \mathbb{N}$. Let also B_H be the unit ball of a complex Hilbert space H.*

- *Is it true that $\Psi_{n,\alpha}\big(E_k^*(\mathbb{U})\big) \subseteq E_k^*(B_H)$?*
- *In particular, is it true that $\Psi_{n,\alpha}(K) \subseteq E_1^*(B_H)$?*

Part III

Contributions in the theory of biholomorphic mappings in complex Banach spaces

Chapter 5

Biholomorphic mappings and Loewner theory in complex Banach spaces

The fifth chapter of this thesis is dedicated to a short study on biholomorphic mappings and Extension operators in complex Banach spaces. We include here extensions of most of the results presented in previous chapters. Among those who have made important contributions in the geometric function theory of complex variables in the infinite dimensional case are J. Mujica, T. Poreda, T.J. Suffridge (see e.g. [106], [117], [118], [127]) and more recently F. Bracci, I. Graham, H. Hamada, G. Kohr and M. Kohr (see e.g. [3], [34], [38], [39], [40], [41]). We start our discussion from the very recent paper published by Graham, Hamada, Kohr and Kohr regarding biholomorphic mappings, Loewner chains and Extention operators in complex Banach spaces (see e.g. [39], [40], [41]). These papers constitute the basis of our study, containing some of the fundamental ideas in obtaining all the other results in this chapter.

First section of this chapter contains basic results and properties of holomorphic functions and holomorphic mappings in infinite dimensions. We present the main notions and results that will be used during this chapter (e.g. the maximum modulus theorem, the Schwarz's lemma). For more details, one may consult [45], [78], [79], [106], [127], [128]. Moreover, we recall here the generalization of the Carathéodory family and the growth results obtained by Gurganus (see [57]), respectively by Bracci, Elin, Shoikhet (see [6]) and Graham, Hamada, Honda, Kohr and Shon (see [31]) in infinite dimensional case.

The next section is dedicated to some particular families of biholomorphic mappings in complex Banach spaces. We present here the classes of starlike, convex, respectively ε-starlike mappings together with their analytical characterization. Important contributions were made by Suffridge (see [127]), Gurganus (see [57]), Hamada and Kohr (see e.g. [68], [74]), Gong and Liu (see [25], [26]).

In §5.3 we focus our attention on general results related to the theory of Loewner chains in complex Banach spaces that will be used in our main results. The study of subordination chains in infinite dimensional spaces was started by Poreda (see e.g. [117], [118]). These ideas were continued and improved by Graham, Hamada, Kohr and Kohr (see e.g. [34], [38], [39], [40], [41]), Hamada and Kohr (see e.g. [70], [72]), Arosio, Bracci, Hamada and Kohr (see e.g. [2], [3]) who obtained important results related to Loewner chains and Loewner PDE in infinite dimensional spaces. The latter part of this section presents results pertaining to the concept of parametric representation in infinite dimensions. This concept was introduced by Graham, Hamada, Kohr and Kohr (see [38]) and represents the generalization of the parametric representation presented in Definition 3.5.11. Also, we discuss in this section about the concept of g-parametric representation, g-Loewner chain and particular families of biholomorphic mappings associated to g-Loewner chains. For details, one may consult [32], [34], [45], [60], [61], [62], [74].

As we have already mentioned, the main bibliographic sources are [3], [34], [38], [39], [40], [41], [106], [117], [118], [127], [131], [132], [133], [134].

5.1 General notions regarding holomorphy in complex Banach spaces

This section is devoted to the study of basic properties of holomorphic functions and holomorphic mappings in infinite dimensions. We present the main notions and results that will be used during this chapter. For more details, one may consult [45], [78], [79], [106], [127], [128].

5.1.1 Holomorphic mappings in complex Banach spaces

Let X be a complex Banach space with respect to the norm $\|\cdot\|$. We denote by

$$\mathbb{B}_X(x_0, r) = \big\{x \in X \,:\, \|x - x_0\| < r\big\}$$

the open ball of center $x_0 \in X$ and radius $r > 0$. In particular, we denote simply by $\mathbb{B}_X = \mathbb{B}_X(0,1)$ *the open unit ball* of X.

We denote by $L(X,Y)$ the set of continuous linear operators from X into another complex Banach space Y with the standard operator norm

$$\|A\| = \sup\big\{\|A(x)\|_Y \,:\, \|x\|_X = 1\big\},$$

for every $A \in L(X,Y)$. When it is clear the norm whose space we are considering, we will omit the inferior indices of the norm. In particular, we denote $L(X,X)$ by $L(X)$ and the identity operator in $L(X)$ by I_X (see e.g. [45], [79], [106], [127], [128]).

Definition 5.1.1. Let $\Omega \subseteq X$ be a domain. The mapping $f : \Omega \to Y$ is called *holomorphic* on Ω if for each $x \in \Omega$ there exists a mapping $Df(x) \in L(X,Y)$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Df(x)(h)\|}{\|h\|} = 0.$$

We denote by $\mathcal{H}(\Omega, Y)$ the set of holomorphic mappings on Ω into Y. If $Y = X$, then we denote $\mathcal{H}(\Omega, Y)$ simply by $\mathcal{H}(\Omega)$.

Remark 5.1.2. Let X be a complex Banach space and let $\Omega \subseteq X$ be a domain such that $0 \in \Omega$. We say that $f \in \mathcal{H}(\Omega)$ is *normalized* if $f(0) = 0$ and $Df(0) = I_X$, where $Df(x)$ is the Fréchet derivative of f at x. We denote the set of normalized holomorphic mappings on Ω by $\mathcal{H}_0(\Omega)$.

The next result is an extension of the Schwarz's lemma (see Lemma 1.1.4 for $n = 1$; see Lemma 3.1.3 for $n \geq 2$) in infinite dimensions (see e.g. [78]).

Lemma 5.1.3. *Let $M > 0$ and let $f \in \mathcal{H}(\mathbb{B}_X)$ be such that $f(0) = 0$ and $\|f(x)\| < M$, for all $x \in \mathbb{B}_X$. Then $\|f(x)\| \leq M\|x\|$, for all $x \in \mathbb{B}_X$. Moreover, if $\exists x_0 \in \mathbb{B}_X \setminus \{0\}$ with $\|f(x_0)\| = M\|x_0\|$, then $\|f(ax_0)\| = M\|ax_0\|$, for all $a \in \mathbb{C}$ with $|a| \leq \frac{1}{\|x_0\|}$.*

5.1.2 Generalizations of the Carathéodory family

In this subsection we present the generalization of the Carathéodory family in complex Banach spaces (see e.g. [45], [57], [74], [106], [127], [128]).

Let X be a complex Banach space and for $x \in X \setminus \{0\}$, we denote by

$$T(x) = \big\{l_x \in L(X, \mathbb{C}) \,:\, l_x(x) = \|x\|_X,\, \|l_x\| = 1\big\}.$$

In view of the Hahn-Banach theorem we know that $T(x) \neq \emptyset$. Note that, if $X = \mathbb{C}^n$ is endowed with a p-norm $\|\cdot\|_p$, $p \geq 1$ (recall that in §3.1.1 the p-norm is defined by $\|x\|_p = \big(\sum_{j=1}^n |x_j|^p\big)^{\frac{1}{p}}$, for all $x \in \mathbb{C}^n$; see e.g. [45], [106]) and $l_x \in L(X, \mathbb{C})$ is given by $l_x(y) = \frac{1}{\|x\|_p^{p-1}} \sum_{j \geq 1, x_j \neq 0} |x_j|^p \frac{y_j}{x_j}$, for all $x, y \in X$ with $x \neq 0$, then $l_x \in T(x)$ (for details, one may consult [45], [106]). It is well-known that this set plays an important role in the study of biholomorphic mappings in complex Banach spaces.

Let $\mathbb{B}_X$ be the open unit ball of X. Then

$$\mathcal{M}(\mathbb{B}_X) = \Big\{h \in \mathcal{H}_0(\mathbb{B}_X) \,:\, \mathfrak{Re}\, l_x\big(h(x)\big) > 0,\, x \in \mathbb{B}_X \setminus \{0\},\, l_x \in T(x)\Big\} \tag{5.1.1}$$

is the Carathéodory family in $\mathcal{H}(\mathbb{B}_X)$. If $X = \mathbb{C}$, it is easy to observe that $f \in \mathcal{M}(\mathbb{U})$ if and only if $\frac{f(x)}{x} \in \mathcal{P}$, where $\mathcal{P}$ is the Carathéodory family on $\mathbb{U}$ defined in Chapter 1.

One of the results that can be extended from the n-dimensional case to the complex Banach spaces is *the growth theorem* for the Carathéodory class (see Theorem 3.2.2). This extension was obtained by Gurganus (see [57]).

Proposition 5.1.4. *Let X be a complex Banach space and let $h \in \mathcal{M}(\mathbb{B}_X)$. Then*

$$\|x\|\frac{1-\|x\|}{1+\|x\|} \leq \mathfrak{Re} l_x\big(h(x)\big) \leq \|x\|\frac{1+\|x\|}{1-\|x\|}, \tag{5.1.2}$$

for all $x \in \mathbb{B}_X \setminus \{0\}$ and $l_x \in T(x)$.

For details and other important results related to the Carathéodory family in higher dimensions, one may consult [32], [36], [38], [45], [118], [128].

5.2 Families of biholomorphic mappings in complex Banach spaces

Let X, Y be two complex Banach spaces and let $\Omega \subseteq X$ be a domain. We say that $f \in \mathcal{H}(\Omega, Y)$ is

- *locally biholomorphic* on Ω if $\forall x \in \Omega$, $\exists r_1, r_2 > 0$ such that f is one-to-one map of $\mathbb{B}_X(x, r_1)$ into $\mathbb{B}_Y(f(x), r_2)$ whose inverse is holomorphic on $\mathbb{B}_Y(f(x), r_2)$;
- *biholomorphic* on Ω if $f(\Omega) \subseteq Y$ is a domain and $\exists f^{-1} \in \mathcal{H}(f(\Omega))$.

As in the finite dimensional case, we denote by

- $\mathcal{LS}(\mathbb{B}_X)$ the family of normalized locally biholomorphic mappings from $\mathbb{B}_X$ into X;
- $S(\mathbb{B}_X)$ the family of normalized biholomorphic mappings from $\mathbb{B}_X$ into X.

In particular, when $X = \mathbb{C}$, we have that $\mathbb{B}_X = \mathbb{U}$ and then $\mathcal{LS}(\mathbb{U}) = \mathcal{LS}$, respectively $S(\mathbb{U}) = S$ as in Chapter 1. For details, one may consult [45], [106], [127], [128].

Remark 5.2.1. It is important to mention here that $f \in \mathcal{LS}(\mathbb{B}_X)$ if and only if the Frechét derivative $Df(x)$ has a bounded inverse at each $x \in \mathbb{B}_X$. If $X = \mathbb{C}^n$, the previous condition reduces to the property that $J_f(z) \neq 0$, for every $z \in \mathbb{B}^n$ (in particular, for $n = 1$, we obtain $f'(\zeta) \neq 0$, for every $\zeta \in \mathbb{U}$).

Remark 5.2.2. Another important remark is that in the case of complex Banach spaces, the notions of univalence and biholomorphy are not equivalent, i.e. there exist univalent mappings which are not biholomoprhic (see e.g. [107], [119], [128]). This result is in contrast with the finite dimensional case (see Definition 3.3.1). For more details and examples, one may consult also [128].

5.2.1 Starlike mappings

In the following, let X, Y be two complex Banach spaces and let $\Omega \subseteq X$ be a domain.

Definition 5.2.3. Let $f : \Omega \to Y$ be a mapping and let $x_0 \in \Omega$. Then f is *starlike with respect to* x_0 on Ω if f is biholomorphic on Ω and $f(\Omega)$ is a starlike domain with respect to $f(x_0)$.

We denote by $S^*(\mathbb{B}_X)$ the family of normalized starlike (with respect to zero) mappings from the open unit ball $\mathbb{B}_X$ into X. The analytical characterization of this family was obtained by Suffridge (see [127]). A simplified form of this result was proved by Gurganus (see [57]); the rectified final form of the characterization is due to Hamada and Kohr (see e.g. [68]).

Theorem 5.2.4. *Let $f : \mathbb{B}_X \to Y$ be a locally biholomorphic mapping such that $f(0) = 0$. Then $f \in S^*(\mathbb{B}_X)$ if and only if there is $h \in \mathcal{M}(\mathbb{B}_X)$ such that*

$$f(x) = Df(x)h(x), \quad x \in \mathbb{B}_X. \tag{5.2.1}$$

For more details and examples of starlike mappings in infinite dimensions, one may consult [45], [75], [127], [128]. Note that the notions of spirallikeness of type $\delta \in (-\pi/2, \pi/2)$, respectively almost starlikeness of order $\alpha \in [0, 1)$ can be extended in the case of complex Banach spaces (see e.g. [45], [74], [127], [128], [133]).

5.2.2 Convex mappings

Let X, Y be two complex Banach spaces and let $\Omega \subseteq X$ be a domain.

Definition 5.2.5. Let $f \in \mathcal{H}(\Omega, Y)$ be a mapping. Then f is *convex on* Ω if f is biholomoprhic on Ω and $f(\Omega)$ is a convex domain in Y.

We denote by $K(\mathbb{B}_X)$ the family of normalized convex mapping from the open unit ball $\mathbb{B}_X$ into X. Next, we present a necessary condition for convexity obtained by Suffridge, respectively by Roper and Suffridge (see [122], [127]). Note that this condition is necessary, but not sufficient for convexity.

Theorem 5.2.6. *If $f : \mathbb{B}_X \to Y$ is convex, then*

$$D^2 f(x)(x, x) + Df(x)x = Df(x)h(x), \tag{5.2.2}$$

for all $x \in \mathbb{B}_X$, where $h \in \mathcal{M}(\mathbb{B}_X)$.

Other important results related to convex mappings and characterization theorems in infinite dimensions can be found in [45], [127], [128].

5.2.3 ε-starlike mappings

An important notion that links the classes presented above is the ε-starlikeness introduced by Gong and Liu (see [25]). We present here the definition, respectively the analytical characterization of ε-starlikeness (see [25]).

Definition 5.2.7. Let $0 \le \varepsilon \le 1$ and let $f : \mathbb{B}_X \to Y$ be a biholomorphic mapping such that $f(0) = 0$. Then f is *ε-starlike on* $\mathbb{B}_X$ if $f(\mathbb{B}_X)$ is starlike with respect to every point in $\varepsilon f(\mathbb{B}_X)$, i.e.

$$(1 - t)f(x) + t\varepsilon f(y) \in f(\mathbb{B}_X), \quad t \in [0, 1], \quad x, y \in \mathbb{B}_X.$$

Remark 5.2.8. It is easy to observe that for $\varepsilon = 0$ we obtain the family of starlike mappings on $\mathbb{B}_X$ and for $\varepsilon = 1$ we obtain the family of convex mappings on $\mathbb{B}_X$.

5.3 The theory of Loewner chains in complex Banach spaces

In this section we present some general results related to the theory of Loewner chains in complex Banach spaces that will be used in our main results. As will be mentioned in the following, the study of subordination chains in infinite dimensional spaces was started by Poreda (see e.g. [117], [118]). These ideas were continued and improved by Graham, Hamada, Kohr and Kohr (see e.g. [34], [38], [39], [40], [41]), Hamada and Kohr (see e.g. [70], [72]), Arosio, Bracci, Hamada and Kohr (see e.g. [2], [3]) who obtained important results related to Loewner chains and Loewner PDE in infinite dimensional spaces. The second part of this section contains results related to the concept of parametric representation in infinite dimensions. This notion is due to Graham, Hamada, Kohr and Kohr (see [38]) and represents the generalization of the parametric representation presented in Definition 3.5.11. Also, we discuss in this part about the concept of g-parametric representation, g-Loewner chain and particular families of biholomorphic mappings associated to g-Loewner chains. For details, one may consult also [32], [34], [45], [60], [61], [62], [74].

5.3.1 Loewner chains and biholomorphic mappings

We start this section with some introductory notions and results related to theory of Loewner chains in infinite dimensions. Poreda was the first who studied subordination chains and the Loewner PDE on the unit ball of a complex Banach space (see e.g. [117], [118]). Later, various important results were obtained by Arosio, Bracci, Hamada and Kohr (see e.g. [2], [3]), Graham, Hamada, Kohr and Kohr (see e.g. [34], [38], [39], [70]). It is important to mention that very recent results were obtained by Graham, Hamada, Kohr and Kohr in [40], [41], [72].

In the following, let X be a complex Banach space with respect to the norm $\|\cdot\|$ and let $\mathbb{B}_X$ be the unit open ball of X.

Definition 5.3.1. Let $f, g, \phi \in \mathcal{H}(\mathbb{B}_X)$. Then

1. ϕ is a *Schwarz mapping* if $\|\phi(x)\| \leq \|x\|$, for all $x \in \mathbb{B}_X$;
2. f *is subordinate to* g and write $f \prec g$ if there exists a Schwarz mapping ϕ such that $f(x) = g(\phi(x))$, for all $x \in \mathbb{B}_X$.

Definition 5.3.2. A mapping $f : \mathbb{B}_X \times [0, \infty) \to X$ is called

- a *univalent subordination chain* if $f(\cdot, t)$ is univalent on $\mathbb{B}_X$, $f(0, t) = 0$ for $t \geq 0$ and $f(\cdot, s) \prec f(\cdot, t)$ for all $0 \leq s \leq t < \infty$;
- in addition, if $f(\cdot, t)$ is biholomorphic on $\mathbb{B}_X$ and $Df(0, t) = e^t I_X$ for all $t \geq 0$, then f is called a *Loewner chain.*

Recall that the previous subordination condition corresponds to the existence of a unique biholomorphic Schwarz mapping $v = v(\cdot, s, t)$ with

$$f(x, s) = f(v(x, s, t), t), \quad x \in \mathbb{B}_X, \quad 0 \leq s \leq t < \infty.$$

The mapping $v = v(\cdot, s, t)$ is called the *transition mapping* associated with $f(x, t)$. The transition mapping v satisfies also the semigroup property (see e.g. [34])

$$v(x, s, t) = v\big(v(x, s, u), u, t\big), \quad x \in \mathbb{B}_X, \quad 0 \leq s \leq u \leq t < \infty.$$

Definition 5.3.3. A mapping $h = h(x, t) : \mathbb{B}_X \times [0, \infty) \to X$ is said to be a *generating vector field* (or *Herglotz vector field*) if:

a) $h(\cdot, t)$ belongs to the class $\mathcal{M}(\mathbb{B}_X)$, for all $t \geq 0$;

b) $h(x, \cdot)$ belongs to the class of strongly measurable functions on $[0, \infty)$, for all $x \in \mathbb{B}_X$.

According to previous definitions, we have the following existence and uniqueness result obtained in [69] (see also [31], [109]). This result is the analogous of the results presented in the one dimensional case (see Theorem 1.6.4), respectively in finite higher dimensions (see Theorems 3.5.4 and 3.5.6).

Lemma 5.3.4. *Let X be a reflexive complex Banach space and let $h = h(x, t) : \mathbb{B}_X \times [0, \infty) \to X$ be a Herglotz vector field. Then for every $s \geq 0$ and $x \in \mathbb{B}_X$, the Cauchy problem*

$$\begin{cases} \dfrac{\partial v}{\partial t} = -h(v, t), \quad a.e. \quad t \geq s \\ v(x, s, s) = x \end{cases} \tag{5.3.1}$$

has a unique solution $v = v(x, s, t)$ such that

- *$v(\cdot, s, t)$ belongs to the family of univalent Schwarz mappings;*
- *$v(x, s, \cdot)$ belongs to the class of Lipschitz continuous functions on $[s, \infty)$ uniformly with respect to $x \in \overline{\mathbb{B}}_X(0, \rho)$, where $\rho \in (0, 1)$;*

- *$Dv(0,s,t) = e^{s-t}I_X$, for $0 \le s \le t$.*

In addition, for all $\rho \in (0,1)$ and $s \ge 0$, there exists the limit $\lim_{t\to\infty} e^t v(x,s,t) = f(x,s)$ uniformly on $\overline{\mathbb{B}}_X(0,\rho)$. Then $f(x,t)$ is a Loewner chain and for each $\rho \in (0,1)$, there exists $M_\rho \le \frac{\rho}{(1-\rho)^2}$ such that

$$\|e^{-t}f(x,t)\| \le M_\rho, \quad \|x\| \le \rho, \quad t \ge 0. \tag{5.3.2}$$

In view of the results obtained by Poreda (see [118]), we know that if $h(x,t)$ is continuous on $\mathbb{B}_X \times [0,\infty)$, then the conclusion of the previous lemma is true in the case of complex Banach spaces, not necessarily reflexive (see e.g. [34]). Other important results related to the theory of Loewner chains in complex Banach spaces can be found in the recent papers of Graham, Hamada, Kohr and Kohr (see e.g. [34], [39], [40], [41], [72]).

5.3.2 Parametric and g-parametric representation

Next, we present the notion of parametric representation in infinite dimensions. This notion is due to Graham, Hamada, Kohr and Kohr (see [38]) and represents the generalization of the idea presented in Definition 3.5.11. Also, we discuss in this part about the g-parametric representation and g-Loewner chains. For details, one may consult [34], [45], [61], [74].

Parametric representation

Let X be a reflexive complex Banach space. Recall that we denote by $\mathcal{H}_0(\mathbb{B}_X)$ the set of normalized holomorphic mappings on $\mathbb{B}_X$.

Definition 5.3.5. A mapping $f \in \mathcal{H}_0(\mathbb{B}_X)$ has *parametric representation* if there is a Herglotz vector field $h(x,t) : \mathbb{B}_X \times [0,\infty) \to X$ such that $f(x) = \lim_{t\to\infty} e^t v(x,t)$ uniformly on $\overline{\mathbb{B}}_X(0,\rho)$, for every $\rho \in (0,1)$, where $v(x,t)$ is the unique Lipschitz continuous solution of the problem (5.3.1) on $[0,\infty)$ for $s = 0$. We denote by $S^0(\mathbb{B}_X)$ the family of mappings which have parametric representation on $\mathbb{B}_X$.

g-parametric representation

In order to present the definition of g-parametric representation, let us consider the following assumption that will play a key role in our study (see e.g. [34], [45]):

Assumption 5.3.6. Let $g : \mathbb{U} \to \mathbb{C}$ be a holomorphic univalent function with $g(0) = 1$ and $\mathfrak{Re}\, g(\zeta) > 0$, for all $\zeta \in \mathbb{U}$.

An important step is to extend the class $\mathcal{M}_g$ in infinite dimensions (see Definition 3.5.16 for $X = \mathbb{C}^n$). For details, one may consult [34], [45], [61], [74].

Definition 5.3.7. Let $g : \mathbb{U} \to \mathbb{C}$ satisfy Assumption 5.3.6 and let $h \in \mathcal{H}_0(\mathbb{B}_X)$. We say that $h \in \mathcal{M}_g(\mathbb{B}_X)$ if $\frac{1}{\|x\|} l_x\big(h(x)\big) \in g(\mathbb{U})$, for all $x \in \mathbb{B}_X \setminus \{0\}$ and $l_x \in T(x)$.

In view of the previous results, we can present the notion of g-parametric representation, respectively g-Loewner chain in infinite dimensions (see e.g. [32], [34], [40]).

Definition 5.3.8. Let X be a reflexive complex Banach space and let $f \in S^0(\mathbb{B}_X)$. We say that f has *g-parametric representation* if $h(\cdot,t) \in \mathcal{M}_g(\mathbb{B}_X)$ and we denote by $S^0_g(\mathbb{B}_X)$ the family of mappings that have g-parametric representation on $\mathbb{B}_X$.

Definition 5.3.9. Let $g : \mathbb{U} \to \mathbb{C}$ satisfy Assumption 5.3.6 and let $f = f(x,t) : \mathbb{B}_X \times [0,\infty) \to X$ be a mapping. We say that $f(x,t)$ is a *g-Loewner chain* if

a) $f(x,t)$ is a Loewner chain such that the family $\left\{e^{-t}f(\cdot,t)\right\}_{t\ge 0}$ is uniformly bounded on $\rho\mathbb{B}_X$, for $\rho \in (0,1)$;

b) $\exists A \subseteq [0,\infty)$ null set such that $\exists \frac{\partial f}{\partial t}(x,t)$ for $t \in [0,\infty) \setminus A$ and for all $x \in \mathbb{B}_X$, and $\exists h = h(x,t) : \mathbb{B}_X \times [0,\infty) \to X$ a generating vector field with the property that $h(\cdot,t)$ belongs to the class $\mathcal{M}_g(\mathbb{B}_X)$ for $t \in [0,\infty) \setminus A$ and

$$\frac{\partial f}{\partial t}(x,t) = Df(x,t)h(x,t), \quad t \in [0,\infty) \setminus A, \quad \forall\, x \in \mathbb{B}_X. \tag{5.3.3}$$

Remark 5.3.10. It is important to highlight here that in [39] (see also [40]) the authors mention that in general, if X is a complex Banach space and if $f(x,t)$ satisfies the first condition from Definition 5.3.9, then it is not known whether $\exists \frac{\partial f}{\partial t}(x,t)$ for $x \in \mathbb{B}_X$ and $t \in [0,\infty) \setminus A$, where $A \subset [0,\infty)$ is a null set. Also, if $\exists \frac{\partial f}{\partial t}(x,t)$ in the same hypothesis, it is not known whether $\exists h(x,t)$ a generating vector field such that the differential Loewner equation (5.3.3) holds. However, in the case of separable reflexive complex Banach spaces, we obtain positive answers to these questions (see e.g. [39], [40]).

5.3.3 Biholomorphic mappings associated to g-Loewner chains

The last part of this section is dedicated to some particular families of biholomorphic mappings associated to g-Loewner chains. These notions were studied in both in finite and infinite dimensions by several authors (see e.g. [32], [45], [60], [61], [62], [74]). In the following, let $g : \mathbb{U} \to \mathbb{C}$ be a function that satisfies Assumption 5.3.6.

Definition 5.3.11. Let $d \in (0,1]$ and $\mu \in [0,1)$. A mapping $f \in \mathcal{LS}(\mathbb{B}_X)$ is called

a) *g-starlike* mapping on $\mathbb{B}_X$ if $h \in \mathcal{M}_g(\mathbb{B}_X)$, where

$$h(x) = \big[Df(x)\big]^{-1} f(x), \quad x \in \mathbb{B}_X. \tag{5.3.4}$$

We denote by $S_g^*(\mathbb{B}_X)$ the family of normalized g-starlike mappings on $\mathbb{B}_X$.

b) *strongly starlike of order d* if $h \in \mathcal{M}_g(\mathbb{B}_X)$, where h is given by (5.3.4) and

$$g(\zeta) = \left(\frac{1-\zeta}{1+\zeta}\right)^d, \quad \zeta \in \mathbb{U}. \tag{5.3.5}$$

Note that the branch of the power function is chosen such that $\left(\frac{1-\zeta}{1+\zeta}\right)^d\Big|_{\zeta=0} = 1$.

c) *almost starlike of order μ* if $h \in \mathcal{M}_g(\mathbb{B}_X)$, where h is given by (5.3.4) and

$$g(\zeta) = (1-\mu)\frac{1-\zeta}{1+\zeta} + \mu, \quad \zeta \in \mathbb{U}. \tag{5.3.6}$$

d) *parabolic starlike of order μ* if $h \in \mathcal{M}_g(\mathbb{B}_X)$, where h is given by (5.3.4), $g = 1/q_\mu$ and

$$q_\mu(\zeta) = 1 + \frac{4(1-\mu)}{\pi^2}\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2, \quad \zeta \in \mathbb{U}$$

Note that we choose the branch of the logarithm function such that $\log 1 = 0$.

Theorem 5.3.12. *Let $f \in \mathcal{LS}(\mathbb{B}_X)$ and let $g : \mathbb{U} \to \mathbb{C}$ be a univalent function that fulfil the conditions from Assumption 5.3.6. Then $f \in S_g^*(\mathbb{B}_X)$ if and only if $f(x,t) = e^t f(x)$ is a g-Loewner chain on $\mathbb{B}_X \times [0,\infty)$.*

For details and results related to other subclasses of biholomorphic mappings associated to g-Loewner chains, one may consult [39], [40], [74]. For the finite dimensional case, see [32], [34], [60] (see also [61], [62], [98]).

Chapter 6

New results related to Loewner chains and Extension operators in complex Banach spaces

This last chapter of the thesis contains original results related to Loewner chains and Extension operators in complex Banachs spaces based on the ideas presented by Graham, Hamada, Kohr and Kohr in [39] and [40]. Part of the original results have been obtained by Grigoriciuc in [55]. First, we consider the Graham-Kohr extension operator Ψ_α on the domain $\Omega_{p,r} = \big\{(z_1, w) \in \mathcal{Y} = \mathbb{C} \times X \, : \, |z_1|^p + \|w\|_X^r < 1\big\}$, where X is a complex Banach space, $\alpha \geq 0$ and $p, r \geq 1$. Based on the results proved by Graham, Hamada, Kohr and Kohr in [39] for $p = 2$ (see also [40]), we try to obtain extension properties for the general case $p \in [1, \infty)$.

The second section is dedicated to study of preservation of Loewner chains by the Pfaltzgraff-Suffridge extension operator from one dimension to infinite dimensional complex Banach spaces. Recently, Graham, Hamada, Kohr and Kohr (see e.g. [40]) proved that the Pfaltzgraff-Suffridge extension operator preserves the first elements of Loewner chains from the open unit ball $\mathbb{B}_X$ of an n-dimensional JB*-triple X into a domain $\mathbb{D}_\alpha \subseteq \mathbb{B}_X \times \mathbb{B}_Y$, where Y is a complex Banach space (for the complete results and their proofs, one may consult [33], [35] and [40]). Similar results were obtained for the finite dimensional case in [21], [33], [43], [49]. Inspired by the results obtained by Graham, Hamada, Kohr and Kohr, we prove that the Pfaltzgraff-Suffridge type extension operator preserve the first elements of Loewner chains from the unit ball B^n of $\mathbb{C}^n$ (with respect to different norms) into the unit ball of $\mathcal{W} = \mathbb{C}^n \times Y$, where Y is a complex Banach space.

The main bibliographic sources are the recent papers [34], [38], [39], [40], [41], [131], [132], [133], [134].

6.1 g-Loewner chains and the Graham-Kohr extension operator

Let X be a complex Banach space and let $p, r \geq 1$. Also, let

$$\Omega_{p,r} = \big\{(z_1, w) \in \mathcal{Y} = \mathbb{C} \times X \, : \, |z_1|^p + \|w\|_X^r < 1\big\}. \tag{6.1.1}$$

where $z_1 \in \mathbb{C}$ and $w \in X$. Then the Minkowski functional of $\Omega_{p,r}$ is a complete norm $\|\cdot\|_{\mathcal{Y}}$ on $\mathcal{Y}$ and $\Omega_{p,r}$ is the unit ball of $\mathcal{Y} = \mathbb{C} \times X$ with respect to this norm, where

- the Minkowski functional of $\Omega_{p,r}$ is given by $\rho(z) = \inf\{t > 0 : \frac{1}{t} z \in \Omega_{p,r}\}$, $z \in \Omega_{p,r}$;
- the norm $\|\cdot\|_{\mathcal{Y}}$ on $\mathcal{Y}$ is given by $\|z\|_{\mathcal{Y}} = |z_1|^p + \|w\|_X^r$, for all $z = (z_1, w) \in \mathcal{Y} = \mathbb{C} \times X$.

Let $\alpha \geq 0$ and let $\Psi_\alpha : \mathcal{LS} \to \mathcal{LS}(\Omega_{p,r})$ be the Graham-Kohr extension operator given by

$$\Psi_\alpha(f)(z_1, w) = \Big(f(z_1), \Big(\frac{f(z_1)}{z_1}\Big)^\alpha w\Big), \quad z = (z_1, w) \in \Omega_{p,r}, \tag{6.1.2}$$

where we take the branch of the power function such that $\left(\frac{f(z_1)}{z_1}\right)^\alpha\big|_{z_1=0} = 1$. Recall that for $\alpha, \beta \geq 0$, we denote by

$$\Phi_{\alpha,\beta}(f)(z_1, w) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^\alpha \left(f'(z_1)\right)^\beta w\right), \quad z = (z_1, w) \in \Omega_{p,r} \tag{6.1.3}$$

the Roper-Suffridge type extension operator. We consider the branches of the power functions such that $\left(\frac{f(z_1)}{z_1}\right)^\alpha\big|_{z_1=0} = 1$ and $\left(f'(z_1)\right)^\beta\big|_{z_1=0} = 1$.

Recently, Graham, Hamada, Kohr and Kohr (see e.g. [39], [40]) proved that the Roper-Suffridge type extension operator $\Phi_{\alpha,\beta}$ preserves the first elements of g-Loewner chains on $\mathbb{U}$ to the first elements of g-Loewner chains on $\Omega_{2,r}$ with $r \geq 1$, where $\alpha \in [0, 1]$ and $\beta \in [0, 1/r]$ such that $\alpha + \beta \leq 1$ and g is a convex function on $\mathbb{U}$ that satifies Assumption 5.3.6. As consequences of this result, the authors proved that $\Phi_{\alpha,\beta}$ preserves the notions of g-starlikness, strongly starlikeness of order $d \in (0, 1]$ and almost starlikeness of order $\mu \in [0, 1)$ in the same setting (see e.g. [39], [40]).

If $\beta = 0$ and $g \in \mathcal{H}_u(\mathbb{U})$ satisfies Assumption 5.3.6 such that $g(\mathbb{U})$ is a starlike domain with respect to 1, then we obtain the preservation of the first elements of g-Loewner chains and of the notion of parabolic starlikeness of order $\mu \in [0, 1)$ under the extension operator $\Phi_{\alpha,0} = \Psi_\alpha$ on $\Omega_{2,r}$, for $\alpha \in [0, 1]$ and $r \geq 1$ (see e.g. [39], [40]). For the finite dimension version of these results, one may consult [33], [42], [47].

6.1.1 Preliminaries

In this section we consider the Graham-Kohr extension operator Ψ_α on the domain $\Omega_{p,r}$, with $\alpha \geq 0$ and $p, r \geq 1$. Based on the results proved by Graham, Hamada, Kohr and Kohr in [39] (see also [40]) we try to obtain extension properties for the case $p \in [1, \infty)$. The original results presented in this section have been obtained by Grigoriciuc in [55].

For this, let X be a complex Banach space and let $\Omega_{p,r}$ given by (6.1.1) be the unit ball of $\mathcal{Y} = \mathbb{C} \times X$, for all $p, r \geq 1$. In order to prove our main results, we will use the following lemma (that is a generalization of Lemma 2.15 proved by Graham, Hamada, Kohr and Kohr in [39] in the case $p = 2$). Very recently, J. Wang proved the same result (see [131]) on the domain $\Omega_{p,r}$, where $p, r \geq 1$. We present here the detailed proof of the result (in the general case) based on the ideas given by Graham, Hamada, Kohr and Kohr in [39].

Lemma 6.1.1. *Let X be a complex Banach space and let $\Omega_{p,r}$ given by (6.1.1) be the unit ball of $\mathcal{Y} = \mathbb{C}\times X$, where $p, r \geq 1$. Let $z = (z_1, w) \neq 0$. Then we have*

$$l_z\big((z_1, 0)\big) = \frac{p|z_1|^p \|z\|_{\mathcal{Y}}}{p|z_1|^p + r(\|z\|_{\mathcal{Y}}^p - |z_1|^p)} \tag{6.1.4}$$

and

$$l_z\big((0, w)\big) = \frac{r(\|z\|_{\mathcal{Y}}^p - |z_1|^p)\|z\|_{\mathcal{Y}}}{p|z_1|^p + r(\|z\|_{\mathcal{Y}}^p - |z_1|^p)}, \tag{6.1.5}$$

for every $l_z \in T(z)$.

The second result proved in this section is related to the holomorphy of the mapping $\Psi_\alpha(f)$ on the domain $\Omega_{p,r}$.

Lemma 6.1.2. *Let $\alpha \in [0, 1]$ and $p, r \geq 1$. Also, let $f \in \mathcal{H}_u(\mathbb{U})$ be such that $f(\mathbb{U}) \subseteq \mathbb{U}$ and $f(0) = 0$. Then*

$$F(z) = \Psi_\alpha(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^\alpha w\right), \quad z = (z_1, w) \in \Omega_{p,r}$$

is a holomorphic mapping from $\Omega_{p,r}$ to $\Omega_{p,r}$.

The last result of this subsection is related to the subordination principle on $\Omega_{p,r}$. Its proof follows the main ideas presented by Wang and Zhang in [134] (see Theorem 3.6) for $p \in [1, 2]$ and $r \geq 1$.

Proposition 6.1.3. *Let $f, g \in S$ and let $\Omega_{p,r}$ be the domain given by (6.1.1), where $p, r \geq 1$. If $\alpha \in (0, \infty)$, then $f \prec g$ on $\mathbb{U}$ if and only if $\Psi_\alpha(f) \prec \Psi_\alpha(g)$ on $\Omega_{p,r}$.*

6.1.2 Extension results on $\Omega_{p,r}$

We continue this section by studying the preservation of the first elements of g-Loewner chains by the Graham-Kohr extension operator Ψ_α given by (6.1.2) on the domain $\Omega_{p,r}$, where $p, r \geq 1$. For now, $g : \mathbb{U} \to \mathbb{C}$ is a convex univalent function which satisfies Assumption 5.3.6. Note that in the proof of this result we follow arguments similar to those for Theorem 3.1 in [39]. It is important to mention here that for $X = \mathbb{C}^{n-1}$, $r = 2$ and $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, one may consult [33, Corollary 2.9], [42, Theorem 2.1] and [47, Theorem 2.1].

Moreover, for similar results related to the Roper-Suffridge type extension operator $\Phi_{\alpha,\beta}$, see [39] for $p = 2$ and $r \geq 1$, respectively [131] for $p, r \geq 1$. The following results were obtained by Grigoriciuc in [55].

Theorem 6.1.4. *Let X be a complex Banach space and let $\Omega_{p,r}$ be the domain given by (6.1.1), where $p, r \geq 1$. Also let g be a convex function on $\mathbb{U}$ which satisfies Assumption 5.3.6. If $f \in S$ is the first element of the g-Loewner chain $f(\cdot, t)$ on $\mathbb{U}$ and $F(\cdot, t)$ is a g-Loewner chain on $\Omega_{p,r}$, for all $t \geq 0$, then $F(\cdot, 0) = \Psi_\alpha(f)$, for all $\alpha \in [0, 1]$.*

In particular, from the previous result, we obtain that the first elements of Loewner chains are preserved from the unit disc $\mathbb{U}$ to the domain $\Omega_{p,r}$, for $p, r \geq 1$, under the Graham-Kohr extension operator (for the case $p = 2$ and the operator given by (6.1.3), see Corollary 3.2 from [39]). Also, see Theorem 2.1 in [42] and Theorem 2.1 in [47] for the case $X = \mathbb{C}^{n-1}$ and $p = r = 2$.

Corollary 6.1.5. *Let $\Omega_{p,r}$ and g be as in Theorem 6.1.4, where $p, r \geq 1$. If $f \in S$ and $F(\cdot, t)$ is a Loewner chain on $\Omega_{p,r}$, for all $t \geq 0$, then $F(\cdot, 0) = \Psi_\alpha(f)$, for all $\alpha \in [0, 1]$.*

In the second consequence of Theorem 6.1.4 we obtain that the Graham-Kohr extension operator Ψ_α preserves g-starlike mappings from $\mathbb{U}$ into $\Omega_{p,r}$ for $p, r \geq 1$. The proof of this result is based on ideas similar to those for Corollary 3.3 in [39], where the authors considered $p = 2$ and the extension operator given by (6.1.3). For the case $X = \mathbb{C}^{n-1}$, $p = r = 2$ and $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, where $\zeta \in \mathbb{U}$, one may consult [43, Theorem 2.2]; see also [11, Corollary 2.3] in the case $X = \mathbb{C}^{n-1}$, $p = r = 2$ and $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$ for $\zeta \in \mathbb{U}$ and $\gamma \in (0, 1)$.

Corollary 6.1.6. *Let $\Omega_{p,r}$ and g be as in Theorem 6.1.4, where $p, r \geq 1$. If $f \in S_g^*(\mathbb{U})$, then $F = \Psi_\alpha(f) \in S_g^*(\Omega_{p,r})$, for all $\alpha \in [0, 1]$, where $S_g^*(\Omega_{p,r})$ is the family of g-starlike mappings on $\Omega_{p,r}$.*

Corollary 6.1.7. *Let $\Omega_{p,r}$ and g be as in Theorem 6.1.4, where $p, r \geq 1$. If $f \in sS_d^*(\mathbb{U})$, for $d \in (0, 1]$, then $F = \Psi_\alpha(f) \in sS_d^*(\Omega_{p,r})$, for $\alpha \in [0, 1]$, where $sS_d^*(\Omega_{p,r})$ is the family of strongly starlike mappings of order d on $\Omega_{p,r}$.*

Another important result is related to the preservation of the first elements of g-Loewner chains by the Graham-Kohr extension operator, where g is a starlike (univalent) function with respect to 1 that satifies Assumption 5.3.6. This result is an extension of [39, Theorem 3.4] and its proof follows arguments similar to those used by authors in [39] for $p = 2$.

Theorem 6.1.8. *Let X be a complex Banach space and let $\Omega_{p,r}$ be the domain given by (6.1.1), where $p, r \geq 1$. Also let g be a function that satisfies Assumption 5.3.6 such that $g(\mathbb{U})$ is starlike with respect to 1. If $f \in S$ is the first element of the g-Loewner chain $f(\cdot, t)$ on $\mathbb{U}$ and $F(\cdot, t)$ is a g-Loewner chain on $\Omega_{p,r}$, for all $t \geq 0$, then $F(\cdot, 0) = \Psi_\alpha(f)$, for all $\alpha \in [0, 1]$.*

From the previous theorem we obtain a consequnece (see [39, Corollary 3.5] for the case $p = 2$) related to the property of parabolic starlikeness of order $\mu \in [0, 1)$. The proof of this result follows the ideas presented by the authors in [60, Theorem 5.1] and [62, Theorem 5.3].

Corollary 6.1.9. *Let $\Omega_{p,r}$ and g be as in Theorem 6.1.8, where $p, r \geq 1$. If $f \in \mathcal{P}S^*(\mathbb{U})$, for $\mu \in [0, 1)$, then $F = \Psi_\alpha(f) \in \mathcal{P}S^*(\Omega_{p,r})$, for all $\alpha \in [0, 1]$, where $\mathcal{P}S^*(\Omega_{p,r})$ is the family of parabolic starlike mappings of order μ on $\Omega_{p,r}$.*

The third theorem is devoted to the preservation of the almost starlikeness of complex order λ, where $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\lambda \leq 0$, under the Graham-Kohr extension operator. We can prove that Ψ_α preserves the almost starlikeness of complex order λ from $\mathbb{U}$ into $\Omega_{p,r}$ for all $p, r \geq 1$ and $\alpha \in [0,1]$. In the proof of this theorem we follow the main ideas as in the proof of [134, Theorem 3.8] given by Wang and Zhang for $p \in [1,2]$ and for the extension operator $\Phi_{\alpha,\beta}$ given by relation (6.1.3). In our result we consider $\alpha \in [0,1]$, $\beta = 0$ and $p, r \in [1, \infty)$.

Remark 6.1.10. Recall that the property of almost starlikeness of complex order $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\lambda \leq 0$ can be described via Loewner chains (see e.g. [134, Lemma 3.5]) as: $f \in \mathcal{LS}(\mathbb{B}_X)$ is an almost starlike mapping of complex order λ on $\mathbb{B}_X$ if and only if $F(z,t)$ is a Loewner chain, where

$$F(z,t) = e^{(1-\lambda)t} f(e^{\lambda t} z), \quad z \in \mathbb{B}_X, t \in [0, \infty). \tag{6.1.6}$$

Theorem 6.1.11. *Let X be a complex Banach space and let $\Omega_{p,r}$ be the domain given by (6.1.1), where $p, r \geq 1$. Let also $\lambda \in \mathbb{C}$ be such that $\mathfrak{Re}\lambda \leq 0$. If $f \in \mathcal{A}sc^*_\lambda(\mathbb{U})$, then $F = \Psi_\alpha(f) \in \mathcal{A}sc^*_\lambda(\Omega_{p,r})$ for all $\alpha \in [0,1]$, where we denote by $\mathcal{A}sc^*_\lambda(\Omega_{p,r})$ the family of almost starlike mappings of complex order λ on $\Omega_{p,r}$.*

Remark 6.1.12. If $p \in [1,2]$, then Theorem 6.1.11 reduces to [134, Theorem 3.8]. Moreover, if $X = \mathbb{C}^{n-1}$ and $p = r = 2$, then $\Omega_{p,r}$ is the Euclidean unit ball $\mathbb{B}^n$. For $\lambda = 0$ we obtain that $F(z,t) = e^t \Psi_\alpha(f)(z)$ is a Loewner chain and hence, $\Psi_\alpha(f)$ is a starlike mapping on $\mathbb{B}^n$. This result is well-known since was proved by Graham and Kohr in [44] (see also [45] or [133]).

Finally, we present two consequences of Theorem 6.1.11 related to the preservation of the almost starlikeness of order $\mu \in [0,1)$, respectively the spirallikeness of order $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ through the Graham-Kohr extension operator Ψ_α from the unit disc $\mathbb{U}$ to the domain $\Omega_{p,r}$ for $p, r \geq 1$ and $\alpha \in [0,1]$.

Remark 6.1.13. According to [134, Definition 2.1] if $\lambda = \frac{\mu}{\mu-1}$ for $\mu \in [0,1)$, then the almost starlikeness of complex order λ reduces to the almost starlikeness of order $\mu \in [0,1)$. On the other hand, if $\lambda = i \tan \delta$, for $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then the almost starlikeness of complex order λ reduces to the spirallikeness of order δ.

Based on Theorem 6.1.11 and Remark 6.1.13 we obtain immediately the following results:

Corollary 6.1.14. *Let $\Omega_{p,r}$ with $p, r \geq 1$ be as in Theorem 6.1.11. If $f \in \mathcal{AS}^*_\mu(\mathbb{U})$, where $\mu \in [0,1)$, then $F = \Psi_\alpha(f) \in \mathcal{AS}^*_\mu(\Omega_{p,r})$, for $\alpha \in [0,1]$.*

Corollary 6.1.15. *Let $\Omega_{p,r}$ with $p, r \geq 1$ be as in Theorem 6.1.11. If $f \in \hat{S}_\delta(\mathbb{U})$, where $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $F = \Psi_\alpha(f) \in \hat{S}_\delta(\Omega_{p,r})$, for $\alpha \in [0,1]$.*

6.2 Loewner chains and the Pfaltzgraff-Suffridge extension operator

Section 6.2 is dedicated to the study of preservation of Loewner chains by the Pfaltzgraff-Suffridge extension operator from one dimension to infinite dimensional complex Banach spaces. Recently, Graham, Hamada, Kohr and Kohr (see e.g. [40]) proved that the Pfaltzgraff-Suffridge extension operator preserves the first elements of Loewner chains from the open unit ball $\mathbb{B}_X$ of an n-dimensional JB*-triple X into a domain $\mathbb{D}_\alpha \subseteq \mathbb{B}_X \times \mathbb{B}_Y$, where Y is a complex Banach space (for the complete results and their proofs, one may consult [33], [35] and [40]). Another important results proved in this general setting are related to the preservation of starlikeness and spirallikeness of order $\gamma \in (-\pi/2, \pi/2)$ from $\mathbb{B}_X$ into $\mathbb{D}_\alpha$ under the Pfaltzgraff-Suffridge type extension operator. Similar results were obtained for the finite dimensional case in [21], [33], [43], [49].

6.2.1 Preliminaries

In this section we consider two particular cases of the operator studied by Graham, Hamada, Kohr and Kohr in [40] on special domains in complex Banach spaces (similar to those in [55], [131] and [134]). The presented results are original.

The main idea is to prove that the Pfaltzgraff-Suffridge type extension operator preserves the first elements of Loewner chains from the unit ball B^n of $\mathbb{C}^n$ (with respect to different norms) to the unit ball of $\mathcal{W} = \mathbb{C}^n \times Y$, where Y is a complex Banach space.

It is important to mention here that throughout this section we will use different norms on the space $\mathbb{C}^n$ (e.g. Euclidean $\|x\| = \sqrt{\sum_{j=1}^n |x_j|^2}$, maximum norm $\|x\|_\infty = \max\{|x_j| : j = \overline{1,n}\}$, p-norm $\|x\|_p = \big(\sum_{j=1}^n |x_j|^p\big)^{1/p}$, for all $x = (x_1, ..., x_n) \in \mathbb{C}^n$, $p \geq 1$; for details, see §3.1.1) which are equivalent since the space $\mathbb{C}^n$ is finite dimensional.

Remark 6.2.1. Let $p, r \geq 1$ and let Y be a complex Banach space with the norm $\|\cdot\|_Y$. If the space $\mathbb{C}^n$ is equipped with the Euclidean norm $\|\cdot\|$, then we denote by

$$\Omega_{n,p,r} = \big\{(x, y) \in \mathcal{W} = \mathbb{C}^n \times Y : \|x\|^p + \|y\|_Y^r < 1\big\} \tag{6.2.1}$$

the unit ball of $\mathbb{C}^n \times Y$, for every $p, r \geq 1$, respectively the Pfaltzgraff-Suffridge extension operator by

$$\Psi_{n,r}(f)(z) = \Big(f(x), \big[J_f(x)\big]^{\frac{2}{r(n+1)}} y\Big), \quad z = (x, y) \in \Omega_{n,p,r}, \tag{6.2.2}$$

where the branch of the power function has the property $\big[J_f(x)\big]^{\frac{2}{r(n+1)}}\big|_{x=0} = 1$. In particular, if $p = 2$, then we denote $\Omega_{n,2,r}$ by $\Omega_{n,r}$, where

$$\Omega_{n,r} = \big\{(x, y) \in \mathcal{W} = \mathbb{C}^n \times Y : \|x\|^2 + \|y\|_Y^r < 1\big\}, \quad r \geq 1.$$

Remark 6.2.2. Let $p, r \geq 1$, let Y be a complex Banach space with the norm $\|\cdot\|_Y$ and let $\mathbb{B}_Y$ be the unit ball of Y. If we endowed the space $\mathbb{C}^n$ with the maximum norm $\|\cdot\|_\infty$, then we denote by

$$\Delta_{n,p,r} = \Big\{(x, y) \in \mathbb{U}^n \times \mathbb{B}_Y : \|y\|_Y < \prod_{j=1}^n (1 - |x_j|^p)^{\frac{1}{rn}}\Big\} \tag{6.2.3}$$

the unit ball of $\mathcal{W} = \mathbb{C}^n \times Y$, for every $p, r \geq 1$, respectively the Pfaltzgraff-Suffridge extension operator by

$$\Gamma_{n,r}(f)(z) = \Big(f(x), \big[J_f(x)\big]^{\frac{1}{rn}} y\Big), \quad z = (x, y) \in \Delta_{n,p,r}, \tag{6.2.4}$$

where we consider the branch of the power function such that $\big[J_f(x)\big]^{\frac{1}{rn}}\big|_{x=0} = 1$. In particular, if $p = 2$, then we denote $\Delta_{n,2,r}$ simply by $\Delta_{n,r}$, where

$$\Delta_{n,r} = \Big\{(x, y) \in \mathbb{U}^n \times \mathbb{B}_Y : \|y\|_Y < \prod_{j=1}^n (1 - |x_j|^2)^{\frac{1}{rn}}\Big\}, \quad r \geq 1.$$

Note that the definition of the domain $\Delta_{n,p,r}$ in (6.2.3) is based on the ideas presented by Graham, Hamada, Kohr and Kohr in [40] (see also [63]). Starting from an n-dimensional JB*-triple X and a complex Banach space Y they defined the set $\mathbb{D}_r = \{(x, y) \in \mathbb{B}_X \times Y : \|y\|_Y < [\det B(x, x)]^{1/(2rc(\mathbb{B}_X))}\}$, where $r \geq 1$, $\mathbb{B}_X$ is the unit ball of X, $B(x, y) \in L(X)$ is the Bergman operator, $x, y \in X$, and $c(\mathbb{B}_X)$ is a constant that depends on the Bergman metric on X (for details, see [40], [63]). If X is the space $\mathbb{C}^n$ with the maximum norm $\|\cdot\|_\infty$, then $\mathbb{B}_X = \mathbb{U}^n$, $c(\mathbb{U}^n) = n$ and $\det B(x, x) = \prod_{j=1}^n (1 - |x_j|^2)^2$, for all $x \in \mathbb{U}^n$ (see [63]). Hence, by simple computations we obtain the domain $\mathbb{D}_r = \Delta_{n,r}$. Finally, if we consider $p \in [1, 2]$, then we obtain the more general domain $\Delta_{n,p,r}$ defined in (6.2.3).

Remark 6.2.3. Let $p \in [1, 2]$ and let $\phi : [0, 1] \to \mathbb{R}$ be given by $\phi(t) = \frac{1-t^2}{1-t^p}$, for all $t \in [0, 1]$. Then ϕ is increasing on $[0, 1]$, for every $p \in [1, 2]$. This result (considered first by Wang in [131, Lemma 3.2] and [134]) will be used in the proofs of the main results in this section.

For the extension operators presented above we can prove the following two lemmas related to the holomorphy of $\Psi_{n,r}(f)$ (respectively $\Gamma_{n,r}(f)$) on $\Omega_{n,p,r}$ (respectively on $\Delta_{n,r,p}$), where $p \in [1, 2]$ and $r \geq 1$.

Lemma 6.2.4. *Let $p \in [1,2]$ and $r \geq 1$. Also let $f \in \mathcal{H}_u(\mathbb{B}^n)$ be such that $f(\mathbb{B}^n) \subseteq \mathbb{B}^n$ and $f(0) = 0$. Then*

$$F(z) = \Psi_{n,r}(f)(z) = \left(f(x), [J_f(x)]^{\frac{2}{r(n+1)}} y\right), \quad z = (x,y) \in \Omega_{n,p,r}$$

is a holomorphic mapping from $\Omega_{n,p,r}$ to $\Omega_{n,p,r}$.

Lemma 6.2.5. *Let $p \in [1,2]$ and $r \geq 1$. Also let $f \in \mathcal{H}_u(\mathbb{U}^n)$ be such that $f(\mathbb{U}^n) \subseteq \mathbb{U}^n$ and $f(0) = 0$. Then*

$$\Gamma_{n,r}(f)(z) = \left(f(x), [J_f(x)]^{\frac{1}{rn}} y\right), \quad z = (x,y) \in \Delta_{n,p,r}$$

is a holomorphic mapping from $\Delta_{n,p,r}$ to $\Delta_{n,p,r}$.

6.2.2 Extension results on $\Omega_{n,p,r}$

In the subsequent part, we outline the main result of this subsection related to the preservation of the first elements of a Loewner chain from the Euclidean unit ball $\mathbb{B}^n$ into the domain $\Omega_{n,p,r}$ under the Pfaltzgraff-Suffridge type extension operator $\Psi_{n,r}$ for $p \in [1,2]$ and $r \geq 1$. This result is strongly related to [35, Theorem 3.1] and [49, Theorem 2.1], where the authors treated the same problem on different domains. The results stated here are original and were proved by the author.

Theorem 6.2.6. *Let $r \geq \frac{2n}{n+1}$, $p \in [1,2]$ and let f belongs to S be the first element of the Loewner chain $f(\cdot,t)$ on $\mathbb{B}^n$. Let also $F(\cdot,t)$ be a Loewner chain on $\Omega_{n,p,r}$, for all $t \geq 0$. Then $F(\cdot,0) = \Psi_{n,r}(f)$.*

From the previous theorem we obtain the following consequences related to the preservation of almost starlikeness of complex order λ, almost starlikeness of order α, spirallikeness of order γ and starlikeness from $\mathbb{B}^n$ into $\Omega_{n,p,r}$ for $p \in [1,2]$. Notice that if $p = 2$, then our results reduces to the one proved by Graham, Hamada, Kohr and Kohr in [40, Corollary 5.4], Graham, Kohr and Pfaltzgraff in [49, Corollary 2.4], respectively by Wang and Zhang in [134, Theorem 3.12, Corollaries 3.13 and 3.14].

Corollary 6.2.7. *Let $r \geq \frac{2n}{n+1}$, $p \in [1,2]$ and $\lambda \in \mathbb{C}$ be such that $\mathfrak{Re}\lambda \leq 0$. If $f \in \mathcal{A}sc^*_\lambda(\mathbb{B}^n)$, then $F = \Psi_{n,r}(f) \in \mathcal{A}sc^*_\lambda(\Omega_{n,p,r})$, where we denote by $\mathcal{A}sc^*_\lambda(\Omega_{n,p,r})$ the family of almost starlike mappings of complex order λ on $\Omega_{n,p,r}$.*

Corollary 6.2.8. *Let $r \geq \frac{2n}{n+1}$, $p \in [1,2]$ and $\alpha \in [0,1)$. If $f \in \mathcal{AS}^*_\alpha(\mathbb{B}^n)$, then $F = \Psi_{n,r}(f) \in \mathcal{AS}^*_\alpha(\Omega_{n,p,r})$, where $\mathcal{AS}^*_\alpha(\Omega_{n,p,r})$ is the class of almost starlike mappings of order α on $\Omega_{n,p,r}$.*

Corollary 6.2.9. *Let $r \geq \frac{2n}{n+1}$, $p \in [1,2]$ and $|\gamma| < \frac{\pi}{2}$. If f belongs to the family $\hat{S}_\gamma(\mathbb{B}^n)$, then $F = \Psi_{n,r}(f) \in \hat{S}_\gamma(\Omega_{n,p,r})$, where $\hat{S}_\gamma(\Omega_{n,p,r})$ is the family of spirallike mappings of order γ on $\Omega_{n,p,r}$.*

Corollary 6.2.10. *Let $r \geq \frac{2n}{n+1}$, $p \in [1,2]$ and $|\gamma| < \frac{\pi}{2}$. If f belongs to the family $S^*(\mathbb{B}^n)$, then $F = \Psi_{n,r}(f) \in S^*(\Omega_{n,p,r})$.*

6.2.3 Remarks on ε-starlikeness

Another important property of the operator $\Psi_{n,r}$ is related to the preservation of the ε-starlikeness on the domain $\Omega_{n,2,r}$. Our first result from this subsection is a generalization of Theorem 2.6 proved by Wang and Wang in [133]. Notice that in the following result we consider $p = 2$.

Theorem 6.2.11. *Let $r \geq \frac{2n}{n+1}$. If $f \in S(\mathbb{B}^n)$ is an ε-starlike mapping on $\mathbb{B}^n$, then $F = \Psi_{n,r}(f) \in S(\Omega_{n,2,r})$ is an ε-starlike mapping on $\Omega_{n,2,r}$.*

Remark 6.2.12. If $Y = \mathbb{C}$ and $r = 2$, then for $\varepsilon = 0$, Theorem 6.2.11 reduces to [49, Corollary 2.4]. On the other hand, if $\varepsilon = 1$, then Theorem 6.2.11 treats the case of convex mappings associated with the operator $\Psi_{n,r}$ conjectured by Pfaltzgraff and Suffridge in [111] (see also [49]).

6.2.4 Remarks on convexity

Let Y be a complex Banach space. For $a \in (0,1]$ and $r \geq 1$, let us denote by

$$D_{a,r} = \left\{(x,y) \in \mathcal{W} = \mathbb{C}^n \times Y \,:\, \|y\|_Y^r < a^{\frac{2n}{n+1}}(1-\|x\|^2)\right\}. \tag{6.2.5}$$

Then $D_{a,r} \subseteq \Omega_{n,p,r}$ and $D_{1,r} = \Omega_{n,2,r}$, where $\Omega_{n,2,r}$ is a particular case of the domain defined by formula (6.2.1) for $p = 2$. Following the arguments presented by Graham, Kohr and Pfaltzgraff in [49], we can extend some of their result as follows:

Theorem 6.2.13. *Let $r \geq \frac{2n}{n+1}$, $f \in K(\mathbb{B}^n)$ and $\alpha_1, \alpha_2 \in \mathbb{R}_+^*$ be such that $\alpha_1 + \alpha_2 \leq 1$. Also, let $F = \Psi_{n,r}(f)$ be the extension operator given by (6.2.2). Then $(1-\lambda)F(x,y) + \lambda F(\tilde{x},\tilde{y}) \in F(D_{\alpha_1+\alpha_2,r})$, where $(x,y) \in D_{\alpha_1,r}$, $(\tilde{x},\tilde{y}) \in D_{\alpha_2,r}$ and $\lambda \in [0,1]$.*

Corollary 6.2.14. *Let $r \geq \frac{2n}{n+1}$, $f \in K(\mathbb{B}^n)$ and $F = \Psi_{n,r}(f)$. Then $(1-\lambda)F(x,y)+\lambda F(\tilde{x},\tilde{y}) \in F(\Omega_{n,2,r})$, where $(x,y), (\tilde{x},\tilde{y}) \in D_{1/2,r}$ and $\lambda \in [0,1]$.*

Remark 6.2.15. As we mentioned above, in the particular case $Y = \mathbb{C}$ and $r = 2$, Theorem 6.2.13 and Corollary 6.2.14 reduce to the results proved by Graham, Kohr and Pfaltzgraff in [49, Theorem 2.2], respectively in [49, Corollary 2.5].

6.2.5 Extension results on $\Delta_{n,p,r}$

The last part of this section is devoted to the study of the Pfaltzgraff-Suffridge type extension operator $\Gamma_{n,r}$ on the domain $\Delta_{n,p,r}$ (for details, one may consult [35] and [40]). Let us consider again Y a complex Banach space, $p \in [1,2]$ and $r \geq 1$. Recall that if $X = \mathbb{C}^n$ is equipped with the maximum norm $\|\cdot\|_\infty$, then we denote by

$$\Delta_{n,p,r} = \left\{(x,y) \in \mathbb{U}^n \times \mathbb{B}_Y \,:\, \|y\|_Y < \prod_{j=1}^n (1-|x_j|^p)^{\frac{1}{rn}}\right\},$$

where $\mathbb{U}^n$ is the unit polydisc in $\mathbb{C}^n$ and $\mathbb{B}_Y$ is the open unit ball in the complex Banach space Y. Moreover, the Pfaltzgraff-Suffridge type extension operator $\Gamma_{n,r}$ is given by

$$\Gamma_{n,r}(f)(z) = \left(f(x), \left[J_f(x)\right]^{\frac{1}{rn}} y\right), \quad z = (x,y) \in \Delta_{n,p,r},$$

where we take the branch of the power function with the property $\left[J_f(x)\right]^{\frac{1}{rn}}\big|_{x=0} = 1$.

Based on the ideas presented by Graham, Hamada and Kohr in [35] we obtain a result similar to the one in Theorem 6.2.6 that describes the preservation of Loewner chains under the operator $\Gamma_{n,r}$ on the domain $\Delta_{n,p,r}$ (see [35, Corollary 3.4] for the case $p = 2$; see also [33, Theorem 2.1], [35, Theorem 3.1] and [49, Theorem 2.1]).

Theorem 6.2.16. *Let $r \geq 1$, $p \in [1,2]$ and let $f \in S$ be the first element of the Loewner chain $f(\cdot,t)$ on $\mathbb{U}^n$. Let also $G(\cdot,t)$ be a Loewner chain on $\Delta_{n,p,r}$, for all $t \geq 0$. Then $G(\cdot,0) = \Gamma_{n,r}(f)$.*

Finally, we present two results that derive from Theorem 6.2.16 that describe the preservation of spirallikeness, respectively of starlikeness under the extension operator $\Gamma_{n,r}$ on the domain $\Delta_{n,p,r}$ for $r \geq 1$ and $p \in [1,2]$. In the proof of these results we follow the main ideas as in the proof of [35, Corollary 3.2] given by Graham, Hamada and Kohr. For the case $p = 2$, see [35, Corollary 3.4].

Corollary 6.2.17. *Let $r \geq 1$, $p \in [1,2]$ and $|\gamma| < \frac{\pi}{2}$. If f belongs to the family $\hat{S}_\gamma(\mathbb{U}^n)$, then $\Gamma_{n,r}(f) \in \hat{S}_\gamma(\Delta_{n,p,r})$.*

Corollary 6.2.18. *Let $r \geq 1$ and $p \in [1,2]$. If f belongs to the family $S^*(\mathbb{U}^n)$, then $\Gamma_{n,r}(f) \in S^*(\Delta_{n,p,r})$.*

It is important to mention that it would be of interest to consider extensions of the results presented above for the Pfaltzgraff-Suffridge type extension operator on the domain $\Omega_{n,p,r}$ for all $p, r \geq 1$ (see e.g. the results presented in [55] for the Graham-Kohr extension operator or in [131] for the Roper-Sufridge type extension operator). On the other hand, another interesting problem would be to study similar results for the Muir extension operator (see [39], [40], [103]).

Conclusions

During this thesis we discussed several results related to the geometric function theory of complex variables in one and higher dimensions. In this last part we want to recall and highlight the main original results we obtained in each chapter.

The first part of the thesis was devoted to the study of classical results and families of univalent functions of one complex variable. The main bibliographical sources used to prepare this part were [19], [29], [45], [77], [85], [87], [102], [114]. Even if **Chapter 1** is an introductory one, it also contains original results obtained by the author in [50] and [51]. First, in section §1.4 we proved some general distortion theorems for starlike (see Theorem 1.4.9), respectively convex functions of order α (see Theorem 1.4.19). These results were obtained by Grigoriciuc in [51].

Another important family of univalent functions in $\mathbb{C}$ was studied in §1.5. Here, we have extended the class $\mathcal{R}$ of functions whose derivative has positive real part introduced by MacGregor in [96]. Starting from $\mathcal{R}$ we defined two new subclasses, $\mathcal{R}_p$ and $\mathcal{R}_p(\alpha)$, and studied some of their properties. The original results were obtained in [50].

In **Chapter 2** we introduced a new differential operator that was used to define two new subclasses of univalent functions on the unit disc in $\mathbb{C}$. In §2.1 we have obtained general properties of the operator $\mathcal{G}_k$ related to linearity, convolution product and a sufficient condition of univalence (see Propositions 2.1.3 – 2.1.6).

The second section of this chapter was dedicated to the study of subclasses $E_k(\alpha)$ and $E_k^*(\alpha)$, where $k \in \mathbb{N}$ and $\alpha \in [0, 1)$. Together with general properties of these subclasses (growth and distortion theorems, coefficient estimations, analytical characterization, connection with Loewner chains) we also studied particular cases (e.g. $k = 1$ and $\alpha = 0$) that were of interest being in close connection with the classes of univalent functions mentioned in the first chapter (see e.g. Propositions 2.2.25 and 2.2.26 in §2.2.2). All the results in this chapter are original and were obtained by the author in [54].

The second part of this thesis begins with **Chapter 3** and was devoted to the study of univalent mappings of several complex variables in $\mathbb{C}^n$, where $n \geq 2$. This part is based on several important books (e.g. [45], [83], [107], [119], [123]) and papers (e.g. [32], [37], [44], [128]). Besides the classical results of the geometric function theory of several complex variables, in this chapter we addressed a new problem: the biholomorphy of convex combinations of biholomorphic mappings on the Euclidean unit ball in $\mathbb{C}^n$. It is known that if $f, g \in S(\mathbb{B}^n)$, then $(1 - \lambda)f + \lambda g$ is not necessary biholomorphic on $\mathbb{B}^n$, where $\lambda \in (0, 1)$. However, in §3.6 we proved some results (see e.g. Proposition 3.6.6 and Theorem 3.6.8) that solved partially this problem (based on the idea proposed by Chichra and Singh in [9]). The original results presented here were obtained in [52].

A natural extension of the previous results that was considered in this chapter (see §3.8) is related to the convex combination of two Graham-Kohr type extension operators (see Definiton 3.8.3). The extension operator that is mentioned here was defined by I. Graham and G. Kohr in [44] (see also [43]). They also proved that their operator preserves the notions of starlikeness, spirallikeness, parametric representation (see e.g. [44], [60], [62]). We used this type of extension operator to prove several properties of a new extension operator in $\mathbb{C}^n$ (see Theorems 3.8.6 – 3.8.9). The results presented in §3.8 are also original are were obtained by the author.

In **Chapter 4** we introduced the n-dimensional form of the differential operator $\mathcal{G}_k$ defined in Chapter 2. Using this operator we have generalized the subclasses E_k^*, respectively E_k from the unit disc $\mathbb{U}$ to the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. In §4.2 we proved some important results related to the connection between subclasses E_1^* (respectively E_1) on $\mathbb{U}$ and the class of convex mappings $K(\mathbb{B}^n)$ in $\mathbb{C}^n$ (see Theorems 4.2.1, 4.2.3 and 4.2.5). Recall that the results from the case $n = 1$ are not longer true in the case of several complex variables (see e.g. Theorem 4.2.1). All the results obtained in this section are original and were obtained by the author in [53].

The second part of this chapter was dedicated to study the preservation of the class $E_1^* = K$ under the Graham-Kohr extension operator $\Psi_{n,\alpha}$ mentioned above. Although the operator $\Psi_{n,\alpha}$ does not preserve the notion of convexity (see e.g. [44]), we managed to prove that $\Psi_{n,\alpha}(K) \subseteq E_1^*(\mathbb{B}^n)$ for $\alpha \in \{0, 1\}$. The results presented in Proposition 4.3.1 and Theorem 4.3.4 are original and partially answer Question 4.3.7 proposed in [53].

The last part of the thesis contains results related to biholomorphic mappings and Extension operators in complex Banach spaces. In **Chapter 5** we have included a short presentation of the main results that were used in the last chapter (e.g. families of biholomorphic mappings, Extension operators and the Loewner theory in infinite dimensional case). This part is mainly based on the references [39], [40], [41], [45], [106], [117], [127].

In **Chapter 6** we proved several results related to extension of Loewner chains under the Graham-Kohr, respectively Pfaltzgraff-Suffridge extension operator on particular domains in infinite dimensions. In §6.1 we obtained some extensions (see e.g. Lemma 6.1.1, Theorems 6.1.4, 6.1.8 and 6.1.11) of the results presented by Graham, Hamada, Kohr and Kohr in [39] and [40]. Starting from their results, we proved that g-Loewner chains are preserved under the Graham-Kohr extension operator from the unit disc $\mathbb{U}$ to the unit ball $\Omega_{p,r}$ defined by (6.1.1) for every $p, r \geq 1$ (for the case $p = 2$, see [39], [40]).

In §6.2 we obtained results related to the preservation of the first elements of Loewner chains from the Euclidean unit ball $\mathbb{B}^n$ into the domain $\Omega_{n,p,r}$ under the Pfaltzgraff-Suffridge type extension operator $\Psi_{n,r}$ for $p \in [1, 2]$ and $r \geq 1$ (for details, see §6.2.1). The original results presented here were obtained using similar arguments to those used by the authors in [35] and [49]. In Theorem 6.2.11 we obtained the preservation of the ε-starlikeness on the domain $\Omega_{n,2,r}$ under the Pfaltzgraff-Suffridge type extension operator. This result is a generalization of Theorem 2.6 proved by Wang and Wang in [133]. We ended this section with some results related to convexity (see e.g. Theorem 6.2.13), respectively the preservation of the first elements of Loewner chains from the unit polydisc $\mathbb{U}^n$ into the domain $\Delta_{n,p,r}$ under the Pfaltzgraff-Suffridge type extension operator $\Gamma_{n,r}$ for $p \in [1, 2]$ and $r \geq 1$ (see Theorem 6.2.16, Corollaries 6.2.17 and 6.2.18). Part of the results presented in this chapter are original and were obtained by the author in [55].

Finally, let us mention that all the original results presented in this thesis have been obtained using classic and modern methods from the geometric function theory of one and several complex variables, with a particular emphasis on the theory of Loewner chains and Extension operators. Note that I. Graham, H. Hamada, G. Kohr and M. Kohr had special and very important contributions in this modern and dynamic field with multiple applications in Fluid Mechanics (e.g. Hele-Shaw flow problems), Probability theory (e.g. Schramm-Loewner evolution, non-commutative probability) or Optimal control.

Further Research Directions

We conclude this thesis with a list of potential research directions that could be explored to enhance or extend the results presented in each chapter.

- Related to the notions defined in **Chapter 2** it would be of interest to study the compactness of the new subclasses E_k and E_k^* (see Question 2.2.35), the relation between two consecutive families, i.e. $E_{k+1} \subset E_k$, for all $k \in \mathbb{N}$ (see Question 2.2.34), radius of starlikeness, respectively convexity for these classes and extensions of presented results in the n-dimensional case and complex Banach spaces.

- In **Chapter 3** we discussed about convex combinations of biholomorphic mappings (with real coefficients) in several complex variables. A natural extension of our results is to consider the problem of convex combinations with complex coefficients.

  An interesting problem related to the convex combinations of two Graham-Kohr type extension operators is the preservation of the notion of parametric representation under the operators $\mathcal{K}_\lambda^\beta$, respectively $\mathcal{K}_\lambda^{\alpha,\beta}$, for all $0 \leq \alpha < \beta \leq 1$ and $\lambda \in (0,1)$. As a consequence of this result, we can study the preservation of the starlikeness under the same extension operators.

  Obviously, although they are not studied in this thesis, the operators $\mathcal{K}_\lambda^\beta$ and $\mathcal{K}_\lambda^{\alpha,\beta}$ can also be considered in infinite dimensional case.

- **Chapter 4** is dedicated to study of families of biholomorphic mappings that generalize the subclasses introduced in Chapter 2. It would be of interest to study properties of the subclasses $E_k(\mathbb{B}^n)$ and $E_k^*(\mathbb{B}^n)$ in $\mathbb{C}^n$ and complex Banach spaces. Beside these, the problem of preservation of the class $K = E_1^*$ through the Graham-Kohr extension operator remains an open problem, as can be seen in Question 4.3.7 (the n-dimensional case) and Question 4.3.8 (the infinite dimensional case). Surely, other important extension operators (e.g. the generalized Roper-Suffridge, the Pfaltzgraff-Suffridge, the Muir extension operator) can be considered for the same problem.

  In view of the results obtained by Hamada, Iancu and Kohr in [65] and [66] (see also [59], [64]) we can also address problems of approximation and density for our new subclasses of biholomorphic mappings in $\mathbb{C}^n$.

- Taking into account the results obtained in §6.2 from **Chapter 6**, it would be of interest to consider extension results under the Pfaltzgraff-Suffridge operator on the domain $\Omega_{n,p,r}$ for $r \geq 1$ and $p > 2$.

  Another problem related to the Pfaltzgraff-Suffridge extension operator is the preservation of g-parametric representation from $\mathbb{B}_X$ to $\mathbb{D}_\alpha$ (see e.g. [40]). In this case, $\mathbb{B}_X$ is the open unit ball of an n-dimensional JB*-triple X, $\mathbb{B}_Y$ is the open unit ball of a complex Banach space Y and $\mathbb{D}_\alpha \subseteq \mathbb{B}_X \times \mathbb{B}_Y$ is a domain such that $\mathbb{B}_X \times \{0\} \subset \mathbb{D}_\alpha$ for $\alpha > 0$. Extension results related to the Pfaltzgraff-Suffridge type extension operator in this abstract setting were obtained by Graham, Hamada, Kohr and Kohr in [40] (see also [132], [134]).

- Together with the operators introduced by Roper and Suffridge, Graham and Kohr, Pfaltzgraff and Suffridge, respectively generalizations of these operators, we can also study the Muir extension operator $\Phi_{n,Q}$. The operator $\Phi_{n,Q}$ was introduced by Muir Jr. (see [103]) as a different generalization of the Roper-Suffridge extension operator. Here, $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 and $\Phi_{n,Q} : \mathcal{LS} \to \mathcal{LS}_n(\mathbb{B}^n)$ is given by

$$\Phi_{n,Q}(f)(z) = \Big(f(z_1) + Q(\tilde{z})f'(z_1), \tilde{z}\sqrt{f'(z_1)}\Big), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where the branch of the square root is considered such that $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$. The Muir extension operator and its generalizations have been intensively studied by several authors (see e.g. [11], [39], [76], [84], [98], [139]) in finite and infinite dimensions. An interesting problem is to study the preservation of the notion of g-parametric representation by the generalized Muir extension operator (defined by Graham, Hamada, Kohr and Kohr in [39])

$$\Phi_{P_k}(f)(z) = \Big(f(x) + P_k(y)f'(x), \big(f'(x)\big)^{\frac{1}{k}}y\Big), \quad z = (x, y) \in \Omega_{p,k},$$

where $P_k : Y \to \mathbb{C}$ is a homogeneous polynomial of degree $k \geq 2$, on the domain $\Omega_{p,k} = \big\{(x, y) \in \mathbb{C} \times Y : |x|^p + \|y\|_Y^k < 1\big\}$, where $p \geq 1$, $k \geq 2$ and Y is a complex Banach space. Note that for $p = 2$ it has been proved in [39] that Φ_{P_k} preserve g-parametric representation and Bloch functions, where g satisfies Assumption 5.3.6.

Recently, Muir (see [104], [105]) considered a new direction in the theory of Loewner chains. He studied Loewner chains that have a locally uniform L^p-continuity property in t. This type of mappings were considered by Muir to construct a new concept of spirallikeness, related to a locally integrable operator-valued function on $[0, \infty)$. In this setting it would be interesting to construct this type of mappings using other extension operators, i.e. the Graham-Kohr extension operator or the generalized Roper-Suffridge type extension operator.

- An important tool to generate extension operators is the semigroup theory studied by Elin (see e.g. [21]). This new approach can be used also in the case of the extension operators listed above.
- Another strong development related to Loewner chains is presented by Arosio, Bracci, Hamada and Kohr in [3]. They considered Loewner chains in the setting of complete hyperbolic complex manifolds and generate a one-to-one equivalence between L^d-Loewner chains and L^d-evolution families. Moreover, they used the Roper-Suffridge extension operator to construct L^d-Loewner chains. A similar study can be considered using another extension operators (e.g. the Graham-Kohr extension operator).
- The most recent approach to the Loewner theory was introduced by Hamada and Kohr in [72]. They studied a new concept, namely the inverse Loewner chain, in infinite dimensional case. Their important work represents a new way of studying the results related to the Loewner chains and extension operators (e.g. the Graham-Kohr extension operator).

Bibliography - selective list

[1] Alexander J.W., *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. **17** (1915), 12–22.

[2] Arosio L., *Resonances in Loewner equations*, Adv. Math. **227** (2011), 1413–1435.

[3] Arosio L., Bracci F., Hamada H., Kohr G., *An abstract approach to Loewner's chains*, J. Anal. Math. **119** (2013), 89–114.

[4] Bieberbach L., *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Preuss. Akad. Wiss. Sitzungsb. **138** (1916), 940–955.

[5] Bracci F., Contreras M.D., Diaz-Madrigal S., *Evolution families and the Loewner equation I: the unit disk*, J. Reine Angew. Math. **672** (2012), 1–37.

[6] Bracci F., Elin M., Shoikhet D., *Growth estimates for pseudo-dissipative holomorphic maps in Banach spaces*, J. Nonlinear Convex Anal. **15** (2014), 191–198.

[7] Cartan H., *Sur la posibilité d'étendre aux fonctions de plusiers varibales complexes la théorie des fonctions univalentes*, Note added to P. Montel, Leçons sur les Fonctions Univalentes ou Multivalentes, 129–155, Gauthier-Villars, Paris, 1933.

[8] Chabat B., *Introduction á l'Analyse Complexe*, I-II, Ed. MIR, Moscow, 1990.

[9] Chichra P., Singh R., *Convex sum of univalent functions*, J. Austral. Math. Soc. **14** (1972), 503–507.

[10] Chirilă T., *An extension operator and Loewner chains on the Euclidean unit ball in* $\mathbb{C}^n$, Mathematica (Cluj) **54**(77) (2012), 116–125.

[11] Chirilă T., *Analytic and geometric properties associated with some extension operators*, Complex Var. Elliptic Equ. **59**(3) (2014), 427–442.

[12] Conway J.B., *Functions of One Complex Variable II*, Springer-Verlag, New-York, 1995.

[13] Curt P., *A Marx-Strohhäcker theorem in several complex variables*, Mathematica (Cluj) **39**(62) (1997), 59–70.

[14] Curt P., *Capitole Speciale de Teoria Geometrică a Funcţiilor de mai multe Variabile Complexe*, Editura Albastră, Cluj-Napoca, 2001.

[15] Curt P., Kohr G., *Subordination chains and Loewner differential equation in several complex variables*, Ann. Univ. Mariae Curie-Skl. Sect. A **57** (2003), 35–43.

[16] Cristea M., *Univalence criteria starting from the method of Loewner chains*, Complex. Anal. Oper. Theory **5**(3) (2011), 863–880.

[17] Cristea M., *The method of Loewner chains in the study of univalence of* C^2 *mappings*, Mathematica (Cluj) **55**(78) (2013), 22–38.

[18] de Branges L., *A proof of the Bieberbach conjecture*, Acta. Math. **154**(1-2) (1985), 137–152.

[19] Duren P.L., *Univalent Functions*, Springer-Verlag, New York, 1983.

[20] Duren P.L., Graham I., Hamada H., Kohr G., *Solutions for the generalized Loewner differential equation in several complex variables*, Math. Ann. **347** (2010), 411–435.

[21] Elin M., *Extension operators via semigroups*, J. Math. Anal. Appl. **377** (2011), 239–250.

[22] Feng S.X., *Some classes of holomorphic mappings in several complex variables*, University of Science and Technology of China, Doctoral Thesis, 2004.

[23] Goluzim G.M., *Geometric Theory of Functions of a Complex Variable*, Moscow, 1952.

[24] Gong S., *Convex and Starlike Mappings in Several Complex Variables*, Kluwer Acad. Publ., Dordrecht, 1998.

[25] Gong S., Liu T.S., *On the Roper-Suffridge extension operator*, J. Anal. Math. **88** (2002), 397–404.

[26] Gong S., Liu T.S., *Criterion for the family of ε starlike mappings*, J. Math. Anal. Appl. **274** (2002), 696–704.

[27] Gong S., Liu T.S., *The generalized Roper-Suffridge extension operator*, J. Math. Anal. Appl. **284** (2003), 425–434.

[28] Gong S., Wang S., Yu Q., *Biholomorphic convex mappings of ball in $\mathbb{C}^n$*, Pacif. J. Math. **161** (1993), 287–306.

[29] Goodman A.W., *Univalent Functions*, Mariner Publ. Comp., Tampa, Florida, 1984.

[30] Graham I., *Growth and covering theorems associated with the Roper-Suffridge extension operator*, Proc. Amer. Math. Soc. **127** (1999), 3215–3220.

[31] Graham I., Hamada H., Honda T., Kohr G., Shon K.H., *Growth, distortion and coefficient bounds for Carathéodory families in $\mathbb{C}^n$ and complex Banach spaces*, J. Math. Anal. Appl. **416** (2014), 449–469.

[32] Graham I., Hamada H., Kohr G., *Parametric representation of univalent mappings in several complex variables*, Canadian J. Math. **54** (2002), 324–351.

[33] Graham I., Hamada H., Kohr G., *Extension operators and subordination chains*, J. Math. Anal. Appl. **386** (2012), 278–289.

[34] Graham I., Hamada H., Kohr G., *Extremal problems and g-Loewner chains in $\mathbb{C}^n$ and reflexive complex Banach spaces*, In: Topics in Mathematical Analysis and Applications (eds. T.M. Rassias and L. Toth), Springer **94** (2014), 387–418.

[35] Graham I., Hamada H., Kohr G., *Loewner chains, Bloch mappings and Pfaltzgraff-Suffridge extension operators on bounded symmetric domains*, Complex Var. Elliptic Equ. **65** (2020), 57–73.

[36] Graham I., Hamada H., Kohr G., Kohr M., *Parametric representation and asymptotic starlikeness in $\mathbb{C}^n$*, Proc. Amer. Math. Soc. **136** (2008), 3963–3973.

[37] Graham I., Hamada H., Kohr G., Kohr M., *Spirallike mappings and univalent subordination chains in $\mathbb{C}^n$*, Ann. Scuola Norm. Sup. Pissa Cl. Sci. (5) **7**(4) (2008), 717–740.

[38] Graham I., Hamada H., Kohr G., Kohr M., *Univalent subordination chains in reflexive complex Banach spaces*, Contemp. Math. (AMS) **591** (2013), 83–111.

[39] Graham I., Hamada H., Kohr G., Kohr M., *g-Loewner chains, Bloch functions and extension operators in complex Banach spaces*, Anal. Math. Phys. **10**(1) (2020), Art. 5, 28 pp.

[40] Graham I., Hamada H., Kohr G., Kohr M., *g-Loewner chains, Bloch functions and extension operators into the family of locally biholomorphic mappings in infinite dimensional spaces*, Stud. Univ. Babeş-Bolyai Math. **67**(2) (2022), 219–236.

[41] Graham I., Hamada H., Kohr G., Kohr M., *Loewner PDE in Infinite Dimensions*, Comput. Methods Funct. Theory (2024). `https://doi.org/10.1007/s40315-024-00536-5`

[42] Graham I., Hamada H., Kohr G., Suffridge T.J., *Extension Operators for Locally Univalent Mappings*, Michigan Math. **50** (2002), 37–55.

[43] Graham I., Kohr G., *Univalent mappings associated with the Roper-Suffridge extension operator*, J. Anal. Math. **81** (2000), 331–342.

[44] Graham I., Kohr G., *An Extension Theorem and Subclasses of Univalent Mappings in Several Complex Variables*, Complex Variables Theory Appl. **47** (2002), 59–72.

[45] Graham I., Kohr G., *Geometric Function Theory in One and Higher Dimensions*, Marcel Deker Inc., New York, 2003.

[46] Graham I., Kohr G., *The Roper-Suffridge extension operator and classes of biholomorphic mappings*, Sci. China Ser. A **49** (2006), 1539–1552.

[47] Graham I., Kohr G., Kohr M., *Loewner chains and the Roper-Suffridge Extension Operator*, J. Math. Anal. Appl. **247** (2000), 448–465.

[48] Graham I., Kohr G., Kohr M., *Loewner chains and parametric representation in several complex variables*, J. Math. Anal. Appl. **281** (2003), 425–438.

[49] Graham I., Kohr G., Pfaltzgarff J.A., *Parametric representation and linear functionals associated with extension operators for biholomorphic mappings*, Rev. Roum. Math. Pures Appl. **52** (2007), 47–68.

[50] **Grigoriciuc E.S.**, *On some classes of holomorphic functions whose derivatives have positive real part*, Stud. Univ. Babeş-Bolyai Math. **66**(3) (2021), 479–490.

[51] **Grigoriciuc E.S.**, *Some general distortion results for $K(\alpha)$ and $S^*(\alpha)$*, Mathematica **64**(87) (2022), 222–232.

[52] **Grigoriciuc E.S.**, *On some convex combinations of biholomorphic mappings in several complex variables*, Filomat **36**(16) (2022), 5503–5519.

[53] **Grigoriciuc E.S.**, *New subclasses of univalent mappings in several complex cariables: Extension operators and applications*, Comput. Methods Funct. Theory **23**(3) (2023), 533–555.

[54] **Grigoriciuc E.S.**, *New subclasses of univalent functions on the unit disc in $\mathbb{C}$*, Stud. Univ. Babeş-Bolyai Math. **69**(4) (2024), 769–787.

[55] **Grigoriciuc E.S.**, *g-Loewner chains and the Graham-Kohr extension operator in complex Banach spaces*, Comput. Methods Funct. Theory, accepted

[56] Gronwall T., *Some remarks on conformal representation*, Ann. Math. **16** (1914-1915), 72–76.

[57] Gurganus K., *Φ-like holomorphic functions in $\mathbb{C}^n$ and Banach spaces*, Trans. Amer. Math. Soc. **205** (1975), 389–406.

[58] Hallenbeck D.J., MacGregor T.H., *Linear Problems and Convexity Techniques In Geometric Function Theory*, Pitman, Boston, 1984.

[59] Hamada H., *Approximation properties on spirallike domains of $\mathbb{C}^n$*, Adv. Math. **268** (2015), 467–477.

[60] Hamada H., Honda T., *Sharp growth theorems and coefficient bounds for starlike mappings in several complex variables*, Chin. Ann. Math. Ser. B. **29** (2008), 353–368.

[61] Hamada H., Honda T., Kohr G., *Growth theorems and coefficient bounds for univalent holomorphic mappings which have parametric representation*, J. Math. Anal. Appl. **317** (2006), 302–319.

[62] Hamada H., Honda T., Kohr G., *Parabolic starlike mappings in several complex variables*, Manuscripta Math. **123** (2007), 301–324.

[63] Hamada H., Honda T., Kohr G., *Trace-order and a distortion theorem for linearly invariant families on the unit ball of a finite dimensional JB*-triple*, J. Math. Anal. Appl. **396** (2012), 829–843.

[64] Hamada H., Iancu M., Kohr G., Schleißinger S., *Approximation properties of univalent mappings on the unit ball in $\mathbb{C}^n$*, J. Approx. Theory **226** (2018), 14–33.

[65] Hamada H., Iancu M., Kohr G., *Approximation of univalent mappings by automorphisms and quasiconformal diffeomorphisms in $\mathbb{C}^n$*, J. Approx. Theory **240** (2019), 129–144.

[66] Hamada H., Iancu M., Kohr G., *A survey on Loewner chains, approximation results, and related problems for univalent mappings on the unit ball in $\mathbb{C}^n$*, Rev. Roumaine Math. Pures Appl. **66** (2021), 709–723.

[67] Hamada H., Kohr G., *Subordination chains and the growth theorem of spirallike mappings*, Mathematica (Cluj) **42**(65) (2000), 153–161.

[68] Hamada H., Kohr G., *Φ-like and convex mappings in infinite dimensional spaces*, Rev. Roum. Math. Pures Appl. **47** (2002), 315–328.

[69] Hamada H., Kohr G., *Roper-Suffridge extension operator and the lower bound for the distortion*, J. Math. Anal. Appl. **300** (2004), 454–463.

[70] Hamada H., Kohr G., *Loewner chains and the Loewner differential equation in reflexive complex Banach spaces*, Rev. Roumanie Math. Pures Appl. **49** (2004), 247–264.

[71] Hamada H., Kohr G., *Quasiconformal extension of biholomorphic mappings in several complex variables*, J. Anal. Math. **96** (2005), 269–282.

[72] Hamada H., Kohr G., *The Loewner PDE, inverse Loewner chains and nonlinear resolvents of the Carathéodory family in infinite dimensions*, Ann. Scuola Norm. Sup. Pissa Cl. Sci. (5) **24**(4) (2023), 2431–2475.

[73] Hamada H., Kohr G., Kohr M., *Parametric representation and extension operators for biholomorphic mappings on some Reinhardt domains*, Complex Var. **50** (2005), 507–519.

[74] Hamada H., Kohr G., Kohr M., *The Fekete-Szegö problem for starlike mappings and nonlinear resolvents of the Carathéodory family on the unit balls of complex Banach spaces*, Anal. Math. Phys. **11**(3) (2021), 115–137.

[75] Hamada H., Kohr G., Liczberski P., *Starlike mappings of order α on the unit ball in complex Banach spaces*, Glasnik Matematiki **36**(56) (2001), 39–48.

[76] Hamada H., Kohr G., Muir Jr. J.R., *Extensions of L^d-Loewner chains to higher dimensions*, J. Anal. Math. **120** (2013), 357–392.

[77] Hamburg P., Mocanu P.T., Negoescu N., *Analiză Matematică (Funcţii Complexe)*, Editura Didactică şi Pedagogică, Bucureşti, 1982 (in romanian).

[78] Harris L.A., *Schwarz's lemma in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **62** (1969), 1014–1017.

[79] Hille E., Phillips R.S., *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ. **31**, Providence, R.I., 1957.

[80] Kikuchi K., *Starlike and convex mappings in several complex varibales*, Pacif. J. Math. **44** (1973), 569–580.

[81] Kohr G., *Certain partial differential inequalities and applications for holomorphic mappings defined on the unit ball of* $\mathbb{C}^n$, Ann. Univ. Mariae Curie-Skl. Sect. A **50** (1996), 87–94.

[82] Kohr G., *Using the method of Löwner chains to introduce some subclasses of biholomorphic mappings in* $\mathbb{C}^n$, Rev. Roum. Math. Pures Appl. **46** (2001), 743–760.

[83] Kohr G., *Basic Topics in Holomorphic Functions of Several Complex Variables*, Presa Universitară Clujeană, Cluj-Napoca, 2003.

[84] Kohr G., *Loewner chains and a modification of the Roper-Suffridge extension operator*, Mathematica (Cluj) **48**(71) (2006), 41–48.

[85] Kohr G., Mocanu P.T., *Capitole speciale de Analiză Complexă*, Presa Universitară Clujeană, Cluj-Napoca, 2003 (in romanian).

[86] Kubicka E., Poreda T., *On the parametric representation of starlike maps of the unit ball in* $\mathbb{C}^n$ *into* $\mathbb{C}^n$, Demonstratio Math. **21** (1988), 345–355.

[87] Krantz S.G., *Handbook of Complex Variables*, Birkhäuser Verlag, Boston, 1999.

[88] Krantz S.G., *Function Theory of Several Complex Variables*, Reprint of the 1992 Edition, AMS Chelsea Publishing, Providence, 2001.

[89] Krishna D.V., RamReddy T., *Coefficient inequality for a function whose derivative has a positive real part of order* α, Math. Bohem. **140** (2015), 43–52.

[90] Krishna D.V., Venkateswarlu B., RamReddy T., *Third Hankel determinant for bounded turning functions of order alpha*, J. Nigerian Math. Soc. **34** (2015), 121–127.

[91] Liu T., *The growth theorems and covering theorems for biholomorphic mappings on classical domains*, University of Science and Technology of China, Doctoral Thesis, 1989.

[92] Liu X., *The generalized Roper-Suffridge extension operator for some biholomorphic mappings*, J. Math. Anal. Appl. **324**(1) (2006), 604–614.

[93] Liu X., Liu T., *The generalized Roper-Suffridge extension operator for locally biholomorphic mappings*, Chin. Quart. J. Math. **18** (2003), 221–229.

[94] Liu X.S., Liu T.S., *The generalized Roper-Suffridge extension operator for spirallike mappings of type* β *and order* α, Chin. Ann. Math. Ser. A **27** (2006), 789–798.

[95] Loewner K., *Untersuchungen über schlichte Abbildungen des Einheritskkreises*, Math. Ann. **89** (1923), 103–121.

[96] MacGregor T.H., *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. **104** (1962), 532–537.

[97] MacGregor T.H., *The univalence of a linear combination of convex mappings*, J. London Math. Soc. **44** (1969), 210–212.

[98] Manu A., *Contributions in the theory of univalent functions of one and several complex variables*, Babeş-Bolyai University of Cluj-Napoca, Doctoral Thesis, 2022.

[99] Matsuno T., *Star-like theorems and convex-like theorems in the complex vector space*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A **5** (1955), 88–95.

[100] Merkes E.P., *On the convex sum of certain univalent functions and the identity function*, Rev. Colombiana Math. **21** (1987), 5–12.

[101] Merkes E.P., Robertson M.S., Scott W.T., *On products of starlike functions*, Proc. Amer. Math. Soc. **13** (1962), 960–964.

[102] Mocanu P.T., Bulboacă T., Sălăgean G.Ş., *Teoria geometrică a funcţiilor univalente*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006 (in romanian).

[103] Muir Jr. J.R., *A modification of the Roper-Suffridge extension operator*, Comput. Methods Funct. Theory **5** (2005), 237–251.

[104] Muir Jr. J.R., *Extensions of Abstract Loewner Chains and Spirallikeness*, J. Geom. Anal. **32**(7) (2022), no. 192, 46 pp.

[105] Muir Jr. J.R., *Generalized parametric representation and extension operators*, J. Math. Anal. Appl. **531** (2024), 127767.

[106] Mujica J., *Complex Analysis in Banach Spaces. Holomorphic Functions and Domains of Holomorphy in Finite and Infinite Dimensions*, North-Holland, Amsterdam, 1986.

[107] Narasimhan R., *Several Complex Variables*, The University of Chicago Press, Chicago, 1971.

[108] Noshiro K., *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Univ. **2** (1934-1935), 129–155.

[109] Pfaltzgraff J.A., *Subordination chains and univalence of holomorphic mappings in* $\mathbb{C}^n$, Math. Ann. **210** (1974), 55–68.

[110] Pfaltzgraff J.A., Suffridge T.J., *Close-to-starlike holomorphic functions of several complex variables*, Pacif. J. Math. **57** (1975), 271–279.

[111] Pfaltzgraff J.A., Suffridge T.J., *An extension theorem and linear invariant families generated by starlike maps*, Ann. Univ. Mariae Curie-Skl. Sect. A **53** (1999), 193–207.

[112] Poincaré H., *Les fonctions analytiques de deux variables et la représentation conforme*, Rend. Circ. Mat. Palermo **23** (1907), 185–220.

[113] Pommereneke C., *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math. **218** (1965), 159–173.

[114] Pommerenke C., *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[115] Poreda T., *On the univalent holomorphic maps of the unit polydisc in* $\mathbb{C}^n$ *which have the parametric representation, I - the geometrical properties*, Ann. Univ. Mariae Curie-Skl. Sect. A **41** (1987), 105–113.

[116] Poreda T., *On the univalent holomorphic maps of the unit polydisc in* $\mathbb{C}^n$ *which have the parametric representation, II - necessary and sufficient conditions*, Ann. Univ. Mariae Curie-Skl. Sect. A **41** (1987), 114–121.

[117] Poreda T., *On the univalent subordination chains of holomorphic mappings in Banach spaces*, Comment. Math. Prace Mat. **28** (1989), 295–304.

[118] Poreda T., *On generalized differential equations in Banach spaces*, Dissertationes Math. **310** (1991), 1–50.

[119] Range M., *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer-Verlag, New-York, 1986.

[120] Robertson M.S., *On the theory of univalent functions*, Ann. Math. **37** (1936), 374–408.

[121] Roper K., Suffridge T.J., *Convex mappings on the unit ball in* $\mathbb{C}^n$, J. Analyse Math. **65** (1995), 333–347.

[122] Roper K., Suffridge T.J., *Convexity properties of holomorphic mappings in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **351** (1999), 1803–1833.

[123] Rudin W., *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New-York, 1980.

[124] Sălăgean G.Ş., *Subclasses of univalent functions*, Lecture Notes in Math., Springer-Verlag, Berlin, **1013** (1983), 362–372.

[125] Špaček L., *Contribution á la theorie des functions univalentes*, Časopia Pěst. Mat. **62** (1932), 12–19.

[126] Suffridge T.J., *The principle of subordination applied to functions of several complex variables*, Pacif. J. Math. **33** (1970), 241–248.

[127] Suffridge T.J., *Starlike and convex maps in Banach spaces*, Pacif. J. Math. **46** (1973), 575–589.

[128] Suffridge T.J., *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, Lecture Notes in Math., Springer-Verlag, New York, **599** (1976), 146–159.

[129] Thomas D.K., *On functions whose derivative has positive real part*, Proc. Amer. Math. Soc. **98** (1986), 68–70.

[130] Vodă M., *Solution of a Loewner chain equation in several complex variables*, J. Math. Anal. Appl. **375** (2011), 58–74.

[131] Wang J., *Extension operators and support points associated with g-Loewner chains on complex Banach spaces*, J. Math. Anal. Appl. **527**(1) (2023), 127411.

[132] Wang J., Liu T., Zhang Y., *Geometric and Analytic Properties Associated With Extension Operators*, J. Geom. Anal. **34** (2024), no. 156, 40 pp.

[133] Wang J., Wang J., *Generalized Roper-Suffridge operator for* ϵ *starlike and boundary starlike mappings*, Acta Math. Sci. Ser. B **40**(6) (2020), 1753–1764.

[134] Wang J., Zhang X., *The subordination principle and its application to the generalized Roper-Suffridge extension operator*, Acta Math. Sci. Ser. B **42** (2022), 611–622.

[135] Warschawksi S.E., *On the higher derivatives at the boundary in conformal mapping*, Trans. Amer. Math. Soc. **38** (1935), 310–340.

[136] Wolff J., *L'intégrale d'une fonction holomorphe et à partie réelle positive dans un demiplan est univalente*, C.R. Acad. Sci. Paris **198** (1934), 1209-1210.

[137] Xu Q.H., Liu T.S., *Löwner chains and a subclass of biholomorphic mappings*, J. Math. Appl. **334** (2007), 1096–1105.

[138] Zhang X., Feng S., Li Y., *Loewner Chain Associated with the Modified Roper-Suffridge Extension Operator*, Comput. Methods Funct. Theory. **16**(2) (2016), 265–281.

[139] Zhang X., Tang Y., *g-Loewner chain associated with Muir-type extension operator*, Complex Var. Elliptic Equ. **69**(4) (2024), 641–654.