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METHODS FOR APPROXIMATING MULTIVARIATE FUNCTIONS

Ph. D. Thesis Summary

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Contents

List of publications and attended conferences

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3. T. Cătinaş, **A. Malina**, *Spherical interpolation of scattered data using least squares thin-plate spline and inverse multiquadric functions*, Numer. Algor. **97** (2024), pp. 1397–1414, DOI: `10.1007/s11075-024-01755-6`, **WoS-SCIE**, **IF** (June 2024): **1.7** (Q2 quartile).

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6. T. Cătinaş, **A. Malina**, *The combined Shepard operator of inverse quadratic and inverse multiquadric type*, Stud. Univ. Babeş-Bolyai Math. **67** (2022), pp. 579—589, DOI: `10.24193/subbmath.2022.3.09`, **WoS-ESCI**, **IF** (June 2024): **0.3**.

7. T. Cătinaş, **A. Malina**, *Shepard operator of least squares thin-plate spline type*, Stud. Univ. Babeş-Bolyai Math. **66** (2021), pp. 257–265, DOI: `10.24193/subbmath.2021.2.02`, **WoS-ESCI**, **IF** (June 2024): **0.3**.

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2. *26th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)*, *Timișoara* (16-19.09.2024), "An application of the Shepard operator in image reconstruction".

3. *Le 16ème Colloque Franco-Roumain, Bucharest* (26-30.08.2024), "Spherical interpolation of scattered data using least squares thin-plate spline and inverse multiquadric functions".

4. *Sesiunea de Comunicări Ştiinţifice ale Studenţilor – Matematică, Facultatea de Matematică și Informatică, UBB Cluj-Napoca* (25.05.2024), "Spherical interpolation of scattered data using least squares thin-plate spline and inverse multiquadric functions".

5. *Numerical Analysis, Numerical Modeling, Approximation Theory (NA-NM-AT), Zilele Academice Clujene, Cluj-Napoca* (06-09.11.2023), "Combined Shepard operators applied in image reconstruction".

6. *9th International Conference on Mathematics and Informatics (MathInfo), Târgu Mureș* (07-08.09.2023), "An application of the Shepard operator in image reconstruction".

7. *14th Joint Conference on Mathematics and Computer Science (MaCS), Cluj-Napoca* (24-27.11.2022), "New Shepard operators in the univariate case".

8. *Numerical Analysis, Numerical Modeling, Approximation Theory (NA-NM-AT), Zilele Academice Clujene, Cluj-Napoca* (26-28.10.2022), "New Shepard operators in the univariate case".

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Introduction

This thesis aims to introduce and study new interpolation operators for three common cases: univariate, bivariate and spherical. Interpolation is an important branch of Approximation Theory, widely used in practical applications where reconstructing a surface from some known data is necessary. This data is usually referred to as scattered due to its lack of a regular structure.

This kind of problem can be formulated as follows. Given some known data consisting of data sites $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}^n,\ i = 1, \ldots, K\}$ and data values $f_i = f(\mathbf{x}_i) \in \mathbb{R},\ i = 1, \ldots, K$, find $s : \mathbb{R}^n \to \mathbb{R}$ such that it either interpolates the given data, i.e., $s(\mathbf{x}_i) = f(\mathbf{x}_i)$, or it approximates it, i.e., $s(\mathbf{x}_i) \approx f(\mathbf{x}_i)$.

The applications of this type of data problem come from many fields, such as mathematics, computer science, biology, geology or engineering, a reason why scattered data approximation became a fast-growing and important research subject. Some concrete examples of these applications mentioned by M. D. Buhmann [7], G. E. Fasshauer [44] and H. Wendland [93] include surface reconstruction (archaeology artifacts, sculptures, machine parts), image restoration and inpainting, terrain modeling, measurements for physical phenomena (pressure, ground or sea level temperature, rainfall, gravitational forces), modeling closed surfaces in CAGD, solutions to partial differential equations and geophysical problems (topography, magnetic field intensity, gravitational potential), kernel approximation and support vector machines in neural networks or data mining.

One of the main tools used in scattered data interpolation is the operator introduced by D. Shepard in 1968, in [82], which is considered one of the best-suited methods for approximating large datasets. In its original form, it was obtained as a combination of the available data values and global weights constructed using the distance between the data sites, involving a positive control parameter. Despite its advantages, such as small storage requirements and easy generalization to additional independent variables, it suffers from poor reproduction quality, low accuracy and a high computational cost. Due to these drawbacks, many authors proposed several alternatives to improve its efficiency and to obtain an increased accuracy. Some pioneers with fundamental results in this study include R. Franke and G. Nielson (see, e.g., [46], [47]), R. J. Renka and A. K. Cline (see, e.g., [72], [74], [76], [77]), R. Farwig (see, e.g., [43]), R. E. Barnhill, R. P. Dube and F. F. Little (see, e.g., [4]). The Approximation School in Cluj achieved significant advancements in the study of this method, in both univariate and bivariate cases. Some of the authors that worked extensively in this field are Gh. Coman, T. Cătinaş and R.

Trîmbiţaş (see, e.g., [10]–[13], [24]–[31], [90]). Another School with substantial contributions is the Italian one. In the univariate and bivariate cases, we mention R. Caira, F. Dell'Accio and F. Di Tomasso (see, e.g. [8], [9], [37]–[39]), while for the spherical case, many improvements were developed by G. Allasia, R. Cavoretto and A. De Rossi (see, e.g., [2], [18]–[21], [35], [36]).

The thesis is organized into four chapters, each containing several sections and subsections.

Chapter 1 primarily focuses on an overview of existing results and notions concerning the Shepard method. Section 1.1 introduces key notations and fundamental results essential throughout this thesis. Section 1.2 briefly discusses radial basis functions, another important tool in approximating scattered data, which will be used to improve the Shepard operator. Section 1.3 presents fundamental results for refined variants of the Shepard operator in the univariate case obtained from its combination with various interpolation polynomials, such as Lagrange [29], Taylor [29], Hermite [29], Birkhoff [29, 31], Abel-Goncharov [12] and Bernoulli [8]. Section 1.4 revisits important contributions in the theory of the bivariate Shepard method. We detail here the local variant of this method which ensures that data points farther away from the approximation point have less influence, obtaining in this manner an improved accuracy compared to the initial method, which is global. This local approach was initially proposed by R. Franke and G. Nielson in [47] and further developed by R. Franke in [46] and R. J. Renka in [76]. As in the previous case, to increase the degree of exactness and obtain smaller approximation errors, several polynomial combinations were proposed, such as Lagrange [30], Taylor [28], Hermite [24], Birkhoff [25], complete Hermite-Birkhoff [38], Lidstone [9], [13] and Bernoulli [11], [37]. Finally, we discuss the spherical Shepard operator in Section 1.5. We first recall some results related to radial basis functions on the sphere (see, e.g. [5], [54], [58], [84]), with the purpose of presenting spherical Shepard-like methods combined with radial basis functions (see, e.g., [18]–[21], [35], [36]). The case involving available data about the function's partial derivatives is addressed in [2], where the Shepard operator of Hermite-Birkhoff-type is studied.

The aim of Chapter 2 is to introduce a new type of Shepard operator in the univariate case, obtained using polynomials constructed based on the weighted least squares method. In Section 2.1, we detail how we constructed these polynomials and study afterward some of their properties. In Theorem 2.1.1, we prove that these polynomials interpolate the function on the set of given data sites. Theorem 2.1.2 shows that the degree of exactness of these operators coincides with the degree of the polynomial used. In the last result of this section, Theorem 2.1.3, we prove the operators' linearity. Section 2.2 focuses on the combination between the Shepard operator and the polynomials previously introduced. In Theorems 2.2.2, 2.2.3 and 2.2.4, respectively, we show that the new Shepard operator inherits the properties regarding the interpolation on the set of given nodes, the degree of exactness and linearity, respectively. Based on Peano's Theorem, we study the remainder of the interpolation formula in Theorem 2.2.5. We end this chapter with some numerical examples performed on two sets of nodes (equidistant and Chebyshev type) and four test functions, comparing our results with the ones obtained using other Shepard operators of different types, such as Lagrange, Taylor and Bernoulli. All the results presented in this chapter are original and they were published in the paper A. Malina

[62].

The goal of Chapter 3 is to present new combined Shepard operators in the two-dimensional case, using three radial basis functions: thin-plate spline, inverse quadratic and inverse multiquadric. The methods proposed aim to improve the original one, due to the proven efficacy of radial basis functions in both practical and theoretical contexts. Besides the classical (global) and the modified (local) forms of the operator, we also use an iterative method that is comparable to the latter one, as proposed by A. V. Masjukov and V. V. Masjukov in [63]. To enhance the accuracy of the methods, we follow some ideas of J. R. McMahon (see, e.g., [64], [65]) and compute some sets of representative knot points for the initial set of nodes. The use of these sets reduces the computational cost and also increases the accuracy of the methods, as shown by several numerical examples on three test functions. The steps are outlined in Algorithm 1. The results obtained using the least squares thin-plate spline, presented in Section 3.1, were structured and published in two articles: T. Cătinaş and A. Malina [14] and A. Malina [61], while the ones achieved in the cases of inverse quadratic and inverse multiquadric were published in the paper T. Cătinaş and A. Malina [17]. Finally, Section 3.3 deals with a practical application of the Shepard operators previously mentioned in image processing, namely the reconstruction of both black-and-white (Subsection 3.3.1) and color (Subsection 3.3.2) images. In our approach, we detail two methods, a global (Algorithm 2) and a local one (Algorithm 3), performing several experiments on three images with different resolutions. The need for image reconstruction may arise in cases such as inpainting or noise reduction. Since we focus on restoration and not on detecting the damage, we consider that the area needed to be reconstructed is known. The results presented in this section are included in the paper A. Malina [60].

Chapter 4 is dedicated to spherical interpolation of scattered data using some new combined Shepard operators. As emphasized in [19] and [84], these types of data fitting problems, where the underlying domain is the sphere, arise in many areas, as, in general, the sphere is taken as a model of the Earth. Section 4.1 presents new local Shepard operators combined with the least squares-thin plate spline and the inverse multiquadric functions. Combined polynomial and radial basis function approximations have often been studied in the context of radial basis functions constructed from conditionally positive definite kernels, in which case a polynomial part is needed to make the theory work. Here, we restrict our attention to the case of (conditionally) strictly positive definite kernels, considering also the inclusion of a polynomial component, motivated by the fact that approximations of this kind offer real advantages. In Theorem 4.1.3 we study the interpolation error, while in Theorem 4.1.4, we prove that our operators are of class C^1. Finally, in Theorem 4.1.5, we provide an error bound based on the modulus of continuity. The new results obtained in this section are contained in the paper T. Cătinaş and A. Malina [15]. In Section 4.2, we derive a second Shepard-type method for the spherical case using the Bernoulli operator, suitable when information about the function's derivatives is known. After performing the Delaunay triangulation of the sphere, we consider two approaches in applying this kind of operator. The first approach, based on the global Shepard method, involves constructing the Bernoulli operator for each node $\mathbf{x}_i$, after selecting as a representative triangle the one that has $\mathbf{x}_i$ as a vertex and for which the Bernoulli approximation error is minimum. We study the interpolation error in this case in Theorem 4.2.6. The second approach, proposed in [97] and [98], involves constructing a Bernoulli operator on each triangle from the sphere's triangulation. We prove in Theorem 4.2.8 that this type of operator interpolates the function at the nodes. The original results of this section have been published in the paper T. Cătinaş and A. Malina [16]. Both sections end with numerical examples performed on test functions, using different sets of data (random points, spiral points, Halton nodes), demonstrating the efficiency of the methods proposed. Moreover, two physical phenomena are investigated in the final part of this chapter. Sections 4.3 and 4.4 present applications of the new spherical Shepard operators introduced in temperature predictions on the Earth's surface and topographic data approximation. The results show that these interpolants offer significant advantages in solving numerous problems that model real-life phenomena.

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Keywords: Shepard operator, interpolation of scattered data, weighted least squares approximation, thin-plate spline, inverse quadratic, inverse multiquadric, degree of exactness, remainder term, error estimations, iterative multiscale method, univariate interpolation, bivariate approximation, spherical interpolation, knot points, spiral points, Halton nodes, Bernoulli operator, spherical harmonics, image reconstruction.

MSC: 41A05, 41A25, 41A80, 65D05, 65D12, 65D15, 33C55.

Chapter 1

Preliminaries

The first chapter is dedicated to existing results and notions from Numerical Analysis, mostly approximation and interpolation theory. After listing some notations and fundamental concepts that will be used throughout this thesis in Section 1.1 and providing a brief introduction to the theory of radial basis functions in Section 1.2, we turn our attention to the main focus of this thesis: the study of the Shepard operator, first introduced in [82]. The pioneers in the study of this method include Franke and Nielson [46], [47], Renka and Cline [72], [74], [76], [77], Farwig [43], Barnhill, Dube and Little [4]. Section 1.3 revisits fundamental results in the univariate case, while Section 1.4 presents the work of several authors in the bivariate case. Notable references for these two cases include studies by Caira, Dell'Accio and Di Tommaso [8], [9], [37], [38], [39], Coman, Cătinaş and Trîmbiţaş [10]–[13], [24]–[31], [90]. Finally, Section 1.5 presents the Shepard operator constructed on the unit sphere. Important results in this area were obtained by Allasia, Cavoretto and De Rossi [2], [18], [19], [20], [21], [35], [36].

1.1 Main notions and notations

Theorem 1.1.1. *[79] (Peano's Theorem in $\mathbb{R}$)*

Let $L : H^n[a,b] \to \mathbb{R}$ be a linear functional such that it commutes with the definite integral operator. If $\ker L = \mathbb{P}_{n-1}$, then

$$L[f] = \int_a^b \phi_n(t) f^{(n)}(t)\ dt,$$

with

$$\phi_n(t) = \frac{1}{(n-1)!} L^x\left[(x-t)_+^{n-1}\right],$$

$L^x[f]$ denoting the fact that L is applied to f in regard to the variable x.

Theorem 1.1.2. *[79] (Peano's Theorem in $\mathbb{R}^2$)*

Consider $X = [a,b] \times [c,d]$ and let $L : B_{\alpha,\beta}(a,c) \to \mathbb{R}$ be a linear functional. If $\ker L = \mathbb{P}^2_{n-1}$, *then*

$$
\begin{aligned}
L[f] = &\sum_{j<\beta} \int_a^b \phi_{n-j,j}(s) \frac{\partial^n f}{\partial x^{n-j} \partial y^j}(s,c)\, ds \\
&+ \sum_{i<\alpha} \int_c^d \phi_{i,n-i}(t) \frac{\partial^n f}{\partial x^i \partial y^{n-i}}(a,t)\, dt \\
&+ \iint\limits_X \phi_{\alpha,\beta}(s,t) \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(s,t)\, ds\, dt,
\end{aligned}
$$

with

$$
\begin{aligned}
\phi_{n-j,j}(s) &= \frac{1}{j! \cdot (n-j-1)!} L^{x,y} \left[(x-s)_+^{n-j-1} \cdot (y-c)^j \right], \\
\phi_{i,n-i}(t) &= \frac{1}{i! \cdot (n-i-1)!} L^{x,y} \left[(x-a)^i \cdot (y-t)_+^{n-i-1} \right], \\
\phi_{\alpha,\beta}(s,t) &= \frac{1}{(\alpha-1)! \cdot (\beta-1)!} L^{x,y} \left[(x-s)_+^{\alpha-1} \cdot (y-t)_+^{\beta-1} \right].
\end{aligned}
$$

1.2 Radial basis functions

Definition 1.2.1. [44] The function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is *radial* if there is a function $\psi : \mathbb{R}_+ \to \mathbb{R}$ such that $\Psi(\mathbf{x}) = \psi(\|\mathbf{x}\|)$, with $\|\cdot\|$ usually denoting the Euclidean norm.

In the RBF case, the interpolant s is considered to have the form [44]

$$
s(\mathbf{x}) = \sum_{i=1}^{K} a_i \Psi(\mathbf{x}, \mathbf{x}_i), \tag{1.2.1}
$$

for $\mathbf{x} \in \mathbb{R}^n,\ \mathbf{x}_i \in X,\ i = 1, \ldots, K$, and $\Psi(\mathbf{x}, \mathbf{x}_i) = \psi(d(\mathbf{x}, \mathbf{x}_i))$, with $\psi : \mathbb{R}_+ \to \mathbb{R}$.

The interpolation conditions required to be satisfied [44]

$$
s(\mathbf{x}_i) = f(\mathbf{x}_i),\ i = 1, \ldots, K, \tag{1.2.2}
$$

lead to the linear system

$$
Ma = f, \tag{1.2.3}
$$

where

$$
\begin{aligned}
&M \in \mathbb{R}^{K \times K},\ M_{ij} = \psi(d(\mathbf{x}_i, \mathbf{x}_j)),\ i,j = 1, \ldots, K, \\
&a = (a_1, \ldots, a_K)^T,\ f = (f_1, \ldots, f_K)^T.
\end{aligned}
$$

To obtain a unique solution for the system (1.2.3), the interpolation matrix M should be invertible. As pointed out in [93], from a numerical point of view, a requirement that comes out naturally is that M should be strictly positive definite.

Often, the study of the strictly positive definiteness of the matrix M will be reduced to the same study applied to the function Ψ, which we will follow in the sequel.

Theorem 1.2.2. *[93] Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then, Ψ is strictly positive definite if and only if it is even and for all $K \in \mathbb{N}$ and for all $a = (a_1, \dots, a_K)^T \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, we have*

$$\sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j \Psi(\mathbf{x}_i, \mathbf{x}_j) > 0,$$

with $\mathbf{x}_1, \dots, \mathbf{x}_K \in \mathbb{R}^n$, distinct.

If in the above settings, we let $a \in \mathbb{R}^K$ and " $>$ " is substituted by " $\geq$ ", the term strictly positive definite becomes positive definite.

Since dealing with RBFs, it is often encountered in literature to also refer to the univariate function ψ as radial (strictly) positive definite function, as emphasized in [44].

Lemma 1.2.3. *[44] Given that $\Psi = \psi \circ d$, we say that ψ is radial (strictly) positive definite on $\mathbb{R}^m$, for any $m \leq n$, if Ψ is radial (strictly) positive definite on $\mathbb{R}^n$.*

Often, a polynomial is added to the interpolant s given in (1.2.1), so it becomes [44]

$$s(\mathbf{x}) = \sum_{i=1}^{K} a_i \Psi(\mathbf{x}, \mathbf{x}_i) + \sum_{i=1}^{D} A_i y_i(\mathbf{x}), \tag{1.2.4}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\{y_1, \dots, y_D\}$ forms a basis of $\mathbb{P}^n_{m-1}$, with $D = \dim \mathbb{P}^n_{m-1} = \binom{m+n-1}{m-1}$.

Besides the interpolatory conditions (1.2.2), one usually adds D constraints of the form

$$\sum_{i=1}^{K} a_i y_j(\mathbf{x}_i) = 0, \quad j = 1, \dots, D, \tag{1.2.5}$$

to obtain a unique solution. The polynomial precision is strongly linked to the condition that $\mathcal{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_K\}$ forms a $(m-1)-$unisolvent set.

Definition 1.2.4. [44] We say that $\mathcal{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_K\} \subset \mathbb{R}^n$ is $m-$unisolvent if the only polynomial of maximum total degree m that vanishes on $\mathcal{X}$ is the zero polynomial.

This problem is reduced to solving the following linear system [44]

$$\begin{pmatrix} M & Y \\ Y^T & O_D \end{pmatrix} \cdot \begin{pmatrix} a \\ A \end{pmatrix} = \begin{pmatrix} f \\ \mathbf{0} \end{pmatrix}, \tag{1.2.6}$$

where

$$\begin{aligned}
&M \in \mathbb{R}^{K \times K},\ M_{ij} = \psi(d(\mathbf{x}_i, \mathbf{x}_j)),\ i, j = 1, \dots, K,\\
&Y \in \mathbb{R}^{K \times D},\ Y_{ij} = y_j(\mathbf{x}_i),\ i = 1, \dots, K,\ j = 1, \dots, D,\\
&a = (a_1, \dots, a_K)^T,\ A = (A_1, \dots, A_D)^T,\\
&f = (f_1, \dots, f_K)^T, \text{ with } f_i = f(\mathbf{x}_i).
\end{aligned}$$

Theorem 1.2.5. *[93] Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be an even, continuous function. Then, Ψ is strictly conditionally positive definite of order m if and only if for all $K \in \mathbb{N}$ and for all $a = (a_1, \dots, a_K)^T \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ with*

$$\sum_{i=1}^{K} a_i y(\mathbf{x}_i) = 0,$$

where y is any real-valued polynomial of degree less than m, we have

$$\sum_{i=1}^{K}\sum_{j=1}^{K} a_i a_j \Psi(\mathbf{x}_i, \mathbf{x}_j) > 0,$$

with $\mathbf{x}_1, \ldots, \mathbf{x}_K \in \mathbb{R}^n$, distinct.

If, in the theorem above, we consider $a \in \mathbb{R}^K$ and substitute $'' >''$ with $'' \geq''$, we obtain the case of functions that are conditionally positive definite of order m.

Lemma 1.2.6. *[44] Given that $\Psi = \psi \circ d$, we say that ψ is radial (strictly) conditionally positive definite on $\mathbb{R}^m$, for any $m \leq n$, if Ψ is radial (strictly) conditionally positive definite on $\mathbb{R}^n$.*

Theorem 1.2.7. *[44] If $\Psi = \psi \circ d$ is an even, radial strictly conditionally positive definite of order m function on $\mathbb{R}^n$ and the distinct data sites $\{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$ form an $(m-1)-$unisolvent set, then, the linear system given in (1.2.6) has a unique solution.*

1.3 Univariate Shepard operators

This section is devoted to revising certain univariate Shepard operators combined with various interpolation polynomials, such as Lagrange [29], Taylor [29], Hermite [29], Birkhoff [29], [31], Abel-Goncharov [12] and Bernoulli [8].

Consider $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ a real-valued function such that for some given K interpolation nodes, $x_i \in X,\ i = 1, \ldots, K$, the values of f are available. Then, for $x \in X$, we can define the *classical univariate Shepard operator* as [82]

$$S_\mu f(x) = \sum_{i=1}^{K} A_{i,\mu}(x) f(x_i), \tag{1.3.1}$$

with the basis functions $A_{i,\mu}$ given by

$$A_{i,\mu}(x) = \frac{|x - x_i|^{-\mu}}{\sum\limits_{j=1}^{K} |x - x_j|^{-\mu}}, \quad i = 1, \ldots, K,\ x_i \neq x_j, \text{ for } i \neq j,\ j = 1, \ldots, K, \tag{1.3.2}$$

and $\mu > 0$ an arbitrary parameter.

1.3.1 Univariate Shepard-Lagrange operators

This operator was studied by Coman and Trîmbiţaş in [29]. The *univariate Shepard-Lagrange operator* is given by [29]

$$SL_k[f](x) = \sum_{i=1}^{K} A_{i,\mu}(x) L_k^i[f](x), \tag{1.3.3}$$

with

$$L_k^i[f](x) = \sum_{j=0}^{k} \prod_{\alpha=0,\ \alpha\neq j}^{k} \frac{(x - x_{i+\alpha})}{(x_{i+j} - x_{i+\alpha})} f(x_{i+j}), \tag{1.3.4}$$

$x_{K+j} = x_j,\ j = 1, \ldots, k.$

1.3.2 Univariate Shepard-Taylor operators

This operator was studied by Coman and Trîmbiţaş in [29]. Consider the sets

$$\Delta = \{\eta_{i,j}(f) = f^{(j)}(x_i) : \ i = 1, \ldots, K, \ j = 0, \ldots, k, \ k \in \mathbb{N}^*\},$$

and

$$\Delta_i = \{\eta_{i,p}(f) : \ p = 0, \ldots, k\},$$

such that Δ_i is a subset of Δ associated to η_i, having $\eta_i \in \Delta_i$, for all $i = 1, \ldots, K$.

Under these assumptions, the *univariate Shepard-Taylor operator* is defined as [29]

$$ST_k[f](x) = \sum_{i=1}^{K} A_{i,\mu}(x) T_k^i[f](x), \tag{1.3.5}$$

with

$$T_k^i[f](x) = \sum_{j=0}^{k} \frac{(x - x_i)^j}{j!} f^{(j)}(x_i). \tag{1.3.6}$$

1.3.3 Univariate Shepard-Hermite operators

The *univariate Shepard-Hermite interpolant* is defined as [29]

$$SH_k[f](x) = \sum_{i=1}^{K} A_{i,\mu}(x) H_k^i[f](x), \tag{1.3.7}$$

with $H_k^i[f]$ being a Hermite-type operator of the form

$$H_k^i[f](x) = \sum_{j=i}^{i+k} \sum_{p=0}^{r_j} h_{jp}(x) f^{(p)}(x_j),$$

with h_{jp} denoting the fundamental Hermite polynomials, considering $x_{K+i} = x_i, \ i \in \mathbb{N}, \ i \leq k$ and $1 \leq k \leq K$.

1.3.4 Univariate Shepard-Birkhoff operators

Coman and Trîmbiţaş studied the *Shepard-Birkhoff operator in the univariate case* in [29] and [31]. It is defined as

$$SBH_k[f](x) = \sum_{i=1}^{K} A_{i,\mu}(x) BH_k^i[f](x), \tag{1.3.8}$$

with $BH_k^i[f]$, for $1 \leq k \leq K$, being the kth degree Birkhoff polynomial

$$BH_k^i[f](x) = \sum_{j=i}^{i+k} \sum_{p \in I_j} b_{jp}(x) f^{(p)}(x_j),$$

considering $x_{K+i} = x_i, \ 1 \leq i \leq k$.

In contrast to the Hermite case, this polynomial does not always exist, nor is it always unique.

1.3.5 Univariate Shepard-Abel-Goncharov operators

The Abel-Goncharov operator has the advantage that it always exists and is unique, by contrast to the last polynomial. We will consider a set of $K+1$ interpolation nodes, $x_i,\ i = 0,\dots,K$. The problem that appears in this case is to find an interpolation polynomial $P_K[f]$ of degree K, such that we have

$$f^{(i)}(x_i) = P_K^{(i)}[f](x_i),\ i = 0,\dots,K.$$

This polynomial can be expressed as [42]

$$P_K[f](x) = \sum_{i=0}^{K} g_i(x) f^{(i)}(x_i),$$

g_i being the Goncharov polynomials [42].

The univariate Shepard-Abel-Goncharov operator was studied by Cătinaş in [12]. Let $k \in \mathbb{N},\ k \le K$ and for each node $x_i,\ i = 0,\dots,K$, consider the following set [12]

$$\mathcal{X}_{i,k} = \{x_i,\dots,x_{i+k}\},$$

with $x_{K+i+1} = x_i,\ i = 0,\dots,k$, and its associated Abel-Goncharov operator

$$P_k^i[f](x) = \sum_{j=i}^{i+k} g_{j-i}^i(x) f^{(j-i)}(x_j),\ i = 0,\dots,K,$$

where

$$\begin{aligned}
g_0^i(x) &= 1,\\
g_1^i(x) &= x - x_i,\\
g_j^i(x) &= \frac{1}{j!}\left[x^j - \sum_{p=0}^{j-1} g_p(x)\binom{j}{p} x_{p+i}^{j-p}\right],\ j \ge 1.
\end{aligned}$$

We can define the *univariate Shepard-Abel-Goncharov* operator as [12]

$$SAG_k[f](x) = \sum_{i=0}^{K} A_{i,\mu}(x) P_k^i[f](x). \tag{1.3.9}$$

1.3.6 Univariate Shepard-Bernoulli operators

Consider $X = [a,b],\ f \in C^k[a,b],\ k \ge 1$ and K distinct interpolation nodes $x_i \in X,\ i \in \{1,\dots,K\}$, arranged in ascending order. Caira and Dell'Accio defined the *univariate Shepard-Bernoulli operator* as follows [8]

$$SB_k[f](x) = \sum_{i=1}^{K} A_{i,\mu}(x) B_k[f; x_i, x_{i+1}](x), \tag{1.3.10}$$

with the Bernoulli operators $B_k[f; x_i; x_{i+1}]$ expressed as

$$B_k[f; x_i, x_{i+1}](x) = f(x_i) + \sum_{j=1}^{k} \frac{h^{j-1}}{j!}\left(B_j\left(\frac{x - x_i}{h}\right) - B_j\right)\left(f^{(j-1)}(x_{i+1}) - f^{(j-1)}(x_i)\right),$$

for $h = x_{i+1} - x_i$ and $x_{K+1} = x_{K-1}$.

In general, for $n \in \mathbb{N}$, the Bernoulli polynomials $B_n(\cdot)$ are defined recursively as [55]

$$\begin{cases} B_0(x) = 1, \\ B_n'(x) = nB_{n-1}(x),\ n \geq 1, \\ \int_0^1 B_n(x)\,dx = 0,\ n \geq 1. \end{cases} \tag{1.3.11}$$

For $x = 0$, one obtains the Bernoulli numbers B_n, i.e., $B_n = B_n(0)$.

1.3.7 Modified univariate Shepard operators

Franke and Nielson proposed in [47] a local procedure to compute the operator, that consists of substituting the weight functions $A_{i,\mu}$ by

$$w_{i,\mu}(x) = \left(\frac{R_w - |x - x_i|}{R_w|x - x_i|}\right)_+^\mu, \tag{1.3.12}$$

with R_w a radius of influence that varies with i, resulting in the so-called *modified univariate Shepard operator* [47]

$$S_W f\left(x\right) = \frac{\sum\limits_{i=1}^{K} w_{i,\mu}\left(x\right) f\left(x_i\right)}{\sum\limits_{i=1}^{K} w_{i,\mu}\left(x\right)}. \tag{1.3.13}$$

1.4 Bivariate Shepard operators

In this section, we direct our attention to the two-dimensional case Shepard studied in [82]. We will focus on presenting this method in its classical (global) form, together with some modified (local) versions developed in [46], [47], [76]. Afterward, we shall discuss some well-known Shepard interpolants combined with different operators, such as Lagrange [30], Taylor [28], Hermite [24], Birkhoff [25], complete Hermite-Birkhoff [38], Lidstone [9], [13] and Bernoulli [11], [37].

Consider an objective function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ whose values $f_i = f(x_i, y_i),\ i = 1, \ldots, K$, are known on a set of scattered data $\mathcal{X} = \{(x_i, y_i),\ i = 1, \ldots, K\} \subset X$. Classically, as proposed by Shepard, this interpolation scheme is given by [82]

$$S_\mu f(x, y) = \sum_{i=1}^{K} A_{i,\mu}(x, y) f(x_i, y_i), \tag{1.4.1}$$

with the weight functions $A_{i,\mu}$ defined as

$$A_{i,\mu}\left(x, y\right) = \frac{\prod\limits_{\substack{j=1 \\ j \neq i}}^{K} d_j^\mu\left(x, y\right)}{\sum\limits_{k=1}^{K} \prod\limits_{\substack{j=1 \\ j \neq k}}^{K} d_j^\mu\left(x, y\right)}, \tag{1.4.2}$$

considering the control parameter $\mu > 0$ and $d_i\,(x, y)$ the distances between $(x, y) \in X$ and the scattered points $(x_i, y_i) \in \mathcal{X},\ i = 1, ..., K$. As in the previous cases discussed in this thesis, the Euclidean distance is considered.

To improve the accuracy of this global method, Franke and Nielson proposed a local approach in [47] which was further discussed and developed by Franke in [46] and by Renka in [76]. Known as the *modified Shepard operator*, it is given as [47]

$$S_W f(x,y) = \sum_{i=1}^{K} \overline{w}_{i,\mu}(x,y) f(x_i, y_i), \tag{1.4.3}$$

with the compact support basis function $\overline{w}_{i,\mu}$ of the form

$$\overline{w}_{i,\mu}(x,y) = \frac{w_{i,\mu}(x,y)}{\sum\limits_{j=1}^{K} w_{j,\mu}(x,y)}, \tag{1.4.4}$$

for

$$w_{i,\mu}(x,y) = \left[\frac{(R_w - d_i(x,y))_+}{R_w d_i(x,y)}\right]^{\mu},\ \mu > 0,$$

considering $d_i(x, y)$ as the Euclidean distance between the ith node and the point (x, y), and R_w a radius of influence that varies with the index i.

Shepard himself considered in [82] an improvement of the method by adding the first-order partial derivatives of the function f, obtaining

$$S'_\mu f(x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y) \left[f(x_i,y_i) + (x - x_i)\frac{\partial f}{\partial x}(x_i,y_i) + (y - y_i)\frac{\partial f}{\partial y}(x_i,y_i)\right]. \tag{1.4.5}$$

In this manner, the degree of exactness is 1 and the interpolation properties hold also for the first-order partial derivatives.

1.4.1 Bivariate Shepard-Lagrange operators

This kind of operator was studied by Coman and Trîmbiţaş in [30]. For the sake of brevity, denote the node (x_i, y_i) by $\mathbf{x}_i,\ i = 1, \dots, K$. If we associate to each node $\mathbf{x}_i$ a set of m points

$$\mathcal{X}_{i,m} = \{\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+m-1}\},\ i = 1, \dots, K,\ m < K,$$

where $\mathbf{x}_{K+i} = \mathbf{x}_i,\ i = 1, \dots, m-1$, then we can introduce the following bivariate Lagrange polynomial of degree n, $L_n^i[f]$, as [30]

$$L_n^i[f](x,y) = \sum_{j=i}^{i+m-1} l_j(x,y) f(x_j, y_j),\ \text{for each } i = 1, \dots, K, \tag{1.4.6}$$

with l_j denoting the corresponding cardinal polynomials, i.e.,

$$l_j(\mathbf{x}_k) = \delta_{jk},\ \text{for each } j,k = i, \dots, i+m-1.$$

The existence and uniqueness of $L_n^i[f],\ i = 1, \dots, K$, are conditioned by the requirement that the points $\mathbf{x}_j,\ j = i, \dots, i+m-1$, should not lie on an algebraic curve of degree n, represented as $\sum\limits_{\alpha+\beta\leq n} \lambda_{\alpha,\beta} x^\alpha y^\beta = 0$. Furthermore, the degree n of the Lagrange polynomial $L_n^i[f]$ should be chosen such that $m = (n+1)(n+2)/2 < K$.

Under these assumptions, the *bivariate Shepard-Lagrange operator* is written as [30]

$$SL_n[f](x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y) L_n^i[f](x,y). \tag{1.4.7}$$

1.4.2 Bivariate Shepard-Taylor operators

When certain information regarding the partial derivatives of the function f are known, we can improve the bivariate Shepard operator by combining it with a Taylor polynomial $T_n^i[f]$ of degree n attached to a node $\mathbf{x}_i = (x_i, y_i),\ i = 1, \ldots, K$, which is defined as [28]

$$T_n^i[f](x,y) = \sum_{\alpha+\beta\leq n} \frac{(x-x_i)^\alpha (y-y_i)^\beta}{\alpha!\beta!} \cdot \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(x_i, y_i). \tag{1.4.8}$$

In this manner, the *Shepard interpolant of Taylor-type* obtained has the form [28]

$$ST_n[f](x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y) T_n^i[f](x,y). \tag{1.4.9}$$

1.4.3 Bivariate Shepard-Hermite operators

In this subsection, we will present the main results regarding the Shepard operator combined with a Hermite-type polynomial, published in [24] by Coman. Consider the following Hermite data known for f:

$$\Delta_H(f) = \left\{ \eta_i^{(\alpha,\beta)} f : \eta_i^{(\alpha,\beta)} f = \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(x_i, y_i),\ \alpha, \beta \in \mathbb{N},\ \alpha+\beta \leq r_i,\ i = 1, \ldots, K \right\}.$$

To each sample point $\mathbf{x}_i$, we attach a set of m_i points, $i = 1, \ldots, K$, denoted by $\mathcal{X}_{i,m_i}$ and defined as

$$\mathcal{X}_{i,m_i} = \{\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_{i+m_i-1}\},\ m_i \in \mathbb{N}^*, m_i < K-1,$$

setting $\mathbf{x}_{K+i} = \mathbf{x}_i,\ i \in \mathbb{N}$. Let us denote by

$$\Delta_i(f) = \{\eta_i^{(\alpha,\beta)} f : \alpha+\beta \leq r_i\}$$

the information we know about the node $\mathbf{x}_i,\ i = 1, \ldots, K$, and by

$$\Delta_{i,m_i}(f) = \bigcup_{j=0}^{m_i-1} \Delta_{i+j}(f)$$

the union of the sets containing the information about the nodes in $\mathcal{X}_{i,m_i}$. The bivariate Hermite polynomial $H_{n_i}^i[f],\ i = 1, \ldots, K$, of degree n_i is the polynomial that fulfills

$$\frac{\partial^{\alpha+\beta} H_{n_i}^i[f]}{\partial x^\alpha \partial y^\beta}(x_j, y_j) = \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(x_j, y_j) \tag{1.4.10}$$

for each $(x_j, y_j) \in \mathcal{X}_{i,m_i}$ with $\alpha+\beta \leq r_j$.

The degree n_i should be chosen such that $\text{card}(\Delta_{i,m_i}(f)) = \frac{(n_i+1)(n_i+2)}{2}$, for each $i = 1, \ldots, K$.

If the Hermite polynomial $H^i_{n_i}$ exists, under all the above assumptions, we can define the *combined Shepard-Hermite operator* as [24]

$$SH_{n_1,\ldots,n_K}[f](x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y) H^i_{n_i}[f](x,y).$$

1.4.4 Bivariate Shepard-Birkhoff operators

A more general case that extends the operators presented in the previous subsections is the Birkhoff-type problem, which was studied by Coman in [25].

Consider the subsequent Birkhoff-type data about a real-valued function $f : X \to \mathbb{R}$

$$\Delta_B(f) = \left\{ \eta_i^{(\alpha,\beta)} f : \eta_i^{(\alpha,\beta)} f = \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(x_i, y_i),\ (\alpha,\beta) \in I_i \subseteq \mathbb{N}^2,\ i = 1, \ldots, K \right\}.$$

As in the Hermite case, to each node $\mathbf{x}_i$ we associate a set $\mathcal{X}_{i,m_i}$ of m_i points, $m_i \in \mathbb{N}^*,\ m_i < K-1,\ i = 1, \ldots, K$, of the form

$$\mathcal{X}_{i,m_i} = \{\mathbf{x}_i, \mathbf{x}_{i+1}, ..., \mathbf{x}_{i+m_i-1}\},$$

and taking into account that $\Delta_i(f)$ represents the known data of f at $\mathbf{x}_i$, i.e.,

$$\Delta_i(f) = \left\{ \eta_i^{(\alpha,\beta)} f : (\alpha,\beta) \in I_i \right\},$$

we also define the following information data set for the nodes in $\mathcal{X}_{i,m_i}$

$$\Delta_{i,m_i}(f) = \bigcup_{j=0}^{m_i-1} \Delta_{i+j}(f),$$

agreeing that $\mathbf{x}_{K+i} = x_i,\ i \in \mathbb{N}$.

The generalization of the Lagrange, Taylor and Hermite problems consists of finding the polynomial $BH^i_{n_i}$ of total degree n_i that meets the interpolation conditions

$$\frac{\partial^{\alpha+\beta} BH^i_{n_i}[f]}{\partial x^\alpha \partial y^\beta}(x_j, y_j) = \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(x_j, y_j), \tag{1.4.11}$$

for every $(x_j, y_j) \in \mathcal{X}_{i,m_i}$, with $(\alpha,\beta) \in I_j$.

When $r_i := \text{card}(\Delta_{i,m_i}(f)) = \frac{(n_i+1)(n_i+2)}{2}$, the matrix of the system resulting from (1.4.11) is square, following that the system has a unique solution in the case of a non-zero determinant. The *Shepard operator of Birkhoff type* in this case is defined as [25]

$$SBH_{n_1,\ldots,n_K}[f](x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y) BH^i_{n_i}[f](x,y). \tag{1.4.12}$$

As in the Hermite case, the difficulty that arises is to correctly select subsets $\mathcal{X}_{i,m_i}$ such that $\text{card}(\Delta_{i,m_i}(f)) = r_i$ and also that the matrix associated to the system (1.4.11) has a non-zero determinant, for every $i = 1, \ldots, K$.

Definition 1.4.1. [25] A sequence of subsets $\mathcal{X}_{i,m_i},\ i = 1, \ldots, K$, is called *admissible* if $\text{card}(\Delta_{i,m_i}(f)) = r_i$ and the matrix associated to the system consisting of the corresponding interpolation equations has a non-zero determinant.

1.4.5 Bivariate Shepard operator of complete Hermite-Birkhoff type

Another approach to constructing a combined Shepard operator involves using complete Hermite-Birkhoff polynomials, as Dell'Accio and Di Tommaso proposed in [38]. For an open, convex and bounded set X and the data sites $\mathcal{X} = \{\mathbf{x}_i,\ i = 1, \ldots, K\} \subset X$, they considered the local Shepard operator, i.e., the modified form of it introduced in (1.4.3).

Assume that for each sample point $\mathbf{x}_i,\ i = 1, \ldots, K$, the following Hermite-Birkhoff data-type is known

$$\Delta_i(f) = \left\{ \frac{\partial^{\alpha+\beta} f}{\partial^\alpha \partial^\beta}(\mathbf{x}_i) : (\alpha, \beta) \in I_i \subset \mathbb{N}^2,\ \alpha + \beta \leq r_i \right\},$$

imposing that, if the value of a partial derivative of order γ of f is available at $\mathbf{x}_i$, then all the other partial derivatives of the same order γ are known.

According to [38], the union of these information sets $\Delta(f) = \bigcup_{i=1}^{K} \Delta_i(f)$ is called a set of *complete Hermite-Birkhoff data*.

We are going to introduce several notations from [38]. First, consider $T(i)$ to be the triangle that has a vertex in $\mathbf{x}_i$ and the other two in $\mathbf{x}_j$ and $\mathbf{x}_k$, such that the latter two are contained in the closed ball of radius R_i, centered at $\mathbf{x}_i$. The point $\mathbf{x}_i$ is considered to be the *referring vertex* of $T(i)$. In a counterclockwise movement, for simplicity of notations, we denote the three vertices as follows: $\mathbf{x}_1 := \mathbf{x}_i,\ \mathbf{x}_2 := \mathbf{x}_j,\ \mathbf{x}_3 := \mathbf{x}_k$.

Denoting by $\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ the signed area of the triangle with vertices $\mathbf{x}, \mathbf{y}, \mathbf{z}$, we introduce the barycentric coordinates with respect to the triangle $T(i)$ as

$$\lambda_1(\mathbf{x}) = \frac{\mathcal{A}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_3)}{\mathcal{A}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)}, \quad \lambda_2(\mathbf{x}) = \frac{\mathcal{A}(\mathbf{x}_1, \mathbf{x}, \mathbf{x}_3)}{\mathcal{A}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)}, \quad \lambda_3(\mathbf{x}) = \frac{\mathcal{A}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x})}{\mathcal{A}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)}. \tag{1.4.13}$$

In addition, we need to use the directional derivatives along the sides of the triangle $T(i)$. Considering $''\cdot''$ to be the Euclidean inner product, they are defined as

$$\begin{aligned} D_{ij} f(\mathbf{x}) &= (\mathbf{x}_i - \mathbf{x}_j) \cdot \nabla f(\mathbf{x}) \\ &= (x_i - x_j)\frac{\partial f}{\partial x}(\mathbf{x}) + (y_i - y_j)\frac{\partial f}{\partial y}(\mathbf{x}),\ i, j = 1, 2, 3,\ i \neq j. \end{aligned} \tag{1.4.14}$$

The composition of these directional derivatives is expressed as

$$D_1^{(\gamma_1,\gamma_2)} = D_{21}^{\gamma_1} D_{31}^{\gamma_2}, \quad D_2^{(\gamma_1,\gamma_2)} = D_{12}^{\gamma_1} D_{32}^{\gamma_2}, \quad D_3^{(\gamma_1,\gamma_2)} = D_{13}^{\gamma_1} D_{23}^{\gamma_2}, \tag{1.4.15}$$

for $(\gamma_1, \gamma_2) \in \mathbb{N}^2$.

Now, we can write the complete Hermite-Birkhoff polynomial $HB^{T(i)}[f]$ as a combination between the values of the directional derivatives and some polynomials in the variable $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ such as [38]

$$\begin{aligned} HB^{T(i)}[f](\boldsymbol{\lambda}) = &\sum_m q_{1,m}(\boldsymbol{\lambda}) D_1^{(\alpha_m^1,\alpha_m^2)} f(\mathbf{x}_1) + \sum_p q_{2,p}(\boldsymbol{\lambda}) D_2^{(\beta_p^1,\beta_p^2)} f(\mathbf{x}_2) \\ &+ \sum_s q_{3,s}(\boldsymbol{\lambda}) D_3^{(\gamma_s^1,\gamma_s^2)} f(\mathbf{x}_3), \end{aligned} \tag{1.4.16}$$

with

$$\{(\alpha_m^1, \alpha_m^2)\}_m = I_i, \quad \{(\beta_p^1, \beta_p^2)\}_p \subset I_j, \quad \{(\gamma_s^1, \gamma_s^2)\}_s \subset I_k,$$

and

$$\mathrm{card}(\{m\}) + \mathrm{card}(\{p\}) + \mathrm{card}(\{s\}) = \frac{(n_i+1)(n_i+2)}{2}.$$

Remark 1.4.2. We have that $BH^{T(i)}[f] \in \mathbb{P}^2_{n_i}$.

After completing the steps of the algorithm consisting of determining $HB^{T(i)}[f]$, the *modified Shepard operator of Hermite-Birkhoff-type* can be written as [38]

$$S_{HB}[f](x,y) = \sum_{i=1}^{K} \overline{w}_{i,\mu}(x,y) HB^{T(i)}[f](x,y). \tag{1.4.17}$$

1.4.6 Bivariate Shepard-Lidstone operators

The Lidstone operator is constructed based on the Lidstone polynomials [1]

$$\begin{cases} \Lambda_0(x) = x, \\ \Lambda_n''(x) = \Lambda_{n-1}(x),\ n \geq 1 \\ \Lambda_n(0) = \Lambda_n(1),\ n \geq 1. \end{cases} \tag{1.4.18}$$

In this subsection, we present two approaches to constructing combined Shepard-Lidstone operators. The first approach, proposed by Cătinaş in [13], extends the Lidstone polynomials on a rectangular domain. The second one, developed by Caira, Dell'Accio and Di Tommaso in [9], relies on the approximation formula over a triangular domain utilizing this kind of polynomials, as presented in [34].

For the rectangular case, we shall first define the Lidstone interpolants in the univariate case and extend them afterward to the two-dimensional case. Let $[a,b],\ [c,d]$ be two intervals of real numbers and denote by $\Delta_1:\ a = x_1 < x_2 < \ldots < x_{K_1} = b$ and $\Delta_2:\ c = y_1 < y_2 < \ldots < y_{K_2} = d$, the uniform partitions of $[a,b]$ and $[c,d]$, with stepsizes $h_1 = \frac{b-a}{K_1-1}$ and $h_2 = \frac{d-c}{K_2-1}$, respectively. We also consider

$$L_n(\Delta_1) = \{p \in C[a,b]:\ p \in \mathbb{P}_{2n-1} \text{ for each subinterval } [x_i, x_{i+1}],\ i = 1, \ldots, K_1 - 1\}.$$

For $f \in C^{2n-2}[a,b]$, the Lidstone interpolant can be written as [1]

$$L_n^{\Delta_1}[f](x) = \sum_{i=1}^{K_1} \sum_{j=0}^{n-1} l_{i,j}^n(x) f^{(2j)}(x_i), \tag{1.4.19}$$

with

$$l_{i,j}^n(x) = \begin{cases} \Lambda_j\left(\frac{x-x_{i-1}}{h_1}\right) h_1^{2j}, & x \in [x_{i-1}, x_i],\ i = 2, \ldots, K_1, \\ \Lambda_j\left(\frac{x_{i+1}-x}{h_1}\right) h_1^{2j}, & x \in [x_i, x_{i+1}],\ i = 1, \ldots, K_1 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

The operator (1.4.19) satisfies the following interpolation properties [1]

$$L_n^{\Delta_1}[f]^{(2j)}(x_i) = f^{(2j)}(x_i),\ j = 0, \ldots, n-1,\ i = 1, \ldots, K_1.$$

In the bivariate case, for $f \in C^{2n-2,2n-2}([a,b] \times [c,d])$ and a rectangular partition $\Delta = \Delta_1 \times \Delta_2$ of $[a,b] \times [c,d]$, the unique Lidstone interpolant is expressed as [1]

$$L_n^{\Delta}[f](x,y) = \sum_{i=1}^{K_1}\sum_{\alpha=0}^{n-1}\sum_{j=1}^{K_2}\sum_{\beta=0}^{n-1} l_{i,\alpha}^n(x) l_{j,\beta}^n(y) \frac{\partial^{2\alpha+2\beta} f}{\partial x^{2\alpha}\partial y^{2\beta}}(x_i,y_j), \tag{1.4.20}$$

possessing the similar interpolation properties of the univariate case

$$\begin{aligned} &\frac{\partial^{2\alpha+2\beta} L_n^{\Delta}[f]}{\partial x^{2\alpha}\partial y^{2\beta}}(x_i,y_j) = \frac{\partial^{2\alpha+2\beta} f}{\partial x^{2\alpha}\partial y^{2\beta}}(x_i,y_j), \\ &\alpha,\beta = 0,\dots,n-1,\ i=1,\dots,K_1,\ j=1,\dots,K_2. \end{aligned} \tag{1.4.21}$$

We confine ourselves to the case of $K_1 = K_2 =: K$ and consider on each subrectangle $\Delta_i = [x_i, x_{i+i}] \times [y_i, y_{i+1}] \subseteq [a,b]\times[c,d],\ i=1,\dots,K$, the following Lidstone-type data

$$\mathcal{L}_i(f) = \left\{ \frac{\partial^{4\alpha} f}{\partial x^{2\alpha}\partial y^{2\alpha}}(x_i,y_i), \frac{\partial^{4\alpha} f}{\partial x^{2\alpha}\partial y^{2\alpha}}(x_{i+1},y_{i+1}),\ \alpha=0,\dots,n-1 \right\}.$$

Considering $L_n^{\Delta_i}[f]$ the restriction of the operator given in (1.4.20) to the rectangle Δ_i, we define the *bivariate Shepard-Lidstone operator* as [13]

$$SL_n^{\Delta}[f](x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y) L_n^{\Delta_i}[f](x,y). \tag{1.4.22}$$

In the last part of this section, we discuss a different method to construct combined Shepard-Lidstone operators, proposed in [9]. As in the previous section, the approach is based on considering a three-point extension of the univariate Lidstone polynomial to a triangle $T(i),\ i = 1,\dots,K$, with referring vertex $\mathbf{x}_i$ and the other two vertices $\mathbf{x}_j,\ \mathbf{x}_k$, situated in the closed ball B_i of center $\mathbf{x}_i$ and radius R_i, denoted as $\mathbf{x}_1 := \mathbf{x}_i,\ \mathbf{x}_2 := \mathbf{x}_j,\ \mathbf{x}_3 := \mathbf{x}_k$. As before, we use the barycentric coordinates $(\lambda_1,\lambda_2,\lambda_3)$ of $\mathbf{x} = (x,y)$ with respect to the vertices of the triangle $T(i)$, defined as in (1.4.13). Moreover, we set

$$\begin{aligned} \mathbf{v}_1 &= (x_2 - x_3, y_2 - y_3), \quad v_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \\ \mathbf{v}_2 &= (x_3 - x_1, y_3 - y_1), \quad v_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \\ \mathbf{v}_3 &= (x_2 - x_1, y_2 - y_1), \quad v_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}. \end{aligned}$$

The bivariate Lidstone interpolant corresponding to the triangle $T(i)$ has the form [9]

$$\begin{aligned} \tilde{L}_n^{T(i)}[f](x,y) = &\sum_{\alpha=0}^{n-1} \left[\sum_{\beta=0}^{n-1-\alpha} \|\mathbf{v}_2\|^{2\beta} \left(\Lambda_\beta(1-\lambda_2-\lambda_3) \frac{\partial^{2\alpha+2\beta} f}{\partial v_2^{2\beta}\partial v_1^{2\alpha}}(\mathbf{x}_1) \right.\right. \\ &\left. + \Lambda_\beta(\lambda_2+\lambda_3) \frac{\partial^{2\alpha+2\beta} f}{\partial v_2^{2\beta}\partial v_1^{2\alpha}}(\mathbf{x}_3) \right) \cdot \Lambda_\alpha\left(\frac{\lambda_3}{\lambda_2+\lambda_3}\right) \\ &+ \sum_{\beta=0}^{n-1-\alpha} \|\mathbf{v}_3\|^{2\beta} \left(\Lambda_\beta(1-\lambda_2-\lambda_3) \frac{\partial^{2\alpha+2\beta} f}{\partial v_3^{2\beta}\partial v_1^{2\alpha}}(\mathbf{x}_1) \right. \\ &\left.\left. + \Lambda_\beta(\lambda_2+\lambda_3) \frac{\partial^{2\alpha+2\beta} f}{\partial v_3^{2\beta}\partial v_1^{2\alpha}}(\mathbf{x}_2) \right) \cdot \Lambda_\alpha\left(\frac{\lambda_2}{\lambda_2+\lambda_3}\right) \right] \cdot \|\mathbf{v}_1\|^{2\alpha}(\lambda_2+\lambda_3)^{2\alpha}, \end{aligned} \tag{1.4.23}$$

for $f \in C^{2n}(X)$, X convex domain, $(x,y) \in X$.

The *modified form of the Shepard-Lidstone operator* in this case is given by [9]

$$S\tilde{L}_n[f](x,y) = \sum_{i=1}^{K} \overline{w}_{i,\mu}(x,y)\tilde{L}_n^{T(i)}[f](x,y). \tag{1.4.24}$$

For each node $\mathbf{x}_i,\ i = 1,\dots,K$, the corresponding triangle $T(i)$ is chosen such that it is contained in the closed ball of center $\mathbf{x}_i$ and radius R_i and in addition, minimizes the quantity $e_{\max}^{2n}(i)\left[e_{\max}^2(i)A(i)\right]^{2n-1}$, where $e_{\max}(i) = \max\{\|\mathbf{x}_1 - \mathbf{x}_2\|, \|\mathbf{x}_2 - \mathbf{x}_3\|, \|\mathbf{x}_3 - \mathbf{x}_1\|\}$ and $A(i) = |2\mathcal{A}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)|^{-1}$.

1.4.7 Bivariate Shepard-Bernoulli operators

An extension of real functions in Bernoulli polynomials over a rectangle was proposed by Costabile and Dell'Accio in [32]. Based on these operators, Cătinaş introduced the bivariate Shepard-Bernoulli interpolant in [11]. We present the main results of these two articles in the first part of the subsequent section.

Consider the rectangle $X = [a,b]\times[c,d] \subseteq \mathbb{R}^2$ and $f \in C^{m,n}(X)$. Assuming these conditions, we can define the Bernoulli-type interpolant as [32]

$$\begin{aligned} B_{m,n}[f](x,y) =& f(a,c) + \sum_{i=1}^{m} \frac{h^{i-1}}{i!} S_i\left(\frac{x-a}{h}\right)\Delta_{(h,0)}\frac{\partial^{i-1}f}{\partial x^{i-1}}(a,c) \\ &+ \sum_{j=1}^{n} \frac{k^{j-1}}{j!} S_j\left(\frac{y-c}{k}\right)\Delta_{(0,k)}\frac{\partial^{j-1}f}{\partial y^{j-1}}(a,c) \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{h^{i-1}k^{j-1}}{i!j!} S_i\left(\frac{x-a}{h}\right) S_j\left(\frac{y-c}{k}\right)\Delta_{(h,k)}\frac{\partial^{i+j-2}f}{\partial x^{i-1}\partial y^{j-1}}(a,c), \end{aligned} \tag{1.4.25}$$

considering the notations

$$\begin{aligned} \Delta_{(h,0)}f(x,y) &= f(x+h,y) - f(x,y), \\ \Delta_{(0,k)}f(x,y) &= f(x,y+k) - f(x,y), \\ \Delta_{(h,k)}f(x,y) &= f(x,y) - f(x+h,y) + f(x+h,y+k) - f(x,y+k), \end{aligned}$$

for $h = b-a$ and $k = d-c$.

We also have that

$$S_i\left(\frac{x-a}{h}\right) = B_i\left(\frac{x-a}{h}\right) - B_i,$$

with $B_i(\cdot)$ being the Bernoulli polynomials and $B_i = B_i(0)$ the Bernoulli numbers, defined in (1.3.11).

For the sake of brevity, we denote the operator introduced in (1.4.25) by $B_{m,n}[f;(a,c),(h,k)]$. Taking into account everything introduced above and considering a set of K sample points $\mathbf{x}_i = (x_i,y_i) \in X,\ i = 1,\dots,K$, we define the *combined Shepard operator of Bernoulli type in the bivariate case* as [11]

$$SB_{m,n}[f](x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y)B_{m,n}^{i}[f](x,y), \tag{1.4.26}$$

$$B_{m,n}^i[f](x,y) = B_{m,n}[f;(x_i,y_i),(h_i,k_i)],\ i=1,\dots,K,$$

considering the rectangle $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$, with $h_i = x_{i+1} - x_i$ and $k_i = y_{i+1} - y_i$, $i = 1, \dots, K$, where $\mathbf{x}_{K+1} = \mathbf{x}_{K-1}$.

The three-point extended Bernoulli operator is obtained by assigning to each node $\mathbf{x}_i,\ i = 1, \dots, K$, a triangle $T(i)$ with other two data samples $\mathbf{x}_j$ and $\mathbf{x}_k$ as vertices, such that they are contained within the closed ball centered at $\mathbf{x}_i$ with radius R_i. For ease of notation, in each triangle $T(i)$ we adhere again to the convention that $\mathbf{x}_1 := \mathbf{x}_i,\ \mathbf{x}_2 := \mathbf{x}_j,\ \mathbf{x}_3 := \mathbf{x}_k$. We consider $(\lambda_1, \lambda_2, \lambda_3)$ the barycentric coordinates of a point $\mathbf{x} \in X$ relative to the triangle $T(i)$, computed as in (1.4.13). Moreover, we require that $X \subseteq \mathbb{R}^2$ is a compact convex domain.

Assuming all of the above, for a real-valued function f of class $C^n(X)$, the Bernoulli operator of order n introduced in [37] is of the form

$$\begin{aligned}\tilde{B}_n^{T(i)}[f](x,y) =& f(\mathbf{x}_1) + \sum_{i=1}^{n} \frac{S_i(\lambda_2+\lambda_3)}{i!}\left(D_1^{(0,i-1)} f(\mathbf{x}_3) - D_1^{(0,j-1)} f(\mathbf{x}_1)\right) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n-i+1} \frac{(\lambda_2+\lambda_3)^{i-1} S_i\left(\frac{\lambda_2}{\lambda_2+\lambda_3}\right) S_j(\lambda_2+\lambda_3)}{i!j!} \\ &\cdot \Big[(-1)^{i+j}\left(D_2^{(j-1,i-1)} f(\mathbf{x}_2) - D_2^{(j-1,i-1)} f(\mathbf{x}_1)\right) \\ &+ (-1)^j \left(D_3^{(j-1,i-1)} f(\mathbf{x}_3) - D_3^{(j-1,i-1)} f(\mathbf{x}_1)\right)\Big],\end{aligned} \tag{1.4.27}$$

with the expressions for the directional derivatives and their compositions $D_i^{(\gamma_1,\gamma_2)},\ i = 1, 2, 3$, provided in (1.4.14) and (1.4.15), respectively.

For each data sample $\mathbf{x}_i$, the choice of the corresponding triangle $T(i)$ is made similarly as in the previous two cases.

We define the *bivariate Shepard-Bernoulli operator of order* n as [37]

$$S\tilde{B}_n[f](x,y) = \sum_{i=1}^{K} \overline{w}_{i,\mu}(x,y) \tilde{B}_n^{T(i)}[f](x,y). \tag{1.4.28}$$

1.5 Spherical data interpolation using the Shepard operator

This section intends to present some literature results regarding the interpolation problem on the d-dimensional unit sphere S^d. The case we are interested in is $d = 2$, since this kind of problem appears in many types of practical areas, where the data represents, for example, some physical phenomena measured on the surface of the Earth, so the unit sphere S^2 is suitable as a fitting model, as noted in [19], [53], [84]. Other topics that use spherical interpolation include modeling closed surfaces in CAGD, as mentioned in [36], or solving some geophysical problems (topography, magnetic field intensity, gravitational potential [53]).

The spherical approximation of functions was the subject of many authors' investigations. A frequent topic that was studied is based on the theory of radial basis functions (see, e.g., [5], [54], [58], [84]). Other authors focused on Shepard-like methods combined with radial basis functions (see, e.g., [18], [19], [20], [21], [35], [36]) or, when data about derivatives are available, combined, for instance, with Hermite-Birkhoff polynomials [2]. Additionally, Shepard-type methods based on a spherical triangulation of the scattered data points have been proposed in [97], [98].

1.5.1 Spherical radial basis functions

The unit sphere S^2 is defined as

$$S^2 = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : \ \|\mathbf{x}\| = 1\},$$

with $\|\cdot\|$ denoting the Euclidean norm. The surface area of S^2 is $\omega_2 = 4\pi$.

The geodesic metric between $\mathbf{x}$, $\mathbf{y} \in S^2$, denoted by $g(\mathbf{x}, \mathbf{y})$, is computed as [54]

$$g(\mathbf{x}, \mathbf{y}) = \arccos(\mathbf{x} \cdot \mathbf{y}), \tag{1.5.1}$$

with $''\cdot''$ denoting the Euclidean inner product. This distance measures the arc length of the shortest path between $\mathbf{x}$ and $\mathbf{y}$.

The spherical analog of the radial basis functions problem (1.2.1) uses instead of the Euclidean distance, the geodesic distance g and a spherical basis function (SBF) $\psi : [0, \pi] \to \mathbb{R}$. For a set of scattered data samples $\mathcal{X} = \{\mathbf{x}_i \in S^2 : i = 1, \ldots, K\}$ with known values of a function $f : S^2 \to \mathbb{R}$ on $\mathcal{X}$, the target is to find an interpolant s of the form [54]

$$s(\mathbf{x}) = \sum_{i=1}^{K} a_i \psi(g(\mathbf{x}, \mathbf{x}_i)), \tag{1.5.2}$$

such that $s(\mathbf{x}_i) = f(\mathbf{x}_i), \ i = 1, \ldots, K$.

The resulting linear system is [54]

$$Ma = f, \tag{1.5.3}$$

where

$$\begin{aligned} &M \in \mathbb{R}^{K \times K}, \ M_{ij} = \psi(g(\mathbf{x}_i, \mathbf{x}_j)), \ i, j = 1, \ldots, K, \\ &a = (a_1, \ldots, a_K)^T, \ f = (f_1, \ldots, f_K)^T, \text{ with } f_i = f(\mathbf{x}_i). \end{aligned}$$

Often, as mentioned in [84], it is considered the approximation by spherical radial basis functions plus spherical harmonics (which are the analog of polynomials). In this situation, the interpolant has the form [54]

$$s_h(\mathbf{x}) = \sum_{i=1}^{K} a_i \psi(g(\mathbf{x}, \mathbf{x}_i)) + y(\mathbf{x}), \tag{1.5.4}$$

with y being a fitting spherical harmonic.

Definition 1.5.1. [54] A polynomial $p : \mathbb{R}^3 \to \mathbb{R}$ of degree $d, \ d \geq 0$, is *homogeneous of degree* d if $p(t\mathbf{x}) = t^d p(\mathbf{x})$, for any $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$.

Definition 1.5.2. [54] A polynomial $p : \mathbb{R}^3 \to \mathbb{R}$ is *harmonic* if $\Delta p(\mathbf{x}) = 0$, for any $\mathbf{x} \in \mathbb{R}^3$, where Δ is the Laplace operator, i.e.,

$$\Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}.$$

Definition 1.5.3. [54] If $\mathcal{P}_d$ is the set of polynomials of degree d in $\mathbb{R}^3$ that are harmonic and homogeneous of order d, the set of *spherical harmonics of exact order d* is given by the linear space

$$\mathcal{H}_d^*(S^2) = \{p|_{S^2} : \ p \in \mathcal{P}_d\}.$$

Definition 1.5.4. [54] The space of *spherical harmonics of maximum order d*, denoted by $\mathcal{H}_d(S^2)$, is defined as

$$\mathcal{H}_d(S^2) = \bigoplus_{j=0}^d \mathcal{H}_j^*(S^2).$$

If y in (1.5.4) is a spherical harmonic of order d, after expanding it, we obtain the following form of the interpolant s_h [54]

$$s_h(\mathbf{x}) = \sum_{i=1}^{K} a_i \psi(g(\mathbf{x}, \mathbf{x}_i)) + \sum_{i=1}^{D} A_i y_i(\mathbf{x}), \tag{1.5.5}$$

with $D = \dim \mathcal{H}_d(S^2)$ and $\{y_1, \ldots, y_D\}$ a basis of $\mathcal{H}_d(S^2)$.

Besides the interpolation conditions

$$s_h(\mathbf{x}_i) = f(\mathbf{x}_i), \ i = 1, \ldots, K, \tag{1.5.6}$$

the following linear constraints should be satisfied [54]

$$\sum_{k=1}^{K} a_k y_i(\mathbf{x}_k) = 0, \ i = 1, \ldots, D. \tag{1.5.7}$$

Now we have an augmented linear system of $K + D$ unknowns and $K + D$ equations, that can be written in matrix form as [54]

$$\begin{pmatrix} M & Y \\ Y^T & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} a \\ A \end{pmatrix} = \begin{pmatrix} f \\ \mathbf{0} \end{pmatrix}, \tag{1.5.8}$$

where

$$\begin{aligned}
&M \in \mathbb{R}^{K \times K}, \ M_{ij} = \psi(g(\mathbf{x}_i, \mathbf{x}_j)), \ i, j = 1, \ldots, K, \\
&Y \in \mathbb{R}^{K \times D}, \ Y_{ij} = y_j(\mathbf{x}_i), \ i = 1, \ldots, K, \ j = 1, \ldots, D, \\
&a = (a_1, \ldots, a_K)^T, \ A = (A_1, \ldots, A_D)^T, \\
&f = (f_1, \ldots, f_K)^T, \text{ with } f_i = f(\mathbf{x}_i).
\end{aligned}$$

According to [54], the SBF interpolants s and s_h uniquely exist if and only if the matrices in (1.5.3) and (1.5.8), respectively, are non-singular. Similar to the RBF case, this occurs for strictly positive and conditionally strictly positive definite functions on the sphere.

Definition 1.5.5. [54] Let $\psi : [0, \pi] \to \mathbb{R}$ be a continuous function. We say that ψ is *strictly positive definite* on S^2 ($\psi \in$ SPD) if, for any set of K distinct data sites $\mathbf{x}_i \in S^2, \ i = 1, ..., K$, the quadratic form

$$\sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j \psi(g(\mathbf{x}_i, \mathbf{x}_j)) \tag{1.5.9}$$

is positive on $\mathbb{R}^K \setminus \{\mathbf{0}\}$.

Theorem 1.5.6. *[54] Any function $\psi \in SPD$ can provide a unique interpolant s as in (1.5.2).*

Definition 1.5.7. [54] Let $\psi : [0, \pi] \to \mathbb{R}$ be a continuous function and $m \in \mathbb{N}$. We say that ψ is *conditionally strictly positive definite* on S^2 of order m ($\psi \in \mathrm{CSPD}(m)$) if, for any set of K distinct data sites $\mathbf{x}_i \in S^2,\ i = 1, ..., K$, the quadratic form (1.5.9) is positive on

$$W_{m-1} = \Big\{\mathbf{a} \in \mathbb{R}^K \setminus \{\mathbf{0}\} : \sum_{i=1}^{K} a_i y(\mathbf{x}_i) = 0 \text{ for all } y \in \mathcal{H}_{m-1}(S^2)\Big\}.$$

Definition 1.5.8. [54] Consider $m \in \mathbb{N}$ and $D = \dim \mathcal{H}_{m-1}(S^2)$. A set of distinct points $\mathcal{X} = \{\mathbf{x}_i \in S^2 : i = 1, \ldots, D\}$ is called *unisolvent* with respect to $\mathcal{H}_{m-1}(S^2)$ if the only element that vanishes for each $\mathbf{x}_i$ is the zero spherical harmonic.

Theorem 1.5.9. *[54] Any function $\psi \in CSPD(m)$ can provide a unique interpolant s_h as in (1.5.5) if the unisolvency condition on the sphere is satisfied.*

1.5.2 Shepard operator combined with spherical basis functions

The spherical interpolation of data using Shepard-type operators combined with radial basis functions has been thoroughly studied by Cavoretto and De Rossi (see, e.g., [18], [19], [20], [21], [35], [36]).

Let us consider the set of distinct nodes $\mathcal{X} = \{\mathbf{x}_i = (x_i, y_i, z_i),\ i = 1, ..., K\}$, lying on the unit sphere S^2 along with their corresponding function values $f_i = f(\mathbf{x}_i),\ i = 1, ..., K$, with $f : S^2 \to \mathbb{R}$. For $\mathbf{x} = (x, y, z) \in S^2$ the *modified spherical Shepard operator* is given by [36]

$$S(\mathbf{x}) = \sum_{j=1}^{K} \overline{w}_j(\mathbf{x}) f_j, \tag{1.5.10}$$

with

$$\overline{w}_j(\mathbf{x}) = \frac{w_j(\mathbf{x})}{\sum\limits_{k=1}^{K} w_k(\mathbf{x})}. \tag{1.5.11}$$

The weights w_j are defined as

$$w_j(\mathbf{x}) = \left[\frac{(R_j^w - g(\mathbf{x}, \mathbf{x}_j))_+}{R_j^w g(\mathbf{x}, \mathbf{x}_j)}\right]^{\mu}, \tag{1.5.12}$$

with R_j^w a radius of influence about the node j and g the geodesic distance (1.5.1).

Definition 1.5.10. [36] The *zonal basis function interpolant* $s^{(1)} : S^2 \to \mathbb{R}$ associated to $\mathcal{X}$ and to the corresponding function values of f on $\mathcal{X}$ is defined as

$$s^{(1)}(\mathbf{x}) = \sum_{j=1}^{K} a_j \psi(g(\mathbf{x}, \mathbf{x}_j)), \tag{1.5.13}$$

with the coefficients $a_j,\ j = 1, ..., K$, obtained from the interpolation relations

$$s^{(1)}(\mathbf{x}_i) = f_i,\ i = 1, ..., K,$$

where $\psi : [0, \pi] \to \mathbb{R}$ is a spherical radial basis function.

Definition 1.5.11. [35] The *augmented zonal basis function interpolant* $s^{(2)} : S^2 \to \mathbb{R}$ associated to $\mathcal{X}$ and to the corresponding function values of f on $\mathcal{X}$ is defined as

$$s^{(2)}(\mathbf{x}) = \sum_{j=1}^{K} a_j \psi(g(\mathbf{x}, \mathbf{x}_j)) + \sum_{k=1}^{D} A_k y_k(\mathbf{x}), \tag{1.5.14}$$

where $y_k \in \mathcal{H}_d(S^2),\ k = 1, \dots, D$, are spherical harmonics of maximum order d, $D = \dim \mathcal{H}_d(S^2)$ and $\{y_1, ..., y_D\}$ forms a basis for $\mathcal{H}_d(S^2)$.

The coefficients $a_j,\ j = 1, ..., K$ and $A_k,\ k = 1, ..., D$ are obtained from

$$s^{(2)}(\mathbf{x}_i) = f_i,\ i = 1, ..., K,$$

and imposing the constraints

$$\sum_{i=1}^{K} a_i y_k(\mathbf{x}_i) = 0,\ k = 1, ..., D.$$

Definition 1.5.12. [36] Attaching to each point $\mathbf{x}_j,\ j = 1, ..., K$, a set I_j of indices of n_Z closest neighbors of $\mathbf{x}_j$, we define a *local zonal basis function interpolant* $s_j^{(1)} : S^2 \to \mathbb{R}$ as

$$s_j^{(1)}(\mathbf{x}) = \sum_{i \in I_j} a_i^j \psi(g(\mathbf{x}, \mathbf{x}_j)), \tag{1.5.15}$$

with $a_i^j,\ i \in I_j,\ j = 1, ..., K$, resulting from imposing

$$s_j^{(1)}(\mathbf{x}_i) = f_i,\ i \in I_j,\ j = 1, ..., K.$$

Definition 1.5.13. [35] Attaching to each point $\mathbf{x}_j,\ j = 1, ..., K$, a set I_j of indices of n_Z closest neighbors of $\mathbf{x}_j$, we define the *augmented local zonal basis function interpolant* $s_j^{(2)} : S^2 \to \mathbb{R}$ as

$$s_j^{(2)}(\mathbf{x}) = \sum_{i \in I_j} a_i^j \psi(g(\mathbf{x}, \mathbf{x}_j)) + \sum_{k=1}^{D} A_k^j y_k(\mathbf{x}), \tag{1.5.16}$$

with $D = \dim \mathcal{H}_d(S^2),\ D \le n_Z,\ \{y_1, \dots, y_D\}$ basis for $\mathcal{H}_d(S^2)$.

The coefficients $a_i^j,\ i \in I_j,\ j = 1, ..., K$ and $A_k^j, k = 1, ..., D$, are obtained imposing

$$s_j^{(2)}(\mathbf{x}_i) = f_i,\ i \in I_j,\ j = 1, ..., K,$$

and

$$\sum_{i \in I_j} a_i^j y_k(\mathbf{x}_i) = 0,\ j = 1, ..., K,\ k = 1, ..., D.$$

Definition 1.5.14. [35, 36] For a set of distinct nodes $\mathcal{X} = \{\mathbf{x}_i \in S^2,\ i = 1, \dots, K\}$ and the associated function values $f_i = f(\mathbf{x}_i),\ i = 1, \dots, K$, for $f : S^2 \to \mathbb{R}$, *the modified spherical Shepard operator combined with a zonal basis function* is given as

$$S^{(k)}(\mathbf{x}) = \sum_{j=1}^{K} \overline{w}_j(\mathbf{x}) s_j^{(k)}(\mathbf{x}),\ k = 1, 2. \tag{1.5.17}$$

1.5.3 Spherical Shepard operator of Hermite-Birkhoff type

Allasia, Cavoretto and De Rossi proposed in [2] a method for spherical interpolation of scattered data based on Shepard-like cardinal basis functions and Hermite-Birkhoff-type operators.

The combined Shepard operator of Hermite-Birkhoff type is defined as [2]

$$H(\mathbf{x}) = \sum_{i=1}^{K} A_i(\mathbf{x}) T[f](\mathbf{x}, \mathbf{x}_i, \Delta_i), \tag{1.5.18}$$

where

$$T[f](\mathbf{x}, \mathbf{x}_i, \Delta_i) = \sum_{\gamma \in \Delta_i} \frac{\partial^{|\gamma|} f}{\partial x^{\gamma_1} \partial y^{\gamma_2} \partial z^{\gamma_3}} (\mathbf{x}_i) \cdot \prod_{j=1}^{3} \frac{(u_j - u_j(\mathbf{x}_i))^{\gamma_j}}{\gamma_j !}, \tag{1.5.19}$$

and A_i are some cardinal basis functions that satisfy the following conditions

$$A_i \in C^k(X),\ A_i(\mathbf{x}) \geq 0,\ \sum_{i=1}^{K} A_i(\mathbf{x}) = 1,\ A_i(\mathbf{x}_j) = \delta_{ij}, \tag{1.5.20}$$

and, moreover,

$$\frac{\partial^{|\gamma|} A_i}{\partial x^{\gamma_1} \partial y^{\gamma_2} \partial z^{\gamma_3}} (\mathbf{x}_j) = 0,\ \gamma \in \Delta_i,\ |\gamma| \in \{1, \ldots, k\},\ j = 1, \ldots, K.$$

To obtain a Shepard-like method, the cardinal basis functions can be expressed as [2]

$$A_i(\mathbf{x}) = \frac{(\alpha(\mathbf{x}, \mathbf{x}_i))^{-1}}{\sum\limits_{j=1}^{K} (\alpha(\mathbf{x}, \mathbf{x}_j))^{-1}},\ A_i(\mathbf{x}_i) = 1,\ i = 1, \ldots, K. \tag{1.5.21}$$

The function $\alpha : X \times X \to \mathbb{R}_+$ should be continuous and bounded, with $\alpha(\mathbf{x}, \mathbf{x}_i) > 0,\ \forall \mathbf{x} \in X,\ \mathbf{x} \neq \mathbf{x}_i$ and $\alpha(\mathbf{x}_i, \mathbf{x}_i) = 0,\ \forall \mathbf{x}_i \in \mathcal{X}$. In addition, α should be $k-$times continuously differentiable on X with

$$\left. \frac{\partial^{|\gamma|} \alpha}{\partial x^{\gamma_1} \partial y^{\gamma_2} \partial z^{\gamma_3}} (\mathbf{x}, \mathbf{x}_i) \right|_{\mathbf{x} = \mathbf{x}_i} = 0,\ 0 < |\gamma| \leq k,\ i = 1, \ldots, K.$$

Based on the geodesic distance g defined in (1.5.1), α can be written as

$$\alpha(\mathbf{x}, \mathbf{y}) = (g(\mathbf{x}, \mathbf{y}))^{\mu},\ \mu \in \mathbb{R}_+,\ \mu \geq k,\ \mathbf{x}, \mathbf{y} \in X. \tag{1.5.22}$$

Another approach developed in [2] implies a local method to construct the cardinal basis functions, of the form

$$\overline{w}_i(\mathbf{x}) = \frac{\tau(\mathbf{x}, \mathbf{x}_i) \left(g(\mathbf{x}, \mathbf{x}_i)\right)^{-\mu}}{\sum\limits_{k=1}^{K} \tau(\mathbf{x}, \mathbf{x}_k) \left(g(\mathbf{x}, \mathbf{x}_k)\right)^{-\mu}}, \tag{1.5.23}$$

such that $\overline{w}_i$ vanishes in the exterior of $\mathbf{x}_i$'s neighborhood, so

$$\tau(\mathbf{x}, \mathbf{x}_i) = \left(1 - \frac{g(\mathbf{x}, \mathbf{x}_i)}{R}\right)_+^{k+1},\ \mathbf{x} \in X,\ i = 1, \ldots, K.$$

With this choice, the function $\tau : X \to \mathbb{R}_+$ is of class C^k on X and, for a proper value of $R > 0$, it vanishes outside the ball of center $\mathbf{x}_i$ and radius R.

Now, the new *local Hermite-Birkhoff interpolation operator* can be written as [2]

$$\tilde{H}(\mathbf{x}) = \sum_{i=1}^{K} \overline{w}_i(\mathbf{x}) T[f](\mathbf{x}, \mathbf{x}_i, \Delta_i). \tag{1.5.24}$$

Chapter 2

Univariate Shepard operators obtained using least-squares polynomials

This chapter introduces a new univariate Shepard operator which is combined with polynomials constructed using the weighted least squares method. Section 2.1 presents the construction of these polynomials along with some of their properties, such as the interpolation properties, the degree of exactness and linearity. Section 2.2 deals with the resulting Shepard operator, proving that it inherits the previously mentioned properties. Finally, we analyze the error based on Peano's Theorem and provide some numerical examples that demonstrate the benefits of using these Shepard operators. The results obtained in this chapter were published in the paper Malina [62].

2.1 Construction and properties of univariate least-squares fitting polynomials

Renka introduced in [72], in 1988, an algorithm for improving the bivariate Shepard operator, considering a quadratic polynomial that interpolates the function f on a set of given nodes and also approximates the data in a weighted least squares way. Later on, in 1999, in [74], he improved this method by replacing the quadratic polynomial with a cubic one. In 2010, W. I. Thacker et al. [89] emphasized the main disadvantages of these two methods and suggested the combination of the Shepard operator with a linear polynomial that still fits the data in a weighted least squares sense.

Consider $X \subset \mathbb{R}$, $f : X \to \mathbb{R}$ and K given real nodes, denoted by x_j, $j = 1, ..., K$. The values of the function f on the given nodes are known and denoted by $f_j = f(x_j),\ j = 1, ..., K$.

Under these assumptions, for a point $x \in X$, let us define the nth degree polynomial function

$C_j^n[f]$, $j = 1, ..., K$, $n \in \mathbb{N}$, as

$$C_j^n[f](x) = f_j + \sum_{k=1}^{n} a_{j,k}(x - x_j)^k, \tag{2.1.1}$$

where the coefficients $a_{j,k}$ are found such that they minimize

$$E_j = \sum_{\substack{i=1 \\ i \neq j}}^{K} \lambda_{i,j} \left[C_j^n[f](x_i) - f_i \right]^2, \tag{2.1.2}$$

where

$$\lambda_{i,j} = \frac{|x_i - x_j|^{-\mu}}{\sum\limits_{\substack{k=1 \\ k \neq i}}^{K} |x_i - x_k|^{-\mu}}, \tag{2.1.3}$$

for $i, j = 1, ..., K$ and $\mu > 0$.

To find the coefficients $a_{j,k}$ (i.e., obtain the minimum of expression (2.1.2)), we solve the following system

$$\frac{\partial E_j}{\partial a_{j,k}} = 0, \text{ for each } k = 1, ..., n, \text{ and } j = 1, ..., K.$$

Further, for every $j = 1, ..., K$, one obtains

$$\frac{\partial E_j}{\partial a_{j,k}} = 2 \sum_{\substack{i=1 \\ i \neq j}}^{K} \lambda_{i,j} \left[\sum_{p=1}^{n} a_{j,p}(x_i - x_j)^p + (f_j - f_i) \right] (x_i - x_j)^k = 0, \text{ for each } k = 1, ..., n.$$

For every $j = 1, ..., K$, we can write the normal equations that result above in matrix form as

$$M_j \cdot a_j = b_j, \tag{2.1.4}$$

where M_j is a $n \times n$ matrix having on the entry (r, s) the element $\sum\limits_{\substack{i=1 \\ i \neq j}}^{K} \lambda_{i,j}(x_i - x_j)^{r+s}$, b_j is a column vector of n elements with $\sum\limits_{\substack{i=1 \\ i \neq j}}^{K} \lambda_{i,j}(x_i - x_j)^k (f_i - f_j)$ on the kth entry and $a_j = (a_{j,1},\ a_{j,2},\ ..., a_{j,n})^T$ is the vector of unknowns.

Theorem 2.1.1. *The operator $C_j^n[f]$ defined in (2.1.1) satisfies the following interpolation properties*

$$C_j^n[f](x_j) = f_j, \ j = 1, ..., K.$$

Theorem 2.1.2. *The operator $C_j^n[f]$, $j = 1, ..., K$, has the degree of exactness n, i.e.,*

$$\mathrm{dex}(C_j^n[f]) = n, \ j = 1, ..., K.$$

Theorem 2.1.3. *The operator $C_j^n[f]$ is linear.*

2.2 Univariate Shepard operators combined with least squares-fitting polynomials

Definition 2.2.1. For $f : X \subset \mathbb{R} \to \mathbb{R}$ and a set $\mathcal{X} = \{x_j : j = 1, \ldots, K\} \subset X$ of K interpolation nodes, such that the values of f are known on $\mathcal{X}$, we can define the *univariate Shepard operator combined with a nth degree polynomial* as

$$SP_n[f](x) = \sum_{j=1}^{K} A_{j,\mu}(x) C_j^n[f](x), \tag{2.2.1}$$

with $C_j^n[f]$ defined in (2.1.1), $A_{j,\mu}$ given by (1.3.2) and $\mu > 0$.

Theorem 2.2.2. *The following interpolation properties hold*

$$SP_n[f](x_j) = f(x_j), \ \ j = 1, ..., K.$$

Theorem 2.2.3. *The operator SP_n is linear.*

Theorem 2.2.4. *The Shepard operator SP_n has the degree of exactness n.*

The interpolation formula for the univariate Shepard operator combined with a polynomial is given by

$$f = SP_n[f] + R_n[f],$$

with $R_n[f]$ denoting the remainder.

Theorem 2.2.5. *If $f \in H^{n+1}[a, b]$, then*

$$R_n[f](x) = \int_a^b \phi_n(x, t) f^{(n+1)}(t) \ dt,$$

where

$$\phi_n(x, t) = \frac{(x - t)_+^n}{n!} - \sum_{j=1}^{K} A_{j,\mu}(x) \left[\frac{(x_j - t)_+^n}{n!} + \sum_{k=1}^{n} a_{j,k}(x - x_j)^k \right], \tag{2.2.2}$$

with $a_{j,k}$ given as solutions of $\frac{\partial E_j}{\partial a_{j,k}} = 0$, for each $k = 1, ..., n$, for

$$E_j = \sum_{\substack{i=1 \\ i \neq j}}^{K} \lambda_{i,j} \left[\frac{(x_j - t)_+^n}{n!} + \sum_{k=1}^{n} a_{j,k}(x_i - x_j)^k - \frac{(x_i - t)_+^n}{n!} \right]^2,$$

and $\lambda_{i,j}$ given in (2.1.3), $j = 1, ..., K$.

Chapter 3

Bivariate Shepard operators obtained using radial basis functions

This chapter introduces new combined Shepard operators in the two-dimensional case, using three radial basis functions: least squares thin-plate spline, inverse quadratic and inverse multiquadric. This approach aims to achieve better approximation results, due to the proven efficacy of radial basis functions in both practical and theoretical contexts. Besides the classical and the modified forms of the operator, we also use an iterative method that is comparable to the latter one. The original results presented in Sections 3.1 and 3.2 were published in three articles: Cătinaş and Malina [14], [17] and Malina [61].

At the end of the chapter, in Section 3.3, we consider an application of some of these operators in the reconstruction of damaged black-and-white and color images. The results obtained in this section are contained in Malina [60].

3.1 The combined Shepard operator of least squares thin-plate spline type

Consider f a real-valued function defined on $X \subset \mathbb{R}^2$, and $(x_i, y_i) \in X$ some distinct points, such that $f(x_i, y_i),\ i = 1, \ldots, K$, are known.

Definition 3.1.1. The *classical Shepard operator of least squares thin-plate spline type* is expressed as

$$S_\mu^m[f](x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y) F_i(x,y), \tag{3.1.1}$$

where $A_{i,\mu}$, $i = 1, ..., K$, are defined by (1.4.2), for a given parameter $\mu > 0$ and the least squares thin-plate splines are given by

$$F_i(x,y) = \sum_{j=1}^{i} \alpha_j d_j^2 \log(d_j) + ax + by + c, \quad i = 1, ..., K, \tag{3.1.2}$$

with $d_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$.

The coefficients α_j, a, b, c of F_i are found such that they minimize the expression

$$E = \sum_{i=1}^{K} [F_i(x_i, y_i) - f(x_i, y_i)]^2,$$

so, as solutions for systems of the following form (see, e.g., [64])

$$\begin{pmatrix} 0 & d_{12}^2 \log d_{12} & \cdots & d_{1K}^2 \log d_{1K} & x_1 & y_1 & 1 \\ d_{21}^2 \log d_{21} & 0 & \cdots & d_{2K}^2 \log d_{2K} & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{K1}^2 \log d_{K1} & d_{K2}^2 \log d_{K2} & \cdots & 0 & x_K & y_K & 1 \\ x_1 & x_2 & \cdots & x_K & 0 & 0 & 0 \\ y_1 & y_2 & \cdots & y_K & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_K \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_K \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.1.3}$$

with $d_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$, $f_i = f(x_i, y_i)$, $i, j = 1, \ldots, K$.

Definition 3.1.2. We define the *modified Shepard operator of least squares thin-plate spline type* as

$$S_W^m[f](x, y) = \sum_{i=1}^{K} \overline{w}_{i,\mu}(x, y) F_i(x, y), \tag{3.1.4}$$

with $\overline{w}_{i,\mu}$ given by (1.4.4) and F_i given by (3.1.2), for $i = 1, ..., K$.

To improve upon the methods described above, we consider a smaller set of $k \in \mathbb{N}^*$ knot points $(\hat{x}_j, \hat{y}_j)$, $j = 1, ..., k$, that will be representative for the original set of K interpolation nodes. This set is obtained following an idea proposed by J. R. McMahon in 1986. The steps of the algorithm are outlined below (see, e.g., [64], [65]):

1. Generate k random knot points, with $k < K$;
2. Assign to each point the closest knot point, based on the Euclidean distance ;
3. If there are knots with no point assigned, replace them with the nearest point ;
4. Compute the next set of knots as the arithmetic mean of all corresponding points ;
5. Repeat steps 2-4 until the sets of knots remain unchanged for two successive iterations.

Algorithm 1: *Generation of representative knot points for a given set of data.*

Remark 3.1.3. Using the set of K interpolation nodes is indicated by setting $m = 1$, while using the representative set of k knot points is indicated by $m = 2$.

The modified Shepard operator, introduced by Franke and Nielson [47], requires some artificial parameters such as the number of closest nodes or a radius of influence. An alternative approach was proposed in [63], consisting of an iterative method that is free of these setup parameters and performs a reduction of the current interpolation result's residue at each iteration. The accuracy of this method is comparable to that of the modified Shepard procedure as shown in [63], although there are cases where one method is preferred to the other.

For $(x,y) \in X$, the iterative Shepard operator introduced in [63] is of the following form

$$u(x,y) = \sum_{i=0}^{M} \sum_{j=1}^{K} \left[u_j^{(k)} \omega\left((x - x_j, y - y_j)/\tau_i\right) / \sum_{p=1}^{K} \omega\left((x_p - x_j, y_p - y_j)/\tau_i\right) \right], \tag{3.1.5}$$

where $(x_i, y_i) \in X,\ i = 1, ..., K$, are the interpolation nodes and ω is a continuously differentiable weight function, satisfying the properties:

$$\omega(x,y) \geq 0,\ \forall (x,y) \in \mathbb{R}^2,\ \omega(0,0) > 0,\ \text{and } \omega(x,y) = 0 \text{ if } \|(x,y)\| > 1.$$

In the equation above, $u_j^{(k)}$ denotes the interpolation residuals at the kth step, with $u_j^{(0)} \equiv u_j$.

Using the ideas and method described in [63], we introduce below a new Shepard operator of least squares thin-plate spline type.

Definition 3.1.4. The *iterative Shepard operator of least squares thin-plate spline type* is represented as

$$u_L^m[f](x,y) = \sum_{i=0}^{M} \sum_{j=1}^{K} \left[u_{F_j}^{(i)} \omega\left((x - x_j, y - y_j)/\tau_i\right) / \sum_{p=1}^{K} \omega\left((x_p - x_j, y_p - y_j)/\tau_i\right) \right], \tag{3.1.6}$$

with the interpolation residuals at the ith step $u_{F_j}^{(i)}$ given by

$$u_{F_j}^{(0)} = F_j(x_j, y_j),\ (x_j, y_j) \in X,\ j = 1, ..., K,$$

and

$$u_{F_j}^{(i+1)} = u_{F_j}^{(i)} - \sum_{q=1}^{K} \left[u_{F_q}^{(i)} \omega\left((x_j - x_q, y_j - y_q)/\tau_i\right) / \sum_{p=1}^{K} \omega\left((x_p - x_q, y_p - y_q)/\tau_i\right) \right].$$

The weight function ω is defined as

$$\omega(x,y) = \omega(x)\omega(y),$$

with

$$\omega(x) = \begin{cases} 5(1-|x|)^4 - 4(1-|x|)^5, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}.$$

The functions F_j are the least squares thin-plate splines given in (3.1.2). The parameter τ_i is chosen as in [63] and it decreases from a given value τ_0, which can be, for example,

$$\tau_0 > \sup_{(x,y)\in X} \max_{1\leq j\leq K} \|(x - x_j, y - y_j)\|$$

to

$$\tau_M < \min_{k\neq j} \|(x_k - x_j, y_k - y_j)\|.$$

The sequence $\{\tau_i\}$ of scale factors is given by

$$\tau_i = \tau_0 \cdot \gamma^i, \qquad \gamma \in (0,1).$$

Applying the concept of multiscale analysis, it was demonstrated in [63] that the behavior of the interpolant remains relatively stable for $\gamma \in [0.6, 0.95]$. Smaller values of γ can also be selected to reduce the computational time, especially in the case of sparsed interpolation nodes.

3.2 The combined Shepard operator of inverse quadratic and inverse multiquadric type

For a function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$, with known values only on a set of K interpolation nodes $(x_i, y_i) \in X,\ i = 1, \ldots, K$, let us consider the functions

$$\phi_i^{\beta}(x,y) = \sum_{j=1}^{i} \alpha_j \left[1 + (\epsilon d_j)^2\right]^{\beta} + ax + by + c, \quad i = 1, ..., K, \tag{3.2.1}$$

with ϵ being a shape parameter and $d_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$.

For $\beta = -1$, we get the case of inverse quadratic RBF and for $\beta = -1/2$, the case of inverse multiquadric RBF.

The coefficients $\alpha_j,\ a,\ b,\ c$ are obtained as solutions of systems that have a similar form to the ones in (3.1.3). Shortly, they can be written as

$$\begin{pmatrix} A & X^T \\ X & O_3 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \tag{3.2.2}$$

considering the following notations:

$$A \in \mathbb{R}^{K \times K},\ a_{ij} = \left[1 + (\epsilon d_{ij})^2\right]^{\beta},\ j = 1, ..., K,\ \beta \in \{-1,\ -1/2\};$$

$$X \in \mathbb{R}^{3 \times K}(\mathbb{R}),\ X = \begin{pmatrix} x_1 & \ldots & x_K \\ y_1 & \ldots & y_K \\ 1 & \ldots & 1 \end{pmatrix};$$

$$\mathbf{u} = (a,\ b,\ c)^T,\ \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K)^T, \mathbf{0} = (0,\ 0,\ 0)^T;$$

$$\mathbf{f} = (f_1, \ldots, f_K)^T, f_i = f(x_i, y_i),\ i = 1, \ldots, K.$$

As in the thin-plate spline case, we will consider two sets of nodes: an initial set with K interpolation nodes ($m = 1$) and a second set with k representative knot points, obtained using Algorithm 1 ($m = 2$).

Definition 3.2.1. The *classical Shepard operators combined with the inverse quadratic and inverse multiquadric RBFs* are defined as

$$S_{\mu}^{\beta,m}[f](x,y) = \sum_{i=1}^{K} A_{i,\mu}(x,y) \phi_i^{\beta}(x,y), \tag{3.2.3}$$

where $A_{i,\mu}$, $i = 1, ..., K$, are defined by (1.4.2), for a given parameter $\mu > 0$ and ϕ_i^{β} are given in (3.2.1), for $\beta \in \{-1, -1/2\}$.

Definition 3.2.2. We define the *modified Shepard operators combined with the inverse quadratic and inverse multiquadric RBFs* as

$$S_{W}^{\beta,m}[f](x,y) = \sum_{i=1}^{K} \overline{w}_{i,\mu}\,(x,y)\, \phi_i^{\beta}(x,y), \tag{3.2.4}$$

with $\overline{w}_{i\ \mu}$, $i = 1, ..., K$, given by (1.4.4) and ϕ_i^{β} defined in (3.2.1), for $\beta \in \{-1, -1/2\}$.

Definition 3.2.3. The *iterative Shepard operators combined with the inverse quadratic and inverse multiquadric RBFs* are expressed as

$$u_{\phi}^{\beta,m}[f](x,y)=\sum_{i=0}^{M}\sum_{j=1}^{K}\left[u_{\phi_j^{\beta}}^{(i)}\omega\left((x-x_j,y-y_j)/\tau_i\right)/\sum_{p=1}^{K}\omega\left((x_p-x_j,y_p-y_j)/\tau_i\right)\right], \qquad (3.2.5)$$

with $\beta\in\{-1,-1/2\}$ and $u_{\phi_j^{\beta}}^{(i)}$ the interpolation residuals at the ith step given by

$$u_{\phi_j^{\beta}}^{(0)}=\phi_j^{\beta}(x_j,y_j),\ (x_j,y_j)\in X,\ j=1,...,K,$$

and

$$u_{\phi_j^{\beta}}^{(i+1)}=u_{\phi_j^{\beta}}^{(i)}-\sum_{q=1}^{K}\left[u_{\phi_q^{\beta}}^{(i)}\omega\left((x_j-x_q,y_j-y_q)/\tau_i\right)/\sum_{p=1}^{K}w\left((x_p-x_q,y_p-y_q)/\tau_i\right)\right].$$

The functions ϕ_i^{β} are given in (3.2.1). For the choice of parameters in the iterative approach, we follow the ideas from [63], as detailed in the preceding section.

3.3 Application in image reconstruction

In the subsequent part, we focus on an application of a combined-type Shepard operator, specifically in image reconstruction. This problem has been intensively studied based on radial basis functions approaches (see, e.g., [69], [80], [83], [91], [92], [95], [96]), but few results have been reported for the Shepard operator. Here, we will reconstruct damaged images, both black-and-white and color, using the combined Shepard operator of inverse quadratic and inverse multiquadric type. Image reconstruction is typically required in cases involving inpainting or noise. Since our focus is on restoration rather than damage detection, we assume that the area to be reconstructed has already been identified.

3.3.1 Reconstruction of damaged black-and-white images

Consider an original, uncorrupted black-and-white image whose matrix representation is denoted by M, with each entry $M(x,y)$ storing a pixel of the image. Let us denote the matrix representation of the corrupted image by $\widehat{M}$. The coordinates of a valid pixel $\mathbf{f}_i$ are denoted by $\mathbf{x}_i=(x_i,y_i)$, i.e., $\mathbf{f}_i=\widehat{M}(x_i,y_i)$ and the coordinates of a defective pixel $\widehat{\mathbf{f}}_i$ are $\widehat{\mathbf{x}}_i=(\widehat{x}_i,\widehat{y}_i)$, i.e., $\widehat{\mathbf{f}}_i=\widehat{M}(\widehat{x}_i,\widehat{y}_i)$. In our approach, we will deliberately corrupt several pixels using the "Salt-and-Pepper" noise. This consists of changing the values of a specific percentage of pixels into 0 (black = pepper) or 255 (white = salt).

- **Global case**

Following some ideas proposed in [69] we consider the reconstruction of a damaged black-and-white image using the bivariate Shepard operators $S_{\mu}^{\beta}[f]$ introduced in (3.2.3), specifically of inverse quadratic ($\beta=-1$) and inverse multiquadric ($\beta=-1/2$) types. For the reconstruction, we will use a global approach, where the value of a defective pixel is computed based on all the information provided by the set of correct pixels. Consider the matrix $\widehat{M}$ associated with an image of resolution $m \times n$ that contains a percentage $p\%$ of damaged pixels. Let us denote by K the number of valid pixels and by $\widehat{K}$ the number of defective pixels.

Let $\mathcal{X}$ be the set of interpolation nodes, $\mathcal{X} = \{\mathbf{x}_i = (x_i, y_i),\ i = 1, ..., K\}$ where x_i and y_i represent the matrix coordinates of the correct pixels $\mathbf{f}_i$, $i = 1, \ldots, K$. We apply the Shepard operator given in (3.2.3) to reconstruct the set of the defective pixels, $\widehat{\mathcal{X}} = \{\widehat{\mathbf{x}}_i = (\widehat{x}_i, \widehat{y}_i),\ i = 1, ..., \widehat{K}\}$. The reconstructed values of the damaged pixels are obtained as $\widehat{\mathbf{f}}_i = S_\mu^\beta[\mathbf{f}](\widehat{\mathbf{x}}_i),\ i = 1, \ldots, \widehat{K}$. The pseudo-code for this approach is given in Algorithm 2.

Data: damaged matrix $\widehat{M}$
Result: reconstructed matrix M'
$\mathcal{X} = \{\mathbf{x}_i = (x_i, y_i),\ i = 1, ..., K\}$; `/* correct pixels' coordinates */`
$\widehat{\mathcal{X}} = \{\widehat{\mathbf{x}}_i = (\widehat{x}_i, \widehat{y}_i),\ i = 1, ..., \widehat{K}\}$; `/* defective pixels' coordinates */`
$\mathbf{f} = \{\widehat{M}(\mathbf{x}_i),\ \mathbf{x}_i \in \mathcal{X},\ i = 1, ..., K\}$; `/* correct pixels' values */`
$M' = \widehat{M}$;
for $i = 1 \ldots \widehat{K}$ **do**
  $M'(\widehat{\mathbf{x}}_i) = S_\mu^\beta[\mathbf{f}](\widehat{\mathbf{x}}_i)$; `/* reconstruction of damaged pixels */`
end

Algorithm 2: *Global reconstruction of black-and-white images.*

- **Local case**

In the global method, an incorrect pixel's value is reconstructed based on all the information provided by the correct pixels, but this approach does not always produce the best results for the reconstructed image, since the pixels of an image have strongly local properties, as emphasized in [83]. It would be more efficient to approximate the value of an incorrect pixel based on a local procedure. This approach leads to a smaller computational time compared to the global case, especially for high-resolution images because the systems (3.2.2) have a significantly reduced size. The pseudo-code is presented in Algorithm 3.

3.3.2 Reconstruction of damaged color images

Consider a multivalued function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^m,\ f = (f_1, \ldots, f_m)$ and a set of K interpolation nodes $\mathbf{x}_i = (x_i, y_i) \in X,\ i = 1, \ldots, K$. Somogyi and Soos introduced in [88] the Shepard-type multivalued interpolation operator as

$$\bigcup_{k=1}^{m} S_{\mu,k}[f](\mathbf{x}) = \bigcup_{k=1}^{m} \sum_{i=1}^{K} A_{i,\mu}(\mathbf{x}) f_k(\mathbf{x}_i), \tag{3.3.1}$$

for $\mathbf{x} \in X$, $\mu > 0$, $A_{i,\mu}$ given in (1.4.2).

Based on this, we can define the multivalued Shepard operator combined with inverse quadratic and inverse multiquadric RBFs in the bivariate case as

$$\bigcup_{k=1}^{m} S_{\mu,k}^\beta[f](\mathbf{x}) = \bigcup_{k=1}^{m} \sum_{i=1}^{K} A_{i,\mu}(\mathbf{x}) \phi_{i,k}^\beta(\mathbf{x}), \tag{3.3.2}$$

Data: damaged matrix $\widehat{M}$, initial tolerance ε
Result: reconstructed matrix M'
$\widehat{\mathcal{X}} = \{\widehat{\mathbf{x}}_i = (\widehat{x}_i, \widehat{y}_i),\ i = 1, ..., \widehat{K}\}$; `/* defective pixels' coordinates */`
$M = \widehat{M}$;
while $\widehat{\mathcal{X}}$ *not empty* **do**
  $\widehat{K} = \text{size}(\widehat{\mathcal{X}})$;
  for $i = 1 \ldots \widehat{K}$ **do**
    Define $\mathcal{X}_i$ = neighborhood of $\widehat{\mathbf{x}}_i$;
    $X_i = \{\mathbf{x}_j \in \mathcal{X}_i,\ j = 1, ..., K_i\}$; `/* correct pixels in` $\mathcal{X}_i$ `*/`
    if *nr_wrong_pixels_neighborhood* / *nr_pixels_neighborhood* $< \varepsilon$ **then**
      $\mathbf{f}_i = \{\widehat{M}(\mathbf{x}_j),\ \mathbf{x}_j \in X_i,\ j = 1, ..., K_i\}$;
      $M(\widehat{\mathbf{x}}_i) = S_\mu^\beta[\mathbf{f}_i](\widehat{\mathbf{x}}_i)$; `/* reconstruction of damaged pixels */`
    end
  end
  Update $\widehat{\mathcal{X}}$;
  $\varepsilon = \varepsilon + 0.01$;
end

Algorithm 3: *Local reconstruction of black-and-white images.*

for $\mathbf{x} \in X$, $\mu > 0$, $A_{i,\mu}$ given in (1.4.2) and

$$\phi_{i,k}^\beta(\mathbf{x}) = \sum_{j=1}^{i} \alpha_{j,k} \left[1 + (\epsilon d_j)^2\right]^\beta + a_k x + b_k y + c_k,\ k = 1, \ldots, m. \tag{3.3.3}$$

The coefficients $\alpha_{j,k}, a_k, b_k, c_k$ are found solving similar systems as the one in (3.2.2).

A color image is represented as an $m \times n \times 3$ structure with each component defining the colors red, green and blue (RGB) of every pixel. Using the multivalued Shepard operator (3.3.2) with $m = 3$, we can reconstruct colored images using similar ideas as in the case of black-and-white images. Computational, we apply the local method described in Algorithm 3 for each of the three components: red, green and blue. Since the three component reconstructions are independent, we considered parallel computing, performed in Matlab.

Chapter 4

Spherical interpolation using some new Shepard operators

The last chapter is dedicated to spherical interpolation of scattered data using combined Shepard operators. As mentioned in Section 1.5, this interpolation problem is important as it appears in solving some problems related to physical phenomena. The new results obtained in Section 4.1 and published in the paper Cătinaş and Malina [15] are derived using two spherical radial basis functions: the least squares thin-plate spline and the inverse multiquadric.

The second Shepard method is obtained using the Bernoulli operators, suitable when information about the function's derivatives is known. After performing the Delaunay triangulation of the sphere, we consider two approaches in applying this kind of operator, detailed in Section 4.2. The original results have been published in the paper Cătinaş and Malina [16].

Two physical phenomena are investigated in Sections 4.3 and 4.4: temperature prediction on the Earth's surface and topographic data approximation. The results show that these methods represent a powerful instrument for solving various problems that model real-life phenomena.

4.1 Spherical Shepard operators combined with radial basis functions

4.1.1 Combined spherical Shepard operators of least squares thin-plate spline and inverse multiquadric type

We consider S^2 to be the unit sphere in $\mathbb{R}^3$ centered at the origin and a set of given interpolation nodes $\mathcal{X} = \{\mathbf{x}_i = (x_i, y_i, z_i) \in S^2 : \ i = 1, \ldots, K\}$ together with the corresponding function values $f_i = f(\mathbf{x}_i), \ i = 1, \ldots, K$, of $f : S^2 \to \mathbb{R}$. Using the local zonal basis functions (1.5.15) and (1.5.16), built upon the thin-plate spline and the inverse multiquadric functions, we intend to improve the modified spherical Shepard operator (1.5.10), considering $\mu = 2$.

Definition 4.1.1. Let $\mathcal{X} = \{\mathbf{x}_i, \ i = 1, \ldots, K\} \subset S^2$ be a set of given data samples and I_j a set that contains the indices that correspond to n_Z closest neighbors of $\mathbf{x}_j$. We define the new *local*

Shepard interpolants of thin-plate spline type as

$$S_1(\mathbf{x}) = \sum_{j=1}^{K} \overline{w}_j(\mathbf{x}) Z_j^{(1)}(\mathbf{x}), \tag{4.1.1}$$

with the local zonal basis function given by

$$Z_j^{(1)}(\mathbf{x}) = \sum_{i \in I_j} a_i^j \psi_1(g(\mathbf{x}, \mathbf{x}_i)),$$

and

$$S_2(\mathbf{x}) = \sum_{j=1}^{K} \overline{w}_j(\mathbf{x}) Z_j^{(2)}(\mathbf{x}), \tag{4.1.2}$$

with the augmented zonal basis function given by

$$Z_j^{(2)}(\mathbf{x}) = \sum_{i \in I_j} a_i^j \psi_1(g(\mathbf{x}, \mathbf{x}_i)) + \sum_{k=1}^{D} A_k^j y_k(\mathbf{x}),$$

considering the thin-plate spline spherical radial basis function (see, e.g., [5])

$$\psi_1(r) = r^2 \log r,\ r = 2 \sin \frac{g(\mathbf{x}, \mathbf{y})}{2},\ g(\mathbf{x}, \mathbf{y}) = \arccos(\mathbf{x} \cdot \mathbf{y}).$$

In both cases, the coefficients that appear are found from

$$Z_j^{(k)}(\mathbf{x}_i) = f_i,\ i \in I_j,\ j = 1, \ldots, K,\ k = 1, 2, \tag{4.1.3}$$

along with additional constraints for $Z_j^{(2)}$, which are imposed on the set that contains spherical harmonics of maximum order d on S^2, $\mathcal{Y} = \{y_1, \ldots, y_D\}$. Here, $\mathcal{Y}$ constitutes a basis for $\mathcal{H}_d(S^2)$ and $D = \dim \mathcal{H}_d(S^2) \leq n_Z$. These conditions are

$$\sum_{i \in I_j} a_i^j y_k(\mathbf{x}_i) = 0,\ k = 1, \ldots, D,\ j = 1, \ldots, K. \tag{4.1.4}$$

Definition 4.1.2. Consider $\mathcal{X} = \{\mathbf{x}_i,\ i = 1, \ldots, K\} \subset S^2$ a set of interpolation nodes lying on S^2 and I_j a set that consists of the indices of n_Z neighbors for the node $\mathbf{x}_j$. We define the new *local Shepard interpolants of inverse multiquadric type* as

$$S_3(\mathbf{x}) = \sum_{j=1}^{K} \overline{w}_j(\mathbf{x}) Z_j^{(3)}(\mathbf{x}), \tag{4.1.5}$$

with the local zonal basis function given by

$$Z_j^{(3)}(\mathbf{x}) = \sum_{i \in I_j} a_i^j \psi_2(g(\mathbf{x}, \mathbf{x}_i)),$$

and

$$S_4(\mathbf{x}) = \sum_{j=1}^{K} \overline{w}_j(\mathbf{x}) Z_j^{(4)}(\mathbf{x}), \tag{4.1.6}$$

with the augmented local zonal basis function given by

$$Z_j^{(4)}(\mathbf{x}) = \sum_{i \in I_j} a_i^j \psi_2(g(\mathbf{x}, \mathbf{x}_i)) + \sum_{k=1}^{D} A_k^j y_k(\mathbf{x}), \tag{4.1.7}$$

considering the inverse multiquadric spherical radial basis function (see, e.g., [5])

$$\psi_2(r) = (r^2 + c^2)^{-\frac{1}{2}},\ r = 2\sin\frac{g(\mathbf{x},\mathbf{y})}{2},\ g(\mathbf{x},\mathbf{y}) = \arccos(\mathbf{x}\cdot\mathbf{y}).$$

As in the previous case, the coefficients of $Z_j^{(3)}$ and $Z_j^{(4)}$ are obtained from the interpolation conditions (4.1.3), together with the additional constraints (4.1.4) for $Z_j^{(4)}$.

Theorem 4.1.3. *Consider a set of distinct nodes $\mathcal{X} = \{\mathbf{x}_i,\ i = 1, ..., K\}$, lying on S^2 and the corresponding function values $f_i, i = 1, ..., K$, with $f : S^2 \to \mathbb{R}$. For each $\mathbf{x} \in S^2$, we obtain the following approximation of the error of the Shepard operators $S_i,\ i = 1, ..., 4$, given by (4.1.1), (4.1.2), (4.1.5) and (4.1.6):*

$$E_i(x) = |f(\mathbf{x}) - S_i(\mathbf{x})| \le \sum_{j=1}^{K} \overline{w}_j(\mathbf{x}) e_j(\mathbf{x}), \quad i = 1, 2, 3, 4,$$

with $e_j(\mathbf{x}) = \left| f(\mathbf{x}) - Z_j^{(i)}(\mathbf{x}) \right|$ being the interpolation error of the local basis functions $Z_j^{(i)},\ i = 1, ..., 4$, on the set of nodes $\mathbf{x}_k,\ k \in I_j,\ I_j$ containing the indices of n_Z closest neighbours of $\mathbf{x}_j$, $j = 1, ..., K$. In addition, we have

$$E_i(\mathbf{x}) \le \max_{j=1,...,K} e_j(\mathbf{x}),\ \textit{for } i = 1, ..., 4,\ \textit{and } \mathbf{x} \in S^2.$$

Theorem 4.1.4. *Consider a set of distinct nodes $\mathcal{X} = \{\mathbf{x}_i,\ i = 1, ..., K\}$, lying on S^2 and the corresponding function values $f_i, i = 1, ..., K$, with $f : S^2 \to \mathbb{R}$. For the Shepard operators $S_1,\ S_3$ given in (4.1.1) and (4.1.5), respectively, we have $S_i(\mathbf{x}) \in C^1(S^2)$, $i = 1, 3$.*

Theorem 4.1.5. *For $\mathcal{X} = \{\mathbf{x}_i,\ i = 1, \dots, K\}$ a set of distinct nodes in S^2 and $f \in C(S^2)$, the following estimation holds*

$$|S_i(\mathbf{x}) - f(\mathbf{x})| \le \sum_{j=1}^{K} \overline{w}_j(\mathbf{x}) |Z_j^{(i)}(\mathbf{x}) - Z_j^{(i)}(\mathbf{x}_j)| + \omega(f, h_{\mathcal{X}}),\ \textit{for } i = 1, ..., 4,$$

where $\omega(f, h_{\mathcal{X}}) = \sup\limits_{d(\mathbf{x},\mathbf{y}) \le h_{\mathcal{X}}} |f(\mathbf{x}) - f(\mathbf{y})|$ is the modulus of continuity of f and $h_{\mathcal{X}}$ is the mesh norm, i.e., $h_{\mathcal{X}} = \sup\limits_{\mathbf{x} \in S^2} g(\mathbf{x}, \mathcal{X})$.

4.2 Spherical Shepard-Bernoulli operators

4.2.1 Delaunay triangulation of a sphere

The problem of data interpolation on the sphere, with the aid of triangulation methods, has been addressed and solved, for example, in [68], [75].

Definition 4.2.1. [73] A triangulation T of $\mathcal{X}$ is a set of triangles that have the following properties:

1. The vertices of the triangles in T are formed by the nodes in $\mathcal{X}$;

2. The only nodes contained in a triangle are the ones that form its vertices;

3. The interiors of the triangles are pairwise disjoint;

4. The union of all triangles covers the convex hull of $\mathcal{X}$.

Remark 4.2.2. The vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ of a triangle are specified in counterclockwise order (i.e, the determinant that has as rows/columns $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, in this order, is non-negative).

Definition 4.2.3. [73] We say that a triangulation T has the *empty circumcircle interior property* if the circumcircle corresponding to each triangle of T does not contain any nodes in its interior. This kind of triangulation is called *Delaunay triangulation.*

4.2.2 Combined spherical Shepard-Bernoulli method

To obtain the new combined Shepard operators of Bernoulli type, we will use the spherical coordinates (ϕ, θ) corresponding to $\mathbf{x} \in S^2$, given in cartesian coordinates (x, y, z).

Similarly to the planar case (1.4.14), one can obtain the directional derivatives of a function f with respect to ϕ and θ (see, e.g., [68]), so, $B_m^{T(i)}[\tilde{f}](\phi, \theta)$ can be written similarly to (1.4.27), considering $\tilde{f}(\phi, \theta) = f(x, y, z)$.

Moreover, in this case, the barycentric coordinates λ_1, λ_2, λ_3 given in (1.4.13) are computed based on the signed area $\mathcal{A}$ of a spherical triangle of vertices $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S^2$, that is

$$\tan\left(\frac{\mathcal{A}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)}{2}\right) = \frac{\mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3)}{1 + \mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_2 \cdot \mathbf{x}_3 + \mathbf{x}_1 \cdot \mathbf{x}_3},$$

with $''\cdot''$ denoting the Euclidean inner product and $''\times''$ the vector cross product.

Remark 4.2.4. The barycentric coordinates λ_1, λ_2, λ_3 satisfy the relation $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Definition 4.2.5. Let T be the Delaunay triangulation of a set $\mathcal{X}$ containing K data samples $\mathbf{x}_i \in S^2$, $i = 1, \ldots, K$, and $\mathcal{T}_i$ the set of all triangles that have a vertex in $\mathbf{x}_i$, for each $i = 1, \ldots, K$. Choosing the representative triangle $T(i) \subset \mathcal{T}_i$, on which the operator $B_m^{T(i)}[\tilde{f}]$ is constructed, such that the approximation error is minimum, we define the *spherical Shepard-Bernoulli operator* as

$$S_{B_m}^1[f](\mathbf{x}) = \sum_{i=1}^{K} A_{i,\mu}(\mathbf{x}) B_m^{T(i)}[\tilde{f}](\phi, \theta), \quad \mathbf{x} \in S^2, \tag{4.2.1}$$

with $B_m^{T(i)}[\tilde{f}](\phi, \theta)$ given in (1.4.27), $\tilde{f}(\phi, \theta) = f(x, y, z)$ and $A_{i,\mu}$ given in barycentric form as

$$A_{i,\mu}(\mathbf{x}) = \frac{(g(\mathbf{x}, \mathbf{x}_i))^{-\mu}}{\sum\limits_{k=1}^{K} (g(\mathbf{x}, \mathbf{x}_k))^{-\mu}}, \tag{4.2.2}$$

for $\mu \in \mathbb{R}_+$ a control parameter and g the geodesic distance (1.5.1).

Theorem 4.2.6. *Consider a set of distinct nodes $\mathbf{x}_i$, $i = 1, ..., K$, lying on the unit sphere S^2 and $f : S^2 \to \mathbb{R}$. For each $\mathbf{x} \in S^2$, we have the following estimation of the error of the Shepard operator $S_{B_m}^1$ given by (4.2.1)*

$$E(\mathbf{x}) = \left|f(\mathbf{x}) - S_{B_m}^1[f](\mathbf{x})\right| \leq \sum_{i=1}^{K} A_{i,\mu}(\mathbf{x}) e_i(\mathbf{x}),$$

with $e_i(\mathbf{x}) = \left| f(\mathbf{x}) - B_m^{T(i)}[f](\phi,\theta) \right|, \quad i = 1, ..., K$. *In addition, we have*

$$E(\mathbf{x}) \leq \max_{i=1,\ldots,K} e_i(\mathbf{x}), \text{ for all } \mathbf{x} \in S^2.$$

In the sequel, we consider another approach, proposed in [97] and [98], that consists of constructing a Bernoulli operator on each triangle from the triangulation T of S^2. Here, we use the Delaunay triangulation. Let N be the number of triangles that form the triangulation. In this part, $\mathcal{T}_i,\ i = 1, \ldots, N$, will denote a triangle from the triangulation, with vertices $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \mathbf{x}_{i_3}$, such that each node $\mathbf{x}_i,\ i = 1, \ldots, K$, is a vertex of at least one triangle, i.e.,

$$\bigcup_i \{i_1, i_2, i_3\} = \{1, \ldots, K\}.$$

Each basis function $\Phi_{i,\mu}$ corresponding to $\mathcal{T}_i$, $i = 1, \ldots, N$, will have the form [97]

$$\Phi_{i,\mu}(\mathbf{x}) = \frac{\prod_{j=1}^{3} \left(g(\mathbf{x}, \mathbf{x}_{i_j})\right)^{-\mu}}{\sum_{k=1}^{N} \prod_{j=1}^{3} \left(g(\mathbf{x}, \mathbf{x}_{k_j})\right)^{-\mu}}, \quad i = 1, \ldots, N, \tag{4.2.3}$$

for $\mu \in \mathbb{R}_+$ a control parameter.

Definition 4.2.7. Let T be a Delaunay triangulation of the unit sphere S^2 and $\mathcal{T}_i$, $i = 1, \ldots, N$, a triangle of T, such that each point from a set of K distinct nodes $\mathbf{x}_i \in S^2,\ i = 1, \ldots, K$, is the vertex of at least one triangle from T. Using the basis functions $\Phi_{i,\mu},\ i = 1, \ldots, N$, given by (4.2.3), we define *the modified spherical Shepard-Bernoulli operator* as

$$S_{B_m}^2[f](\mathbf{x}) = \sum_{i=1}^{N} \Phi_{i,\mu}(\mathbf{x}) B_m^{\mathcal{T}_i}[\tilde{f}](\phi,\theta), \text{for all } \mathbf{x} \in S^2, \tag{4.2.4}$$

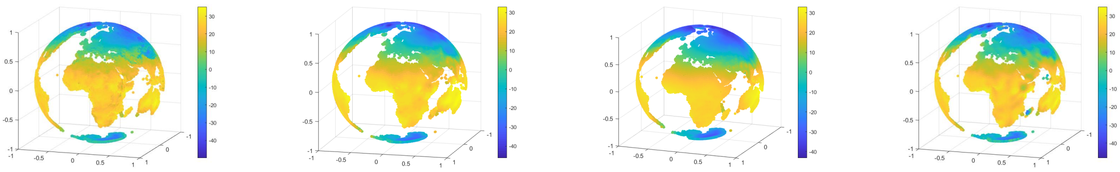
with $B_m^{\mathcal{T}_i}[\tilde{f}](\phi,\theta)$ given in (1.4.27) and $\tilde{f}(\phi,\theta) = f(x,y,z)$.

Theorem 4.2.8. *Considering* $\mathbf{x}_j,\ j = 1, ..., K$, *the vertices of at least one triangle* $\mathcal{T}_i,\ i = 1, \ldots, N$, *we have the following interpolation properties of* $S_{B_m}^2[f]$,

$$S_{B_m}^2[f](\mathbf{x}_j) = f(\mathbf{x}_j), \quad j = 1, ..., K. \tag{4.2.5}$$

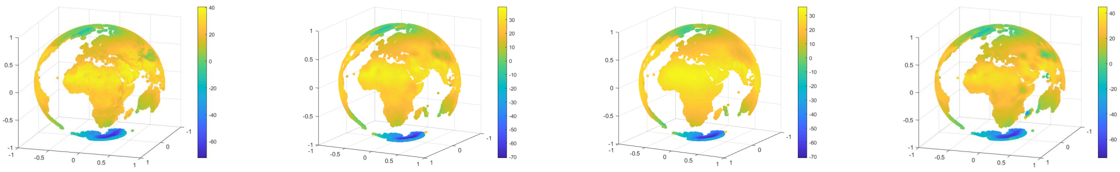
4.3 Application in monthly mean temperature predictions

We consider an example of approximation with the operators introduced in the previous two sections, using some real data, namely the monthly mean temperatures on the Globe in January 2010 and June 2010. The set of data was selected from `https://www.kaggle.com/datasets/shishu1421/global-temperature?select=air_temp.2010`. For our numerical tests, we considered 1073 nodes and we reconstructed the temperature values for 21449. For the Shepard operator combined with the two spherical RBFs, we considered the case of zonal basis functions combined with a spherical harmonic, S_2 and S_4, as given in (4.1.2) and (4.1.6). In the case of the Shepard-Bernoulli operator, we considered its second variant, as in (4.2.4), using the Bernoulli operator of order 1, i.e., $S_{B_1}^2$. The values from January are displayed in Figure 4.1 and the values from June in Figure 4.2.



(a) Real values. (b) S_2 (c) S_4 (d) $S_{B_1}^2$

Figure 4.1: Temperatures in January 2010.

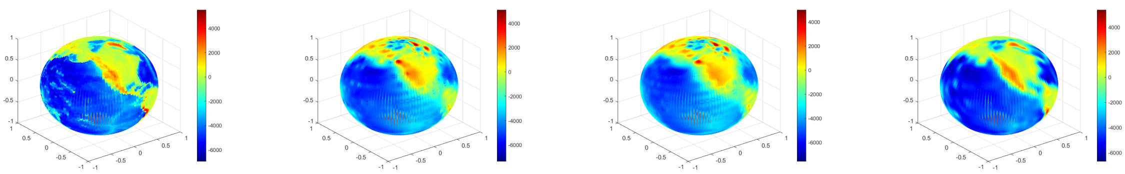


(a) Real values. (b) S_2 (c) S_4 (d) $S_{B_1}^2$

Figure 4.2: Temperatures in June 2010.

4.4 Application to topographic data problem

To illustrate other practical benefits of these operators, we use topographic data from the National Geophysical Data Center, NOAA US Department of Commerce, available in Matlab using the command `load topo`. For the reconstruction of 21600 data values using the operators S_2, S_4 and $S_{B_1}^2$, given in (4.1.2), (4.1.6) and (4.2.4), respectively, we have used 1063 nodes. The graphical results are displayed in Figure 4.3.



(a) Exact values. (b) S_2 (c) S_4 (d) $S_{B_1}^2$

Figure 4.3: Topographic data values.

Conclusions

The main objective of this thesis was to introduce and study new Shepard operators in the univariate, bivariate and spherical cases. Given the importance of the scattered data interpolation problem nowadays, it is understood that finding new ways to improve this kind of interpolant is needed due to its numerous practical applications.

The primary goal was to introduce a new type of Shepard operator in the univariate case, obtained using polynomials constructed based on the weighted least squares method. We detailed the construction of these kinds of polynomials and investigated some of their properties, including the interpolation properties, the degree of exactness and linearity. Subsequently, with the aid of these polynomial functions, we derived some new Shepard interpolants and proved that they inherit the properties mentioned before. Additionally, we investigated the remainders of the interpolation formulas, using Peano's Theorem.

The second research direction concerned the bivariate case. Using three radial basis functions (thin-plate spline, inverse quadratic and inverse multiquadric), we developed new methods of approximation, using the classical, modified and iterative forms of the Shepard operator. The first one is a global method, as proposed in its original form [82], in 1968. The second one [47] is a local approach that ensures only the closest neighbors of a point have a significant influence on the approximation data. The latter one [63] is free of the setup parameters necessary in the first two approaches and performs a reduction of the current interpolation result's residue at each iteration. We demonstrated that these operators can be successfully used in image reconstruction of damaged black-and-white and color images.

The last part of this thesis focused on the spherical interpolation of scattered data. We introduced two types of Shepard operators. The first one was constructed using a local method and two spherical radial basis functions: the thin-plate spline and the inverse multiquadric. Additionally, we used another approach, based on the addition of a polynomial component, specifically spherical harmonics, motivated by the fact that approximations of this kind offer real advantages. We studied the interpolation error, proved that our operators are of class C^1 and provided an error bound based on the modulus of continuity. For the second type of Shepard interpolant, we considered its combination with the Bernoulli operator, suitable when information about the function's partial derivatives are available. The new operators were obtained after we performed the Delaunay triangulation of the sphere, using two types of basis functions. We investigated the interpolation error and the interpolatory properties. Finally, we presented two real-life applications of the operators introduced in the last chapter, namely temperature prediction on the Earth's surface and topographic data approximation.

The accuracy of our methods has been investigated throughout all three chapters using several test functions and datasets.

As future research directions following this thesis, we mention, for example, further improvements of the spherical Shepard operator, which is a recent topic in the literature. Combinations with operators such as Lagrange or Lidstone could be performed. Moreover, optimization of the algorithm implementation is needed, as real-data problems involve large numbers of datasets, requiring speed, efficiency and parallelization of algorithms in all three cases. Given the current demand for research in artificial intelligence and machine learning, we also intend to explore potential applications of Shepard-type operators in these fields. Recently, novel techniques based on the Shepard interpolation have been developed for neural networks. They have been successfully tested in solving different tasks like time series classification, image classification, image recognition or inpainting, emphasizing the potential of this kind of operator in such applications.

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