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Evolutionary Techniques in Computational Game Theory

Summary of the Ph.D. Thesis

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Publications related to the Thesis

- D. Dumitrescu, Rodica Ioana Lung, Tudor Mihoc, **Réka Nagy**. Fuzzy Nash-Pareto equilibrium: Concepts and Evolutionary detection. (EvoGames 2010), *Lecture Notes in Computer Science*, Volume 6024/2010, Pages 71-79, DOI 10.1007/978-3-642-12239-2_8.

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- Ligia Cremene, Dumitru Dumitrescu, **Réka Nagy**, Marcel Cremene. Game Theoretic Modelling for Dynamic Spectrum Access in TV Whitespace. (CROWNCOM 2011), *Sixth International ICST Conference on Cognitive Radio Oriented Wireless Networks and Communications, IEEE*, 2011, Pages 336-340, ISBN: 978-1-4577-0140-5.

- **Réka Nagy**, Mihai Suciu, D. Dumitrescu. Lorenz Equilibrium: Equitability in Non-Cooperative Games. (GECCO 2012),*ACM Proceedings of the 14th annual conference on Genetic and evolutionary computation*, 2012, Pages: 489-496, ISBN 978-1-4503-1177-9, DOI 10.1145/2330163.2330233.

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Contents

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Introduction

Game Theory is a method of studying strategic interactions that helps in understanding what happens when decision-makers interact. The main area of Game Theory is non-cooperative games, that models simple forms of interactions between rational players. In non-cooperative games each player has a payoff function to maximize. The value of this function depends on the decisions taken simultaneously by all players.

Game theoretical models are abstract representations of real world situations. Their abstractness allow them to be used in the study of a wide range of phenomena. Game Theory has become a basic tool in Economics for analyzing various economic processes, such as competition, cooperation, strategic behavior, bargaining, etc. Besides Economics, Game Theory is successfully used also in Politics, Biology, Sports, Psychology, Sociology. With the emergence of the Internet, Game Theory has become increasingly important in Computer Science too.

Problem Statement

Standard game theory relies on the assumption that players are rational agents acting to maximize their profit. A solution of a game is a state where each player has adopted a strategy that they are unlikely to change. Such a game situation is called a *game equilibrium.*

The most popular solution concept in game theory is Nash equilibrium. A game state is a Nash equilibrium, if no player has the incentive to unilateral deviate from her strategy. However, Nash equilibrium can be inefficient when applied in real world situations. Nash equilibrium solutions are not always optimal; in many cases there exist other solutions that offer higher payoffs for all players. Moreover, in many situations the Nash equilibrium is not unique, there might be more than one Nash equilibrium, which makes a prediction of the outcome hard.

Besides Nash equilibrium other equilibrium concepts, such as Pareto equilibrium or Berge-Zhukovskii equilibrium, have been proposed. The Pareto equilibrium consists of the set of Pareto-optimal solutions. Pareto equilibrium models a state where no player can change her strategy without decreasing the strategy of another player. The Berge-Zhukovskii equilibrium of a game is a state where no player or group of player can switch their strategy in order to improve the payoff of any other player.

Each equilibrium captures a type of behavior or *rationality.* For example Nash rationality corresponds to a behavior when a player maximizes only her own payoff disregarding the payoffs of the other players. Pareto or Berge-Zhukovskii rationality on the other hand corresponds to a more other regarding rationality, where players take in consideration the gains of the other players too.

Standard game theoretic models allow interactions only between players that have the same rationality types. This is an unrealistic restriction; in real life situations players within a game rarely think and act in the same way. Moreover, rationality types captures by standard equilibria often model extreme behaviors that is rarely observed in case of human players. Human players might be more, less selfish, more or less other regarding, more or less cooperative, etc. The irrational nature of real players is neglected in standard game theory.

Several studies show that human players rarely follow the theoretical predictions. Trust games are a class of games that reveal the inefficiencies of the Nash equilibrium. In trust games usually players win more by choosing a cooperative strategy than by mutual defection. In most of trust games the highest payoffs are assured by defecting against a cooperator, but if everyone defects then the payoffs are very low. Rational play suggest that the best strategy choice is defection, so the Nash equilibrium in most trust games is the state when everyone defects. However, several studies show that human players rarely follow the theoretical predictions, they tend to choose partial cooperation.

In real life decision making usually there are more criteria to consider. These criteria are often contradictory and can not be aggregated into one single criterion. Games with multiple criteria offer a more accurate real life models. Many multicriteria equilibrium concepts have been proposed and vast research addressed their existence but the detection of these equilibria did not receive much attention. Our aim is to extend our evolutionary equilibrium detection method to be suitable for multicriteria games too. We consider that the interaction and decision-making of human players can be modeled more accurately by multicriteria games.

The aim of this Thesis in general is to develop new concepts that and build more realistic models that offer a more accurate model of real life situations.

Contributions

The main contributions of the thesis are the following:

- The concept of fuzzy equilibrium for non-cooperative games

  Fuzzy equilibrium allows players to have fuzzy rationalities, meaning that a rationality of a player can be somewhere in between two extreme rationality types. Players can be more or less biased towards a certain rationality. This bias may be expressed as a membership degree to the certain equilibria type. By tuning these membership degrees, many types of rationalities can be modeled. This way players having different fuzzy rationalities can interact within the same game. This yields to several new equilibrium types, such as Fuzzy Nash-Pareto or Fuzzy Nash–Berge-Zhukovskii equilibrium.

  Fuzzy equilibrium offers a more realistic modeling of human players. Numerical experiments show that fuzzy equilibrium may capture the manner real people play trust games.

- The concept of Lorenz equilibrium for non-cooperative games

  The Lorenz equilibrium of the game consists of only those optimal strategies that assure the most balanced and most equitable payoffs for all players. Numerical experiments show that Lorenz equilibrium overcomes the disadvantages of Nash and Pareto equilibria. The Lorenz equilibrium always consists of optimal solutions, and the size of this solution set in most of the cases is considerably smaller the size of the Pareto equilibrium set. Moreover, Lorenz equilibrium can be successfully applied in the process of selecting the most optimal Nash equilibrium in case of multiple equilibria.

- An evolutionary technique for detecting equilibria in multicriteria games

  Multicriteria games are games that have vector payoffs. Several equilibrium concepts have been proposed for solving multicriteria games. Vast research addressed the existence of these equilibria, but the equilibria detection in multicriteria games received less attention. The evolutionary method for equilibrium detection in single criterion games is extended for multicriteria games also.

- Multicriteria dilemmas with identity payoffs

  With the help of multicriteria games a more realistic model is built that considers the identity of the players as a second criterion.

- Modeling cognitive radio spectrum access scenarios as non-cooperative games

  Oligopoly games can be reformulated to be suitable to model different spectrum access scenarios. The Nash equilibruim as a solution for spectrum access problems has some drawbacks: it does not assure optimal solutions and there can be multiple equilibria. Joint Nash-Pareto and Lorenz equilibrium are proposed as alternative solution concepts.

Chapter 1

Background and Related Work

A solution of the game is a state of equilibrium where all players are happy with their gain and none of the players wants to deviate from her strategy. Solving a game can be viewed as an optimization problem with multiple objectives: the payoff for each player needs to be maximized. Evolutionary algorithms offer a powerful tool for solving multiobjective optimization problems. Thus evolutionary techniques can be successfully used also in addressing Game Theoretic problems such as equilibrium detection.

1.1 Game Theoretical Introduction

Game Theory is a mathematical tool for studying strategic interactions between profit maximizing agents. The gain of an agent depends not only on her choice but on the choice of the other players too.

Mathematical Game Theory was launched by John von Neumann and Oskar Morgenstern in 1944 when they published their seminar book Theory of Games and Economic Behavior [Neumann and Morgenstern, 1944]. Followed by John Nash's contribution [Nash, 1950], citenash51, Game Theory has become a basic tool in fields as Economics, Computer Science, Politics, Psychology, Sociology, etc.

The purpose of this section is to provide a simple introduction to the most common concepts used in Game Theory: non-cooperative games, game equilibria. Various game equilibrium concepts are presented and some examples for different classes of non-cooperative games are provided.

1.1.1 Non-cooperative Games

The main area of Game Theory is the field of Non-cooperative Games, also called Strategic Games. In Non-Cooperative Game Theory the goal of each player is to maximize her payoff. The value of the payoff function depends on the decisions taken simultaneously by all players.

Definition 1 *A finite strategic game is defined as a system*

$$\Gamma = ((N, S_i, u_i), i = 1, ..., n),$$

where:

- *N represents a set of n players, $N = \{1, ..., n\}$;*

- *for each player* $i \in N$*,* S_i *represents the set of actions available to her,* $S_i = \{s_{i_1}, s_{i_2}, ..., s_{i_m}\}$*;*

$$S = S_1 \times S_2 \times ... \times S_n$$

is the set of all possible strategies (situations of the game);

- *for each player* $i \in N$*,* $u_i : S \to R$ *represents the payoff function.*

Denote by (s_{i_j}, s^*_{-i}) the strategy profile obtained from s^* by replacing the strategy of player i with s_{i_j} i.e.

$$(s_{i_j}, s^*_{-i}) = (s^*_1, s^*_2, ..., s^*_{i-1}, s_{i_j}, s^*_{i+1}, ..., s^*_n).$$

In classical Computational Game Theory the following propositions are assumed:

- Players choose their strategies simultaneously, without collaborating with each other. The profit of each player is affected by the strategies chosen by the other players as well.
- All players are rational, meaning that the objective of each player is to maximize her payoff.
- Players have common knowledge of the game and the rationality of the other players.

These assumptions are unrealistic, that rarely appear in real life situations. Real life players can be driven by many other motives besides the maximization of their profit. They might care for their reputation, they might be more altruistic, more cooperative or they might simply act irrational. Our goal is to develop solution concepts that are able to capture a more realistic model of real life situations.

1.2 Solution Concepts in Game Theory

A desirable game solution is a state where each player is satisfied with the outcome and has no incentive to deviate from her strategy.

1.2.1 Nash Equilibrium

The central solution concept in computational game theory is Nash equilibrium, introduced in [Nash, 1951]. Nash equilibrium captures a state in which individual players act according to their incentives, maximizing their payoff.

Definition 2 *The strategy* s^* *is a Nash equilibrium if and only if the inequality*

$$u_i(s_i, s^*_{-i}) - u_i(s^*) \leq 0,\ \forall s_i \in S_i, \forall i \in N$$

holds, where $(s_{i_j}, s^*_{-i}) = (s^*_1, s^*_2, ..., s^*_{i-1}, s_{i_j}, s^*_{i+1}, ..., s^*_n)$*.*

In other words, a strategy is a Nash equilibrium [Nash, 1951] [Bade et al., 2009] [McKelvey and McLe if no player has the incentive to unilaterally deviate. Once all players are playing Nash equilibrium, it is in interest of every player to stick to her strategy.

Even though Nash equilibrium is the most popular solution concept in computational game theory, it has some shortcomings. Firstly, even though all players goal is to maximize their payoffs, Nash equilibrium may mot always be an optimal solution. Moreover, Nash equilibrium may not be unique. The existence of multiple Nash equilibria makes it hard to make predictions about the outcome of the game.

1.2.2 Pareto Equilibrium

The concept of Pareto equilibrium is inspired from the solution of Multiobjective Optimization problems. Pareto equilibrium consists of the Pareto-optimal outcomes of the game, and is based on the Pareto dominance relation.

A strategy profile s Pareto dominates the strategy profile s^* if and only if

$$\begin{aligned} s \succ_P s^* \Leftrightarrow \quad & \forall i = 1, ..., n u_i(s) \geq u_i(s^*) \text{and} \\ & \exists j : u_j(s) > u_i(s^*). \end{aligned}$$

A strategy profile s^* is Pareto non-dominated, or Pareto-efficient, if there exists no $s \in S$ such that $s \succ_P s^*$.

In other words, a strategy profile is Pareto non-dominated if no player can increase her payoff without decreasing the payoff of other players.

Definition 3 *The Pareto equilibrium of the game is the set of Pareto non-dominated strategy profiles.*

Thus, the Pareto-equilibrium of the game consists of the optimal outcomes. Very often this is an infinite set of solutions.

1.2.3 Berge-Zhukovskii Equilibrium

In case of Nash equilibrium players are self regarding, acting only to maximize their own payoffs without taking in consideration other players. In contrast the Berge-Zhukovskii equilibrium [Zhukovskii, 1994] is a state where no player, or group of players, can improve the payoff of any other player by changing their strategy.

Definition 4 *Let $N - i$ denote any group of players that excludes player i. The strategy profile s^* is a Berge-Zhukovskii equilibrium if the inequality*

$$u_i(s^*) \geq u_i(s_i^*, s_{N-i})$$

holds for each player $i = 1, ..., n$, and $s_{N-i} \in S_{N-i}$.

Berge-Zhukovskii equilibrium allows reaching cooperative features making it possible to determine cooperation in a non-cooperative game.

Chapter 2

Evolutionary Equilibria Detection

One of the most important open problems in Computational Game Theory is finding the Nash equilibrium of the game. Detecting the Nash equilibrium is a computationally hard problem [Papadimitriou, 1994].

Equilibria detection can be viewed as a multiobjective optimization problem, where the payoff of each player is to be maximized. Since evolutionary algorithms are a powerful tool for solving multiobjective optimization problems, they can also be used for finding game equilibrium. In an equilibrium detection problem the functions to be optimized are the payoff functions of the players, so the number of variables and objectives are given by the number of players in the game.

This Chapter presents an evolutionary method for detecting various equilibrium types in non-cooperative games. This method is based on an evolutionary algorithm, a population of strategy profiles is evolved and continuously improved. A key concept in the algorithm for equilibria detection is the notion of *generative relation*.

Most of the equilibrium types can be characterized by a generative relation. Generative relations are algebraic tools that allow to compare two strategy profiles with respect to a certain equilibrium. The evolutionary algorithm for equilibrium detection is based on generative relations that guide the search towards the certain equilibrium.

This Section is based on the following publications: [Lung and Dumitrescu, 2008, Dumitrescu et al., Dumitrescu et al., 2010b, Gaskó et al., 2012] and some unpublished work of the author with D. Dumitrescu.

2.1 Generative Relations

Game equilibria may be characterized by generative relations on the set of game strategies [Lung and Dumitrescu, 2008]. The idea is that the non-dominated strategies with respect to the generative relation equals (or approximate) the equilibrium set.

Let us consider a relation $\mathcal{R}$ over $S \times S$.
A strategy x is non dominated with respect to relation $\mathcal{R}$ if

$$\nexists y \in S : (x, y) \in \mathcal{R}.$$

Let us denote by NDR the set of non-dominated strategies with respect to relation $\mathcal{R}$. A subset $S' \subset S$ is non-dominated with respect to $\mathcal{R}$ if and only if $\forall s \in S'$, $s \in NDR$.

Relation $\mathcal{R}$ is said to be a *generative relation* for the equilibrium E if and only if the set of non-dominated strategies with respect to $\mathcal{R}$ equals the set E of strategies i.e. $NDR = E$.

2.1.1 Generative Relation for Nash Equilibrium

A strategy profile is a Nash equilibria if there are no players that can improve their payoff by unilateral deviation.

Let $x, y \in S$ be two strategy profiles. Let $k(x, y)$ denote the number of players which benefit by deviating from x towards y [Lung and Dumitrescu, 2008]:

$$k(x, y) = card\{i \in N, u_i(y_i, x_{-i}) > u_i(x), x_i \neq y_i\}.$$

The value $k(x, y)$ is a relative quality measure of strategy profile x with respect to strategy profile y with respect to the Nash equilibrium.

Definition 5 *We say the strategy profile x is better than y with respect to Nash equilibrium ($x \succ_N y$) if and only if fewer players benefit by deviating from x towards y than from y towards x. More formally:*

$$x \succ_N y \Leftrightarrow k(x, y) < k(y, x).$$

Definition 6 *Strategy profile x is called Nash non-dominated, if and only if there is no strategy profile $y \in S$ such that*

$$y \succ_N x.$$

The relation $\succ_N$ can be considered as the *generative relation of Nash equilibrium*, meaning that the set of non-dominated strategies with respect to $\succ_N$ induces the *Nash equilibrium* [Lung and Dumitrescu, 2008].

2.1.2 Generative Relation for Pareto Equilibrium

The Pareto equilibrium of a non-cooperative game consists of the optimal solutions, meaning that no player can improve her payoff by switching her strategy without decreasing the ayoff of another player.

Let $x, y \in S$ be two strategy profiles.

Definition 7 *We say the strategy profile x is better than y with respect to Pareto equilibrium if x Pareto-dominates y, i.e*

$$x \succ_P y.$$

Definition 8 *Strategy profile x is called Pareto non-dominated, if and only if there is no strategy profile $y \in S$ such that*

$$y \succ_P x.$$

The relation $\succ_P$ can be considered as the *generative relation for Pareto equilibrium.* Otherwise stated the non-dominated strategies with respect to the relation $\succ_P$ induces the Pareto equilibrium.

2.1.3 Generative Relation for Berge-Zhukovskii Equilibrium

Let $x, y \in S$ be two strategy profiles. Let $b(x, y)$ denote the number of players who lose by remaining to the initial strategy x, while the other players are allowed to play the corresponding strategies from y and at least one player switches from x to y. We may express $b(x, y)$ as [Gaskó et al., 2012]:

$$b(x, y) = card[i \in N, u_i(x) < u_i(x, y_{N-i})].$$

Definition 9 *We say the strategy x is better than strategy y with respect to Berge-Zhukovskii equilibrium, and we write $x \succ_{BZ} y$, if and only if the inequality*

$$b(x, y) < b(y, x)$$

holds.

Definition 10 *Strategy profile x is called Berge-Zhukovskii non-dominated, if and only if there is no strategy profile $y \in S$ such that*

$$y \succ_{BZ} x.$$

We may consider relation $\succ_{BZ}$ as a *generative relation of the Berge-Zhukovskii equilibrium*. Otherwise stated the set of the non-dominant strategies with respect to the relation $\succ_{BZ}$ equals the set of Berge-Zhukovskii equilibrium.

2.2 Evolutionary Equilibria Detection

Games can be viewed as multiobjective optimization problem, where the payoffs of the participating players are to be maximized. All of the objectives to be optimized are uniform and equally important. A solution of the game is called an equilibrium. At equilibrium all players are happy with their outcome, and they are not willing to switch their strategies.

An appealing technique is the use of generative relations and evolutionary algorithms for detecting equilibrium strategies. The payoff of each player is treated as an objective and the generative relation induces an appropriate dominance concept, which is used for fitness assignment purpose. Evolutionary multiobjective algorithms are thus suitable tools in searching for game equilibria.

A population of strategies is evolved. A chromosome is an n-dimensional vector representing a strategy profile $s \in S$. The initial population is randomly generated. Population model is generational. The non-dominated individuals from the population of strategy profiles at iteration t may be regarded as the current equilibrium approximation. Subsequent application of the search operators is guided by a specific selection operator induced by the generative relation. Successive populations produce new approximations of the equilibrium front, which hopefully are better than the previous ones.

In case the game does not have a certain equilibrium (in the sense of strict mathematical characterization) the proposed evolutionary technique may allow to detect a game situation which is a suitable approximation of this equilibrium.

For evolutionary equilibria detection any state of the art evolutionary multiobjective algorithm can be used. Our goal is to focus on the detected equilibria types and not on the algorithm used.

Chapter 3

Joint and Fuzzy Equilibria

In standard non-cooperative games it is usually considered that players act according to an unique equilibrium concept, i.e. only players acting according to the same type of equilibrium are allowed to interact. The underlying solution concept is Nash equilibrium or one of its variants, backward induction, iterated dominance, etc. This normative theory has some limitations when applied to explain social life and fails to account for some aspects of economic transitions.

One simple step towards a more realistic and flexible approach is to relax the rationality principle and look for the corresponding equilibrium concepts. In order to cope with more complex situations a concept of generalized game is presented. Players are allowed to have different behaviors/rationality types resulting in an adequate meta-strategy concept [Dumitrescu et al., 2009].

This Section presents the concept of joint and fuzzy equilibrium in non-cooperative games. The concept of generalized game is presented. In generalized games the interaction of players with different rationalities within the same game is allowed. For example a selfish player (that has a Nash rationality) can play with a more other regarding player (that has a Pareto rationality). Having players with different rationality types in the same game yields to new equilibrium concepts, such as Joint Nash-Pareto or Joint Nash–Berge-Zhukovskii equilibrium.

Most crisp equilibria model an extreme behavior neglecting the human nature. In real life, players can be more or less cooperative, more or less competitive and more or less rational. In case of fuzzy equilibrium players need not have crisp rationalities. Players are allowed to be somewhere in between two behavior types, thus having fuzzy rationalities. Each player has different biases towards a certain rationality type. This bias may be expressed by a fuzzy membership degree. This generalization yields to new equilibrium concepts such as Fuzzy Nash-Pareto of Fuzzy Nash–Berge-Zhukovskii equilibria.

This section is based on the following publications: [Dumitrescu et al., 2009, Dumitrescu et al., 2010a, Nagy et al., 2011a].

3.1 Fuzzy Equilibria

Each concept of equilibrium may be associated with a rationality type. These rationalities describe extreme behaviors which are rarely met in real-life situations. For example Nash equilibrium models an extreme selfish behavior, where a player acts only to maximize her own payoff without taking in consideration other players. On the other hand, Berge-Zhukovskii equilibrium models extreme altruism where a player takes in consideration only the well-being of other players. Players playing according to Pareto rationality seek for optimal solutions and they choose to take a more benefiting strategy only if this does not decrease the payoff for any other player.

Real life players rarely have crisp rationalities, they can be more or less biased towards certain rationality types. They can be more or less selfish, more or less altruist, more or less other regarding, etc.

Fuzzy equilibria [Dumitrescu et al., 2010a] allows each player to have different biases towards a certain rationality type. This bias may be expressed by a fuzzy membership degree. A player may have for instance the membership degree 0.7 to Nash and the membership 0.3 to Pareto. This means that the rationality of the player is a mixture of Nash and Pareto rationality being closer to Nash than to Pareto rationality.

This way several new equilibria types, like fuzzy Nash-Pareto equilibrium or fuzzy Nash–Berge-Zhukovskii equilibrium, can be introduced.

3.1.1 Fuzzy Nash-Pareto Equilibrium

Let us consider a fuzzy set A_N on the player set N i.e.

$$A_N : N \to [0, 1]$$

. $A_N(i)$ expresses the membership degree of the player i to the class of Nash-biased players. Therefore A_N is the class of Nash-biased players. Similar a fuzzy set

$$A_P : N \to [0, 1]$$

may describe the fuzzy class of Pareto-biased players.

A fuzzy Nash-Pareto equilibrium concept [Dumitrescu et al., 2010a] using an appropriate generative relation is considered in this section. The concept is a natural generalization of the sharp Nash-Pareto equilibrium characterization by generative relations and it is completely different from the notion considered in [Herings et al., 2004].

Let us consider a game involving both Nash and Pareto-biased players. It is natural to assume that $\{A_N, A_P\}$ represents a fuzzy partition of the player set. Therefore the condition

$$A_N(i) + A_P(i) = 1$$

holds for each player i.

Values $A_N(i)$ and $A_P(i)$ indicate where the rationality of the Player i is in between Nash and Pareto rationality. The value $A_N(i) = 1$ (thus $A_P(i) = 0$) indicates that Player i is a crisp Nash player, while a crisp Pareto player has membership values of $A_N(i) = 0$ and $A_P(i) = 1$. If in a game there are only pure Nash and pure Pareto players, than the corresponding equilibrium concept is called joint Nash-Pareto equilibruim.

Generative Relation for fuzzy Nash-Pareto Equilibrium

The relative quality measure of two strategies needs to involve the fuzzy membership degrees. Let us consider the threshold function:

$$t(a) = \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{otherwise} \end{cases}$$

The fuzzy version of the quality measure $k(x, y)$ is denoted by $E_N(x, y)$ and may be defined as

$$E_N(x, y) = \sum_{i=1}^{n} A_N(i) t(u_i(y_i, x_{-i}) - u_i(x)).$$

$E_N(x, y)$ expresses the relative quality of the strategies x and y with respect to the fuzzy class of Nash-biased players.

The relative quality measure of strategy profiles x and y with respect to the fuzzy class of Pareto players is given by

$$E_P(x, y) = \sum_{i=1}^{n} A_P(i) t(u_i(y) - u_i(x)),$$

where A_P is the fuzzy set of the Pareto-biased players.

The relative quality measure of the strategies x and y with respect to fuzzy Nash-Pareto rationality may be defined as

$$E_{fNP}(x, y) = E_N(x, y) + E_p(x, y).$$

Using the relative quality measure E_{fNP} we can compare two strategy profiles.

Definition 11 *Let us introduce the relation $\succ_{fNP}$ defined as $x \succ_{fNP} y$ if and only if the strict inequality $E(x, y) < E(y, x)$ holds.*

We may consider the relation $\succ_{fNP}$ as a generative relation for *fuzzy Nash-Pareto equilibrium*. Otherwise stated the set of the non-dominated strategies with respect to the relation $\succ_{fNP}$ equals the joint fuzzy Nash-Pareto equilibrium.

3.1.2 Fuzzy Nash–Berge-Zhukovskii Equilibrium

A fuzzy set

$$A_{BZ} : N \to [0, 1]$$

may describe the fuzzy class of Berge-Zhukovskii-biased players.

Let us consider a game involving both Nash and Berge-Zhukovskii-biased players. It is natural to assume that $\{A_N, A_{BZ}\}$ represents a fuzzy partition of the player set. Therefore the condition

$$A_N(i) + A_{BZ}(i) = 1$$

holds for each player i.

GENERATIVE RELATION FOR FUZZY NASH–BERGE-ZHUKOVSKII EQUILIBRIUM

The relative quality measure of two strategies needs to involve the fuzzy membership degrees. Let us consider the threshold function:

$$t(a) = \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{otherwise} \end{cases}$$

The fuzzy version of the quality measure $k(x, y)$ is denoted by $E_N(x, y)$ and may be defined as

$$E_N(x, y) = \sum_{i=1}^{n} A_N(i) t(u_i(y_i, x_{-i}) - u_i(x)).$$

$E_N(x, y)$ expresses the relative quality of the strategies x and y with respect to the fuzzy class of Nash-biased players.

The fuzzy version of $b(x, y)$ may be defined as

$$E_{BZ}(x, y) = \sum_{i=1}^{n} A_{BZ}(i) t(u_i(y, x_{N-i}) - u_i(x)).$$

$E_{BZ}(x, y)$ expresses the relative quality of the strategies x and y with respect to the fuzzy class of Berge-Zhukovskii-biased players.

The relative quality measure of the strategies x and y with respect to fuzzy Nash–Berge-Zhukovskii rationality may be defined as

$$E_{fNBZ}(x, y) = E_N(x, y) + E_{BZ}(x, y).$$

Using the relative quality measure E_{fNBZ} we can compare two strategy profiles.

Definition 12 *Let us introduce the relation $\succ_{fNBZ}$ defined as $x \succ_{fNBZ} y$ if and only if the strict inequality $E(x, y) < E(y, x)$ holds.*

Fuzzy Nash–Berge-Zhukovskii equilibrium is the set of non-dominated strategies with respect to the relation $\succ_{fNBZ}$.

Chapter 4

Modelling Human Behavior

In many cases standard game theory fails to accurately model real life situations. The most popular solution concept, Nash equilibrium, assumes that players are rational agents that act to maximize their payoffs. Nash equilibrium models an extreme self-concerning and rational behavior that is rarely followed by human players.

In case of trust games, players can adopt either a cooperator or a defector strategy. In most of the cases players can achieve maximal payoffs by mutual cooperation. Also, since the payoff for defecting against a cooperative opponent is considerably larger then the payoff for cooperation, the temptation for defection is high. Meanwhile cooperating with a defector yields to very small payoff. Therefore, if a player would not cooperate than the other player would do better is she would defect too. Rational thinking would lead to a situation where both of the players defect, which is the Nash equilibrium in most of the trust games. Nash equilibrium rarely assures optimal payoffs for the players, moreover mutual defection is usually the worst possible outcome for all players.

Other solution concepts, like Pareto or Berge-Zhukovskii equilibrium, are often better choices in case of trust games. Pareto equilibrium assures the optimal outcomes, where no player can improve her payoff without decreasing the payoff of another player. Berge-Zhukovskii equilibrium models a type of altruism. Berge-Zhukovskii players when choosing their strategy beyond their gain also take in consideration the gain of their opponent. Both Pareto and Berge-Zhukovskii equilibria usually assure greater payoffs for all players then Nash equilibrium. In most trust games both Pareto and Berg-Zhukovskii equilibrium is mutual cooperation, which is the most favorable outcome for all players.

However, both mutual cooperation and mutual defection are two extremes. Our intuition is, that human behavior is somewhere in between: they do not fully cooperate but neither do they defect.

In this section we consider two studies about how human player play trust games. The first study involves several discrete versions of the Centipede Game, while the second study involves the continuous version. Both studies show that human players tend to choose a strategy somewhere in between Nash and Pareto or Berge-Zhukovskii equilibrium.

We use Fuzzy Nash-Pareto and Fuzzy Nash–Berge-Zhukovskii equilibrium to reproduce the human results. Numerical experiments show that fuzzy equilibrium is a suitable tool to model the human behavior in case of the studied trust games.

This section is based on two publications. Various equilibria types for the discrete centipede game is presented in [Dumitrescu et al., 2010b]. [Nagy et al., 2012a] studies continuous trust games and Fuzzy Nash–Berge-Zhukovskii equilibria.

4.1 How People Play Trust Games?

4.1.1 Centipede Game

McKelvey and Palfrey [McKelvey and Palfrey, 1992] studied actual behavior in three different versions of the centipede game [Rosenthal, 1981].

Rational play suggests that players should choose to stop at the first move. However subjects do not follow the theoretical predictions. Experiments involve human players without a game theoretical background (they are not informed and they do not make inferences about Nash equilibrium). Very few players stopped in the first round, choosing a secure win, and despite the high payoffs, very few players risked to wait until the last round of the game.

Studies carried out with human players show, that only a very small percentage of the people tends to stop at the first round or waits until the last round. The majority of the people stop at an intermediate round.

4.1.2 Continuous Centipede Game

The symmetric real time trust game (SRTT game) [Murphy et al., 2006] is a continuous version of the centipede game. Similarly to the discrete centipede game, the Nash equilibrium the SRTT game is when all players stop the clock at zero seconds, so all players end up with minimal payoffs.

Human behavior for the SRTT game has been studied in [Murphy et al., 2006]. The studied SRTT game has the following parameter settings: $T = 45$, $\theta = 5$, $\lambda = 5$, $\delta = 0.5$ and $g = 0$. Thus the players who loose receive 10% of the winner's payoff and if no one stops the clock before 45 seconds the payoff for all players is zero.

The experiment was carried out with the help of 21 participants, who had no knowledge about the theoretical predictions. A random grouping procedure was used, the players were not informed about their opponents, thus no collaboration was possible. The game was repeated for 47 rounds, and the stopping time for each game was recorded. Results show that real life players in the first rounds tend to stop the timer between 25 and 42 seconds.

4.2 Human Behavior and Fuzzy Equilibria

Crisp equilibria in many cases fail to model human behavior for trust games. Nash equilibrium models an extreme selfishness that usually yield to mutual defection. Other equilibria types as Pareto or Berge-Zhukovskii equilibrium in most of the cases is mutual cooperation, that is a more favorable outcome then mutual defection. Berge-Zhukovskii equilibrium is a model for extreme altruism, where players take in consideration only the opponents payoffs. Pareto equilibrium models a state where players make decisions taking in consideration the payoffs for all players.

Neither of these extremes are realistic. Experiments on human players, as presented in Section 4.1, show that the majority of people play according to a rationality that is between these extremes.

Fuzzy equilibrium is suitable to capture intermediate states, thus it might be used successfully in modeling human behavior. Our intuition is that combining a selfish rationality with an other-regarding rationality type might lead to more realistic results.

Numerical experiments show that fuzzy equilibria can be successfuly applied to reproduce the human results.

Chapter 5

Equitability in Games

The most commonly used solution concepts in game theory are Nash and Pareto equilibria. Nash equilibrium presumes that all the players in the game are completely rational agents, that have common knowledge of the structure of the game and they choose their strategies as best responses to the strategies of the adversary players. The outcome assured by Nash equilibrium is not always Pareto optimal. In many cases a more favorable outcome could be reached if all the players changed their strategies. On the other hand, Pareto equilibrium assures the greatest possible payoffs for the players, but the distribution of the payoffs is uneven and rarely equitable. Also, the set of Pareto optimal solutions is often a very large or an infinite set.

The Lorenz dominance, also called equitable dominance relation, has been introduced by Kostreva and Ogryczak in [Kostreva and Ogryczak, 1999]. Lorenz domination, a refinement of the Pareto domination, is used in decision theory and fair optimization problems. Considering models with equitable efficiency relieves some of the burden from the decision maker by shrinking the solution set. The set of equitably efficient solutions is contained within the set of efficient solutions for the same problem.

A problem frequently encountered in classical multi-criteria optimization is the existence of a large (often infinite) set of optimal solutions. The decision making based on selecting a unique preferred solution becomes difficult. In this Section we study the Lorenz dominance relation for multiobjective optimization problems. We propose a differential evolution algorithm based on Lorenz dominance. The detected solutions are those optimal solutions that are the most balanced and most equitable when all objectives are considered. Compared to the same evolutionary algorithm based on Pareto dominance, the Lorenz based algorithm is more scalable to the number of objectives [Nagy et al., 2012b].

Having noticed the favorable properties of the Lorenz dominance, we consider that the Lorenz dominance relation could be successfully applied to address some limitations of the classical Game Theory. Based on the Lorenz dominance relation, the concept of Lorenz equilibrium is proposed [Nagy et al., 2011b]. Lorenz equilibrium provides optimal solutions that assure maximal payoffs for all players and an equitable payoff distribution.

Another drawback of classical Game Theory is the multiplicity of Nash equilibrium. The goal of Game Theory is to predict the outcome of games but when it comes to games with multiple equilibria it is impossible to give such a prediction. We apply Lorenz equilibrium in the process of selecting the most favorable Nash equilibrium in case of multiple equilibria.

This Chapter is based on the following publications [Nagy et al., 2011b], [Nagy et al., 2012c] and [Nagy et al., 2012b].

5.1 The Lorenz Dominance Relation

The equitable dominance relation, has been defined in [Kostreva and Ogryczak, 1999] and extended in [Kostreva et al., 2004].

Let us consider a multi-criteria maximization problem with m objectives and n decision parameters. The problem can be formulated as follows:

$$\begin{cases} \text{Maximize} & f(x) \\ & x \in X \\ \text{where:} & f_i : X \to Y, \\ & f(x) = (f_1(x), ..., f_m(x)), \\ & x = (x_1, ..., x_n). \end{cases} \tag{5.1}$$

where x is the decision vector, X is the parameter space, $f(x)$ is the objective vector and Y the objective space.

Multicriteria optimization solution concepts may be defined by properties of the corresponding preference model. Let the relation of weak preference be denoted by $\succeq$. The corresponding relations of strict preference $\succ$ and indifference $\cong$ are defined as follows:

$$y' \succ y'' \Leftrightarrow (y' \succeq y'' \text{ and not } y'' \succeq y').$$

$$y' \cong y'' \Leftrightarrow (y' \succeq y'' \text{ and } y'' \succeq y').$$

The most common multicriteria optimization solution concept is Pareto-optimality. Lorenz dominance is a refinement of Pareto dominance used in fair optimization problems. In addition to the initial objective aiming at maximizing individual utilities, fairness refers to the idea of favoring well-balanced solutions [Kostreva et al., 2004]. Hence, in fair optimization problems, we are interested in working with a preference relations $\succeq$ and $\succ$ satisfying the following axioms:

(1) P-Monotonicity: For all $x', x'' \in X$:

$$f(x') \succeq_P f(x'') \Rightarrow f(x') \succeq f(x'')$$

and

$$f(x') \succ_P f(x'') \Rightarrow f(x') \succ f(x'').$$

(2) Impartiality: While dealing with uniform criteria, we want to focus on the distribution of outcome values while ignoring their ordering. In other words, a solution generating individual outcomes: 4, 2, 0 for criteria f_1, f_2 and f_3 respectively, should be considered equally good as a solution generating outcomes 0, 2 and 4. Hence we assume that the preference model is impartial (anonymous, symmetric). More formally:

$$(f_{\tau(1)}(x), f_{\tau(2)}(x), ..., f_{\tau(m)}(x)) \cong (f_1(x), f_2(x), ..., f_m(x))$$

for any permutation τ of $\{1, 2, ...m\}$, $x \in X$.

(3) Principle of transfers: The (Pigou-Dalton) principle of transfers [Stephen et al., 1999] states that a transfer of any small amount from an outcome to any other relatively worse-off outcome results in a more preferred outcome vector. More formally:

$$f_i(x) > f_j(x) \Rightarrow$$
$$((f_1(x), ..., f_i(x) - \epsilon, ..., f_j(x) + \epsilon, ..., f_m(x)) \succ (f_1(x), ..., f_m(x))$$

for $0 < \epsilon < f_{i'}(x) - f_{i''}(x)$.

Thus a solution generating all three outcomes equal to 2 is better than any solution generating individual outcomes 4, 2 and 0.

The preference relation satisfying axioms (1)-(3) are called equitable (Lorenz) preference relations [Kostreva and Ogryczak, 1999], [Kostreva et al., 2004].

Definition 13 *Let us consider*

$$f_{(1)}(x) \leq f_{(2)}(x) \leq ... \leq f_{(m)}(x)$$

as the components of

$$f = (f_1(x), f_2(x), ..., f_m(x))$$

sorted by increasing order. Let $x \in X$ be a solution vector.

The generalized Lorenz vector associated to x is

$$L(x) = (l_1, ..., l_m),$$

where

$l_1 = f_{(1)}(x),$

$l_2 = f_{(1)}(x) + f_{(2)}(x),$

$...$

$l_m = \sum_{i=1}^{m} f_{(i)}(x).$

Let $x', x'' \in X$ two solutions.

Definition 14 *The solution x' weakly Lorenz dominates the solution x'' if:*

$$x' \succeq_L x'' \Leftrightarrow L(x') \succeq_P L(x'').$$

Definition 15 *The solution x' Lorenz dominates x'' if:*

$$x' \succ_L x'' \Leftrightarrow L(x') \succ_P L(x'').$$

Remark 1 *If a solution $x \in X$ is a Lorenz-optimal solution (or simply Lorenz solution) of a multiple criteria problem, it is also a Pareto-optimal solution [Kostreva et al., 2004].*

5.2 Lorenz Equilibrium

Lorenz equilibrium is defined in the framework of non-cooperative game theory using Lorenz dominance as its generative relation.

5.2.1 Generative Relation of Lorenz Equilibrium

Game equilibria may be characterized by generative relations on the set of game strategies [Lung and Dumitrescu, 2008], [Dumitrescu et al., 2009]. The idea is that the non-dominated strategies with respect to the generative relation equals (or approximate) the equilibrium set.

Definition 16 *The* Lorenz equilibrium *of a game is the the set of non-dominated strategies with respect to the* $\succ_L$ *relation.*

Therefore we may consider $\succ_L$ as a *generative relation for Lorenz equilibrium.*

5.2.2 Properties of Lorenz Equilibrium

The most studied equilibrium concepts in game theory are the Nash and Pareto equilibria. However, when applied in real-world situations these theoretical concepts have some limitations. Nash equilibrium presumes that players are rational agents choosing their strategies as best response to strategies chosen by other players. This equilibrium rarely assures maximal payoffs for all players. In contrary, Pareto equilibrium assures the optimal payoffs for the players but the set of Pareto-optimal solutions is often too large, while in game theory it is useful to find a unique preferred solution of the game. Moreover, payoffs of Pareto solutions may be highly unequal.

The advantage of the Lorenz equilibrium is that it preserves the qualities of Pareto equilibrium and the set of solutions is considerably smaller. Moreover the resulting solutions assure the maximal payoffs that are equitable for all players.

5.2.3 Lorenz Equilibrium in Games with Multiple Nash Equilibria

An interesting question arises when there are multiple Nash equilibria in a game [Bade et al., 2009, Sekiguchi et al., 2009]. In this case, classical game theory can say nothing about the outcome of the game and in case of multiple equilibria there is no particular reason to eliminate one out of these equilibria. We propose the use of Lorenz equilibrium for the selection of one Nash equilibrium.

Our assumption is, that because of its favorable properties, Lorenz equilibrium can be applied in the process of selecting the most equitable Nash equilibrium. Lorenz equilibrium is Pareto-optimal (it assures maximal payoffs) and provides the most fair payoff distribution to the players. Our assumption is, that in case of multiple equilibria, selecting the Nash equilibrium that is the closest to the Lorenz equilibrium is a natural choice. The distance is considered int the payoff space.

In the case when the Lorenz equilibrium is a subset of the Nash solutions, the selection is trivial. Since this is not always the case, the closest distance in the payoff space is considered.

Chapter 6

Multicriteria Games

In standard non-cooperative games players are agents that act to maximize their payoffs. This is an overly simplified model of real life situations. In real life situations several, more complicated, scenarios can be found where players have to make decisions considering more than one criteria. These criteria in most of the cases are not measured by the same unit, they can not be just aggregated into one single criteria.

Multicriteria games (or games with vector payoffs) are natural extensions of single criterion games and offer a more realistic model for real life situations.

A finite strategic multicriteria game is defined as a system

$$\Gamma = ((N, S_i, u_i), i = \overline{1, n}),$$

where:

- N represents a set of n players, $N = \{1, ..., n\}$;
- for each player $i \in N$, S_i represents the set of pure strategies available to her, $S_i = \{s_{i_1}, s_{i_2}, ..., s_{i_m}\}$;
$$S = S_1 \times S_2 \times ... \times S_n$$
is the set of all possible strategies (situations of the game);
- for each player $i \in N$
$$u_i : S \to R^{r(i)}$$
represents the vector payoff function, where $r(i) \in N$ is the number of criteria for player i.

We consider that each player is a maximizer.

6.1 Equilibria in Multicriteria Games

The equilibrium concepts in multicriteria games has been studied by many authors [Borm et al., 1988] [Zhao, 1991] [Wang, 1993] [Borm et al., 1999].

6.1.1 Pareto-Nash Equilibrium

The most studied equilibrium concept is the Pareto-Nash equilibrium introduced in [Shapley and Rigby, 195
The Pareto-Nash equilibrium concept is an extension of the Nash equilibrium for single-criterion games that is based on Pareto domination. We can distinguish weak and strong Pareto-Nash equilibria.

Definition 17 *A strategy profile $s^* \in S$ weak Pareto-Nash equilibrium, if and only if the following condition holds*

$$u_i(s^*) \geq_P u_i(s_i s^*_{-i}),\ \forall s_i \in S_i, \forall i \in N$$

Definition 18 *A strategy profile $s^* \in S$ strong Pareto-Nash equilibrium, if and only if the following condition holds*

$$u_i(s^*) >_P u_i(s_i s^*_{-i}),\ \forall s_i \in S_i, \forall i \in N$$

6.1.2 Ideal Nash Equilibrium

The ideal Nash equilibrium was introduced in [Voorneveld et al., 1999], and studied also in [Radjef and Fahem, 2008]. Multicriteria games are viewed here as an organization; each criteria corresponds to the concerns of a different member of the organization. Thus, each player i corresponds to an organization with r_i members.

The idea of ideal Nash equilibrium captures the following reasoning: a choice of strategy of the organization $i$is supposed to be taken by common agreement of all the r_i members with the objective to maximize the payoff for each member of the organization. Also, the payoff of each members depend also on the strategy choices of other organizations.

The idea of viewing players as organization is realistic, since in many real life situations decisions are influenced by several individuals with different objectives.

Definition 19 *The ideal Nash equilibrium of a multicriteria game, G, consists of those solutions that are Nash equilibria in the single-criterion games, that constitute the multicriteria game G.*

6.1.3 Pareto Equilibrium

6.2 Evolutionary Equilibrium Detection in Multicriteria Games

Vast research addressed the existence of these equilibria, but the detection of equilibria did not receive much attention. We consider that various game equilibria may be characterized by generative relations on the set of game strategies. The idea is that the non-dominated strategies with respect to the generative relation equal (or approximate) the equilibrium set.

6.2.1 Generative Relation for Nash-Pareto Equilibrium

Let $x, y \in S$ be two strategy profiles. Let $k_{PN}(x, y)$ denote the number of players which by deviating from x towards y can increase a payoff for a criterion without decreasing the payoffs for the other criteria:

$$k_{PN}(x, y) = card\{i \in N, u_i(y_i, x_{-i}) \geq_P u_i(x), x_i \neq y_i\}.$$

Consider the relation $\prec_P N$ defined as

$$x \prec_{PN} y \text{ if and only if } k_{PN}(x, y) < k_{PN}(y, x).$$

$k_{PN}(x, y)$ is a relative quality measure of y and x - with respect to the Nash equilibrium. The relation $\prec_{PN}$ can be considered as the *generative relation of Pareto-Nash equilibrium*, i.e. that the set of non-dominated strategies with respect to $\prec_{PN}$ induces the *Pareto-Nash equilibrium*.

6.2.2 Generative Relation for Ideal Nash Equilibrium

Let $x, y \in S$ be two strategy profiles. Let $k_{iN}(x, y)$ denote the number of players which by deviating from x towards y can increase their payoff for any criteria:

$$k_{iN}(x, y) = card\{i \in N, u_i^j(y_i, x_{-i}) > u_i^j(x) \forall j \in M, x_i \neq y_i\}.$$

Consider the relation $\prec_{iN}$ defined as

$$x \prec_{iN} y \text{ if and only if } k_{iN}(x, y) < k_{iN}(y, x).$$

$k_{iN}(x, y)$ is a relative quality measure of y and x - with respect to the ideal Nash equilibrium. The relation $\prec_{iN}$ can be considered as the *generative relation of ideal Nash equilibrium*, i.e. that the set of non-dominated strategies with respect to $\prec_{iN}$ induces the *ideal Nash equilibrium*.

6.2.3 Generative Relation for Pareto Equilibrium

6.2.4 Evolutionary Equilibrium Detection in Multicriteria Games

The proposed evolutionary equilibrium detection algorithm for multicriteria games is based on the evolutionary equilibrium detection algorithm presented in Section 2.2. The method is based on an evolutionary multiobjective optimization algorithm. A population of strategy profiles is evolved. The objectives are the payoffs of the n players, but unlike in case of standard single-criterion games, the payoff for each player i is an r_i dimensional vector.

Instead of Pareto-domination, that is used in multiobjective optimization algorithms, the strategy profiles are compared with the help of generative relations for multicriteria game equilibria. Successive iterations yield to hopefully better and better equilibrium approximations. The process will finally converge to the multicriteria equilibrium induced by the generative relation.

Just as in the case of single-criterion games, for equilibrium detection any state of the art multiobjective optimization algorithm based on domination can be used.

6.3 Multicriteria Games and Identity Payoffs

An advantage of the multicriteria games over standard, single criterion games is that it they offer a more realistic modeling of real life situations. In most of the cases players make decisions considering more than one criterion, and these criteria can not be unified into one criterion.

Standard single criterion games take in consideration only the actual payoffs of the game. However, in human decision making there are many other factors that play an important role. Human players, besides actual payoffs, take in consideration other criteria such as morality, selfishness, cooperation, altruism. These factors are hard to quantify and can not be added to the actual payoffs.

Multicriteria games allow us to model the above mentioned human factors as a second criteria. This criteria can capture the human thinking without taking in consideration the actual payoffs. This way a human identity can be modeled, therefore we refer to this criteria as identity payoff.

So standard single criterion games can be transformed into multicriteria games. The first criterion is the actual payoff while the second criterion is the above described identity payoff.

Multicriteria games with identity payoffs offer a more realistic picture?! of human decision making.

6.3.1 Two Player Dilemmas and Identity Payoffs

Let us consider the two-player discrete games. In all three situations, prisoner's dilemma, hawk-dove and coordination games, players can choose between cooperation and defection. In most if the cases when both players interests are considered mutual cooperation is the most preferable outcome. Still, theoretical predictions are that the players will defect.

Human players do not make their decision based only on actual payoffs. Besides considering the concrete payoffs human players also take in consideration other criteria as solidarity, conscience, morality, etc. Besides the actual payoffs their identity determines their choices. For example, a person who has a cooperator identity is less likely to defect even though rational thinking implies defection regardless of the opponent strategy.

In case of two player dilemmas the most important question regarding human players is, whether they are willing to act in a way that is favorable for the community or they act only to maximize their own gain. In all two-player dilemmas both of the players benefit by mutual cooperation, but in the same time, the temptation to defect against a cooperator is high. However, fair play suggests the best strategy would be to cooperate. With the help of the second criterion, this fair decision making can be modeled. In case of a "cooperator" the identity-payoff for cooperating is considerably larger than for defecting.

6.4 Spatial Model

Chapter 7

Game Theoretic Modeling for Cognitive Radio Environments

Cognitive radios [Mitola, 2000] are radios that are aware of their environment. They are able to detect changes in their environment and can adapt to these changes. Cognitive radio technology is seen as the key enabler for next generation communication networks, which will be spectrum-aware, dynamic spectrum access networks [Akyildiz et al., 2006, de M. Cordeiro et al., 2006]. Cognitive radios hold the promise for an efficient use of the radio resources and are seen as the solution to the current low usage of the radio spectrum. In a cognitive radio environment users strategically compete for spectrum resources in dynamic scenarios.

In this Chapter the problem of spectrum access and resource allocation is addressed from a game theoretical perspective. Game Theory provides a fertile framework and the computational tools for cognitive radio interaction analysis. By devising GT simulations, insight may be gained on unanticipated situations that may arise in spectrum access. Cognitive radio interactions are strategic interactions [Neel, 2006]: the utility of one cognitive radio agent/player depends on the actions of all the other radios in the area. The proposed approach relies on the following assumptions:

- cognitive radios have perfect channel sensing and radio frequency reconfiguration capabilities
- cognitive radios are myopic, self-regarding players,
- repeated interaction among the same radios is not likely to occur on a regular basis, and
- cognitive radios do not know in advance what actions the other radios will choose.

These are reasons to consider one-shot, non-cooperative games for the open spectrum access analysis.

In this Chapter we focus on applications of Game Theory in Telecommunication. We approach the problem of cognitive radio spectrum access and resource allocation from a game theoretical perspective. Oligopoly models are reformulated in terms of spectrum access. Continuous and discrete instances of the game are analyzed. Nash and Pareto equilibria are revisited for the discrete instance of the games. Heterogeneity of players is captured by joint Nash-Pareto equilibria, allowing cognitive radios to be biased toward different types of equilibrium. Also, the Lorenz equilibrium is proposed to overcome the disadvantages of Nash and Pareto equilibria.

This Chapter is based on the following published work: [Cremene et al., 2010, Cremene et al., 2011, Cremene et al., 2012].

7.1 Oligopoly Models for Cognitive Radio Environments

The problem of spectrum access is modeled as a non-cooperative, one-shot game. We consider oligopoly models reformulated in terms of radio resource access. Cognitive radio simultaneous access situations are considered and modeled as one-shot games. As simultaneous spectrum access scenarios do not imply large numbers of users, two and three-player games are considered relevant. Continuous and discrete instances of the game are analyzed. We analyze different types of game equilibria, as they describe several types of strategic interactions between cognitive agents. The action of each cognitive radio directly affects the other the payoffs of the other cognitive radios.

7.1.1 Cournot spectrum access modeling

We consider a general open spectrum access scenario that can be modeled as a reformulation of the Cournot Oligopoly. In the Cournot Game players are firms that produce the same good. They simultaneously choose quantities they will produce, the price than depends on the demand for the good and the total quantity, produced by all firms. This model can be reformulated such as to model open spectrum access.

Suppose there are n cognitive radios attempting to access the same set of available channels, simultaneously. Each cognitive radio i may decide the number of simultaneous channels to access, c_i. The question is how many simultaneous channels should each CR access in order to maximize its operation efficiency? The more channel a cognitive radio accesses, the better the quality????, but more bandwith implies more power consumption and more processing resources.

For a general open access scenario the Cournot competition may be reformulated as follows [Neel, 2006]:

- The players are cognitive radios simultaneously attempting to access a certain set of channels W;
- The strategy of each player i is the number of simultaneous accessed channels, c_i; A strategy profile is a vector
$$c = (c_1, , c_n)$$
.
- The payoff of each player i is given by the difference between a function of goodput $P(C)c_i$ and the cost of accessing c_i simultaneous channels K_{c_i}.

The goodput??????? is given by function P and it depends on the number of available channels and the aggregate number of accessed channels by all radios, C.????? The goodput of radio i is thus given by
$$P(C)c_i.$$
Let the cost of radio i for accessing c_i simultaneuos channels be $C_i(c_i)$. Then the payoff of radio i may be written as:
$$u_i(c) = P(C)c_i - C_i(c_i).$$

In order to focus on the emergent phenomena, we consider a simplified payoff function. Let the function P be defined as:

$$P(C) = \begin{cases} W - C, & \text{if } W > C, \\ 0, & \text{otherwise} \end{cases}$$

where

- $W > 0$ is the whitespace (the set of available channels)
- $C = \sum_{i=1}^{n} c_i$ is the aggregate number of accessed channels.

n general, P decreases with the total number of implemented channels.

Suppose that each radio has constant costs for accessing a channel. So the cost of radio i for accessing c_i channels is

$$C_i(c_i) = Kc_i.$$

The payoff function of radio i can be thus rewritten as:

$$u_i(c) = (W - \sum_{k=1}^{n} c_k)c_i - Kc_i,$$

It is important to note that this is a overly simplified model of open spectrum access. The computational model allows for more complex payoff functions to be implemented, accounting for various parameters. However the model described above is suitable for out studies.

The Nash equilibrium is considered as the solution of this game and can be calculated as follows:

$$c_i^* = \frac{W - K}{n - 1}, \forall i \in N.$$

7.1.2 Bertrand Spectrum Access Modeling

In the Bertrand spectrum access model we consider a scenario where several cognitive radios are competing for resources in a crowded spectrum. Each radio can decide on the target number of non-interfered symbols. The objective of each radio is to activate a subset of channels in order to satisfy its current demand level (e.g. target throughput).

The Bertrand competition for crowded spectrum access may be reformulated as follows:

- The players are the cognitive radios attempting to access the whitespace.
- The strategy of each player i is a target number p_i of non-interfered symbols;
- The payoff of each player i is given by the difference between a function of goodput and the cost of accessing c_i simultaneous channels.

Suppose the radios set different target numbers of non-interfered symbols. The lower this target is the higher the chances are for the radio to access one or several channels. On the other hand, as the number of non-interfered symbols decreases, the demand for channels increases. In a crowded spectrum where the available resources are reduced the radio setting the lowest target number of non-interfered symbols has the highest chances of activating a subset of channels, while the other radios might not be able to access the spectrum. If more radios set the same lowest target number, than they share the resources.

For the sake of simplicity, we consider a linear demand function and constant cost functions for all radios. The demand function for a given target number of non-interfered symbols, p, can be defined as follows:

$$D(p) = \begin{cases} W - p, & \text{if } p \leq W, \\ 0, & \text{otherwise} \end{cases}$$

where W is a parameter.

The cost of radio i for activating c_i simultaneous channels is given by

$$C_i(c_i) = Kc_i.$$

The payoff function for radio i can thus be defined as:

$$u_i(p_1, ..., p_n) = \begin{cases} \frac{1}{m}(p_i - K)(W - p_i), & \text{if } p = min(p_1, ..., p_n) \\ 0, & \text{otherwise} \end{cases}$$

where m is the number of radios that set the lowest price.

The Nash equilibrium of the game is when all radios choose a target number that is equal to the unit cost:

$$p_i = K, i = 1, ..., n.$$

The payoffs of the cognitive radios in Nash equilibrium is zero, which is obviously the worst possible outcome.

!!!!!!!!! Numerical experiments indicate that in many cases the joint equilibria assure more favorable outcomes than the Nash equilibrium. The Lorenz equilibrium overcomes the main disadvantages of Nash equilibrium: it is Pareto optimal and in case of the studied models there is one single Lorenz equilibrium solution.

Chapter 8

Conclusions and Future Work

This Chapter summarizes the results presented in this thesis about building more realistic models in Non-cooperative Game Theory. Also future research directions are also discussed.

8.1 Summary of Results

Standard Game Theory has some limitations when applied in real life situations. In a standard non-cooperative game only players with same rationality types are allowed to interact. In real life situations this is not the case; human players involved in a game act and think differently.

The central solution concept in Game Theory is Nash equilibrium. Nash equilibrium is the state where no player can improve her payoff by unilateral deviation. Nash equilibrium does not always assure the most optimal outcome for all. Moreover, it models an extreme selfish behavior that is rarely encountered in case of real players.

Other equilibria types have been proposed such as Pareto or Berge-Zhukovskii equilibrium. Pareto equilibrium is the state where no player can improve her player without decreasing the payoff of another player. Berge-Zhukovskii equilibrium is reached where no player or group of players can switch their strategy as to improve the payoff on any other player. Both Pareto and Berge-Zhukovskii equilibria describe an other regarding rationality that is the opposite of the Nash rationality.

Human players can rarely are characterised by such strict rationalities. They can act irrationally, they can be more or less selfish, more or less competitive, more or less cooperative, etc. Thus standard equilibria concepts do not always model the human decision making accurately.

In Chapter 3 the concept of Fuzzy Equilibrium is presented. Fuzzy equilibrium allows each player to have different biases towards different rationality types. This bias is expressed by a fuzzy membership degree to the certain equilibrium. For example a player can have a membership degree of 0.7 to Nash rationality and 0.3 to Pareto rationality. This way several new equilibrium concepts are proposed such as fuzzy Nash-Pareto or fuzzy Nash–Berge-Zhukovskii. Fuzzy equilibria is illustrated using the oligopoly models. Both fuzzy Nash-Pareto and fuzzy Nash–Berge-Zhukovskii equilibrium capture an intermediate state between Nash and Pareto equilibria, and Nash and Berge-Zhukovskii equilibria respectively.

Two studies about how people play the discrete and continuous version of the centipede game are considered in Chapter 4. These studies show that when human players are involved the outcome of the game rarely corresponds to the known equilibria concepts. Fuzzy equilibrium is used to model the human decision making. Numerical experiments validate the hypothesis that fuzzy Nash-Pareto and fuzzy Nash–Berge-Zhukovskii equilibrium is suitable to capture the human results.

In Chapter 5 the Lorenz dominance relation is investigated for multiobjective optimization problems and games. Experiments are carried out on randomly generated solutions in order to have a better comparison of Pareto and Lorenz dominance. These experiments indicate that the size of the Lorenz non-dominated set is considerably smaller than the size of the Pareto non-dominated set. Moreover the cardinality of the set of Lorenz non-dominated solutions remains relatively constant with the increasing number of objectives, while the set of Pareto non-dominated solutions grows drastically.

A modified version of GDE3 algorithm called L-GDE3 is used to evolutionary detect the Lorenz front. The Lorenz front is a subset of the Pareto front that usually consists only of the most balanced and equitable solutions for all objectives. The effect of Lorenz dominance on the scalability of the evolutionary techniques with respect to the number of objectives is also addressed. Numerical experiments show that the algorithm based on Lorenz dominance is scalable up to ten objectives.

Based on the Lorenz dominance relation, a new solution concept for non-cooperative games, Lorenz equilibrium, is presented. This solution concept overcomes some limitations of the Nash equilibrium. The Lorenz equilibrium solutions are optimal and fair for all players. In our numerical experiments we illustrate the Lorenz equilibrium on different types of games. The results show, that Lorenz equilibrium is a powerful and natural solution concept: it is both Pareto-optimal and assures a fair payoff distribution for all players.

In Chapter 6 an evolutionary method for detecting mulitcriteria game equilibrium is proposed. In real life decision making usually there are more criteria to consider. These criteria are often contradictory and can not be aggregated into one single criterion. Multicriteria games thus offer a more realistic modeling of real life situations. Basic single criterion cooperation dilemmas are extended to multicriteria games. An identity is added as a second criterion, that allows the modeling of a psychological factor. Numerical results show that the Nash equilibrium of games with identity payoffs is closer to mutual cooperation. The Prisoner's Dilemma with identity payoffs is analyzed as a repeated spatial game. Results show that the introduction of the identity payoff helps in the emergence of cooperation.

Chapter 7 studies a telecommunication problem from a game theoretic perspective. Various cognitive radio spectrum access scenarios are reformulated as non-cooperative games. The Cournot game is used to model an open spectrum access scenario, while the Bertrand game models a situation where there is a crowded spectrum with limited bandwidth. The Nash equilibrium for these games are not Pareto optimal, which leads to inefficient network usage. Moreover, in some instances of these models there are multiple Nash equilibria, that makes hard to predict the outcome of the game. The joint Nash-Pareto and Lorenz equilibria are proposed as possible solution concepts. Numerical experiments indicate that in many cases the joint equilibria assure more favorable outcomes than the Nash equilibrium. The Lorenz equilibrium overcomes the main disadvantages of Nash equilibrium: it is Pareto optimal and in case of the studied models there is one single Lorenz equilibrium solution.

8.2 Future Work

Future work will consider extending the proposed models for more general situations. Our goal is to extend the evolutionary algorithm for equilibrium detection for games involving many players. Parallelizing the underlying evolutionary algorithm will speed up the search.

Future work involves the further investigation of the fuzzy Nash–Berge-Zhukovskii equilibrium. The altruism of the Berge-Zhukovskii equilibrium compensates the selfishness of the Nash equilibrium so this combination leads to realistic outcomes. Our numerical experiments indicate that fuzzy Nash–Berge-Zhukovskii is able to capture the manner real people make decisions. Our aim is to investigate the human behavior in other trust games, and check whether the results can be reproduced with fuzzy Nash–Berge-Zhukovskii equilibrium.

A challenging possibility is applying the joint and fuzzy equilibria in economical models. The continuous centipede game has some analogies with stock market models. An interesting direction is the analysis of these models and the investigation of the outcomes of joint and fuzzy equilibria.

Future research will address the investigation of multicriteria games with identity payoffs. Identity payoffs are suitable to add a human factor to the game, thus they are suitable to model different player types (more or less ethical, more or less cooperative, etc). We consider that a spatial multicriteria Prisoner's dilemma involving players with different identity payoffs would offer a more realistic simulation.

The reformulation of oligopoly models for different spectrum access scenarios also offer many future research paths. A possibility would be to consider more complex models instead of linear demand functions and unit costs. Also, other oligopoly models might be considered to model various spectrum access scenarios. For example the Stackelberg game can be reformulated as for capturing a hierarchical resource allocation between cognitive radios. Some of this work is captured in [Cremene et al., 2011].

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