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DOCTORAL SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE

PhD THESIS - SUMMARY

CONTRIBUTIONS TO THE THEORY OF ELLIPTIC BOUNDARY VALUE PROBLEMS AND THEIR APPLICATIONS IN FLUID MECHANICS

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Cluj-Napoca, 2024

Contents

Acknowledgements

First of all, I would like to express my sincerest and deepest gratitude to **Prof. Mirela Kohr**. I would like to thank her for her continued guidance, constant encouragement throughout these years and I deeply appreciate her unshakable belief in me. Without her endless patience and loving care this thesis wouldn't have been possible. I am grateful that she introduced me to this domain and for the act of sharing with me her experience and expertise.

Secondly, I am grateful to **Prof. Gabriela Kohr**. I had the honor of attending her lectures and I had the unique opportunity to assist her as a seminar teacher in her didactic activities in Complex Analysis during my studies. Due to her and **Prof. Mirela Kohr**, I was able to learn the most important principles in order to be a true teacher and I have learned from both of them what it takes to be a true researcher. **Prof. Gabriela Kohr** is deeply missed.

Thirdly, I would like to offer sincere thanks to **Prof. Radu Precup**, **Prof. Teodor Groşan** and **Assoc. Prof. Cornel Pintea** for their useful insights they have imparted with me throughout my studies.

I would like to express my gratitude to **Prof. Massimo Lanza de Cristoforis** for the opportunity to visit and to study in the Department of Mathematics "Tullio Levi-Civita" at the Universita degli Studi di Padova, Italy. I am also grateful to **Prof. Wolfgang L. Wendland** for the insightful discussions and assistance.

I am grateful to **Prof. Ioan Pop** for the financial support that I have received as a scientific research assistant in the first two years of my studies, under the project PN-III-P4-ID-PCE-2016-0036.

I would like to thank **Ph.D Student Ioan Papuc** for the help he has provided with the numerical simulation included in this thesis. I would like to also thank **Emanuela Paul** for her assistance with some of the figures included in this thesis.

Throughout my studies I have been supported by my family and my friends. They have assisted me many times and I am grateful for their help. Also, I wish to express my gratitude to my greatest supporters, my parents, who always believed in me and were by my side, both in action and in thought.

Introduction

The purpose of this PhD thesis is the treatment of important elliptic boundary value problems for systems of partial differential equations (PDEs) that arise in Fluid Mechanics by using the methods of potential theory and a fixed point theorem. We have treated various boundary value problems as the Dirichlet, Robin-Dirichlet, transmission, Robin-transmission in the linear case as well as the non-linear case. We have provided suggestive numerical examples for a practical problem with multiple applications, while the objective is to complete the theoretical study, which is presented in the first three chapters.

In what follows, let $\mathsf{D} \subset \mathbb{R}^n$, $n \geq 2$ be a bounded Lipschitz domain and we denote its boundary by Γ. Let us consider $\mathcal{P}$, a matrix-valued function, whose entries are essentially bounded functions. We introduce the generalized Brinkman system by

$$\Delta \mathbf{v} - \mathcal{P}\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \text{ in } \mathsf{D}, \tag{0.0.1}$$

where the pair $(\mathbf{v}, p)$ represents the velocity and pressure fields of the considered fluid flow and $\mathbf{f}$ is a given, external force which acts on the fluid flow. In the special case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha > 0$ is a given constant, the system (0.0.1) becomes the classical Brinkman system,

$$\Delta \mathbf{v} - \alpha\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}. \tag{0.0.2}$$

If we consider $\mathcal{P} = 0$ in the system (0.0.1), we obtain the well-known Stokes system,

$$\Delta \mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}. \tag{0.0.3}$$

Now, let us also consider the generalized Darcy-Forchheimer-Brinkman system

$$\Delta \mathbf{v} - \mathcal{P}\mathbf{v} - k|\mathbf{v}|\mathbf{v} - \beta(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}, \tag{0.0.4}$$

where k, β are positive, essentially bounded functions on D. In the special case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha > 0$ is a given constant and $k, \beta > 0$ are given constants, the system (0.0.4) reduces to the classical Darcy-Forchheimer-Brinkman system

$$\Delta \mathbf{v} - \alpha\mathbf{v} - k|\mathbf{v}|\mathbf{v} - \beta(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}. \tag{0.0.5}$$

Let us mention the fact that the Darcy-Forchheimer-Brinkman system is used in problems in which the inertia of the fluid is not negligible (see, e.g., [86]).

Finally, for $\mathcal{P} = 0$, $k = 0$ and $\beta > 0$ a given constant, the system (0.0.4) becomes the Navier-Stokes system

$$\Delta \mathbf{v} - \beta(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}. \tag{0.0.6}$$

For additional details regarding the Navier-Stokes equations we refer the reader to [38], [100], [97], [102].

In this thesis, we will concern ourselves with the coupling of these aforementioned PDE systems. In these transmission problems, we will deal specifically with two types of configurations. The geometry of these configurations is thoroughly specified in Chapter 1. Moreover, in these problems, we consider the following boundary conditions

$$\mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}_- = \mathbf{g},\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}_+, p_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{v}_-, p_-, \mathbf{f}_-) + \mathbf{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}_+ = \mathbf{h},\ \text{on } \Gamma, \tag{0.0.7}$$

which will be referred as *transmission conditions*, where the trace operator Tr, the conormal derivative operator $\mathbf{t}$ and the matrix-valued function $\mathbf{L}$ are described in the latter.

In our thesis, we have considered the generalized Brinkman system (0.0.1), which we obtained by substituting the constant $\alpha > 0$ (in the system (0.0.2)) with a matrix-valued function $\mathcal{P}$ whose entries are essentially bounded functions. In this case, by this aforementioned generalization, we move towards the concept of an anisotropic Brinkman system. The purpose that we have in mind is that of investigating fluid flow in porous media, in the case that our porous medium has variable porosity or permeability. For additional details, see, e.g., [58], [59], [60].

Let us provide some insight for the practical motivation for the study of transmission problems. Note that, transmission problems appear as a mathematical model for the study of environmental problems where free air flow is interacting with evaporation from soils and or the transvascular exchange between blood flow in vessels and the surrounding tissue (for additional details [52] and the references therein). The anisotropic Stokes system is used to describe certain processes (for example, processes in physics, engineering, industry) in which the flow of immiscible fluids or the flow of nonhomogeneous fluids with density dependent viscosity are involved (cf. [18], see also [60]).

In order to study such problems, many techniques can be employed. For linear boundary value problems, we emphasize two approaches, namely, layer potential methods and variational methods, respectively. Also, for the study of nonlinear boundary value problems, one can employ either fixed point theory or topological degree theory.

In the latter, we shall provide a historical overview of the scientific literature that concerns boundary problems.

Let us explore previous works that are concerned with the study of boundary problems in Euclidean setting. We begin with the work of Verchota [107], who established the invertibility property of the classical layer potentials for Laplace's equation, on $L^2(\partial\Omega)$ and subspaces of $L^2(\partial\Omega)$, in the case of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Dahlberg, Kenig and Verchota [21] have obtained well-posedness results for the Dirichlet and traction boundary problems for the Lame system in an arbitary Lipschitz domain in $\mathbb{R}^n$ with L^2-boundary data. They have also investigated the 'slip condition' for the Stokes equations, for boundary data belonging to L^2 boundary spaces accompanied by optimal estimates (see also [22]). Amrouche, Girault and Girore [12] have solved the Dirichlet and Neumann boundary value problems for the Laplacian in exterior domains of $\mathbb{R}^n$, $n \geq 2$, while working in weighted Sobolev spaces. Fabes, Mendez and Mitrea [32] have used boundary integral methods for the investigation of inhomogeneous boundary problems for the Laplacian in arbitrary Lipschitz domains with data in Besov spaces. Escauriaza and Mitrea [30] have established existence and uniqueness results for the transmission problem for the Laplacian in the setting of complementary Lipschitz domains in $\mathbb{R}^n$ for $n \geq 2$, while the boundary data was considered in Lebesgue and Hardy spaces.

In what follows, let us name a few papers in which the studies on the Stokes system (0.0.3) were conducted. Nevertheless, the list of publication where this subject is discussed is much more longer. The work of Fabes, Kenig and Verchota [31] is an important contribution of the field of layer potential theory. The authors have used layer potentials in order to obtain existence and uniqueness results for the Dirichlet problem for the Stokes system in an arbitrary Lipschitz domain in $\mathbb{R}^n$, in the case of boundary data in L^2. Dauge [24] has studied the H^s-regularity of solutions of the Stokes system in domains with corners. Girault and Sequeira [40] have investigated the Dirichlet problem for the Stokes system in exterior Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$. Power [89] has extended the method used in [90] to that of the Stokes flow problem in multiple cylinders, in the two-dimensional setting of bounded and unbounded domains. Shen [98] has considered the L^p Dirichlet problem for the Stokes system in bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, and has provided well-posedness results for such a problem. Alliot and Amrouche [9] have devoted a study to the Stokes problem in $\mathbb{R}^n$, $n \geq 2$, in weighted Sobolev spaces. This approach allows the authors discuss the decay or growth of solutions at infinity. Alliot and Amrouche [11] have investigated the nonhomogeneous Dirichlet problem for the Stokes system in an exterior, connected, Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$ in weighted Sobolev spaces, in order to account the behavior of the solution at infinity. Russo and Tartaglione [94] have provided existence and uniqueness results for the Robin type problem associated to the Stokes system and also for the Navier-Stokes system, in a bounded Lipschitz domain in Euclidean setting.

The linear, elliptic Brinkman system (0.0.2) was also investigated by a great deal of researchers. McCracken [72] has studied the Dirichlet problem for the Stokes resolvent system on the half-space of in $\mathbb{R}^3$ and provided the well-posedness of the Dirichlet problem in some L^p spaces. Deuring [25] has constructed solutions in L^p-spaces for the Dirichlet problem for the resolvent Stokes system in the exterior of a bounded domain with $\mathcal{C}^2$ boundary belonging to $\mathbb{R}^3$. Farwig and Sohr [33] have shown that the Dirichlet problem for the Stokes resolvent system admits a unique solution in weighted Sobolev spaces, in the setting of an exterior $\mathcal{C}^{1,1}$ domain of $\mathbb{R}^n$, $n \geq 2$. Shen [99] has obtained L^p resolvent estimates for the Stokes system in the setting of Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, by employing layer potential methods in his study. Kohr, Lanza de Cristoforis and Wendland [53] have investigated Robin type boundary problems for the Brinkman system and the Darcy-Forchheimer-Brinkman system in Lipschitz domains in Euclidean setting. They treat also mixed Dirichlet-Robin and transmission boundary value problems for the Brinkman systems in the setting of bounded creased Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, as well as the Navier problem for the Brinkman system in a bounded Lipschitz domain of $\mathbb{R}^3$. Kohr, Lanza de Cristoforis and Wendland [55] have obtained an existence result for the Poisson problem for a semilinear Brinkman system on a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$ with Dirichlet or Robin conditions on the boundary. Medkova [76] has investigated the Dirichlet problem for the resolvent Stokes system in the setting of bounded and unbounded domains with compact Lyapunov boundary.

Now, let us focus on previous studies that aim to investigate boundary value problems for non-linear equations, such as the Navier-Stokes equations (0.0.6) or the Darcy-Forchheimer-Brinkman equations (0.0.5). We mention the contribution of Alliot and Amrouche [10], who have studied regularity properties of the weak solutions of the steady-state Navier-Stokes system in exterior domains of $\mathbb{R}^3$. Russo and Tartaglione [95] have studied the Robin problem for the Oseen and Navier-Stokes systems in an C^1-class, exterior domain of $\mathbb{R}^3$. They have used a layer potential approach in order to show the existence of a solution for the Robin problem for the Oseen system and for existence result for the Robin problem for the Navier-Stokes system, they have employed a fixed point method. Amrouche and Nguyen [13] have investigated the exterior, homogeneous, Dirichlet problem for the Navier-Stokes system in an exterior Lipschitz domain in $\mathbb{R}^3$, in the setting of weighted Sobolev spaces. Russo and Tartaglione [96] have used a variational approach and fixed point theorems to obtain existence results for the Navier problem for the Navier-Stokes system in bounded Lipschitz domains and exterior Lipschitz domains in $\mathbb{R}^3$. Kohr, Lanza de Cristoforis and Wendland [55] obtained an existence and uniqueness result for the Dirichlet problem for the semilinear Darcy-Forchheimer-Brinkman system in the case of small boundary data.

Researchers have also devoted themselves to the investigation of boundary problems in the setting of manifolds. We highlight some works in the later. Let us begin by noting that Mitrea, Mitrea, Mitrea and Taylor [80] have treated boundary problems for the Hodge-Laplacian in the setting of Riemannian manifolds. Also, Dindos and Mitrea [26] employed the method of boundary integral equations to obtain the well-posedness of the Poisson problem for the Stokes system in Lipschitz domains in the setting of smooth, compact Riemannian manifolds. In [63], Kohr, Pintea and Wendland have used a layer potential approach in order to investigate a certain type of general pseudodifferential matrix operators defined on Lipschitz domains in compact Riemannian manifolds. The authors have proposed a useful approach, by which, well-posedness results of certain boundary value problems can be derived by using well-posedness results for transmission-type problems. Kohr, Mikhailov and Wendland [57] have investigated transmission-type boundary value problems for the Navier-Stokes and Darcy-Forchheimer-Brinkman systems in complementary Lipschitz domains in a compact Riemannian manifold of dimension m, $m = 2, 3$. Their approach is based on layer potential techniques combined with fixed point arguments.

Let us point out some papers that deal with transmission-type problems. Mitrea and Taylor [83] have developed layer potential methods for partial differential equations on Lipschitz domains in smooth, connected and compact Riemannian manifolds of dimension $m \geq 3$. Mitrea and Taylor [84] have provided well-posedness results for the Dirichlet problem for the Stokes system and for the initial boundary value problem for the Navier-Stokes system with Dirichlet boundary condition. Kohr, Lanza de Cristoforis and Wendland have [54] investigated the existence of a solution for the nonlinear Neumann-transmission problem for the Stokes and Brinkman systems in Lipschitz domains in Euclidean setting. Medkova [74] has employed the method of integral equations in order to provide well-posedness results of transmission problems, Robin-transmission problem and Dirichlet-transmission problem for the Brinkman system in the setting of complementary Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. The author in [75] has used the method of integral equations to find well-posedness results for transmission problems associated to the Stokes equations in complementary domains of $\mathbb{R}^3$ with Lipschitz boundaries. Kohr, Lanza de Cristoforis, Mikhailov and Wendland [52] have obtained well-posedness results for a transmission problem for the Darcy-Forchheimer-Brinkman and Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$. Their approach proposes a layer potential technique combined with a fixed point theorem. Kohr, Lanza and Wendland [56] have investigated a Robin-transmission problem for the Darcy-Forchheimer-Brinkman and Navier-Stokes systems in two adjacent and bounded Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$. The authors in [56] have studied a Robin-transmission boundary value problem for the Darcy-Forchheimer-Brinkman and Navier-Stokes systems in two adjacent Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$, with linear transmission and linear Robin boundary conditions.

Let us also mention important works that concern the investigation of variable-coefficient PDE systems and boundary value problems for such systems. Duffy [29] has provided a model for an anisotropic incompressible viscous fluid. In this case, the equations of state of such a fluid involve an anisotropic physical constant tensor. Mitrea, Mitrea and Shi [81] have investigated variable coefficient transmission boundary value problems in the setting of bounded Lipschitz domains defined on non-smooth manifolds of dimension $n \geq 2$. Choi and Yang [19] have studied the fundamental solution of the measurable-coefficient stationary Stokes system in $\mathbb{R}^n$, $n \geq 3$. Choi, Dong and Kim [18] have investigated the conormal derivative problem for the stationary Stokes equations with irregular coefficients in Sobolev spaces defined on Reifenberg flat domains. Dong and Kim [28] have studied the stationary Stokes system with variable coefficients, which are measurable in one direction, in a Reifenberg flat domain. In addition, they establish well-posedness results in standard Sobolev spaces and in Muckenhoupt type weighted Sobolev spaces as well. Dong and Kim [27] have investigated solutions of the stationary Stokes system with variable coefficients in bounded Lipschitz domains. Kohr and Wendland [67] have obtained, in the setting of Lipschitz domains on compact Riemannian manifolds, well-posedness results for the Dirichlet boundary value problems for the L^∞-variable coefficients Stokes and Navier-Stokes PDE systems.

Kohr, Mikhailov and Wendland [59] have investigated transmission problems for the anisotropic Stokes and Navier-Stokes systems with L^∞ strongly elliptic coefficient tensor in the setting of complementary Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. The well-posedness of transmission-type problems that involve the anisotropic Stokes system was extracted by a variational method, and, as a consequence, the author have introduced volume and layer potentials for the anisotropic Stokes system with L^∞ strongly elliptic coefficient tensor and mapping properties for these operators were also established. These aforementioned potentials were used to establish the well-posedness of certain linear transmission problems. The well-posedness results in the linear case, together with a fixed point argument, have led the authors to obtain well-posedness results in the non-linear case as well. Kohr, Mikhailov and Wendland [60] have studied the anisotropic Stokes system with L^∞ viscosity tensor coefficient which fulfills an ellipticity condition for symmetric matrices such that their trace is equal to zero. They have provided a layer potential theory for this PDE system, in L^2-based weighted Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. Their approach is rooted in the investigation of particular transmission problems for the anisotropic Stokes system. After introducing the layer potentials and the volume potential, they employ these potentials to analyze Dirichlet and Neumann boundary value problems for the anisotropic Stokes system.

Kohr, Mikhailov and Wendland [58] have investigated the anisotropic Stokes system with L^∞ viscosity tensor coefficient which satisfies an ellipticity condition in terms of symmetric matrices with zero matrix trace. For such a system, they have obtained well-posedness results for Dirichlet and transmission problems in Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, with data belonging to standard and weighted Sobolev spaces. Moreover, the authors also treat Dirichlet and transmission problems for the anisotropic Navier-Stokes system in bounded Lipschitz domains in $\mathbb{R}^3$. Kohr and Precup [65] have provided a theoretical analysis for coupled systems of Navier-Stokes type with non-homogeneous reaction-type terms. Kohr and Precup [66] have used a variational approach and fixed point index theory in order to analyze a Dirichlet boundary value problem for a general coupled systems of stationary Navier-Stokes type equations with variable coefficients and non-homogeneous reaction type terms in a bounded domain of $\mathbb{R}^n$, $n \leq 3$.

Note that boundary value problems can be investigated also from a numerical point of view. This has lead to the development of diverse numerical methods (finite differences, finite volumes, finite element) whose purpose is to find numerical solutions for various boundary value problems (see also [93]). In the latter we discuss studies that are concerned with the numerical treatment of these problems. Ghia, Ghia and Shin [39] have used the vorticity-stream function formulation for the incompressible Navier-Stokes equations in dimension $n = 2$. The model problem that they have employed is the driven flow in a square cavity. Vafai [104] has analyzed the effects that occur in the case of variable porosity and inertial forces on convective flow and heat transfer in porous media. Guo and Zhao [43] have proposed a lattice Boltzmann model for an isothermal incompressible flow in porous media and they have included the porosity into the equilibrium distribution and a force term to the evolution equation (to account for the drag forces of the medium), i.e., the Darcy term and the Forchheimer term. Yang, Xue and Mahias [109] have concerned themselves with the investigation of the lid-driven rectangular cavity containing a porous Brinkman-Forchheimer medium. AlAmiri [3] has investigated an incompressible, laminar mixed-convection heat transfer in square lid-driven cavity in the presence of a porous block. Gutt and Groşan [44] have studied the flow of an incompressible viscous fluid through a porous medium in a square cavity of dimension

$n = 2$. They analyze this problem theoretically and numerically, as well. Groşan, Pătrulescu and Pop [42] have proposed a mathematical model which contains the Brinkman PDE system in order to discuss the steady free convection in a square differentially heated cavity which is filled by a bidisperse porous medium.

The thesis consists of four chapters.

- **Chapter 1** contains an overview of the notions that are used in this thesis. We define the concept of a Lipschitz domain, we discuss some notations that we use in this thesis. Also, we provide two assumptions (see Assumption 1.1.6 and Assumption 1.1.7, respectively) which describe the geometric setting in which we investigate our boundary problems. These problems are analyzed in the following chapters. Next, we introduce the function spaces that we use in this thesis, namely Sobolev spaces in Lipschitz domains in the Euclidean setting, Sobolev spaces on Lipschitz boundaries in the Euclidean setting, weighted Sobolev spaces in $\mathbb{R}^3$ (see [47]). We discuss the (Gagliardo) trace operator in the case of classical Sobolev spaces and also in the case of weighted Sobolev spaces. Next, we describe the Stokes operator and the Brinkman operator. For each of these operators, we give their corresponding conormal derivative operators. We also introduce a generalized version of the Brinkman system and provide its associated conormal derivate operator (see Definition 1.2.14 and Lemma 1.2.15). Furthermore, we provide the fundamental solution of the Stokes system and we give the Newtonian potentials and layer potentials for the Stokes system together with their mapping properties, their jump properties and their growth conditions. A similar approach is made also for the Brinkman system, we give the fundamental solution of the Brinkman system, we provide the Newtonian potentials and layer potentials for the Brinkman system, their mapping properties, their jump properties and their growth properties.

- **Chapter 2** is concerned with existence and uniqueness results of transmission type problems for linear PDE systems. We begin this chapter by providing a well-posedness result for the Dirichlet-type problem for the Brinkman system in an exterior Lipschitz domain in $\mathbb{R}^3$ (see Theorem 2.1.2). Next, an existence and uniqueness result is given for the transmission problem for the generalized Brinkman equations and Stokes equations in $\mathbb{R}^3$ (see Theorem 2.2.2 and Theorem 2.2.3). We continue by providing a well-posedness result for the transmission problem for the classical and generalized Brinkman equations in $\mathbb{R}^3$ (see Theorem 2.3.1). In the last section of this chapter, we have the well-posedness result for a Robin-transmission problem for the Brinkman equations in $\mathbb{R}^n$, $n \geq 2$ (see Theorem 2.4.1). In addition, by using a similar procedure as in the case of Theorem 2.4.1, we provide an existence and uniqueness result for a limiting Robin-transmission problem for the Brinkman equations in $\mathbb{R}^n$, $n \geq 2$ (see Theorem 2.4.2). As a consequence of Theorem 2.4.2, we are able to derive an existence and uniqueness result for the Robin-Dirichlet problem for the Brinkman system (see Corollary 2.4.3). The content of this chapter is based on the papers [6], [7], [8].

- In **Chapter 3** we discuss a generalization of the Darcy-Forchheimer-Brinkman equations (see Relation (3.1.1)). Also, we have provided a useful lemma (see Lemma 3.1.3). Then, we give an existence and uniqueness result for the transmission problem for the generalized Darcy-Forchheimer-Brinkman and Stokes equations in $\mathbb{R}^3$ (see Theorem 3.2.1). Next, we present an existence and uniqueness result for the transmission problem for the generalized Darcy-Forchheimer-Brinkman and Brinkman equations in $\mathbb{R}^3$ (see Theorem 3.3.1). We also have an existence and uniqueness result for the Robin-transmission problem for the Darcy-Forchheimer-Brinkman equations in $\mathbb{R}^n$, $n = 2, 3$ (see Theorem 3.4.1). Similar arguments are employed in order to get a well-posedness result for a limiting Robin-transmission problem Darcy-Forchheimer-Brinkman equations in $\mathbb{R}^n$, $n = 2, 3$ (see Theorem 3.4.2). Finally, due to Theorem 3.4.2, we are able to obtain an existence result for the Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman equations in $\mathbb{R}^n$, $n = 2, 3$ (see Corollary 3.4.3). The content of this chapter is based on the papers [4], [5], [8].

- Lastly, the goal of **Chapter 4** is to give a numerical analysis in order to determine a numerical solution for the Robin-Dirichlet boundary problem for the Darcy-Forchheimer-Brinkman equations. This numerical study concerns the lid-driven porous cavity problem with Navier slip boundary condition in the presence of a solid body. The geometric setting of this problem can be seen in Figure 4.1. In order to solve this problem, first we write our mathematical model (see Relation (4.1.1)), we conduct a non-dimensional analysis (see Relation (4.1.2)). To get a numerical solution, we use a numerical software, namely COMSOL Multiphysics. Then, we determine the optimal grid for our analysis (see Table 4.1) and we validate our model via comparison with existent results (see Figure 4.2). Finally, we investigate the impact of dimensionless slip length (see Subsection 4.1.4). The content of this chapter is based on the paper [8].

The following list contains the papers in which we have included the original results that appear in this thesis. The papers are:

- **Albişoru, A.F.**, *A note on a transmission problem for the Brinkman system and the generalized Darcy-Forchheimer-Brinkman system in Lipschitz domains in* $\mathbb{R}^3$, **Studia Universitatis Babeş-Bolyai, Series Mathematica**, **64**(3), 2019, 399-412. **WOS-ESCI**.

- **Albişoru, A.F.**, *On transmission-type problems for the generalized Darcy-Forchheimer-Brinkman and Stokes systems in complementary Lipschitz domains in* $\mathbb{R}^3$, **Filomat**, **33**(11), 2019, 3361-3373. **ISI**, **IF**(November 2022): **0.988**.

- **Albişoru, A.F.**, *A layer potential analysis for transmission problems for Brinkman-type systems in Lipschitz domains in* $\mathbb{R}^3$, **Mathematische Nachrichten**, **292**(9), 2019, 1876-1896. **ISI**, **IF**(November 2022): **1.199**.

- **Albişoru, A.F.**, *A Poisson Problem of Transmission-type for the Stokes and Generalized Brinkman Systems in Complementary Lipschitz Domains in* $\mathbb{R}^3$, **Taiwanese Journal of Mathematics**, **24**(2), 2020, 331-354. **ISI**, **IF**(November 2022): **0.87**.

- **Albişoru, A.F.**, Kohr, M., Papuc, I., Wendland, W.L., *On some Robin-transmission problems for the Brinkman system and a Navier-Stokes type system*, **Mathematical Methods in Applied Sciences**, DOI:https://doi.org/10.1002/mma.10170, 2024, published online: May 2024. **ISI**, **IF**(June 2023): **2.9**.

Keywords: transmission problems, elliptic boundary value problems, Sobolev spaces, weighted Sobolev spaces, fundamental solution, potential theory, Lipschitz domains, trace operator, conormal derivative operator, Stokes system, Brinkman system, Navier-Stokes system, Darcy-Forchheimer-Brinkman system, finite element method, lid-driven cavity flow problem.

MSC: Primary 35J25, 35Q35, 42B20, 46E35; Secondary 76D, 76M.

1

Layer Potential Methods for the Stokes and Brinkman systems in Lipschitz domains

This chapter establishes the functional setting in which we will analyze our boundary values problems for the Stokes, Brinkman, Navier-Stokes and Darcy-Forchheimer-Brinkman equations. To this end, we recall definitions, notations and properties that we will use throughout this work.

Hence, we will introduce the concepts of a bounded Lipschitz domain and an unbounded (or exterior) Lipschitz domain in (the Euclidean setting of) $\mathbb{R}^n$, where $n \geq 2$. We will also place an emphasis on the case $n = 3$ in whose setting, we have obtained many of our well-posedness results. Next, we will recall the definitions of the Sobolev spaces in the Euclidean setting and their properties, which are most relevant to our study. In addition, we will also discuss the Gagliardo Trace Lemma which allows us to define the trace operator in the setting of Sobolev spaces. This previous operator is involved in the boundary conditions of the boundary value problems that we study.

Further, we will study the Stokes and Brinkman systems. In the case of these two systems, we will discuss their associated conormal derivative operators. These operators, again, will appear in the boundary conditions of the boundary value problems that we treat in the latter.

One important aspect that we wish to point out is that, in this chapter, we deal with a generalized version of the Brinkman system. Our original results involve these particular systems of PDEs.

Finally, we conclude this chapter with two very important sections. These sections contain the layer potential operators associated to the Stokes and Brinkman equations, respectively. These operators are used in the proof of our well-posedness results, due to the fact that with their help, we are able to construct solutions for our boundary value problems. The sources that were used in the preparation of this chapter are [1], [2], [45], [49], [51], [73], [91], [101], [103], [108].

1.1 Functional Setting

This section is dedicated to the description of the main notions that are used all through this work. First of all, we define the concept of Lipschitz domain and we describe important notations that we use throughout this thesis. Also, we describe the geometry of the Lipschitz domains that are involved in the boundary problems that we will study in the latter. Next, we provide an overview of Sobolev spaces in $\mathbb{R}^n$, on Lipschitz domains and Lipschitz boundaries. Some properties of these Sobolev spaces are also given. Moreover, we recall the concept of a weighted Sobolev space in the exterior of a bounded Lipschitz domain in $\mathbb{R}^3$. We end this section with the useful Gagliardo trace lemma.

1.1.1 Lipschitz domains

We will review the definition of a bounded Lipschitz domain and introduce the spaces in which we seek our solutions for our boundary value problems. Also, we will discuss the systems that are encountered in our study, and describe the operators that appear in our boundary conditions. Let us provide in the latter the definition of the concept of a Lipschitz domain (see also [46, Def. 2.1]).

Definition 1.1.1. *Let* $\mathsf{D} \subset \mathbb{R}^n$, $n \geq 2$ *be a nonempty, open and bounded set. Denote by* Γ *the boundary of the set* D. *We say that* D *is a bounded Lipschitz domain if for any* $\boldsymbol{x} \in \Gamma$, *there are some constants* $r_1, r_2 > 0$, *a coordinate system* $(y_1, ..., y_n) = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ *that is isometric to the canonical one and has its origin at* $\boldsymbol{x}$, *and a Lipschitz function* $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$, *such that*

$$\mathsf{D} \cap \mathcal{C}(r_1, r_2) = \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y'| < r_1 \text{ and } \psi(y') < y_n < r_2\},$$

where

$$\mathcal{C}(r_1, r_2) := \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y'| < r_1,\ |y_n| < r_2\} \subseteq \mathbb{R}^n.$$

Next, we state some useful remarks.

Remark 1.1.2. *In this thesis, we will use the repeated index summation convention.*

Remark 1.1.3. *In this thesis, we use the notation a.e. instead of almost everywhere.*

Remark 1.1.4. *If* X *denotes a Banach space, its topological dual is denoted by* X'.

Remark 1.1.5. *If* Y *is an open subset of* $\mathbb{R}^n$, $n \geq 2$, *then we denote the duality pairing between two dual spaces defined on* Y *by* $\langle \cdot, \cdot \rangle_Y$.

In the latter, we will state some assumptions that allow us to represent the geometry of the Lipschitz domains, the setting where our problems will be formulated.

Assumption 1.1.6. *Let* $\mathsf{D}_+ := \mathsf{D} \subseteq \mathbb{R}^n$, $n \geq 2$, *be a bounded Lipschitz domain with connected boundary* Γ. *Denote by* $\mathsf{D}_- := \mathbb{R}^n \setminus \overline{\mathsf{D}}$ *the complementary (exterior) Lipschitz domain (see Figure 1.1).*

Assumption 1.1.7. *Let* $\mathsf{D} \subseteq \mathbb{R}^n$, $n \geq 2$, *be a bounded Lipschitz domain with connected boundary* Γ_-. *Assume that* D_+ *is a bounded Lipschitz domain, with connected boundary denoted by* Γ_+, *such that* $\overline{\mathsf{D}}_+ \subset \mathsf{D}$ *and let* $\mathsf{D}_- := \mathsf{D} \setminus \overline{\mathsf{D}}_+$. *Hence, the boundary of* D_- *has two connected components, namely,* Γ_+ *and* Γ_- *(see Figure 1.2).*

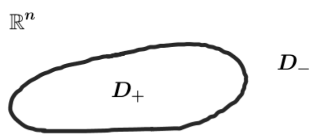


Figure 1.1: The complementary Lipschitz domains D_+ and D_- in $\mathbb{R}^n$.

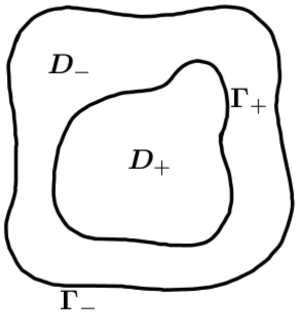


Figure 1.2: A bounded Lipschitz domain $\mathsf{D} = \overline{\mathsf{D}}_+ \cup \mathsf{D}_-$ which satisfies Assumption 1.1.7

1.1.2 On Sobolev spaces in Lipschitz domains

The purpose of this section is to provide an overview of Sobolev spaces in an Euclidean setting in $\mathbb{R}^n$. These spaces are used in the investigation of (weak) solutions of certain PDEs, for which no classical solution can be found. We will use these spaces throughout this thesis.

In the latter, $\mathbb{Z}_+$ denotes the set of non-negative integers and the vector $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$ is called a multi-index. Let us set $|\alpha| = \sum_{i=1}^n \alpha_j$. We introduce the differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} ... \partial {x_n}^{\alpha_n}}. \tag{1.1.1}$$

Moreover, we also introduce the differential operator

$$D_k := \frac{1}{i}\frac{\partial}{\partial x^k},\ i^2 = -1. \tag{1.1.2}$$

Now, we denote by $\mathsf{D} \subseteq \mathbb{R}^n$, $n \geq 2$, either a bounded Lipschitz domain or an exterior Lipschitz domain or $\mathbb{R}^n$. In the case of a bounded Lipschitz domain or an exterior Lipschitz domain D, we denote the boundary of such domains by Γ.

Note that space $\mathcal{C}(\overline{\mathsf{D}})$ is the space of continuous functions on $\overline{\mathsf{D}}$ and it is endowed with the sup-norm.

For a function $g : \mathsf{D} \to \mathbb{R}$, we define the support of g by

$$\text{supp } g := \overline{\{x \in \mathsf{D} \mid g(x) \neq 0\}}. \tag{1.1.3}$$

We denote by $\mathcal{C}^\infty(\mathsf{D})$ the space of infinitely differentiable functions defined on D. We also denote by $\mathcal{C}_0^\infty(\mathsf{D})$ the space of infinitely differentiable functions, that vanish in some neighborhood of Γ. Let us note that if $g \in \mathcal{C}_0^\infty(\mathsf{D})$ then $g|_\Gamma = 0$. Also, if $g \in \mathcal{C}_0^\infty(\mathsf{D})$ then the set (1.1.3) is compact in D. We also introduce the vector function spaces $\mathcal{C}^\infty(\mathsf{D})^n$ and $\mathcal{C}_0^\infty(\mathsf{D})^n$ by

$$\begin{aligned} \mathcal{C}^\infty(\mathsf{D})^n &:= \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in \mathcal{C}^\infty(\mathsf{D}), i = \overline{1, n}\}, \\ \mathcal{C}_0^\infty(\mathsf{D})^n &:= \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in \mathcal{C}_0^\infty(\mathsf{D}), i = \overline{1, n}\}. \end{aligned} \tag{1.1.4}$$

For $p \in [1, \infty)$, the Lebesgue space $L^p(\mathsf{D})$ of (equivalence classes of) measurable functions, p-th power, absolute value Lebesgue integrable on D is given by

$$L^p(\mathsf{D}) := \left\{ u : \mathsf{D} \to \mathbb{R} \ \middle|\ \int_\mathsf{D} |u(x)|^p \mathrm{d}x < \infty \right\} \tag{1.1.5}$$

and its norm is given by

$$||u||_{L^p(\mathsf{D})} := \left(\int_{\mathsf{D}} |u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{1.1.6}$$

for $1 \leq p < \infty$. Also, we define the space of vector functions $L^p(\mathsf{D})^n$ by

$$L^p(\mathsf{D})^n := \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in L^p(\mathsf{D}), i = \overline{1,n}\}. \tag{1.1.7}$$

Note that the space $L^\infty(\mathsf{D})$ is the space of (equivalence classes of) essentially bounded functions on D. Its norm is given by

$$||u||_{L^\infty(\mathsf{D})} := \text{esssup}_{x\in\mathsf{D}}|u(x)|. \tag{1.1.8}$$

The quantity in the right hand side of relation (1.1.8) is called the essential supremum of u. It is the smallest number ϵ such that the set $\{x \in \mathsf{D} \mid u(x) > \epsilon\}$ has Lebesgue measure equal to zero. In addition, we define the space $L^\infty(\mathsf{D})^n$ by

$$L^\infty(\mathsf{D})^n := \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in L^\infty(\mathsf{D}), i = \overline{1,n}\}. \tag{1.1.9}$$

In the latter, we will also use the space

$$L^\infty(\mathsf{D})^{n\times n} := \{\mathbf{U} : \mathsf{D} \to \mathbb{R}^n \times \mathbb{R}^n \mid \mathbf{U} = (u_{ij}), u_{ij} \in L^\infty(\mathsf{D}),\ i,j = \overline{1,n}\} \tag{1.1.10}$$

Note that, for $p \in (1,\infty)$, the topological dual of the space $L^p(\mathsf{D})$ is the space $L^q(\mathsf{D})$, where $\frac{1}{p} + \frac{1}{q} = 1$. In addition the dual of the space $L^1(\mathsf{D})$ is the space $L^\infty(\mathsf{D})$. Let us note that, for $1 \leq p \leq \infty$, the space $L^p(\mathsf{D})$ is a Banach space. In addition, $L^2(\mathsf{D})$ is a Hilbert space.

In the latter, let us view the space $\mathcal{C}_0^\infty(\mathsf{D})$ as a topological vector space. Then, let us introduce the spaces $\mathcal{D}(\mathsf{D})$ and $\mathcal{D}'(\mathsf{D})$.

Definition 1.1.8. *The Schwarz space of test functions $\mathcal{D}(\mathsf{D})$ is the space $\mathcal{C}_0^\infty(\mathsf{D})$ endowed with the inductive limit topology.*

Note that, the space $\mathcal{D}(\mathsf{D})^n$ can be defined in a similar way, namely,

$$\mathcal{D}(\mathsf{D})^n := \{\boldsymbol{\psi} : \mathsf{D} \to \mathbb{R}^n \mid \boldsymbol{\psi} = (\psi_1, ..., \psi_n), \psi_i \in \mathcal{D}(\mathsf{D}), i = \overline{1,n}\}. \tag{1.1.11}$$

Definition 1.1.9. *The space of distributions $\mathcal{D}'(\mathsf{D})$ is the space of all linear and continuous functionals on $\mathcal{D}(\mathsf{D})$.*

The space of vector functions $\mathcal{D}'(\mathsf{D})^n$ is given by

$$\mathcal{D}'(\mathsf{D})^n := \{\boldsymbol{\Psi} : \mathsf{D} \to \mathbb{R}^n \mid \boldsymbol{\Psi} = (\Psi_1, ..., \Psi_n), \Psi_i \in \mathcal{D}'(\mathsf{D}), i = \overline{1,n}\}. \tag{1.1.12}$$

Next, we describe the notion of a Sobolev space. Note that, in this thesis, we use L^2-based Sobolev spaces that are defined on D. Consequently, we introduce the integer order L^2-based Sobolev spaces as follows.

Definition 1.1.10. *Assume that $k \in \mathbb{Z}_+$. Then, the Sobolev space $H^k(\mathsf{D})$ is defined by*

$$H^k(\mathsf{D}) := \{u \in L^2(\mathsf{D}) \mid D^\alpha u \in L^2(\mathsf{D}), \forall\ \alpha \in \mathbb{Z}_+^n, |\alpha| \leq k\}, \tag{1.1.13}$$

and its norm is given by

$$||u||_{H^k(\mathsf{D})} := \left(\sum_{|\alpha|\leq k} ||D^\alpha u||^2_{L^2(\mathsf{D})}\right)^{\frac{1}{2}}. \tag{1.1.14}$$

We also introduce the Sobolev space $H^k(\mathsf{D})^n$ by

$$H^k(\mathsf{D})^n := \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in H^k(\mathsf{D}), i = \overline{1, n}\}. \tag{1.1.15}$$

Let us also introduce the space $H_0^k(\mathsf{D}) \equiv \mathring{H}^k(\mathsf{D})$ as the closure of $\mathcal{D}(\mathsf{D})$ in $H^k(\mathsf{D})$ with respect to the norm $||\cdot||_{H^k(\mathsf{D})}$. Similarly, we can introduce the space $H_0^k(\mathsf{D})^n \equiv \mathring{H}^k(\mathsf{D})^n$. Moreover, $\mathring{H}^k(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ and $\mathring{H}^k(\mathbb{R}^n)^n = H^k(\mathbb{R}^n)^n$.

The spaces $H^k(\mathsf{D})$ and $\mathring{H}^k(\mathsf{D})$ are Hilbert spaces. Also, let us mention that the Hilbert space $H^k(\mathsf{D})$ is endowed with the inner product

$$(u, v)_{H^k(\mathsf{D})} := \sum_{|\alpha| \geq k} (D^\alpha u, D^\alpha v)^{\frac{1}{2}}_{L^2(\mathsf{D})}. \tag{1.1.16}$$

The following definitions allow us to introduce the fractional order L^2-based Sobolev spaces.

Definition 1.1.11. *Assume that $0 < s < 1$. The fractional order Sobolev space $H^s(\mathsf{D})$ is defined by*

$$H^s(\mathsf{D}) := \left\{ u \in L^2(\mathsf{D}) \;\middle|\; \int_\mathsf{D}\int_\mathsf{D} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \mathrm{d}x\mathrm{d}y < \infty \right\} \tag{1.1.17}$$

and its norm is given by

$$||u||_{H^s(\mathsf{D})} = \left(\int_\mathsf{D} |u(x)|^2 \mathrm{d}x + \int_\mathsf{D}\int_\mathsf{D} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}. \tag{1.1.18}$$

Definition 1.1.12. *Assume that $0 < s < 1$ and $k \in \mathbb{Z}_+$. Let $\sigma = k + s$. The fractional order Sobolev space $H^\sigma(\mathsf{D})$ is defined by*

$$H^\sigma(\mathsf{D}) := \{u \in H^k(\mathsf{D}) \mid D^\alpha u \in H^s(\mathsf{D}), \forall \alpha \in \mathbb{Z}_+^n,\ 0 \leq |\alpha| \leq k\} \tag{1.1.19}$$

and its norm is given by

$$||u||_{H^\sigma(\mathsf{D})} := \left(\sum_{|\alpha| \leq k} ||D^\alpha u||^2_{H^s(\mathsf{D})} \right)^{\frac{1}{2}}. \tag{1.1.20}$$

By taking into account Definition 1.1.11 and Definition 1.1.12, one can introduce the spaces of vector-valued functions $H^s(\mathsf{D})^n$ and $H^\sigma(\mathsf{D})^n$ component-wise. Moreover, the fractional order Sobolev space $H^\sigma(\mathsf{D})$ is a Hilbert space.

Let us discuss the negative order L^2-based Sobolev spaces. Let $k \in \mathbb{Z}_+$. In order to introduce these Sobolev spaces, let us note that the space $H_0^k(\mathsf{D})$ is the closure of $\mathcal{C}_0^\infty(\mathsf{D})$ in the space $H^k(\mathsf{D})$. In addition,

$$H_0^k(\mathbb{R}^n) = H^k(\mathbb{R}^n). \tag{1.1.21}$$

Similarly, we can define the space of vector functions $H_0^k(\mathsf{D})^n$ component-wise and relation (1.1.21) can be written also for the vector-valued space $H_0^k(\mathbb{R}^n)^n$.

Let us now define the negative order L^2-Sobolev spaces.

Definition 1.1.13. *Assume that $k \in \mathbb{Z}_+$. Then, the negative order Sobolev space $H^{-k}(\mathsf{D})$ is the dual of the space $H_0^k(\mathsf{D})$, i.e.,*

$$H^{-k}(\mathsf{D}) := (H_0^k(\mathsf{D}))', \tag{1.1.22}$$

and its norm is given by

$$||h||_{H^{-k}(\mathsf{D})} := \sup_{u \in H_0^k(\mathsf{D}), u \neq 0} \frac{|\langle h, u\rangle|}{||u||_{H_0^k(\mathsf{D})}}. \tag{1.1.23}$$

Let us note that the vector-valued space $H^{-k}(\mathsf{D})^n$ is defined component-wise. Moreover, we have that $H^{-k}(\mathsf{D})^n = (H_0^k(\mathsf{D})^n)'$. In addition, we have that the density of $\mathcal{C}_0^\infty(\mathsf{D})$ in $H_0^k(\mathsf{D})$ implies the inclusion $H^{-k}(\mathsf{D}) \subset \mathcal{D}'(\mathsf{D})$. Note that the space $H^{-k}(\mathsf{D})$ is a Hilbert space.

Since $C_0^\infty(\mathsf{D})$ is not dense in the space $H^k(\mathsf{D})$, for $k \in \mathbb{Z}_+$, the dual of $H^k(\mathsf{D})$ cannot be embedded as a subspace of the space of distributions $\mathcal{D}'(\mathsf{D})$.

Definition 1.1.14. *Assume that $s \in \mathbb{R}$. Assume that D is a Lipschitz domain in $\mathbb{R}^n$. The space $\tilde{H}^s(\mathsf{D})$ is defined as the closure of $\mathcal{D}(\mathsf{D})$ in $H^s(\mathbb{R}^n)$.*

Moreover, the vector-valued space $\tilde{H}^s(\mathsf{D})^n$ is given by

$$\tilde{H}^s(\mathsf{D})^n := \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in \tilde{H}^s(\mathsf{D}), i = \overline{1, n}\}. \tag{1.1.24}$$

The space $\tilde{H}^s(\mathsf{D})$ can be characterized as

$$\tilde{H}^s(\mathsf{D}) = \{u \in H^s(\mathbb{R}^n) \mid \text{supp } u \subseteq \mathsf{D}\}. \tag{1.1.25}$$

In addition, $\tilde{H}^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

We have the following duality relations

$$(H_0^k(\mathsf{D}))' = H^{-k}(\mathsf{D}),\ (H^k(\mathsf{D}))' = \tilde{H}^{-k}(\mathsf{D}), \tag{1.1.26}$$

for $k \in \mathbb{Z}_+$. Let us mention that the duality relations in (1.1.26) hold also in the case of the vector function Sobolev spaces.

Next, we provide the Sobolev embedding theorem (see, e.g., [1, Theorem 4.12], [2]).

Theorem 1.1.15. *Assume that $k \in \mathbb{Z}_+$. Let $\mathsf{D} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We have that*

(i) the embedding $H^k(\mathsf{D}) \hookrightarrow \mathcal{C}(\overline{\mathsf{D}})$ is continuous if $k > \frac{n}{2}$.

(ii) the embedding $H^k(\mathsf{D}) \hookrightarrow L^q(\mathsf{D})$ is continuous and compact, for all $q \in [1, \infty)$, if $k = \frac{n}{2}$.

(iii) the embedding $H^k(\mathsf{D}) \hookrightarrow L^q(\mathsf{D})$ is continuous for $\frac{1}{q} = \frac{1}{2} - \frac{k}{n}$, if $k < \frac{n}{2}$.

(iv) the embedding $H^k(\mathsf{D}) \hookrightarrow L^r(\mathsf{D})$ is compact for $1 < r < q$, $\frac{1}{q} = \frac{1}{2} - \frac{k}{n}$, if $k < \frac{n}{2}$.

1.1.3 Sobolev spaces on Lipschitz boundaries

Define the space $L^2(\Gamma)$ of (equivalence classes of) square-power integrable functions on Γ as the completion of the space $\mathcal{C}^0(\Gamma)$ with respect to the norm

$$||g||_{L^2(\Gamma)} := \left(\int_\Gamma |g(y)|^2 \mathrm{d}\sigma\right)^{\frac{1}{2}}.$$

Let $s \in (0, 1)$. Define the boundary Sobolev space $H^s(\Gamma)$ as the completion of the space

$$\mathcal{C}_2^0 := \{f \in \mathcal{C}^0(\Gamma) \mid ||f||_{H^s(\Gamma)} < \infty\},$$

with respect to the norm

$$||f||_{H^s(\Gamma)} := \left\{||f||_{L^2(\Gamma)}^2 + \int_\Gamma \int_\Gamma \frac{|f(x) - f(y)|^2}{|x - y|^{n-1+2s}} \mathrm{d}\sigma_x \mathrm{d}\sigma_y\right\}^{\frac{1}{2}}.$$

Let us conclude this part by taking into account that, for $s \in (-1, 0)$, we define the Sobolev spaces of negative index by duality, that is, $H^{-s}(\Gamma) = (H^s(\Gamma))'$. As usual, we have $H^0(\Gamma) = L^2(\Gamma)$. The vector-valued versions of the spaces introduced in the former are defined component-wise.

1.1.4 Weighted Sobolev spaces

In this subsection, we will consider the setting provided by Assumption 1.1.6 in the case $n = 3$. We point out that, in this particular case, we work with an exterior (or complementary) Lipschitz domain D_- in $\mathbb{R}^3$. This fact brings an issue to the forefront. Some of our considered transmission problems contain the Stokes system in this complementary Lipschitz domain D_-. Our purpose will be that of taking into account the behavior at infinity of the solutions of our studied boundary value problems. As such, the behavior of these solutions must be included in the spaces that will be used in our analysis and this can be done in terms of weights. Hence, in the setting of $\mathbb{R}^3$, we introduce the weighted Sobolev spaces, as in the work of Hanouzet (see [47]).

Let Assumption 1.1.6 be satisfied for $n = 3$. Let us consider the weight function

$$\rho(\mathbf{x}) := (1 + |\mathbf{x}|)^{\frac{1}{2}}, \text{ for } \mathbf{x} \in \mathbb{R}^3.$$

We introduce the weighted Lebesgue space

$$L^2(\rho^{-1}; \mathsf{D}) := \{f : \mathsf{D}_- \to \mathbb{R} \mid \rho^{-1} f \in L^2(\mathsf{D}_-)\}$$

and with its help, we are able to define the weighted Sobolev space

$$\mathcal{H}^1(\mathsf{D}_-) := \{f \in \mathcal{D}(\mathsf{D}_-) \mid \rho^{-1} f \in L^2(\mathsf{D}_-), \nabla f \in L^2(\mathsf{D}_-)^3\},$$

where the vector-function space $L^2(\mathsf{D}_-)^3$ can be described component-wise (as in (1.1.7)). The weighted Sobolev space $\mathcal{H}^1(\mathsf{D}_-)$ is a Hilbert space with respect to the norm

$$||f||_{\mathcal{H}^1(\mathsf{D}_-)} := \left(||\rho^{-1} f||^2_{L^2(\mathsf{D}_-)} + ||\nabla f||^2_{L^2(\mathsf{D})^3}\right)^{\frac{1}{2}}. \tag{1.1.27}$$

Let us introduce also the space

$$\tilde{\mathcal{H}}^1(\mathsf{D}_-) \text{ as the closure of } \mathcal{D}(\mathsf{D}_-) \text{ in } \mathcal{H}^1(\mathbb{R}^3).$$

We introduce the spaces

$$\mathcal{H}^{-1}(\mathsf{D}_-) = (\tilde{\mathcal{H}}^1(\mathsf{D}_-))', \quad \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-) = (\mathcal{H}^1(\mathsf{D}_-))'.$$

Let us remark that $\mathcal{D}(\mathsf{D}_-)$ is dense in the space $\mathcal{H}^1(\mathsf{D}_-)$ and the space $\mathcal{D}(\overline{\mathsf{D}}_-)$ is dense in $\tilde{\mathcal{H}}^1(\mathsf{D}_-)$. In view of the fact that the seminorm

$$|g|_{\mathcal{H}^1(\mathsf{D}_-)} := ||\nabla g||_{L^2(\mathsf{D}_-)^3}$$

is equivalent to the norm (1.1.27) and by the Sobolev inequality (see [1, Theorem 4.31]) we have the embedding

$$\mathcal{H}^1(\mathsf{D}_-) \hookrightarrow L^6(\mathsf{D}_-).$$

Note that the vector-value weighted Sobolev spaces $\mathcal{H}^1(\mathsf{D}_-)^3$ and $\tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3$ are given by

$$\begin{aligned} &\mathcal{H}^1(\mathsf{D}_-)^3 := \{\mathbf{u} : \mathsf{D}_- \to \mathbb{R} \mid \mathbf{u} = (u_1, u_2, u_3), u_i \in \mathcal{H}^1(\mathsf{D}_-), i = \overline{1,3}\}, \\ &\tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3 := \{\mathbf{u} : \mathsf{D}_- \to \mathbb{R} \mid \mathbf{u} = (u_1, u_2, u_3), u_i \in \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-), i = \overline{1,3}\}. \end{aligned} \tag{1.1.28}$$

Finally, let us describe the notion of a function that tends to a constant at infinity in the sense of Leray and a particular result. These concepts will be used in the following chapters (see, e.g., [52, Definition 2.3 and Corollary 2.4] and the references therein).

Definition 1.1.16. *A function* $\mathbf{u}$ *tends to a constant* $\mathbf{u}_\infty$ *at* ∞*, in the sense of Leray if*

$$\lim_{r \to \infty} \int_{\mathcal{S}^2} |\mathbf{u}(ry) - \mathbf{u}_\infty| \mathrm{d}\sigma_y = 0,$$

where $\mathcal{S}^2$ *denotes the unit sphere in* $\mathbb{R}^3$.

Corollary 1.1.17. *If* $u \in \mathcal{H}^1(\mathsf{D}_-)$*, then* u *tends to zero at* ∞ *in the sense of Leray.*

1.1.5 The trace operator on Sobolev spaces

In this subsection, our aim is to introduce an operator which appears in the boundary conditions of our transmission-type problems that are studied in this thesis.

The connection between the Sobolev spaces defined on Lipschitz domains and the Sobolev spaces defined on Lipschitz boundaries is given by the following result known as the Gagliardo Trace Lemma (see, e.g., [20], [36], [50, Proposition 3.3], [77, Lemma 2.6]).

Lemma 1.1.18. *(The Gagliardo Trace Lemma) Let Assumption 1.1.6 be satisfied. Then, there exist linear and bounded operators*

$$\mathrm{Tr}_{\mathsf{D}_\pm} : H^1(\mathsf{D}_\pm) \to H^{\frac{1}{2}}(\Gamma), \tag{1.1.29}$$

called the (Gagliardo) trace operators, such that

$$\mathrm{Tr}_{\mathsf{D}_\pm} v = v|_\Gamma, \tag{1.1.30}$$

for all $v \in \mathcal{D}(\overline{\mathsf{D}}_\pm)$. *Moreover, these operators are surjective and have (non-unique) linear and bounded right inverse operators*

$$\mathrm{Tr}^{-1}_{\mathsf{D}_\pm} : H^{\frac{1}{2}}(\Gamma) \to H^1(\mathsf{D}_\pm), \tag{1.1.31}$$

that is $\mathrm{Tr}_{\mathsf{D}_\pm} \circ \mathrm{Tr}^{-1}_{\mathsf{D}_\pm} = \mathbb{I}$.

We end this subsection by pointing out some useful remarks.

Remark 1.1.19. *Similar to Lemma 1.1.18, one can define the exterior trace operator on the weighted Sobolev space* $\mathcal{H}^1(\mathsf{D}_-)$, *that is,* $\mathrm{Tr}_{\mathsf{D}_-} : \mathcal{H}^1(\mathsf{D}_-) \to H^{\frac{1}{2}}(\Gamma)$ *(for additional details, see, e.g., [77, Theorem 2.3, Lemma 2.6], [52, Lemma 2.2]).*

Remark 1.1.20. *Lemma 1.1.18 holds also in the case of vector-valued and matrix-valued functions. For the sake of brevity, we keep the notations* $\mathrm{Tr}_{\mathsf{D}_\pm}$ *and* $\mathrm{Tr}^{-1}_{\mathsf{D}_\pm}$ *in the setting of vector-valued or matrix-valued functions.*

1.2 The Stokes, classical Brinkman and generalized Brinkman operators

In this section we will discuss the operators that appear in this work. These operators are involved in the transmission problems that we study. Recall the $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions and recall that its dual, denoted by $\mathcal{S}'(\mathbb{R}^n)$, is the space of tempered distributions. The vector function spaces $\mathcal{S}(\mathbb{R}^n)^n$ and $\mathcal{S}'(\mathbb{R}^n)^n$ are defined component-wise.

The *Stokes operator* is given by

$$\mathbb{S} := \begin{bmatrix} \Delta & -\nabla \\ \mathrm{div} & 0 \end{bmatrix} : \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n) \tag{1.2.1}$$

and the operator

$$\mathsf{L}_0 : \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n)^n \to \mathcal{S}(\mathbb{R}^n)^n, \quad \mathsf{L}_0(\mathbf{v}, p) := \Delta \mathbf{v} - \nabla p. \tag{1.2.2}$$

Let us note that the operator $\mathbb{S}$ introduced in relation (1.2.1) is Agmon-Douglis-Nirenberg elliptic (see also [49], [108]) and this operator $\mathbb{S}$ together with the operator L_0 can be extended to linear and bounded operators, that is,

$$\mathbb{S} : H^1(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n)^n \to H^{-1}(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n), \quad \mathsf{L}_0 : H^1(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n)^n \to H^{-1}(\mathbb{R}^n)^n. \quad (1.2.3)$$

Let $\alpha > 0$ be a given constant. Let us introduce also the *Brinkman operator* as follows

$$\mathcal{B}_\alpha := \begin{bmatrix} (\Delta - \alpha\mathbb{I}) & -\nabla \\ \mathrm{div} & 0 \end{bmatrix} : \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n) \quad (1.2.4)$$

and it associated operator

$$\mathsf{L}_\alpha : \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n)^n \to \mathcal{S}(\mathbb{R}^n)^n, \quad \mathsf{L}_\alpha(\mathbf{v}, p) := (\Delta - \alpha\mathbb{I})\mathbf{v} - \nabla p. \quad (1.2.5)$$

The operator B_α introduced in relation (1.2.4) is Agmon-Douglis-Nirenberg elliptic (see also [49], [108]) and together with its associated operator L_α are extended to linear and bounded operators, as follows

$$\mathcal{B}_\alpha : H^1(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n)^n \to H^{-1}(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n), \quad \mathsf{L}_\alpha : H^1(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n)^n \to H^{-1}(\mathbb{R}^n)^n. \quad (1.2.6)$$

Finally, we address some notations that we will employ from now on, throughout this thesis.

Notation 1.2.1. *Consider the spaces of divergence free vector fields*

$$H^1_{\mathrm{div}}(\mathsf{D})^n = \{\mathbf{u} \in H^1(\mathsf{D})^n \mid \mathrm{div}\, \mathbf{u} = 0 \text{ in } \mathsf{D}\}, \quad (1.2.7)$$

and

$$\mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3 := \{\mathbf{u} \in \mathcal{H}^1(\mathsf{D}_-)^3 \mid \mathrm{div}\, \mathbf{u} = 0 \text{ in } \mathsf{D}_-\}. \quad (1.2.8)$$

Notation 1.2.2. *Throughout this thesis, we introduce the operator $\mathring{\mathsf{E}}_\pm$, which represents the extension by zero operator outside $\mathsf{D}_\pm$. More specifically, it allows us to extend functions from $\mathring{H}^1(\mathsf{D}_\pm)$ by zero to $\mathbb{R}^n \setminus \mathsf{D}_\pm$. We keep the same notation $\mathring{\mathsf{E}}_\pm$ in the case of vector-valued spaces.*

1.2.1 The conormal derivate operator associated to the Stokes and Brinkman systems

This subsection is dedicated to the introduction of the conormal derivative operators associated to the Stokes and Brinkman systems. We discuss the classical derivative operator and the generalized conormal derivative operator associated for these systems. In the latter, let $\mathsf{D} \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with connected boundary Γ.

We will introduce the classical conormal derivative operator as follows. For a pair $(\mathbf{v}, p) \in C^1(\overline{\mathsf{D}}_\pm)^n \times C^0(\overline{\mathsf{D}}_\pm)$ satisfying $\mathrm{div}\, \mathbf{v} = 0$ in $\mathsf{D}_\pm$ we have that classical derivative operator (or traction field) associated to the Stokes or Brinkman operator is provided by the constitutive equation of the Newtonian (viscous) incompressible fluid, i.e.,

$$\mathbf{t}^\pm(\mathbf{v}, p) := \mathrm{Tr}_{\mathsf{D}}\boldsymbol{\sigma}(\mathbf{v}, p)\boldsymbol{\nu}, \quad (1.2.9)$$

where

$$\boldsymbol{\sigma}(\mathbf{v}, p) := -p\mathbb{I} + 2\mathbb{E}(\mathbf{v}) \quad (1.2.10)$$

is the stress tensor and $\mathbb{E}(\mathbf{v})$ is the symmetric part of $\nabla\mathbf{v}$, that is $\mathbb{E}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^t)$, where the superscript t denotes the transpose. The symbol $\boldsymbol{\nu}$ represents the outward unit normal to D, which is defined a.e. on Γ.

Note that, for $\boldsymbol{\phi} \in \mathcal{D}(\mathbb{R}^n)^n$, we have the following Green identity for the Brinkman system,

$$\pm\langle \mathbf{t}_\alpha^\pm(\mathbf{v},p), \boldsymbol{\phi}\rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{v}), \mathbb{E}(\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} + \alpha\langle \mathbf{v}, \boldsymbol{\phi}\rangle_{\mathsf{D}_\pm} - \langle p, \operatorname{div}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm} + \langle \mathsf{L}_\alpha(\mathbf{v},p), \boldsymbol{\phi}\rangle_{\mathsf{D}_\pm}, \tag{1.2.11}$$

where $\alpha > 0$ is a given constant. In particular, for $\alpha = 0$, we obtain the Green identity for the Stokes system,

$$\pm\langle \mathbf{t}^\pm(\mathbf{v},p), \boldsymbol{\phi}\rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{v}), \mathbb{E}(\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} - \langle p, \operatorname{div}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm} + \langle \mathsf{L}_0(\mathbf{v},p), \boldsymbol{\phi}\rangle_{\mathsf{D}_\pm}. \tag{1.2.12}$$

Formulas (1.2.11) and (1.2.12) follow after repeated integration by parts.

Formula (1.2.12) suggests the definition of the generalized conormal derivative operator associated to the Stokes system, and the corresponding Green formula in the setting of Sobolev spaces (see, e.g., [85, Theorem 10.4.1], [20, Lemma 3.2], [77, Defintion 3.1, Theorem 3.2]).

Definition 1.2.3. *Let* $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^n$*, be a bounded Lipschitz domain and let* $\mathsf{D}_- := \mathbb{R}^n \setminus \overline{\mathsf{D}}$*. Define the space* $\boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0)$ *by*

$$\boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0) := \{(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in H^1(\mathsf{D}_\pm)^n \times L^2(\mathsf{D}_\pm) \times \tilde{H}^{-1}(\mathsf{D}_\pm)^n : \mathsf{L}_0(\mathbf{v}_\pm, p_\pm) = \mathbf{g}_\pm|_{\mathsf{D}_\pm} \text{ and } \operatorname{div}\mathbf{v}_\pm = 0 \text{ in } \mathsf{D}_\pm\}.$$

Then, the generalized conormal derivative operators $\mathbf{t}_{\mathsf{D}_\pm}$ *for the Stokes system in* $\mathsf{D}_\pm$ *are defined on each* $(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0)$ *by the following relation:*

$$\begin{aligned}\pm\langle \mathbf{t}_{\mathsf{D}_\pm}(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm), \boldsymbol{\phi}\rangle_\Gamma := {} & 2\langle \mathbb{E}(\mathbf{v}_\pm), \mathbb{E}(\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} - \langle p_\pm, \operatorname{div}(\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} \\ & + \langle \mathbf{g}_\pm, \mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm}, \forall\, \boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n.\end{aligned} \tag{1.2.13}$$

Lemma 1.2.4. *In the setting of Definition* 1.2.3*, the generalized conormal derivative operators*

$$\mathbf{t}_{\mathsf{D}_\pm} : \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0) \to H^{-\frac{1}{2}}(\Gamma)^n \tag{1.2.14}$$

are linear and bounded and Definition 1.2.3 *is independent of the choice of a right inverse* $\mathrm{Tr}_{\mathsf{D}_\pm}^{-1} : H^{\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n$ *of the trace operator* $\mathrm{Tr}_{\mathsf{D}_\pm} : H^1(\mathsf{D}_+)^n \to H^{\frac{1}{2}}(\Gamma)^n$*. Moreover, the following Green formulas hold*

$$\begin{aligned}\pm\langle \mathbf{t}_{\mathsf{D}_\pm}(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm), \mathrm{Tr}_{\mathsf{D}_\pm}\boldsymbol{\psi}_\pm\rangle_\Gamma := {} & 2\langle \mathbb{E}(\mathbf{v}_\pm), \mathbb{E}(\boldsymbol{\psi}_\pm)\rangle_{\mathsf{D}_\pm} - \langle p_\pm, \operatorname{div}\boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm} \\ & + \langle \mathbf{g}_\pm, \boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm},\end{aligned} \tag{1.2.15}$$

for all $(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0)$ *and for any* $\boldsymbol{\psi}_\pm \in H^1(\mathsf{D}_\pm)^n$*.*

Similarly, formula (1.2.11) suggests the definition of the generalized conormal derivative operator associated to the Brinkman system, (see, e.g., [20, Lemma 3.2], [56, Lemma 2.2], [52, Lemma 2.5]).

Definition 1.2.5. *Let* $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^n$*, be a bounded Lipschitz domain and let* $\mathsf{D}_- := \mathbb{R}^n \setminus \overline{\mathsf{D}}$*. Define the space* $\boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha)$ *by*

$$\boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha) := \{(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in H^1(\mathsf{D}_\pm)^n \times L^2(\mathsf{D}_\pm) \times \tilde{H}^{-1}(\mathsf{D}_\pm)^n : \mathsf{L}_\alpha(\mathbf{v}_\pm, p_\pm) = \mathbf{g}_\pm|_{\mathsf{D}_\pm} \text{ and } \operatorname{div}\mathbf{v}_\pm = 0 \text{ in } \mathsf{D}_\pm\}.$$

Then, the generalized conormal derivative operators $\mathbf{t}_{\alpha,\mathsf{D}_\pm}$ *for the Brinkman system in* $\mathsf{D}_\pm$ *are defined on each* $(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha)$ *by the following relation:*

$$\begin{aligned}\pm\langle \mathbf{t}_{\alpha,\mathsf{D}_\pm}(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm), \boldsymbol{\phi}\rangle_\Gamma := {} & 2\langle \mathbb{E}(\mathbf{v}_\pm), \mathbb{E}(\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} + \alpha\langle \mathbf{v}_\pm, \mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm} \\ & - \langle p_\pm, \operatorname{div}(\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} + \langle \mathbf{g}_\pm, \mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm}, \forall\, \boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n.\end{aligned} \tag{1.2.16}$$

Lemma 1.2.6. *In the setting of Definition* 1.2.5*, the generalized conormal derivative operators*

$$\mathbf{t}_{\alpha,\mathsf{D}_\pm} : \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha) \to H^{-\frac{1}{2}}(\Gamma)^n \tag{1.2.17}$$

are linear and bounded and Definition 1.2.5 *is independent of the choice of a right inverse* $\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}$: $H^{\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n$ *of the trace operator* $\mathrm{Tr}_{\mathsf{D}_\pm} : H^1(\mathsf{D}_+)^n \to H^{\frac{1}{2}}(\Gamma)^n$*. Moreover, the following Green formulas hold*

$$\begin{aligned}\pm\langle \mathbf{t}_{\alpha,\mathsf{D}_\pm}(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm), \mathrm{Tr}_{\mathsf{D}_\pm}\boldsymbol{\psi}_\pm\rangle_\Gamma := {}& 2\langle \mathbb{E}(\mathbf{v}_\pm), \mathbb{E}(\boldsymbol{\psi}_\pm)\rangle_{\mathsf{D}_\pm} + \alpha\langle \mathbf{v}_\pm, \boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm} \\ &-\langle p_\pm, \mathrm{div}\ \boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm} + \langle \mathbf{g}_\pm, \boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm},\end{aligned} \tag{1.2.18}$$

for all $(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha)$ *and for any* $\boldsymbol{\psi}_\pm \in H^1(\mathsf{D}_\pm)^n$.

Let us end this subsection by pointing out some useful remarks (see also [52, Remark 2.6, Lemma 2.9], [56, Remark 2.4]).

Remark 1.2.7. *For* $\alpha = 0$*, the conormal derivative for the Brinkman system (see Definition 1.2.5) reduces to the conormal derivative for the Stokes system (see Definition 1.2.3).*

Remark 1.2.8. *Let* $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^3$*, be a bounded Lipschitz domain and let* $\mathsf{D}_- := \mathbb{R}^3 \setminus \overline{\mathsf{D}}$*. For* $(\mathbf{v}_-, p_-, \mathbf{g}_-) \in \mathcal{H}^1(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-) \times \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3$ *satisfying* $\mathsf{L}_0(\mathbf{v}_-, p_-) = \mathbf{g}_-|_{\mathsf{D}_-}$*, the conormal derivative operator* $\mathbf{t}_{\mathsf{D}_-}(\mathbf{v}_-, p_-, \mathbf{g}_-)$ *is well-defined by relation (1.2.13) and a corresponding Green formula similar to relation (1.2.15) holds true in* D_-*.*

Remark 1.2.9. *Let* $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^3$*, be a bounded Lipschitz domain and let* $\mathsf{D}_- := \mathbb{R}^3 \setminus \overline{\mathsf{D}}$*. Then for* $(\mathbf{v}_-, p_-, \mathbf{g}_-) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-) \times \tilde{H}^{-1}(\mathsf{D}_-)^3$*, such that* $\mathsf{L}_\alpha(\mathbf{v}_-, p_-) = \mathbf{g}_-|_{\mathsf{D}_-}$*, the conormal derivative* $\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}_-, p_-, \mathbf{g}_-)$ *is well-defined by relation (1.2.16). In addition, in this case, the Green formula (1.2.18) also holds, in* D_-*. The space* $\mathfrak{M}(\mathsf{D}_-)$ *is provided by Definition 2.1.1.*

Remark 1.2.10. *Let* $\mathsf{D} \subset \mathbb{R}^n$*,* $n \geq 2$*, be a bounded Lipschitz domain and denote its boundary by* Γ*. In the case* $\Gamma = \Gamma_1 \cup \Gamma_2$*, where* Γ_1 *and* Γ_2 *are connected components of* Γ *such that* $\Gamma_1 \cap \Gamma_2 = \emptyset$*, we define the operator*

$$(\mathbf{t}_{\alpha,\mathsf{D}}(\cdot,\cdot,\cdot))|_{\Gamma_1} : \boldsymbol{H}^1(\mathsf{D}, \mathsf{L}_\alpha) \to H^{-\frac{1}{2}}(\Gamma_1)^n, \tag{1.2.19}$$

by the relation

$$\langle \mathbf{t}_{\alpha,\mathsf{D}}(\mathbf{v}, p, \mathbf{g})|_{\Gamma_1}, \boldsymbol{\Phi}\rangle_{\Gamma_1} := \langle \mathbf{t}_{\alpha,\mathsf{D}}(\mathbf{v}, p, \mathbf{g}), \boldsymbol{\Phi}\rangle_\Gamma, \tag{1.2.20}$$

for all $\boldsymbol{\Phi} \in C^\infty(\mathbb{R}^n)^n$ *which vanish in an open neighborhood of* Γ_2*.*

Remark 1.2.11. *Throughout this thesis, we will write* $\mathbf{t}_{\alpha,\mathsf{D}}(\mathbf{v}, p)$ *instead of* $\mathbf{t}_{\alpha,\mathsf{D}}(\mathbf{v}, p, \mathbf{0})$*.*

1.2.2 The generalized Brinkman system and related results

In this thesis we consider a generalized type Brinkman system. Indeed, the term $\alpha\mathbb{I}$ which appears in the classical Brinkman operator (see relations (1.2.4) and (1.2.5)) has been replaced by another, much more general term. Part of the original results that are included in the thesis are transmission problem in which this generalized version of the Brinkman system is involved. More recently, this generalized type Brinkman system has also been treated in the much more general setting of variable coefficient PDE systems (see, e.g., [58], [59], [60], [67]).

Hence, for the introduction of this generalized version of the Brinkman system, we consider a bounded Lipschitz domain $\mathsf{D} \subseteq \mathbb{R}^3$. The generalized Brinkman system is given by

$$\mathsf{L}_{\mathcal{P}}(\mathbf{v}, p) := \Delta\mathbf{v} - \mathcal{P}\mathbf{v} - \nabla p = \mathbf{g} \text{ in } \mathsf{D},\ \mathrm{div}\ \mathbf{v} = 0 \text{ in } \mathsf{D}, \tag{1.2.21}$$

where $\mathcal{P} \in L^\infty(\mathsf{D})^{3\times 3}$ such that $\mathcal{P}$ satisfies the following non-negativity condition

$$\langle \mathcal{P}\mathbf{v}, \mathbf{v}\rangle_{\mathsf{D}} \geq c_{\mathcal{P}} \|\mathbf{v}\|^2_{L^2(\mathsf{D})^3},\ \forall\ \mathbf{v} \in L^2(\mathsf{D})^3, \tag{1.2.22}$$

where $c_{\mathcal{P}} > 0$ is a constant.

The system (1.2.21) is viewed in a distributional sense, that is, for $(\mathbf{v}, p) \in H^1(\mathsf{D})^3 \times L^2(\mathsf{D})$, we have

$$\langle \mathsf{L}_{\mathcal{P}}(\mathbf{v}, p), \boldsymbol{\psi}\rangle_{D_+} = \langle \mathbf{g}, \boldsymbol{\psi}\rangle_{\mathsf{D}},\ \langle \operatorname{div} \mathbf{v}, g_0\rangle_{\mathsf{D}} = 0, \tag{1.2.23}$$

for all $(\boldsymbol{\psi}, g_0) \in \mathcal{D}(\mathsf{D})^3 \times \mathcal{D}(\mathsf{D})$, where

$$\langle \mathsf{L}_{\mathcal{P}}(\mathbf{v}, p), \boldsymbol{\psi}\rangle_{\mathsf{D}} := \langle \Delta\mathbf{v} - \mathcal{P}\mathbf{v} - \nabla p, \boldsymbol{\psi}\rangle_{\mathsf{D}} = -\langle \nabla\mathbf{v}, \nabla\boldsymbol{\psi}\rangle_{\mathsf{D}} - \langle \mathcal{P}\mathbf{v}, \boldsymbol{\psi}\rangle_{\mathsf{D}} + \langle p, \operatorname{div} \boldsymbol{\psi}\rangle_{\mathsf{D}}.$$

Also, the continuous embedding $L^2(\mathsf{D}) \hookrightarrow H^{-1}(\mathsf{D})$ implies the linearity and boundedness of the operator

$$\mathsf{L}_{\mathcal{P}} : H^1(\mathsf{D})^3 \times L^2(\mathsf{D}) \to H^{-1}(\mathsf{D})^3 = (\mathring{H}^1(\mathsf{D})^3)'. \tag{1.2.24}$$

Note that, we are able to extract from this generalized version of the Brinkman system the classical Stokes or Brinkman systems, respectively. This fact is emphasized in the following remarks.

Remark 1.2.12. *For $\mathcal{P} \equiv 0$, the system (1.2.21) is the classical Stokes system.*

Remark 1.2.13. *For $\mathcal{P} \equiv \alpha\mathbb{I}$, where $\alpha > 0$ is a constant, the system (1.2.21) is the classical Brinkman system.*

For this generalized version of the Brinkman system, we introduce its associated conormal derivative operator (see, e.g., [6, Lemma 2.4]).

Definition 1.2.14. *Let $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^3$, be a bounded Lipschitz domain and denote its boundary by Γ. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ such that condition (1.2.22) is satisfied. Define the space $\boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}})$ by*

$$\boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}}) := \{(\mathbf{v}_+, p_+, \mathbf{g}_+) \in H^1(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times \tilde{H}^{-1}(\mathsf{D}_+)^3 : \mathsf{L}_{\mathcal{P}}(\boldsymbol{v}_+, p_+) = \mathbf{g}_+|_{\mathsf{D}_+}$$
$$\text{and } \operatorname{div} \mathbf{v}_+ = 0 \text{ in } \mathsf{D}_+\}.$$

Then, the conormal derivative operator

$$\mathbf{t}_{\mathcal{P},\mathsf{D}_+} : \boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}}) \to H^{-\frac{1}{2}}(\Gamma)^3 \tag{1.2.25}$$

for the generalized Brinkman system in D_+ is defined on each $(\mathbf{v}_+, p_+, \mathbf{g}_+) \in \boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}})$ by the following relation:

$$\begin{aligned}\langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}_+, p_+, \mathbf{g}_+), \boldsymbol{\phi}\rangle_\Gamma := {} & 2\langle \mathbb{E}(\mathbf{v}_+), \mathbb{E}(\mathrm{Tr}^{-1}_{\mathsf{D}_+}\boldsymbol{\phi})\rangle_{\mathsf{D}_+} + \langle \mathcal{P}\mathbf{v}_+, \mathrm{Tr}^{-1}_{\mathsf{D}_+}\boldsymbol{\phi}\rangle_{\mathsf{D}_+} \\ & - \langle p_+, \operatorname{div}(\mathrm{Tr}^{-1}_{\mathsf{D}_+}\boldsymbol{\phi})\rangle_{\mathsf{D}_+} + \langle \mathbf{g}_+, \mathrm{Tr}^{-1}_{\mathsf{D}_+}\boldsymbol{\phi}\rangle_{\mathsf{D}_+}, \forall\ \boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^3.\end{aligned} \tag{1.2.26}$$

Lemma 1.2.15. *In the setting of Definition* 1.2.14, *the conormal derivative operator for the generalized Brinkman system,*

$$\mathbf{t}_{\mathcal{P},\mathsf{D}_+} : \boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}}) \to H^{-\frac{1}{2}}(\Gamma)^3 \tag{1.2.27}$$

is linear and bounded, and Definition 1.2.14 *is independent of the choice of a right inverse $\mathrm{Tr}^{-1}_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^3 \to H^1(\mathsf{D}_+)^3$ of the trace operator $\mathrm{Tr}_{\mathsf{D}_+} : H^1(\mathsf{D}_+)^3 \to H^{\frac{1}{2}}(\Gamma)^3$. Moreover, the following Green formula holds*

$$\begin{aligned}\langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}_+, p_+, \mathbf{g}_+), \mathrm{Tr}_{\mathsf{D}_+}\boldsymbol{\psi}_+\rangle_\Gamma := {} & 2\langle \mathbb{E}(\mathbf{v}_+), \mathbb{E}(\boldsymbol{\psi}_+)\rangle_{\mathsf{D}_+} + \langle \mathcal{P}\mathbf{v}_+, \boldsymbol{\psi}_+\rangle_{\mathsf{D}_+} \\ & - \langle p_+, \operatorname{div} \boldsymbol{\psi}_+\rangle_{\mathsf{D}_+} + \langle \mathbf{g}_+, \boldsymbol{\psi}_+\rangle_{\mathsf{D}_+},\end{aligned} \tag{1.2.28}$$

for all $(\mathbf{v}_+, p_+, \mathbf{g}_+) \in \boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}})$ and for any $\boldsymbol{\psi}_+ \in H^1_{\operatorname{div}}(\mathsf{D}_+)^n$.

The proof of Lemma 1.2.15 follows similar ideas to those used in the proof of Lemma 1.2.6, i.e., the special case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha > 0$ is a given constant. For additional details, we refer the reader to [56, Lemma 2.2].

Remark 1.2.16. *Taking into account the definitions of the conormal derivative operators for the Stokes and generalized Brinkman systems, given by (1.2.13) and (1.2.26), we deduce that*

$$\mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}, p, \mathbf{g}) = \mathbf{t}_{\mathsf{D}_+}(\mathbf{v}, p, \mathbf{g} + \mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{v})), \tag{1.2.29}$$

where $\mathring{\mathsf{E}}_+$ *denotes the operator of extension by zero outside* D_+*.*

1.3 Stokes layer potentials and their properties

In this section we give the fundamental solution for the Stokes system in $\mathbb{R}^n$, $n \geq 2$, and with its help we define the layer operators that are involved in the solutions of our transmission problems. The sources that we used for the preparation of this section are [49], [52], [64], [85].

1.3.1 The Stokes system and its fundamental solution

Let $(\mathbf{G}(\cdot,\cdot), \mathbf{P}(\cdot,\cdot)) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^{n\times n} \times \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^n$ denote the fundamental solution of the Stokes system. By $\mathbf{G}(\cdot,\cdot)$ we denote the fundamental velocity tensor and by $\mathbf{P}(\cdot,\cdot)$ we denote the fundamental pressure vector for the Stokes system in $\mathbb{R}^n$.

Note that the fundamental solution of the Stokes system satisfies the equations

$$\Delta_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\mathbf{P}(\mathbf{x},\mathbf{y}) = -\delta_{\mathbf{y}}(\mathbf{x})\mathbb{I},\ \mathrm{div}_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{y}) = 0, \tag{1.3.1}$$

where the symbol $\delta_{\mathbf{y}}$ denotes the Dirac distribution with mass at $\mathbf{y}$. Also, the differential operators $\Delta_{\mathbf{x}}$, $\nabla_{\mathbf{x}}$ and $\mathrm{div}_{\mathbf{x}}$ act with respect to the variable $\mathbf{x}$.

The components of the fundamental solution $(\mathbf{G}(\mathsf{G}_{jk}), \mathbf{P}(\mathsf{P}_k))$ are given by (see, e.g., [64, p. 38-39], [85, Relation (4.19), Relation (4.20), Relation (4.21)], [106])

$$\mathsf{G}_{jk}(\mathbf{x},\mathbf{y}) := \frac{1}{2\omega_n}\left\{\frac{\delta_{jk}}{(n-2)|\mathbf{y}-\mathbf{x}|^{n-2}} + \frac{x_jx_k}{|\mathbf{y}-\mathbf{x}|^n}\right\}, \quad \mathsf{P}_k(\mathbf{x},\mathbf{y}) = \frac{1}{\omega_n}\frac{x_k}{|\mathbf{y}-\mathbf{x}|^n}, \tag{1.3.2}$$

for $n \geq 3$, and

$$\mathsf{G}_{jk}(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi}\left\{\frac{x_jx_k}{|\mathbf{y}-\mathbf{x}|^2} - \delta_{jk}\log|\mathbf{y}-\mathbf{x}|^{n-2}\right\}, \quad \mathsf{P}_k(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\frac{x_k}{|\mathbf{y}-\mathbf{x}|^2}, \tag{1.3.3}$$

for $n \geq 2$. Note that δ_{jk} denotes the Kronecher symbol and ω_n is the surface measure of the unit sphere $\mathcal{S}^{n-1}$ in $\mathbb{R}^n$.

Let also $\mathbf{S}(\mathsf{S}_{jkl})$ and $\mathbf{R}(\mathsf{R}_{jk})$ denote the associated stress and pressure tensors for the Stokes system. Their components are given by (see, e.g., [64, Chapter 2], [85])

$$\mathsf{S}_{jkl}(\mathbf{x},\mathbf{y}) := \frac{n}{\omega_n}\frac{x_ix_jx_k}{|\mathbf{y}-\mathbf{x}|^{n+2}}, \quad \mathsf{R}_{jk}(\mathbf{x},\mathbf{y}) := -\frac{2}{\omega_n}\left\{-\frac{\delta_{jk}}{|\mathbf{y}-\mathbf{x}|^n} + n\frac{x_jx_k}{|\mathbf{y}-\mathbf{x}|^{n+2}}\right\}, \tag{1.3.4}$$

for $n \geq 2$.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{y}$, the pair $(\mathbf{S}(\mathsf{S}_{jkl}), \mathbf{R}(\mathsf{R}_{jk}))$ satisfies the Stokes system

$$\Delta_{\mathbf{x}}\mathsf{S}_{jkl}(\mathbf{x},\mathbf{y}) - \frac{\partial\mathsf{R}_{jk}(\mathbf{y},\mathbf{x})}{\partial x_k} = 0,\ \frac{\partial\mathsf{S}_{jkl}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0. \tag{1.3.5}$$

1.3.2 The volume potential for the Stokes system and its properties

The purpose of this subsection is to introduce the Newtonian (volume) potential operators associated to the Stokes system and to give their mapping properties. To this end, we consider the Lipschitz domains $\mathsf{D}_\pm$ as described in Assumption 1.1.6 and we will take into account the fundamental solution of the Stokes system, that is, the pair $(\mathbf{G}(\cdot,\cdot),\mathbf{P}(\cdot,\cdot)) \in \mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)^{n\times n}\times\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)^n$ given by formula (1.3.2) or (1.3.3).

Definition 1.3.1. *For $\mathbf{f}\in H^{-1}(\mathbb{R}^n)^n$, define the Newtonian (volume) velocity and pressure potentials for the Stokes system, by*

$$(\mathcal{N}_{\mathbb{R}^n}\mathbf{f})(\boldsymbol{x}) := -\langle \mathbf{G}(\boldsymbol{x},\cdot),\mathbf{f}\rangle_{\mathbb{R}^n},\ \ (\mathcal{Q}_{\mathbb{R}^n}\mathbf{f})(\boldsymbol{x}) := -\langle \mathbf{P}(\boldsymbol{x},\cdot),\mathbf{f}\rangle_{\mathbb{R}^n}. \tag{1.3.6}$$

Moreover, the Newtonian (volume) velocity and pressure potentials for the Stokes system corresponding to $\mathsf{D}_\pm$, are given by

$$\mathcal{N}_{\mathsf{D}_\pm}\mathbf{f} := (\mathcal{N}_{\mathbb{R}^n}\mathbf{f})|_{\mathsf{D}_\pm},\ \ \mathcal{Q}_{\mathsf{D}_\pm}\mathbf{f} := (\mathcal{Q}_{\mathbb{R}^n}\mathbf{f})|_{\mathsf{D}_\pm}, \tag{1.3.7}$$

where $|_{\mathsf{D}_\pm}$ is the restriction operator to $\mathsf{D}_\pm$, which acts on vector-valued or scalar-valued functions in $\mathbb{R}^n$.

The following lemma describes the mapping properties of the Newtonian (volume) layer potential operators in the setting of Sobolev spaces (see, e.g., [52, Lemma A.3]).

Theorem 1.3.2. *The Newtonian (volume) velocity and pressure potential operators for the Stokes system, introduced in relation (1.3.6),*

$$\begin{aligned}&\mathcal{N}_{\mathbb{R}^n} : H^{-1}(\mathbb{R}^n)^n \to H^1(\mathbb{R}^n)^n,\ \ \mathcal{Q}_{\mathbb{R}^n} : H^{-1}(\mathbb{R}^n)^n \to L^2(\mathbb{R}^n),\\ &\mathcal{N}_{\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \to \mathcal{H}^1(\mathbb{R}^3)^3,\ \ \mathcal{Q}_{\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)\end{aligned} \tag{1.3.8}$$

are linear and continuous operators. Moreover, the Newtonian (volume) velocity and pressure potentials for the Stokes system, introduced in relation (1.3.7),

$$\mathcal{N}_{\mathsf{D}_+} : \tilde{H}^{-1}(\mathsf{D}_+)^n \to H^1(\mathsf{D}_+)^n,\ \ \mathcal{Q}_{\mathsf{D}_+} : \tilde{H}^{-1}(\mathsf{D}_+)^n \to L^2(\mathsf{D}_+), \tag{1.3.9}$$

and

$$\mathcal{N}_{\mathsf{D}_-} : \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3 \to \mathcal{H}^1(\mathsf{D}_-)^3,\ \ \mathcal{Q}_{\mathsf{D}_-} : \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3 \to L^2(\mathsf{D}_-), \tag{1.3.10}$$

in the case $n = 3$, are linear and continuous operators as well.

Finally, by taking into account relation (1.3.1), we have that the Newtonian potentials satisfy the following equations (in the sense of distributions):

$$\Delta(\mathcal{N}_{\mathbb{R}^n}\mathbf{f}) - \nabla(\mathcal{Q}_{\mathbb{R}^n}\mathbf{f}) = \mathbf{f},\ \text{div}\ (\mathcal{N}_{\mathbb{R}^n}\mathsf{f}) = 0,\ \text{in}\ \mathbb{R}^n, \tag{1.3.11}$$

and

$$\Delta(\mathcal{N}_{\mathsf{D}_\pm}\mathbf{f}) - \nabla(\mathcal{Q}_{\mathsf{D}_\pm}\mathbf{f}) = \mathbf{f},\ \text{div}\ (\mathcal{N}_{\mathsf{D}_\pm}\mathsf{f}) = 0,\ \text{in}\ \mathsf{D}_\pm, \tag{1.3.12}$$

respectively.

1.3.3 Stokes layer potentials and related results

In this subsection, we concern ourselves with the single layer potential and the double layer potential operators associated to the Stokes system. Our purpose is to give their definitions, their mapping properties their jump relations and specify their behavior at infinity. From now on, let Assumption 1.1.6 be satisfied and in addition, we assume that the bounded Lipschitz domain D_+ has a connected boundary Γ.

Firstly, let us focus on the single-layer velocity and pressure potentials, associated to the Stokes system (see, e.g., [85, Relation (4.24), Relation (4.27)]).

Definition 1.3.3. *Let Assumption* 1.1.6 *be satisfied. Let* $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\Gamma)^n$. *Define the single-layer velocity potential* $\mathbf{V}_\Gamma \boldsymbol{\varphi}$ *and its associated pressure potential* $\mathcal{Q}^s_\Gamma \boldsymbol{\varphi}$ *for the Stokes system, by*

$$(\mathbf{V}_\Gamma \boldsymbol{\varphi}) := \langle \mathbf{G}(\boldsymbol{x}, \cdot), \boldsymbol{\varphi} \rangle_\Gamma, \ (\mathcal{Q}^s_\Gamma \boldsymbol{\varphi}) := \langle \mathbf{P}(\boldsymbol{x}, \cdot), \boldsymbol{\varphi} \rangle_\Gamma, \ \forall \ \boldsymbol{x} \in \mathbb{R}^n \setminus \Gamma. \tag{1.3.13}$$

By taking into account relation (1.3.1), we have that the pair $(\mathbf{V}_\Gamma \boldsymbol{\varphi}, \mathcal{Q}^s_\Gamma \boldsymbol{\varphi})$ satisfies the homogeneous Stokes system

$$\Delta(\mathbf{V}_\Gamma \boldsymbol{\varphi}) - \nabla(\mathcal{Q}^s_\Gamma \boldsymbol{\varphi}) = 0, \quad \mathrm{div}\ (\mathbf{V}_\Gamma \boldsymbol{\varphi}) = 0 \tag{1.3.14}$$

in $\mathbb{R}^n \setminus \Gamma$.

The following theorem gives some useful mapping properties for the single layer potential operators associated to the Stokes system (see, e.g., [52, Lemma A.4]).

Theorem 1.3.4. *Let Assumption 1.1.6 be satisfied. Then the following operators*

$$(\mathbf{V}_\Gamma)|_{\mathsf{D}_+} : H^{-\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n, \ (\mathcal{Q}^s_\Gamma)|_{\mathsf{D}_+} : H^{-\frac{1}{2}}(\Gamma)^n \to L^2(\mathsf{D}_+) \tag{1.3.15}$$

are linear and bounded. Moreover, for $n = 3$, *we have that the operators*

$$(\mathbf{V}_\Gamma)|_{\mathsf{D}_-} : H^{-\frac{1}{2}}(\Gamma)^3 \to \mathcal{H}^1(\mathsf{D}_-)^3, \ (\mathcal{Q}^d_\Gamma)|_{\mathsf{D}_-} : H^{-\frac{1}{2}}(\Gamma)^3 \to L^2(\mathsf{D}_-) \tag{1.3.16}$$

are linear and bounded as well, where the weighted Sobolev space $\mathcal{H}^1(\mathsf{D}_-)^3$ *is given in relation (1.1.28).*

Secondly, we focus on the double-layer velocity and pressure potentials, associated to the Stokes system (see, e.g., [85, Relation (4.25), Relation (4.28)]).

Definition 1.3.5. *Let Assumption* 1.1.6 *be satisfied. Let* $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n$. *Then, the double-layer velocity potential* $\mathbf{W}_\Gamma \boldsymbol{\phi}$ *and its associated pressure potential* $\mathcal{Q}^d_\Gamma \boldsymbol{\phi}$ *for the Stokes system are defined by*

$$\begin{aligned} (\mathbf{W}_\Gamma \boldsymbol{\phi})_k(\boldsymbol{x}) &:= \int_\Gamma \mathsf{S}_{jkl}(\boldsymbol{y}, \boldsymbol{x}) \nu_l(\boldsymbol{y}) \phi_j(\boldsymbol{y}) d\sigma_{\boldsymbol{y}}, \ \forall \ \boldsymbol{x} \in \mathbb{R}^n \setminus \Gamma, \\ (\mathcal{Q}^d_\Gamma \boldsymbol{\phi})(\boldsymbol{x}) &:= \int_\Gamma \mathsf{R}_{jl}(\boldsymbol{x}, \boldsymbol{y}) \nu_l(\boldsymbol{y}) \phi_j(\boldsymbol{y}) d\sigma_{\boldsymbol{y}}, \ \forall \ \boldsymbol{x} \in \mathbb{R}^n \setminus \Gamma, \end{aligned} \tag{1.3.17}$$

where $\boldsymbol{\nu}(\nu_l)_{l=\overline{1,n}}$ *is the outward unit normal to* D_+, *defined a.e. on* Γ.

Note that, in view of relation (1.3.5), we have that the pair $(\mathbf{W}_\Gamma \boldsymbol{\phi}, \mathcal{Q}^d_\Gamma \boldsymbol{\phi})$ satisfies the homogeneous Stokes system

$$\Delta(\mathbf{W}_\Gamma \boldsymbol{\phi}) - \nabla(\mathcal{Q}^d_\Gamma \boldsymbol{\phi}) = 0, \quad \mathrm{div}\ (\mathbf{W}_\Gamma \boldsymbol{\phi}) = 0 \tag{1.3.18}$$

in $\mathbb{R}^n \setminus \Gamma$.

In addition, let us introduce the boundary version of the Stokes double layer velocity potential in the sense of the principal value, as follows (see, e.g., [85, Relation (4.44)]).

Definition 1.3.6. *Define the principal value of $\mathbf{W}_\Gamma\boldsymbol{\phi}$, denoted by $\mathbb{K}_\Gamma\boldsymbol{\phi}$ and given by:*

$$\begin{aligned}(\mathbb{K}_\Gamma\boldsymbol{\phi})_k(\boldsymbol{x}) :&= \text{p.v.}\int_\Gamma S_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}}\\ &= \lim_{\varepsilon\to 0}\int_{\Gamma\setminus(\Gamma\cap\overline{B}(x,\varepsilon))} S_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\end{aligned} \tag{1.3.19}$$

for $\boldsymbol{x}\in\Gamma$, where this limit makes sense.

Also, the following result provides us with useful mapping properties of the double layer potential operators for the Stokes system (see, e.g., [52, Lemma A.4]).

Theorem 1.3.7. *Let Assumption* 1.1.6 *be satisfied. Then, the following operators*

$$(\mathbf{W}_\Gamma)|_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n,\ (\mathcal{Q}^d_\Gamma)|_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^n \to L^2(\mathsf{D}_+), \tag{1.3.20}$$

are linear and bounded. Moreover, for $n = 3$, we have that the operators

$$(\mathbf{W}_\Gamma)|_{\mathsf{D}_-} : H^{\frac{1}{2}}(\Gamma)^3 \to \mathcal{H}^1(\mathsf{D}_-)^3,\ (\mathcal{Q}^d_\Gamma)|_{\mathsf{D}_-} : H^{\frac{1}{2}}(\Gamma)^3 \to L^2(\mathsf{D}_-) \tag{1.3.21}$$

are linear and bounded as well.

Let us also provide the lemma which describes the jump relations of the single and double layer potentials for the Stokes system, in the setting of Sobolev spaces (see, e.g., [52, Lemma A.4], [85, Proposition 4.2.2, Proposition 4.2.5, Proposition 4.2.9, Corollary 4.3.2, Theorem 5.3.6, Theorem 5.4.1]).

Lemma 1.3.8. *Let Assumption* 1.1.6 *be satisfied.*

(i) For $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\Gamma)^n$ and $\boldsymbol{\phi}\in H^{\frac{1}{2}}(\Gamma)^n$, the following jump relations

$$\begin{aligned}&\mathrm{Tr}_{\mathsf{D}_+}(\mathbf{V}_\Gamma\boldsymbol{\varphi}) = \mathrm{Tr}_{\mathsf{D}_-}(\mathbf{V}_\Gamma\boldsymbol{\varphi}) =: \mathcal{V}_\Gamma\boldsymbol{\varphi},\\ &\mathrm{Tr}_{\mathsf{D}_\pm}(\mathbf{W}_\Gamma\boldsymbol{\phi}) = \left(\mp\frac{1}{2}\mathbb{I}+\mathbb{K}_\Gamma\right)\boldsymbol{\phi},\\ &\mathbf{t}_{\mathsf{D}_\pm}(\mathbf{V}_\Gamma\boldsymbol{\varphi},\mathcal{Q}^s_\Gamma\boldsymbol{\varphi}) = \left(\pm\frac{1}{2}\mathbb{I}+\mathbb{K}^*_\Gamma\right)\boldsymbol{\varphi},\\ &\mathbf{t}_{\mathsf{D}_+}(\mathbf{W}_\Gamma\boldsymbol{\phi},\mathcal{Q}^d_\Gamma\boldsymbol{\phi}) = \mathbf{t}_{\mathsf{D}_-}(\mathbf{W}_\Gamma\boldsymbol{\phi},\mathcal{Q}^d_\Gamma\boldsymbol{\phi}) =: \mathbb{D}_\Gamma\boldsymbol{\phi}\end{aligned} \tag{1.3.22}$$

hold a.e. on Γ, where $\mathbb{K}^*_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n$ is the adjoint of the double layer potential operator $\mathbb{K}_\Gamma : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n$.

(ii) The following Stokes layer potential operators

$$\begin{aligned}&\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n,\ \mathbb{K}_\Gamma : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n,\\ &\mathbb{K}^*_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n,\ \mathbb{D}_\Gamma : H^{\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n,\end{aligned} \tag{1.3.23}$$

are linear and bounded. Moreover, the operator $\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n$ is a Fredholm operator of index zero and its kernel, denoted by Ker $\mathcal{V}_\Gamma$*, is given by*

$$\text{Ker}\ \{\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n\} = \mathbb{R}\boldsymbol{\nu}. \tag{1.3.24}$$

We have the following useful remark.

Remark 1.3.9. *If f and g are two functions defined in a neighborhood of a point $\boldsymbol{x}$ (which could also be ∞), then*

$$f(\boldsymbol{x}) = O(g(\boldsymbol{x})) \Leftrightarrow \frac{|f(\boldsymbol{x})|}{|g(\boldsymbol{x})|} \text{ is bounded.} \tag{1.3.25}$$

Let us end this subsection by stating the following asymptotic formulas which are satisfied by the Stokes layer potential at infinity (see, e.g., [54, Relation (3.14)])

$$\begin{aligned}
&(\mathbf{V}_\Gamma\boldsymbol{\varphi})(\mathbf{x}) = O(\ln|\mathbf{x}|), \quad (\mathcal{Q}^s_\Gamma\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty,\ n = 2\\
&(\mathbf{V}_\Gamma\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{2-n}), \quad (\mathcal{Q}^s_\Gamma\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty,\ n \geq 3\\
&(\mathbf{W}_\Gamma\boldsymbol{\phi})(\mathbf{x}) = O(|\mathbf{x}|^{1-n}), \quad (\mathcal{Q}^d_\Gamma\boldsymbol{\phi})(\mathbf{x}) = O(|\mathbf{x}|^{-n}) \text{ as } |\mathbf{x}| \to \infty,\ n \geq 2.
\end{aligned} \tag{1.3.26}$$

1.4 Brinkman layer potentials and their properties

In this section we consider the fundamental solution for the Brinkman system in $\mathbb{R}^n$, $n \geq 2$, and then we define the layer potential operators that are useful in the analysis of the transmission problems in the next chapters. The sources used in the preparation of this section are [49], [54], [56], [52].

1.4.1 The Brinkman system and its fundamental solution

Let $\alpha > 0$ be a given constant. Let $(\mathbf{G}^\alpha(\cdot,\cdot), \mathbf{P}^\alpha(\cdot,\cdot)) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^{n\times n} \times \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^n$ denote the fundamental solution of the Brinkman system, where $\mathbf{G}^\alpha(\cdot,\cdot)$ is the fundamental velocity tensor and by $\mathbf{P}^\alpha(\cdot,\cdot)$ is the fundamental pressure vector for the Brinkman system in $\mathbb{R}^n$. Therefore, the pair $(\mathbf{G}^\alpha(\cdot,\cdot), \mathbf{P}^\alpha(\cdot,\cdot))$ satisfies the following equations

$$\Delta_{\mathbf{x}}\mathbf{G}^\alpha(\mathbf{x},\mathbf{y}) - \alpha\mathbf{G}^\alpha(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\mathbf{P}^\alpha(\mathbf{x},\mathbf{y}) = -\delta_{\mathbf{y}}(\mathbf{x})\mathbb{I},\ \operatorname{div}_{\mathbf{x}}\mathbf{G}^\alpha(\mathbf{x},\mathbf{y}) = 0. \tag{1.4.1}$$

Recall that $\delta_{\mathbf{y}}$ denotes the Dirac distribution with mass at $\mathbf{y}$ and the differential operators $\Delta_{\mathbf{x}}$, $\nabla_{\mathbf{x}}$ and $\operatorname{div}_{\mathbf{x}}$ act with respect to the variable $\mathbf{x}$.

The components of the fundamental solution $(\mathbf{G}^\alpha(\mathsf{G}^\alpha_{jk}), \mathbf{P}^\alpha(\mathsf{P}^\alpha_k))$ are given by (see, e.g., [54, Relation (2.29)], [106])

$$\begin{aligned}
\mathsf{G}^\alpha_{jk}(\mathbf{x},\mathbf{y}) &:= \frac{1}{2\omega_n}\left\{\frac{\delta_{jk}}{|\mathbf{y}-\mathbf{x}|^{n-2}}E_1(\alpha|\mathbf{y}-\mathbf{x}|) + \frac{x_j x_k}{|\mathbf{y}-\mathbf{x}|^n}E_2(\alpha|\mathbf{y}-\mathbf{x}|)\right\},\\
\mathsf{P}^\alpha_k(\mathbf{x},\mathbf{y}) &= \frac{1}{\omega_n}\frac{x_k}{|\mathbf{y}-\mathbf{x}|^n},
\end{aligned} \tag{1.4.2}$$

where

$$\begin{aligned}
E_1(s) &:= \frac{\left(\frac{s}{2}\right)^{\frac{n}{2}-1}K_{\frac{n}{2}-1}(s)}{\Gamma\left(\frac{n}{2}\right)} + 2\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}}K_{\frac{n}{2}}(s)}{s^2\cdot\boldsymbol{\Gamma}\left(\frac{n}{2}\right)} - \frac{1}{s^2},\\
E_2(s) &:= \frac{n}{s^2} - 4\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(s)}{s^2\cdot\Gamma\left(\frac{n}{2}\right)},
\end{aligned} \tag{1.4.3}$$

and K_β is the second kind Bessel function of order $\beta \geq 0$, $\Gamma(\cdot)$ is the Euler Gamma function. Recall that δ_{jk} is the Kronecher symbol and ω_n is the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$, $n \geq 2$.

In addition, let $\mathbf{S}^\alpha(\mathsf{S}^\alpha_{jkl})$ and $\mathbf{R}^\alpha(\mathsf{R}^\alpha_{jk})$ be the associated stress and pressure tensors for the Brinkman system. Their components are given by (see, e.g., [54, Relation (2.31) and Relation (2.32)])

$$\begin{aligned}
\mathsf{S}^\alpha_{jkl}(\mathbf{x},\mathbf{y}) &:= \frac{1}{\omega_n}\left\{\delta_{jl}\frac{x_j}{|\mathbf{y}-\mathbf{x}|^n}E_1(\alpha|\mathbf{y}-\mathbf{x}|) + \frac{\delta_{jl}x_i+\delta_{ij}x_l}{|\mathbf{y}-\mathbf{x}|^n}E_2(\alpha|\mathbf{y}-\mathbf{x}|) + \frac{x_ix_jx_l}{|\mathbf{x}|^{n+2}}E_3(\alpha|\mathbf{y}-\mathbf{x}|)\right\},\\
\mathsf{R}^\alpha_{jk}(\mathbf{x},\mathbf{y}) &:= \frac{1}{2\pi}\left\{-(y_i-x_i)\frac{4(y_k-x_k)}{|\mathbf{y}-\mathbf{x}|^4} - (\alpha|\mathbf{y}-\mathbf{x}|^2\log|\mathbf{y}-\mathbf{x}|+2)\frac{\delta_{ik}}{|\mathbf{y}-\mathbf{x}|^2}\right\}, n=2,\\
\mathsf{R}^\alpha_{jk}(\mathbf{x},\mathbf{y}) &:= \frac{1}{\omega_n}\left\{-(y_i-x_i)\frac{2n(y_k-x_k)}{|\mathbf{y}-\mathbf{x}|^{n+2}} + \frac{2\delta_{ik}}{|\mathbf{y}-\mathbf{x}|^n} - \alpha\frac{1}{n-2}\frac{1}{|\mathbf{y}-\mathbf{x}|^{n-2}}\delta_{ik}\right\}, n\geq 3,
\end{aligned} \tag{1.4.4}$$

where

$$\begin{aligned}
E_1(s) &:= 8\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(s)}{s^2\cdot\Gamma\left(\frac{n}{2}\right)} - \frac{2n}{z^2}+1\\
E_2(s) &:= 8\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(s)}{s^2\cdot\Gamma\left(\frac{n}{2}\right)} + 2\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}}K_{\frac{n}{2}}(s)}{\Gamma\left(\frac{n}{2}\right)} - \frac{2n}{s^2}\\
E_3(s) &:= -16\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}+2}K_{\frac{n}{2}+2}(s)}{s^2\cdot\Gamma\left(\frac{n}{2}\right)} + \frac{2n(n+2)}{s^2}.
\end{aligned} \tag{1.4.5}$$

For $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, $\mathbf{x}\neq\mathbf{y}$, the pair $(\mathbf{S}^\alpha(\mathsf{S}^\alpha_{jkl}),\mathbf{R}^\alpha(\mathsf{R}^\alpha_{jk}))$ satisfies the Brinkman system

$$\Delta_{\mathbf{x}}\mathsf{S}^\alpha_{jkl}(\mathbf{x},\mathbf{y}) - \alpha\mathsf{S}^\alpha_{jkl}(\mathbf{x},\mathbf{y}) - \frac{\partial\mathsf{R}^\alpha_{jk}(\mathbf{y},\mathbf{x})}{\partial x_k} = 0,\ \frac{\partial\mathsf{S}^\alpha_{jkl}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0. \tag{1.4.6}$$

1.4.2 The volume potential for the Brinkman system and its properties

The purpose of this subsection is to introduce the Newtonian (volume) potential operators associated to the Brinkman system and to give their mapping properties. To this end, we consider the Lipschitz domains $\mathsf{D}_\pm$ as described in Assumption 1.1.6 and we will take into account the fundamental solution of the Brinkman system, that is, the pair $(\mathbf{G}^\alpha(\cdot,\cdot),\mathbf{P}^\alpha(\cdot,\cdot))\in\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)^{n\times n}\times\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)^n$ given by relation (1.4.2).

Definition 1.4.1. *Let $\alpha>0$ be a given constant. For $\mathbf{f}\in H^{-1}(\mathbb{R}^n)^n$, define the Newtonian (volume) velocity and pressure potentials for the Brinkman system, by*

$$(\mathcal{N}_{\alpha,\mathbb{R}^n}\mathbf{f})(\boldsymbol{x}) := -\langle\mathbf{G}^\alpha(\boldsymbol{x},\cdot),\mathbf{f}\rangle_{\mathbb{R}^n},\ (\mathcal{Q}_{\alpha,\mathbb{R}^n}\mathbf{f})(\boldsymbol{x}) := -\langle\mathbf{P}^\alpha(\boldsymbol{x},\cdot),\mathbf{f}\rangle_{\mathbb{R}^n}. \tag{1.4.7}$$

Moreover, the Newtonian (volume) velocity and pressure potentials for the Brinkman system corresponding to $\mathsf{D}_\pm$, are given by

$$\mathcal{N}_{\alpha,\mathsf{D}_\pm}\mathbf{f} := (\mathcal{N}_{\alpha,\mathbb{R}^n}\mathbf{f})|_{\mathsf{D}_\pm},\ \mathcal{Q}_{\alpha,\mathsf{D}_\pm}\mathbf{f} := (\mathcal{Q}_{\alpha,\mathbb{R}^n}\mathbf{f})|_{\mathsf{D}_\pm}. \tag{1.4.8}$$

Recall that $|_{\mathsf{D}_\pm}$ is the restriction to $\mathsf{D}_\pm$ operator, which acts on vector-valued or scalar-valued distributions in $\mathbb{R}^n$.

The following result describes the mapping properties of the Newtonian (volume) layer potential operators in the setting of Sobolev spaces (see, e.g., [52, Lemma A.3]).

Theorem 1.4.2. *Let $\alpha > 0$ be a given constant. The Newtonian (volume) velocity and pressure potential operators for the Brinkman system, given by relation (1.4.7),*

$$\begin{aligned} \mathcal{N}_{\alpha,\mathbb{R}^n} : H^{-1}(\mathbb{R}^n)^n \to H^1(\mathbb{R}^n)^n, \ \mathcal{Q}_{\alpha,\mathbb{R}^n} &: H^{-1}(\mathbb{R}^n)^n \to L^2(\mathbb{R}^n), \\ \mathcal{Q}_{\alpha,\mathbb{R}^3} &: H^{-1}(\mathbb{R}^3)^3 \to \mathfrak{M}(\mathbb{R}^3) \end{aligned} \tag{1.4.9}$$

are linear and continuous operators. Moreover, the Newtonian (volume) velocity and pressure potentials operators for the Brinkman system, introduced in relation (1.4.8),

$$\mathcal{N}_{\alpha,\mathsf{D}_+} : \tilde{H}^{-1}(\mathsf{D}_+)^n \to H^1(\mathsf{D}_+)^n, \ \mathcal{Q}_{\alpha,\mathsf{D}_+} : \tilde{H}^{-1}(\mathsf{D}_+)^n \to L^2(\mathsf{D}_+) \tag{1.4.10}$$

and

$$\mathcal{N}_{\alpha,\mathsf{D}_-} : \tilde{H}^{-1}(\mathsf{D}_-)^3 \to H^1(\mathsf{D}_-)^3, \ \mathcal{Q}_{\alpha,\mathsf{D}_-} : \tilde{H}^{-1}(\mathsf{D}_-)^3 \to \mathfrak{M}(\mathsf{D}_-), \tag{1.4.11}$$

in the case $n = 3$, are linear and continuous operators as well, while the spaces $\mathfrak{M}(\mathbb{R}^3)$ and $\mathfrak{M}(\mathsf{D}_-)$ are given by Definition 2.1.1.

Finally, by taking into account relation (1.4.1), we have that the Newtonian (volume) potentials for the Brinkman system, introduced in Definition 1.4.1 satisfy the equations (in the sense of distributions)

$$\Delta(\mathcal{N}_{\alpha,\mathbb{R}^n}\mathbf{f}) - \alpha(\mathcal{N}_{\alpha,\mathbb{R}^n}\mathbf{f}) - \nabla(\mathcal{Q}_{\alpha,\mathbb{R}^n}\mathbf{f}) = \mathbf{f}, \ \text{div } (\mathcal{N}_{\alpha,\mathbb{R}^n}\mathrm{f}) = 0, \ \text{in } \mathbb{R}^n, \tag{1.4.12}$$

and

$$\Delta(\mathcal{N}_{\alpha,\mathsf{D}_\pm}\mathbf{f}) - \alpha(\mathcal{N}_{\alpha,\mathsf{D}_\pm}\mathbf{f}) - \nabla(\mathcal{Q}_{\alpha,\mathsf{D}_\pm}\mathbf{f}) = \mathbf{f}, \ \text{div } (\mathcal{N}_{\alpha,\mathsf{D}_\pm}\mathbf{f}) = 0, \ \text{in } \mathsf{D}_\pm, \tag{1.4.13}$$

respectively.

1.4.3 Brinkman layer potentials and related results

In this subsection, we consider the single layer potential and the double layer potential operators associated to the Brinkman system. We give their definitions, their mapping properties, their jump relations and we describe their behavior at infinity. As in the previous section, let Assumption 1.1.6 be satisfied and let D_+ be a bounded Lipschitz domain with connected boundary Γ.

Firstly, let us focus on the single-layer velocity and pressure potentials, associated to the Brinkman system (see, e.g., [56, Relation (3.6)], [54, Relation (3.1)]).

Definition 1.4.3. *Let Assumption* 1.1.6 *be satisfied. Let $\alpha > 0$ be a given constant. Let $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\Gamma)^n$. Define the single-layer velocity potential $\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}$ and its associated pressure potential $\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi}$ for the Brinkman system, by*

$$(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) := \langle \mathbf{G}^\alpha(\boldsymbol{x},\cdot), \boldsymbol{\varphi}\rangle_\Gamma, \ (\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi}) := \langle \mathbf{P}^\alpha(\boldsymbol{x},\cdot), \boldsymbol{\varphi}\rangle_\Gamma, \ \forall \ \boldsymbol{x} \in \mathbb{R}^n \setminus \Gamma. \tag{1.4.14}$$

By taking into account relation (1.4.1), we have that the pair $(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}, \mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi})$ satisfies the homogeneous Brinkman system

$$\Delta(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) - \alpha(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) - \nabla(\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi}) = 0, \quad \text{div } \mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi} = 0 \tag{1.4.15}$$

in $\mathbb{R}^n \setminus \Gamma$.

The following theorem provides some useful mapping properties of the single-layer potentials associated to the Brinkman system (see, e.g., [54, Lemma 3.1], [52, Lemma A.8]).

Theorem 1.4.4. *Let Assumption* 1.1.6 *be satisfied. Let $\alpha > 0$ be a given constant. Then the following operators*

$$(\mathbf{V}_{\alpha,\Gamma})|_{\mathsf{D}_+} : H^{-\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n,\ (\mathcal{Q}^s_{\alpha,\Gamma})|_{\mathsf{D}_+} : H^{-\frac{1}{2}}(\Gamma)^n \to L^2(\mathsf{D}_+), \tag{1.4.16}$$

are linear and bounded. Moreover, for $n = 3$, we have that the operators

$$(\mathbf{V}_{\alpha,\Gamma})|_{\mathsf{D}_-} : H^{-\frac{1}{2}}(\Gamma)^3 \to H^1(\mathsf{D}_-)^3,\ (\mathcal{Q}^s_{\alpha,\Gamma})|_{\mathsf{D}_-} : H^{-\frac{1}{2}}(\Gamma)^3 \to L^2(\mathsf{D}_-) \tag{1.4.17}$$

are linear and bounded as well.

Secondly, we focus on the double-layer velocity and pressure potentials, associated to the Brinkman system (see, e.g., [56, Relation (3.7)]).

Definition 1.4.5. *Let Assumption* 1.1.6 *be satisfied. Let $\alpha > 0$ be a given constant. Let $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n$. We define the double-layer potential $\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}$ and its associated pressure potential $\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}$ for the Brinkman system, by*

$$\begin{aligned}(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi})_k(\boldsymbol{x}) &:= \int_\Gamma \mathcal{S}^\alpha_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\ \forall\ \boldsymbol{x} \in \mathbb{R}^n \setminus \Gamma,\\ (\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi})(\boldsymbol{x}) &:= \int_\Gamma \mathcal{R}^\alpha_{jl}(\boldsymbol{x},\boldsymbol{y})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\ \forall\ \boldsymbol{x} \in \mathbb{R}^n \setminus \Gamma.\end{aligned} \tag{1.4.18}$$

Recall that $\boldsymbol{\nu}(\nu_l)_{l=\overline{1,n}}$ is the outward unit normal to D_+, defined a.e. on Γ.

Note that, in view of relation (1.4.6), the pair $(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}, \mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi})$ satisfies the homogeneous Brinkman system

$$\Delta(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}) - \alpha(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}) - \nabla(\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}) = 0, \quad \text{div}\ (\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}) = 0 \tag{1.4.19}$$

in $\mathbb{R}^n \setminus \Gamma$.

Moreover, we introduce the boundary version of the Brinkman double layer velocity potential in the sense of principal value, as follows (see, e.g., [56, Relation (3.8)]).

Definition 1.4.6. *Let $\alpha > 0$ be a given constant. Define the principal value of $\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}$, denoted by $\mathbb{K}_{\alpha,\Gamma}\boldsymbol{\phi}$ and given by:*

$$\begin{aligned}(\mathbb{K}_{\alpha,\Gamma}\boldsymbol{\phi})_k(\boldsymbol{x}) : &= \text{p.v.} \int_\Gamma \mathcal{S}^\alpha_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}}\\ &= \lim_{\varepsilon\to 0} \int_{\Gamma\setminus(\Gamma\cap\overline{B}(x,\varepsilon))} \mathcal{S}^\alpha_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\end{aligned} \tag{1.4.20}$$

for $\boldsymbol{x} \in \Gamma$, where this limit makes sense.

Also, we state some useful mapping properties of the double-layer potentials in the following statement (see, e.g., [54, Lemma 3.1], [52, Lemma A.8]).

Theorem 1.4.7. *Let Assumption* 1.1.6 *be satisfied. Let $\alpha > 0$ be a given constant. Then the following operators*

$$(\mathbf{W}_{\alpha,\Gamma})|_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n,\ (\mathcal{Q}^d_{\alpha,\Gamma})|_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^n \to L^2(\mathsf{D}_+) \tag{1.4.21}$$

are linear and bounded. Moreover, for $n = 3$, we have that the operators

$$(\mathbf{W}_{\alpha,\Gamma})|_{\mathsf{D}_-} : H^{\frac{1}{2}}(\Gamma)^3 \to H^1(\mathsf{D}_-)^3,\ (\mathcal{Q}^d_{\alpha,\Gamma})|_{\mathsf{D}_-} : H^{\frac{1}{2}}(\Gamma)^3 \to \mathfrak{M}(\mathsf{D}_-) \tag{1.4.22}$$

are linear and bounded as well and the space $\mathfrak{M}(\mathsf{D}_-)$ is given by Definition 2.1.1.

Next, we concern ourselves with the jump relations of the single and double layer potentials for the Brinkman system, in the setting of Sobolev spaces (see, e.g., [54, Lemma 3.1], [52, Lemma A.4]).

Lemma 1.4.8. *Let Assumption* 1.1.6 *be satisfied. Let* $\alpha > 0$ *be a given constant.*

(i) Let $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\Gamma)^n$ *and* $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n$*. Then, the following jump formulas*

$$\begin{aligned}
&\mathrm{Tr}_{\mathsf{D}_+}(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) = \mathrm{Tr}_{\mathsf{D}_-}(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) := \mathcal{V}_{\alpha,\Gamma}\boldsymbol{\varphi},\\
&\mathrm{Tr}_{\mathsf{D}_\pm}(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}) = \left(\mp\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma}\right)\boldsymbol{\phi},\\
&\mathbf{t}_{\alpha,\mathsf{D}_\pm}(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}, \mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi}) = \left(\pm\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma}\right)\boldsymbol{\varphi},\\
&\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}, \mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}) = \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}, \mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}) = \mathbb{D}_{\alpha,\Gamma}\boldsymbol{\phi}
\end{aligned} \tag{1.4.23}$$

hold a.e. on Γ*, where* $\mathbb{K}^*_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n$ *is the adjoint of the double layer potential operator* $\mathbb{K}_{\alpha,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n$*.*

(ii) The following operators

$$\begin{aligned}
&\mathcal{V}_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n, \quad \mathbb{K}_{\alpha,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n,\\
&\mathbb{K}^*_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n, \quad \mathbb{D}_{\alpha,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n
\end{aligned} \tag{1.4.24}$$

are well-defined, linear and continuous. Moreover, the operator $\mathcal{V}_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n$ *is a Fredholm operator of index zero and its kernel, denoted by* Ker $\mathcal{V}_{\alpha,\Gamma}$*, is given by*

$$\mathrm{Ker}\ \{\mathcal{V}_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n\} = \mathbb{R}\boldsymbol{\nu}. \tag{1.4.25}$$

Now, we introduce the operators

$$\begin{aligned}
&\mathcal{V}_{\alpha,0,\Gamma} := \mathcal{V}_{\alpha,\Gamma} - \mathcal{V}_\Gamma, \quad \mathbb{K}_{\alpha,0,\Gamma} := \mathbb{K}_{\alpha,\Gamma} - \mathbb{K}_\Gamma,\\
&\mathbb{K}^*_{\alpha,0,\Gamma} := \mathbb{K}^*_{\alpha,\Gamma} - \mathbb{K}^*_\Gamma, \quad \mathbb{D}_{\alpha,0,\Gamma} := \mathbb{D}_{\alpha,\Gamma} - \mathbb{D}_\Gamma,
\end{aligned} \tag{1.4.26}$$

which will be called complementary layer potential operators. Note that the operators $\mathcal{V}_\Gamma$, $\mathbb{K}_\Gamma$, $\mathbb{K}^*_\Gamma$ and $\mathbb{D}_\Gamma$ are introduced in Lemma 1.3.8 which concerns the jump formulas for the single layer and double layer potentials associated to the Stokes system. We have the following lemma (see, e.g., [54, Theorem 3.1]).

Lemma 1.4.9. *The complementary layer potential operators*

$$\begin{aligned}
&\mathcal{V}_{\alpha,0,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n, \quad \mathbb{K}_{\alpha,0,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n,\\
&\mathbb{K}^*_{\alpha,0,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n, \quad \mathbb{D}_{\alpha,0,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n,
\end{aligned} \tag{1.4.27}$$

are compact a.e. on Γ*.*

Now, we provide the asymptotic formulas (see Remark 1.3.9) which specify the behavior at infinity for the Brinkman layer potentials (see, e.g., [54, Relation (3.12)])

$$\begin{aligned}
&(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{-n}), \quad (\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi})(\mathbf{x}) = O(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty,\ n \geq 2\\
&(\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad (\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi})(\mathbf{x}) = O(\ln|\mathbf{x}|) \text{ as } |\mathbf{x}| \to \infty,\ n = 2\\
&(\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{1-n}), \quad (\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi})(\mathbf{x}) = O(|\mathbf{x}|^{2-n}) \text{ as } |\mathbf{x}| \to \infty,\ n \geq 3.
\end{aligned} \tag{1.4.28}$$

Remark 1.4.10. *The results presented in Section* 1.3 *and in Section* 1.4*, including the definitions of the layer potentials, can be extended to the case of the Stokes (or Brinkman) system with variable coefficients (which belong to* L^∞*) by using a variational approach (see, e.g., [58], [59], [60], [67]).*

2

Linear Boundary Value Problems of Transmission-type related to the Stokes and Brinkman systems

This chapter deals with certain linear transmission-type problems which involve the Stokes system, the Brinkman system and a generalized version of the Brinkman system (see relation (1.2.21)) in the Lipschitz domains in Euclidean setting (see Assumption 1.1.6 and Assumption 1.1.7). The content of this chapter follows the results that were obtained in the papers [6], [7], [8].

We present and prove well-posedness results for the following boundary value problems. Firstly, we treat the Poisson problem of transmission-type for the generalized Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$. Secondly, we analyze the Poisson problem of transmission-type for the generalized Brinkman and classical Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$. Next, we look at the Poisson problem of Robin-transmission-type for the Brinkman system in Euclidean setting provided by Assumption 1.1.7.

Let us note that, the Stokes system can be seen as a particular case of the Brinkman system. Even so, we have separated the study of transmission problems involving the Brinkman and Stokes system from the transmission problems involving only the Brinkman system. Such a distinction can be justified in view of different practical applications (see, e.g., [15], [52]). Moreover, we use different solution spaces for each of the studied transmission problems, namely, if we work with the Stokes system in an unbounded, exterior domain, we use weighted Sobolev spaces, while if we work with the Brinkman system in an unbounded, exterior domain, we use the usual Sobolev spaces.

In order to obtain the results that are presented in this chapter, the main tools of investigation that we have employed are layer potential theory and Fredholm operator theory. Indeed, using the Stokes layer potentials, the Brinkman layer potentials, results regarding Fredholm operators and Green formulas we construct unique solutions to our considered boundary problems.

In the latter, let us mention some past works that have contributed to investigation of elliptic boundary value problems. Firstly, let us note the paper of Fabes, Kenig and Verchota [31] which concerns the investigation of the Dirichlet problem for the Stokes system in a Lipschitz domain in $\mathbb{R}^n$ and they provided representation formulas in terms of layer potential for the solution. Costabel [20] has studied simple and double layer potentials for second order linear elliptic differential operators on Lipschitz domains in Euclidean setting and has provided continuity and regularity results. Dalla Riva, Lanza de Cristoforis and Musolino [23] have analyzed basic boundary value problems for the Laplace equation in singularly perturbed domains, with an emphasis on domains with small holes.

Varnhorn [105] has used potential theory to construct an explicit solution of the Stokes rezolven system in a bounded domain in $\mathbb{R}^3$ with C^2-boundary. Also, Varnhorn [106] has provided a theory of solvability for the Stokes system in exterior domains and has analyzed the existence of strong solutions in Sobolev spaces and further properties. Medkova [74] has used the integral equation method in order to obtain L^2-solutions for the transmission problem, Robin-transmission problem and the Dirichlet-transmission problem for the Brinkman system, while in [75], she used the same method in order to study the transmission problem for the Stokes system, in homogeneous Sobolev spaces in Lipschitz domains in $\mathbb{R}^3$. Medkova [76] has also studied the Dirichlet problem for the resolvent Stokes system with bounded boundary data in the setting of bounder and unbounded domains with compact Lyapunov boundary. Chkadua, Mikhailov and Natroshvili [16] have used localized integral potentials associated with the Laplace operator in order to reduce boundary value problems for variable-coefficient divergence-from second-order elliptic PDEs to systems of localized boundary-domain singular integral equations. Escauriaza and Mitrea [30] have used layer potential methods to obtain the well-posedness of the transmission problem for the Laplacian in the presence of a Lipschitz interface with boundary data belonging to Lebesgue and Hardy spaces. The work of Mitrea and Wright [85] concerns also transmission boundary value problems for the Stokes system in Lipschitz domains in the Euclidean setting, for $n \geq 2$.

The authors in [56] have obtained a well-posedness result for a linear Robin-transmission problem for the Stokes and Brinkman systems in adjacent Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$ with linear transmission conditions on the Lipschitz interface and Robin condition on the remaining boundary. Kohr, Wendland and Lanza de Cristoforis [54] have analyzed a nonlinear Neumann-transmission problems for the Stokes and Brinkman systems in Euclidean Lipschitz domains with boundary data in L^p, Sobolev and Besov spaces. Fericean, Groşan, Kohr and Wendland [34] have treated interface boundary value problems of Robin-transmission type for the Stokes and Brinkman systems in Lipschitz domains in $\mathbb{R}^n$ for $n \geq 3$ and with boundary data in L^p or Sobolev spaces. Fericean and Wendland [35] have used layer potential theory in order to obtain well-posedness results for a Dirichlet-transmission problem for the Stokes and Brinkman systems in Lipschitz domains in $\mathbb{R}^n$ for $n \geq 3$. The authors in [52] have obtained a well-posedness result for a transmission problem for the Stokes and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$ in weighted Sobolev spaces by making use of layer potential techniques. In [78], Mikhailov and Portillo have studied a mixed boundary value problem for the stationary compressible Stokes system with variable viscosity in an exterior domain of $\mathbb{R}^3$ by the means of boundary-domain integral equations (BDIEs). Mitrea, Mitrea and Mitrea [79] have proved well-posedness and Fredholm solvability results for boundary value problems for elliptic second-order homogeneous constant coefficient systems in domains of general geometric nature.

Regarding the setting of manifolds, let us mention that Kohr, Pintea and Wendland [63] have developed a potential analysis for certain pseudodifferential matrix operators on Lipschitz domains in compact Riemmanian manifolds and they have studied Dirichlet-transmission problems for Brinkman operators in Lipschitz domains in compact Riemmanian manifolds. Also, Kohr, Mikhailov and Wendland [57] have investigated a linear transmission problem for the Stokes and generalized Brinkman system in two complementary Lipschitz domains in a compact Riemannian manifold of dimension $m \geq 2$.

More recently, a great deal of attention has been directed to the variable coefficient PDEs. Note that the works of Kohr, Mikhailov and Wendland [58], [59], [60] concern the analysis of the anisotropic Stokes system with L^∞ coefficient tensor. They investigate diverse boundary problems, Dirichlet type, transmission type and they have also discussed potentials for this anisotropic system.

2.1 Dirichlet type problem for the Brinkman system in an exterior domain

The goal of this section is twofold. First, we introduce a special function space which is involved in the mapping properties of the Newtonial layer potentials for the Brinkman system (see relation (1.4.10) of Theorem 1.4.2). Secondly, we study an exterior Dirichlet boundary value problem for the Brinkman system, which is involved in the proof of our well-posedness results (see Theorem 2.3.1, Theorem 2.3.3, Theorem 3.3.1).

In the latter, let D be either of the domains $\mathbb{R}^3$, D_+ or D_-, which are described in Assumption 1.1.6, for $n = 3$. Let us introduce the space $H^{-1}_{\text{curl}}(\mathsf{D})^3$ by

$$H^{-1}_{\text{curl}}(\mathsf{D})^3 := \{\mathbf{h} \in H^{-1}(\mathsf{D})^3 : \text{curl } \mathbf{h} = 0\}.$$

Definition 2.1.1. *Let* D *be either of the domains described in Assumption* 1.1.6*, for* $n = 3$*. Define the space* $\mathfrak{M}(\mathsf{D})$ *by*

$$\mathfrak{M}(\mathsf{D}) := \{g \in L^2(\rho^{-1}, \mathsf{D}) : \nabla g \in H^{-1}_{\text{curl}}(\mathsf{D})^3\}. \tag{2.1.1}$$

Now, denote by $\mathfrak{M}'(\mathsf{D})$ the dual of the space $\mathfrak{M}(\mathsf{D})$, we have the following continuous chain of embeddings (see, e.g., [52, (A.24)])

$$L^2(\rho, \mathsf{D}) \subset \mathfrak{M}'(\mathsf{D}) \subset L^2(\mathsf{D}) \subset \mathfrak{M}(\mathsf{D}) \subset L^2(\rho^{-1}, \mathsf{D}) \subset L^2_{\text{loc}}(\mathsf{D}). \tag{2.1.2}$$

In the latter, we concern ourselves with two important results. First, we analyze exterior Dirichlet problem for the classical Brinkman system in the space $H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$, where D_- is the exterior Lipschitz domain introduced in Assumption 1.1.6, in the case $n = 3$. The well-posedness result is as follows (see, [6, Theorem A.1], [52, Lemma A.2], and [105, Prop. 4.5] in the case of an exterior domain with a $\mathcal{C}^2$-boundary).

Theorem 2.1.2. *Let Assumption 1.1.6 be satisfied for* $n = 3$*. Let* $\alpha > 0$ *be a given constant. Then, the exterior Dirichlet problem for the Brinkman system*

$$\begin{cases} \Delta \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = 0 \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{u} = 0 \text{ in } \mathsf{D}_-, \\ \text{Tr}_{\mathsf{D}_-} \mathbf{u} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma)^3 \text{ on } \Gamma, \\ \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \nabla \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \pi(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty, \end{cases} \tag{2.1.3}$$

has a unique solution in the space $H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$.

Lastly, we conclude this section with a result, which shows that if we have a pair $(\mathbf{v}, p) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$ satisfies the Brinkman system in an exterior Lipschitz domain, then the far field conditions described in problem (2.1.3) are also satisfied (see [6, Lemma A.2], [52, Lemma A.2]).

Lemma 2.1.3. *Let Assumption 1.1.6 be satisfied for* $n = 3$*. Let* $\alpha > 0$ *be a given constant. If the pair* $(\mathbf{v}, \mathrm{p}) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$ *satisfies*

$$\Delta \mathbf{v} - \alpha \mathbf{v} - \nabla \mathrm{p} = 0, \text{ div } \mathbf{v} = 0 \text{ in } \mathsf{D}_-, \tag{2.1.4}$$

then

$$\mathbf{v}(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \nabla \mathbf{v}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \mathrm{p}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty. \tag{2.1.5}$$

2.2 Transmission problem for a Brinkman type system and the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$

In this section we aim to state and prove a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.6 for $n = 3$, i.e., complementary Lipschitz domains in $\mathbb{R}^3$. We consider a generalized version of the Brinkman system in the bounded Lipschitz domain D_+ and the Stokes system in the complementary Lipschitz set D_-. Also, we have the following assumption that we will use in the latter.

Assumption 2.2.1. *Let $n \geq 2$. Assume that $\mathfrak{L} \in L^\infty(\Gamma)^{n\times n}$ be a symmetric matrix valued function, which satisfies the following non-negativity condition*

$$\langle \mathfrak{L}\boldsymbol{u}, \boldsymbol{u}\rangle_\Gamma \geq 0, \tag{2.2.1}$$

for all $\boldsymbol{u} \in L^2(\Gamma)^n$.

We consider the following spaces, namely the space of solutions,

$$\mathsf{X}_w := H^1_{\mathrm{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times \mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-) \tag{2.2.2}$$

and the space of given data,

$$\mathsf{Y}_w := \tilde{H}^{-1}(\mathsf{D}_+)^3 \times \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3 \times H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3, \tag{2.2.3}$$

respectively.

The considered transmission problem of Poisson type for the Stokes and generalized Brinkman systems is

$$\begin{cases} \Delta\mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - \nabla\pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+,\\ \Delta\mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-,\\ \mathrm{div}\ \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm,\\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma,\\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{2.2.4}$$

with the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$. Note that, the presence of the Stokes system in the unbounded Lipschitz domain D_- justifies the inclusion of the space $\mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3$ in the solution space X_w provided in relation (2.2.2).

Hence, we begin with the well-posedness result that was obtained for the transmission problem (2.2.4), in the case $\mathbf{u}_\infty = 0$ (see [7, Theorem 4.5], [52, Theorem 4.2], [57, Theorem 4.4]).

Theorem 2.2.2. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ such that condition (1.2.22) holds. Then, for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_w$ given, the Poisson problem of transmission type for Stokes and generalized Brinkman systems (2.2.4) has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$. Moreover, there is a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ such that*

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_w} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_w}, \tag{2.2.5}$$

and $\mathbf{u}_-$ vanishes at infinity in the sense of Leray.

Now, we provide the well-posedness result for the transmission problem (2.2.4), in the case $\mathbf{u}_\infty \neq 0$ (see, [7, Remark 4.6], [52, Theorem 4.4]).

Theorem 2.2.3. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for* $n = 3$. *Let* $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ *such that condition (1.2.22) holds. Then, for* $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$*, the Poisson problem of transmission-type for the generalized Brinkman and Stokes systems (2.2.4) has a unique solution*

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \tag{2.2.6}$$

satisfying the condition

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w. \tag{2.2.7}$$

In addition, the corresponding solution operator,

$$\mathsf{T} : \mathsf{Y}_w \times \mathbb{R}^3 \to \mathsf{X}_w, \tag{2.2.8}$$

is linear and bounded, and hence, there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ *such that the unique solution of (2.2.4) satisfies the estimate*

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-)||_{\mathsf{X}_w} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w\times\mathbb{R}^3}, \tag{2.2.9}$$

and $\mathbf{u}_- - \mathbf{u}_\infty$ *vanishes at infinity in the sense of Leray.*

2.2.1 Transmission problem for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$

This subsection is dedicated to the study of the transmission problem for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$, i.e., the setting of Assumption 1.1.6 for $n = 3$. In addition, let Assumption 2.2.1 be satisfied, for $n = 3$. The considered transmission-type problem for the Stokes system reads as follows

$$\begin{cases} \Delta \mathbf{u}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \operatorname{div} \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{2.2.10}$$

Let us describe the steps that we follow in order to show that the transmission problem (2.2.10), is well-posed. Firstly, we will state and prove the following lemma (see, [7, Lemma 4.1, Corollary 4.2]).

Lemma 2.2.4. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for* $n = 3$. *Then, the operators*

$$\begin{aligned} &\mathbb{I} + \mathcal{V}_\Gamma \mathfrak{L} : H^{\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3, \\ &\mathbb{I} + \mathfrak{L}\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3, \end{aligned} \tag{2.2.11}$$

are isomorphisms.

Secondly, we state and prove a lemma that shows that our transmission problem (2.2.10) has at most one solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w$, where $\mathbf{u}_\infty \in \mathbb{R}^3$ is a constant vector (see [7, Lemma 4.1], [52, Lemma 4.1]).

Lemma 2.2.5. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for* $n = 3$. *Then, for* $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$, *the Poisson problem of transmission-type for the Stokes system (2.2.10) has at most one solution* $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)$ *which satisfies* $(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w$.

Let us now state and prove the well-posedness result, that we have obtained, for our transmission problem for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$ in the case $\mathbf{u}_\infty = 0$ (see [7, Theorem 4.3], [52, Theorem 4.2], [75, Theorem 5.1]).

Theorem 2.2.6. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for* $n = 3$. *Then, for* $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_w$, *the Poisson problem of transmission-type for the Stokes system (2.2.10) has a unique solution* $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}$. *Moreover, there exists a linear and continuous operator*

$$\mathsf{S} : \mathsf{Y}_w \to \mathsf{X}_w, \tag{2.2.12}$$

that maps the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_w$ *to the unique solution* $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$ *of the problem (2.2.10), in the sense that, there is a constant* $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathfrak{L}) > 0$ *such that*

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_w} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_w}. \tag{2.2.13}$$

In addition, $\mathbf{u}_-$ *vanishes at infinity in the sense of Leray.*

Lastly, we provide the well-posendess result of the transmission problem (2.2.10) in the case $\mathbf{u}_\infty \neq 0$ (see, e.g., [7, Theorem 4.4], [52, Theorem 4.4]).

Theorem 2.2.7. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied for* $n = 3$. *Then, for* $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$, *the Poisson problem of transmission-type for the Stokes system (2.2.10) has a unique solution* $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)$ *satisfying the condition* $(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w$. *Moreover, there is a constant* $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathfrak{L}) > 0$ *such that*

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-)||_{\mathsf{X}_w} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3}, \tag{2.2.14}$$

and $\mathbf{u}_- - \mathbf{u}_\infty$ *vanishes at infinity in the sense of Leray.*

2.3 Transmission problem for the generalized Brinkman and classical Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$

In this section we will state and prove a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.6 for $n = 3$, i.e., complementary Lipschitz domains in $\mathbb{R}^3$. We have considered a generalized version of the Brinkman in the bounded Lipschitz domain D_+ and the Brinkman system in the complementary Lipschitz set D_-. Also, let Assumption 2.2.1 be satisfied, for $n = 3$.

Let us introduce the following spaces, namely the space of solutions,

$$\mathsf{X}_{\mathcal{B}} := H^1_{\text{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times H^1_{\text{div}}(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-) \tag{2.3.1}$$

and the space of given data,

$$\mathsf{Y}_{\mathcal{B}} := \tilde{H}^{-1}(\mathsf{D}_+)^3 \times \tilde{H}^{-1}(\mathsf{D}_-)^3 \times H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3, \tag{2.3.2}$$

respectively. We point out the fact that the space $\mathfrak{M}(\mathsf{D}_-)$ is introduced in Definition 2.1.1.

Let us emphasize the fact that the presence of the Brinkman system in the exterior Lipschitz domain D_-, provided in Assumption 1.1.6, $n = 3$, leads to *the use of classical Sobolev spaces, instead of the weighted Sobolev spaces* (as in the case of the Stokes system in exterior Lipschitz domains), in order to find the velocity field in D_-. This is a consequence of the behavior of the fundamental solution of the Brinkman system at infinity, in the case $n = 3$.

We turn our attention to the transmission problem for the generalized Brinkman and classical Brinkman system, which is given by

$$\begin{cases} \Delta \mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{u}_- - \alpha \mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \text{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \text{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\text{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{2.3.3}$$

where unknown fields are $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$.

Let us state and prove the well-posedness result that we have obtained for problem (2.3.3) (see also [6, Theorem 3.3] and [57, Theorem 4.4] in the case of compact Riemannian manifolds). In addition, the following well-posedness result also provides the far field conditions that our solution satisfies (see Remark 1.3.9).

Theorem 2.3.1. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied for $n = 3$. Let $\alpha > 0$ be a constant. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ be such that the condition (1.2.22) holds. Then, for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$ given, the Poisson problem of transmission-type for the generalized and classical Brinkman systems (2.3.3) has a unique solution*

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}. \tag{2.3.4}$$

In addition, the corresponding solution operator,

$$\mathsf{T}_{\mathcal{B}} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}, \tag{2.3.5}$$

is linear and bounded, and hence, there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ such that the unique solution of (2.3.3) satisfies

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{\mathcal{B}}} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}. \tag{2.3.6}$$

Moreover, the pair $(\mathbf{u}_-, \pi_-)$ satisfies the following far field conditions

$$\mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{-2}),\ \nabla \mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{-1}),\ \pi_-(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \tag{2.3.7}$$

as $|\mathbf{x}| \to \infty$.

2.3.1 Transmission problem for the Stokes and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$

This subsection is dedicated to the study of the transmission problem for the Stokes and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$, i.e., the setting of Assumption 1.1.6

for $n = 3$. In addition, let Assumption 2.2.1 be satisfied, for $n = 3$. The considered transmission-type problem for the Stokes and Brinkman systems is given by

$$\begin{cases} \Delta \mathbf{u}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{u}_- - \alpha \mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \text{Tr}_{\mathsf{D}_+} \mathbf{u}_+ - \text{Tr}_{\mathsf{D}_-} \mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\alpha, \mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L} \text{Tr}_{\mathsf{D}_+} \mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{2.3.8}$$

First of all, we have a preliminary result in which we will show that the transmission problem (2.3.8) has at most one solution (see [6, Lemma 3.1], [52, Lemma 4.1]).

Lemma 2.3.2. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied for $n = 3$. Let $\alpha > 0$ be a constant. Then, for the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$, the Poisson problem of transmission-type for the Stokes and Brinkman systems (2.3.8) has at most one solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$.*

We are now able to state and prove the well-posedness result for the transmission problem (2.3.8) (see, [6, Theorem 3.2], [52, Theorem 4.2]).

Theorem 2.3.3. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied for $n = 3$. Let $\alpha > 0$ be a constant. Then, for the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$, the Poisson problem of transmission-type for the Stokes and Brinkman systems (2.3.8) has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$. In addition, the operator*

$$\mathsf{S} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}, \tag{2.3.9}$$

which maps the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$ to the corresponding solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$ of the transmission problem (2.3.8) is linear and continuous. Consequently, there is a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathfrak{L}) > 0$ such that:

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{\mathcal{B}}} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}. \tag{2.3.10}$$

Moreover, $\mathbf{u}_-$, π_- satisfy the far field conditions

$$\mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{-2}),\ \nabla \mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{-1}),\ \pi_-(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \tag{2.3.11}$$

as $|\mathbf{x}| \to \infty$.

2.4 On a Robin-Transmission problem for the Brinkman system

In this section we aim to state and prove a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.7. Before we state the transmission problem, let us mention that such problems are used to model the fluid flow in the exterior of a cavity or in cavities filled with porous media, in the case of the jump of either tensions or velocity on the interface. Another idea is to analyze the fluid flow in a porous media in reservoirs whose boundary has two parts, the first one that of a solid surface and the second, an interface between the fluid and another fluid or viscoelastic material (for additional details, see, e.g., [53]). From a practical point of view, Baber [15] has analyzed applications of transmission problems, such as the water management in fuel cells or the processing of nutrients between two domains, one containing blood, the other porous tissue.

The transmission-type problem that we wish to treat will be called the Robin-transmission problem for the Brinkman system (see problem (2.4.3)). In addition, let Assumption 2.2.1 be satisfied.

We consider the following spaces, namely the space of solutions,

$$\mathsf{X}_{RT} := H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times L^2(\mathsf{D}_+) \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-), \tag{2.4.1}$$

and the space of given data,

$$\mathsf{Y}_{RT} := \tilde{H}^{-1}(\mathsf{D}_+)^n \times \tilde{H}^{-1}(\mathsf{D}_-)^n \times H^{\frac{1}{2}}_{\nu}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_-)^n, \tag{2.4.2}$$

respectively.

We will study the Robin-transmission problem for the Brinkman system, which is given by

$$\begin{cases} \Delta \mathbf{u}_\pm - \alpha \mathbf{u}_\pm - \nabla \pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+) - \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-)\right)|_{\Gamma_+} = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-)\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{2.4.3}$$

where $\alpha > 0$ and $\lambda \in (0, 1]$ are given constants. We aim to determine the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$.

Let us state and prove the well-posedness result that was obtained for the Robin-transmission problem (2.4.3) (see also [8, Theorem 1], [56, Theorem 4.1], [63, Theorem 5.8]).

Theorem 2.4.1. *Let $\alpha > 0$ and $\lambda \in (0, 1]$ be given constants. Let Assumption 1.1.7 and Assumption 2.2.1 be satisfied. Then, for all data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, the Poisson problem of Robin-transmission type for the Brinkman system (2.4.3) has a unique solution*

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}. \tag{2.4.4}$$

In addition, the corresponding solution operator,

$$\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}, \tag{2.4.5}$$

is linear and bounded, and hence, there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$ such that the unique solution of (2.4.3) satisfies

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{2.4.6}$$

Proof. We prove this result in a similar way as to the one used in the proof of Theorem 4.1 in [56] and Theorem 5.8 in [63]. This approach uses layer potential methods. In order to preserve the simplicity of our arguments, let us introduce the space

$$\mathbb{Y} := H^{\frac{1}{2}}_{\nu}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_-)^n, \tag{2.4.7}$$

which will appear in the latter.

We divide our arguments into two separate cases. The first case concerns the situation $\lambda \in (0,1)$ and the second case refers to the situation $\lambda = 1$.

Case 1: Assume that $\lambda \in (0,1)$. Firstly, we show that our problem admits a solution, (i.e., the existence of a solution) and we aim to construct it by using a layer potential approach. To this end, let us seek a solution in the form

$$\begin{aligned}
\mathbf{u}_+ &= \mathcal{N}_{\alpha,\mathsf{D}_+}\mathbf{f}_+ + \mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi} + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi},\\
\pi_+ &= \mathcal{Q}_{\alpha,\mathsf{D}_+}\mathbf{f}_+ + \mathbf{Q}^d_{\alpha,\Gamma_+}\mathbf{\Phi} + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi},\\
\mathbf{u}_- &= \mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_- + \mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi} + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi} + \mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi},\\
\pi_- &= \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_- + \mathbf{Q}^d_{\alpha,\Gamma_+}\mathbf{\Phi} + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi} + \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi},
\end{aligned} \tag{2.4.8}$$

where $(\mathbf{\Phi},\boldsymbol{\varphi},\boldsymbol{\psi}) \in \mathbb{Y}$ are unknown densities and the space $\mathbb{Y}$ is given in relation (2.4.7).

Note that, the mapping properites of the Newtonian, simple and double layer potentials for the Brinkman system (see Theorem 1.4.2, Theorem 1.4.4, Theorem 1.4.7) imply that $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-) \in \mathsf{X}_{RT}$.

Next, by taking into account the jump formulas for single and double layer potentials (see relation (1.4.23) of Lemma 1.4.8) and by substitution into relation $(2.4.3)_3$, we get

$$\left(\lambda\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) - \left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right)\right)\mathbf{\Phi} + \left(\lambda\mathcal{V}_{\alpha,\Gamma_+} - \mathcal{V}_{\alpha,\Gamma_+}\right)\boldsymbol{\varphi} - \mathcal{V}_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} = \mathbf{g}_{01}, \tag{2.4.9}$$

and $\mathbf{g}_{01}$ is given by

$$\mathbf{g}_{01} = \mathbf{g}_1 - \lambda(\mathrm{Tr}_{\mathsf{D}_+}(\mathcal{N}_{\alpha,\mathsf{D}_+}\mathbf{f}_+)) + (\mathrm{Tr}_{\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-))|_{\Gamma_+}. \tag{2.4.10}$$

Note that, the operator

$$\mathcal{V}_{\Gamma_-,\Gamma_+} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{\frac{1}{2}}(\Gamma_+)^n,\ \mathcal{V}_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} := \left(\mathrm{Tr}_{\mathsf{D}_-}\left(\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}\right)\right)|_{\Gamma_+}, \tag{2.4.11}$$

is compact, as an integral operator with real analytic kernel (see [23, Theorem A.28, Statement (ii)] which deals with the properties of integral operators with real analytic kernels) and due to the compact embedding $H^1(\Gamma_+)^n \hookrightarrow H^{\frac{1}{2}}(\Gamma_+)^n$. Also, we have $\mathbf{g}_{01} \in H^{\frac{1}{2}}_{\nu}(\Gamma_+)^n$. This assertion holds true after the application of the Divergence Theorem while taking into account relation (1.4.13).

Now, let us take into account again the jump formulas for single and double layer potentials, and by substitution into relation $(2.4.3)_4$, we get

$$\boldsymbol{\varphi} - \mathbb{K}^*_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} = \mathbf{h}_{01}, \tag{2.4.12}$$

and $\mathbf{h}_{01}$ is given by

$$\mathbf{h}_{01} := \mathbf{h}_1 - \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathcal{N}_{\alpha,\mathsf{D}_+}\mathbf{f}_+, \mathcal{Q}_{\alpha,\mathsf{D}_+}\mathbf{f}_+, \mathbf{f}_+) + (\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathbf{f}_-))|_{\Gamma_+}. \tag{2.4.13}$$

Let us notice that the operator

$$\mathbb{K}^*_{\Gamma_-,\Gamma_+} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{-\frac{1}{2}}(\Gamma_+)^n,\ \mathbb{K}^*_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} := \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}, \mathbf{Q}_{\alpha,\Gamma_-}\boldsymbol{\psi})\right)|_{\Gamma_+}, \tag{2.4.14}$$

is a compact operator, based on [23, Theorem A.28, Statement (ii)] and the compactness of the embedding $L^2(\Gamma_+)^n \hookrightarrow H^{-\frac{1}{2}}(\Gamma_+)^n$.

It remains now to apply, again, the jump properties of the single-layer and double-layer potentials for the Brinkman system (see Lemma 1.4.8) and by substitution into relation $(2.4.3)_5$ (i.e., the Robin boundary condition), we get the equation

$$\left(\mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-}\right)\mathbf{\Phi} + \left(\mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-}\right)\boldsymbol{\varphi} + \left(\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-}\right)\boldsymbol{\psi} = \mathbf{g}_{02}, \tag{2.4.15}$$

and $\mathbf{g}_{02}$ is given by

$$\mathbf{g}_{02} := \mathbf{g}_2 - (\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathbf{f}_-))|_{\Gamma_-} - \mathfrak{L}(\mathrm{Tr}_{\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-))|_{\Gamma_-}. \tag{2.4.16}$$

We emphasise the fact that the following operators

$$\begin{aligned}
&\mathbb{D}_{\Gamma_+,\Gamma_-} : H^{\frac{1}{2}}(\Gamma_+)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n,\ \mathbb{D}_{\Gamma_+,\Gamma_-}\mathbf{\Phi} := \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi}, \mathbf{Q}^d_{\alpha,\Gamma_+}\mathbf{\Phi})\right)|_{\Gamma_-},\\
&\mathbb{K}_{\Gamma_+,\Gamma_-} : H^{\frac{1}{2}}(\Gamma_+)^n \to H^{\frac{1}{2}}(\Gamma_-)^n,\ \mathbb{K}_{\Gamma_+,\Gamma_-}\mathbf{\Phi} := \left(\mathrm{Tr}_{\mathsf{D}_-}\left(\mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi}\right)\right)|_{\Gamma_-},\\
&\mathbb{K}^*_{\Gamma_+,\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_+)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n,\ \mathbb{K}^*_{\Gamma_+,\Gamma_-}\boldsymbol{\varphi} := \left(\mathbf{t}_{\alpha,\mathsf{D}_-}\left(\mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}, \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}\right)\right)|_{\Gamma_-},\\
&\mathcal{V}_{\Gamma_+,\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_+)^n \to H^{\frac{1}{2}}(\Gamma_-)^n,\ \mathcal{V}_{\Gamma_+,\Gamma_-}\boldsymbol{\varphi} := \left(\mathrm{Tr}_{\mathsf{D}_-}\left(\mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}\right)\right)|_{\Gamma_-},
\end{aligned} \tag{2.4.17}$$

which are present in relation (2.4.15) are compact operators. This assertion holds true if we apply [23, Theorem A.28, Statement (ii)] and if we take into account the compactness of the embeddings $H^1(\Gamma_+)^n \hookrightarrow H^{\frac{1}{2}}(\Gamma_+)^n$ and $L^2(\Gamma_+)^n \hookrightarrow H^{-\frac{1}{2}}(\Gamma_+)^n$.

Consequently, the Robin-transmission problem (2.4.3) reduces to the equations given by relations (2.4.9), (2.4.12), (2.4.15). Let us write these equations in matrix form as follows

$$\mathsf{A}(\mathbf{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t = (\mathbf{g}_{01}, \mathbf{h}_{01}, \mathbf{g}_{02}) \text{ in } \mathbb{Y}, \tag{2.4.18}$$

in the unknown $(\mathbf{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t \in \mathbb{Y}$. The matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ involved in relation (2.4.18) is given by

$$\mathsf{A} := \begin{bmatrix} \lambda\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) - \left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) & \lambda\mathcal{V}_{\alpha,\Gamma_+} - \mathcal{V}_{\alpha,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+}\\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+}\\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.19}$$

Let us write the matrix operator (2.4.19) in the following equivalent form,

$$\mathsf{A} := \begin{bmatrix} -\frac{\lambda+1}{2}\mathbb{I} + (\lambda-1)\mathbb{K}_{\alpha,\Gamma_+} & (\lambda-1)\mathcal{V}_{\alpha,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+}\\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+}\\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.20}$$

We claim that the matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is an isomorphism. In order to prove this claim, we will prove operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is a Fredholm operator of index zero, for $\lambda \in (0, 1]$ and that it is also an injective operator.

Let us proceed by showing, first of all, that $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is a Fredholm operator of index zero. A simple rearrangement allows us to rewrite operator (2.4.20) in the following form

$$\mathsf{A} := \begin{bmatrix} (\lambda-1)\left(\frac{1}{2}\frac{1+\lambda}{1-\lambda}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) & (\lambda-1)\mathcal{V}_{\alpha,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+}\\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+}\\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.21}$$

It is immediate that the matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is well-defined, linear and continuous.

Let us now recall the definition of the complementary layer-potential operators (see relation (1.4.26)) and with their help we are decompose the matrix operator (2.4.21) as

$$\mathsf{A} = \mathsf{A}_0 + \mathsf{A}_\mathsf{C} : \mathbb{Y} \to \mathbb{Y}, \tag{2.4.22}$$

where the operators $\mathsf{A}_0 : \mathbb{Y} \to \mathbb{Y}$ and $\mathsf{A}_\mathsf{C} : \mathbb{Y} \to \mathbb{Y}$ are defined by

$$\mathsf{A}_0 := \begin{bmatrix} (\lambda-1)\left(\frac{1}{2}\frac{1+\lambda}{1-\lambda}\mathbb{I} + \mathbb{K}_{\Gamma_+}\right) & (\lambda-1)\mathcal{V}_{\Gamma_+} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_-} \end{bmatrix} \tag{2.4.23}$$

and

$$\mathsf{A}_\mathsf{C} := \begin{bmatrix} (\lambda-1)\mathbb{K}_{\alpha,0,\Gamma_+} & (\lambda-1)\mathcal{V}_{\alpha,0,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+} \\ \mathbf{0} & \mathbf{0} & -\mathbb{K}^*_{\Gamma_-,\Gamma} \\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\alpha,0,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,0,\Gamma_-} \end{bmatrix}. \tag{2.4.24}$$

Let us analyze the properties of the operator $\mathsf{A}_0 : \mathbb{Y} \to \mathbb{Y}$ given by relation (2.4.23). Let us take into account the fact the operator

$$\frac{1}{2}\frac{1+\lambda}{1-\lambda}\mathbb{I} + \mathbb{K}_{\Gamma_+} : H^{\frac{1}{2}}(\Gamma_+)^n \to H^{\frac{1}{2}}(\Gamma_+)^n, \tag{2.4.25}$$

is a Fredholm operator of index zero (see, e.g., [85, Corollary 9.1.2], [63, Lemma 5.3]). Next, by the second statement of Lemma 1.3.8, the operator

$$\mathcal{V}_{\Gamma_+} : H^{-\frac{1}{2}}(\Gamma_+)^n \to H^{\frac{1}{2}}(\Gamma_+)^n, \tag{2.4.26}$$

is also a Fredholm operator of index zero. Moreover, the operator

$$\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n, \tag{2.4.27}$$

is another Fredholm operator of index zero, since

$$\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n \tag{2.4.28}$$

is Fredholm operator of index zero and the operator

$$\mathfrak{L}\mathcal{V}_{\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n \tag{2.4.29}$$

is a compact operator in view of the compact embeddings $H^{\frac{1}{2}}(\Gamma_-)^n \hookrightarrow L^2(\Gamma_-)^n$ and $L^2(\Gamma_-)^n \hookrightarrow H^{-\frac{1}{2}}(\Gamma_-)^n$ (for additional details see, e.g., [56, Theorem 4.1]).

By the arguments in the former, we have shown that the operators (2.4.25), (2.4.26) and (2.4.27) are Fredholm operators of index zero and it follows that the operator $\mathsf{A}_0 : \mathbb{Y} \to \mathbb{Y}$ is Fredholm of index zero.

Let us now focus on the operator $\mathsf{A}_\mathsf{C} : \mathbb{Y} \to \mathbb{Y}$ provided in relation (2.4.24). In view of the compactness of the complementary layer-potential operators $\mathbb{K}_{\alpha,0,\Gamma_+}$, $\mathcal{V}_{\alpha,0,\Gamma_+}$, $\mathbb{K}^*_{\alpha,0,\Gamma_-}$, $\mathcal{V}_{\alpha,0,\Gamma_-}$ (see relation (1.4.27) of Lemma 1.4.9) and also the compactness of the operators (2.4.11), (2.4.14) and (2.4.17), we have that, in turn, the operator $\mathsf{A}_\mathsf{C} : \mathbb{Y} \to \mathbb{Y}$ is a compact operator.

Since our operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is a sum of a Fredholm operator of index zero and a compact operator, we deduce that $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is a Fredholm operator of index zero, for $\lambda \in (0,1)$.

In order to fully prove our claim, we show that the operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is injective, or equivalently, we show that the kernel of the operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is the null space, i.e.,

$$\operatorname{Ker}\{\mathsf{A} : \mathbb{Y} \to \mathbb{Y}\} = \{\mathbf{0}\}. \tag{2.4.30}$$

To achieve this, we consider $(\boldsymbol{\Phi}_0, \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0)^t \in \operatorname{Ker}\{\mathsf{A} : \mathbb{Y} \to \mathbb{Y}\}$. Then, we construct the fields $(\mathbf{u}_+^0, \pi_+^0)$ and $(\mathbf{u}_-^0, \pi_-^0)$ as follows

$$\begin{aligned} &\mathbf{u}_+^0 := \mathbf{W}_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0 + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0 \quad \pi_+^0 := \mathbf{Q}^d_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0 + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0 \\ &\mathbf{u}_-^0 := \mathbf{W}_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0 + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0 + \mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 \quad \pi_-^0 := \mathbf{Q}^d_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0 + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0 + \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0. \end{aligned} \tag{2.4.31}$$

Let us note that these fields $(\mathbf{u}_+^0, \pi_+^0)$ and $(\mathbf{u}_-^0, \pi_-^0)$ satisfy

$$\begin{aligned} &\lambda(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0) = \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\right)|_{\Gamma_+} \text{ a.e. on } \Gamma_+, \\ &\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0) = \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0)\right)|_{\Gamma_+} \text{ a.e. on } \Gamma_+, \\ &\left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0)\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\right)|_{\Gamma_-} = 0, \text{ a.e. on } \Gamma_-. \end{aligned} \tag{2.4.32}$$

Now, we apply the Green formula (1.2.18) to the fields introduced in relation (2.4.31) and we get

$$\begin{aligned} 2\left\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{u}_+^0, \mathbf{u}_+^0\right\rangle_{\mathsf{D}_+} &= \left\langle \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0\right\rangle_{\Gamma_+}, \\ 2\left\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_-} + \alpha\left\langle \mathbf{u}_-^0, \mathbf{u}_-^0\right\rangle_{\mathsf{D}_-} &= -\left\langle \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0)\right)|_{\Gamma_+}, \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\right)|_{\Gamma_+}\right\rangle_{\Gamma_+} \\ &\quad + \left\langle \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0)\right)|_{\Gamma_-}, \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\right)|_{\Gamma_-}\right\rangle_{\Gamma_-}. \end{aligned} \tag{2.4.33}$$

Let us now multiply relation $(2.4.33)_1$ by λ and we add the resulting quantities to relation $(2.4.33)_2$ and by using relations $(2.4.32)_1$, $(2.4.32)_2$ and $(2.4.32)_3$ we have

$$\begin{aligned} &\lambda\left(2\left\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{u}_+^0, \mathbf{u}_+^0\right\rangle_{\mathsf{D}_+}\right) + 2\left\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_-} + \alpha\left\langle \mathbf{u}_-^0, \mathbf{u}_-^0\right\rangle_{\mathsf{D}_-} \\ &\quad = -\left\langle \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\right)|_{\Gamma_-}, \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\right)|_{\Gamma_-}\right\rangle_{\Gamma_-}. \end{aligned} \tag{2.4.34}$$

Note that, the left hand side of the equality (2.4.34) is non-negative and the right hand side of the equality (2.4.34) is non-positive (due to the fact that $\mathfrak{L}$ satisfies condition (2.2.1)). This leads to the fact that

$$\lambda\left(2\left\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{u}_+^0, \mathbf{u}_+^0\right\rangle_{\mathsf{D}_+}\right) + 2\left\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_-} + \alpha\left\langle \mathbf{u}_-^0, \mathbf{u}_-^0\right\rangle_{\mathsf{D}_-} = 0. \tag{2.4.35}$$

Consequently, we get $\mathbf{u}_\pm^0 = \mathbf{0}$ in $\mathsf{D}_\pm$, which, in turn, implies that $\pi_\pm^0 = c_\pm^0$, where $c_\pm^0 \in \mathbb{R}$ are constants. Also, relations $(2.4.32)_2$ and $(2.4.32)_3$ imply $c_+^0 = c_-^0 = 0$. Hence, we have that

$$\mathbf{u}_\pm^0 = \mathbf{0}, \text{ in } \mathsf{D}_\pm, \ \pi_\pm^0 = 0 \text{ in } \mathsf{D}_\pm. \tag{2.4.36}$$

Let us now apply relation (1.4.23) of Lemma 1.4.8 in order to get

$$\begin{aligned} &\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_+^0 = \boldsymbol{\Phi}_0, \ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_-^0 = -\boldsymbol{\Phi}_0, \text{ a.e. on } \Gamma_+, \\ &\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_+^0, \pi_+^0) = -\boldsymbol{\varphi}_0, \ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_-^0, \pi_-^0) = \boldsymbol{\varphi}_0 \text{ a.e. on } \Gamma_+. \end{aligned} \tag{2.4.37}$$

In addition, the membership $\boldsymbol{\Phi}_0 \in H_\nu^{\frac{1}{2}}(\Gamma_+)^n$ implies that $(\mathbf{W}_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0)(\mathbf{x}) = O(|\mathbf{x}|^{-n})$ as $|\mathbf{x}| \to \infty$, (see [106, Lemma 2.12]). Let us mention that the single-layer potential $\mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0$ behaves in a similar manner at infinity (see [106, Lemma 2.12]). Hence, $\mathbf{u}_+^0(\mathbf{x}) = O(|\mathbf{x}|^{-n})$ as $|\mathbf{x}| \to \infty$. Consequently, the fields $(\mathbf{u}_+^0, \pi_+^0)$ satisfy the Green formula (1.2.18) corresponding to the domain $\mathbb{R}^n \setminus \overline{\mathsf{D}}_+$. Let us apply the Green formula (1.2.18) for the fields $(\mathbf{u}_+^0, \pi_+^0)$ in $\mathbb{R}^n \setminus \overline{\mathsf{D}}_+$, while taking into account relation (2.4.37). We get

$$\begin{aligned} &2\left\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+} + \alpha\left\langle \mathbf{u}_+^0, \mathbf{u}_+^0\right\rangle_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+} \\ &\quad = -\left\langle \mathbf{t}_{\alpha,\nu,\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}(\mathbf{u}_+^0, \pi_+^0), \mathrm{Tr}_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}\mathbf{u}_+^0\right\rangle_{\Gamma_+} = \langle \boldsymbol{\varphi}_0, \boldsymbol{\Phi}_0\rangle_{\Gamma_+}. \end{aligned} \tag{2.4.38}$$

Moreover, we apply the Green formula (1.2.18) for $(\mathbf{u}_-^0, \pi_-^0)$ in D_+, while taking into account relation (2.4.37) and we obtain

$$2\left\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{u}_-^0, \mathbf{u}_-^0\right\rangle_{\mathsf{D}_+} = \left\langle \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_-^0, \pi_-^0), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_-^0\right\rangle_{\Gamma_+} = -\langle \boldsymbol{\varphi}_0, \boldsymbol{\Phi}_0\rangle_{\Gamma_+}. \tag{2.4.39}$$

Let us now add relations (2.4.38) and (2.4.39). We obtain the following

$$2\left\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+} + \alpha\left\langle \mathbf{u}_+^0, \mathbf{u}_+^0\right\rangle_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+} + 2\left\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{u}_-^0, \mathbf{u}_-^0\right\rangle_{\mathsf{D}_+} = 0, \tag{2.4.40}$$

which shows that

$$\mathbf{u}_+^0 = \mathbf{0},\ \pi_+^0 = 0 \text{ in } \mathbb{R}^n \setminus \overline{\mathsf{D}}_+,\ \mathbf{u}_-^0 = \mathbf{0},\ \pi_-^0 = 0 \text{ in } \mathsf{D}_+. \tag{2.4.41}$$

Let us stress the fact that $\pi_+^0 = 0$ in $\mathbb{R}^n \setminus \overline{\mathsf{D}}_+$ is a consequence of the fact that the pair $(\mathbf{u}_+^0, \pi_+^0)$ satisfies the homogeneous Brinkman equation in $\mathbb{R}^n \setminus \overline{\mathsf{D}}_+$ and also the fact that $\pi_+^0(\mathbf{x}) = O(|\mathbf{x}|^{1-n})$ as $|\mathbf{x}| \to \infty$ (see [54, Relations (3.12), (3.13)]).

Now, by relations (2.4.37) and (2.4.41) we are able to deduce that

$$\boldsymbol{\Phi}_0 = \mathbf{0},\ \boldsymbol{\varphi}_0 = \mathbf{0}. \tag{2.4.42}$$

Relation (2.4.42) together with the fact that $\mathbf{u}_-^0 = \mathbf{0}$ in D_+, implies that $\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 = \mathbf{0}$ in D_+. The continuity of the single layer potential for the Brinkman system on Γ_- (see Theorem 1.4.4) implies that

$$\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 = \mathbf{0} \text{ in } \mathbb{R}^n \setminus \overline{\mathsf{D}}_+, \tag{2.4.43}$$

while the behavior at infinity of the single layer pressure potential (namely, that $\mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 = O(|\mathbf{x}|^{1-n})$ for $n \geq 2$, as it can be seen in relation (3.12) of [54]) leads to the fact that

$$\mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 = 0 \text{ in } \mathbb{R}^n \setminus \overline{\mathsf{D}}_+. \tag{2.4.44}$$

Therefore, by relations (2.4.43) and (2.4.44) we get

$$\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0, \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0) = \mathbf{0}, \text{ on } \Gamma_-, \quad \mathbf{t}_{\alpha,\nu,\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}(\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0, \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0) = \mathbf{0}, \text{ on } \Gamma_-. \tag{2.4.45}$$

Let us subtract $(2.4.45)_2$ from $(2.4.45)_1$ and by using the jump formulas (1.4.23) of Lemma 1.4.8 we obtain

$$\boldsymbol{\psi}_0 = \mathbf{0}. \tag{2.4.46}$$

To conclude our argument, we have that, in view of relations (2.4.42) and (2.4.46), we have that property (2.4.30) is satisfied, namely the kernel of the matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is the null space, or equivalently, $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is injective.

It follows that our matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is an isomorphism and equation (2.4.18) has a unique solution $(\boldsymbol{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t \in \mathbb{Y}$. The unique solution of the equation (2.4.18) together with the layer potential representations provided in relation (2.4.8) give a solution of the Robin-transmission problem (2.4.3) in the space X_{RT}.

Next, we are concerned about the *uniqueness of the solution of the problem* (2.4.3). In order to show this property, let us assume that the problem (2.4.3) admits two solutions and we denote their difference by $(\mathbf{v}^0_\pm, \pi^0_\pm)$. Hence, the fields $(\mathbf{v}^0_+, \pi^0_+, \mathbf{v}^0_-, \pi^0_-) \in \mathsf{X}_{RT}$ satisfy

$$\begin{cases} \Delta \mathbf{v}^0_\pm + \alpha \mathbf{v}^0_\pm - \nabla p^0_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{v}^0_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda(\text{Tr}_{\mathsf{D}_+}\mathbf{v}^0_+) - \left(\text{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_+} = 0 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{v}^0_+, \pi^0_+) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-)\right)|_{\Gamma_+} = 0 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-)\right)|_{\Gamma_-} + \mathfrak{L}\left(\text{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_-} = 0 \text{ on } \Gamma_-, \end{cases} \tag{2.4.47}$$

i.e., the homogenenous version of (2.4.3).

Let us now use Green's formula (1.2.18) in the domains $\mathsf{D}_\pm$ in order to get the following relations

$$\begin{aligned} 2\left\langle \mathbb{E}(\mathbf{v}^0_+), \mathbb{E}(\mathbf{v}^0_+)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{v}^0_+, \mathbf{v}^0_+\right\rangle_{\mathsf{D}_+} &= \left\langle \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{v}^0_+, \pi^0_+), \text{Tr}_{\mathsf{D}_+}\mathbf{v}^0_+\right\rangle_{\Gamma_+} \\ 2\left\langle \mathbb{E}(\mathbf{v}^0_-), \mathbb{E}(\mathbf{v}^0_-)\right\rangle_{\mathsf{D}_-} + \alpha\left\langle \mathbf{v}^0_-, \mathbf{v}^0_-\right\rangle_{\mathsf{D}_-} &= -\left\langle \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-)|_{\Gamma_+}, \left(\text{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_+}\right\rangle_{\Gamma_+} \\ &\quad + \left\langle \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-)\right)|_{\Gamma_-}, \left(\text{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_-}\right\rangle_{\Gamma_-}. \end{aligned} \tag{2.4.48}$$

Let us multiply relation $(2.4.48)_1$ by λ and to the result we will add $(2.4.48)_2$, while taking into account the boundary conditions in problem (2.4.47). After computations, we get

$$\begin{aligned} &\lambda\left(2\left\langle \mathbb{E}(\mathbf{v}^0_+), \mathbb{E}(\mathbf{v}^0_+)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{v}^0_+, \mathbf{v}^0_+\right\rangle_{\mathsf{D}_+}\right) + 2\left\langle \mathbb{E}(\mathbf{v}^0_-), \mathbb{E}(\mathbf{v}^0_-)\right\rangle_{\mathsf{D}_-} + \alpha\left\langle \mathbf{v}^0_-, \mathbf{v}^0_-\right\rangle_{\mathsf{D}_-} \\ &\quad = -\left\langle \mathfrak{L}\left(\text{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_-}, \left(\text{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_-}\right\rangle_{\Gamma_-}. \end{aligned} \tag{2.4.49}$$

Let us note that, left hand side of (2.4.49) is non-negative and since $\mathfrak{L}$ satisfies condition (2.2.1), the right hand side of (2.4.49) is non-positive. It follows that

$$\mathbf{v}^0_\pm = \mathbf{0},\ \pi^0_\pm = c^0_\pm \in \mathbb{R} \text{ in } \mathsf{D}_\pm. \tag{2.4.50}$$

Now, in view of relation (2.4.50) and the boundary conditions in (2.4.47), we get $c^0_\pm = 0$ in $\mathsf{D}_\pm$. This shows the uniqueness of the solution of the problem (2.4.3).

Finally, the continuity of the potentials involved in relation (2.4.8) implies the existence of some constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$, such that the solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ of the problem (2.4.3) satisfies (2.4.6).

Case 2: Assume that $\lambda = 1$. In this particular case, the matrix operator A in (2.4.20) becomes

$$\mathsf{A} = \begin{bmatrix} -\mathbb{I} & \mathbf{0} & -\mathcal{V}_{\Gamma_-,\Gamma_+} \\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+} \\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.51}$$

By using similar steps as presented in the case $\lambda \in (0,1)$, we are able to prove that the Robin-transmission problem (2.4.3) admits a unique solution which depends continuously on the given data for $\lambda = 1$. This concludes our proof. □

2.4.1 The Brinkman system and a related Limiting Robin-Transmission problem in the case $\lambda = 0$

In the latter, let Assumption 1.1.7 be satisfied. We dedicate our efforts to the treatment of the Robin-transmission problem of the Brinkman system (2.4.3) in the special case $\lambda = 0$. This particular choice leads to the problem (2.4.52) which contains a special transmission condition on the boundary Γ_+, namely, that it contains just a trace of the unknown velocity $\mathbf{u}_-$ on Γ_+. Hence, we will call problem (2.4.52) the limiting Robin-transmission problem for the Brinkman system. We treat this case separately due to the fact that the Robin-transmission problem (2.4.3) is not the same problem as the limiting Robin-transmission problem (2.4.52). These problems are different because they have different transmission conditions on the interior boundary. The analysis of the Robin-transmission problem for the Brinkman system (2.4.52) is very useful as its well-posedness provides the well-posedness of the Dirichlet-Robin problem for the same system. This analysis comes from the idea to find well-posedness results for Dirichlet, Neumann, and Robin problems, and of their combination, from well-posedness results for transmission problems (see [63]).

We consider now $\lambda = 0$ in our Robin-transmission problem (2.4.3) and we obtain the following limiting transmission problem

$$\begin{cases} \Delta \mathbf{u}_\pm - \alpha \mathbf{u}_\pm - \nabla \pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-\right)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-)\right)|_{\Gamma_+} = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-)\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{2.4.52}$$

where $\alpha > 0$ is a given constant. We aim to determine the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$.

In this special case, we have obtained the following well-posedness result (see also [8, Theorem 2], [56, Theorem 4.1], [63, Theorem 6.1]).

Theorem 2.4.2. *Let $\alpha > 0$ be a given constant. Let Assumption 1.1.7 and Assumption 2.2.1 be satisfied. Then, for all data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, the limiting Poisson problem of Robin-transmission type (2.4.52) has a unique solution*

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}. \tag{2.4.53}$$

In addition, the corresponding solution operator,

$$\mathsf{T}_{lim} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}, \tag{2.4.54}$$

is linear and bounded, and hence, there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$ such that the unique solution of (2.4.52) satisfies

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{2.4.55}$$

2.4.2 The Brinkman system and a related Robin-Dirichlet problem

In this subsection, we aim to emphasize the special role that a transmission-type problem fulfills. In the latter, let $\alpha > 0$ be a given constant and let Assumption 1.1.7 be satisfied. Let us mention that, we will be focusing on the Lipschitz domain D_- and we use similar arguments to those presented in [63, p. 4581]. We point out the fact that the problem (2.4.52) is well-posed, as it was established in Theorem 2.4.2. This means that we get a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$, of the problem (2.4.52). This solution produces a pair $(\mathbf{u}_-, \pi_-) \in H^1_{\text{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-)$ that satisfies another boundary value problem, namely, the following Robin-Dirichlet problem for the Brinkman system

$$\begin{cases} \Delta \mathbf{u}_- - \alpha \mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{u}_- = 0 \text{ in } \mathsf{D}_-, \\ (\text{Tr}_{\mathsf{D}_-} \mathbf{u}_-)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ (\mathbf{t}_{\alpha, \mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-))|_{\Gamma_-} + \mathfrak{L}(\text{Tr}_{\mathsf{D}_-} \mathbf{u})|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-. \end{cases} \tag{2.4.56}$$

In other words, we are able to determine the solution to a boundary value problem (namely, problem (2.4.56)) by extracting it from the solution of a transmission-type problem (namely, problem (2.4.52)). Consequently, the pair $(\mathbf{u}_-, \pi_-)$ is a solution of the Robin-Dirichlet problem (2.4.56).

Moreover, an uniqueness argument, similar to that presented in the proof of Theorem 2.4.2, will lead to the fact that, the Robin-Dirichlet problem for the Brinkman system (2.4.56) is, in turn, well-posed. Under the assumption of Theorem 2.4.2, we obtain the following result (see [8, Corollary 1], [63, p. 4581]).

Corollary 2.4.3. *The Robin-Dirichlet problem for the Brinkman system (2.4.56) has a unique solution* $(\mathbf{u}_-, \pi_-) \in H^1_{\text{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-)$, *for* $n \in \mathbb{N}$, $n \geq 2$.

3

Nonlinear Boundary Value Problems of Transmission-type related to the Navier-Stokes and Darcy-Forchheimer-Brinkman systems

This purpose of this chapter is to treat nonlinear transmission-type problems which contain a generalized version of the Darcy-Forchheimer-Brinkman system or the classical Darcy-Forchheimer-Brinkman system (see relation (3.1.1) in Lipschitz domains in Euclidean setting (see Assumption 1.1.6 and Assumption 1.1.7). All these problems are important for their practical applications (see, e.g., [37], [86]). The content of this chapter follows the results that were obtained in the papers [4], [5], [8].

Let us briefly describe the content of this chapter. We give existence and uniqueness results for the following boundary problems. First of all, we analyze Poisson problem of transmission-type for the generalized Darcy-Forchheimer-Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$. Next, we investigate the Poisson problem of transmission-type for the generalized Darcy-Forchheimer-Brinkman and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$. Lastly, we have the the Poisson problem of Robin-transmission-type for the Darcy-Forchheimer-Brinkman system in Euclidean setting provided by Assumption 1.1.7.

The well-posedness results for the linear problems analyzed in Chapter 2 introduce their solution operators, which are linear and continuous. Taking them into account together with the nonliniarities of the PDEs considered in this chapter (Navier-Stokes equations, Darcy-Forcheimer-Brinkman equations), we reduce the analysis of the boundary value problems for such nonlinear PDEs to the study of certain nonlinear operators and of their fixed points in some special cases. Such nonlinear operators appear from the composition of the linear operators mentioned above and the operators that describe the nonlinearities of the nonlinear PDEs. Their fixed points will provide the solutions of our nonlinear boundary problems (see also [70]).

Let us also take note of some works that concern the investigation of boundary problems which involve nonlinear PDE systems. For example, Choe and Kim [17] have obtained the existence and regularity of solutions for the non-homogeneous Dirichlet problem for the Navier-Stokes system in a bounded Lipschitz domain in $\mathbb{R}^3$, whose boundary data possesses minimal regularity. Kohr, Lanza de Cristoforis and Wendland [55] have obtained an existence and uniqueness result for the Dirchlet problem for the semilinear Darcy-Forchheimer-Brinkman system in a bounded Lipschitz domain in $\mathbb{R}^n$, $n \leq 4$. The authors in [52] have obtained an existence and uniqueness result for a transmission-type problem for the Darcy-Forchheimer-Brinkman and Stokes systems in $\mathbb{R}^3$. Also, in [61], the authors have obtained the existence of solutions of a Dirichlet-transmission problem for the anisotropic Navier-Stokes system in Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$ (see also [14], [68], [71]).

3.1 The generalized Darcy-Forchheimer-Brinkman system and related results

In this section, let us consider $\mathsf{D} \subset \mathbb{R}^3$ a bounded Lipschitz domain, unless specified otherwise. We present a generalized version of the Darcy-Forchheimer-Brinkman system, which is given by

$$\Delta \mathbf{v} - \mathcal{P}\mathbf{v} - k|\mathbf{v}|\mathbf{v} - \beta(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p = \mathbf{g} \text{ in } \mathsf{D}, \ \operatorname{div} \mathbf{v} = 0 \text{ in } \mathsf{D}, \tag{3.1.1}$$

where $\mathcal{P} \in L^\infty(\mathsf{D})^{3\times 3}$ such that condition (1.2.22) holds and $k, \beta : \mathsf{D} \to \mathbb{R}_+$ are given functions, such that $k, \beta \in L^\infty(\mathsf{D}, \mathbb{R}_+)$, i.e., essentially bounded, non-negative functions defined on D (for additional details, see also [41]). We have the following useful remarks.

Remark 3.1.1. *For $\mathcal{P} \equiv \alpha\mathbb{I}$ and $\alpha, k, \beta > 0$ given constants, the system (3.1.1) becomes the classical Darcy-Forchheimer-Brinkman system.*

Remark 3.1.2. *For $\mathcal{P} \equiv 0$, $k = 0$ and for $\beta > 0$ a given constant, the system (3.1.1) becomes the well-known Navier-Stokes system.*

Now, let us state and prove a lemma that we will employ in the proofs of our well-posedness results of this chapter. The lemma reads as follows (see also, [52, Lemma 5.1]).

Lemma 3.1.3. *Let $\mathsf{D} \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain and let $k, \beta : \mathsf{D} \to \mathbb{R}_+$ such that $k, \beta \in L^\infty(\mathsf{D}, \mathbb{R}_+)$. Let*

$$\mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{u}) := \mathring{\mathsf{E}}(k|\mathbf{u}|\mathbf{u} + \beta(\mathbf{u} \cdot \nabla)\mathbf{u}), \tag{3.1.2}$$

where $\mathring{\mathsf{E}}$ is the extension by zero operator outside D. Then, the nonlinear operator

$$\mathsf{J}_{k,\beta,\mathsf{D}} : H^1_{\operatorname{div}}(\mathsf{D})^n \to \tilde{H}^{-1}(\mathsf{D})^n, \tag{3.1.3}$$

is continuous and bounded, in the sense that there exists a constant $c_0 = c_0(\mathsf{D}, k, \beta) > 0$ such that

$$||\mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{u})||_{\tilde{H}^{-1}(\mathsf{D})^n} \leq c_0 ||\mathbf{u}||^2_{H^1(\mathsf{D})^n}. \tag{3.1.4}$$

In addition, the following Lipschitz-like relation

$$||\mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{u}) - \mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{v})|| \leq c_0(||\mathbf{u}||_{H^1(\mathsf{D})^n} + ||\mathbf{v}||_{H^1(\mathsf{D})^n})||\mathbf{u} - \mathbf{v}||_{H^1(\mathsf{D})^n}, \tag{3.1.5}$$

holds, where $c_0 = c_0(\mathsf{D}, k, \beta) > 0$ is the constant that is present in relation (3.1.4).

3.2 Transmission problem for the generalized Darcy-Forchheimer-Brinkman and classical Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$

The purpose of this section is to provide a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.6 for $n = 3$, i.e., complementary Lipschitz domains in $\mathbb{R}^3$. We have considered a generalized version of the Darcy-Forchheimer-Brinkman system in the bounded Lipschitz domain D_+ and the Stokes system in the complementary Lipschitz set D_-. Also let Assumption 2.2.1 be satisfied, for $n = 3$.

Let us recall the space in which we seek our solution, that is,

$$\mathsf{X}_w := H^1_{\rm div}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times \mathcal{H}^1_{\rm div}(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-) \tag{3.2.1}$$

and the space of given data,

$$\mathsf{Y}_w := \tilde{H}^{-1}(\mathsf{D}_+)^3 \times \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3 \times H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3. \tag{3.2.2}$$

We study the following transmission problem of Poisson type for the generalized Darcy-Forchheimer-Brinkman and Stokes systems,

$$\begin{cases} \Delta \mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - k|\mathbf{u}_+|\mathbf{u}_+ - \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+ - \nabla\pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+,\\ \Delta \mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-,\\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm,\\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma,\\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+,\pi_+,\mathbf{f}_+ + \mathring{\mathsf{E}}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+))\\ \quad - \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-,\pi_-,\mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{3.2.3}$$

and we aim to determine the unknown fields $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-) \in \mathsf{X}_w$. Once again, since the Stokes system appears in the unbounded Lipschitz domain D_-, we must work with the weighted space $\mathcal{H}^1_{\rm div}(\mathsf{D}_-)^3$, which is included in the solution space X_w.

The following result regarding the well-posedness of the transmission problem (3.2.3) was obtained, for $\mathbf{u}_\infty \in \mathbb{R}^3$ a given constant (see [5, Theorem 3.3] see also [52, Theorem 5.2] in the case $k,\beta > 0$, $\mathcal{P} \equiv \alpha\mathbb{I}$, where $\alpha > 0$ is a constant).

Theorem 3.2.1. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times3}$ such that condition (1.2.22) holds. Let $\mathbf{u}_\infty \in \mathbb{R}^3$ be a constant vector. Then, there exist two constants*

$$\xi = \xi(\mathsf{D}_+,\mathsf{D}_-,\mathcal{P},k,\beta,\mathfrak{L}) > 0, \quad \eta = \eta(\mathsf{D}_+,\mathsf{D}_-,\mathcal{P},k,\beta,\mathfrak{L}) > 0, \tag{3.2.4}$$

such that for all given $(\mathbf{f}_+,\mathbf{f}_-,\mathbf{g},\mathbf{h},\mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$ that satisfy

$$||(\mathbf{f}_+,\mathbf{f}_-,\mathbf{g},\mathbf{h},\mathbf{u}_\infty)||_{\mathsf{Y}_w\times\mathbb{R}^3} \le \xi, \tag{3.2.5}$$

the Poisson problem of transmission-type for the Darcy-Forchheimer-Brinkman and Stokes systems (3.2.3) has a unique solution $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-) \in \mathsf{X}_w$ and

$$||(\mathbf{u}_+,\pi_+,\mathbf{u}_- - \mathbf{u}_\infty,\pi_-)||_{\mathsf{X}_w} \le \eta. \tag{3.2.6}$$

In addition, the solution depends continuously on the given data and satisfies the following estimate

$$||(\mathbf{u}_+,\pi_+,\mathbf{u}_- - \mathbf{u}_\infty,\pi_-)||_{\mathsf{X}_w} \le C_0||(\mathbf{f}_+,\mathbf{f}_-,\mathbf{g},\mathbf{h},\mathbf{u}_\infty)||_{\mathsf{Y}_w\times\mathbb{R}^3}, \tag{3.2.7}$$

where $C_0 = C_0(\mathsf{D}_+,\mathsf{D}_-,\mathcal{P},\mathfrak{L}) > 0$ is a constant and $\mathbf{u}_- - \mathbf{u}_\infty$ vanishes at infinity in the sense of Leray.

We end this section by stating some important remarks which show the particular situations that are also treated by Theorem 3.2.1.

Remark 3.2.2. *In the case $k = 0$ and $\beta : \mathsf{D}_+ \to \mathbb{R}_+$ such that $\beta \in L^\infty(\mathsf{D}_+,\mathbb{R}_+)$, Theorem 3.2.1 gives a well-posedness result for the nonlinear transmission problem for the generalized Navier-Stokes and Stokes systems.*

Remark 3.2.3. *In the case $k : \mathsf{D}_+ \to \mathbb{R}_+$ such that $k \in L^\infty(\mathsf{D}_+,\mathbb{R}_+)$ and $\beta = 0$, Theorem 3.2.1 gives a well-posedness result for a semilinear transmission problem for a semilinear Darcy-Forchheimer-Brinkman system and Stokes system.*

3.3 Transmission problem for the generalized Darcy-Forchheimer-Brinkman and classical Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$

In this section, our goal is to provide a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.6 for $n = 3$, i.e., complementary Lipschitz domains in $\mathbb{R}^3$. We have considered a generalized version of the Darcy-Forchheimer-Brinkman system in the bounded Lipschitz domain D_+ and the Brinkman system in the complementary Lipschitz set D_-. Also, let Assumption 2.2.1 be satisfied, for $n = 3$.

Let us recall the space

$$\mathsf{X}_{\mathcal{B}} := H^1_{\text{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times H^1_{\text{div}}(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-) \tag{3.3.1}$$

that is, the space in which we seek our solution and

$$\mathsf{Y}_{\mathcal{B}} := \tilde{H}^{-1}(\mathsf{D}_+)^3 \times \tilde{H}^{-1}(\mathsf{D}_-)^3 \times H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3, \tag{3.3.2}$$

the space of given data. Note that $\mathfrak{M}(\mathsf{D}_-)$ is the space provided by Definition 2.1.1.

Since we are dealing with the Brinkman system in the exterior Lipschitz domain D_- (see Assumption 1.1.6 in the case $n = 3$), it follows that we are able to use the classical Sobolev space $H^1_{\text{div}}(\mathsf{D}_-)^3$, instead of the weighted Sobolev space $\mathcal{H}^1_{\text{div}}(\mathsf{D}_-)^3$, as the space in which we seek the velocity field in D_-. This is due to the behavior of the fundamental solution of the Brinkman system at infinity, in the case $n = 3$.

Now, we consider the transmission problem for the generalized Darcy-Forchheimer-Brinkman and classical Brinkman systems, which is given by

$$\begin{cases} \Delta \mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - k|\mathbf{u}_+|\mathbf{u}_+ - \beta(\mathbf{u}_+ \cdot \nabla)\mathbf{u}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{u}_- - \alpha \mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \text{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \text{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathring{\mathsf{E}}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+ \cdot \nabla)\mathbf{u}_+)) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) \\ \quad + \boldsymbol{\mathfrak{L}}\text{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{3.3.3}$$

in the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$.

The well-posedness result that we have obtained is as follows (see e.g., [4, Theorem 3.2], and [52, Theorem 5.2] in the case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha, k, \beta > 0$ are constants).

Theorem 3.3.1. *Let $\alpha > 0$ be a given constant. Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ such that condition (1.2.22) holds. Then, there exist two constants,*

$$\xi = \xi(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, k, \beta, \boldsymbol{\mathfrak{L}}) > 0 \quad \eta = \eta(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, k, \beta, \boldsymbol{\mathfrak{L}}) > 0 \tag{3.3.4}$$

such that, for all given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$ that satisfy the condition

$$||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}} \le \xi, \tag{3.3.5}$$

the Poisson problem of transmission-type for the generalized Darcy-Forchheimer-Brinkman and Stokes systems (3.3.3) has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$ such that

$$||\mathbf{u}_+||_{H^1_{\text{div}}(\mathsf{D}_+)^3} \le \eta. \tag{3.3.6}$$

In addition, the solution depends continuously on the given data, which means that there exists a given constant $C_0 = C_0(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \boldsymbol{\mathfrak{L}}) > 0$ *such that*

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{\mathcal{B}}} \leq C_0||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}. \tag{3.3.7}$$

We end this section by stating some useful remarks that are derived from our well-posedness result, that is, Theorem 3.3.1.

Remark 3.3.2. *If* $k = 0$ *and* $\beta : \mathsf{D}_+ \to \mathbb{R}_+$ *such that* $\beta \in L^\infty(\mathsf{D}_+, \mathbb{R}_+)$ *in Theorem* 3.3.1, *then we get the well-posedness result for the nonlinear transmission problem for the generalized Navier-Stokes and Brinkman systems in complementary Lipschitz domains in* $\mathbb{R}^3$.

Remark 3.3.3. *If* $k : \mathsf{D}_+ \to \mathbb{R}_+$ *such that* $k \in L^\infty(\mathsf{D}_+, \mathbb{R}_+)$ *and* $\beta = 0$ *in Theorem* 3.3.1, *then we get the well-posedness result for a semilinear transmission problem for a semilinear Darcy-Forchheimer-Brinkman system and the Brinkman system in complementary Lipschitz domains in* $\mathbb{R}^3$.

3.4 On a Robin-Transmission problem for the Darcy-Forchheimer-Brinkman system

In this section, we give an existence and uniqueness result for a transmission-type problem, which was obtained in the setting of Assumption 1.1.7. This particular transmission-type problem that we study will be called the Robin-transmission problem for the Darcy-Forchheimer-Brinkman system (see problem (3.4.3)). In addition, let $\lambda \in (0, 1]$ be a constant and let Assumption 2.2.1 be satisfied, for $n = 2, 3$.

Let us recall the space in which we seek our solution,

$$\mathsf{X}_{RT} := H^1_{\text{div}}(\mathsf{D}_+)^n \times L^2(\mathsf{D}_+) \times H^1_{\text{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-), \tag{3.4.1}$$

and the space of given data,

$$\mathsf{Y}_{RT} := \tilde{H}^{-1}(\mathsf{D}_+)^n \times \tilde{H}^{-1}(\mathsf{D}_-)^n \times H^{\frac{1}{2}}_{\nu}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_-)^n. \tag{3.4.2}$$

The Robin-transmission problem for the Darcy-Forchheimer-Brinkman system is given by

$$\begin{cases} \Delta\mathbf{u}_\pm - \alpha\mathbf{u}_\pm - k|\mathbf{u}_\pm|\mathbf{u}_\pm - \beta(\mathbf{u}_\pm\cdot\nabla)\mathbf{u}_\pm - \nabla\pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda\left(\text{Tr}_{\mathsf{D}_+}\mathbf{u}_+\right) - \left(\text{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathring{\mathsf{E}}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+)) \\ \quad - \Big(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-))\Big)|_{\Gamma_+} = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \Big(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-))\Big)|_{\Gamma_-} \\ \quad + \boldsymbol{\mathfrak{L}}\left(\text{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{3.4.3}$$

in the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{XT}$. Note that $\mathring{\mathsf{E}}_\pm$ is the extension by zero-operator outside $\overline{\mathsf{D}}_\pm$.

We have obtained the following well-posedness result (see also, [52, Theorem 5.2]).

Theorem 3.4.1. *Let $\alpha > 0$, $k, \beta \in \mathbb{R}^*$ and $\lambda \in (0,1]$ be given constants. Let Assumption* 1.1.7 *and Assumption 2.2.1 be satisfied, for $n = 2, 3$. Then, there exist two constants,*

$$\xi \equiv \xi(\mathsf{D}_+, \mathsf{D}_-, \alpha, k, \beta, \lambda, \mathfrak{L}) > 0, \quad \eta \equiv \eta(\mathsf{D}_+, \mathsf{D}_-, \alpha, k, \beta, \lambda, \mathfrak{L}) > 0, \tag{3.4.4}$$

such that, for every $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, which satisfies the condition

$$||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} \leq \xi, \tag{3.4.5}$$

the Poisson problem of Robin-transmission type (3.4.3) for the Darcy-Forchheimer-Brinkman system has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ with the property

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq \eta. \tag{3.4.6}$$

Moreover, there exists a constant $C_0 \equiv C_0(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$ such that the unique solution satisfies

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq C_0 ||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{h}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{3.4.7}$$

Proof. We prove this result by employing similar arguments to those presented in the proof of [52, Theorem 5.2]. We divide our arguments into three steps.

Step 1. We will show that *a solution of the problem* (3.4.3) *exists.* We rewrite the nonlinear transmission problem (3.4.3) as

$$\begin{cases} \Delta \mathbf{u}_\pm - \alpha \mathbf{u}_\pm - \nabla \pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_\pm)|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda \left(\mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}_+\right) - \left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_+)) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-. \end{cases} \tag{3.4.8}$$

Next, we aim to construct a nonlinear operator H that maps a closed ball B_η of the space $H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$ into itself, and also is a contraction on B_η. Hence, the unique fixed point of H will provide a solution of the problem (3.4.8).

Let us construct our nonlinear operator in the following way. Recall that the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$ which appears in (3.4.8) is fixed. In addition, we fix

$$(\mathbf{u}_+, \mathbf{u}_-) \in H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n. \tag{3.4.9}$$

Let us consider the following linear Poisson problem of transmission type for the Brinkman system in the unknowns $(\mathbf{u}^0_+, \pi^0_+, \mathbf{u}^0_-, \pi^0_-)$

$$\begin{cases} \Delta \mathbf{u}^0_\pm - \alpha \mathbf{u}^0_\pm - \nabla \pi^0_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_\pm)|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}^0_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda \left(\mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}^0_+\right) - \left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}^0_-\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+, \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_+)) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}^0_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-. \end{cases} \tag{3.4.10}$$

In addition, the membership $\mathring{\mathsf{E}}(k|\mathbf{u}_\pm|\mathbf{u}_\pm + \beta(\mathbf{u}_\pm \cdot \nabla)\mathbf{u}_\pm) \in \tilde{H}^{-1}(\mathsf{D}_\pm)^n$ holds in view of Lemma 3.1.3.

Let us apply Theorem 2.4.1. Consequently we deduce that the transmission problem (3.4.10) has a unique solution

$$\begin{aligned}(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0) &:= \mathsf{T}_{RT}(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-} + \mathsf{J}_{k,\beta,\mathsf{D}_-}(\mathbf{u}_-)|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{X}_{RT} \\ &= (\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-)).\end{aligned} \tag{3.4.11}$$

Let us note that, the operator $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ which is involved in relation (3.4.11) is the solution operator given by relation (2.4.5). Let us recall that $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ is the well-defined, linear and continuous operator, which maps the given data (belonging to the space Y_{RT}) to the unique solution of the Poisson problem of Robin-transmission type (2.4.3) for the Brinkman system in the setting of Assumption 1.1.7, for $n = 2, 3$. Also, $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ satisfies the estimate (2.4.6) of Theorem 2.4.1.

Furthermore, by Lemma 3.1.3 and Theorem 2.4.1 and for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, the nonlinear operators given by relation (3.4.11),

$$(\mathsf{U}_+, \mathsf{R}_+, \mathsf{U}_-, \mathsf{R}_-) : H^1_{\rm div}(\mathsf{D}_+)^n \times H^1_{\rm div}(\mathsf{D}_-)^n \to \mathsf{X}_{RT}, \tag{3.4.12}$$

are continuous and there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \alpha, \lambda, \mathfrak{L}) > 0$ such that

$$\begin{aligned}&||(\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-))||_{\mathsf{X}_{RT}} \\ &\leq C||(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)|_{\mathsf{D}_+}, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_-}(\mathbf{u}_-)|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} \\ &\leq C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} + ||\mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)||_{\tilde{H}^{-1}(\mathsf{D}_+)^n} + ||\mathsf{J}_{k,\beta,\mathsf{D}_-}(\mathbf{u}_-)||_{\tilde{H}^{-1}(\mathsf{D}_-)^n} \\ &\leq C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} + c_1^+ C||\mathbf{u}_+||^2_{H^1_{\rm div}(\mathsf{D}_+)^n} + c_1^- C||\mathbf{u}_-||^2_{H^1_{\rm div}(\mathsf{D}_-)^n},\end{aligned} \tag{3.4.13}$$

for all $(\mathbf{u}_+, \mathbf{u}_-) \in H^1_{\rm div}(\mathsf{D}_+)^n \times H^1_{\rm div}(\mathsf{D}_-)^n$, where c_1^+ and c_1^- are the constants provided by Lemma 3.1.3, corresponding to D_+ and D_-, respectively.

By taking into account (3.4.10), we have

$$\begin{cases}\Delta \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) - \alpha \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) - \nabla \mathsf{R}_\pm(\mathbf{u}_+, \mathbf{u}_-) \\ \quad = \mathbf{f}_\pm|_{\mathsf{D}_\pm} + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_\pm)|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda\left(\mathrm{Tr}_{\mathsf{D}_+}\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-)\right) - \left(\mathrm{Tr}_{\mathsf{D}_-}\mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_+)) \\ \quad - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_-} \\ \quad + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-.\end{cases} \tag{3.4.14}$$

Let us introduce the nonlinear operator

$$\mathsf{H} : H^1_{\rm div}(\mathsf{D}_+)^n \times H^1_{\rm div}(\mathsf{D}_-)^n \to H^1_{\rm div}(\mathsf{D}_+)^n \times H^1_{\rm div}(\mathsf{D}_-)^n$$

by

$$\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) := (\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)). \tag{3.4.15}$$

Now, if we prove that the nonlinear operator H possesses a fixed point $(\mathbf{u}_+.\mathbf{u}_-) \in H^1_{\rm div}(\mathsf{D}_+)^n \times H^1_{\rm div}(\mathsf{D}_-)^n$, this fixed point will solve the equation $\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) = (\mathbf{u}_+, \mathbf{u}_-)$ and together with $\pi_\pm = \mathsf{R}_\pm(\mathbf{u}_+, \mathbf{u}_-)$ provides a solution of the problem (3.4.8) in X_{RT}.

In order to justify our claim, we show that H maps a closed ball $\mathsf{B}_\eta \subseteq H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$ to itself and also is a contraction on the ball B_η.

Let us introduce the constants

$$\xi := \frac{3}{16C^2 \max\{c_1^+, c_1^-\}} > 0, \quad \eta := \frac{1}{4C \max\{c_1^+, c_1^-\}} > 0, \tag{3.4.16}$$

and the closed ball

$$\mathsf{B}_\eta := \{(\mathbf{u}_+, \mathbf{u}_-) \in H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n : ||(\mathbf{u}_+, \mathbf{u}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \le \eta\}, \tag{3.4.17}$$

while the constants c_1^+ and c_1^+ are the same constants that appear in relation (3.4.13). In addition, we assume that the given data satisfies

$$||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} \le \xi. \tag{3.4.18}$$

In view of relations (3.4.13), (3.4.16), (3.4.17), (3.4.18), we get

$$||(\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-))||_{\mathsf{X}_{RT}} \le \eta, \tag{3.4.19}$$

for all $(\mathbf{u}_+, \mathbf{u}_-) \in \mathsf{B}_\eta$, which shows that $||\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \le \eta$. Consequently H maps B_η to B_η.

Let us prove that H is a contraction on B_η. To achieve this, let us fix the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$. If $(\mathbf{v}_+, \mathbf{v}_-), (\mathbf{w}_+, \mathbf{w}_-) \in \mathsf{B}_\eta$ are arbitrary fields, we obtain

$$\begin{aligned}
&||\mathsf{H}(\mathbf{v}_+, \mathbf{v}_-) - \mathsf{H}(\mathbf{w}_+, \mathbf{w}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \\
&\le C||(\mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+) - \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{w}_+), \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_-) - \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{w}_-))||_{\tilde{H}^{-1}(\mathsf{D}_+)^n \times \tilde{H}^{-1}(\mathsf{D}_-)^n} \\
&\le Cc_1^+(||\mathbf{v}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + ||\mathbf{w}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n})||\mathbf{v}_+ - \mathbf{w}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} \\
&\qquad + Cc_1^-(||\mathbf{v}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n} + ||\mathbf{w}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n})||\mathbf{v}_- - \mathbf{w}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \\
&\le 2\eta C \max\{c_1^+, c_1^-\}||(\mathbf{v}_+ - \mathbf{w}_+, \mathbf{v}_- - \mathbf{w}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \\
&= \frac{1}{2}||(\mathbf{v}_+ - \mathbf{w}_+, \mathbf{v}_- - \mathbf{w}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n}.
\end{aligned} \tag{3.4.20}$$

In (3.4.20) we have used the linearity and continuity of the operator $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ (see relation (2.4.5)) together with relation (3.1.5) of Lemma 3.1.3. Hence we have that the operator $\mathsf{H} : \mathsf{B}_\eta \to \mathsf{B}_\eta$ is a $\frac{1}{2}$-contraction.

Due to Banach's fixed point theorem we get the existence of a unique fixed point $(\mathbf{u}_+, \mathbf{u}_-) \in \mathsf{B}_\eta$ of the operator H, namely, $\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) = (\mathbf{u}_+, \mathbf{u}_-)$. The pair $(\mathbf{u}_+, \mathbf{u}_-)$ together with the functions $\pi_\pm = \mathsf{R}_\pm(\mathbf{u}_+, \mathbf{u}_-)$ given by (3.4.11), determine a solution of the nonlinear problem (3.4.8) in the space X_{RT}. Hence, $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)$ is a solution of the nonlinear transmission problem (3.4.3) in X_{RT}.

In view of the membership $(\mathbf{u}_+, \mathbf{u}_-) \in \mathsf{B}_\eta$, we get

$$Cc_1^+||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} \le Cc_1^+\eta \le \frac{1}{4}, \quad Cc_1^-||\mathbf{u}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} \le Cc_1^-\eta \le \frac{1}{4}. \tag{3.4.21}$$

Then, we apply inequality (3.4.13) to obtain

$$\begin{aligned}
&||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + ||\pi_+||_{L^2(\mathsf{D}_+)} + ||\mathbf{u}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n} + ||\pi_-||_{L^2(\mathsf{D}_-)} = ||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \\
&\le C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} + \frac{1}{4}||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + \frac{1}{4}||\mathbf{u}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n},
\end{aligned} \tag{3.4.22}$$

hence

$$||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + ||\mathbf{u}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \leq \frac{4}{3}C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{3.4.23}$$

By substituting relation (3.4.23) into relation (3.4.22), we get the desired estimate (3.4.7) with $C_0 = \frac{4}{3}C$.

Step 2. We want to show the *uniqueness property of the solution of the nonlinear transmssion problem* (3.4.3). The Banach fixed point theorem implies the uniqueness property of the solution of problem (3.4.3) inside the ball B_η. Since the arguments that are involved in the proof of this step are similar to those in the proof of Theorem 3.2.1, we omit them for the sake of brevity.

Step 3. It remains to show that *the solution of our problem* (3.4.3) *depends continuously on the given data.* To this end, the continuity of the nonlinear operator $\mathsf{H} : \mathsf{B}_\eta \to \mathsf{B}_\eta$ and the continuity of the solution operator $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ (see relation (2.4.5)) show that the unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ depends continuously on the given data and the estimate (3.4.7) holds with the choice of constant $C_0 = \frac{4}{3}C$. This concludes our proof. □

3.4.1 The Darcy-Forchheimer-Brinkman system and a related Limiting Robin-Transmission Problem in the case $\lambda = 0$

In this subsection, we will work in the setting of Assumption 1.1.7. We wish to discuss a special Robin-transmission problem of the Darcy-Forchheimer-Brinkman system. This new transmission-type problem is obtained by choosing $\lambda = 0$ in the transmission problem (3.4.3). Consequently, we get the problem (3.4.24) which includes a particular transmission condition on the boundary Γ_+, that is, it contains just a trace of the unknown velocity $\mathbf{u}_-$ on Γ_+. Due to this fact, problem (3.4.24) will be called the limiting Robin-transmission problem for the Darcy-Forchheimer-Brinkman system. Note that, this limiting Robin-transmission problem contains a Robin-Dirichlet boundary value problem for the Darcy-Forchheimer-Brinkman system in D_-. Our purpose is to state the well-posedness of the limiting Robin-transmission problem for the Darcy-Forchheimer-Brinkman system and, as a consequence, obtain a well-posedness result for the Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system. Equivalently, we isolate the solution of the Robin-Dirichlet problem from the solution of the limiting Robin-transmission problem. This original method emphasizes the fact that the solutions of certain boundary value problems can be determined by considering, first of all, particular transmission problems.

Let us consider $\lambda = 0$ in the Robin-transmission problem for the Darcy-Forchheimer-Brinkman system (3.4.3). We get the following limiting Robin-transmission problem for the Darcy-Forchheimer-Brinkman system,

$$\begin{cases} \Delta\mathbf{u}_\pm - \alpha\mathbf{u}_\pm - k|\mathbf{u}_\pm|\mathbf{u}_\pm - \beta(\mathbf{u}_\pm\cdot\nabla)\mathbf{u}_\pm - \nabla\pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \operatorname{div}\mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathring{\mathsf{E}}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+)) \\ \quad - \Big(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-))\Big)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \Big(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-))\Big)|_{\Gamma_-} \\ \quad + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{3.4.24}$$

in the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{XT}$. Note that, $\mathring{\mathsf{E}}_\pm$ is (the extension by zero)-operator outside $\overline{\mathsf{D}}_\pm$. In addition, let Assumption 2.2.1 be fulfilled, for $n = 2, 3$.

The well-posedness result that was obtained is as follows (see, e.g., [52, Theorem 5.2]).

Theorem 3.4.2. *Let $\alpha > 0$, $k, \beta \in \mathbb{R}^*$ be given constants. Let Assumption 1.1.7 and Assumption 2.2.1 be satisfied for $n = 2, 3$.s Then, there exist two constants,*

$$\xi \equiv \xi(\mathsf{D}_+, \mathsf{D}_-, \alpha, k, \beta, \mathfrak{L}) > 0, \quad \eta \equiv \eta(\mathsf{D}_+, \mathsf{D}_-, \alpha, k, \beta, \mathfrak{L}) > 0, \tag{3.4.25}$$

such that, for every $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, which satisfies the condition

$$||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} \leq \xi, \tag{3.4.26}$$

the limiting Poisson problem of Robin-transmission type (3.4.24) for the Darcy-Forchheimer-Brinkman system has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ with the property

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq \eta. \tag{3.4.27}$$

Moreover, there exists a constant $C_0 \equiv C_0(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$ such that the unique solution satisfies the estimate

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq C_0 ||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{3.4.28}$$

3.4.2 The Darcy-Forchheimer-Brinkman system and a related Robin-Dirichlet problem

The goal of this subsection is to highlight the particular role that a transmission-type problem satisfies. In the latter, let $\alpha, k, \beta > 0$ be given constants and let Assumption 1.1.7 be satisfied. Note that, we consider the Lipschitz domain D_- and we use similar arguments as those described in [63, p. 4581]. Let us proceed by stating the fact that the problem (3.4.24) is well-posed (see Theorem 3.4.2). Consequently, we obtain a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ of the problem (3.4.24). From it, we extract the pair $(\mathbf{u}_-, \pi_-) \in H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-)$ and we note that this particular pair satisfies another boundary value problem, namely, the following Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system in D_-. This boundary value problem is given by

$$\begin{cases} \Delta \mathbf{u}_- - \alpha \mathbf{u}_- - k|\mathbf{u}_-|\mathbf{u}_- - \beta(\mathbf{u}_- \cdot \nabla)\mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{u}_- = 0 \text{ in } \mathsf{D}_-, \\ (\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ (\mathbf{t}_{\alpha, \mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_- \cdot \nabla)\mathbf{u}_-)))|_{\Gamma_-} + \mathfrak{L}(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u})|_{\Gamma_-} = \mathbf{g}_2, \text{ on } \Gamma_-. \end{cases} \tag{3.4.29}$$

To summarize, we can obtain the solution for a boundary value problem (that is, problem (3.4.29)) by extracting it from the solution of a transmission-type problem (that is, problem (3.4.24)). It follows that the pair $(\mathbf{u}_-, \pi_-)$ is a solution of the Robin-Dirichlet problem (3.4.29) for the Darcy-Forchheimer-Brinkman system.

Let $(\mathbf{f}_-, \mathbf{g}_1, \mathbf{g}_2) \in \tilde{H}^{-1}(\mathsf{D}_-)^n \times H^{\frac{1}{2}}_\nu(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_-)^n$ satisfying conditon (3.4.26) of Theorem 3.4.2. Then, we have the following consequence (see [63, p. 4581]).

Corollary 3.4.3. *The Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system (3.4.29) has a solution in the space $H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-)$, where $n = 2, 3$.*

4

A numerical approach related to the Darcy-Forchheimer-Brinkman system with Robin-Dirichlet conditions

The aim of this chapter is to study, numerically, the Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system, namely problem (3.4.29). In addition, we have an existence result for the problem (3.4.29), which is Corollary 3.4.3. We solve numerically a lid-driven cavity problem. This problem consists of a square cavity which contains a solid square. Consequently, we have an interior boundary (that is, the boundary of the internal solid square) and an exterior boundary (that is, the exterior walls of the cavity). The interior walls are considered to be fixed. The exterior walls slide at different constant velocities. In addition, the domain contained between the exterior and the interior boundary is filled with a porous media and is saturated by a viscous Newtonian incompressible fluid, which is modelled by the Darcy-Forchheimer-Brinkman system (see Relation (4.1.1)). The geometry is given in Figure 4.1. The content of this chapter follows the results that were obtained in the paper [8].

We note that our previous approaches in Chapter 2 and Chapter 3 have focused on obtaining a unique solution for our transmission-type problems. Indeed, we have used layer potential theory to construct a solution in the linear problems. We have also used the Banach fixed point Theorem in order to get a solution in the non-linear setting. In addition, we have seen that we may obtain a solution to other boundary value problems by extracting it from a transmission problem. We present another approach to finding a solution for a boundary value problem which is rooted in some devices that stem from Numerical Analysis.

In the latter, we take note of some past works that concern the lid-driven cavity flow problem. Firstly, let us emphasize the contribution of Ghia, Ghia and Shin [39]. The authors have obtained numerical results for a driven flow in a square cavity. These results provide a useful test case by which other numerical methods can be checked against. In [69], the authors note that the lid-driven cavity flow problem is a test problem, in two or three dimensions, through which diverse numerical schemes can be validated or invalidated. The attractiveness of such a problem consists of its simple geometry and its perceived flow structure. Gutt and Groşan [44] have investigated numerically a mixed Dirichlet-Robin boundary problem for the Darcy-Brinkman system in the setting of the lid-driven porous cavity problem. They also analyze the influence of various parameters on the fluid flow.

4.1 Numerical study of the lid-driven cavity flow problem in a 2-dimensional cavity with Navier slip boundary condition in the presence of a solid body

4.1.1 Statement of the problem and remarks

Let us describe the mathematical model of our lid-driven problem in a two-dimensional cavity with Navier slip boundary condition, in the presence of a solid body. Our goal is to study the flow of a viscous Newtonian incompressible fluid in a porous medium in a special Lipschitz domain denoted by D_-, as seen in Figure 4.1, while we consider Dirichlet boundary condition on the interior boundary and Robin boundary conditions on the exterior boundary. Let us describe the geometry of our problem. We consider $\mathsf{D} \subset \mathbb{R}^2$ a square cavity of length L which contains a solid square obstacle, denoted by D_+, of length l such that $l < L$. Let us define $\mathsf{D}_- := \mathsf{D} \setminus \mathsf{D}_+$. The interior boundary, denoted by Γ_+ is considered to be fixed, while the exterior boundary Γ_-, consist of four walls Γ_-^t, Γ_-^l, Γ_-^b, Γ_-^r which are sliding at different constant velocities (see Figure 4.1).

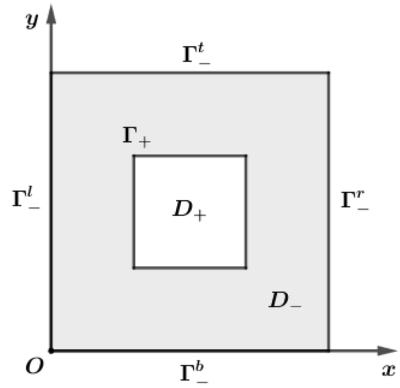


Figure 4.1: The porous cavity with internal square block

4.1.2 Mathematical model of the problem

Inside the porous cavity, i.e., Fig 4.1, the fluid flow is described by the Darcy-Forchheimer-Brinkman system (see, e.g., [3], [43], [109]). On the exterior boundary Γ_-, we impose the Navier-slip condition which is a Robin type boundary condition (see, [48], [92]) and on the interior boundary Γ_+, we impose the Dirichlet boundary condition. The mathematical model for our problem is

$$\begin{cases} \Delta \mathbf{u}_- - \dfrac{\kappa}{K}\mathbf{u}_- - \dfrac{\kappa}{\nu\rho}\nabla\pi_- = \dfrac{1}{\nu}(\mathbf{u}_- \cdot \nabla)\dfrac{\mathbf{u}_-}{\kappa} + \dfrac{\kappa C_f}{\nu\sqrt{K}}|\mathbf{u}_-|\mathbf{u}_- & \text{in } \mathsf{D}_- \\ \operatorname{div} \mathbf{u}_- = 0 \quad \text{in } \mathsf{D}_- \\ \mathbf{u}_- = \mathbf{g}_1 \quad \text{on } \Gamma_+ \\ \mathbf{u}_- + s_l \dfrac{\partial \mathbf{u}_-}{\partial \mathbf{n}_-} = \mathbf{g}_2 \quad \text{on } \Gamma_-. \end{cases} \tag{4.1.1}$$

Now, in order to conduct the non-dimensional analysis, let us replace the dimensional variables in (4.1.1) with the dimensionless variables

$$X = \frac{x}{L}, \qquad Y = \frac{y}{L}, \qquad S_l = \frac{s_l}{L}, \qquad U_x = \frac{u_x}{u^t}, \qquad U_y = \frac{u_y}{u^t}, \qquad \Pi = \frac{\pi}{\rho(u^t)^2}.$$

. Hence, we obtain

$$\begin{cases} \Delta \mathbf{U}_- - \dfrac{\kappa}{Da}\mathbf{U}_- - Re\,\kappa \nabla \Pi = Re\,(\mathbf{U}_- \cdot \nabla)\dfrac{\mathbf{U}_-}{\kappa} + \dfrac{Re\,\kappa\, C_f}{\sqrt{Da}}|\mathbf{U}_-|\mathbf{U}_- & \text{in } \mathsf{D}_- \\ \operatorname{div} \mathbf{U}_- = 0 & \text{in } \mathsf{D}_- \\ \mathbf{U}_- = (0,0) & \text{on } \Gamma_+ \\ \mathbf{U}_- + S_l \dfrac{\partial \mathbf{U}_-}{\partial \mathbf{n}_-} = \mathbf{G}_2 & \text{on } \Gamma_- \end{cases} \tag{4.1.2}$$

and

$$\mathbf{G}_2 = \begin{cases} (1,0) & \text{on } \Gamma_-^t \\ (0,U^r) & \text{on } \Gamma_-^r \\ (U^b,0) & \text{on } \Gamma_-^b \\ (0,U^l) & \text{on } \Gamma_-^l. \end{cases} \tag{4.1.3}$$

In our analysis, we also consider the stream function Ψ which is given by

$$U_x = \frac{\partial \Psi}{\partial Y},\; U_y = -\frac{\partial \Psi}{\partial X}. \tag{4.1.4}$$

We use this function to compute the maximum stream function value reached inside the cavity, Ψ_{max}. Also, we use the stream function Ψ in order to visualize the fluid flow pattern, which is observed in the form of the stream lines.

4.1.3 Numerical method and validation of the model

We use the finite element based software COMSOL Multiphysics (see [110]) in order to solve the system (4.1.2) together with the equation

$$\Delta \Psi = \frac{\partial U_x}{\partial Y} - \frac{\partial U_y}{\partial X}, \tag{4.1.5}$$

Note that equation (4.1.5) is derived from relation (4.1.4).

In order to discretize the domain in Figure 4.1, we consider a free quad mesh. The mesh was constructed as follows. We starting with a fixed number of elements, N, which established on either side of Γ_-. On the side of the Γ_+ we have $N\dfrac{L}{l}$ elements. The maximum size of an element inside the cavity is set to $\dfrac{1}{N}$. To get a numerical solution, the nonlinear solver iterates until the relative error is less than $\epsilon = 10^{-6}$.

Next, we perform a convergence test for the maximum value of the stream function, Ψ_{max}, depending on the refinement level of the mesh. Then, for our problem (4.1.2) together with (4.1.5) we have the following default settings

$$L = 1,\; l = 0.4,\; \kappa = 0.3,\; Re = 100,\; Da = 0.01,\; U^r = U^b = -0.1,\; U^l = 0.1. \tag{4.1.6}$$

In view of (4.1.6) we have obtained Table 4.1, which contains the computed values of Ψ_{max} for different values N. From Table 4.1 we determine that the choice of the mesh containing 80 elements on each side of Γ_- of in Figure 4.1 is appropriate for our simulations.

N (elements on exterior side)	Ψ_{max}	**Error**$_{\Psi_{max}}$
20	0.04366673	
40	0.04358508	0.000081465
60	0.04358398	0.0000011
80	0.04358332	0.00000066

Table 4.1: Mesh dependence

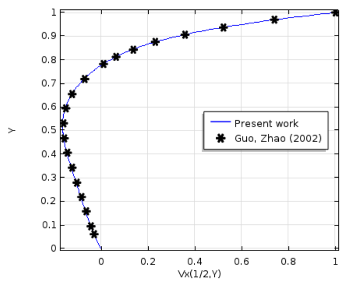
Now we compare our numerical solutions with previous established results in order to validate our approach. To this end, we have the following settings

$$U^r = U^b = U^l = 0,\ \mu = 1,\ l = 0,\ S_l = 0, \tag{4.1.7}$$

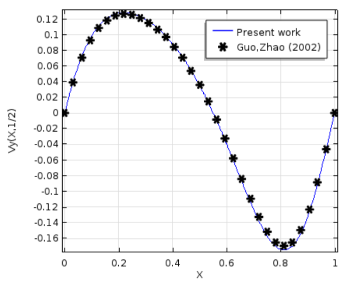
which is the case of the porous lid-driven square cavity problem with vanishing obstacle and no-slip boundary condition. Next, for the values

$$\kappa = 0.1,\ Re = 10,\ Da = 0.01, \tag{4.1.8}$$

we plot the x component of the velocity, U_x, along the vertical line through the cavity center and the y component of the velocity, U_y, along the horizontal line through the cavity center. We compare the obtained velocity profiles that we determined with the data obtained in [43]. Both graphs in Figure 4.2 show a good agreement.



(a) U_x along vertical centerline



(b) U_y along horizontal centerline

Figure 4.2: The components of the velocity along vertical and horizontal center-lines of the squared cavity, compared with [43].

4.1.4 Results and discussion

We aim to determine the impact of the dimensionless slip length, S_l, on the fluid flow inside the porous cavity. To this aim, we set the parameters

$$l = 0.4,\ \kappa = 0.3,\ Re = 100,\ Da = 0.01,\ U^r = U^b = -0.1,\ U^l = 0.1 \tag{4.1.9}$$

and we study the flow properties for $S_l \in (0, 0.003)$.

The computed values Ψ_{max} inside the cavity for different values of the dimensionless parameter $S_l \in (0, 0.003)$ are displayed in Table 4.2. These values are also represented in Figure 4.3. Figure 4.3 shows the linear decrease of Ψ_{max} between $S_l = 0.0005$ and $S_l = 0.003$. The fluid displacement

S_l	Ψ_{max}
0	0.04371041
0.0005	0.04317579
0.001	0.04290319
0.0015	0.04262631
0.002	0.04234790
0.0025	0.04206969
0.003	0.04179283

Table 4.2: Maximum stream function values for different S_l

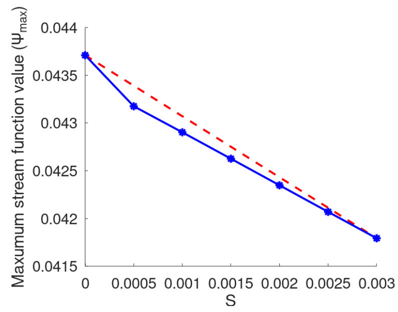
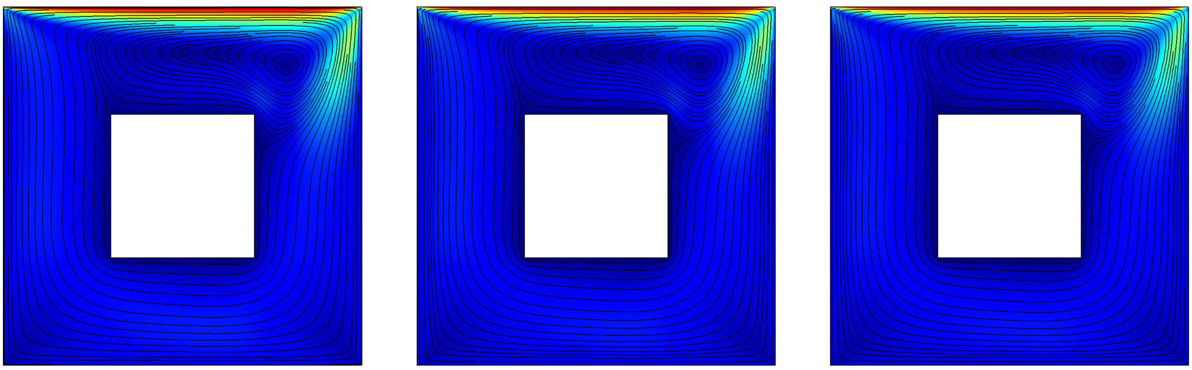


Figure 4.3: Dimensionless slip length effect

inside the porous cavity is highlighted in Figure 4.4. An important remark that can be made here is that the variation of the dimensionless slip parameter S_l does not suddenly change the flow pattern. This can be seen in the similarity of all three images in Figure 4.4 being quite similar. Even if the stream lines and the velocity profile are different in each case, these differences are negligible and not so obvious.



(a) $S^l = 0$ (b) $S^l = 0.0015$ (c) $S^l = 0.003$

Figure 4.4: Streamlines and Velocity profiles for different values of slidding parameter S_l.

We continue our analysis and we set $S_l = 0.0005$. We consider

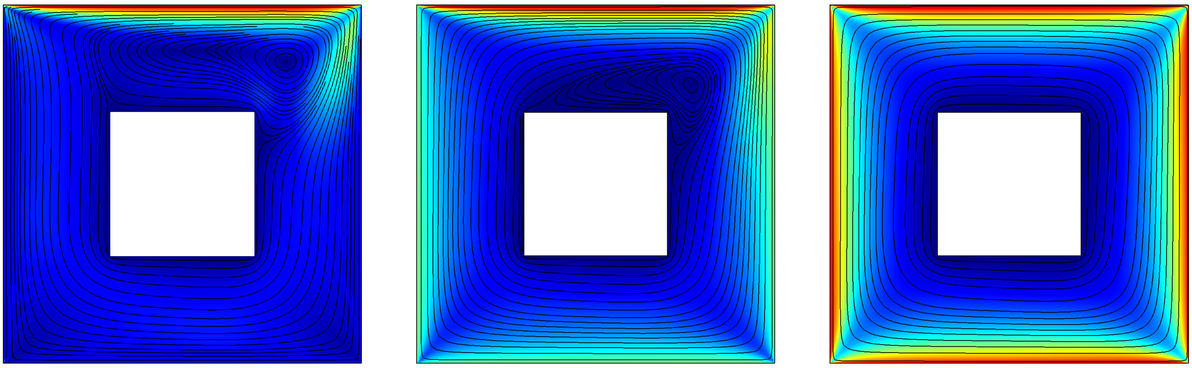
$$U^r = U^b = U,\ U^l = -U, \tag{4.1.10}$$

where U is a constant which takes the values

$$U = 0.1, 0.3, 0.5, 0.7, 0.9, \tag{4.1.11}$$

respectively. Hence, we want to see the how fluid flow behaves inside the cavity, whether the velocity of the vertical walls and the bottom one increases towards the velocity of the top lid. The other parameters remain the same as in relation (4.1.9). In this situation, the stream lines and the velocity profiles for the fluid particles for $U = 0.1, 0.5, 0.9$ are provided in Figure 4.5. Let us note that, for increasing values of U, the center of the secondary vortex, which is initially close to the top side, tends to approach the center of the cavity, eventually being assimilated by the main vortex rotating around the obstacle. This is due to the balance of forces generated by the four walls arranged symmetrically. In Table 4.3 we see how Ψ_{max} varies, while its minimum is reached for

$U = 0.3$. Beyond $U = 0.3$, Ψ_{max} tends to increase as U approaches the velocity of the upper wall, reaching a maximum value for $U = 0.9$.



(a) $U = 0.1$ (b) $U = 0.5$ (c) $U = 0.9$

Figure 4.5: Streamlines and Velocity profiles for $S_l = 0.0005$ and $U = 0.1, 0.5, 0.9$.

U	Ψ_{max}
0.1	0.04317579
0.3	0.04286458
0.5	0.04294202
0.7	0.04330634
0.9	0.04392932

Table 4.3: Maximum stream function values for variation of U

Further research directions

We would like to point out some research directions that could be followed after this thesis.

Extension of the obtained results

As a first future direction, we aim to extend the original results that were presented in this thesis to more general function spaces such as L^p-based Sobolev space, for $p \in (1, \infty)$, Besov spaces, Bessel potential spaces, Triebel-Lizorkin spaces. We can also consider our boundary value problems in certain domains whose geometry is more general or more complex, for example, polyhedral domains, domains with cusps. In addition, we intend to obtain such results by using other techniques such variational methods and the fixed point index theory. In addition, we can pursue a practical study such as the investigation of the correlation between physical parameters (for example, the Reynolds number) and the existence of vortexes in some viscous fluid flows in the presence of solid obstacles. In such a study we can formulate boundary value problems which are similar to the ones investigated in this thesis.

Variable coefficients

In recent years, a great deal of work has been devoted (see, e.g., [59], [60], [67]) to the generalization of the Stokes equations. Namely, instead of the Laplacian, one can consider another divergence form, second-order elliptic differential operator. Consequently, this approach leads to the anisotropic Stokes system and anisotropic Navier-Stokes system, respectively. These generalizations account of the possibility of the modeling of a incompressible fluid with variable viscosity.

This new perspective leads to the future idea of studying boundary problems for more general Brinkman or Darcy-Forchheimer-Brinkman equations, in various configurations, while all the coefficients that appear in these systems are variable (see, e.g., [66]).

Bidisperse (Multidisperse) Porous Media Models

Another possible development that can be pursued is the theoretical and/or numerical study of bidisperse porous media.

The authors in [65] have developed a theoretical analysis for a general system of coupled Navier-Stokes-type equations in the incompressible case in the setting of a bounded domain, where a homogeneous Dirichlet condition was considered. Their approach is based on the model proposed by Nield and Kuznetsov in the papers [87] and [88]. Kohr and Precup [66] have studied a general class of coupled anisotropic Navier-Stokes-like equations with variable coefficients that describe viscous fluid flows in multidisperse anisotropic porous media. They have considered also non-homogeneous reaction-type terms in the incompressible case. The authors have employed a variational technique and fixed point index theory in order to obtain existence results.

The papers [65] and [66] suggest an a possible research direction, that of the investigation of other models that appear in the study of flows in anisotropic bidisperse (or multidisperse) porous media with the goal of obtaining existence results for other boundary problems associated to the underlying PDE systems.

Moreover, another point of exploration can be the diversification of the numerical methods that can be employed in the study of boundary problems suggested by applications in Fluid Mechanics and porous media. Let us mention that, in addition to the classical approaches as finite difference methods (e.g., employed in [42]), finite volume methods, there are powerful PDE solvers such as FreeFem++, Ansys, Comsol that can be employed in order to obtain numerical results for future studies of various boundary value problems.

Boundary value problems on manifolds

Finally, we want to specify the results included in the thesis have all been obtained in the Euclidean setting of $\mathbb{R}^n$. There are also many works devoted to the investigation of boundary problems on compact manifolds (see, e.g., [57], [63], [67], [83], [84]). A natural step would be to consider similar boundary value problems with those treated in this thesis in the setting of compact Riemannian manifolds or non-compact Riemannian manifolds.

More recently, a new concept has been developed. We want to highlight a contribution made by Kohr, Nistor and Wendland in [62], in which they obtained the results needed to introduce and investigate layer potentials on manifolds with conical or cylindrical ends. They devoted their study to the introduction of classes of pseudodifferential operators that are defined on these manifolds, called 'translation invariant at infinity' and 'essentially translation invariant' operators and studied their properties, having in view applications to the Stokes system. As a future research direction that can be inferred, the work [62] (see also [82]) provides an opening for the analysis of various boundary problems for other elliptic PDE systems in the setting of manifolds with cylindrical ends.

Conclusions

The aim of this thesis is to provide existence and uniqueness results for transmission-type boundary value problems for certain constant-coefficient and variable-coefficient elliptic systems. Such systems can be found in the field of Fluid Mechanics, while others are involved in certain models of porous media. The aforementioned transmission-type problems are investigated in the Euclidean setting using the means of potential theory and fixed point methods and we complement the theoretical results with a numerical investigation of a boundary value problem.

We begin by describing all notions that we use throughout this thesis. We introduce the (Gagliardo) trace operator in the classical Sobolev spaces as well in the weighted Sobolev spaces. We analyze the Stokes, Brinkman and generalized Brinkman equations and we provide their associated conormal derivative operators. For the Stokes and Brinkman systems, respectively, we give their respective fundamental solution, we introduce their associated single layer, double layer and Newtonian potentials. For each of these potentials we have given their mapping properties, their jump properties and their growth conditions at infinity.

The following chapter is concerned with well-posedness results for transmission problems for linear PDE systems. First, we have a well-posedness result for the exterior Dirichlet problem for the Brinkman system in $\mathbb{R}^3$ (see Theorem 2.1.2). Next, we have a well-posedness result for the transmission problem for the generalized Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$ (see Theorem 2.2.2). Another well-posedness result for the transmission problem for the generalized Brinkman and classical Brinkman systems is also obtained in complementary Lipschitz domains in $\mathbb{R}^3$ (see Theorem 2.3.1). Moreover, we have a well-posedness result for the Robin-transmission problem for the classical Brinkman system (see Theorem 2.4.1). Also, we show that, the limiting Robin-transmission for the classical Brinkman system is also well-posed (see Theorem 2.4.2) and as a consequence, we get the well-posedness of the Robin-Dirichlet problem for the Brinkman system (see Corollary 2.4.3).

The next chapter contains the generalization of the Darcy-Forchheimer-Brinkman system and a useful lemma. Here, we have the well-posedness result for the transmission problem the generalized Darcy-Forchheimer-Brinkman and Stokes systems in weighted Sobolev spaces in $\mathbb{R}^3$ (see Theorem 3.2.1). Next, we have the well-posedness result for the transmission problem for the generalized Darcy-Forchheimer-Brinkman and Brinkman systems in $\mathbb{R}^3$ (see Theorem 3.3.1). Also, another well-posedness result is obtained for the Robin-transmission problem for the classical Darcy-Forchheimer-Brinkman system (see Theorem 3.4.1). In addition, the limiting Robin-transmission problem for the classical Darcy-Forchheimer-Brinkman system is also well-posed (see Theorem 3.4.2) and this result also gives an existence result for the Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system (see Corollary 3.4.3).

The final chapter consists of a numerical investigation for the lid-driven cavity flow problem in two dimensions for the Darcy-Forchheimer-Brinkman system. We consider Dirichlet boundary conditions on the interior wall and Robin boundary conditions on the exterior wall. We analyze the impact of the dimensionless slip length on the behavior of the fluid flow inside the porous cavity.

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