Babeş–Bolyai University of Cluj-Napoca

NEW APPROACH FOR HARDY–RELLICH INEQUALITIES

PhD Thesis – summary

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Abstract

This PhD thesis presents alternative approaches to Hardy and Rellich inequalities on Riemannian manifolds. Regarding Hardy inequalities, we establish a generic functional inequality, both in *additive* and *multiplicative* forms, which produces well-known and genuinely new inequalities. For the additive version we introduce the notion of *Riccati pairs*, which enables us to give short/elegant proofs for several celebrated functional inequalities on Riemannian manifolds with sectional curvature bounded from above, by simply solving a Riccati-type ODE. The multiplicative version allows us to cover *uncertainty principles* as well. Concerning the Rellich inequalities, we establish two generic functional inequalities on Riemannian manifolds of the same type as before. As applications, on the one hand, we prove *sharp spectral gap estimates* for various higher-order eigenvalue problems. On the other hand, we provide extensions for some well-known Rellich inequalities. The latter methods differ from the approach of Riccati pairs, thus, as a final point, we discuss the applicability of Riccati pairs in the context of Rellich inequalities. The elegance of our approaches lies in their simplicity: the proofs are based on convexity arguments and applications of divergence/comparison theorems; moreover, they are symmetrization-free. Consequently, the validity of the generalized Cartan–Hadamard conjecture is not required, which broadens the range of applicability of our results.

Keywords and phrases

Hardy inequalities, Rellich inequalities, Riemannian manifolds, Riccati pairs, Spectral gap estimates, Symmetrization-free approach

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Relevant publications

The present PhD thesis is built upon the following articles:

[56] **S. Kajántó**, A. Kristály, I. R. Peter, and W. Zhao. A generic functional inequality and Riccati pairs: an alternative approach to Hardy-type inequalities.
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[38] C. Farkas, **S. Kajántó**, and A. Kristály. Sharp spectral gap estimates for higher-order operators on Cartan–Hadamard manifolds.
Commun. Contemp. Math., 2024. accepted. (see Chapter 4).
https://doi.org/10.1142/S0219199724500135
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[55] **S. Kajántó**. Rellich inequalities via Riccati pairs on model space forms.
J. Math. Anal. Appl., 531(2, Part 1):13, Paper No. 127870, 2024. (see Chapter 5).
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Journal metrics (JCR 2022): RIS: **1.088**, AIS: **0.671**, IF: **1.3**.

The following articles are tangentially related to the thesis:

[57] **S. Kajántó** and A. Kristály. Unexpected Behaviour of Flag and S-Curvatures on the Interpolated Poincaré Metric.
J. Geom. Anal., 31:10246–10262, 2021.
https://doi.org/10.1007/s12220-021-00644-x
Journal metrics (JCR 2022): RIS: **1.385**, AIS: **0.854**, IF: **1.1**.

[58] **S. Kajántó** and A. Kristály. Saturation phenomena of a nonlocal eigenvalue problem: the Riemannian case.
Optimization, 1–24, 2023.
https://doi.org/10.1080/02331934.2023.2239881
Journal metrics (JCR 2022): RIS: **1.097**, AIS: **0.708**, IF: **2.2**.

[39] C. Farkas, **S. Kajántó**, and C. Varga. Lower semicontinuity of Kirchhoff-type energy functionals and spectral gaps on (sub)Riemannian manifolds.
Topol. Methods Nonlinear Anal., 61(2):743–760, 2023.
https://doi.org/10.12775/TMNA.2022.034
Journal metrics (JCR 2022): RIS: **0.685**, AIS: **0.422**, IF: **0.7**.

Conventions on citations

The primary results of the thesis are *generic functional inequalities* that provide Hardy and Rellich inequalities. These results are also presented in one of the above papers. In these cases, we provide citations in the following format:

Theorem X.Y (see [REF]).

The secondary results of the thesis are *applications* of the primary results. These typically involve extensions and alternative proofs of some *formally* well-known Hardy and Rellich inequalities. These results are also present in one of the above papers; we indicate this fact by using the following format:

Theorem X.Y (see [REF])**.**

We note that particular versions of the secondary results can also be found in other publications beyond the papers mentioned above; additional references are provided in the text surrounding the corresponding result in the following manner:

see e.g., Author1, Author2 [REF].

Citations of auxiliary results taken from other publications than the above-mentioned papers are provided in the text surrounding the corresponding result, using the latter format. Due to their explicatory role, the citations of remarks, which are partially or entirely present in the above papers, are omitted.

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Sincerely,
the author.

Chapter 1

Introduction

More than one hundred years elapsed since the celebrated (one-dimensional) *Hardy inequality* appeared (see Hardy [52]). A multi-dimensional version of the original result can be stated as follows: Given a smooth, compactly supported function u on $\Omega \subseteq \mathbb{R}^n$, the L^2-norm of the singular term $u(x)/|x|$ is controlled by the L^2-norm of $|\nabla u(x)|$, more precisely, one has

$$\int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x \geq \frac{(n-2)^2}{4} \int_\Omega \frac{|u(x)|^2}{|x|^2} \, \mathrm{d}x, \qquad \forall u \in C_0^\infty(\Omega).$$

Several extensions and improvements of this inequality can be found by now in the literature, involving more general weights, additional correction terms, and/or various underlying geometrical settings; sometimes in alternative formulation, often referred to as *uncertainty principles*. In general, they can be written in the following forms:

$$\int_\Omega V|\nabla u|^p \, \mathsf{dm} \geq \int_\Omega W|u|^p \, \mathsf{dm}, \qquad \forall u \in C_0^\infty(\Omega), \tag{\textbf{H}}$$

$$\left(\int_\Omega V|\nabla u|^p \, \mathsf{dm}\right)^{\frac{1}{p}} \left(\int_\Omega \overline{W}|u|^p \, \mathsf{dm}\right)^{\frac{1}{p'}} \geq \int_\Omega \widetilde{W}|u|^p \, \mathsf{dm}, \qquad \forall u \in C_0^\infty(\Omega), \tag{\textbf{UP}}$$

respectively, where $p > 1$ and $p' \overset{\text{def}}{=} \frac{p}{p-1}$ is the *conjugate* of p; Ω is an open subset of an ambient space M, which could be the Euclidean space $\mathbb{R}^n$, any Riemannian/Finsler manifold, or a stratified group; m is a measure on M, while $V, W, \overline{W}, \widetilde{W} : \Omega \to (0, \infty)$ are certain potentials, possibly containing singular terms.

These inequalities became indispensable from the point of view of applications. On the one hand, solutions of a large class of elliptic problems involving singular terms are based on the validity of corresponding Hardy-type inequalities; hence, they appear in several fields of mathematics. In particular, they play a crucial role in the spectral theory of the *fixed membrane problem*, describing the fundamental tones.

On the other hand, uncertainty principles have an especially important role in quantum mechanics, formulating one of its most fundamental and yet most surprising facts: Given an arbitrary particle of the Universe, one cannot determine precisely both its position and momentum, i.e., the more precisely we know its position, the less precisely we know its momentum, and vice versa. In this study we restrict ourselves to the mathematical point of view of such uncertainty principles.

Key observations related to Hardy-type inequalities have been made (mainly for $p = 2$ and the Euclidean setting) e.g., by Adimurthi, Chaudhuri, and Ramaswamy [1], Brezis and Marcus [16], Brezis and Vázquez [17], Devyver, Fraas, and Pinchover [33], Fefferman [40], Filippas, Maz'ya, and Tertikas [41, 42], Filippas and Tertikas [43], Muckenhoupt [81], Ruzhansky and Suragan [91] and Tertikas and Zographopoulos [94].

The higher-order variants of the inequality (**H**) are the following two Rellich-type inequalities:

$$\int_\Omega U|\Delta u|^p \,\mathsf{dm} \geq \int_\Omega W|u|^p \,\mathsf{dm}, \qquad \forall u \in C_0^\infty(\Omega), \tag{$\mathbf{R}_1$}$$

$$\int_\Omega U|\Delta u|^p \,\mathsf{dm} \geq \int_\Omega V|\nabla u|^p \,\mathsf{dm}, \qquad \forall u \in C_0^\infty(\Omega), \tag{$\mathbf{R}_2$}$$

where $p > 1$ and U, V, W are given positive potentials on Ω. Both inequalities of types ($\mathbf{R}_1$) and ($\mathbf{R}_2$) have numerous applications; here we highlight the spectral theory of the *clamped plate problem* and the *buckling problem*, respectively. The first problem describes the vibrations within a thin elastic plate with a clamped boundary. The second problem investigates a similar plate that is subjected to a compressive load. We also notice that inequality ($\mathbf{R}_1$) can be obtained by combining either ($\mathbf{R}_2$) and (**H**), or their alternative versions, where the classical gradient ∇u is replaced by the directional derivative $\nabla^{\mathrm{rad}} u = \langle \nabla u, \nabla d_{x_0} \rangle$ and d_{x_0} is the distance from a fixed point $x_0 \in \Omega$.

For both problems, it is increasingly true that they are mainly considered in the Euclidean setting for $p = 2$. The classical version of ($\mathbf{R}_1$) dates back to the 1950s and is due to Rellich [90]. Surprisingly, as claimed by the authors, the corresponding version of ($\mathbf{R}_2$) only appeared relatively recently (in the 2000s) in the paper by Tertikas and Zographopoulos [94]. These facts suggest inequality ($\mathbf{R}_2$) being more problematic. Further pioneering results in the Euclidean setting can be found e.g., in the papers by Davies and Hinz [31] and Mitidieri [78]. For results on more general structures, see e.g., Kombe and Özaydin [61, 62] and Kristály and Repovš [70]. Comprehensive discussions about Hardy and Rellich inequalities can also be found in the monographs by Balinsky, Evans, and Lewis [7], Ghoussoub and Moradifam [50], and Ruzhansky and Suragan [92].

A milestone result – concerning the problems (**H**) and ($\mathbf{R}_2$) – has been provided by Ghoussoub and Moradifam [48, 49], again for $p = 2$ and the Euclidean setting. On the one hand, the authors showed that the inequality (**H**) holds if and only if (V, W) is a *Bessel pair*. The latter notion is based on the solvability of a second-order linear Bessel-type ordinary differential equation (ODE for short) containing the potentials V and W. On the other hand, they showed that under certain conditions, inequality ($\mathbf{R}_2$) holds if and only if $(U, \widetilde{V})$ is a Bessel pair, where $\widetilde{V}$ is a potential involving V and an additional correction term. The proofs are heavily based on the technique of *spherical harmonics decomposition*, which works well on the model space forms (Euclidean space, hyperbolic space, and the sphere), but it cannot be applied to general Riemann manifolds. The concept of Bessel pairs was extended to general $p > 1$ (see Duy, Lam, and Lu [35]), and has applications on non-positively curved Riemannian manifolds (see Flynn, Lam, Lu, and Mazumdar [45] and Berchio, Ganguly, and Grillo [10]), where still the usual notion of Bessel pairs and fine comparison arguments are used.

In this work, we present an alternative approach to Hardy–Rellich inequalities, which completely avoids the use of spherical harmonics decomposition. As we shall see, our forthcoming proofs are built upon simple convexity arguments, multiple uses of divergence theorems, and comparison theorems that encode curvature information of the ambient manifold. In the sequel, we present our results simultaneously with the structure of this study, as follows.

In Chapter 2 we recall some definitions and preliminary results that are relevant for our presentation. In Section 2.1 we list several concepts and their corresponding notations concerning general Riemannian manifolds. In Section 2.2 we present various notions regarding model space forms and relevant comparison results. In Section 2.3 we address various eigenvalue problems and their spectral gap estimates; namely, the *fixed membrane problem* (Section 2.3.1), the *clamped plate problem* (Section 2.3.2) and the *buckling problem* (Section 2.3.3). Finally, in Section 2.4 we summarize the method of Bessel pairs mentioned earlier. Here, we also discuss the relation between Bessel pairs and the classical method of *supersolutions*, also known as the *Allegretto–Moss–Piepenbrink approach*, which emerges from the early works of Allegretto [4] and Moss and Piepenbrink [80].

In Chapter 3 we restrict our attention to Hardy inequalities; our approach for them is presented in Section 3.1: First, in Theorem 3.1, we provide a general functional inequality on Riemannian manifolds in both *additive* and *multiplicative* forms that turn out to be *equivalent* to each other. Both forms involve several parameters besides the unknown function u; substituting concrete parameters yields inequalities of types (**H**) and (**UP**), respectively. Next, in Section 3.1.1, we make a key observation: Both forms contain the Laplacian of a given potential (implicitly encoding curvature information about the manifold), which suggests the application of an appropriate comparison argument. This comparison furnishes – in the additive form – a Riccati-type ordinary differential inequality (ODI), which leads to the notion of *Riccati pairs* for certain potentials (see Definition 3.2). Incorporating this notion into the additive form yields Theorem 3.3, which turns out to be extremely efficient in proving inequalities of type (**H**). Indeed, to prove an inequality of this type, it is enough to solve a Riccati-type ODI, and to apply Theorem 3.3. Finally, in Proposition 3.4 and Remark 3.5 we show that Riccati pairs extend Bessel pairs (slightly in the Euclidean case); however, the difference is mainly in the underlying technique, not in the ODE/ODI.

In Section 3.2 we present simple alternative proofs for various functional inequalities of type (**H**) using Theorem 3.1/(i) and Theorem 3.3. We highlight that a part of these results is formally well-known in the Euclidean setting. Our method, however, extends them to *Cartan–Hadamard manifolds*, which are complete, simply connected Riemannian manifolds, with non-positive sectional curvature. This demonstrates the efficiency of our main results, mostly based on Riccati pairs. We shall consider the following inequalities:

- In Section 3.2.1 we present two *L^p-Caccioppoli-type inequalities* on Riemannian manifolds, providing alternative proofs for the results obtained by D'Ambrosio and Dipierro [29] (see Theorems 3.6 & 3.8). Some new improvements are also established in the case $p \in (1, 2]$.

- In Section 3.2.2 we discuss a number of *improved Hardy-type inequalities* on Cartan–Hadamard manifolds, including results by Carron [19] and Kombe and Özaydin [61, 62] (see Theorem 3.9), Edmunds and Triebel [36] (see Theorem 3.12), Adimurthi, Chaudhuri, and Ramaswamy [1] (see Theorem 3.13), and Brezis and Vázquez [17] (see Theorem 3.14).

- In Section 3.2.3 we present *spectral estimates* on Riemannian manifolds. First, we establish a simple alternative proof of the celebrated Cheng's comparison result on Riemannian manifolds with sectional curvature bounded from above (see Theorem 3.15). Next, we consider the well-known Faber–Krahn inequality and the McKean spectral gap estimate (see Theorems 3.16 & 3.17). In addition, we give a short proof of the main spectral result of Carvalho and Cavalcante [20] (see Theorem 3.18).

- In Section 3.2.4 we establish an *interpolation inequality* connecting the Hardy inequality and McKean's spectral gap estimate on Cartan–Hadamard manifolds, in the spirit of Berchio, Ganguly, Grillo, and Pinchover [11] (see Theorem 3.19). A simple modification of the latter argument also provides a short alternative proof of the inequality by Akutagawa and Kumura [3] (see Theorem 3.22).

- In Section 3.2.5 we consider two parameter-dependent *Ghoussoub–Moradifam-type weighted inequalities* in the Euclidean case (cf. [49]), where the weights are of non-singular type (see Theorem 3.23). For a certain parameter range, we also extend these inequalities to Cartan–Hadamard manifolds (see Theorem 3.25).

In Section 3.3 we provide alternative proofs for various multiplicative Hardy-type inequalities, which are simple consequences of Theorem 3.1/(ii). We proceed as follows:

- In Section 3.3.1 we establish a sharp parameter-dependent *uncertainty principle* on Cartan–Hadamard manifolds, which implies the Heisenberg–Pauli–Weyl and the Hydrogen uncertainty principles (see Theorem 3.27). We also establish a rigidity result: If the quantitative uncertainty principle holds on an n-dimensional Cartan–Hadamard manifolds with Ricci curvature bounded from below, then the manifold is isometric to the corresponding model space form (see Theorem 3.29).

- In Section 3.3.2 we present two sharp *Caffarelli–Kohn–Nirenberg inequalities* on Cartan–Hadamard manifolds (see Theorems 3.30 & 3.31).

In Chapter 4 we focus on Rellich inequalities. In Section 4.1 we present two general functional inequalities on Riemannian manifolds, built upon convexity arguments and divergence/comparison theorems. The first inequality involves a second-order ODI, and provides inequalities of type ($\mathbf{R}_1$), for general $p > 1$ (see Theorem 4.1). The second inequality involves a second-order *partial* differential inequality (PDI) and produces inequalities of type ($\mathbf{R}_2$), for $p = 2$ (see Theorem 4.2). We notice that for special choices of parameters, the latter PDI reduces to an ODI, becoming easier to deal with.

In the rest of the chapter, several applications are presented on Cartan–Hadamard manifolds. Here we highlight that our proofs are *symmetrization-free*, hence they do not require the validity of the *generalized Cartan–Hadamard conjecture*, i.e., the validity of the *sharp isoperimetric inequality* in this geometrical setting. We note that the conjecture is valid for general Cartan–Hadamard manifolds in dimension $n \in \{2, 3\}$ (see Bol [14] and Kleiner [59]), and for space forms in any dimension (see Dinghas [34]).

In Section 4.2 we establish fourth-order variants of the celebrated spectral gap result by McKean [75]. The latter result states that a strong negative curvature (when it is less than or equal to a negative number) produces a universal, domain-independent spectral gap for the first/principal eigenvalue of the fixed membrane problem, which is in radical contrast to the Euclidean case.

- In Section 4.2.1 we prove a spectral gap estimate and its *sharpness* for the clamped plate problem on Cartan–Hadamard manifolds with strong negative curvature, for $p > 1$ (see Theorem 4.3) We notice that the same estimate was also established by Kristály [65] (on Cartan–Hadamard manifolds satisfying the generalized Cartan–Hadamard conjecture) and Ngô and Nguyen [83] (on space forms) using symmetrization. In this case, our result completes the picture.

- In Section 4.2.2 we provide a *sharp* spectral gap estimate for the buckling problem on Cartan–Hadamard manifolds with strong negative curvature, for $p = 2$ (see Theorem 4.4). The same estimate is known on space forms due to Ngô and Nguyen [83]. In this case, our result adds a new piece of puzzle to the picture; however, the pieces corresponding to the general case when $p > 1$ are still missing.

- In Section 4.2.3 we establish higher-order *sharp* spectral gap estimates by simply combining our previous inequalities. The results are valid on Cartan–Hadamard manifolds with strong negative curvature. The clamped plate-type results hold for $p > 1$, while the buckling-type results hold for $p = 2$ (see Theorems 4.5 & 4.6).

In Section 4.3 we consider additional Rellich-type inequalities on Cartan–Hadamard manifolds, namely:

- In Section 4.3.1 we extend the classical and weighted Rellich inequalities to Cartan–Hadamard manifolds (see Theorem 4.7 and Corollary 4.8). These results are well-known in the Euclidean settings (see e.g., Mitidieri [79]).

- In Section 4.3.2 we provide higher-order variants of the classical Rellich inequality on Cartan–Hadamard manifolds (see Theorem 4.9).

- In Section 4.3.3 we provide further Rellich-type inequalities on general Cartan–Hadamard manifolds: Theorems 4.10 & 4.11 are improvements of type ($\mathbf{R}_1$), the second result is valid for strong negative curvature. In Theorem 4.12 we extend the classical Rellich inequality of type ($\mathbf{R}_2$) to Cartan–Hadamard manifolds, but only in dimensions $n \geq 8$. This constraint arises from technical conditions; for a similar phenomenon, see Kristály and Repovš [70].

In Chapter 5 we present a second alternative approach for establishing Rellich inequalities on space forms, using Riccati pairs. Parallel to presenting the structure of this chapter, which contains the final versions of this approach and its applications, let us briefly describe the development process and the intermediate observations. This additional information is intended to motivate the choice of presentation of our results.

Our initial goal was to develop a general functional inequality built upon Riccati pairs, which provides Rellich inequalities of type ($\mathbf{R}_2$). Having such an inequality and combining it with the results from Chapter 3 providing inequalities of type ($\mathbf{H}$), would automatically yield inequalities of type ($\mathbf{R}_1$). Clearly, we intended to formulate our inequality on the most general manifolds possible. Keeping in mind the requirement of the Laplace comparison in our argument, manifolds with sectional curvature bounded from above seemed to be good candidates in this case as well.

During the study, it turned out that the suitable convexity argument requires both a Laplace and a Hessian comparison, having 'opposite' direction to each other. Thus, one either imposes a lower bound on the sectional curvature as well, or simply considers space forms. We decided on the second option because several applications are also formulated in this setting. Additionally, in this particular case, the results can be presented in a formally simpler and more accessible manner.

In Section 5.1 we present our approach for Rellich inequalities on space forms. In Section 5.1.1 we start with two definitions. First, motivated by the simplicity of the ambient geometrical setting, in Definition 5.1 we introduce *simplified Riccati pairs* including the same ODI, but more straightforward conditions on parameter functions. Next, in Definition 5.2 we introduce *dual Riccati pairs* including an ODI, which is the true driving force behind the Rellich inequalities. We note that the latter ODI differs from the former by a change of function, the introduction of the second concept is a personal decision that hopefully enhances the presentation of the results.

Using these simplified concepts, we established a general functional inequality that furnishes inequalities of type ($\mathbf{R}_2$) (see Theorem 5.3/(ii)). Unfortunately, it turned out that the technical conditions resulting from the convexity argument typically imply a dimension constraint, similarly to Theorem 4.12. At this point, the idea of radial derivatives seemed useful. First, in Theorem 5.3/(i) we developed a general functional inequality, which provides radial versions of inequality ($\mathbf{R}_2$), where the gradient is replaced with the radial derivative. Here, the technical conditions are less restrictive, at least they do not require unnecessary dimensional constraints. Next, in Theorem 5.4 we established a general functional inequality providing radial versions of inequality ($\mathbf{H}$). We note that due to the Cauchy–Schwarz inequality, the latter inequality is stronger than its non-radial variant, i.e., the reformulation of Theorem 3.3 in these geometrical settings. We also notice that this inequality does not require any additional conditions. In the rest of the chapter, we present applications of type ($\mathbf{R}_1$) exploiting the idea of radial derivatives. We note that each of the obtained intermediate radial inequalities admits a non-radial counterpart, whose technical condition is typically more restrictive; for the simplicity of presentation, these inequalities are omitted.

In Section 5.2 we provide applications to our method in Euclidean spaces; here we intend to present both the strengths and limitations of our approach. To do this, we test our result against two recent inequalities proved by Adimurthi, Grossi, and Santra [2].

First, we combine Theorem 5.3/(i) and Theorem 5.4 to provide a general functional inequality (see Theorem 5.5) formulated in terms of Bessel potentials (a notion related to Bessel pairs), which allows us to have a simpler presentation. Next, we show that the first inequality meets the conditions of the latter result (see Corollary 5.6), while the second does not (see Remark 5.7).

In Section 5.3 we apply our approach to hyperbolic spaces. In Theorem 5.8 we provide a radial inequality as a consequence of Theorem 5.3/(i), interpolating between Rellich-type and spectral gap estimate-type inequalities. This result can be seen as a higher-order radial variant of the interpolation inequality from Theorem 3.19. For the extremal values of the parameter, we obtain formally well-known inequalities (see Corollaries 5.9 & 5.10). In Theorem 5.11 lower-order counterparts of Theorem 5.8 are presented, which are simple consequences of Theorem 5.4. Finally, in Theorem 5.13 we combine the previous two results and provide a sophisticated inequality of type ($\mathbf{R}_1$) on hyperbolic spaces.

Chapter 2

Preliminaries and notations

2.1 General manifolds

2.2 Model space forms

2.3 Eigenvalue problems

2.3.1 Fixed membrane problem

2.3.2 Clamped plate problem

2.3.3 Buckling problem

2.4 The approach of Bessel pairs

Chapter 3

Hardy inequalities on general Riemannian manifolds

In this chapter, we consider Hardy inequalities. In § 3.1 we present our abstract approach, while in § 3.2 & 3.3 we provide application of types (**H**) & (**UP**), respectively.

3.1 General functional inequalities

Our first result is a general functional inequality formulated in two *equivalent* forms, which provide inequalities of types (**H**) and (**UP**). It can be stated as follows.

Theorem 3.1 (see [56]). *Let (M, g) be a complete, non-compact, $n \geq 2$-dimensional Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $p > 1$, and let $\rho \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be non-constant and positive with $\mathcal{H}^n_g(\rho^{-1}(\sup_\Omega \rho)) = 0$. Suppose that $w\colon (0, \sup_\Omega \rho) \to (0, \infty)$, $G\colon (0, \sup_\Omega \rho) \to \mathbb{R}$, and $H : \mathbb{R} \to \mathbb{R}$ are C^1 functions such that*

$$(\mathbf{G})_{\rho,w} \ : \ G(\rho)w(\rho)^{\frac{1}{p'}}|\nabla_g \rho|^{p-1} \in L^{p'}_{\mathrm{loc}}(\Omega) \text{ and } (G(\rho)w(\rho))'|\nabla_g\rho|^p, w(\rho) \in L^1_{\mathrm{loc}}(\Omega),$$

and $H(0) = H'(0) = 0$. The following inequalities hold:

(i) (Additive form) *For every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g \geq p\int_\Omega \left[(G(\rho)w(\rho))'|\nabla_g\rho|^p + G(\rho)w(\rho)\Delta_{g,p}\rho\right] H(u)\,\mathrm{d}v_g$$
$$+ (1-p)\int_\Omega |G(\rho)|^{p'}|\nabla_g\rho|^p w(\rho)|H'(u)|^{p'}\,\mathrm{d}v_g. \tag{3.1}$$

(ii) (Multiplicative form) *For every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g \geq \frac{\left|\displaystyle\int_\Omega \left[(G(\rho)w(\rho))'|\nabla_g\rho|^p + G(\rho)w(\rho)\Delta_{g,p}\rho\right] H(u)\,\mathrm{d}v_g\right|^p}{\left(\displaystyle\int_\Omega |G(\rho)|^{p'}|\nabla_g\rho|^p w(\rho)|H'(u)|^{p'}\,\mathrm{d}v_g\right)^{p-1}}, \tag{3.2}$$

provided that there exists a neighborhood $\mathcal{V} \subseteq \mathbb{R}$ of zero satisfying

$$H'(s) \neq 0, \forall s \in \mathcal{V}\setminus\{0\},\ \ G(t) \neq 0, \forall t \in (0, \sup_\Omega \rho), \text{ and } \mathcal{H}^n_g(|\nabla_g\rho|^{-1}(0)) = 0. \tag{3.3}$$

As one shall see, both inequalities in Theorem 3.1 are *generic*, that is, for suitable choices of parameters, they produce various functional inequalities. In the sequel we show that this procedure can be reversed: For a number of Hardy inequalities one can find suitable choices of parameters providing short/elegant proofs for them.

3.1.1 Riccati pairs

Let us focus on the additive form (i) of Theorem 3.1. Let $p > 1$, and observe that if $H(s) = \frac{|s|^p}{p}$, for every $s \in \mathbb{R}$, then $pH(s) = |H'(s)|^{p'} = |s|^p$. This observation, together with the Laplace comparison suggest the following notion.

Definition 3.2 (see [56]). Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $p > 1$, and $\rho \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a positive function with $|\nabla_g \rho| = 1$ $\mathrm{d}v_g$-a.e. in Ω. Fix the continuous functions $L, W\colon (0, \sup_\Omega \rho) \to (0, \infty)$ and the function $w\colon (0, \sup_\Omega \rho) \to (0, \infty)$ of class C^1. We say that the couple (L, W) is a (p, ρ, w)-*Riccati pair in* $(0, \sup_\Omega \rho)$ if there exists a function $G\colon (0, \sup_\Omega \rho) \to \mathbb{R}$ such that

(c1) $(\mathbf{G})_{\rho,w}$ holds (from Theorem 3.1);

(c2) $\Delta_g \rho \geq L(\rho)$ in the distributional sense in Ω, and

$$G \geq 0, \quad \text{if} \quad \mathcal{H}^n_g(\{x \in \Omega : \Delta_g \rho(x) > L(\rho(x))\}) \neq 0;$$

(c3) for every $t \in (0, \sup_\Omega \rho)$ one has

$$(G(t)w(t))' + G(t)w(t)L(t) + (1-p)|G(t)|^{p'} w(t) \geq W(t)w(t). \tag{3.4}$$

A function G satisfying the above conditions is said to be *admissible* for (L, W).

An efficient application of Theorem 3.1/(i) based on the concept of Riccati pairs can be stated as follows.

Theorem 3.3 (see [56]). *Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $p > 1$, and $\rho \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a positive function with $|\nabla_g \rho| = 1$ $\mathrm{d}v_g$-a.e. in Ω. Suppose that $L, W\colon (0, \sup_\Omega \rho) \to (0, \infty)$ are continuous functions and $w\colon (0, \sup_\Omega \rho) \to (0, \infty)$ is of class C^1 such that (L, W) is a (p, ρ, w)-Riccati pair in $(0, \sup_\Omega \rho)$. Then for every $u \in C^\infty_0(\Omega)$ one has*

$$\int_\Omega w(\rho)|\nabla_g u|^p \, \mathrm{d}v_g \geq \int_\Omega W(\rho)w(\rho)|u|^p \, \mathrm{d}v_g. \tag{3.5}$$

To present the efficiency of Theorem 3.3, we sketch a short proof of the celebrated *McKean's sharp spectral gap estimate*: If (M, g) is an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$, then the *essential spectrum* of the Laplace–Beltrami operator on (M, g) is $[K_{\kappa,n}, \infty)$, where

$$K_{\kappa,n} \stackrel{\text{def}}{=} \left(\frac{(n-1)\sqrt{-\kappa}}{2}\right)^2.$$

Indeed, fix $x_0 \in M$ and choose $\rho = d_{g,x_0}$, i.e., the distance from x_0, as well as

$$w \equiv 1, \quad L \equiv (n-1)\sqrt{-\kappa}, \quad W \equiv (n-1)c\sqrt{-\kappa} - c^2, \quad \text{and} \quad G \equiv c,$$

for some $c > 0$ which will be defined later. We obtain that the ODI of (3.4) is verified with equality, thus, Conditions (c1) & (c3) clearly hold. The Laplace comparison implies $\Delta_g \rho \geq L$, hence (c2) holds as well. We obtained that (L, W) is a (p, ρ, w)-Riccati pair in $(0, \infty)$, and G is admissible for (L, W). A simple computation yields

$$\max_{c>0} \left((n-1)\sqrt{-\kappa}c - c^2\right) = K_{\kappa,n},$$

thus, Theorem 3.3 immediately implies the required spectral estimate:

$$\int_M |\nabla_g u|^2 \, \mathrm{d}v_g \geq K_{\kappa,n} \int_M |u|^2 \, \mathrm{d}v_g, \qquad \forall u \in C_0^\infty(M).$$

In the sequel, we establish the connection between Riccati and Bessel pairs. The definition of tha latter is as follows: Let $p > 1$, $R > 0$, and $A, B \colon (0, R) \to \mathbb{R}$ are functions with A being of class C^1. The couple (A, B) is a *p-Bessel pair* in $(0, R)$, if the ODE

$$\left(t^{n-1}A(t)|y'(t)|^{p-2}y'(t)\right)' + t^{n-1}B(t)|y(t)|^{p-2}y(t) = 0 \tag{3.6}$$

has a positive solution in $(0, R)$. This concept is related to Riccati pairs as follows.

Proposition 3.4 (see [56]). *Let $R > 0$ and $w, W \colon (0, R) \to (0, \infty)$ be two potentials with w of class C^1. The function $y > 0$ is a solution of (3.6) on $(0, R)$ for the couple*

$$(A, B) = (w, wW)$$

if and only if

$$G(t) = -\frac{|y'(t)|^{p-2}y'(t)}{y(t)^{p-1}} \tag{3.7}$$

is a solution of

$$(G(t)w(t))' + G(t)w(t)L_0(t) + (1-p)|G(t)|^{p'}w(t) = W(t)w(t), \tag{3.8}$$

on $(0, R)$, that is precisely (3.4) with equality and

$$L(t) = \frac{n-1}{t} = L_0(t).$$

Remark 3.5. In a Riemannian manifold (M, g) with sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa \in \mathbb{R}$, a more appropriate ODE in the definition of a p-Bessel pair (A, B) instead of (3.6), is as follows:

$$\left(\mathbf{s}_\kappa^{n-1}(t)A(t)|y'(t)|^{p-2}y'(t)\right)' + \mathbf{s}_\kappa^{n-1}(t)B(t)|y(t)|^{p-2}y(t) = 0, \qquad \forall t \in (0, R). \tag{3.9}$$

Indeed, when $\kappa = 0$, equation (3.9) reduces to (3.6), while for $\kappa \neq 0$, the density $\mathbf{s}_\kappa$ encodes the curvature and explains the choice of

$$L(t) = (n-1)\mathbf{ct}_\kappa(t) = L_\kappa(t), \qquad \forall t \in (0, R).$$

This observation will be crucial in some functional inequalities in the forthcoming sections, which will be obtained by means of Riccati pairs; see e.g., Cheng's comparison principle for the first eigenvalue in Theorem 3.15.

3.2 Applications I: Additive Hardy-type inequalities

3.2.1 Caccioppoli inequalities

The first simple consequence of Theorem 3.1 is a Caccioppoli-type inequality, proved by D'Ambrosio and Dipierro [29, Theorems 2.1 & 3.1; Corollary 2.3].

Theorem 3.6 (see [56]). *Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $p > 1$, and $\rho \in W^{1,p}_{\rm loc}(\Omega)$ be a non-negative function. If $\alpha \in \mathbb{R}$ such that $-(p-1-\alpha)\Delta_{g,p}\rho \geq 0$ in the distributional sense in Ω and $|\nabla_g \rho|^p \rho^{\alpha-p}, \rho^\alpha \in L^1_{\rm loc}(\Omega)$, then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega \rho^\alpha |\nabla_g u|^p \,\mathrm{d}v_g \geq \left(\frac{|p-1-\alpha|}{p}\right)^p \int_\Omega \rho^\alpha \frac{|u|^p}{\rho^p} |\nabla_g \rho|^p \,\mathrm{d}v_g. \tag{3.10}$$

An immediate consequence of Theorem 3.6 is the estimate of the first Dirichlet eigenvalue of the p-Laplace–Beltrami operator, i.e., of the p-fixed membrane problem. For simplicity, we consider the unweighted case ($\alpha = 0$):

Corollary 3.7 (see [56]). *Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold, $\Omega \subseteq M$ be a bounded domain, $p > 1$, and $\rho(x) = d_{g,\partial\Omega}(x)$ for every $x \in \Omega$. If $-\Delta_g \rho \geq 0$ in the distributional sense in Ω, then the first Dirichlet eigenvalue of the Riemannian p-Laplacian can be estimated as*

$$\Lambda_{\mathrm{m},p}(\Omega) = \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_g u|^p \,\mathrm{d}v_g}{\int_\Omega |u|^p \,\mathrm{d}v_g} \geq \left(\frac{p-1}{p}\right)^p \frac{1}{R_\Omega^p},$$

where $R_\Omega = \sup_{x\in\Omega} \rho(x)$ is the Riemannian-inradius *of the domain $\Omega \subseteq M$.*

Using the notation $R_\Omega = \sup_{x\in\Omega} \rho(x)$ from Corollary 3.7, in the spirit of Brezis and Marcus [16] and Barbatis, Filippas, and Tertikas [8], we provide an improvement of Theorem 3.6 with a suitable reminder term, whenever $1 < p \leq 2$, as follows.

Theorem 3.8 (see [56]). *Under the assumptions of Corollary 3.7, if $1 < p \leq 2$, then one has*

$$\int_\Omega |\nabla_g u|^p \,\mathrm{d}v_g \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{\rho^p}(1 + \mathcal{R}_p(\rho)) \,\mathrm{d}v_g, \tag{3.11}$$

for every $u \in C_0^\infty(\Omega)$, where

$$\mathcal{R}_p(t) = \left(1 + \log^{-1}\left(\frac{t}{eR_\Omega}\right)\right)^{p-2} \left(1 + (2-p)\log^{-1}\left(\frac{t}{eR_\Omega}\right) + \log^{-2}\left(\frac{t}{eR_\Omega}\right)\right) - 1 \geq 0,$$

for every $t \in (0, R_\Omega)$. In particular, if $p = 2$, then we have

$$\int_\Omega |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{1}{4}\int_\Omega \frac{u^2}{\rho^2} \,\mathrm{d}v_g + \frac{1}{4}\int_\Omega \frac{u^2}{\rho^2} \log^{-2}\left(\frac{\rho}{eR_\Omega}\right) \mathrm{d}v_g. \tag{3.12}$$

3.2.2 Improved Hardy inequalities on Cartan–Hadamard manifolds

We present the following L^p-Hardy inequality, which has been first established by Kombe and Özaydin [61, Theorem 2.1] for generic $p > 1$; the initial version for $p = 2$ is the celebrated result of Carron [19, Theorem 1.4].

Theorem 3.9 (see [56]). *Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold. Let $\alpha \in \mathbb{R}$, $p > 1$, and $\rho \colon M \to [0, \infty)$ be a function such that $\rho^{-1}(0) \subseteq M$ is compact, $|\nabla_g \rho| = 1$ and $\Delta_g \rho \geq \frac{C}{\rho}$ in the distributional sense for some $C > 0$ with the property that $C + 1 + \alpha > p > 1$. Then for every $u \in C_0^\infty(M \setminus \rho^{-1}(0))$ one has*

$$\int_M \rho^\alpha |\nabla_g u|^p \, \mathrm{d}v_g \geq \left(\frac{C + 1 + \alpha - p}{p}\right)^p \int_M \rho^\alpha \frac{|u|^p}{\rho^p} \, \mathrm{d}v_g. \tag{3.13}$$

Remark 3.10. We notice that if the p-capacity of the compact set $\rho^{-1}(0) \subseteq M$ is zero, then inequality (3.13) is valid not only in $C_0^\infty(M \setminus \rho^{-1}(0))$ but also in $C_0^\infty(M)$; see e.g., Carron [19] and D'Ambrosio and Dipierro [29]. In particular, if $n \geq p$ and $\mathcal{H}_g^{n-p}(\rho^{-1}(0)) < \infty$, then the p-capacity of $\rho^{-1}(0) \subseteq M$ is zero (see Heinonen, Kilpeläinen, and Martio [54]).

A simple consequence of Theorem 3.9 is the following weighted Hardy inequality.

Corollary 3.11 (see [56]). *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold. Let $x_0 \in M$ be fixed and $p, \alpha \in \mathbb{R}$ such that $1 < p < n + \alpha$. Then for every $u \in C_0^\infty(M \setminus \{x_0\})$ one has*

$$\int_M d_{g,x_0}^\alpha |\nabla_g u|^p \, \mathrm{d}v_g \geq \left(\frac{n + \alpha - p}{p}\right)^p \int_M d_{g,x_0}^\alpha \frac{|u|^p}{d_{g,x_0}^p} \, \mathrm{d}v_g. \tag{3.14}$$

Moreover, the constant $\left(\frac{n+\alpha-p}{p}\right)^p$ is sharp.

When $\alpha = 0$ in Corollary 3.11, the limit case $p = n$ does not provide any reasonable inequality similar to (3.14). In the next result we prove a parameter-dependent Hardy inequality with logarithmic weights, which is valid also in the limit case $p = n$; similar results were established by Edmunds and Triebel [36] in the Euclidean case, as well as by D'Ambrosio and Dipierro [29, Theorem 6.5], Nguyen [84], and Zhao [100, Theorem 1.3] on Riemannian/Finsler manifolds.

Theorem 3.12 (see [56]). *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold, $x_0 \in M$ be a fixed point, $\Omega = B_{g,x_0}(1)$, and $\alpha, p \in \mathbb{R}$ such that $1 < p \leq n$ and $\alpha + 1 < p$. Then for every $u \in C_0^\infty(\Omega \setminus \{x_0\})$ one has*

$$\int_\Omega \log^\alpha(1/d_{g,x_0}) |\nabla_g u|^p \, \mathrm{d}v_g \geq \left(\frac{p - \alpha - 1}{p}\right)^p \int_\Omega \log^{\alpha - p}(1/d_{g,x_0}) \frac{|u|^p}{d_{g,x_0}^p} \, \mathrm{d}v_g. \tag{3.15}$$

Moreover, the constant $\left(\frac{p-\alpha-1}{p}\right)^p$ is sharp.

In the sequel, we provide an alternative approach to establish improved Hardy inequalities on Cartan–Hadamard manifolds. For simplicity of presentation, we shall consider the case when $p = 2$ and $\alpha = 0$. The first such result 'interpolates' between Corollary 3.11 and Theorem 3.12 (see Adimurthi, Chaudhuri, and Ramaswamy [1]).

Theorem 3.13 (see [56]). *Let (M, g) be an $n \geq 3$-dimensional Cartan–Hadamard manifold, and $\Omega \subseteq M$ be a bounded domain. Let $x_0 \in \Omega$ and $D_\Omega = \sup_{x\in\Omega} d_g(x_0, x)$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{d_{g,x_0}^2} \,\mathrm{d}v_g + \frac{1}{4} \int_\Omega \log^{-2}\left(\frac{d_{g,x_0}}{eD_\Omega}\right) \frac{u^2}{d_{g,x_0}^2} \,\mathrm{d}v_g. \tag{3.16}$$

Another relevant improvement of the Hardy inequality in $\mathbb{R}^n$ is due to Brezis and Vázquez [17, Theorem 4.1]; more precisely, if $\Omega \subseteq \mathbb{R}^n$ is a bounded domain ($n \geq 2$), then for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \geq \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} \,\mathrm{d}x + j_{0,1}^2 \left(\frac{\omega_n}{\mathrm{Vol}(\Omega)}\right)^{\frac{2}{n}} \int_\Omega u^2 \,\mathrm{d}x, \tag{3.17}$$

where $j_{0,1} \approx 2.4048$ is the first positive root of the Bessel function J_0, and ω_n is the volume of the unit Euclidean ball. Inequality (3.17) has been obtained by Schwarz symmetrization and an ingenious 1-dimensional analysis. In the sequel, by using our approach, we provide a Riemannian version of the result by Brezis and Vázquez [17], which sheds new light on the appearance of $j_{0,1}$ in inequality (3.17). As before, let $j_{\nu,k}$ be the k^{th} positive root of the Bessel function J_ν of the first kind and order $\nu \in \mathbb{R}$.

Theorem 3.14 (see [56]). *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold and $\Omega \subseteq M$ be a bounded domain. Let $x_0 \in \Omega$ and $D_\Omega = \sup_{x\in\Omega} d_{g,x_0}(x)$. Then for every $\nu \in \left[0, \frac{n-2}{2}\right]$ and $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\nabla_g u|^2 \,\mathrm{d}v_g \geq \left(\frac{(n-2)^2}{4} - \nu^2\right) \int_\Omega \frac{u^2}{d_{g,x_0}^2} \,\mathrm{d}v_g + \frac{j_{\nu,1}^2}{D_\Omega^2} \int_\Omega u^2 \,\mathrm{d}v_g. \tag{3.18}$$

3.2.3 Spectral estimates on Riemannian manifolds

Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 2$. Let $\Omega \subseteq M$ be a domain and $p > 1$. The *first Dirichlet eigenvalue* of Ω for the p-Laplace–Beltrami operator $-\Delta_{g,p}$ on (M, g) is given by

$$\Lambda_{\mathrm{m},p}(\Omega) = \inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_g u|^p \,\mathrm{d}v_g}{\int_\Omega |u|^p \,\mathrm{d}v_g}.$$

The first result of the section is *Cheng's comparison principle* (see Cheng [27]), whose original proof is based on Barta's argument.

Theorem 3.15 (see [56]). *Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa \in \mathbb{R}$. Fix $x_0 \in M$ and suppose that $0 < R < \min(\mathrm{inj}_{x_0}, \pi/\sqrt{\kappa})$ with the convention $\pi/\sqrt{\kappa} \stackrel{def}{=} \infty$, if $\kappa \leq 0$. Then one has*

$$\Lambda_{\mathrm{m},2}(B_{g,x_0}(R)) \geq \Lambda_{\mathrm{m},2}(B_\kappa(R)), \tag{3.19}$$

where $B_\kappa(R)$ is an arbitrary ball of radius R in the model space form $\mathbf{M}_\kappa^n$.

In the case when the domain is not a ball, as in Cheng's result, a more powerful argument is needed. We shall consider only the case when $\kappa = 0$, which corresponds to the famous Faber–Krahn inequality on Cartan–Hadamard manifolds:

Theorem 3.16 (see [56]). *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold, which satisfies the Cartan–Hadamard conjecture, and let $\Omega \subseteq M$ be a bounded domain. Then we have*

$$\Lambda_{\mathrm{m},2}(\Omega) \geq \Lambda_{\mathrm{m},2}(\Omega^*) = j^2_{\frac{n}{2}-1,1}\left(\frac{\omega_n}{\mathrm{Vol}_g(\Omega)}\right)^{2/n}, \tag{3.20}$$

where $\Omega^* \subseteq \mathbb{R}^n$ is a ball with $\mathrm{Vol}(\Omega^*) = \mathrm{Vol}_g(\Omega)$.

The next result is McKean's spectral gap estimate, established by McKean [75] for $p = 2$ by using fine properties of Jacobi fields; our argument is based on Riccati pairs.

Theorem 3.17 (see [56]). *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold, with sectional curvature $\mathbf{K} \leq \kappa < 0$. If $p > 1$, then*

$$\Lambda_{\mathrm{m},p}(M) \geq \left(\frac{n-1}{p}\sqrt{-\kappa}\right)^p. \tag{3.21}$$

The next result by Carvalho and Cavalcante [20, Theorem 1.1] concludes the section.

Theorem 3.18 (see [56]). *Let (M, g) be an n-dimensional Riemannian manifold, $n \geq 2$ and $\Omega \subseteq M$ be a domain. Given $p > 1$, we assume that there exists a function $\rho\colon \Omega \to \mathbb{R}$ such that $|\nabla_g \rho| \leq a$ and $\Delta_{g,p}\rho \geq b$ for some $a, b > 0$. Then*

$$\Lambda_{\mathrm{m},p}(\Omega) \geq \frac{b^p}{p^p a^{p(p-1)}}. \tag{3.22}$$

3.2.4 Interpolation: Hardy inequality versus McKean spectral gap

The main result of this section is to prove an interpolation between the classical Hardy inequality and McKean's spectral gap, established first by Berchio, Ganguly, Grillo, and Pinchover [11, Theorem 2.1] in the hyperbolic space $\mathbb{H}^n_{-1}$.

Theorem 3.19 (see [56]). *Let (M, g) be an $n \geq 3$-dimensional Cartan–Hadamard manifold, having sectional curvature $\mathbf{K} \leq \kappa < 0$, and $x_0 \in M$ be fixed. Then, for every $\lambda \in [n-2, \frac{(n-1)^2}{4}]$ and $u \in C_0^\infty(M \setminus \{x_0\})$ one has*

$$\begin{aligned}\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \lambda|\kappa| \int_M u^2 \,\mathrm{d}v_g + h_n^2(\lambda) \int_M \frac{u^2}{d^2_{g,x_0}} \,\mathrm{d}v_g \\ + |\kappa| \left(\frac{(n-2)^2}{4} - h_n^2(\lambda)\right) \int_M \frac{u^2}{\sinh^2(\sqrt{-\kappa}\, d_{g,x_0})} \,\mathrm{d}v_g \\ + h_n(\lambda)\gamma_n(\lambda) \int_M \frac{\mathbf{D}_\kappa(d_{g,x_0})}{d^2_{g,x_0}} u^2 \,\mathrm{d}v_g,\end{aligned} \tag{3.23}$$

where $\gamma_n(\lambda) = \sqrt{(n-1)^2 - 4\lambda}$ and $h_n(\lambda) = \frac{\gamma_n(\lambda)+1}{2}$.

A direct consequence of Theorem 3.19 can be stated as follows for the two marginal values of λ.

Corollary 3.20 (see [56]). *Let (M,g) be an $n \geq 3$-dimensional Cartan–Hadamard manifold with sectional curvature $\mathbf{K} \leq \kappa < 0$. If $x_0 \in M$ is fixed, then for every $u \in C_0^\infty(M \setminus \{x_0\})$ the following inequalities hold:*

(i) (Hardy improvement)

$$\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(n-2)^2}{4} \int_M \frac{u^2}{d_{g,x_0}^2} \,\mathrm{d}v_g + (n-2)|\kappa| \int_M u^2 \,\mathrm{d}v_g + \frac{(n-2)(n-3)}{2} \int_M \frac{\mathbf{D}_\kappa(d_{g,x_0})}{d_{g,x_0}^2} u^2 \,\mathrm{d}v_g. \tag{3.24}$$

(ii) (McKean spectral gap improvement)

$$\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(n-1)^2}{4}|\kappa| \int_M u^2 \,\mathrm{d}v_g + \frac{1}{4} \int_M \frac{u^2}{d_{g,x_0}^2} \,\mathrm{d}v_g + |\kappa| \frac{(n-1)(n-3)}{4} \int_M \frac{u^2}{\sinh^2(\sqrt{-\kappa} d_{g,x_0})} \,\mathrm{d}v_g. \tag{3.25}$$

Remark 3.21. (a) It is worth mentioning that inequalities from Theorem 3.19 and Corollary 3.20 are known to be *critical* on $\mathbb{H}^n_{-1}$ (see Devyver, Fraas, and Pinchover [33, Definition 2.1]). Classical criticality proofs are usually formulated using the approach of supersolutions; however, they can be adapted to Riccati pairs.

(b) We note that Berchio, Ganguly, and Grillo [10, Theorem 2.5] provided a more general version of the inequality (3.25) under a pointwise curvature assumption. Similar inequalities can also be obtained in terms of Riccati pairs by using an appropriate pointwise Laplace comparison (see e.g., Greene and Wu [51]).

The next result by Akutagawa and Kumura [3, Theorem 1.3/(5)] concludes the section.

Theorem 3.22 (see [56]). *Let (M,g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold with sectional curvature $\mathbf{K} \leq \kappa < 0$. Let $x_0 \in M$ be fixed, $R > 0$, and $\Omega = M \setminus B_{g,x_0}(R)$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\nabla_g u|^2 \,\mathrm{d}v_g \geq \int_\Omega \frac{(n-1)^2}{4}|\kappa| u^2 \,\mathrm{d}v_g + \int_\Omega \frac{u^2}{4\left(d_{g,x_0} - R + \frac{1}{(n-1)\mathbf{ct}_\kappa(R)}\right)^2} \,\mathrm{d}v_g + \int_\Omega |\kappa| \frac{(n-1)(n-3)}{4\sinh^2(\sqrt{-\kappa} d_{g,x_0})} u^2 \,\mathrm{d}v_g.$$

3.2.5 Ghoussoub–Moradifam-type inequalities

In this section, we consider some inequalities established by Ghoussoub and Moradifam [49, Theorem 2.12] (see also [50]), where the weights are of the form

$$(a+b|x|^\alpha)^\beta/|x|^{2m}$$

for some parameters $a, b > 0$. Possible extensions of these inequalities to the case of Cartan–Hadamard manifolds are also discussed; see Remark 3.24 and Theorem 3.25, where some technical difficulties are commented.

Theorem 3.23 (see [56])**.** *Suppose that $a, b > 0$ and $\alpha, \beta, m \in \mathbb{R}$. The following statements hold:*

(i) *If $\alpha\beta > 0$ and $m \leq \frac{n-2}{2}$, then for every $u \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}} |\nabla u|^2 \,\mathrm{d}x \geq \left(\frac{n-2m-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 \,\mathrm{d}x. \tag{3.26}$$

(ii) *If $\alpha\beta < 0$ and $2m - \alpha\beta \leq n-2$, then for every $u \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}} |\nabla u|^2 \,\mathrm{d}x \geq \left(\frac{n-2m+\alpha\beta-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 \,\mathrm{d}x. \tag{3.27}$$

Remark 3.24. One could expect a similar proof on Cartan–Hadamard manifolds as in Theorem 3.23. However, a subtle technical difficulty shows up that comes from the fact that – despite several confirming numerical tests – there is no evidence on *positiveness* of the function G for the full range of parameters.

Theorem 3.25 (see [56])**.** *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold, and $x_0 \in M$ be a point. Let $a, b, \alpha, \beta > 0$ and $m \in \mathbb{R}$, with $m \leq \frac{n-2}{2}$ and*

$$\alpha\beta + \sqrt{\alpha\beta(\alpha\beta + 2(n-2m-2))} \leq 2. \tag{3.28}$$

Then for every $u \in C_0^\infty(M)$ the following inequality holds:

$$\int_M \frac{(a+bd_{g,x_0}^\alpha)^\beta}{d_{g,x_0}^{2m}} |\nabla_g u|^2 \,\mathrm{d}v_g \geq \left(\frac{n-2m-2}{2}\right)^2 \int_M \frac{(a+bd_{g,x_0}^\alpha)^\beta}{d_{g,x_0}^{2m+2}} u^2 \,\mathrm{d}v_g. \tag{3.29}$$

Remark 3.26. As we already pointed out in Remark 3.24, numerical tests confirm the positiveness of G for every $a, b > 0$ and $\alpha, \beta, m \in \mathbb{R}$, whose proof requires some specific arguments from the theory of special functions. At this moment, such an approach is not available. In particular, we expect to cancel the additional hypothesis from Theorem 3.25.

3.3 Applications II: Multiplicative Hardy-type inequalities

3.3.1 Sharp uncertainty principles

Let (M,g) be a complete, non-compact n-dimensional Riemannian manifold ($n \geq 2$), and $x_0 \in M$ be fixed. Suppose that $p, \alpha \in \mathbb{R}$ such that

$$n > p > 1 \quad \text{and} \quad -p+1 < \alpha \leq 1. \tag{3.30}$$

We investigate the following *uncertainty principle*: for every $u \in C_0^\infty(M)$, one has

$$\left(\int_M |\nabla_g u|^p \, \mathrm{d}v_g\right)^{\frac{1}{p}} \left(\int_M d_{g,x_0}^{p'\alpha} |u|^p \, \mathrm{d}v_g\right)^{\frac{1}{p'}} \geq \frac{n+\alpha-1}{p} \int_M \left(1 + \frac{n-1}{n+\alpha-1} \mathbf{D}_\kappa(d_{g,x_0})\right) d_{g,x_0}^{\alpha-1} |u|^p \, \mathrm{d}v_g. \tag{$\mathbf{UP}_\kappa$}$$

We observe that ($\mathbf{UP}_\kappa$) formally reduces to the:

- *Heisenberg–Pauli–Weyl uncertainty principle*, whenever $\alpha = 1$ (see Kombe and Özaydin [61, 62] for $p = 2$ and $\kappa = 0$, Kristály [64] for $p = 2$ and $\kappa \leq 0$, and Nguyen [85] for generic $p > 1$ and $\kappa \leq 0$);
- *Hydrogen uncertainty principle*, whenever $\alpha = 0$ (see Cazacu, Flynn, and Lam [22] and Frank [46] in $\mathbb{R}^n$, thus, for $\kappa = 0$);
- *Hardy inequality* in the limit case when $\alpha \to -p+1$ (see Corollary 3.11).

Our first result shows the validity of ($\mathbf{UP}_\kappa$) on Cartan–Hadamard manifolds, which can be stated as follows.

Theorem 3.27 (see [56])**.** *Let (M,g) be an n-dimensional Cartan–Hadamard manifold ($n \geq 2$), such that $\mathbf{K} \leq \kappa \leq 0$. If the conditions of (3.30) hold, then ($\mathbf{UP}_\kappa$) holds as well; moreover, the constant $\frac{n+\alpha-1}{p}$ is sharp.*

Remark 3.28. If equality holds in ($\mathbf{UP}_\kappa$) for some positive function $u \in W^{1,p}(M)$, then equality $\Delta_g d_{g,x_0} = (n-1)\mathbf{ct}_\kappa(d_{g,x_0})$ also holds, which implies that the manifold (M,g) is isometric to the model space form $\mathbf{M}_\kappa^n$. This rigidity result is known for $\kappa = 0$ from Kristály [64] for $p = 2$, and from Nguyen [85] for $p > 1$.

The following result is a counterpart of Theorem 3.27, providing the *rigidity* of Riemannian manifolds with $\mathbf{Ric} \geq \kappa(n-1)g$ for some $\kappa \leq 0$ supporting the uncertainty principle ($\mathbf{UP}_\kappa$). Similar results have been obtained first by Kristály [64] for $p = 2$ and $\alpha = 1$, and then by Nguyen [85] for generic $p > 1$, both considering only the case $\kappa = 0$. Now, we have a more general result, valid for every $\kappa \leq 0$:

Theorem 3.29 (see [56])**.** *Let (M,g) be an $n \geq 2$-dimensional complete, non-compact Riemannian manifold, with $\mathbf{Ric} \geq \kappa(n-1)g$, for some $\kappa \leq 0$. Suppose that ($\mathbf{UP}_\kappa$) holds for some $x_0 \in M$ and the parameters α, p, n verify either $-p+1 < \alpha \leq 1 < p < n$, when $\kappa = 0$, or $0 < \alpha \leq 1 < p < n$, when $\kappa < 0$. Then (M,g) is isometric to the model space form $\mathbf{M}_\kappa^n$.*

3.3.2 Caffarelli–Kohn–Nirenberg inequalities

In this section, we prove a version of the *Caffarelli–Kohn–Nirenberg inequality* (see [18]), which easily follows from the multiplicative inequality of Theorem 3.1/(ii).

Theorem 3.30 (see [56]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ such that $\mathbf{K} \leq \kappa$ for some $\kappa \leq 0$. Fix $x_0 \in M$ and suppose that $p, \alpha, r \in \mathbb{R}$ satisfy*

$$r > p > 1, \quad \alpha + p > 1, \quad \text{and} \quad p(n + \alpha - 1) > r(n - p) > 0.$$

Then for every $u \in C_0^\infty(M)$ one has

$$\left(\int_M |\nabla_g u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p}} \left(\int_M d_{g,x_0}^{p'\alpha} |u|^{p'(r-1)} \,\mathrm{d}v_g\right)^{\frac{1}{p'}} \geq \frac{n + \alpha - 1}{r} \int_M \left(1 + \frac{n-1}{n + \alpha - 1} \mathbf{D}_\kappa(d_{g,x_0})\right) d_{g,x_0}^{\alpha - 1} |u|^r \,\mathrm{d}v_g.$$

Moreover, the constant $\frac{n+\alpha-1}{r}$ is sharp.

As we already pointed out, various choices of H and G in Theorem 3.1 produce well-known or new functional inequalities. In this spirit, we conclude the section with an unusual Caffarelli–Kohn–Nirenberg-type inequality, which can be the starting point to build further functional inequalities through Theorem 3.1.

Theorem 3.31 (see [56]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold $(n \geq 2)$, such that $\mathbf{K} \leq \kappa < 0$. Then for every $u \in C_0^\infty(M) \setminus \{0\}$ and $c \in \mathbb{R}$, we have*

$$\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{\left(\int_M \mathbf{s}_c^2(u) \,\mathrm{d}v_g\right)^2}{\int_M \mathbf{s}_c^2(2u) \,\mathrm{d}v_g} (n-1)^2 |\kappa|. \tag{3.31}$$

In particular, we also have the McKean spectral gap estimate

$$\lambda_{\mathrm{m}}(M) \geq \frac{(n-1)^2}{4} |\kappa|.$$

Chapter 4

Rellich inequalities on general Riemannian manifolds

In this chapter, we present our first approach to Rellich inequalities. In § 4.1 we present our abstract approach. In § 4.2 we establish spectral gap estimates for various higher-order eigenvalue problems on general Riemannian manifolds, while in § 4.3 we provide alternative proofs for additional Rellich inequalities.

4.1 General functional inequalities

In this section, we present two general functional inequalities. The first inequality connects $|\Delta_g u|^p$ and $|u|^p$, for general $p > 1$. The statement is as follows.

Theorem 4.1 (see [38]). *Let (M, g) be an $n \geq 2$-dimensional, complete, non-compact Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $x_0 \in \Omega$, and $\rho = d_{g,x_0}$. Let $p > 1$ and suppose that $L, W, w, G, H \colon (0, \sup_\Omega \rho) \to (0, \infty)$ satisfy the following conditions:*

(C1) *L, W are continuous, w, G are of class C^2, and H is of class C^1;*

(C2) *$\Delta_g \rho \geq L(\rho)$ in the distributional sense and $(wG)' \leq 0$;*

(C3) *the ordinary differential inequality*

$$(p-1)\left[2(wGH)' + 2wGHL - pwGH^2 - w|G|^{p'}\right] - (wG)'' - (wG)'L \geq W \quad (4.1)$$

holds for the functions $L(t), W(t), w(t), G(t), H(t)$, for all $t \in (0, \sup_\Omega \rho)$.

Then for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega w(\rho)|\Delta_g u|^p \, \mathrm{d}v_g \geq \int_\Omega W(\rho)|u|^p \, \mathrm{d}v_g.$$

The second functional inequality connects $|\Delta_g u|^2$ and $|\nabla_g u|^2$. The statement is as follows.

Theorem 4.2 (see [38]). *Let (M,g) be an $n \geq 2$-dimensional complete, non-compact Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $x_0 \in \Omega$, and $\rho = d_{g,x_0}$. Suppose that $L, W, G, H \colon (0, \sup_\Omega \rho) \to (0, \infty)$ satisfy the following conditions:*

(C1') *L, W are continuous, G is of class C^2, and H is of class C^1;*

(C2') *$\Delta_g \rho \geq L(\rho)$ in the distributional sense;*

(C3') *the following PDI holds for $\rho = d_{g,x_0}(x)$ and for every $x \in \Omega$:*

$$(W(\rho)H(\rho))' + W(\rho)H(\rho)L(\rho) - W(\rho)H(\rho)^2 \geq \Delta_g G(\rho) + G(\rho)^2. \tag{4.2}$$

Then for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega |\Delta_g u|^2 \, dv_g \geq \int_\Omega (2G(\rho) - W(\rho)) |\nabla_g u|^2 \, dv_g.$$

4.2 Applications I: Spectral gap estimates

In this section, we establish sharp spectral estimates on Cartan–Hadamard manifolds for the clamped plate problem (for general $p > 1$), the buckling problem (for $p = 2$) and their higher-order variants. All the proofs are built upon convexity and comparison arguments; moreover, they are symmetrization-free.

4.2.1 Clamped plate problem

Our spectral gap result concerning the clamped plate problem reads as follows.

Theorem 4.3 (see [38]). *Let (M,g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$, for some $\kappa < 0$. Let $p > 1$ and $\Omega \subseteq M$ be a domain. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^p \, dv_g \geq \left(\frac{(n-1)^2 |\kappa| (p-1)}{p^2} \right)^p \int_\Omega |u|^p \, dv_g. \tag{4.3}$$

Moreover, the constant in (4.3) is sharp.

We highlight that the estimate from Theorem 4.3 is a novel result that has not yet been established in such a general context.

4.2.2 Buckling problem

Our spectral gap result concerning the buckling problem reads as follows.

Theorem 4.4 (see [38]). *Let (M,g) be an n-dimensional Cartan–Hadamard manifold, with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$, for some $\kappa < 0$. If $\Omega \subseteq M$ is a domain, then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^2 \, dv_g \geq \frac{(n-1)^2 |\kappa|}{4} \int_\Omega |\nabla_g u|^2 \, dv_g. \tag{4.4}$$

Moreover, the constant in (4.4) is sharp.

We note that Theorem 4.4 is a novel result, which provides the spectral gap estimate in general Cartan–Hadamard manifolds for $p = 2$.

4.2.3 Higher-order estimates

In the sequel, we present higher-order estimates concerning both the clamped plate problem and the buckling problem.

Theorem 4.5 (see [38]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$. Let $p > 1$ and $\Omega \subseteq M$ be a domain. Then for every $u \in C_0^\infty(\Omega)$ and $k \geq 1$ with $k \in \mathbb{N}$, one has*

$$\int_\Omega |\Delta_g^k u|^p \, \mathrm{d}v_g \geq \left(\frac{(n-1)^2(p-1)\kappa^2}{p^2} \right)^{kp} \int_\Omega |u|^p \, \mathrm{d}v_g, \tag{4.5}$$

$$\int_\Omega |\nabla_g \Delta_g^k u|^p \, \mathrm{d}v_g \geq \left(\frac{(n-1)\kappa}{p} \right)^p \left(\frac{(n-1)^2(p-1)\kappa^2}{p^2} \right)^{kp} \int_\Omega |u|^p \, \mathrm{d}v_g. \tag{4.6}$$

Moreover, the constants in (4.5) and (4.6) are sharp.

Theorem 4.6 (see [38]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$. If $\Omega \subseteq M$ is a domain, then for every $u \in C_0^\infty(\Omega)$ and $k \geq 1$ one has*

$$\int_\Omega |\Delta_g^k u|^2 \, \mathrm{d}v_g \geq \left(\frac{(n-1)\kappa}{2} \right)^{4k-2} \int_\Omega |\nabla_g u|^2 \, \mathrm{d}v_g, \tag{4.7}$$

$$\int_\Omega |\nabla_g \Delta_g^k u|^2 \, \mathrm{d}v_g \geq \left(\frac{(n-1)\kappa}{2} \right)^{4k} \int_\Omega |\nabla_g u|^2 \, \mathrm{d}v_g. \tag{4.8}$$

4.3 Applications II: Rellich inequalities

In this section, we present additional applications of our general functional inequalities.

4.3.1 Classical and weighted Rellich inequalities

The extension of the weighted Rellich inequality reads as follow; for the Euclidean version, see Mitidieri [79, Theorem 3.1].

Theorem 4.7 (see [38]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold, with $n \geq 5$. Let $\Omega \subseteq M$ be a domain, and $p, \gamma \in \mathbb{R}$ such that*

$$1 < p < n/2 \quad \text{and} \quad 2 - \frac{n}{p} < \gamma < \frac{n(p-1)}{p}.$$

Fix $x_0 \in \Omega$ and let $\rho = d_{g,x_0}$. Then for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega \rho^{\gamma p} |\Delta_g u|^p \, \mathrm{d}v_g \geq \left(\frac{n}{p} - 2 + \gamma \right)^p \left(\frac{n(p-1)}{p} - \gamma \right)^p \int_\Omega \frac{|u|^p}{\rho^{(2-\gamma)p}} \, \mathrm{d}v_g. \tag{4.9}$$

Corollary 4.8 (see [38]). *By choosing $\gamma = 0$ in Theorem 4.7, we obtain the extension of the classical Rellich inequality; namely, one has for every $u \in C_0^\infty(\Omega)$ that*

$$\int_\Omega |\Delta_g u|^p \, \mathrm{d}v_g \geq \left(\frac{n}{p} - 2\right)^p \left(\frac{n(p-1)}{p}\right)^p \int_\Omega \frac{|u|^p}{\rho^{2p}} \, \mathrm{d}v_g.$$

In particular, for $p = 2$ one has for every $u \in C_0^\infty(\Omega)$ that

$$\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{|u|^2}{\rho^4} \, \mathrm{d}v_g. \tag{4.10}$$

4.3.2 Higher-order variants of the classical Rellich inequality

Extension of higher-order Rellich inequalities can be stated as follows; for the Euclidean versions, see Mitidieri [79, Theorem 3.3].

Theorem 4.9 (see [38]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold, with $n \geq 5$. Let $\Omega \subseteq M$ be a domain, fix $x_0 \in \Omega$ and define $\rho = d_{g,x_0}$. The following inequalities hold:*

(i) *If $k \geq 1$ and $n > 2kp$, then*

$$\int_\Omega |\Delta_g^k u|^p \, \mathrm{d}v_g \geq \Lambda_{\mathrm{r};1}(k,p) \int_\Omega \frac{|u|^p}{\rho^{2kp}} \, \mathrm{d}v_g, \qquad \forall u \in C_0^\infty(\Omega),$$

where

$$\Lambda_{\mathrm{r};1}(k,p) = \prod_{s=1}^{k} \left(\frac{n}{p} - 2s\right)^p \left(\frac{n(p-1)}{p} + 2s - 2\right)^p.$$

(ii) *If $k \geq 1$ and $n > (2k+1)p$, then*

$$\int_\Omega |\nabla_g \Delta_g^k u|^p \, \mathrm{d}v_g \geq \Lambda_{\mathrm{r};2}(k,p) \int_\Omega \frac{|u|^p}{\rho^{(2k+1)p}} \, \mathrm{d}v_g, \qquad \forall u \in C_0^\infty(\Omega)$$

where

$$\Lambda_{\mathrm{r};2}(k,p) = \left(\frac{n-p}{p}\right)^p \prod_{s=1}^{k} \left(\frac{n}{p} - 2s - 1\right)^p \left(\frac{n(p-1)}{p} + 2s - 1\right)^p.$$

4.3.3 Further Rellich inequalities

In this section we present further applications to our general functional inequalities providing short proofs for additional Rellich-type inequalities.

Theorem 4.10 (see [38]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold ($n \geq 5$). Let $\Omega = B_{g,x_0}(1) \subseteq M$ be a ball centered at $x_0 \in M$ with unit radius. Define $\rho = d_{g,x_0}$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g \geq \frac{n^2(n-4)^4}{16} \int_\Omega \frac{u^2}{\rho^4} \, \mathrm{d}v_g + \frac{n(n-4)j_{0,1}^2}{2} \int_\Omega \frac{u^2}{\rho^2} \, \mathrm{d}v_g,$$

where $j_{0,1}$ denotes the first positive zero of the Bessel function J_0.

The second result deals with the case when $\mathbf{K} \leq \kappa$ for some $\kappa < 0$, and can be formulated as follows.

Theorem 4.11 (see [38])**.** *Let (M, g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 5$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$. Let $\Omega \subseteq M$ be a domain, fix $x_0 \in \Omega$ and define $\rho = d_{g,x_0}$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g \geq \frac{(n-1)^4 |\kappa|^2}{16} \int_\Omega u^2 \, \mathrm{d}v_g + \frac{(n-1)^2 |\kappa|}{8} \int_\Omega \frac{u^2}{\rho^2} \, \mathrm{d}v_g + \frac{(n-1)^3 (n-3) |\kappa|^2}{8} \int_\Omega \frac{u^2}{\sinh^2(\kappa \rho)} \, \mathrm{d}v_g.$$

The third result of the section is a simple application of Theorem 4.2.

Theorem 4.12 (see [38])**.** *Let (M, g) be an n-dimensional Cartan–Hadamard manifold $(n \geq 8)$, $\Omega \subseteq M$ be a domain, $x_0 \in M$ be fixed and define $\rho = d_{g,x_0}$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g \geq \frac{n^2}{4} \int_\Omega \frac{|\nabla_g u|^2}{\rho^2} \, \mathrm{d}v_g.$$

Remark 4.13. Note that Theorem 4.12 is expected to hold for every $n \geq 5$; however, the technical condition $n \geq 8$ is required to guarantee the applicability of Theorem 4.2 ($W > 0$ whenever $n \geq 9$, and if $n = 8$, then $W = 0$, in which case the proof of Theorem 4.2 is obvious). A similar restriction also appeared in the Finsler context when proving quantitative Rellich inequalities; see Kristály and Repovš [70], where another approach was applied.

Chapter 5

Rellich inequalities on space forms

In this chapter, using the idea of Riccati pairs, we provide a second approach for proving Rellich inequalities on model space forms: In § 5.1 we present our abstract approach, while in § 5.2 & 5.3 we provide applications in the Euclidean spaces and hyperbolic spaces, respectively.

5.1 General functional inequalities

In this section, we present our abstract approach. Motivated by the simplify of the underlying geometrical setting, we also refactor the definition of Riccati pairs.

5.1.1 Simplified Riccati pairs and dual Riccati pairs

Let $\kappa \leq 0$ and define $L_\kappa \colon (0,\infty) \to (0,\infty)$, by

$$L_\kappa(t) = (n-1)\mathbf{ct}_\kappa(t).$$

A simplified definition of Riccati pairs refactored for space forms is as follows (for the original version, see Definition 3.2).

Definition 5.1 (see [55]). Let $\kappa \leq 0$ and $\Omega \subseteq \mathbf{M}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and let $\rho = d_{\kappa,x_0}$ the distance from x_0. Suppose that $w, W \colon (0, \sup_\Omega \rho) \to [0,\infty)$ are smooth functions. The couple (L_κ, W) is a (ρ, w)-*simplified Riccati pair* on $(0, \sup_\Omega \rho)$ if there exists a smooth function $G \colon (0, \sup_\Omega \rho) \to \mathbb{R}$ such that the following ODI holds:

$$G'(t) + \left(L_\kappa(t) + \frac{w'(t)}{w(t)}\right) G(t) - G(t)^2 \geq W(t), \qquad \forall t \in (0, \sup_\Omega \rho). \tag{5.1}$$

A function G satisfying (5.1) is said to be *admissible* for (L_κ, W).

The above inequality is a driving force for the Hardy-type functional inequalities. However, as we shall see soon, the Rellich-type inequalities are more compatible with its dual version, which can be stated as follows.

Definition 5.2 (see [55]). Let $\kappa \leq 0$ and $\Omega \subseteq \mathbf{M}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and let $\rho = d_{\kappa,x_0}$ the distance from x_0. Suppose that $v, V\colon (0, \sup_\Omega \rho) \to [0,\infty)$ are smooth functions. The couple (L_κ, V) is a (ρ, v)-*dual Riccati pair* on $(0, \sup_\Omega \rho)$ if there exists a smooth function $H\colon (0, \sup_\Omega \rho) \to \mathbb{R}$ such that the following ODI holds:

$$-H'(t) + \left(L_\kappa(t) - \frac{v'(t)}{v(t)}\right) H(t) - H(t)^2 \geq V(t), \qquad \forall t \in (0, \sup_\Omega \rho). \tag{5.2}$$

A function H satisfying (5.2) is said to be *dual admissible* for (L_κ, V).

The first result of the section is a general functional inequality, which can be stated as follows.

Theorem 5.3 (see [55]). *Let $\kappa \leq 0$ and $\Omega \subseteq \mathbf{M}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and define $\rho = d_{\kappa,x_0}$. Suppose that (L_κ, V) is a dual Riccati pair on $(0, \sup_\Omega \rho)$ and H is dual admissible for (L_κ, V). Then the following statements hold.*

(i) *For every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega v(\rho)|\Delta_\kappa u|^2 \,\mathrm{d}x_\kappa \geq \int_\Omega v(\rho)V(\rho)|\nabla^{\mathrm{rad}}_\kappa u|^2 \,\mathrm{d}x_\kappa, \tag{5.3}$$

provided that $E_1(t) = (v(t)H(t))' + v(t)H(t)(L_\kappa(t) - 2\mathbf{ct}_\kappa(t)) \geq 0$, for all $t > 0$.

(ii) *For every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega v(\rho)|\Delta_\kappa u|^2 \,\mathrm{d}x_\kappa \geq \int_\Omega v(\rho)V(\rho)|\nabla_\kappa u|^2 \,\mathrm{d}x_\kappa, \tag{5.4}$$

provided that $E_2(t) = 2(v(t)H(t))' + v(t)H(t)(H(t) - 2\mathbf{ct}_\kappa(t)) \geq 0$, for all $t > 0$.

The second result of the section is tailored to complement Theorem 5.3/(i). It can be stated as follows.

Theorem 5.4 (see [55]). *Let $\kappa \leq 0$ and $\Omega \subseteq \mathbf{M}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and define $\rho = d_{\kappa,x_0}$. Suppose that (L_κ, V) is a simplified Riccati pair on $(0, \sup_\Omega \rho)$ and H is admissible for (L_κ, V). Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega w(\rho)|\nabla^{\mathrm{rad}}_\kappa u|^2 \,\mathrm{d}x_\kappa \geq \int_\Omega w(\rho)W(\rho)u^2 \,\mathrm{d}x_\kappa. \tag{5.5}$$

We note that Theorems 5.3/(i) & 5.4 can be efficiently use to prove Rellich-type inequalities on model space forms. This can also be done by combining Theorem 5.3/(ii) and Theorem 3.3 but at the price of some additional conditions, typically involving restrictive dimension constraints.

5.2 Applications I: Inequalities on Euclidean spaces

In the Euclidean setting, a number of well-known Rellich inequalities are discussed by Ghoussoub and Moradifam [49] in terms of Bessel potentials and/or Bessel pairs. After recalling the related concepts, we discuss two novel inequalities among them, which highlight the strengths and the limitations of our method. For the original proofs of the selected inequalities (5.7) & (5.8), see Adimurthi, Grossi, and Santra [2].

Recall that a function $Z > 0$ is a *Bessel potential* on $(0, R)$ if there exist a constant $c > 0$ and a function $z > 0$ such that following ODE holds:

$$z''(t) + \frac{z'(t)}{t} + c \cdot Z(t) \cdot z(t) = 0, \qquad \forall t \in (0, R). \tag{5.6}$$

Using Theorems 5.3/(i) & 5.4 we obtain the following general result.

Theorem 5.5 (see [55]). *Let $n \geq 5$ and $B \subseteq \mathbb{R}^n$ be a ball centered at the origin with radius $R > 0$. Suppose that Z is a Bessel potential on $(0, R)$ with solution z and best constant c, such that*

$(\mathbf{Z})$: $\frac{Z'(t)}{Z(t)} = -\frac{\lambda}{t} + f(t)$, *where* $\lambda < n - 2$, $f(t) \geq 0$ *and* $\lim_{t\to 0} tf(t) = 0$.

holds. Define $H(t) = \frac{n}{2t} + \frac{z'(t)}{z(t)}$. If $E_1(t) = H'(t) + H(t) \cdot \frac{n-3}{t} \geq 0$ for all $t \in (0, R)$, then for every $u \in C_0^\infty(B)$ one has

$$\int_B |\Delta u|^2 \,\mathrm{d}x \geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} \,\mathrm{d}x + c\left(\frac{n^2}{4} + \frac{(n-\lambda-2)^2}{4}\right) \int_B \frac{Z(|x|)u^2}{|x|^2} \,\mathrm{d}x.$$

In the sequel, for an arbitrary function h define

$$h_{[0]}(t) = t, \quad h_{[1]}(t) = h(t) \quad \text{and} \quad h_{[i]}(t) = h(h_{[i-1]}(t)), \qquad \forall i \geq 2.$$

A corollary of Theorem 5.5 can be stated as follows.

Corollary 5.6 (see [55]). *Let $n \geq 5$ and $B \subseteq \mathbb{R}^n$ be a ball centered at the origin with radius $R > 0$. If $k \geq 1$ and $r = R \cdot \exp_{[k-1]}(e)$, then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_B |\Delta u|^2 \,\mathrm{d}x \geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} \,\mathrm{d}x + \left(1 + \frac{n(n-4)}{8}\right) \sum_{j=1}^{k} \int_B \frac{u^2}{|x|^4} \left(\prod_{i=1}^{j} \log_{[i]}\left(\frac{r}{|x|}\right)\right)^{-2} \mathrm{d}x. \tag{5.7}$$

The limitations of Theorem 5.5 can be illustrated as follows.

Remark 5.7. Let $\ell(t) = \frac{1}{1-\log(t)}$, $k \geq 1$, $R > 0$ and consider the inequality

$$\int_\Omega |\Delta u|^2 \,\mathrm{d}x \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \,\mathrm{d}x + \left(1 + \frac{n(n-4)}{8}\right) \sum_{j=1}^{k} \int_\Omega \frac{u^2}{|x|^4} \prod_{i=1}^{j} \ell_{[i]}^2\left(\frac{|x|}{R}\right) \mathrm{d}x. \tag{5.8}$$

The inequality (5.8) is generated by the Bessel potential with parameters

$$\widetilde{Z}_{k,R}(t)=\sum_{j=1}^{k}\frac{1}{t^2}\prod_{i=1}^{j}\ell_{[i]}^2\left(\frac{t}{R}\right),\quad \widetilde{z}_{k,R}(t)=\left(\prod_{i=1}^{k}\ell_{[i]}\left(\frac{t}{R}\right)\right)^{-\frac{1}{2}},\qquad \forall t\in(0,R),$$

and $c=\frac{1}{4}$. Moreover, Condition (Z) holds for $\lambda=2$. Observe that

$$\lim_{t\to R}E_1(t)=\lim_{t\to R}\left(H'(t)+H(t)\cdot\frac{n-3}{t}\right)=\frac{n-k}{2},\quad \text{where}\quad H(t)=\frac{n}{2t}+\frac{\widetilde{z}_{k,r}'(t)}{\widetilde{z}_{k,r}(t)}.$$

The above relation shows that if $k>n$, the positivity condition does not hold, and therefore, Theorem 5.5 cannot be applied.

5.3 Applications II: Inequalities on hyperbolic spaces

In this section, we present applications to our results on hyperbolic spaces. The first result in this setting is the following interpolation inequality.

Theorem 5.8 (see [55]). *Let $\kappa<0$, $n\geq 5$, and $\Omega\subseteq\mathbb{H}_\kappa^n$ be an open domain. Fix $x_0\in\Omega$ and denote by $\rho=d_{\kappa,x_0}$ the Riemannian distance from x_0. Then for every $u\in C_0^\infty(\Omega)$ one has*

$$\begin{aligned}\int_\Omega|\Delta_\kappa u|^2\,\mathrm{d}x_\kappa\geq{}&|\kappa|\lambda\int_\Omega|\nabla_\kappa^{\mathrm{rad}}u|^2\,\mathrm{d}x_\kappa+h_n^2(\lambda)\int_\Omega\frac{|\nabla_\kappa^{\mathrm{rad}}u|^2}{\rho^2}\,\mathrm{d}x_\kappa\\&+|\kappa|\left(\frac{n^2}{4}-h_n^2(\lambda)\right)\int_\Omega\frac{|\nabla_\kappa^{\mathrm{rad}}u|^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa\\&+\gamma_n(\lambda)h_n(\lambda)\int_\Omega\frac{(\rho\,\mathbf{ct}_\kappa(\rho)-1)}{\rho^2}|\nabla_\kappa^{\mathrm{rad}}u|^2\,\mathrm{d}x_\kappa,\end{aligned}\tag{5.9}$$

where $0\leq\lambda\leq\frac{(n-1)^2}{4}$, $\gamma_n(\lambda)=\sqrt{(n-1)^2-4\lambda}$ and $h_n(\lambda)=\frac{\gamma_n(\lambda)+1}{2}$.

Direct consequences can be stated as follows for the two marginal values of λ.

Corollary 5.9 (see [55]). *Choose $\lambda=0$ in (5.9) to obtain for every $u\in C_0^\infty(\Omega)$ that*

$$\int_\Omega|\Delta_\kappa u|^2\,\mathrm{d}x_\kappa\geq\frac{n^2}{4}\int_\Omega\frac{|\nabla_\kappa^{\mathrm{rad}}u|^2}{\rho^2}\,\mathrm{d}x_\kappa+\frac{n(n-1)}{2}\int_\Omega\frac{(\rho\,\mathbf{ct}_\kappa(\rho)-1)}{\rho^2}|\nabla_\kappa^{\mathrm{rad}}u|^2\,\mathrm{d}x_\kappa.$$

Corollary 5.10 (see [55]). *Choose $\lambda=\frac{(n-1)^2}{4}$ in (5.9) to obtain for every $u\in C_0^\infty(\Omega)$ that*

$$\begin{aligned}\int_\Omega|\Delta_\kappa u|^2\,\mathrm{d}x_\kappa\geq{}&\frac{(n-1)^2|\kappa|}{4}\int_\Omega|\nabla_\kappa^{\mathrm{rad}}u|^2\,\mathrm{d}x_\kappa+\frac{1}{4}\int_\Omega\frac{|\nabla_\kappa^{\mathrm{rad}}u|^2}{\rho^2}\,\mathrm{d}x_\kappa\\&+\frac{(n^2-1)|\kappa|}{4}\int_\Omega\frac{|\nabla_\kappa^{\mathrm{rad}}u|^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa.\end{aligned}$$

Lower-order counterparts of Corollary 5.10 can be stated as follows.

Theorem 5.11 (see [55]). *Let $\kappa < 0$, $n \geq 5$, and $\Omega \subseteq \mathbb{H}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and denote by $\rho = d_{\kappa,x_0}$ the Riemannian distance from x_0. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\nabla^{\mathrm{rad}}_\kappa u|^2 \,\mathrm{d}x_\kappa \geq \frac{(n-1)^2|\kappa|}{4}\int_\Omega u^2\,\mathrm{d}x_\kappa + \frac{1}{4}\int_\Omega \frac{u^2}{\rho^2}\,\mathrm{d}x_\kappa + \frac{(n-1)(n-3)|\kappa|}{4}\int_\Omega \frac{u^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa, \tag{5.10}$$

$$\int_\Omega \frac{|\nabla^{\mathrm{rad}}_\kappa u|^2}{\rho^2}\,\mathrm{d}x_\kappa \geq \frac{9}{4}\int_\Omega \frac{u^2}{\rho^4}\,\mathrm{d}x_\kappa - (n-1)\int_\Omega \frac{\mathbf{ct}_\kappa(\rho)}{\rho^3}\,\mathrm{d}x_\kappa + \frac{(n-1)^2|\kappa|}{4}\int_\Omega \frac{u^2}{\rho^2}\,\mathrm{d}x_\kappa + \frac{(n-1)(n-3)|\kappa|}{4}\int_\Omega \frac{u^2}{\rho^2\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x, \tag{5.11}$$

$$\int_\Omega \frac{|\nabla^{\mathrm{rad}}_\kappa u|^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa \geq \frac{1}{4}\int_\Omega \frac{u^2}{t^2\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa + \frac{(n-3)^2|\kappa|}{4}\int_\Omega \frac{u^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa + \frac{(n-3)(n-5)|\kappa|}{4}\int_\Omega \frac{u^2}{\sinh^4(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa. \tag{5.12}$$

Remark 5.12. We note that non-radial versions of the latter three inequalities can be obtained using Theorem 3.3 by choosing the same parameter functions; see also Section 3.2.4 and the results therein.

We conclude the section with the following result, which combines Theorem 5.11 and Corollary 5.10. This last result can be stated as follows.

Theorem 5.13 (see [55]). *Let $\kappa < 0$, $n \geq 5$, and $\Omega \subseteq \mathbb{H}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and denote $\rho = d_{\kappa,x_0}$ the Riemannian distance from x_0. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\begin{aligned}\int_\Omega |\Delta_\kappa u|^2\,\mathrm{d}x_\kappa \geq{}& \frac{(n-1)^4|\kappa|}{16}\int_\Omega u^2\,\mathrm{d}x_\kappa + \frac{(n-1)^2|\kappa|}{8}\int_\Omega \frac{u^2}{\rho^2}\,\mathrm{d}x_\kappa \\ &+ \frac{(n-1)^2|\kappa|}{8}\int_\Omega \frac{u^2}{\rho^2\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa \\ &+ \frac{(n-1)(n-3)(n^2-2n-1)\kappa^2}{8}\int_\Omega \frac{u^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa \\ &- \frac{(n-1)\kappa}{4}\int_\Omega \frac{\mathbf{ct}_\kappa(\rho)|u^2|}{t^3}\,\mathrm{d}x_\kappa \\ &+ \frac{(n^2-1)(n-3)(n-5)\kappa^2}{16}\int_\Omega \frac{u^2}{\sinh^4(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa + \frac{9}{16}\int_\Omega \frac{u^2}{\rho^4}\,\mathrm{d}x_\kappa.\end{aligned}$$

Bibliography

[1] Adimurthi, N. Chaudhuri, and M. Ramaswamy. An improved Hardy-Sobolev inequality and its applications. *Proc. Amer. Math. Soc.*, 130(2):489–505, 2002.

[2] Adimurthi, M. Grossi, and S. Santra. Optimal Hardy–Rellich inequalities, maximum principle and related eigenvalue problem. *J. Funct. Anal.*, 240(1):36–83, 2006.

[3] K. Akutagawa and H. Kumura. Geometric relative Hardy inequalities and the discrete spectrum of Schrödinger operators on manifolds. *Calc. Var. Partial Differ. Equ.*, 48:67–88, 2013.

[4] W. Allegretto. On the equivalence of two types of oscillation for elliptic operators. *Pac. J. Math.*, 55(2):319–328, 1974.

[5] P.R.S. Antunes, D. Buoso, and P. Freitas. On the behavior of clamped plates under large compression. *SIAM J. Appl. Math.*, 79(5):1872–1891, 2019.

[6] M. Ashbaugh and R. Benguria. On Rayleigh's conjecture for the clamped plate and its generalization to three dimensions. *Duke Math. J.*, 78(1):1–17, 1995.

[7] A. A. Balinsky, W. D. Evans, and R. T. Lewis. *The analysis and geometry of Hardy's inequality.* Springer Verlag Universitext, 2015.

[8] G. Barbatis, S. Filippas, and A. Tertikas. A unified approach to improved L^p Hardy inequalities with best constants. *Trans. Amer. Math. Soc.*, 356(6):2169–2196, 2004.

[9] E. Berchio, L. D'Ambrosio, D. Ganguly, and G. Grillo. Improved L^p-Poincaré inequalities on the hyperbolic space. *Nonlinear Anal.*, 157:146–166, 2017.

[10] E. Berchio, D. Ganguly, and G. Grillo. Sharp Poincaré–Hardy and Poincaré–Rellich inequalities on the hyperbolic space. *J. Funct. Anal.*, 272(4):1661–1703, 2017.

[11] E. Berchio, D. Ganguly, G. Grillo, and Y. Pinchover. An optimal improvement for the Hardy inequality on the hyperbolic space and related manifolds. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(4):1699–1736, 2020.

[12] E. Berchio, D. Ganguly, and P. Roychowdhury. On some strong Poincaré inequalities on Riemannian models and their improvements. *J. Math. Anal. Appl.*, 490(1):124213, 2020.

[13] E. Berchio, D. Ganguly, and P. Roychowdhury. Hardy–Rellich and second order Poincaré identities on the hyperbolic space via Bessel pairs. *Calc. Var. Partial Differ. Equ.*, 61(4):24, Paper No. 130, 2022.

[14] G. Bol. Isoperimetrische Ungleichungen für Bereiche auf Flächen. *Jahresber. Dtsch. Math.-Ver.*, 51:219–257, 1941.

[15] D. Borisov and P. Freitas. The spectrum of geodesic balls on spherically symmetric manifolds. *Comm. Anal. Geom.*, 2017(3):507–544, 2017.

[16] H. Brezis and M. Marcus. Hardy's inequalities revisited. *Ann. Scuola Norm. Sup. Pisa Cl. Sci*, 25:217–237, 1997.

[17] H. Brezis and J. L. Vázquez. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Compl. Madrid*, 10:443–469, 1997.

[18] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. *Compos. Math.*, 53(3):259–275, 1984.

[19] G. Carron. Inégalités de Hardy sur les variétés riemanniennes non-compactes. *J. Math. Pures Appl.*, 76(10):883–891, 1997.

[20] F. G. de S. Carvalho and M. P de A. Cavalcante. On the fundamental tone of the p-Laplacian on Riemannian manifolds and applications. *J. Math. Anal. Appl.*, 506(2):8, Paper No. 125703, 2022.

[21] F. Catrina and Z.-Q. Wang. On the Caffarelli–Kohn–Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.*, 54(2):229–258, 2001.

[22] C. Cazacu, J. Flynn, and N. Lam. Sharp second order uncertainty principles. *J. Funct. Anal.*, 283(10):37, Paper No. 109659, 2022.

[23] I. Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984.

[24] Y. Chen, N. C. Leung, and W. Zhao. Sharp Hardy inequalities via Riemannian submanifolds. *J. Geom. Anal.*, 32(7):204, 2022.

[25] Q.-M. Cheng and H. Yang. Estimates for eigenvalues on Riemannian manifolds. *J. Differ. Equ.*, 247(8):2270–2281, 2009.

[26] Q.-M. Cheng and H. Yang. Universal inequalities for eigenvalues of a clamped plate problem on a hyperbolic space. *Proc. Amer. Math. Soc.*, 139(2):461–471, 2011.

[27] S.-Y. Cheng. Eigenvalue comparison theorems and its geometric applications. *Math. Z.*, 143:289–297, 1975.

[28] C. B. Croke. A sharp four dimensional isoperimetric inequality. *Comment. Math. Helvetici*, 59(1):187–192, 1984.

[29] L. D'Ambrosio and S. Dipierro. Hardy inequalities on Riemannian manifolds and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):449–475, 2014.

[30] E. B. Davies. *Heat kernels and spectral theory.* 92. Cambridge university press, 1989.

[31] E. B. Davies and A. M. Hinz. Explicit constant for Rellich inequalities in $L^p(\Omega)$. *Math. Z.*, 227:511–524, 1998.

[32] F. Della Pietra, G. di Blasio, and N. Gavitone. Anisotropic Hardy inequalities. *Proc. Roy. Soc. Edinburgh Sect. A*, 148(3):483–498, 2018.

[33] B. Devyver, M. Fraas, and Y. Pinchover. Optimal Hardy weight for second-order elliptic operator: An answer to a problem of Agmon. *J. Funct. Anal.*, 266(7):4422–4489, 2014.

[34] A. Dinghas. Einfacher Beweis der isoperimetrischen Eigenschaft der Kugel in Riemannschen Räumen konstanter Krümmung. *Math. Nachr.*, 2(3-4):148–162, 1949.

[35] N. T. Duy, N. Lam, and G. Lu. p-Bessel Pairs, Hardy's Identities and Inequalities and Hardy–Sobolev Inequalities with Monomial Weights. *J. Geom. Anal.*, 32(109):36, 2022.

[36] D. E. Edmunds and H. Triebel. Sharp Sobolev embeddings and related Hardy inequalities: the critical case. *Math. Nachr.*, 207:79–92, 1999.

[37] G. Faber. Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt. *Sitzungsber. Bayer. Akad. Wiss. München, Math.-Phys. Kl.*, pages 169–172, 1923.

[38] C. Farkas, S. Kajántó, and A. Kristály. Sharp spectral gap estimates for higher-order operators on Cartan-Hadamard manifolds. *Commun. Contemp. Math.*, 2024. accepted.

[39] C. Farkas, S. Kajántó, and C. Varga. Lower semicontinuity of Kirchhoff-type energy functionals and spectral gaps on (sub)Riemannian manifolds. *Topol. Methods Nonlinear Anal.*, 61(2):743–760, 2023.

[40] C. L. Fefferman. The uncertainty principle. *Bull. Amer. Math. Soc. (N.S.)*, 9(2):129–206, 1983.

[41] S. Filippas, V. Maz'ya, and A. Tertikas. On a question of Brezis and Marcus. *Calc. Var.*, 25(4):491–501, 2006.

[42] S. Filippas, V. Maz'ya, and A. Tertikas. Critical Hardy–Sobolev inequalities. *J. Math. Pures Appl.*, 87(1):37–56, 2007.

[43] S. Filippas and A. Tertikas. Optimizing improved Hardy inequalities. *J. Funct. Anal.*, 192(1):186–233, 2002.

[44] J. Flynn, N. Lam, and G. Lu. Hardy-Poincaré-Sobolev type inequalities on hyperbolic spaces and related Riemannian manifolds. *J. Funct. Anal.*, 283(12):37, Paper No. 109714, 2022.

[45] J. Flynn, N. Lam, G. Lu, and S. Mazumdar. Hardy's identities and inequalities on Cartan-Hadamard manifolds. *J. Geom. Anal.*, 33(1):34, Paper No. 27, 2023.

[46] R. Frank. *Sobolev inequalities and uncertainty principles in mathematical physics, Part 1, Lectur Notes.* LMU Munich, 2011.

[47] S. Gallot, D. Hulin, and J. Lafontaine. *Riemmanian Geometry.* Universitext Springer-Verlag, Berlin, 2004.

[48] N. Ghoussoub and A. Moradifam. On the best possible remaining term in the Hardy inequality. *Proc. Natl. Acad. Sci. USA*, 105(37):13746–13751, 2008.

[49] N. Ghoussoub and A. Moradifam. Bessel pairs and optimal Hardy and Hardy–Rellich inequalities. *Math. Ann.*, 349(1):1–57, 2011.

[50] N. Ghoussoub and A. Moradifam. *Functional Inequalities: New Perspectives and New Applications*, volume 187 of *Mathematical Surveys and Monographs.* AMS, 2013.

[51] R. E. Greene and H.-H. Wu. *Function theory on manifolds which possess a pole*, volume 699. Springer, 2006.

[52] G. H. Hardy. Note on a theorem of Hilbert. *Math. Z.*, 6(3-4):314–317, 1920.

[53] E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities: Sobolev spaces and inequalities*, volume 5 of *Courant Lect. Notes Math.* AMS, 2000.

[54] J. Heinonen, T. Kilpeläinen, and Martio O. *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Clarendon Press, Oxford, 1993.

[55] S. Kajántó. Rellich inequalities via Riccati pairs on model space forms. *J. Math. Anal. Appl.*, 531(2, Part 1):13, Paper No. 127870, 2024.

[56] S. Kajántó, A. Kristály, I. R. Peter, and W. Zhao. A generic functional inequality and Riccati pairs: an alternative approach to Hardy-type inequalities. *Math. Ann.*, 2024. accepted.

[57] S. Kajántó and A. Kristály. Unexpected Behaviour of Flag and S-Curvatures on the Interpolated Poincaré Metric. *J. Geom. Anal.*, 31:10246–10262, 2021.

[58] S. Kajántó and A. Kristály. Saturation phenomena of a nonlocal eigenvalue problem: the Riemannian case. *Optimization*, pages 1–24, 2023.

[59] B. Kleiner. An isoperimetric comparison theorem. *Invent. Math.*, 108(1):37–47, 1992.

[60] B. R. Kloeckner and G. Kuperberg. The Cartan–Hadamard conjecture and the Little Prince. *Rev. Mat. Iberoam.*, 35(4):1195–1258, 2019.

[61] I. Kombe and M. Özaydin. Improved Hardy and Rellich inequalities on Riemannian manifolds. *Trans. Am. Math. Soc.*, 361(12):6191–6203, 2009.

[62] I. Kombe and M. Özaydin. Hardy-Poincaré, Rellich and uncertainty principle inequalities on Riemannian manifolds. *Trans. Am. Math. Soc.*, 365(10):5035–5050, 2013.

[63] E. Krahn. Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. *Math. Ann.*, 94(1):97–100, 1925.

[64] A. Kristály. Sharp uncertainty principles on Riemannian manifolds: the influence of curvature. *J. Math. Pures Appl.*, 119(9):326–346, 2018.

[65] A. Kristály. Fundamental tones of clamped plates in nonpositively curved spaces. *Adv. Math.*, 367:39, Paper No. 107113, 2020.

[66] A. Kristály. Lord Rayleigh's conjecture for vibrating clamped plates in positively curved spaces. *Geom. Funct. Anal.*, 32(4):881–937, 2022.

[67] A. Kristály. New features of the first eigenvalue on negatively curved spaces. *Adv. Calc. Var.*, 15(3):475–495, 2022.

[68] A. Kristály, Á. Mester, and I. I. Mezei. Sharp Morrey–Sobolev inequalities and eigenvalue problems on Riemannian–Finsler manifolds with nonnegative Ricci curvature. *Commun. Contemp. Math.* Paper No. 2250063, 2022.

[69] A. Kristály and S. Ohta. Caffarelli–Kohn–Nirenberg inequality on metric measure spaces with applications. *Math. Ann.*, 357(2):711–726, 2013.

[70] A. Kristály and D. Repovš. Quantitative Rellich inequalities on Finsler–Hadamard manifolds. *Commun. Contemp. Math.*, 18(6):17, Paper No. 1650020, 2016.

[71] A. Kristály and A. Szakál. Interpolation between Brezis-Vázquez and Poincaré inequalities on nonnegatively curved spaces: sharpness and rigidities. *J. Differ. Equ.*, 266(10):6621–6646, 2019.

[72] R. Küstner. Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order α. *Comput. Methods Funct. Theory*, 2:597–610, 2002.

[73] R. T. Lewis, J. Li, and Y. Li. A geometric characterization of a sharp Hardy inequality. *J. Funct. Anal.*, 262(7):3159–3185, 2012.

[74] X. Li, J. Mao, and L. Zeng. Eigenvalues of the bi-drifting Laplacian on the complete noncompact Riemannian manifolds. *Z. Angew. Math. Phys.*, 73(6):19, Paper No. 240, 2022.

[75] H. McKean. An upper bound to the spectrum of Δ on a manifold of negative curvature. *J. Diff. Geom.*, 4(3):359–366, 1970.

[76] C. Meng, H. Wang, and W. Zhao. Hardy type inequalities on closed manifolds via Ricci curvature. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(3):993–1020, 2021.

[77] Á. Mester, I. R. Peter, and C. Varga. Sufficient criteria for obtaining Hardy inequalities on Finsler manifolds. *Mediterr. J. Math.*, 18(2):1–22, Paper No. 76, 2021.

[78] E. Mitidieri. A Rellich type identity and applications: Identity and applications. *Commun. Partial. Differ. Equ.*, 18(1-2):125–151, 1993.

[79] E. Mitidieri. A simple approach to Hardy inequalities. *Mathematical notes*, 67:479–486, 2000.

[80] W. Moss and J. Piepenbrink. Positive solutions of elliptic equations. *Pac. J. Math.*, 75(1):219–226, 1978.

[81] B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972.

[82] N. S. Nadirashvili. Rayleigh's conjecture on the principal frequency of the clamped plate. *Arch. Rational Mech. Anal.*, 129(1):1–10, 1995.

[83] Q. A. Ngô and V. H. Nguyen. Sharp constant for Poincaré-type inequalities in the hyperbolic space. *Acta Math. Vietnam.*, 44:781–795, 2019.

[84] V. H. Nguyen. New sharp Hardy and Rellich type inequalities on Cartan–Hadamard manifolds and their improvements. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(6):2952–2981, 2020.

[85] V. H. Nguyen. Sharp Caffarelli–Kohn–Nirenberg inequalities on Riemannian manifolds: the influence of curvature. *Proc. Roy. Soc. Edinburgh Sect. A*, 152(1):102–127, 2022.

[86] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds). *NIST Handbook of Mathematical Functions.* Cambridge Univ. Press, Cambridge, 2010.

[87] Y. Pinchover. Criticality and ground states for second-order elliptic equations. *J. Differ. Equ.*, 80(2):237–250, 1989.

[88] Y. Pinchover. On criticality and ground states of second order elliptic equations, II. *J. Differ. Equ.*, 87(2):353–364, 1990.

[89] J. W. S. Rayleigh. The Theory of Sound. 2nd edition, revised and enlarged (in two volumes), 1945.

[90] F. Rellich. Halbbeschränkte Differentialoperatoren höherer Ordnung. In *Proc. Int. Congr. Math.*, volume 3, pages 243–250, 1954.

[91] M. Ruzhansky and D. Suragan. Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups. *Adv. Math.*, 317:799–822, 2017.

[92] M. Ruzhansky and D. Suragan. *Hardy inequalities on homogeneous groups: 100 years of Hardy inequalities.* Birkhäuser Basel, 2019.

[93] A. Savo. On the lowest eigenvalue of the Hodge Laplacian on compact, negatively curved domains. *Ann. Global Anal. Geom.*, 35(1):39–62, 2009.

[94] A. Tertikas and N. B. Zographopoulos. Best constants in the Hardy–Rellich inequalities and related improvements. *Adv. Math.*, 209(2):407–459, 2007.

[95] Z.-Q. Wang and M. Willem. Caffarelli–Kohn–Nirenberg inequalities with remainder terms. *J. Funct. Anal.*, 203(2):550–568, 2003.

[96] G. N. Watson. A Treatise on the Theory of Bessel Functions, reprint of the second (1944) edition. *Cambridge Univ. Press*, 1995.

[97] C. Xia. The Caffarelli–Kohn–Nirenberg inequalities on complete manifolds. *Math. Res. Lett.*, 14(5):875–885, 2007.

[98] Q. Yang, D. Su, and Y. Kong. Hardy inequalities on Riemannian manifolds with negative curvature. *Commun. Contemp. Math.*, 16:24, 2014.

[99] S.-T. Yin and Q. He. The first eigenvalue of Finsler p-Laplacian. *Differ. Geom. Appl.*, 35:30–49, 2014.

[100] W. Zhao. Hardy inequalities with best constants on Finsler metric measure manifolds. *J. Geom. Anal.*, 31(2):1992–2032, 2021.