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NONLINEAR SYSTEMS AND NASH TYPE EQUILIBRIUM

Ph.D. Thesis Summary

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Introduction

In this thesis, we combine the notions of fixed points and critical points to gain a more profound understanding of the qualitative properties of solutions to various nonlinear systems. Specifically, we investigate the equilibrium properties of solutions for some nonlinear systems. Our primary emphasis lies on Nash-type equilibria.

The concept of equilibrium, nowadays understood under the name of Nash equilibrium, has its historical roots in the economic study conducted by A. Cournot in the mid-nineteenth century, in the book *The Mathematical Principles of the Theory of Wealth* [21, Chapter VII]. This study examined the outcomes of two 'proprietors' who were analyzing both the total price per product and the quantity of sales. The analysis assumed that the proprietors were not operating as a monopoly, meaning none of them exerted influence over the others. In other words, the analysis focused on a scenario in which both proprietors were in a state where neither could improve their profit relative to the other.

In 1951, J. Nash [40] examined such equilibria in non-cooperative finite games within game theory and provided a rigorous existence result using Brouwer's fixed-point theorem [11]. This paper's novelty lies in its applicability to any finite game, contrasting with earlier attempts such as the one by J. Neumann and O. Morgenstern in 1944 [39].

A new point of view is to use the notion of a Nash equilibrium more generally for systems of operator equations, specifically for a system of two equations with u and v as unknowns, where each one of the equations has an energy functional $E_1(u, v)$ and $E_2(u, v)$, respectively. A solution (u^*, v^*) is a Nash equilibrium if

$$E_1(u^*, v^*) = \min E_1(\cdot, v^*) \text{ and } E_2(u^*, v^*) = \min E_2(u^*, \cdot).$$

Since 1951, the idea of a Nash equilibrium has been extensively developed not only in the field of game theory but also in various other domains (see, e.g., F. Facchinei and C. Kanzow [29], S. Park [41, 42], J. Li and S. Park [37], J. Krawczyk [36], S. Cacace, E. Cristiani, M. Falcone [15], J.A. Ramos, R. Glowinski and J. Periaux [58, 59]).

Structure of the thesis

Our thesis consists of four chapters, each with several sections within.

Chapter 1 is dedicated to essential preliminary concepts, results, and notations used throughout this work. In Section 1.1, we introduce fundamental results related to the Fréchet derivative and Nemytskii operators. Section 1.2 discusses Ekeland's variational principle and its consequences. In Section 1.3, we review concepts related to matrices converging to zero and their associated properties. Finally, the last three sections provide necessary results used throughout the thesis, covering fixed point theorems, Sobolev spaces, and a new concept of linking introduced by R. Precup.

In **Chapter 2**, we focus on systems where each one of the equations admits a variational structure, that is, each equation is equivalent with a critical point problem. Section 2.1 starts with an existence and uniqueness result for an equation of Kirchhoff type, where we also prove its equivalence with a critical point problem. Subsequently, we investigate a system of Kirchhoff equations, demonstrating the existence of a solution that is also a Nash equilibrium for the associated energy functionals. This result is retrieved in both the entire domain and in a ball. Illustrative examples are provided for each case.

The chapter continues with Section 2.2, where we study an abstract system on reflexive and uniformly convex Banach spaces, under the assumption that each equation possesses a variational form.

All the results from Section 2.1 are original and have been published in R. Precup and A. Stan [54]. In Sections 2.2 and 2.3, our contributions are: Theorem 2.8, Theorem 2.9, Theorem 2.10 and Example 2.3. They have been published in A. Stan [67].

The purpose of **Chapter 3** is to further investigate the existence of solutions that constitute Nash equilibria, even for systems where not all the equations admit a variational structure. In Section 3.1, we study a system of three equations where only the last two of them have this property. We provide sufficient conditions such that the system is solvable and moreover, the last two components from the solution are a Nash equilibrium for the associated energy functionals.

In Section 3.2, we explore a system similar to the one in Section 3.1 but with an arbitrary number of equations. Our assumption is that only the last p equations admit a variational formulation. Our goal is not only to prove the existence of solutions such that the last p components of the solution are a Nash equilibrium for their energy functionals, but also to establish their localization within certain conical sets. Finally, in Section 3.3, we present applications of the results obtained in both Section 3.1 and Section 3.2. Each application is accompanied by an illustrative example.

All the results from this chapter are original and they can be found in A. Stan [65, 66].

Chapter 4 aims to extend the concept of Nash equilibrium discussed in previous chapters. The idea is not only to attain the minimum of the energy functionals but also to capture saddle points, all of this through a unitary theory. Thus, given a critical point system $E_{11}(u,v)=0$ and $E_{22}(u,v)=0$, where E_{ii} $(i=1,2)$ represents the Fréchet derivative of some functional E_i with respect to the ith variable, we aim to obtain a solution (u^*,v^*) such that one of the following situations holds: a) $E_1(u^*,v^*)$ is a minimum for $E_1(u^*,\cdot)$ and $E_2(u^*,v^*)$ is a minimum for $E_2(\cdot,v^*)$ (Nash equilibrium), b) $E_1(u^*,v^*)$ is a minimum for $E_1(u^*,\cdot)$ and $E_2(u^*,v^*)$ is a mountain pass point for $E_2(\cdot,v^*)$ or c) $E_1(u^*,v^*)$ is a mountain pass point for $E_1(\cdot,v^*)$ and $E_2(u^*,v^*)$ is a mountain pass point for $E_2(u^*,\cdot)$. To emphasize the significance of the problem, let us consider the pair of functionals below on $\mathbb{R}^2\times\mathbb{R}^2$:

$$
\begin{aligned}
E_1(x,y,z,w) &= x^2+y^2+z^2+w^2-xz,\ E_2(x,y,z,w)=x^2+2y^2+z^2+w^2-yw,\\
F_1(x,y,z,w) &= x^2+y^2+z^2+w^2-xz,\ F_2(x,y,z,w)=x^2+2y^2+z^2-w^2-yw,\\
G_1(x,y,z,w) &= x^2-y^2+z^2+w^2-xz,\ G_2(x,y,z,w)=x^2+2y^2+z^2-w^2-yw.
\end{aligned}
$$

One easily sees that the pair (u^*,v^*), where $u^*=v^*=(0,0)$, is a critical point for all the functionals above, but with different proprieties. Indeed: u^* minimizes $E_1(\cdot,v^*)=x^2+y^2$ while v^* minimizes $E_2(u^*,\cdot)=z^2+w^2$, u^* minimizes $F_1(\cdot,v^*)=x^2+y^2$ while v^* is a mountain pass point for $F_2(u^*,\cdot)=z^2-w^2$, and finally u^* is a mountain pass point for $G_1(\cdot,v^*)=x^2-y^2$ while v^* is a mountain pass point for $G_2(u^*,\cdot)=z^2-w^2$.

This work significantly complements the paper [53] and expands upon the ideas and techniques presented in M. Bełdzinski and M. Galewski [8], R. Precup and A. Stan [48, 52, 53, 65] (see also G. Kassay and V. D. Rădulescu [34, Ch. 8]). However, the absolute novelty introduced by this work lies in the unified approach to obtain solutions which are generalized Nash equilibrium for the system, i.e., some components of the solution can be mountain pass critical points, while others can be minimum points. The theory applies not just to systems with two equations but can be extended to any number of equations.

All the results from this chapter are included in R. Precup and A. Stan [55].

Author's publications:

1. A. Stan. *Nonlinear systems with a partial Nash type equilibrium.* Studia Univ. Babeş-Bolyai Math., **66**(2):397–408, 2021.

2. R. Precup and A. Stan. *Stationary Kirchhoff equations and systems with reaction terms.* AIMS Math., **7**(8):15258–15281, 2022.

3. A. Stan. *Nash equilibria for componentwise variational systems.* J. Nonlinear Funct. Anal., **6**, 2023.

4. R. Precup and A. Stan. *Linking methods for componentwise variational systems.* Results Math., **78**:246, 2023.

5. A. Stan. *Localization of Nash-type equilibria for systems with a partial variational structure* J. Numer. Anal. Approx. Theory, **52**(2):253–272, 2023.

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Chapter 1

Preliminaries

In this chapter we list some notions and results that we use throughout our thesis. Ekeland's variational principle, fixed point theorems, proprieties of matrices convergent to zero and results from the theory of Sobolev spaces are the primary tools in our research.

The concepts discussed here are well-documented in the literature. Some of the notable references include works by A. I. Perov [44], I. A. Rus [61, 62], F. Browder [12], H. Brezis [10], K. Deimling [22], R. Precup [45, 49], P. G. Ciarlet [16], H. Le Dret [26], C. Zălinescu [70], G. Kassay and V. D. Rădulescu [34], R. S. Varga, [69], A. Granas and J. Dugundji [32], R. Adams and J. Fournier [1].

1.1 Differential calculus in Banach spaces

Definition 1.1. *It is said that E is Fréchet differentiable at $u \in X$, if there exists $E'(u) \in X^*$ such that*

$$E(u+v) - E(u) = \langle E'(u)\,,\, v\rangle + \omega(u,v), \text{ for all } v \in X,$$

where ω is such that

$$\frac{\omega(u,v)}{|v|} \to 0, \quad \text{as } |v| \to 0.$$

Definition 1.2 ([45, Definition 5.1])**.** *A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be of Carathéodory type if*

1. *$f(\cdot, y) : \mathbb{R}^m \to \mathbb{R}^n$ is measurable for every $y \in \mathbb{R}^m$;*

2. *$f(x, \cdot) : \mathbb{R}^m \to \mathbb{R}^n$ is continuous for almost every $x \in \mathbb{R}^m$.*

In the subsequent discussion, $\Omega \subset \mathbb{R}^m$ denotes a bounded open set.

Definition 1.3. *Let $f \colon \Omega \times \mathbb{R}^n \to \mathbb{R}^N$ be a function. The Nemytskii operator associated with f is the map which assign to any function $u \colon \Omega \to \mathbb{R}^n$, the new* *function* $N_f(u)\colon \Omega \to \mathbb{R}^N$*, given by*

$$N_f\,(u)\,(x) = f(x, u(x)),\ \textit{for all } x \in \Omega.$$

Theorem 1.1 ([46, Theorem 9.1])**.** *Let* $p, q \in (1, \infty)$ *and let* $f\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^N$ *be a Carathéodory type function. Assume that there are constants* $c_1, c_2 \in \mathbb{R}_+$ *such that*

$$|f(x, y)| \le c_1|y|^{\frac{p}{q}} + c_2,\ \textit{for all } y \in \mathbb{R}^n \textit{ and almost all } x \in \Omega.$$

Then, the Nemytskii operator N_f *is well defined and continuous from* $L^p\,(\Omega; \mathbb{R}^n)$ *to* $L^q\left(\Omega; \mathbb{R}^N\right)$*.*

Example 1.1. *Let* $F\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ *be a function with the following properties*

(1) $F(\cdot, 0) = 0$*,*

(2) F *is of Carathéodory type,*

(3) $F(x, \cdot)$ *continuously differentiable.*

If $\nabla F(x, \cdot)$ *is also of Carathéodory type, then the functional*

$$E\colon L^p(\Omega; \mathbb{R}^n) \to \mathbb{R},\ E(u) = \int_\Omega F(x, u(x))dx,$$

belongs to $C^1\left(L^p(\Omega, \mathbb{R}^n)\right)$*, and moreover* $E' = N_f$*, i.e.,*

$$\langle E'(u), v\rangle = \int_\Omega \left(\nabla F(x, u(x))\,,\, v(x)\right) dx,\ \ \textit{for all } v \in L^p(\Omega; \mathbb{R}^n).$$

1.2 Ekeland variational principle

First, we recall the weak form of Ekeland's variational principle (see, I. Ekeland [27], D. G. de Figueiredo [30]).

Theorem 1.2 (Ekeland)**.** *Let* (X, d) *be a complete metric space, and* $E : X \to \mathbb{R}$ *a lower semicontinuous functional bounded from below. Then, for every* $\varepsilon > 0$*, there exists an element* $x \in X$ *that satisfies the following two proprieties*

$$E\,(x) \le \inf_{y \in X} E\,(y) + \varepsilon,$$

and

$$E\,(x) \le E\,(y) + \varepsilon d\,(x, y)\,, \quad \textit{for all } y \in X.$$

Proposition 1.3. *Given the assumptions of Theorem 1.2, when X is a Banach space equipped with the norm $|\cdot|_X$ and E is a C^1 functional, there exists a sequence (u_k) from X such that*

$$E(u_k) \to \inf_X E \quad \text{and} \quad E'(u_k) \to 0.$$

The Ekeland variational principle extends to balls or conical sets, notably valuable for demonstrating the existence of almost critical points within bounded sets. Consider H as a Hilbert space with inner product $(\cdot,\cdot)_H$ and induced norm $|\cdot|_H$.

Theorem 1.4 ([63, Theorem 5.3.1])**.** *Let $R > 0$, and let $E : B_R \to \mathbb{R}$ be a C^1 functional that is bounded from below, where B_R denotes the closed ball of radius R centered at the origin. Then, there exists a sequence (u_k) from B_R such that*

$$E(u_k) \to \inf_{B_R} E,$$

and one of the following two situations holds

(a) $E'(u_k) \to 0$;

(b) $|u_k|_H = R$, $(E'(u_k), u_k) \leq 0$, *for all* $k \in \mathbb{N}$, *and*

$$E'(u_k) - \frac{(E'(u_k), u_k)_H}{R^2} u_k \to 0.$$

Let $K \subset H$ be a cone, and let $l \colon K \to \mathbb{R}$ be an upper semicontinuous concave functional. Additionally, assume the existence of an operator $N \colon H \to H$ and a C^1 functional $E \colon H \to \mathbb{R}$ such that $E'(u) = u - N(u)$, for all $u \in H$. For two positive real numbers $0 < r < R$, consider the convex conical set $K_{r,R}$ be defined by

$$K_{r,R} := \{u \in K \,:\, r \leq l(u),\ |u|_H \leq R\}.$$

In subsequent, we recall a variant of the Ekeland variational principle on the set $K_{r,R}$. For the proof and further details we send to R. Precup [52, Lemma 2.1]

Lemma 1.5. *Assume the following conditions are satisfied:*

(i) The functional E is bounded from below on $K_{r,R}$, i.e.,

$$m := \inf_{K_{r,R}} E(\cdot) > -\infty.$$

(ii) There exists $\varepsilon > 0$ such that for all $u \in K_{r,R}$ satisfying both $|u|_H = R$ and $l(u) = r$, we have $E(u) \geq m + \varepsilon$.

(iii) $l(N(u)) \geq r$, *for all* $u \in K_{r,R}$.

Then, there exists a sequence $(u_k) \in K_{r,R}$ *such that*

$$E(u_k) \leq m + \frac{1}{k},$$

and

$$|E'(u_k) + \lambda_n u_k|_H \leq \frac{1}{k},$$

where

$$\lambda_n = \begin{cases} -\frac{1}{R^2}(E'(u_k), u_k)_H, & \textit{when } |u_k|_H = R \textit{ and } (E'(u_k), u_k)_H < 0 \\ 0, & \textit{otherwise.} \end{cases}$$

1.3 Matrices convergent to zero

Definition 1.4. *A square matrix* $A \in \mathbb{M}_{n\times n}(\mathbb{R}_+)$ *is said to be convergent to zero if*

$$A^k \to O_n \textit{ as } k \to \infty,$$

where O_n *denotes the zero matrix of order* n.

For any $r \in \{1, \ldots, n\}$, let us consider the diagonal submatrix $A_r := [a_{ij}]_{1\leq i,j\leq r}$. It is not difficult to see that if A is convergent to zero, then A_r is also convergent to zero, as follows from the subsequent lemma.

Lemma 1.6. *Assume that the matrix* A *is convergent to zero. Then* A_r *is also convergent to zero, for any* $r \in \{1, \ldots, n\}$.

For a square matrix $A \in \mathbb{M}_{n\times n}(\mathbb{R}_+)$, condition that A^k tends to the zero matrix O_n as $k \to \infty$ is equivalent to each one of the following properties from Lemma 1.7 below (see, e.g., A. Berman and R. J. Plemmons [7], R. Precup [47]).

Lemma 1.7. *The following statements are equivalent:*

(i) The matrix A *is convergent to zero.*

(ii) The matrix $I - A$ *is nonsingular, and the entries of its inverse* $(I - A)^{-1}$ *are nonnegative.*

(iii) The spectral radius of A *is less then* 1*, i.e., the maximum magnitude of its eigenvalues is less than 1.*

(iv) There exists a positive diagonal matrix $D = (d_{ii})_{1\leq i\leq n}$ *such that*

$$(D(I - A)x, x) > 0, \textit{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

In case when $n = 2$, the following equivalent characterization holds true (see, e.g., R. Precup [47]).

Lemma 1.8. *Let $A = [a_{ij}]_{1\leq i,j\leq 2}$ be a square matrix of nonnegative real numbers. Then, A is convergent to zero if and only if $a_{11}, a_{22} < 1$ and*

$$a_{11} + a_{22} < 1 + a_{11}a_{22} - a_{12}a_{21}. \tag{1.1}$$

The following result related to matrices convergent to zero is intensively used throughout this thesis.

Lemma 1.9 ([65, Lemma 2.2]). *Let $(x_{k,p})_{k\geq 1}$, $(y_{k,p})_{k\geq 1}$ be two sequences of vectors in $\mathbb{R}^n_+$ (column vectors), both dependent on an parameter p, which additionally satisfy:*

$$x_{k,p} \leq Ax_{k-1,p} + y_{k,p}$$

for all k and p, where $A \in \mathbb{M}_{n\times n}(\mathbb{R}_+)$ is a matrix convergent to zero. If the sequence $(x_{k,p})_{k\geq 1}$ is bounded uniformly with respect to p and $y_{k,p} \to 0_n$ as $k \to \infty$ uniformly with respect to p, then $x_{k,p} \to 0_n$ as $k \to \infty$ uniformly with respect to p.

1.4 Fixed point type theorems

Theorem 1.10 (Perov). *Consider two complete metric spaces (X_i, d_i) $(i = 1, 2)$. Let $N_i : X_1 \times X_2 \to X_i$ be two mappings and assume there exists a square matrix A of size two with nonnegative entries and spectral radius $\rho(A) < 1$ such that the following vector inequality holds*

$$\begin{pmatrix} d_1(N_1(x,y), N_1(u,v)) \\ d_2(N_2(x,y), N_2(u,v)) \end{pmatrix} \leq A \begin{pmatrix} d_1(x,y) \\ d_2(u,v) \end{pmatrix},$$

for all $(x, y), (u, v) \in X_1 \times X_2$. Then, there exists a unique point $(x^*, y^*) \in X_1 \times X_2$ with $x^* = N_1(x^*, y^*)$ and $y^* = N_2(x^*, y^*)$. Furthermore, the point (x^*, y^*) can be attained using the method of successive approximations starting from an arbitrarily initial point (x_0, y_0), since for any $k \in \mathbb{N}$ we have

$$\begin{pmatrix} d_1\left(N_1^k(x_0,y_0), x^*\right) \\ d_2\left(N_2^k(x_0,y_0), y^*\right) \end{pmatrix} \leq A^k (I-A)^{-1} \begin{pmatrix} d_1(x_0, N_1(x_0,y_0)) \\ d_2(y_0, N_2(x_0,y_0)) \end{pmatrix}.$$

Theorem 1.11 (Schauder). *Let X be a Banach space, $D \subset X$ a nonempty closed convex bounded set and $T : D \to D$ a compact operator (i.e., continuous, with $T(D)$ relatively compact). Then, T has at least one fixed point in D.*

Theorem 1.12 (Leray-Schauder). *Let X be a Banach space and $T\colon X \to X$ a continuous compact mapping that satisfies the following condition: there exists $R > 0$ such that the set $\cup_{\lambda\in[0,1]}\{x \in X \,:\, x = \lambda Tx\}$ is contained within a ball of radius R, centered in the origin. Then, T admits at least one fixed point.*

Definition 1.5. *Let $T\colon X \to X^*$ be an operator. It is said that*

(i) T is strongly monotone if there exists $a > 0$ such that

$$\langle T(u) - T(v)\,,\, u - v\rangle \geq a|u - v|_X^2, \text{ for all } u, v \in X.$$

T is said to be monotone if the constant a may take the value 0.

(ii) T is coercive if

$$\frac{\langle T(u), u\rangle}{|u|_X} \to \infty \text{ as } |u|_X \to \infty.$$

(iii) T is demicontinuous if for any $x_n \to x^*$ in X we have that $T(x_n) \to T(x)$ weakly, i.e.,

$$\langle T(x_n)\,,\, y\rangle \to \langle T(x^*)\,,\, y\rangle, \text{ for any } y \in X.$$

Theorem 1.13 (Minty-Browder). *Let X be a real, reflexive and separable Banach space. Assume $T\colon X \to X^*$ is a bounded, demicontinuous, coercive and monotone operator. Then, for any given $v \in X^*$, there exists a unique $u \in X$ such that $T(u) = v$.*

1.5 Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, and let us consider the Sobolev space

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) \,:\, \nabla u \in L^p(\Omega; \mathbb{R}^n)\}\,.$$

Proposition 1.14. *For $1 < p < \infty$, the space $W^{1,p}(\Omega)$ is a reflexive and separable Banach space with the norm $\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|u'\|_{L^p}$. When $p = 2$, the space $H^1(\Omega) := W^{1,2}(\Omega)$ becomes a Hilbert space together with the inner product*

$$(u, v)_{H^1} = (u, v)_{L^2} + (\nabla u, \nabla v)_{L^2}.$$

In the following, our emphasis will be on to the the Sobolev space

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p} \,:\, u|_\Omega = 0 \text{ in the sense of traces }\right\}.$$

Proposition 1.15 (Poincaré Inequality, [10,28,43])**.** *There exists a constant $C > 0$ such that*

$$|u|_{L^p} \leq C|\nabla u|_{L^p}, \text{ for all } u \in W_0^{1,p}(\Omega).$$

Proposition 1.16. *The Sobolev space $\left(W_0^{1,p}(\Omega), |\cdot|_{W_0^{1,p}}\right)$ is a uniformly convex real Banach space.*

Further, let us consider the dual of $W^{1,p}(\Omega)$ denoted with $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. One has the following diagram,

$$W_0^{1,p}(\Omega) \rightrightarrows L^p(\Omega) \xrightarrow{N_f} L^{p'}(\Omega) \hookrightarrow W^{-1,p}(\Omega).$$

The subsequent result establishes an equivalence between p-Laplacian and the duality mapping corresponding to the gauge function $\varphi(t) = t^{p-1}$ on $\left(W_0^{1,p}, |\cdot|_{W_0^{1,p}}\right)$. For details we send to G. Dinca, P. Jebelean and J. Mawhin [25, Theorem 3].

Theorem 1.17. *The operator $-\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is the Fréchet derivative of the functional $\psi \colon W_0^{1,p}(\Omega) \to \mathbb{R}$, where $\psi(u) = \frac{1}{p}|u|^p_{W_0^{1,p}}$. Specifically,*

$$\psi' = -\Delta_p = J_\varphi,$$

where J_φ represents the duality mapping corresponding to the gauge function $\varphi(t) = t^{p-1}$.

Let $H^{-1}(\Omega)$ stand for the dual space of $H_0^1(\Omega)$. For any $f \in H^{-1}(\Omega), u \in H_0^1(\Omega)$, the expression $\langle f, u\rangle$ represents the value at u of the continuous linear functional f. Moreover, one has the Poincaré inequality

$$|u|_{L^2} \leq \frac{1}{\sqrt{\lambda_1}}|u|_{H_0^1} \quad \left(u \in H_0^1\right),$$

where λ_1 is the first eigenvalue of the Dirichlet problem for the operator $-\Delta$. We use the notation $(-\Delta)^{-1}$ for the inverse of the Laplacian with respect to the homogeneous Dirichlet boundary condition.

In case $(0,T) = \Omega \subset \mathbb{R}$, the Poincaré inequality holds with $\lambda_1 = \frac{\pi^2}{T^2}$ (see, e.g., H. Brezis [10], R. Precup [45]), i.e.,

$$|u|_{L^2} \leq \frac{1}{\sqrt{\lambda_1}}|u|_{H_0^1} = \frac{T}{\pi}|u|_{H_0^1}, \left(u \in H_0^1\right),$$

where λ_1 is the first eigenvalue of the Dirichlet problem $-u'' = \lambda u$, $u(0) = u(T) = 0$.

Additionally, there exists a positive constant $c > 0$ exists such that for all $t \in (0,T)$ and $u \in H_0^1(0,T)$, the following inequality holds true

$$|u(t)| \leq c|u|_{H_0^1}.$$

1.6 A unifying notion of linking

The concept of linking, crucial in critical point theory, has widespread application (V. Benci and P.H. Rabinowitz [5], P.H. Rabinowitz [57], M. Schechter [63], M. Struwe [68]). Originating from the mountain pass theorem by Ambrosetti and Rabinowitz [2], it has evolved and adapted to various generalizations. Linking has become a potent tool in analyzing diverse nonlinear problems (D.G. Costa and C.A. Magalh aes [17], N. Costea, M. Csirik and C. Varga [20], R. Filippucci, P. Pucci and F. Robert [31], P. Pucci and V. D. Rădulescu [56], E.A.B. Silva [64]).

Let X be a Banach space, D and Q be two subsets of X with $\emptyset \neq Q \subset D$.

Definition 1.6 ([53]). *It is said that a nonempty set $A \subset D$ links a set $B \subset Q$ via Q (in D) if $\gamma(Q) \cap A \neq \emptyset$ for every $\gamma \in C(Q, D)$ with $\gamma|_B = id_B$.*

Note that, according to the above definition, the entire set $A = D$ links the empty set $B = \emptyset$, via any Q, particularly through any singleton $Q = \{\overline{u}\}$ with $\overline{u} \in D$. As further explained below, this limiting scenario of trivial linking provides us with minima of a functional when using the min-max procedure.

Assume that A links B in D via Q. Let $E : D \to \mathbb{R}$ be a functional, and let

$$\Gamma = \{\gamma \in C(Q, D) : \ \gamma|_B = \ \mathrm{id}_B\}.$$

Denote

$$m := \inf_{v \in D} E(v), \quad a := \inf_{v \in A} E(v), \quad b := \sup_{v \in B} E(v),$$

and

$$c := \inf_{\gamma \in \Gamma} \sup_{q \in Q} E(\gamma(q)).$$

We immediately deduce that

$$m \leq a \leq c \quad \text{and} \quad b \leq c.$$

Also, if $B = \emptyset$ and $A = D$, then

$$m = a, \quad b = -\infty \quad \text{and} \quad c = m.$$

Chapter 2

Nash equilibria for componentwise variational systems

2.1 Kirchhoff type systems

We consider the coupled system of Kirchhoff equations (see, G. Kirchhoff [35])

$$\begin{cases} -\left(a+b\,|u|_{H_0^1}^2\right)\Delta u = f_1 + g_1\,(x,u,v) \\ -\left(a+b\,|v|_{H_0^1}^2\right)\Delta v = f_2 + g_2\,(x,u,v) \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where we are interested in a solution that is also a Nash equilibrium. The main idea is to express the system (2.1) as

$$\begin{cases} N_1(u,v) = u \\ N_2(u,v) = v\,, \end{cases} \tag{2.2}$$

where both equations admit a variational structure. This means that there exist energy functionals $E_1\,(u,v)$ and $E_2\,(u,v)$ such that (2.2) is equivalent with the critical point problem

$$\begin{cases} E_{11}(u,v) = 0 \\ E_{22}(u,v) = 0\,. \end{cases} \tag{2.3}$$

Here, E_{ii} stands for the partial Fréchet derivative of E_i $(i = 1,2)$ with respect to the ith variable.

In the ssequel, let us consider the following Kirchhoff equation with Dirichlet boundary condition

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = h, \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

The first result concerns the existence of a continuous solution operator.

Theorem 2.1. *If $h \in H^{-1}(\Omega)$, the problem (2.4) has a unique weak solution, i.e., there exists a unique $u \in H_0^1(\Omega)$ such that*

$$\left(a+b\int_\Omega |\nabla u|^2\,dx\right)(u,v)_{H_0^1} = \langle h, v\rangle, \qquad v \in H_0^1(\Omega). \tag{2.5}$$

The main idea to guarantee the existence of a solution of (2.5), is to consider the operator,

$$S_h : H_0^1(\Omega) \to H_0^1(\Omega), \quad S_h(v) = \frac{1}{a+b|v|_{H_0^1}^2}(-\Delta)^{-1}h \quad \left(v \in H_0^1(\Omega)\right).$$

Clearly, S_h is compact and moreover,

$$|S_h(v)|_{H_0^1} \le \frac{1}{a}|h|_{H^{-1}}. \tag{2.6}$$

If we define

$$B = \left\{ v \in H_0^1(\Omega) : \ |v|_{H_0^1} \le \frac{1}{a}|h|_{H^{-1}} \right\},$$

one clearly has $S_h(B) \subset B$. Consequently, from Theorem 1.11, there exists at least one u such that $S_h(u) = u$.

Given the monotony of the function $(a+bx^2)\,x$, we deduce that any two solutions u_1, u_2 of (2.5) satisfy $|u_1|_{H_0^1} = |u_2|_{H_0^1}$. Thus, the uniqueness of solution for the Dirichlet problem related to $-\Delta$ provides $u_1 = u_2$.

Theorem 2.2. *(The energy functional) A function $u \in H_0^1(\Omega)$ is a weak solution of the Dirichlet problem if and only if it is a critical point of the C^1 functional $E : H_0^1(\Omega) \to \mathbb{R}$,*

$$E(v) = \frac{1}{4}\left(2a + b|v|_{H_0^1}^2\right)|v|_{H_0^1}^2 - \langle h, v\rangle. \tag{2.7}$$

Theorem 2.3. *A function $u \in H_0^1(\Omega)$ solves the Dirichlet problem if and only if it represents a minimum for the corresponding energy functional (2.7).*

2.1.1 Global solution

We are interested to prove the existence of a solution which is a Nash equilibrium in the entire space $H_0^1(\Omega) \times H_0^1(\Omega)$ for the system (2.1).

For each one of the equation from the system (2.1) we associate the energy functionals $E_1, E_2 \colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, given by

$$\begin{aligned} E_1(u,v) &= \frac{1}{4}\left(2a + b|u|_{H_0^1}^2\right)|u|_{H_0^1}^2 - \langle f_1, u\rangle - \int_\Omega G_1(x, u(x), v(x))\,dx, \\ E_2(u,v) &= \frac{1}{4}\left(2a + b|v|_{H_0^1}^2\right)|v|_{H_0^1}^2 - \langle f_2, v\rangle - \int_\Omega G_2(x, u(x), v(x))\,dx, \end{aligned}$$

where $G_1(x,u,v) = \int_0^u g_1(x,s,v)\,ds$ and $G_2(x,u,v) = \int_0^v g_2(x,u,s)\,ds$. One has,

$$\begin{aligned} E_{11}(u,v) &= \left(a + b|u|_{H_0^1}^2\right)u - (-\Delta)^{-1}(f_1 + g_1(\cdot,u,v)), \\ E_{22}(u,v) &= \left(a + b|v|_{H_0^1}^2\right)v - (-\Delta)^{-1}(f_2 + g_2(\cdot,u,v)), \end{aligned}$$

for every $u, v \in H_0^1(\Omega)$.

Definition 2.1. *A function $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be of coercive-type if the functional $\phi : H_0^1(\Omega) \to \mathbb{R}$,*

$$\phi(v) = \frac{1}{4}\left(2a + b|v|_{H_0^1}^2\right)|v|_{H_0^1}^2 - \langle f_2, v\rangle - \int_\Omega H(x,v)\,dx$$

is coercive, i.e., $\phi(v) \to +\infty$ as $|v|_{H_0^1} \to +\infty$.

Theorem 2.4. *For each $i \in \{1,2\}$, assume that the functions $f_i \in H^{-1}(\Omega)$ and $g_i \colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are of Carathéodory type and $g_i(\cdot,0,0) = 0$. Additionally, let us assume that the following conditions hold:*

(h1) *There are constants $a_{ij} \in \mathbb{R}_+$ $(i,j = 1,2)$ such that*

$$a_{ii} < \lambda_1 a, \quad i = 1,2,$$

$$a_{12}a_{21} < (\lambda_1 a - a_{11})(\lambda_1 a - a_{22}), \tag{2.8}$$

and

$$\begin{aligned} (g_1(t,x,y) - g_1(t,\overline{x},\overline{y}))(x - \overline{x}) &\le a_{11}|x - \overline{x}|^2 + a_{12}|x - \overline{x}||y - \overline{y}|, \\ (g_2(t,x,y) - g_2(t,\overline{x},\overline{y}))(y - \overline{y}) &\le a_{21}|x - \overline{x}||y - \overline{y}| + a_{22}|y - \overline{y}|^2, \end{aligned} \tag{2.9}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$ and a.e. $t \in \Omega$. Here, λ_1 represents the first eigenvalue of the Dirichlet problem $-\Delta u = \lambda u$, $u_{\partial\Omega} = 0$.

(h2) *There exist two functions $H_1,\ H_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ of coercive-type such that*

$$H_1(t,y) \le G_2(t,x,y) \le H_2(t,y), \tag{2.10}$$

for all $x, y \in \mathbb{R}$ and a.e. $t \in \Omega$.

Then, the system (2.1) *has a unique solution which is a Nash equilibrium for the functionals E_1, E_2.*

Remark 2.5 (Classical Lipschitz conditions)**.** *It is clear that the unilateral Lipschitz conditions (2.9) hold when g_1 and g_2 satisfy to the classical Lipschitz conditions:*

$$\begin{aligned} |g_1(t,x,y) - g_1(t,\overline{x},\overline{y})| &\leq a_{11}|x-\overline{x}| + a_{12}|\,|y-\overline{y}|, \\ |g_2(t,x,y) - g_2(t,\overline{x},\overline{y})| &\leq a_{21}|x-\overline{x}| + a_{22}|y-\overline{y}|, \end{aligned}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$ and a.e. $t \in \Omega$. In R. Precup's paper [50], the conditions imposed on the coefficients a_{ij} allow us to directly establish both the existence and uniqueness of the solution to system (2.2) using Perov's fixed-point theorem (Theorem 1.10). It is noteworthy that the application of unilateral Lipschitz conditions to prove the existence of Nash equilibria was initially introduced in R. Precup [51]

Example 2.1. Consider the Dirichlet problem for the system of Kirchhoff type

$$\begin{cases} -\left(1+\int_0^1 |u'|^2\right) u'' = u - \sin v \\ -\left(1+\int_0^1 |v'|^2\right) v'' = v + \sin u & \text{on } (0,1) \\ u(0)=v(0)=u(1)=v(1)=0. \end{cases} \tag{2.11}$$

Theorem 2.4 is employed with

$$\Omega = (0,1)\,,\ a=b=1\,,\ g_1(t,x,y) = x - \sin y\,,\ g_2(t,x,y) = \sin x + y.$$

Note that condition (2.9) is satisfied with $a_{ij} = 1\ (i,j=1,2)$. Also, the first eigenvalue of the Dirichlet problem $-u'' = \lambda u$ on $(0,1)$, $u(0) = u(1) = 0$ has the value π^2 (see, e.g., R. Precup [45, p. 72]), therefore relation (2.8) holds true since $1 < \pi^2$ and $1 < (\pi^2-1)^2$. To verify condition (h2), we calculate

$$G_2(t,x,y) = \int_0^y (s+\sin x)ds = \frac{1}{2}y^2 + y\sin x.$$

Let the coercive-type functions $H_1(t,y) = \frac{1}{2}y^2 - |y|$ and $H_2(t,y) = \frac{1}{2}y^2 + |y|$. One easily sees that

$$H_1(t,y) \leq G_2(t,x,y) \leq H_2(t,y).$$

Henceforth, the Dirichlet problem (2.11) possesses a unique solution $(u^*, v^*) \in H_0^1(0,1) \times H_0^1(0,1)$ that also is a Nash equilibrium for the associated energy functionals.

2.1.2 Solutions in bounded domains

We aim to establish the existence of a solution for the system (2.1) in the bounded domain $B_{R_1} \times B_{R_2}$, where B_{R_i} represents balls of radius R_i $(i = 1, 2)$ centered at the origin of the space $H_0^1(\Omega)$.

We consider the following Leray-Schauder boundary conditions

$$E_{11}(u, v) + \mu\, u \neq 0 \text{ for all } (u, v) \in B_{R_1} \times B_{R_2} \text{ with } |u|_{H_0^1} = R_1 \text{ and all } \mu > 0, \tag{2.12}$$

$$E_{22}(u, v) + \gamma\, v \neq 0 \text{ for all } (u, v) \in B_{R_1} \times B_{R_2} \text{ with } |v|_{H_0^1} = R_2 \text{ and all } \gamma > 0.$$

Theorem 2.6. *Let $f_i \in H^{-1}(\Omega)$ and let $g_i\colon\ \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be Carathéodory type functions with $g_i(\cdot, 0, 0) = 0$ $(i = 1, 2)$ that satisfy the monotony conditions from assumption (h1) of Theorem 2.4. Moreover, let us assume that*

(h2')

$$\begin{aligned} \frac{a_{11}}{\lambda_1} R_1 + \frac{a_{12}}{\lambda_1} R_2 + |f_1|_{H^{-1}} &\leq aR_1 + bR_1^3, \\ \frac{a_{21}}{\lambda_1} R_1 + \frac{a_{22}}{\lambda_1} R_2 + |f_2|_{H^{-1}} &\leq aR_2 + bR_2^3. \end{aligned}$$

Then, the system (2.1) has a unique solution within $B_{R_1} \times B_{R_2}$ which is a Nash equilibrium for the functionals E_1, E_2.

Example 2.2. Let the Dirichlet problem for the system of Kirchhoff type

$$\begin{cases} -\left(2 + \int_0^1 |u'|^2\right) u'' = -u^3 + u - \sin v + \pi^2 \sin(\pi x) \\ -\left(2 + \int_0^1 |v'|^2\right) v'' = -v^3 + v + \sin u & \text{on } (0,1) \\ u(0) = v(0) = u(1) = v(1) = 0. \end{cases} \tag{2.13}$$

Let $R_1 = R_2 = 1$. In the following, we apply Theorem 2.6 with

$$\Omega = (0, 1),\ a = 2,\ b = 1,\ f_1(t) = \pi^2 \sin(\pi t),\ f_2 \equiv 0,$$

$$g_1(t, x, y) = -x^3 + x - \sin y,\ g_2(t, x, y) = -y^3 + y + \sin x.$$

For any $x, \overline{x} \in \mathbb{R}$ one clearly has

$$\begin{aligned} (g_1(t, x, y) - g_1(t, \overline{x}, \overline{y}))(x - \overline{x}) &\leq |x - \overline{x}|^2 + |x - \overline{x}|\,|y - \overline{y}|, \\ (g_2(t, x, y) - g_2(t, \overline{x}, \overline{y}))(y - \overline{y}) &\leq |x - \overline{x}|\,|y - \overline{y}| + |y - \overline{y}|^2. \end{aligned}$$

Therefore, condition (2.9) holds with $a_{ij} = 1$ $(i, j = 1, 2)$. Moreover, since $\lambda_1 = \pi^2$, note that condition (2.8) is also satisfied. Hence, assumption (h1) is verified. Next, we check condition (h2'). Observe that $|f_2|_{H^{-1}} = 0$ and in addition, the function $u_0(t) = \sin(\pi t)$ is the solution of the Dirichlet problem $-u'' = f_1$ in $(0,1)$, $u(0) = u(1) = 0$. Thus,

$$|f_1|_{H^{-1}} = |u_0|_{H_0^1} = |u_0'|_{L^2} = \left(\int_0^1 \pi^2 \cos^2(\pi t)\, dt\right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{2}}.$$

Finally, condition (h2') holds true since

$$\frac{2}{\pi^2} + \frac{\pi}{\sqrt{2}} < 3 \text{ and } \frac{2}{\pi^2} < 3.$$

Henceforth, there is a unique solution

$$(u^*, v^*) \in \left\{u \in H_0^1(0,1) \,:\, |u|_{H_0^1} \leq 1\right\} \times \left\{v \in H_0^1(0,1) \,:\, |v|_{H_0^1} \leq 1\right\},$$

to the Dirichlet problem (2.13) that is also a Nash equilibrium for the corresponding energy functionals.

2.2 Abstract systems in reflexive Banach spaces

In this section, we present some extension of the results obtained by R. Precup [50], within the context of Hilbert spaces, to a broader functional framework.

Unlike previous approaches using Perov contraction conditions and Ekeland's variational principle, our method employs different mathematical tools, including insights from C. Avramescu [3] and techniques with monotone operators like the Minty-Browder theorem (cf. Theorem 1.13) and the Leray-Schauder fixed-point theorem (cf. Theorem 1.12).

We consider the system

$$\begin{cases} N_1(u,v) = J_1(u) \\ N_2(u,v) = J_2(v), \end{cases} \tag{2.14}$$

where N_1, N_2 are continuous operators and J_1, J_2 represent the duality mappings corresponding to suitable Banach spaces.

Consider a real, separable, and uniformly convex Banach space X along with its dual space X^*. Let $\langle \cdot, \cdot \rangle$ denote the dual pairing between X^* and X, and J the duality mapping associated with the gauge function $\varphi(t) := t^{p-1}$, where $p > 1$, i.e.,

$$Jx := \{x^* \in X^* \,:\, \langle x^*, x\rangle = |x|^p,\ |x^*|_{X^*} = |x|^{p-1}\}. \tag{2.15}$$

Lemma 2.7. *The duality mapping* (2.15) *has the following properties:*

i) *J is single valued.*

ii) *J is strictly monotone, i.e., $\langle Jx - Jy,\, x-y\rangle > 0$ for all $x \neq y$.*

iii) *J satisfies the $(S)_+$ condition, i.e., if $x_n \to x$ weakly and $\limsup_{n\to\infty}\langle J\,x_n\,,\, x_n - x\rangle \leq 0$, then $x_n \to x$ strongly.*

iv) *J is demicontinuous, i.e., if $x_n \to x$ strongly, then $Jx_n \to Jx$ weakly.*

v) *J is bijective from X to X^*.*

Let $(X_1, |\cdot|_1)$, $(X_2, |\cdot|_2)$ be two separable and uniformly convex real Banach spaces, together with their dual spaces X_1^* and X_2^*. Denote with $\langle \cdot\,,\,\cdot\rangle_1$, $\langle \cdot\,,\,\cdot\rangle_2$ the dual pairings between X_1^*, X_1 and X_2^*, X_2, respectively. The duality mappings J_1 and J_2 correspond to gauge functions $\varphi_1(t) := t^{p-1}$ and $\varphi_2(t) = t^{q-1}$, respectively, where $p \geq q > 1$.

We assume variational structure for (2.14), with energy functionals E_1 and E_2, such that

$$E_{11}(u,v) = J_1(u) - N_1(u,v), \quad E_{22}(u,v) = J_2(v) - N_2(u,v),$$

where E_{11} and E_{22} are partial Fréchet derivatives. Any $(u^*, v^*) \in X_1 \times X_2$ satisfying $E_{11}(u^*, v^*) = 0$ and $E_{22}(u^*, v^*) = 0$ is a solution of (2.14)..

Let $a_{11}, a_{22} \in [0,1)$ be such that

$$\langle N_1(u,v) - N_1(\overline{u},v)\,,\, u-\overline{u}\rangle_1 \leq a_{11}\langle J_1(u) - J_1(\overline{u})\,,\, u-\overline{u}\rangle_1, \tag{2.16}$$

$$\langle N_2(u,v) - N_2(u,\overline{v})\,,\, v-\overline{v}\rangle_2 \leq a_{22}\langle J_2(v) - J_2(\overline{v})\,,\, u-\overline{v}\rangle_2, \tag{2.17}$$

for all $u, \overline{u} \in X_1$ and $v, \overline{v} \in X_2$.

The problem of finding a Nash equilibrium solution for system (2.14) can be divided into two subproblems:

(i) Proving any solution's status as a Nash equilibrium.

(ii) Ensuring the existence of at least one solution.

This division simplifies analysis while maintaining clarity. Notably, in our case, the equivalence between the original problem and its subproblems (i) and (ii) holds.

Our first result below, ensures that the monotony conditions (2.16) and (2.17) are sufficient to solve the first subproblem (i).

Theorem 2.8. *Given the previous assumptions, if* $(u^*, v^*) \in X_1 \times X_2$ *satisfies both* $E_{11}(u^*, v^*) = 0$ *and* $E_{22}(u^*, v^*) = 0$ *simultaneously, then* $(u^*, v^*) \in X_1 \times X_2$ *is, in fact, a Nash equilibrium for the energy functionals* (E_1, E_2)*, i.e.,*

$$E_1(u^*, v^*) = \inf_{X_1} E_1(\cdot, v^*) \text{ and } E_2(u^*, v^*) = \inf_{X_2} E_2(u^*, \cdot)\,. \tag{2.18}$$

Theorem 2.9. *Assume the following conditions hold true*

(h1) *The operator* $J_2^{-1} \circ N_2 \colon X_1 \times X_2 \to X_2$ *is compact.*

(h2) *There are real numbers* $a_{12}, a_{21} \in (0,1)$ *and* $M_1, M_2 \in \mathbb{R}_+$ *such that*

$$|N_1(0, v)| \le a_{12}|v|_1^{p-1} + M_1, \quad \text{for all } v \in X_2, \tag{2.19}$$

$$|N_2(u, 0)| \le a_{21}|u|_1^{q-1} + M_2, \quad \text{for all } u \in X_1, \tag{2.20}$$

and the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is convergent to zero.

Then, there exists a solution $(u^*, v^*) \in X_1 \times X_2$ *of the system* (2.14).

2.3 Applications

Let us consider the Dirichlet problem

$$\begin{cases} -\Delta_p\, u = f_1(\cdot, u, v) \\ -\Delta_q\, v = f_2(\cdot, u, v) & \text{on } \Omega \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0\,, \end{cases} \tag{2.21}$$

where $p \ge q > 1$ and Ω is some bounded domain from $\mathbb{R}^n$ with Lipschitz boundary. We consider $X_1 = W_0^{1,p}(\Omega)$ and $X_2 = W_0^{1,q}(\Omega)$, equipped with the usual norms $|u|_{1,p} := |\nabla u|_{L^p}$ and $|u|_{1,q} := |\nabla u|_{L^q}$. From Theorem 1.17 we see that the dual mapping $J_1 = -\Delta_p$ and $J_2 = -\Delta_q$. We assume the $f_1, f_2 \colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfy the growth conditions

$$|f_1(t, x, y)| \le C_1|x|^{p-1} + C_2|y|^{p-1} + a(t)\,, \tag{2.22}$$

$$|f_2(t, x, y)| \le C_1|x|^{q-1} + C_2|y|^{q-1} + b(t)\,, \tag{2.23}$$

for all $x, y \in \mathbb{R}$ and $t \in \Omega$, where $C_1, C_2 \in \mathbb{R}$, $a \in L^{p'}(\Omega)$ and $b \in L^{q'}(\Omega)$. Here, p' and q' are such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

From (2.22) and (2.23) we conclude that the Nemytskii operators

$$N_{f_1}(u,v)(t) := f_1(t,u(t),v(t)) \text{ and } N_{f_2}(u,v)(t) := f_2(t,u(t),v(t))$$

are well defined, continuous and bounded from $L^p(\Omega)$ to $L^{p'}(\Omega)$, respectively $L^q(\Omega)$ to $L^{q'}(\Omega)$. The compact embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$ and $W_0^{1,q}(\Omega)$ in $L^q(\Omega)$, guarantees that the operator

$$T = (-\Delta_q)^{-1}N_{f_2}(u,v)\colon W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)\to W_0^{1,q}(\Omega)$$

is compact (see, e.g., G. Dinca and P. Jebelean [24]).

Observe that each equation from (2.21) admits a variational formulation given by the energy functionals $E_1, E_2\colon W^{1,p}(\Omega)\times W^{1,q}(\Omega)\to\mathbb{R}$,

$$E_1(u,v) := \frac{1}{p}|u|_{1,p}^p - \int_\Omega F_1(\cdot,u,v),\;\; E_2(u,v) := \frac{1}{q}|u|_{1,q}^q - \int_\Omega F_2(\cdot,u,v),$$

where

$$F_1(t,u(t),v(t)) := \int_0^{u(t)} f_1(t,s,w(t))ds,\;\; F_2(t,u(t),v(t)) := \int_0^{v(t)} f_2(t,u(t),s)ds.$$

Theorem 2.10. *Let the above assumptions be satisfied. Furthermore, let us assume*

(H1) *There exists positive real numbers* $\overline{a}_{11},\overline{a}_{22}$ *such that*

$$(x-\overline{x})(f_1(\cdot,x,y)-f_1(\cdot,\overline{x},y)) \le \overline{a}_{11}|x-\overline{x}|^p, \tag{2.24}$$
$$(y-\overline{y})(f_2(\cdot,x,y)-f_2(\cdot,x,\overline{y})) \le \overline{a}_{22}|s-\overline{s}|^q, \tag{2.25}$$

for all real numbers $x,\overline{x},y,\overline{y}$.

(H2) *There exists positive real numbers* $\overline{a}_{12},\overline{a}_{21},M_1,M_2$ *such that*

$$|f_1(\cdot,0,y)| \le \overline{a}_{12}|y|^{p-1}+M_1, \tag{2.26}$$
$$|f_2(\cdot,x,0)| \le \overline{a}_{21}|x|^{q-1}+M_2, \tag{2.27}$$

for all real numbers x,y.

(H3) *The matrix*

$$A := \begin{bmatrix} C^p\,\overline{a}_{11} & C^p\,\overline{a}_{12} \\ D^q\,\overline{a}_{21} & D^q\,\overline{a}_{22}\end{bmatrix}$$

is convergent to zero, where C *and* D *represent the constants associated with the Poincaré inequality (Proposition 1.15) in the spaces* $W_0^{1,p}(\Omega)$ *and* $W_0^{1,q}(\Omega)$*, respectively.*

Then, there exists a solution $(u^*, v^*) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *for the system* (2.21)*, such that it is a Nash equilibrium for the energy functionals* E_1, E_2.

In conducting the arguments of the previous results, the following lema is needed

Lemma 2.11. *([23, Proposition 8]) Under the growth conditions (2.26-2.27), the Nemytskii's operators* $(\overline{N}_{f_1}v)(x) := f_1(x, 0, v(x))$ *and* $(\overline{N}_{f_2}u)(x) := f_2(x, u(x), 0)$ *satisfy*

$$\begin{aligned} |\overline{N}_{f_1}v|_{L^{p'}} &\leq \overline{a}_{12}|v|_{L^p}^{p-1} + M_1' \\ |\overline{N}_{f_2}u|_{L^{q'}} &\leq \overline{a}_{21}|u|_{L^q}^{q-1} + M_2'. \end{aligned} \tag{2.28}$$

Example 2.3. *Consider the following second order system of differential equations with Dirichlet boundary conditions*

$$\begin{cases} -u'' = -u + \pi\, \sin(u) + \frac{\pi}{2}v \\ -v'' = u + \cos(v) & \text{on } (0,1) \\ \ u(0) = v(0) = u(1) = v(1) = 0. \end{cases} \tag{2.29}$$

To achieve a solution that is a Nash equilibrium for the associated energy functionals, we will demonstrate that all the assumptions specified in Theorem 2.10 are fulfilled, where

$$\Omega = (0,1),\ p = q = 2,\ n = 1,\ C = \tfrac{1}{\pi}$$

$$f_1(t, x, y) = -x + \pi\, \sin(x) + \tfrac{\pi}{2}y,\ \ f_2(t, x, y) = x + \cos(y).$$

Note that growth conditions (2.22-2.23) holds with $C_1 = 1$, $C_2 = \frac{\pi}{2}$ and $a(t) = \pi$, $b(t) = 1$.

One clearly has

$$\begin{aligned} &(f_1(t, x, y) - f_1(t, \overline{x}, y)\,(x - \overline{x}) \leq \pi|x - \overline{x}|, \\ &(f_2(t, x, y) - f_1(t, x, \overline{y})\,(y - \overline{y}) \leq |y - \overline{y}|. \end{aligned}$$

Hence, we may chose $\overline{a}_{11} = \pi$ and $\overline{a}_{22} = 1$ to satisfy condition (H1). Simple calculations demonstrate that (H2) also holds with $\overline{a}_{12} = \frac{\pi}{2}$, $\overline{a}_{21} = 1$, $M_1 = 0$, and $M_2 = 1$. In the end, it is clear that the matrix

$$A = \begin{bmatrix} \frac{1}{\pi} & \frac{1}{2\pi} \\ \frac{1}{\pi^2} & \frac{1}{\pi^2} \end{bmatrix}$$

is convergent to zero. Therefore, the system (2.29) has a solution $(u^*, v^*) \in W_0^{1,2}(0,1) \times W_0^{1,2}(0,1)$ which is a Nash equilibrium for the corresponding energy functionals.

Chapter 3

Nash equilibria for partial variational systems

In this section we extend the results for systems with three equations, where we aim to find solutions that are a partial Nash type equilibrium.

Related results can be found in R. Precup [52], B. Renata and R. Precup [13], J. R. López, R. Precup and C.I Gheorghiu [60], I. Benedetti, T. Cardinali and R. Precup [6], M. Bełdziński, M. Galewski and D. Barilla [9].

3.1 Global existence

We consider the system

$$\begin{cases} N_1(u, v, w) = u \\ N_2(u, v, w) = v \\ N_3(u, v, w) = w, \end{cases} \tag{3.1}$$

where only the last two equations admit a variational formulation. Our objective is to find a solution (u, v, w) such that the pair (v, w) is a Nash-type equilibrium for the energy functionals associated with the last two equations.

Let (X_1, d) be a complete metric space and $(X_2, |\cdot|_2)$, $(X_3, |\cdot|_3)$ be two real Hilbert spaces which are identified with their duals. Denote $X := X_1 \times X_2 \times X_3$. We assume that there exist two functionals $E_2, E_3 : X \to \mathbb{R}$ such that $E_2(u, \cdot, w)$ is Fréchet differentiable for every $(u, w) \in X_1 \times X_3$, $E_3(u, v, \cdot)$ is Fréchet differentiable for every $(u, v) \in X_1 \times X_2$ and

$$\begin{aligned} E_{22}(u, v, w) &= v - N_2(u, v, w), \\ E_{33}(u, v, w) &= w - N_3(u, v, w). \end{aligned}$$

Here, E_{22} represents the Fréchet derivative of the functional $E_2(u, \cdot, w)$, while E_{33} is the Fréchet derivative of the functional $E_3(u, v, \cdot)$.

In addition, we assume that the operators N_i satisfy the following Lipschitz conditions (Perov contraction condition): there are nonnegative real numbers a_{ij} $(i, j = 1, 2, 3)$ such that

$$\begin{aligned} d\left(N_1(u,v,w),\ N_1(\overline{u},\overline{v},\overline{w})\right) &\leq a_{11}d(u,\overline{u}) + a_{12}|v-\overline{v}|_2 + a_{13}|w-\overline{w}|_3, \\ |N_2(u,v,w),\ N_2(\overline{u},\overline{v},\overline{w})|_2 &\leq a_{21}d(u,\overline{u}) + a_{22}|v-\overline{v}|_2 + a_{23}|w-\overline{w}|_3, \\ |N_3(u,v,w),\ N_3(\overline{u},\overline{v},\overline{w})|_3 &\leq a_{31}d(u,\overline{u}) + a_{32}|v-\overline{v}|_2 + a_{33}|w-\overline{w}|_3, \end{aligned} \tag{3.2}$$

for all $(u, v, w), (\overline{u}, \overline{v}, \overline{w}) \in X$ and the matrix $A = [a_{ij}]_{1\leq i,j\leq 3}$ is convergent to zero.

Theorem 3.1. *Under the previously functional framework, in addition we assume:*

(h1) *For every triple $(u, v, w) \in X$, the functionals $E_2(u, \cdot, w),\ E_3(u, v, \cdot)$ are bounded from below.*

(h2) *There are positive real numbers $R_2, R_3, a > 0$ such that*

$$E_2(u,v,w) \geq \inf_{X_2} E_2(u,\cdot,w) + a \quad \text{for all } (u,w) \in X_1 \times X_3 \text{ and } |v|_2 \geq R_2, \tag{3.3}$$

and

$$E_3(u,v,w) \geq \inf_{X_3} E_3(u,v,\cdot) + a \quad \text{for all } (u,v) \in X_1 \times X_2 \text{ and } |w|_3 \geq R_3. \tag{3.4}$$

Then, the unique fixed point (u^*, v^*, w^*) guaranteed by the Perov contraction theorem has the property that (v^*, w^*) is a Nash type equilibrium for the pair of functionals (E_2, E_3), that is,

$$\begin{aligned} E_2(u^*,v^*,w^*) &= \inf_{X_2} E_2(u^*,\cdot,w^*), \\ E_3(u^*,v^*,w^*) &= \inf_{X_3} E_3(u^*,v^*,\cdot). \end{aligned}$$

3.2 Existence of solutions in conical sets

We consider a system with n equations

$$\begin{cases} N_1(u^1,\ldots,u^n) = u^1 \\ \ldots \\ N_p(u^1,\ldots,u^p,\ldots,u^n) = u^p \\ \ldots \\ N_n(u^1,\ldots,u^n) = u^n, \end{cases} \tag{3.5}$$

having the special property that only the last n-p equations admit a variational structure. We aim to find a solution $(u_*^1, \ldots, u_*^p, \ldots, u_*^n)$ such that $(u_*^{p+1}, \ldots, u_*^n)$ is located within the Cartesian product of some conical sets and moreover, it is a Nash equilibrium for the corresponding energy functionals

Definition 3.1. *Let $x = (x_i), y = (y_i) \in \mathbb{R}^n$ be two vectors. We denote with $\circ \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the Hadamard product, i.e.,*

$$x \circ y = (x_1 y_1, \ldots, x_n y_n)^T.$$

The Hadamard product is related to the inner product by the following relation.

Proposition 3.2. *Let $A = (a_{ij})_{1 \le i,j \le n} \in \mathbb{M}_{n,n}(\mathbb{R}_+)$ be a matrix with positive entries and let $x = (x_i), y = (y_i), z = (z_i) \in \mathbb{R}_+^n$. If*

$$Ax \circ y \le z$$

then

$$Ax \cdot y \le \sqrt{n}|z|.$$

Let $(X_i, |\cdot|_i)$ $(i = 1, \ldots, n)$ be Hilbert spaces identified with their duals. Denote

$$X := X_1 \times \cdots \times X_n \text{ and } X_{1,q} := X_1 \times \cdots \times X_q, (q = 1, \ldots, n),$$

together with the inner products $(u, v)_X = (u_1, v_1)_1 + \ldots + (u_n, v_n)_n$, $(u, v)_{X_{1,q}} = (u_1, v_1)_1 + \ldots + (u_q, v_q)_1$, and induced norms $|u|_X^2 = (u, v)_X$, $|u|_{X_{1,q}}^2 = (u, v)_{X_{1,q}}$. Also, let $\overline{X}_q$ denotes the space obtained from X by excluding X_q, i.e.,

$$\overline{X}_q := X_1 \times \cdots \times X_{q-1} \times X_{q+1} \times \cdots \times X_n.$$

For simplicity, for any $q \in \{1, \ldots, n\}$, we refer to

$$(u^1, \ldots, u^q)^T \text{ as } u^{1,q},\ (u^{q+1}, \ldots, u^n)^T \text{ as } u^{q+1,n}$$

and

$$(N_1(u), \ldots, N_q(u))^T \text{ as } N_{1,q}(u),\ (N_{q+1}(u), \ldots, N_n(u))^T \text{ as } N_{q+1,n}(u).$$

With these notations, we have

$$u = \left(u^{1,p}, u^{p+1,n}\right)^T$$

and

$$(N_1(u), \ldots, N_n(u))^T = (N_{1,q}(u), N_{q+1,n}(u))^T.$$

On $X_{1,q}$, we consider the vector-valued inner product

$$\langle\langle u, v\rangle\rangle = \left((u^1, v^1)_1, \ldots, (u^q, v^q)_q\right)^T \in \mathbb{R}^q,$$

and vector valued norm

$$\|u\| := \left(|u^1|_1, \ldots, |u^q|_q\right)^T \in \mathbb{R}^q,$$

for any $u = (u^1, \ldots, u^q)$, $v = (v^1, \ldots, v^q) \in X_{1,q}$. It is not difficult to see that these notations remain consistent with respect to Hadamard product since $\langle\langle u\,,\,u\rangle\rangle = \|u\| \circ \|u\|$.

For each $q \in \{p+1, \ldots, n\}$, we assume the existence of functionals $E_q\colon X \to \mathbb{R}$ that are Fréchet differentiable with respect to the qth component (this derivative is denoted with E_{qq}), such that

$$E_{qq}(u) = u^q - N_q(u). \tag{3.6}$$

For each $q \in \{p+1, \ldots, n\}$, let $K_q \subset X_q$ be a cone. Also, let $l_q\colon K_q \to \mathbb{R}_+$ be an upper semicontinuous and concave functional with the property that $l_q(0) = 0$. On K_q we consider the convex conical set $(K_q)_{r_q,R_q}$,

$$(K_q)_{r_q,R_q} := \{u^q \in K_q \;:\; r^q \leq l_q(u^q),\; |u^q|_q \leq R_q\},$$

where $0 \leq r_q < R_q \leq \infty$ are nonnegative real numbers. Denote

$$K := (K_{p+1})_{r_{p+1},R_{p+1}} \times \cdots \times (K_n)_{r_n,R_n}$$

and

$$\overline{K}_q := (K_{p+1})_{r_{p+1},R_{p+1}} \times \cdots \times (K_{q-1})_{r_{q-1},R_{q-1}} \times (K_{q+1})_{r_{q+1},R_{q+1}} \times \cdots \times (K_n)_{r_n,R_n}.$$

3.2.1 Existence of a minimizing sequence

Theorem 3.3. *In what follows, we assume:*

(h1) *There exists a matrix $A = [a_{ij}]_{1\leq i,j\leq n}$ convergent to zero such that*

$$\langle\langle N_{1,n}\,(u) - N_{1,n}\,(v)\;,\;u - v\rangle\rangle \leq A\|u - v\| \circ \|u - v\|, \tag{3.7}$$

i.e,

$$\left(N_i(u) - N_i(v), u^i - v^i\right)_i \leq \sum_{j=1}^{n} \left|u^i - v^i\right|_i \sum_{j=1}^{n} a_{ij} \left|u^j - v^j\right|_j, \; (i = 1, \ldots, n), \tag{3.8}$$

for all $u = (u^1, \ldots, u^n), v = (v^1, \ldots, v^n) \in X$.

(h2) *For each* $q \in \{p+1, \ldots, n\}$, *one has*

$$l_q\left(N_q(u)\right) \geq r_q, \text{ for all } u \in X_{1,p} \times K.$$

(h3) *There exists* $m := \inf_{u \in X_{1,p} \times K} E_q(u) > -\infty$ *and* $\varepsilon > 0$ *such that*

$$E_q(u) \geq \inf_{(K_q)_{r_q,R_q}} E_q(u^1, \ldots, u^{q-1}, \cdot, u^{q+1}, \ldots, u^n) + \varepsilon,$$

for all $(u^1, \ldots, u^{q-1}, u^{q+1}, u^n) \in X_{1,p} \times \overline{K}_q$ *that satisfies* $l_q(u^q) = r_q$ *and* $|u^q|_q = R_q$, *simultaneously.*

Then, there exists a sequence $u_k = (u_k^1, \ldots, u_k^p, u_k^{p+1}, \ldots, u_k^n)^T \in X_{1,p} \times K$ *such that*

$$E_q\left(u_k^{1,q}, u_{k-1}^{q+1,n}\right) \leq \inf_{(K_q)_{r_q,R_q}} E_q\left(u_k^{1,q-1}, \cdot, u_{k-1}^{q+1,n}\right) + \frac{1}{k}$$

and

$$\left|E_{qq}\left(u_k^{1,q}, u_{k-1}^{q+1,n}\right) + \lambda_k^q u_k^q\right|_q \leq \frac{1}{k},$$

where

$$\lambda_k^q := \begin{cases} -\frac{1}{R_q^2}\left(E_{qq}\left(u_k^{1,q}, u_{k-1}^{q+1,n}\right), u_k^q\right)_q, & \text{if } |u_k^q|_q = R_q \text{ and } \\ & \left(E_{qq}\left(u_k^{1,q}, u_{k-1}^{q+1,n}\right), u_k^q\right)_q < 0 \\ 0, & \text{otherwise,} \end{cases}$$

for all $q \in \{p, \ldots, n\}$, $k \in \mathbb{N}$.

3.2.2 Convergence of the localized minimizing sequence

Now, we establish conditions ensuring convergence of the minimizing sequence (u_k) generated in Theorem 3.3.

Theorem 3.4. *Let* $u_k = (u_k^1, \ldots, u_k^p, u_k^{p+1}, \ldots, u_k^n)^T \in X_{1,p} \times K$ *be the sequence generated in Theorem 3.3. Additionally, we suppose*

(h2') *For every $q \in \{p+1, \ldots, n\}$, the following Leray-Schauder boundary condition are satisfied:*

$$N_q(u) - u^q - \lambda\, u^q \neq 0, \text{ for all } \lambda > 0 \text{ and } u \in X_{1,p} \times K \text{ with } |u^q|_q = R_q.$$

(h4) *The operator $N_q\left(0_{X_1}, \ldots, 0_{X_p}, \cdot\right)$ is bounded on K.*

Then, the sequence u_k is convergent to $u_* = (u_*^{1,p}, u_*^{p+1,n}) \in X_{1,p} \times K$. Furthermore, u_* is a solution of the system (3.5) and $u_*^{p+1,n}$ is a Nash equilibrium in K for the functionals $(E_{p+1}, \ldots, E_n)$, i.e. ,

$$E_q(u_*) = \inf_{(K_q)_{r_q,R_q}} E_q(u_*^{1,q-1}\,, \cdot, \, u_*^{q+1,n}) \quad (q = p+1, \ldots, n)\,.$$

Remark 3.5 (Limit cases)**.** *In our theory, we do not restrict ourselves to using only nonegative real numbers for r_q and R_q. When we aim for solutions within a ball, we set $r_q = 0$, and when we intend to find unbounded solutions from above, we choose for $R_q = \infty$.*

3.3 Applications

3.3.1 Global existence for a partial gradient type system

Let us consider the problem

$$\begin{cases} -u'' + a_1^2 u = f_1(t, u(t), v(t), w(t), u'(t)) \\ -v'' + a_2^2 v = \nabla_y f_2(t, u(t), v(t), w(t)) \qquad \text{on } (0,T) \\ -w'' + a_3^2 w = \nabla_z f_3(t, u(t), v(t), w(t)), \end{cases} \tag{3.9}$$

with the periodic conditions

$$\begin{aligned} u(0) - u(T) &= u'(0) - u'(T) = 0, \\ v(0) - v(T) &= v'(0) - v'(T) = 0, \\ w(0) - w(T) &= w'(0) - w'(T) = 0, \end{aligned}$$

where $f_{2,3}\colon (0,T)\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}\times\mathbb{R}^{k_3} \to \mathbb{R}$ and $f_1\colon (0,T)\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}\times\mathbb{R}^{k_3}\times\mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$. We assume that $f_i\,(i = 1, 2, 3)$, $\nabla_y f_2$ and $\nabla_z f_3$ are continuous and Carathéodory type functions. Let

$$C_p^1 = \{u \in C^1[0,T] \,:\, u(0) - u(T) = u'(0) - u'(T) = 0\},$$

and denote by $H^1_p(0,T)$ the completion of C^1_p in $H^1(0,T)$.

On $H^1_p(0,T)$, we can define two inner products

$$(u,v)_i = \int_0^T u'v' + a_i^2 uv = (u',v')_{L^2} + a_i^2 (u,v)_{L^2},$$

which give rise to equivalent norms. Now, from Riesz representation theorem, for any $h \in (H^1_p(0,T))'$, there is a unique $u_h \in H^1_p(0,T)$ such that

$$h(v) = (u_h, v)_i, \text{ for any } v \in H^1_p(0,T).$$

Thus, we may define the operators

$$J_i \colon (H^1_p(0,T))' \to (H^1_p(0,T)), J_i(h) = u_h \text{ with } (J_i h, v)_i = h(v), \; (i = 1,2,3).$$

For the second and third equation from (3.18), we associate the functionals

$$E_2, E_3 \colon H^1_p(0,T;\mathbb{R}^{k_1}) \times H^1_p(0,T;\mathbb{R}^{k_2}) \times H^1_p(0,T;\mathbb{R}^{k_3}) \to \mathbb{R},$$

where

$$E_2(u,v,w) = \frac{1}{2}|v|_2^2 - \int_0^T f_2(t,u(t),v(t),w(t)),$$

$$E_3(u,v,w) = \frac{1}{2}|w|_3^2 - \int_0^T f_3(t,u(t),v(t),w(t)).$$

Following J. Mawhin and M. Willem [38, Theorem 1.4], we have

$$(E_{22}(u,v,w,u',w'),\varphi) = (v,\varphi)_2 - (J_2(\nabla_y f_2),\varphi)_2,$$

for any $\varphi \in H^1_p(0,T;\mathbb{R}^{k_2})$. Thus, we may write $E_{22}(u,v,w) = v - J_2(\nabla_y f_2)$. Similarly, we derive the same relation for E_{33}, i.e.,

$$E_{33}(u,v,w,u',v') = w - J_3(\nabla_z f_3).$$

Therefore, we can write our system (3.18) as a fixed point equation,

$$\begin{cases} N_1(u,v,w) = u \\ N_2(u,v,w) = v \\ N_3(u,v,w) = w \end{cases}$$

where

$$\begin{aligned} N_1(u,v,w) &= J_1 f_1(\cdot,u,v,w,u'), \\ N_2(u,v,w) &= J_2 \nabla_y f_2(\cdot,u,v,w), \\ N_3(u,v,w) &= J_3 \nabla_z f_3(\cdot,u,v,w). \end{aligned}$$

Related to f_1, f_2, f_3, we will make the following assumptions

$$|f_1(t,x_1,...,x_4) - f_1(t,\overline{x_1},...,\overline{x_4})| \leq \sum_{i=1}^{4} a_{1i}|x_i - \overline{x_i}|, \tag{3.10}$$

$$|\nabla_y f_2(t,x_1,x_2,x_3) - \nabla_y f_2(t,\overline{x_1},\overline{x_2},\overline{x_3})| \leq \sum_{i=1}^{3} a_{2i}|x_i - \overline{x_i}|, \tag{3.11}$$

$$|\nabla_z f_3(t,x_1,x_2,x_3) - \nabla_z f_3(\overline{x_1},\overline{x_2},\overline{x_3})| \leq \sum_{i=1}^{3} a_{3i}|x_i - \overline{x_i}|, \tag{3.12}$$

where $a_{ij}, a_{14}\,(i,j = 1,2,3)$ are some positive real numbers.

For every $h \in L^2[0,T]$, we can derive the subsequent estimates for the solution operators J_i $(i = 1,2,3)$, $|J_i h|_i \leq \frac{1}{a_i}|h|_{L_2}$. Next, from (3.10), we obtain

$$\begin{aligned} &|N_1(u,v,w) - N_1(\overline{u},\overline{v},\overline{w})|_1 = |J_1\left(f_1(\cdot,u,v,w,u') - f_1(\cdot,\overline{u},\overline{v},\overline{w},\overline{u}')\right)|_1 \\ &\leq \frac{1}{a_1}\left(\left(\frac{a_{11}}{a_1}\right)^2 + a_{14}^2\right)^{\frac{1}{2}} |u-\overline{u}|_1 + \frac{a_{12}}{a_1 a_2}|v-\overline{v}|_2 + \frac{a_{13}}{a_1 a_3}|w-\overline{w}|_3. \end{aligned}$$

For $N_2(u,v,w)$ and $N_3(u,v,w)$, we have

$$\begin{aligned} |N_2(u,v,w) - N_2(\overline{u},\overline{v},\overline{w})|_2 &\leq \frac{a_{21}}{a_2 a_1}|u-\overline{u}|_1 + \frac{a_{22}}{a_2^2}|v-\overline{v}|_2 + \frac{a_{23}}{a_2 a_3}|w-\overline{w}|_3, \\ |N_3(u,v,w) - N_3(\overline{u},\overline{v},\overline{w})|_3 &\leq \frac{a_{31}}{a_3 a_1}|u-\overline{u}|_1 + \frac{a_{32}}{a_3^2}|v-\overline{v}|_2 + \frac{a_{33}}{a_3^2}|w-\overline{w}|_3. \end{aligned}$$

Therefore, the condition related to (3.2) is satisfied if the matrix

$$A = \begin{bmatrix} \frac{1}{a_1}\left(\left(\frac{a_{11}}{a_1}\right)^2 + a_{14}^2\right)^{\frac{1}{2}} & \frac{a_{12}}{a_1 a_2} & \frac{a_{13}}{a_1 a_3} \\ \frac{a_{21}}{a_2 a_1} & \frac{a_{22}}{a_2^2} & \frac{a_{23}}{a_2 a_3} \\ \frac{a_{31}}{a_3 a_1} & \frac{a_{32}}{a_2 a_3} & \frac{a_{33}}{a_3^2} \end{bmatrix} \tag{3.13}$$

is convergent to zero.

Next, we aim to establish conditions that ensure $E_2(u,\cdot,w)$ and $E_3(u,v,\cdot)$ are bounded from below. To achieve this, let us assume that for $i \in \{2,3\}$ and $j \in$

$\{1, 2, 3, 4\}$, there exist $\sigma_{ij} \in L^1(0, T; \mathbb{R}+)$ and $\gamma_i \in \mathbb{R}$ with $\gamma_i^2 < \frac{a_i^2}{2}$, satisfying

$$f_2(t, x, y, z) \leq \gamma_2^2 |y|^2 + \sigma_{21}(t)|x| + \sigma_{22}(t)|y| + \sigma_{23}(t)|z| + \sigma_{24}(t) \tag{3.14}$$

$$f_3(t, x, y, z) \leq \gamma_3^2 |z|^2 + \sigma_{31}(t)|x| + \sigma_{32}(t)|y| + \sigma_{33}(t)|z| + \sigma_{34}(t). \tag{3.15}$$

Considering the continuous embedding of $H_p^1\left(0, T; \mathbb{R}^{k_i}\right)$ into $C\left([0, T]; \mathbb{R}^{k_i}\right)$, we obtain

$$E_2(u, v, w) \geq \left(1 - \frac{2\gamma_2^2}{a_2^2}\right) |v|_2^2 - C_{21}|u|_1 - C_{22}|v|_2 - C_{23}\, |w|_3 - C_{24},$$

for some constants C_{2j} $(j = 1, 2, 3, 4)$. This guarantees that $E_2(u, v, w) \to \infty$ as $|v|_2 \to \infty$. Similarly, $E_3\,(u, v, w) \to \infty$ as $|w|_3 \to \infty$. Consequently, the functionals $E_2(u, \cdot, w)$ and $E_3(u, v, \cdot)$ are coercive, and moreover, in accordance with R. Precup [50, Lemma 4.1], they are also bounded from bellow.

Our final assumption concerns the existence of certain L^1-Carathéodory functions $g_{i1}, g_{i2} \colon (0, T) \times \mathbb{R}^{k_i} \to \mathbb{R}$ $(i = 2, 3)$, of coercive type, such that

$$g_{21}(t, y) \leq f_2(t, x, y, z) \leq g_{22}(t, y), \tag{3.16}$$

$$g_{31}(t, z) \leq f_3(t, x, y, z) \leq g_{32}(t, z), \tag{3.17}$$

for all for all $(x, y, z) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^{k_3}$ and $t \in (0, T)$. Letting $a > 0$ be fixed, we can use the above assumption to conclude that

$$\inf_{v \in H_p^1} E_2(u, \cdot, w) + a \leq \inf_{v \in H_p^1} \left(\frac{1}{2}\, |v|_2^2 - \int_0^T g_{21}\,(t, v)\, dt\right) + a.$$

Moreover, since g_{22} is coercive, there exists $R_2 > 0$ such that

$$\inf_{v \in H_p^1} \left(\frac{1}{2}\, |v|_2^2 - \int_0^T g_{21}\,(t, v)\, dt\right) + a \leq \frac{1}{2}\, |v|_2^2 - \int_0^T g_{22}\,(t, v)\, dt,$$

for all $|v|_2 \geq R_2$. Now, for $|v|_2 \geq R_2$ and all $(u, w) \in H_p^1(0, T; \mathbb{R}^{k_1}) \times H_p^1(0, T; \mathbb{R}^{k_3})$, using again (3.16), we deduce

$$E_2\,(u, v, w) \geq \frac{1}{2}\, |v|_2^2 - \int_0^T g_{22}\,(t, v)\, dt \geq \inf_{v \in H_p^1} E_2(u, \cdot, w) + a,$$

as desired. A similar inequality can be established for E_3.

Under the assumptions (3.10), (3.11),(3.12), (3.14), (3.15), (3.16), (3.17) and if the matrix (3.20) is convergent to zero, then all the hypotheses of Theorem 3.1 are fulfilled.

Next, we move to the second subsection where we aim to present an application of a system of second-order differential equations that satisfies all the assumptions outlined in Theorem 3.3 and Theorem 3.4.

3.3.2 Local existence for a second-order ODE system.

We consider the problem

$$\begin{cases} -u''(t) = f_1(t, u(t), v(t), w(t), u'(t)) \\ -v''(t) = f_2(t, u(t), v(t), w(t)) \\ -w''(t) = f_3(t, u(t), v(t), w(t)), \end{cases} \quad \text{on (0,T)} \tag{3.18}$$

with the Dirichlet boundary conditions

$$\begin{cases} u(0) = u(T) = 0 \\ v(0) = v(T) = 0 \\ w(0) = w(T) = 0, \end{cases}$$

where $f_1\colon (0,T)\times\mathbb{R}^4 \to \mathbb{R}_+$, $f_2, f_3\colon (0,T)\times\mathbb{R}^3 \to \mathbb{R}_+$ are functions of Carathéodory type. We emphasize that the presence of u' in the first equation, unlike equations 2 and 3, disrupts its variational structure. Here, the Hilbert spaces X_1, X_2, X_3 denote the Sobolev space $H_0^1(0,T)$ equipped with the inner product $(u,v)_{H_0^1} = \int_0^T u'v'$ and the norm $|u|_{H_0^1} = \left(\int_0^T (u')^2\right)^{\frac{1}{2}}$.

Let $(H_0^1(0,T))'$ be the dual space of $H_0^1(0,T)$ and let $(\cdot\,,\,\cdot)'$ be the dual pairing between $(H_0^1(0,T))'$ and $H_0^1(0,T)$. From Riesz's representation theorem (see, e.g., G. Bachman and L. Narici [4, Theorem 1.9]), for each $h \in (H_0^1(0,T))'$, there exists a unique $u_h \in H_0^1(0,T)$ such that

$$(h,\phi)' = (u_h,\phi)_{H_0^1}, \text{ for every } \phi \in H_0^1(0,T).$$

Hence, we define the solution operator $S\colon\ (H_0^1(0,T))' \to H_0^1(0,T)$, where $S(h) = u_h$. When $h \in L^2(0,T)$, the expression of $S(h)$ is given by

$$S(h)(t) = \int_0^T G(t,s)h(s)ds,$$

where $G(t,s)\colon (0,T)^2 \to \mathbb{R}_+$ is the Green function (see, e.g., A. Cabada [14, Example 1.8.18]),

$$G(t,s) = \begin{cases} s\left(1 - \frac{t}{T}\right), s \le t \\ t\left(1 - \frac{s}{T}\right), s \ge t. \end{cases}$$

Let $K := K_2 = K_3$ denote the cone of nonnegative functions from $H_0^1(0,T)$ and let $[a,b]$ be a fixed compact subinterval of $(0,T)$. Furthermore, we consider the concave upper semicontinous functionals $l_2, l_3 \colon K \to \mathbb{R}_+$,

$$l_1(u) = l_2(u) = \min_{t\in[a,b]} u(t) \quad (u \in K)\,,$$

and the conical sets

$$(K)_{r_j,R_j} = \{u \in K_j \,|\, r_j \le l_j(u),\ |u|_{H_0^1} \le R_j\}, (j = 2,3)\,,$$

where $0 < r_j < R_j$ are positive real numbers.

We emphasize that the second and third equations from (3.18) admit a variational formulation given by the energy functionals $E_2, E_3 \colon H_0^1(0,T) \times K \times K \to \mathbb{R}$,

$$E_2(u,v,w) := \frac{1}{2}|v|_{H_0^1}^2 - \int_0^T F_2(\cdot,u,v,w),\ E_3(u,v,w) := \frac{1}{2}|w|_{H_0^1}^2 - \int_0^T F_3(\cdot,u,v,w)$$

where

$$F_2(x,u(x),v(x),w(x)) := \int_0^{v(x)} f_2(x,u(x),s,w(x))ds$$
$$F_3(x,u(x),v(x),w(x)) := \int_0^{w(x)} f_2(x,u(x),v(x),s)ds.$$

Additionally, if $H_0^1(0,T)$ is identified with its dual $\left(H_0^1(0,T)\right)'$, we have

$$E_{22}(u,v,w) = v - Sf_2(u,v,w),\ E_{33}(u,v,w) = w - Sf_3(u,v,w).$$

Hence, the system (3.18) is equivalent with the following fixed point equation

$$\begin{cases} N_1(u,v,w) = u \\ N_2(u,v,w) = v \\ N_3(u,v,w) = w, \end{cases}$$

where

$$\begin{cases} N_1(u,v,w) = Sf_1(\cdot,u,v,w,u') \\ N_2(u,v,w) = Sf_2(\cdot,u,v,w) \\ N_3(u,v,w) = Sf_3(\cdot,u,v,w). \end{cases}$$

Let us denote

$$m := \min_{t\in[a,b]} \int_0^T G(t,s)ds = \min_{t\in[a,b]} \frac{t(T-t)}{2} = \min\left\{\frac{a(T-a)}{2}, \frac{b(T-b)}{2}\right\}.$$

Theorem 3.6. *Given the assumptions mentioned earlier, we additionally consider* *the following*

(H1) There exist $a_{ij}, a_{14} > 0$ $(i, j = 1, 2, 3)$ *such that for all real numbers* $x_1, \dots, x_4$ *and* $\overline{x}_1, \dots, \overline{x}_4$, *we have*

$$\begin{aligned}(x_1 - \overline{x}_1)\left(f_1(t, x_1, ..., x_4) - f_1(t, \overline{x}_1, ..., \overline{x}_4)\right) &\le |x_1 - \overline{x}_1| \sum_{j=1}^{4} a_{1j}|x_j - \overline{x}_j|, \\ (x_i - \overline{x}_i)\left(f_i(t, x_1, x_2, x_3) - f_i(t, \overline{x}_1, \overline{x}_2, \overline{x}_3)\right) &\le |x_i - \overline{x}_i| \sum_{j=1}^{3} a_{ij}|x_j - \overline{x}_j|,\end{aligned} \tag{3.19}$$

where $i \in \{2, 3\}$, *and moreover, the matrix*

$$A = \tfrac{T^2}{\pi^2}\begin{bmatrix} \left(a_{11} + \frac{\pi}{T}a_{41}\right) & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{3.20}$$

is convergent to zero.

(H2) The functions $f_i(t, x, y, z)$ $(i = 2, 3)$, *satisfy:*

(i) they are monotonically increasing with respect to the variables y *and* z.

(ii)

$$f_i(t, \cdot, r_2, r_3) \ge \frac{r_i}{m(b - a)} \tag{3.21}$$

and

$$|f_i(t, \cdot, 0, 0)|_{L^2} \le \frac{\pi}{T}R_2 - \frac{T}{\pi}\left(a_{i2}R_2 + a_{i3}R_3\right) \tag{3.22}$$

for all $t \in (0, T)$.

(iii) there are real numbers $M_1, M_2, M_3, M_4 > 0$ *such that*

$$\begin{aligned} f_2(t, \cdot, cR_2, cR_3) &\le M_1, \;\; f_2(t, \cdot, 0, r_3) \ge M_2, \\ f_3(t, \cdot, cR_2, cR_3) &\le M_3, \;\; f_3(t, \cdot, r_2, 0) \ge M_4, \end{aligned}$$

for every $t \in (0, T)$ *and*

$$TcR_2M_1 - \frac{R_2^2}{2} < r_2(b - a)M_2, \; TcR_3M_3 - \frac{R_3^2}{2} < r_3(b - a)M_4.$$

Then, there exists a solution $(u^*, v^*, w^*) \in H_0^1(0, T) \times (K_2)_{r_2,R_2} \times (K_3)_{r_3,R_3}$ *for the system* (3.18) *such that* (v^*, w^*) *is a Nash equilibrium for the energy functionals* E_2 *and* E_3.

Example 3.1. Let the system

$$\begin{cases} -u''(t) = \bar{a}_1 \left(e^{-u^2(t)} + e^{-(u'(t))^2} + e^{-v^2(t)} + e^{-w^2(t)} \right) \\ -v''(t) = \bar{a}_2 \left(e^{-u^2(t)} + \arctan(v(t) + 2w(t)) + \frac{\pi}{2} \right) \\ -w''(t) = \bar{a}_3 \left(e^{-u^2(t)} + \arctan(2v(t) + w(t)) + \frac{\pi}{2} \right), \end{cases} \quad \text{on } (0,3) \tag{3.23}$$

with Dirichlet boundary conditions

$$\begin{cases} u(0) = u(3) = 0 \\ v(0) = v(3) = 0 \\ w(0) = w(3) = 0 \end{cases},$$

where $\bar{a}_i \ (i = 1, 3)$ are positive real numbers.

We apply the results from Theorem 3.6 with,

$$\begin{aligned} f_1(t, x_1, x_2, x_3, x_4) &= \bar{a}_1 \left(e^{-x_1^2} + e^{-x_2^2} + e^{-x_3^2} + e^{-x_4^2} \right) \\ f_2(t, x_1, x_2, x_3) &= \bar{a}_2 \left(e^{-x_1^2} + \arctan(x_2 + x_3) + \frac{\pi}{2} \right) \\ f_3(t, x_1, x_2, x_3) &= \bar{a}_3 \left(e^{-x_2^2} + \arctan(x_2 + x_3) + \frac{\pi}{2} \right) \end{aligned}$$

Here, we set $c = \sqrt{3}$, $r = r_2 = r_3$ and $R_1 = R_2 = \infty$. The value of r is selected in such a way that for each $i = 2, 3$,

$$\bar{a}_i \left(\arctan 2r + \frac{\pi}{2} \right) \geq r. \tag{3.24}$$

The compact interval $[a, b]$ is chosen to be the interval [1,2]. Consequently

$$m = \min \left\{ \frac{1(3-1)}{2}, \frac{2(3-2)}{2} \right\} = 1.$$

If the matrix

$$A = \frac{9}{\pi^2} \begin{bmatrix} \bar{a}_1 \left(\frac{\pi}{3} + 1 \right) & \bar{a}_1 & \bar{a}_1 \\ \bar{a}_2 & \bar{a}_2 & \bar{a}_2 \\ \bar{a}_2 & \bar{a}_3 & \bar{a}_3 \end{bmatrix}$$

is convergent to zero, then the system (2.21) has a solution (u^*, v^*, w^*) such that (v^*, w^*) represents a Nash equilibrium on $(K)_{r,R} \times (K)_{r,R}$ for the energy functionals associated with the second and third equations.

Chapter 4

Equilibrium points for componentwise variational systems

In previous chapters, we explored Nash equilibria for energy functionals, where each minimizes one component while others are fixed. This chapter extends the concept to generalized Nash-type equilibria, combining mountain pass points with points of minimum or maximum.

4.1 The equilibrium problem

In this chapter, we explore critical points (u_1, u_2) for functionals E_1 and E_2, satisfying conditions $E_{11}(u_1, u_2) = 0$ and $E_{22}(u_1, u_2) = 0$. These points can be classified as follows:

(a) Nash equilibria, where u_1 minimizes E_1 and u_2 minimizes E_2.

(b) Min-mountain pass equilibria, with u_1 minimizing E_1 and u_2 as a mountain pass type point for E_2.

(c) Mountain pass-mountain pass equilibria, where u_1 is a mountain pass type point for E_1 and u_2 is a mountain pass type point for E_2.

A solution with one of the above three proprieties is called a generalized Nash equilibrium.

We aim to unify the treatment of these cases, employing the linking concept introduced by R. Precup, which generates both minimizers and mountain pass type critical points. This approach involves constructing an approximation sequence via linking alternately to one component while keeping the other fixed, with subsequent analysis on the convergence of this sequence to the desired critical point.

Let H_i $(i = 1, 2)$ be Hilbert spaces together with inner product $(\cdot, \cdot)_i$ and norm $|\cdot|_i$, identified with their duals. Denote $H = H_1 \times H_2$. For each space H_i, consider a linking giving by two closed sets $A_i, B_i \subset H_i$ and a compact set $Q_i \subset H_i$ with $A_i, Q_i \neq \emptyset$ and $B_i \subset Q_i$. Let

$$\Gamma_i := \{\gamma_i \in C(Q_i, H_i) : \ \gamma_i(u_i) = u_i \text{ for all } u_i \in B_i\}.$$

It is not difficult to see that these sets are complete metric spaces equipped with the metrics d_i,

$$d_i(\gamma_i, \overline{\gamma_i}) := \max_{q \in Q_i} |\gamma_i(q) - \overline{\gamma_i}(q)\ |_i,$$

for any $\gamma_i, \overline{\gamma_i} \in \Gamma_i$.

Let $E_i : H \to \mathbb{R}\,(i = 1, 2)$ be two Fréchet differentiable functionals. For each $(u_1, u_2) \in H$, we define:

$$\begin{array}{ll} m_1(u_2) := \inf_{X_1} E_1(\cdot, u_2)\,; & m_2(u_1) := \inf_{X_2} E_2(u_1, \cdot)\,; \\ a_1(u_2) := \inf_{A_1} E_1(\cdot, u_2)\,; & a_2(u_1) := \inf_{A_2} E_2(u_1, \cdot)\,; \\ b_1(u_2) := \sup_{B_1} E_1(\cdot, u_2)\,; & b_2(u_1) := \sup_{B_2} E_2(u_1, \cdot)\,; \end{array} \tag{4.1}$$

$$c_1(u_2) := \inf_{\mu \in \Gamma_1} \max_{q \in Q_1} E_1(\mu(q), u_2)\,; \tag{4.2}$$

$$c_2(u_1) := \inf_{\mu \in \Gamma_2} \max_{q \in Q_2} E_2(u_1, \mu(q)). \tag{4.3}$$

One easily sees that

$$m_i \leq a_i \leq c_i \quad \text{and} \quad b_i \leq c_i \ (i = 1, 2).$$

4.2 Existence of a minimizing sequence to a generalized Nash equilibrium

Under a particular linking, we will use the Ekeland variational principle to create an approximation sequence of nearly critical points, aiming to converge to a desired critical point falling into categories $(a), , (b), , (c)$, determined by the chosen linking.

Lemma 4.1. *Let $(X, |\cdot|_X)$ be a Banach space, K a compact subset of X and $f \in C(K, X^*)$ a continuous mapping from K to the dual of X. Then, for each $\varepsilon > 0$, we may find a function $\varphi \in C(K, X)$ such that:*

$$|\varphi(x)|_X \leq 1, \ \textit{and} \ \langle f(x), \ \varphi(x) \rangle > |f(x)|_X - \varepsilon,$$

for all $x \in K$.

Theorem 4.2. *Let A_i links B_i via Q_i in H_i, and assume that*

$$b_i < a_i, \quad i = 1, 2.$$

Then, there exist two sequences $\left(u_1^k\right) \in H_1$ and $\left(u_2^k\right) \in H_2$ such that

$$0 \leq E_1(u_1^k, u_2^{k-1}) - c_1\left(u_2^{k-1}\right) \to 0, \qquad 0 \leq E_2(u_1^k, u_2^k) - c_2\left(u_1^k\right) \to 0 \tag{4.4}$$

and

$$E_{11}\left(u_1^k, u_2^{k-1}\right) \to 0, \qquad E_{22}\left(u_1^k, u_2^k\right) \to 0, \tag{4.5}$$

as $k \to \infty$.

4.3 Exploring the limiting case.

In the preceding section, we examined the prerequisites for forming an approximation sequence. Yet, the possibility of non-convergence in this sequence necessitates careful consideration. In the subsequent section, we delve into the characteristics of limit points, assuming their existence.

Theorem 4.3. *Let $\left(u_1^k\right), \left(u_2^k\right)$ be the sequences obtained in Theorem 4.2. Assume that they are convergent, i.e., there exists u^*, v^* such that $u_1^k \to u^*$ and $u_2^k \to v^*$. Then*

$$E_{11}(u^*, v^*) = 0\,, \quad E_{22}(u^*, v^*) = 0, \tag{4.6}$$

$$c_1\left(u_2^k\right) \to c_1\left(v^*\right), \quad c_2\left(u_1^k\right) \to c_2\left(u^*\right) \tag{4.7}$$

and

$$E_1(u^*, v^*) = c_1\left(v^*\right)\,, \quad E_2(u^*, v^*) = c_2\left(u^*\right). \tag{4.8}$$

Remark 4.4. *In the light of the conclusions of Theorem 4.3, we can distinguish between the following scenarios:*

(a) If both linkings of the spaces H_1 and H_2 are trivial, then u^* is a minimizer of the functional $E_2(\cdot, v^*)$ and v^* is a minimizer of the functional $E_2(u^*, \cdot)$. In other words, the pair (u^*, v^*) represents a Nash equilibrium for the functionals E_1 and E_2.

(b) If only the linking of the space H_2 is trivial, then u^* is a mountain pass type point for $E_1(\cdot, v^*)$, while v^* serves as a minimizer for the functional $E_2(u^*, \cdot)$.

(b) If both linkings of the spaces H_1 and H_2 are nontrivial, then u^* is a mountain pass type point for the functional $E_2(\cdot, v^*)$, and likewise, v^* is a mountain pass type point for the functional $E_2(u^*, \cdot)$.

Remark 4.5. *Our theory applies in particular to a single functional E defined on a product space $H_1 \times H_2$, when we can take either*

(1^0) $E_1 = E_2 = E$; *or*

(2^0) $E_1 = E$ *and* $E_2 = -E$.

If the two sequences (u_1^k) *and* (u_2^k) *converge to* u_1^* *and* u_2^**, respectively, one can obtain critical points* (u_1^*, u_2^*) *of* E *with one of the following properties*

$$\begin{aligned}
E(u_1^*, u_2^*) &= \min E(\cdot, u_2^*) = \max E(u_1^*, \cdot);\\
E(u_1^*, u_2^*) &= \min E(\cdot, u_2^*) = \sup_{\mu \in \Gamma_2} \min_{q \in Q_2} E(u_1^*, \mu(q));\\
E(u_1^*, u_2^*) &= \inf_{\mu \in \Gamma_1} \max_{q \in Q_1} E(\mu(q), u_2^*) = \max E(u_1^*, \cdot);\\
E(u_1^*, u_2^*) &= \inf_{\mu \in \Gamma_1} \max_{q \in Q_1} E(\mu(q), u_2^*) = \sup_{\mu \in \Gamma_2} \min_{q \in Q_2} E(u_1^*, \mu(q)).
\end{aligned}$$

4.4 Conditions for convergence.

In previous sections, we discussed the properties of the limits of the generated sequences. Now, we address the challenge of ensuring these limits exist by presenting conditions for their existence. To achieve this, we impose monotonicity conditions on derivatives E_{11} and E_{22}.

Theorem 4.6. *Let* $L = (L_1, L_2) : H \to H,\ L_i : H \to H_i\ (i = 1, 2)$ *be a continuous operator and let* $N = (N_1, N_2) : H \to H,\ N_i : H \to H_i\ (i = 1, 2)$, *be defined by*

$$N(u) = u - L(E_{11}(u), E_{22}(u)). \tag{4.9}$$

Suppose the following conditions hold

(i) *There are nonnegative constants* $a_{ij}\ (i, j = 1, 2)$ *such that*

$$\begin{aligned}
&(N_1(u_1, u_2) - N_1(\overline{u}_1, \overline{u}_2),\ u_1 - \overline{u}_1)_1 \qquad (4.10)\\
&\le a_{11} |u_1 - \overline{u}_1|_1^2 + a_{12} |u_1 - \overline{u}_1|_1\ |u_2 - \overline{u}_2|_2,\\
&(N_2(u_1, u_2) - N_2(\overline{u}_1, \overline{u}_2),\ u_2 - \overline{u}_2)_2 \qquad (4.11)\\
&\le a_{22} |u_2 - \overline{u}_2|_2^2 + a_{21} |u_1 - \overline{u}_1|_1\ |u_2 - \overline{u}_2|_2,
\end{aligned}$$

for all $u_1, \overline{u}_1 \in H_1$ *and* $u_2, \overline{u}_2 \in H_2$;

(ii) *The matrix* $A = [a_{ij}]_{1 \le i,j \le 2}$ *is convergent to zero;*

(iii) *The sequence* $\left(u_2^k\right)$ *(equivalently* $\left(u_1^k\right)$*) is bounded.*

Then, the sequences $\left(u_1^k\right)$ *and* $\left(u_2^k\right)$ *are convergent.*

Remark 4.7. *The utilization of a continuous operator L enables us to attain a continuous transformation of the derivatives, to which we can then apply the necessary monotonicity conditions. Without this transformation, meeting the required monotonicity conditions appears challenging due to the nature of the mountain pass geometry. It is worth noting that in our previous works focused on Nash-type equilibria, the need for a specialized operator like L was avoided, and in those cases, the identity operator sufficed.*

Condition (iii) (boundedness of one sequence) is not assumed beforehand and needs to be ensured. Here, we outline sufficient conditions based on the chosen linking to guarantee this condition.

Theorem 4.8. *The sequence $\left(u_2^k\right)$ remains bounded in each one of the following scenarios:*

(a) *The linking in H_2 is trivial. There exists $w \in H_2$ such that*

$$E_2\left(\cdot, w\right) \quad \text{is bounded on } H_1; \tag{4.12}$$

and

$$E_2\left(u, \cdot\right) \;\; \text{is coercive uniformly with respect to } u. \tag{4.13}$$

(b) *The linking in H_2 is nontrivial. There exists $w \in B_2$, such that*

$$-E_2\left(\cdot, w\right) \quad \text{is bounded on } H_1; \tag{4.14}$$

and

$$-E_2\left(u, \cdot\right) \;\; \text{is coercive uniformly with respect to } u. \tag{4.15}$$

Remark 4.9. *It is worth to note that in practical applications, additional specific conditions, such as growth and coercivity conditions or the Ambrosetti-Rabinowitz condition, can be employed to ensure the boundedness of $\left(u_2^k\right)$.*

4.5 Application

We consider the following Dirichlet problem

$$\begin{cases} -\Delta v_1 = \nabla_{v_1} F(v_1, w_1, v_2, w_2) \\ -\Delta w_1 = \nabla_{w_1} F(v_1, w_1, v_2, w_2) \\ -\Delta v_2 = \nabla_{v_2} G(v_1, w_1, v_2, w_2) \\ -\Delta w_2 = \nabla_{w_2} G(v_1, w_1, v_2, w_2) \quad \text{on } \Omega \\ v_1|_{\partial\Omega} = w_1|_{\partial\Omega} = v_2|_{\partial\Omega} = w_2|_{\partial\Omega} = 0. \end{cases} \tag{4.16}$$

Here, Ω represents a bounded open set in $\mathbb{R}^n$ $(n \geq 3)$. We emphasize that such problems are well-known in the literature and are commonly employed to model real-world processes, including stationary diffusion or wave propagation.

Assume the following conditions on the potentials F and G:

(H1) The functions $F, G \colon \mathbb{R}^4 \to \mathbb{R}$ are of C^1 class and

$$F(0, x_2) = 0 \quad \text{and} \quad G(x_1, 0) = 0,$$

for all $x_1, x_2 \in \mathbb{R}^2$. In addition, for some $2 \leq p \leq 2^* = \frac{2n}{n-2}$, they satisfy the growth conditions

$$\begin{aligned} |F(x_1, x_2)| &\leq C_F \left(|x_1|^p + 1\right), \\ |G(x_1, x_2)| &\leq C_G \left(|x_2|^p + 1\right), \end{aligned} \tag{4.17}$$

for all $x_1, x_2 \in \mathbb{R}^2$ and some positive constants C_F, C_G.

Here, $H_1 = H_2 := \left(H_0^1(\Omega)\right)^2 = H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with the inner product

$$(u, \overline{u})_{H_0^1 \times H_0^1} = (v, \overline{v})_{H_0^1} + (w, \overline{w})_{H_0^1},$$

and the norm

$$|u|_{H_0^1 \times H_0^1} = \left(|v|^2_{H_0^1} + |w|^2_{H_0^1}\right)^{1/2},$$

for $u = (v, w), \overline{u} = (\overline{v}, \overline{w})$.

The distinctive characteristic of the system (4.16) is that the first two equations and the last two equations coupled together, permit a variational formulation that can be expressed through the energy functionals $E_1, E_2 \colon \left(H_0^1(\Omega)\right)^2 \times \left(H_0^1(\Omega)\right)^2 \to \mathbb{R}$,

$$\begin{aligned} E_1(u_1, u_2) &= \tfrac{1}{2}|u_1|^2_{H_0^1 \times H_0^1} - \int_\Omega F(u_1, u_2), \\ E_2(u_1, u_2) &= \tfrac{1}{2}|u_2|^2_{H_0^1 \times H_0^1} - \int_\Omega G(u_1, u_2), \end{aligned}$$

where $u_1 = (v_1, w_1)\,,\ u_2 = (v_2, w_2) \in \left(H_0^1(\Omega)\right)^2$.

Let us denote

$$\begin{aligned} f_1(y_1, z_1, y_2, z_2) &= \nabla_{y_1} F(y_1, z_1, y_2, z_2), \\ f_2(y_1, z_1, y_2, z_2) &= \nabla_{z_1} F(y_1, z_1, y_2, z_2), \\ g_1(y_1, z_1, y_2, z_2) &= \nabla_{y_2} G(y_1, z_1, y_2, z_2), \\ g_2(y_1, z_1, y_2, z_2) &= \nabla_{z_2} G(y_1, z_1, y_2, z_2). \end{aligned}$$

If we identify $H_0^1(\Omega)$ with its dual $H^{-1}(\Omega)$ via $-\Delta$, then the partial derivatives of

E_1 and E_2 with respect to the first and second component, respectively, are given by

$$\begin{aligned} E_{11}(u_1,u_2) &= u_1 - \left((-\Delta)^{-1}f_1(u_1,u_2)\,,\,(-\Delta)^{-1}f_2(u_1,u_2)\right), \\ E_{22}(u_1,u_2) &= u_2 - \left((-\Delta)^{-1}g_1(u_1,u_2)\,,\,(-\Delta)^{-1}g_2(u_1,u_2)\right). \end{aligned}$$

Note that under the growth conditions (4.17), the Nemytskii's operators

$$\mathcal{N}_{f_i}(u_1,u_2)(x) := f_i(u_1(x),u_2(x)), \quad \mathcal{N}_{g_i}(u_1,u_2)(x) := g_i(u_1(x),u_2(x)),$$

(i=1,2), are well defined from $\left(L^{2^*}(\Omega)\right)^4$ to $\left(L^{(2^*)'}(\Omega)\right)^2$, bounded (map bounded sets into bounded sets) and continuous. Hence, the operators

$$\begin{aligned} N_1(u_1,u_2) &= \left((-\Delta)^{-1}f_1(u_1,u_2)),(-\Delta)^{-1}f_2(u_1,u_2))\right) \\ N_2(u_1,u_2) &= \left((-\Delta)^{-1}g_1(u_1,u_2)),(-\Delta)^{-1}g_2(u_1,u_2))\right) \end{aligned}$$

are well-defined and continuous from $\left(H_0^1(\Omega)\right)^4$ to $\left(H_0^1(\Omega)\right)^2$.

(H2) The inequalities (see, e.g., [2, 17–19, 33]).

$$\limsup_{|x_1|\to 0} \frac{F(x_1,x_2)}{|x_1|^2} < \frac{\lambda_1}{2} < \liminf_{|y_1|\to\infty} \frac{F\left((y_1,0),x_2\right)}{{y_1}^2},$$

hold for all $y_1 \in \mathbb{R}$ and uniformly with respect to $x_2 \in \mathbb{R}^2$.

From (4.17) and (H2), we find an r_0' such that

$$E_1(u_1,u_2) \geq c > 0 \quad \text{for all} \quad |u_1|_{H_0^1\times H_0^1} = r_0'. \tag{4.18}$$

Also, there exists $\alpha_0 > r_0'$ such that

$$E_1\left((\alpha_0\phi_1,0),u_2\right) < 0 \quad \text{for all} \quad u_2 \in \left(H_0^1(\Omega)\right)^2. \tag{4.19}$$

Moreover, one clearly has

$$E_1((0,0),u_2) = 0. \tag{4.20}$$

On $\left(H_0^1(\Omega)\right)^2$, we consider the sets

$$\begin{aligned} A_1 &= \left\{u_1 \in \left(H_0^1(\Omega)\right)^2 : \, |u_1|_{H_0^1\times H_0^1} = r_0'\right\}, \\ Q_1 &= \left\{s\,(\phi_1,0) \in \left(H_0^1(\Omega)\right)^2 : \, 0 \leq s \leq \alpha_0\right\}, \\ B_1 &= \left\{((0,0)\,,\ (s_0\phi_1,0))\right\}. \end{aligned}$$

From (4.18), (4.19) and (4.20), we see that A_1 links B_1 via Q_1, and in addition

$$\inf_{A_1} E_1(\cdot, u_2) \geq c > \sup_{B_1} E_1(\cdot, u_2),$$

for all $u_2 \in (H_0^1(\Omega))^2$, i.e., $b_1 < a_1$.

Let us consider

$$A_2 = \left(H_0^1(\Omega)\right)^2, \quad B_2 = \emptyset \quad \text{and} \quad Q_2 = \{(0,0)\}.$$

This corresponds to trivial linking. To ensure $b_2 < a_2$ (or equivalently, $-\infty < m_2$), the functional $E_2(\cdot, u_2)$ must be uniformly bounded from below with respect to u_1. This requirement can be achieved if we assume the following unilateral growth condition on G:

(H3) There exists $0 \leq \sigma < \lambda_1$ with

$$G\left(x_1, x_2\right) \leq \frac{\sigma}{2}\left|x_2\right|^2 + C, \text{ for all } x_1, x_2 \in \mathbb{R}^2. \tag{4.21}$$

Based on Theorem 4.2, it can be inferred that there are two sequences, $\left(u_1^k\right)$ and $\left(u_2^k\right)$, that satisfy (4.4) and (4.5). Referring to Theorem 4.6, let us consider the linear operator $L = \left(L_1, L_2\right)$, where $L_1, L_2\colon (H_0^1(\Omega))^2 \to (H_0^1(\Omega))^2$ are given by

$$L_1(v_1, w_1) = L_1\left(u_1\right) = \beta(v_1 - w_1, v_1 - w_1), \qquad L_2(v_2, w_2) = L_2\left(u_2\right) = u_2, \tag{4.22}$$

for $u_1 = \left(v_1, w_1\right),\ u_2 = (v_2, w_2) \in (H_0^1(\Omega))^2$ and some $\beta > 0$. Consequently, we can express the operators N_1 and N_2 in terms of L as follows

$$\begin{aligned} N_1(u_1, u_2) = &((1-\beta)v_1 + \beta w_1, (1-\beta)w_1 - \beta v_1) \\ &+ \beta\left((-\Delta)^{-1}\left(f_1(u_1,u_2) - f_2(u_1,u_2)\right),\ (-\Delta)^{-1}\left(f_1(u_1,u_2) - f_2(u_1,u_2)\right)\right). \end{aligned}$$

$$N_2(u_1, u_2) = u_2 - L_2\ \left(E_{22}(u_1, u_2)\right) = \left((-\Delta)^{-1} g_1(u_1, u_2),\ (-\Delta)^{-1} g_2(u_1, u_2)\right)$$

Next, we will discuss some conditions related to the monotonicity of the functions $\widetilde{f} := f_1 - f_2$, g_1 and g_2 that appear in the expressions for N_1 and N_2.

(H4) There are nonnegative numbers m_{ij} $(i, j = 1, 4)$ such that

$$\begin{aligned}
& \left(\widetilde{f}(x_1, x_2) - \widetilde{f}(\overline{x}_1, \overline{x}_2)\right)(y_1 - \overline{y}_1) \qquad (4.23)\\
\leq \; & |y_1 - \overline{y}_1| \left(m_{11}|y_1 - \overline{y}_1| + m_{12}|z_1 - \overline{z}_1| + m_{13}|y_2 - \overline{y}_2| + m_{14}|z_2 - \overline{z}_2|\right), \\
& \left(\widetilde{f}(x_1, x_2) - \widetilde{f}(\overline{x}_1, \overline{x}_2)\right)(z_1 - \overline{z}_1) \\
\leq \; & |z_1 - \overline{z}_1| \left(m_{21}|y_1 - \overline{y}_1| + m_{22}|z_1 - \overline{z}_1| + m_{23}|y_2 - \overline{y}_2| + m_{24}|z_2 - \overline{z}_2|\right), \\
& \left(g_1(x_1, x_2) - g_1(\overline{x}_1, \overline{x}_2)\right)(y_2 - \overline{y}_2) \\
\leq \; & |y_2 - \overline{y}_2|(m_{31}|y_1 - \overline{y}_1| + m_{32}|z_1 - \overline{z}_1| + m_{33}|y_2 - \overline{y}_2| + m_{34}|z_2 - \overline{z}_2|), \\
& \left(g_2(x_1, x_2) - g_2(\overline{x}_1, \overline{x}_2)\right)(z_2 - \overline{z}_2) \\
\leq \; & |z_2 - \overline{z}_2| \left(m_{41}|y_1 - \overline{y}_1| + m_{42}|z_1 - \overline{z}_1| + m_{43}|y_2 - \overline{y}_2| + m_{44}|z_2 - \overline{z}_2|\right),
\end{aligned}$$

for all $x_1 = (y_1, z_1),\ \overline{x}_1 = (\overline{y}_1, \overline{z}_1),\ x_2 = (y_2, z_2),\ \overline{x}_2 = (\overline{y}_2, \overline{z}_2) \in \mathbb{R}^2$.

Assuming hypothesis (H4), the operators N_1, N_2 fulfill the conditions of monotonicity (2.16) and (2.17), with the coefficients

$$a_{11} = 1 - \beta + \tfrac{\beta}{\lambda_1} \max\{m_{11},\ m_{22}\} + \tfrac{\beta}{2\lambda_1}\left(m_{12} + m_{21}\right), \qquad (4.24)$$

$$a_{12} = \tfrac{\beta}{\lambda_1} \max\left\{\sqrt{m_{13}^2 + m_{23}^2},\ \sqrt{m_{14}^2 + m_{24}^2}\right\}, \qquad (4.25)$$

$$a_{21} = \frac{1}{\lambda_1} \max\left\{\sqrt{m_{31}^2 + m_{32}^2},\ \sqrt{m_{41}^2 + m_{42}^2}\right\}, \qquad (4.26)$$

$$a_{22} = \frac{m_{34} + m_{43}}{2\lambda_1} + \max\{m_{33},\ m_{44}\}. \qquad (4.27)$$

Now it is clear that the first two conditions from Theorem 4.6 are are fulfilled if

(H5) The matrix $M := [a_{ij}]_{1 \leq i,j \leq 2}$ is convergent to zero.

It remains to show that the sequence $\left(u_2^k\right)$ is bounded. To do this, we apply Theorem 4.8 (a). From $G(\cdot, 0) = 0$ and the growth condition (4.21), we obtain

$$E_2(u_1, u_2) \geq \left(\frac{1}{2} - \frac{\sigma}{2\lambda_1}\right) |u_2|^2_{H_0^1 \times H_0^1} - C \operatorname{meas}(\Omega) \to \infty,$$

as $|u_2|_{H_0^1 \times H_0^1} \to \infty$, uniformly with respect to u_1. Therefore, since all conditions from Theorem 4.6 are fulfilled, we infer that the sequences $\left(u_1^k\right)$ and $\left(u_2^k\right)$ are convergent in $\left(H_0^1(\Omega)\right)^2$.

Therefore, relying on Theorem 4.2, we can formulate the following result.

Theorem 4.10. *Under the assumptions (H1)-(H5), we conclude that the problem (2.21) has a mountain pass-min solution. That is, there exists a solution* $(u_1^*, u_2^*) \in \left(H_0^1(\Omega)\right)^2 \times \left(H_0^1(\Omega)\right)^2$ *such that* u_1^* *is a mountain pass type critical point of the functional* $E_1(\cdot, u_2^*)$ *and* u_2^* *is a minimizer of the functional* $E_2(u_1^*, \cdot)$.

To achieve a mountain pass solution, we follow a similar approach to Theorem 1, with key clarifications. Firstly, both functions F and G must satisfy conditions (H2)' for nontrivial linkings. By imposing (H3)' with $-G$ instead of G, we ensure boundedness of u_2^k (cf. Theorem 3(b)).

Secondly, we use an alternative operator L_2 instead of the identity operator (for simplicity we choose $L_2 = L_1$). Condition (H4) requires a monotonicity condition for $\tilde{g} = g_1 - g_2$ instead of g_1 and g_2 (denoted as (H4)'). Changing L_2 necessitates revising coefficients a_{21} and a_{22} (cf. equations (5) and (6)), where a_{21} corresponds to a_{12} and a_{22} corresponds to a_{11}.

Theorem 4.11. *Let the conditions (H1), (H2)'-(H4)', (H5) be fulfilled. Then the problem* (2.21) *has a mountain pass-mountain pass solution, i.e., there is a solution* $(u_1^*, u_2^*) \in (H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2$ *such that* u_1^* *is a mountain pass critical point of the functional* $E_1(\cdot, u_2^*)$ *and* u_2^* *mountain pass critical point of the functional* $E_2(u_1^*, \cdot)$.

Example 1. Consider the Dirichlet problem

$$\begin{cases} -\Delta v_1 = a(v_1 + w_1)^3 + \tilde{a} v_1 + a(v_1 + w_1)\frac{1}{v_2^2 + w_2^2 + 1} \\ -\Delta w_1 = a(v_1 + w_1)^3 - \tilde{a} w_1 + a(v_1 + w_1)\frac{1}{v_2^2 + w_2^2 + 1} \\ -\Delta v_2 = b v_2 + \frac{1}{v_1^2 + c^2} \\ -\Delta w_2 = b w_2 + \frac{1}{v_2^2 + c^2} \text{ on } \Omega \\ v_1|_\Omega = w_1|_\Omega = v_2|_\Omega = w_2|_\Omega = 0. \end{cases} \tag{4.28}$$

We apply Theorem 4.10, where

$$\begin{aligned} &\Omega \subset \mathbb{R}^3, \quad a \le \frac{\lambda_1}{4}, \quad \tilde{a} < \frac{\lambda_1}{2}, \quad b < 1, \quad b + \tfrac{4}{c} < \lambda_1, \quad c > 1, \\ &F(y_1, z_1, y_2, z_2) = \tfrac{a}{4}(y_1 + z_1)^4 + \tfrac{\tilde{a}}{2}\left(y_1^2 - z_1^2\right) + \tfrac{a}{2}(y_1 + z_1)^2 \frac{1}{y_2^2 + z_2^2 + 1}, \\ &G(y_1, z_1, y_2, z_2) = \tfrac{b}{2}\left(y_2^2 + z_2^2\right) + \frac{y_2}{y_1^2 + c^2} + \frac{z_2}{z_1^2 + c^2}. \end{aligned}$$

The absolute value of $F(x_1, x_2)$ $(x_1, x_2 \in \mathbb{R}^2)$ is bounded from above by a fourth-degree polynomial in $|x_1|$ and

$$|G(y_1, z_1, y_2, z_2)| \le \left(\tfrac{b}{2} + \tfrac{2}{c}\right) |(y_2, z_2)|^2 + \tfrac{2}{c}.$$

Hence, the condition (H1) is guaranteed. In addition, condition (H3) also is satisfied since $\frac{b}{2} + \frac{2}{c} < \frac{\lambda_1}{2}$. Simple computations yields

$$\lim_{|y_1| + |z_1| \to 0} \frac{F(y_1, z_1, y_2, z_2)}{{y_1}^2 + {z_1}^2} \le \tfrac{\tilde{a}}{2} + a < \frac{\lambda_1}{2} \text{ and } \lim_{|y_1| \to \infty} \frac{F((y_1, 0), x_2)}{{y_1}^2} \ge \lim_{|y_1| \to \infty} \tfrac{a}{4} y_1^2 = \infty,$$

which guarantees that (H2) is satisfied. We see that

$$\begin{aligned} f_1(y_1,z_1,y_2,z_2) &= a(y_1+z_1)^3 + \tilde{a}y_1 + a(y_1+z_1)\frac{1}{y_2^2+z_2^2+1}, \\ f_2(y_1,z_1,y_2,z_2) &= a(y_1+z_1)^3 - \tilde{a}z_1 + a(y_1+z_1)\frac{1}{y_2^2+z_2^2+1}, \\ g_1(y_1,z_1,y_2,z_2) &= by_2 + \frac{1}{y_1^2+c^2}, \\ g_2(y_1,z_1,y_2,z_2) &= bz_2 + \frac{1}{z_1^2+c^2}, \end{aligned}$$

which yields

$$\widetilde{f}(y_1,z_1,y_2,z_2) = \tilde{a}y_1 + \tilde{a}z_1.$$

The linearity of $\widetilde{f}$ and the Lipschitz property $\left|\frac{1}{x^2+c^2} - \frac{1}{\overline{x}^2+c^2}\right| \le \frac{1}{c}|x-\overline{x}|$ yields

$$\begin{aligned} &\left(\widetilde{f}(y_1,z_1,y_2,z_2) - \widetilde{f}(\overline{y}_1,\overline{z}_1,\overline{y}_2,\overline{z}_2)\right)(y_1-\overline{y}_1) \le \tilde{a}|y_1-\overline{y}_1|^2 + \tilde{a}\,|y_1-\overline{y}_1|\,|z_1-\overline{z}_1|, \\ &\left(\widetilde{f}(y_1,z_1,y_2,z_2) - \widetilde{f}(\overline{y}_1,\overline{z}_1,\overline{y}_2,\overline{z}_2)\right)(z_1-\overline{z}_1) \le \tilde{a}|z_1-\overline{z}_1|^2 + \tilde{a}\,|y_1-\overline{y}_1|\,|z_1-\overline{z}_1|, \\ &(g_1(y_1,z_1,y_2,z_2) - g_1(\overline{y}_1,\overline{z}_1,\overline{y}_2,\overline{z}_2))(y_2-\overline{y}_2) \le b|y_2-\overline{y}_2|^2 + \tfrac{1}{c}|y_2-\overline{y}_2||y_1-\overline{y}_1|, \\ &(g_2(y_1,z_1,y_2,z_2) - g(\overline{y}_1,\overline{z}_1,\overline{y}_2,\overline{z}_2))(z_2-\overline{z}_2) \le b|z_2-\overline{z}_2|^2 + \tfrac{1}{c}|z_1-\overline{z}_1||z_2-\overline{z}_2|. \end{aligned}$$

Thus, the monotony conditions (4.23) hold with

$$\begin{aligned} m_{11} &= \tilde{a}, \quad m_{12} = \tilde{a}, \quad m_{13} = 0, \quad m_{14} = 0, \\ m_{21} &= \tilde{a}, \quad m_{22} = \tilde{a}, \quad m_{23} = 0, \quad m_{24} = 0, \\ m_{31} &= \frac{1}{c}, \quad m_{32} = 0, \quad m_{33} = b, \quad m_{34} = 0, \\ m_{41} &= 0, \quad m_{42} = \frac{1}{c}, \quad m_{43} = 0, \quad m_{44} = b. \end{aligned}$$

After straightforward calculations, we obtain

$$M = \begin{bmatrix} 1-\beta\left(1-2\frac{\tilde{a}}{\lambda_1}\right) & 0 \\ \frac{1}{c\lambda_1} & b \end{bmatrix}.$$

Given that $b<1$ and $1-2\frac{\tilde{a}}{\lambda_1}>0$, we can select $\beta>0$ in (4.22) small enough that the matrix M converges to zero.

Therefore, as all the hypothesis of Theorem 2.10 are fulfilled, the problem (4.28) has a solution $(v_1^*, w_1^*, v_2^*, w_2^*)$. Moreover, $u_1^* := (v_1^*, w_1^*)$ and $u_2^* := (v_2^*, w_2^*)$ are such that u_1^* is a mountain pass critical point for the energy functional $E_1(\cdot, u_2^*)$, and u_2^* is a minimizer for the energy functional $E_2(u_1^*, \cdot)$.

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