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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

PhD THESIS SUMMARY

Contributions to the Study of Convex-Type Functions and Fractional Optimization

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Introduction

This thesis commenced under the supervision of the late Professor Nicolae Popovici

Optimization is a branch of mathematics that involves finding the best solution within a set of possible solutions, typically by maximizing or minimizing an objective function while taking constraints into account. It is a highly important and influential field within mathematics, and it has significant relevance not only for mathematicians but also for various other fields and industries, due to its ability to find optimal solutions in complex decision-making scenarios.

Among the great list of fields that rely upon Optimization, we mention the following (see, e.g., Frenk and Schaible [26], Schaible [72], Shen and Yu [75, 76], Hillier and Lieberman [41], Khisty and Lall [46] and the references therein).

Operations Research: Optimization is at the core of operations research, helping businesses make decisions related to resource allocation, production planning, inventory management, and logistics;

Engineering: Engineers use optimization techniques to design efficient systems, structures, and processes, such as optimizing the shape of aircraft wings or designing energy-efficient buildings;

Economics and Finance: In economics, optimization models are used to study and predict economic behavior. In finance, it's used for portfolio optimization and risk management;

Transportation and Logistics: Optimization is essential for route planning, vehicle scheduling, and supply chain management, leading to cost reductions and improved efficiency;

Energy and Environment: Optimization helps optimize energy consumption, reduce emissions, and design sustainable systems in areas like renewable energy integration and environmental modeling.

There are various types of optimization problems, including linear programming, convex programming, nonlinear programming, integer programming, and dynamic programming, among others. Like many other mathematical domains, optimization draws upon and incorporates concepts and results from various related fields, including mathematical analysis, functional analysis, convex analysis, game theory etc and is, nontheless, open to algorithms and numerical modeling.

Following the trend observed in many other mathematical fields, optimization has evolved to encompass linearity, a development exemplified by the emergence of linear programming as a significant component within the field.

The next step in this field, crucial for addressing practical problems effectively, involved tackling nonlinearity and, by extension, nonlinear functions. This still presents a crucial challenge, which would not have been possible to address without the intermediate step of incorporating convex analysis. Convexity plays a fundamental and critical role in optimization because it simplifies the problem-solving process, guarantees global optimality, ensures robustness in real-world applications, and enables efficient algorithms. These advantages make it a cornerstone of modern optimization theory and practice.

A popular branch within the domain of Optimization is Fractional Optimization. This specialized subfield deals with optimizing functions involving fractions or rational expressions and is commonly used in economics (fractional programming), engineering, healthcare (resource allocation), network design (optimal routes for data transmission), optimal control systems (for processes that involve fractional-order dynamics, such as electrical circuits, chemical reactors, or mechanical systems), supply chain management, mathematics (Fractional Differential Equations), environmental studies (optimizing land use, allocating conservation funds, or designing wildlife corridors) among other areas, where such problems typically arise. In these domains, fractional optimization techniques are actively researched and applied.

The typical fractional optimization problem (usually called fractional programming problem or fractional program) is defined as follows:

$$\frac{A(x)}{B(x)} \longrightarrow \min_{x \in D},$$

where A and B are two functions defined on a nonempty set D and $B(x) \neq 0$ for all $x \in D$. Solving the problem involves obtaining a pair (λ^*, x^*), where λ^* represents the minimum value of A/B, and x^* corresponds to a minimum point., i.e.,

$$\lambda^* = \frac{A(x^*)}{B(x^*)} = \min_D \frac{A}{B}.$$

From a history perspective, as referenced in Frenk and Schaible [26], it is noteworthy that one of the earliest instances of fractional programming, although not explicitly termed as such, emerged in 1937 with John von Neumann's equilibrium model for a growing economy. The model calculates the economic growth rate by selecting the highest value among the smallest output-input ratios from multiple sources. Nonetheless, with the exception of a handful of individual papers, such as those by von Neumann, a comprehensive study of fractional programming started significantly later. The influence of several authors who significantly contributed to the field during that period is also highlighted in Frenk and Schaible [26].

The first monograph on fractional programming, titled *Analyse und Anwendungen von Quotientenprogrammen, Ein Beitrag zur Planung mit Hilfe der nichtlinearen Programmierung*, was written by Schaible in 1978. As mentioned in Frenk and Schaible [26], two additional monographs focused exclusively on fractional programming followed. One was authored by Craven in 1988, and the other, by Stancu-Minasian in 1997.

Since then, many researchers have invested substantial effort in this topic. Notable contributions include the papers of Avriel et al. [2], Cambini and Martein [14], Stancu-Minasian [79], Elbenani and Ferland [21], Crouzeix [15, 16], Hadjisavvas [26], Schaible [71, 72], Rodenas [69], Shi [77], Boţ et al. [10, 11], and Tammer [81], among others, as referenced therein.

The methods and techniques developed within the field of fractional optimization have generated significant interest regarding the properties of the functions f and g, such as convexity, concavity, boundedness, linearity etc. Consequently, the first two chapters of this thesis explore this particular aspect.

To begin with, we introduce a new concept of semistrict quasiconvexity for vector-valued functions defined on a nonempty convex set in a real linear space X and taking values in some real topological linear space Y, partially ordered by a proper solid convex cone C. Subsequently, we provide a characterization of these functions using the nonlinear scalarization function introduced by Gerstewitz (Tammer) in 1983. In addition, we give some characterizations of two special classes of fractional-type set-valued functions in terms of convexity-preserving properties of sets by direct and inverse images.

An important area of research within the field of fractional optimization focuses on the development of algorithms for solving fractional programs. Among many algorithms addressing fractional problems, old and new (see, for instance, Boţ and Csetnek [10], Boţ, Dao and Li [11], Boyd et al. [12], Geissler et al. [28], Goldstein et al. [31], Kleinert and Schmidt [47] and the references therein), the Dinkelbach Algorithm (Dinkelbach [20]) stands as one of the most renowned. As mentioned in Tammer and Ohlendorf [81, Sec. 3], it is based on the algorithm of Jagannathan from 1966 for linear fractional problems. This procedure is a technique used to solve fractional programming problems by transforming the initial fractional problem into a non-fractional parametric problem and, the second part of the thesis is dedicated to this subject.

In this second part, we initially introduced two approximated variations of the original Dinkelbach algorithm (with a given error $\varepsilon > 0$ and with errors decreasing to zero), and delivered a new version of the algorithm, the Dinkelbach-Ekeland algorithm, which incorporates Ekeland's variational principle.

In the final chapter of the thesis, we presented a modified version of Dinkelbach's algorithm, which we have named the Componentwise Dinkelbach algorithm. Unlike the original version, this variant is designed to address fractional objective functions that depend on two variables. A noteworthy accomplishment of our work lies in establishing the convergence of this algorithm. However, it's essential to note that its success relies on additional prerequisites concerning the spaces and functions involved, encompassing conditions like Lipschitz-type continuity, partial Fréchet differentiability, and coercivity.

The thesis is divided into four chapters, with each chapter containing multiple sections and subsections for structured organization.

Chapter 1: Generalized convexity for vector functions

In the first chapter, we introduce the new concept of semistrict quasiconvexity for vector functions defined on a convex set in a real linear space X and mapped to a real topological linear space Y ordered by a convex cone C. Section 1.1 focuses on presenting the general framework and notions that will prove to be of great interest in the sequel. In Section 1.2, we review some classical generalized convexity notions for real-valued and vector-valued functions, and state our new semistrictly quasiconvexity concept for vector-valued functions. Finally, in Section 1.3, we present our main result, which characterizes semistrictly C-quasiconvex vector-valued functions using nonlinear scalarization functions. Additionally, within the specific framework of finite-dimensional real Euclidean space $Y = \mathbb{R}^m$, ordered by the standard cone $C = \mathbb{R}^m_+$, we establish a relationship between our concept of semistrict C-quasiconvexity and well-established concepts, namely, componentwise semistrict quasiconvexity and componentwise explicit quasiconvexity.

Our contribution in this chapter is as follows. In Section 1.2 we have: Definition 1.2.7 (semistrict C- quasiconvexity), Lemma 1.2.8, Lemma 1.2.9 (relationships between semistrict C-quasiconvexity and other concepts of generalized convexity), Theorem 1.2.10 (characterization of explicitly C-quasiconvex vector functions), Lemma 1.2.11, Theorem 1.2.13 (sufficient conditions for C-quasiconvexity), Theorem 1.2.15 (relationship between semistrict C-quasiconvexity and *semistrictly* $\langle P\rangle$*-quasiconvex* in the sense of Flores-Bazán). In Section 1.3 we present: Theorem 1.3.1 (characterization of explicitly C-quasiconvex vector functions by means of the nonlinear scalarization functions) and Corollary 1.3.2 (characterization of componentwise explicitly quasiconvex functions).

All of the above results are original and have been included into the paper Günther, Orzan, and Popovici [35].

Chapter 2: Properties of fractional functions

In this chapter, we provide characterization results for two particular classes of fractional-type set-valued functions, based on the preservation of convexity properties of sets through both direct and inverse mappings. The chapter is divided into four sections.

Section 2.1 is devoted to presenting the overall framework and essential tools of set-valued and convex analysis.

Section 2.2 studies the concept of set-valued affine functions, as defined by Tan [84]. We emphasize that the inverse of this function shares an affine characteristic, setting it apart from other concepts of affine set-valued functions, as shown by Kuroiwa *et al.* [48, Ex. 2]. Furthermore, we extend the classic results established by Rothblum [70] from finite-dimensional Euclidean spaces to encompass real linear spaces. In the last part of this section, we introduce a class of set-valued ratios derived from affine functions and provide a series of convexity preserving properties.

In Section 2.3, we examine an alternative notion of affine set-valued functions as introduced in the literature by Gorokhovik [33, 34] and present a series of results that will be of significant importance for the subsequent section.

The last Section 2.4 delivers a series of results concerning convexity-preserving properties of sets through direct and inverse images for general set-valued maps, and investigates the scenario involving set-valued ratios of affine functions, where it extends and generalizes certain convexity-preserving findings presented in the preceding sections.

Our contribution in this chapter is as follows. In Section 2.2 we present:

Proposition 2.2.2 (condition for affinity of set-valued functions in the sense of Tan), Theorem 2.2.3, Corollary 2.2.4 (concerning the inverse of a set-valued affine function), Propositions 2.2.6, 2.2.7, 2.2.8 (extensions to real linear spaces of results obtained by Rothblum in the particular framework of finite-dimensional Euclidean spaces), Theorem 2.2.10 (characterization of affine set-valued functions), Theorems 2.2.11, 2.2.12, Corollary 2.2.13 (convexity preserving results for set-valued ratios). In Section 2.3 we provide: Lemma 2.3.4, Proposition 2.3.5 (properties of set-valued ratios in the sense of Gorokhovik). In the last Section 2.4, we have: Propositions 2.4.1, 2.4.3, Corollary 2.4.2 (convexity-preserving results for general set-valued functions), Theorem 2.4.4, 2.4.6, Corollary 2.4.5, Theorem 2.4.7, Corollaries 2.4.8, 2.4.9 (convexity-preserving results for set-valued ratios of affine functions).

All of the above results are original and have been included in the papers authored by Orzan and Popovici [60, 61], as well as in Orzan's publication [59].

Chapter 3: Dinkelbach type approximation algorithms for fractional problems

In the third chapter of the thesis, we present approximate versions of the classical Dinkelbach algorithm for nonlinear fractional optimization problems within the general framework of Banach spaces. The chapter is structured into five sections.

Section 3.1 serves as an introduction to our overall framework and the original Dinkelbach algorithm. In Section 3.2, we address the case where the minimizer point for our optimization problem can only be determined with a specified error ($\varepsilon > 0$), and also establish a sufficient condition for achieving the minimum value of the functional A/B. In Section 3.3, we deliver the Dinkelbach algorithm with error decreasing to zero, demonstrating its convergence to the solution of our fractional optimization problem under the Palais-Smale compactness condition.

Section 3.4 provides our new algorithm, the Dinkelbach-Ekeland approximation algorithm, for fractional optimization problems, which utilizes Ekeland's variational principle to generate the sequence of points (x_k) involved in the iterative process.

The final section Section 3.5, offers sufficient conditions that enable the fulfillment of the Palais-Smale requirement and, in turn, establishes our ultimate result regarding the convergence of the Dinkelbach-Ekeland algorithm.

Our contribution in this chapter is as follows. In Section 3.2 we present: Theorems 3.2.2, 3.2.3 (for the case when the Dinkelbach algorithm incorporates a specific predetermined error threshold $\varepsilon > 0$). Section 3.3 provides: Theorems 3.3.2 and 3.3.3 (concerning the convergence of the Dinkelbach Algorithm when the error ε_k, accepted at any step, decreases to zero). In Section 3.4 we have: Algorithm 3.4.1 (the Dinkelbach-Ekeland algorithm) and Theorem 3.4.2 (with regard to the convergence of the algorithm). The concluding Section 3.5 delivers: Lemmas 3.5.1, 3.5.2 , Proposition 3.5.3 and Theorem 3.5.4 (all results representing important steps in ensuring the sufficient conditions under which a functional F satisfies the (PS) condition, which is crucial for the convergence of your algorithm).

All of the above results are original and have been included in the paper Orzan and Precup [62].

Chapter 4: Componentwise Dinkelbach algorithms for fractional problems

In the last chapter of the thesis, we introduce a Dinkelbach-type approximation algorithm, specifically developed for calculating partial minimizers in fractional optimization problems. The chapter is divided into four sections.

Section 4.1 presents the background essential for understanding fractional optimization problems where the objective function involved (a ratio of two functions) is defined on the Cartesian product of two real normed spaces, namely X and Y. Another objective of this section is to identify "partial minimizers", points in $X \times Y$ where one variable minimizes the objective function while the other remains constant. In Section 4.2 we establish some connections between global minimizers and partial minimizers, and also explore the case where $A(x, y)$ and $B(x, y)$ have separate variables, demonstrating that in this context, the partial minimizer coincides with the global minimizer. Additionally, we provide a similar result for finite-dimensional Euclidean spaces towards the end of this section. Section 4.3 begins by introducing the original Dinkelbach algorithm, applied to functions with two variables. Additionally, it presents our componentwise variant of the Dinkelbach algorithm, and investigates its limitations.

The final Section 4.4 focuses on the algorithm's convergence. It demonstrates that by imposing assumptions on the spaces and functions involved, such as Lipschitz-type continuity, partial Fréchet differentiability, and coercivity, the algorithm converges to a partial minimizer.

Our contribution in this chapter is as follows. In Section 4.2 we present: Propositions 4.2.1, 4.2.2 and 4.2.3 (outlying connections between global minimizers, partial minimizers, and critical points in different settings). Section 4.3 provides: Algorithm 4.3.2 (Componentwise Dinkelbach algorithm), Proposition 4.3.3 (existence results for the solutions of the problems (1.31) and (1.32) involved in the algorithm). The last Section 4.4 delivers: Theorems 4.4.1, 4.4.2 and 4.4.4 (vital results that ensure the convergence of our Algorithm 4.3.2).

All of the above results are original and have been included in the paper Günther, Orzan and Precup [36].

* * *

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Keywords

Generalized convexity, set-valued ratio of affine functions, convexity-preserving functions, fractional optimization, Dinkelbach type algorithm, Ekeland's principle, partial minimizer.

Chapter 1
Generalized convexity for vector functions

The conventional understanding of function convexity is acknowledged as a limiting factor in many practical scenarios, as demonstrated in works like Cambini and Martein [14]. In response to this limitation, several researchers have introduced broader classes of functions known as generalized convex functions. These generalized convex and concave functions encompass the class of convex functions while also inheriting valuable properties from them. Generalized convex and concave functions find applications across diverse fields, including economics for representing preferences and utility functions, finance for tasks like portfolio optimization, risk management, and asset pricing, as well as engineering for applications in structural design, process optimization, and signal processing, among others.

In terms of mathematical applications, these functions are widely recognized and extensively studied, particularly in the context of scalar, vector, and set optimization. Notable references in this field include works by Avriel *et al.* [2], Bagdasar and Popovici [3], Crouzeix, Martínez-Legaz and Volle [17], Flores-Bazán [23], Flores-Bazán and Vera [25], Göpfert *et al.* [32], Günther and Tammer [39, 40], Jahn [43], Khan, Tammer and Zălinescu [45], La Torre and Popovici [49], Luc [53], Luc and Schaible [54], Popovici [64], along with additional references cited therein.

Widely recognized concepts of generalized convexity for vector functions, defined on a nonempty convex subset D of a real linear space X and mapping to a real linear topological space Y equipped with a convex cone C, encompass the notions of C-convexity, C-quasiconvexity, and explicit C-quasiconvexity, which were introduced by Luenberger [55], Borwein [9], Luc [53], and Popovici [63].

These concepts of C-convexity, C-quasiconvexity, and explicit C-quasiconvexity represent natural extensions of the classical concepts of convexity, quasiconvexity, and explicit quasiconvexity, which are typically applied to real-valued functions. This extension arises from the fact that within the specific context of the finite-dimensional real Euclidean space $Y = \mathbb{R}^m$, which is partially ordered by the standard cone $C = \mathbb{R}^m_+$, a vector function $f = (f_1, \dots, f_m)$ is C-convex (C-quasiconvex, explicit C-quasiconvex) if and only if its scalar component functions conform to the classical definition of being convex (quasiconvex, explicit quasiconvex).

Certainly, applying such a componentwise approach to general image spaces is not feasible. Therefore, an intriguing research area within vector optimization focuses on characterizing the generalized convexity of vector functions by leveraging the classical generalized convexity principles of specific real-valued functions. This approach has been explored by researchers such as Benoist, Borwein, and Popovici [6], La Torre, Popovici, and Rocca [50], Luc [53], and Günther and Popovici [37, 38].

In this first chapter, we present a new concept of semistrict quasiconvexity for vector functions defined on a nonempty convex set in a real linear space X and mapping to a real topological linear space Y, partially ordered by a proper solid convex cone C. The main result of this chapter is the characterization of semistrictly C-quasiconvex functions using semistrictly quasiconvex scalar functions $\sigma_a \circ f$. Based on this, it appears that our notion of vector semistrict C-quasiconvexity serves as a natural vector counterpart for the corresponding scalar concept of semistrict quasiconvexity.

The chapter is divided into three sections. **Section 1.1** is focused on presenting the foundational framework and essential tools that will prove to be of great interest in the sequel.

In **Section 1.2**, we reiterate some classical notions of generalized convexity for both real-valued and vector-valued functions and state our new concept of semistrictly quasiconvexity for vector functions.

In the concluding **Section 1.3**, we deliver our central result, which provides a characterization of semistrictly C-quasiconvex vector functions through the use of nonlinear scalarization functions. Furthermore, within the context of finite-dimensional real Euclidean space $Y = \mathbb{R}^m$, partially ordered by the standard cone $C = \mathbb{R}^m_+$, we establish a connection between our new concept of semistrict C-quasiconvexity and well-established notions, namely, componentwise semistrict quasiconvexity and componentwise explicit quasiconvexity.

The results of this part of the thesis were included in the paper Günther, Orzan and Popovici [35].

1.1 Preliminaries and properties of convex cones

In the first section of this chapter we will present a set of fundamental definitions and key results related to generalized convexity and convex cones. These insights will serve as valuable building blocks as we delve deeper into the subject. To facilitate our understanding, let us begin by considering that X is a real linear space, Y is a real topological linear space, $D \subseteq X$ is a nonempty convex set, and $C \subseteq Y$ is a convex cone, i.e., $0 \in C = \mathbb{R}_+ \cdot C = C + C$, where 0 stands for the origin of Y while $\mathbb{R}_+$ is the set of all nonnegative real numbers. The lineality space of C is given by $\ell(C) := C \cap (-C)$, which is in fact the largest linear subspace of Y contained by C. For any point $y \in Y$ and any set $A \subseteq Y$, we denote by $\mathcal{V}(y)$ the family of all neighborhoods of y, while by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$ the interior, the closure and the boundary of A, respectively. It is well known that

$$\operatorname{int} A \subseteq \{x \in A \mid \forall\, d \in Y,\ \exists\, r \in \mathbb{R}_+^* \text{ s.t. } x + [-r, r] \cdot d \subseteq A\}, \tag{1.1}$$

where $\mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\}$ is the set of all positive real numbers.

We will investigate several concepts of generalized convexity for vector functions $f : D \to Y$ with respect to C, which naturally extend the corresponding classical notions of generalized convexity known for scalar functions of type $\varphi : D \to \mathbb{R}$. To this aim, we may occasionally need to introduce additional assumptions concerning the convex cone C, the specific context will dictate the nature of these assumptions. We recall that C is said to be: proper, if $C \neq Y$; pointed, if $\ell(C) = \{0\}$; solid, if $\operatorname{int} C \neq \emptyset$; closed, if $\operatorname{cl} C = C$.

In the subsequent part of our preliminaries, we review some basic properties of the convex cone C (see e.g., the books by Jahn [43], and Tammer and Weidner [82]):

$$\operatorname{int} C = \mathbb{R}_+^* \cdot \operatorname{int} C = C + \operatorname{int} C = \operatorname{cl} C + \operatorname{int} C = C + \mathbb{R}_+^* \cdot e,\ \forall\, e \in \operatorname{int} C, \tag{1.2}$$

$$Y = \operatorname{int} C - \mathbb{R}_+^* \cdot e,\ \forall\, e \in \operatorname{int} C, \tag{1.3}$$

$$\operatorname{cl} C = \mathbb{R}_+^* \cdot \operatorname{cl} C,$$

$$\operatorname{bd} C = \mathbb{R}_+^* \cdot \operatorname{bd} C. \tag{1.4}$$

If C is solid, then

$$\operatorname{int} C = \operatorname{int}(\operatorname{cl} C),$$
$$\operatorname{cl} C = \operatorname{cl}(\operatorname{int} C).$$

If C is proper, then

$$0 \in \operatorname{bd} C.$$

Lemma 1.1.1. *Assume that C is solid. Then, for any $e \in \operatorname{int} C$ we have*

$$Y \setminus \operatorname{cl} C = \operatorname{bd} C - \mathbb{R}^*_+ \cdot e = \operatorname{bd} C - \operatorname{int} C. \tag{1.5}$$

Lemma 1.1.2. *Assume that C is solid and let $e \in \operatorname{int} C$. Then, for any $v \in Y$ the following assertions are equivalent:*

1° $v \in \operatorname{cl} C$.

2° $(v + \operatorname{int} C) \cap \operatorname{bd} C = \emptyset$.

3° $(v + \mathbb{R}^*_+ \cdot e) \cap \operatorname{bd} C = \emptyset$.

1.2 Generalized convexity

1.2.1 Generalized convex scalar functions

In this subsection we will revisit the classical concept of convexity, in addition to three well-established concepts of generalized convexity for real-valued functions, which are widely recognized for their significance in scalar optimization (refer to, for example, Avriel *et al.* [2], Cambini and Martein [14]).

Definition 1.2.1. *A function $\varphi : D \to \mathbb{R}$ is called:*

- *convex, if for all $x, x' \in D$ and $t \in (0,1)$ we have*

$$\varphi((1-t)x + tx') \le (1-t)\varphi(x) + t\varphi(x').$$

- *quasiconvex, if for all $x, x' \in D$ and $t \in (0,1)$ we have*

$$\varphi((1-t)x + tx') \le \max\big\{\varphi(x), \varphi(x')\big\}.$$

- *semistrictly quasiconvex, if for any $x, x' \in D$ and $t \in (0,1)$,*

$$\varphi(x) \ne \varphi(x') \quad \Longrightarrow \quad \varphi((1-t)x + tx') < \max\big\{\varphi(x), \varphi(x')\big\}.$$

- *explicitly quasiconvex, if φ is both quasiconvex and semistrictly quasiconvex.*

The next result follows immediately by Definition 1.2.1 and the characterization of explicit quasiconvexity given by Popovici [63, Rem. 3.1] (see also Günther and Popovici [37, Prop. 2.2]).

Proposition 1.2.2. *For any function $\varphi : D \to \mathbb{R}$ the following characterizations hold:*

a) φ is convex if and only if for any $\lambda, \lambda' \in \mathbb{R}$, $x, x' \in D$ and $t \in (0,1)$,

$$\varphi(x) \leq \lambda \ \text{ and } \ \varphi(x') \leq \lambda' \quad \Longrightarrow \quad \varphi((1-t)x + tx') \leq (1-t)\lambda + t\lambda'.$$

b) φ is quasiconvex if and only if for any $\lambda \in \mathbb{R}$, $x, x' \in D$ and $t \in (0,1)$,

$$\varphi(x) \leq \lambda \ \text{ and } \ \varphi(x') \leq \lambda \quad \Longrightarrow \quad \varphi((1-t)x + tx') \leq \lambda.$$

c) φ is semistrictly quasiconvex if and only if for any $\lambda \in \mathbb{R}$, $x, x' \in D$ and $t \in (0,1)$,

$$\varphi(x) = \lambda \ \text{ and } \ \varphi(x') < \lambda \quad \Longrightarrow \quad \varphi((1-t)x + tx') < \lambda.$$

d) φ is explicitly quasiconvex if and only if for any $\lambda \in \mathbb{R}$, $x, x' \in D$ and $t \in (0,1)$,

$$\varphi(x) \leq \lambda \ \text{ and } \ \varphi(x') < \lambda \quad \Longrightarrow \quad \varphi((1-t)x + tx') < \lambda.$$

Remark 1. *The following statements regarding real-valued functions are well-known:*

a) Convex functions are explicitly quasiconvex.

b) Semistrictly quasiconvex functions, that are lower semicontinuous along line segments (cf. Popovici [64]), are also explicitly quasiconvex.

c) Quasiconvexity and semistrict quasiconvexity do not necessarily imply each other, as shown by the next examples.

In the subsequent part, for any $n \in \mathbb{N}$, will be convenient to denote

$$I_n := \{1, \ldots, n\}.$$

In the upcoming sections, we will present and examine concepts of generalized convexity for vector functions that map to the real topological linear space Y. We would like to point out that in the finite-dimensional framework when $Y = \mathbb{R}^n$ ($n \geq 2$) one can also define such concepts by a componentwise approach as follows.

Definition 1.2.3. *A vector function $f = (f_1, \ldots, f_n) : D \to \mathbb{R}^n$ is said to be componentwise convex (quasiconvex, semistrictly quasiconvex, explicitly quasiconvex) if for every $i \in I_n$ the scalar function $f_i : D \to \mathbb{R}$ is convex (quasiconvex, semistrictly quasiconvex, explicitly quasiconvex, respectively) in the classical sense.*

The following result collects some notable characterizations of componentwise generalized convex functions (see, e.g., Günther and Popovici [37, Ex. 4.7], La Torre, Popovici and Rocca [50, Cor. 8], Luc [53, Cor. 6.6]).

Proposition 1.2.4. *For any vector function $f = (f_1, \ldots, f_n) : D \to \mathbb{R}^n$ the following assertions are equivalent:*

1° *f is componentwise convex (quasiconvex, explicitly quasiconvex).*

2° *For any numbers $a_1, \ldots, a_n \in \mathbb{R}$, the scalar function*

$$\max\{f_i(\cdot) - a_i \mid i \in I_n\} : D \to \mathbb{R}$$

is convex (quasiconvex, explicitly quasiconvex, respectively) .

1.2.2 C-convex, C-quasiconvex, and explicitly C-quasiconvex vector functions

Subsequently, we will recapitulate three important concepts of generalized convexity for vector functions. These concepts hold contemporary relevance in the domain of vector optimization and its associated fields, as evidenced by their incorporation in works such as those authored by Bagdasar and Popovici [3, 4], Luc [53], Popovici [63, 64], and the references therein.

Definition 1.2.5. *A function $f : D \to Y$ is called*

- *C-convex, if for all $x, x' \in D$ and $t \in (0, 1)$ we have*

  $$f((1-t)x + tx') \in (1-t)f(x) + tf(x') - C, \text{ i.e.,}$$

  for any $y, y' \in Y$, $x, x' \in D$ and $t \in (0, 1)$,

  $$f(x) \in y - C \text{ and } f(x') \in y' - C \implies f((1-t)x + tx') \in (1-t)y + ty' - C.$$

- *C-quasiconvex, if for any $y \in Y$, $x, x' \in D$ and $t \in (0, 1)$,*

  $$f(x) \in y - C \text{ and } f(x') \in y - C \implies f((1-t)x + tx') \in y - C.$$

- *explicitly C-quasiconvex, if for any $y \in Y$, $x, x' \in D$ and $t \in (0, 1)$,*

  $$f(x) \in y - C \text{ and } f(x') \in y - \operatorname{int} C \implies f((1-t)x + tx') \in y - \operatorname{int} C.$$

The subsequent result compiles various recognized characterizations of generalized convex vector functions (see, e.g., Luc [53] and Popovici [63]).

Proposition 1.2.6. *For any function $f : D \to Y$ the following assertions hold:*

1° *f is C-convex iff its epigraph $\operatorname{epi}_C(f) = \{(x, y) \in D \times Y \mid y \in f(x)+C\}$ is convex.*

2° *f is C-quasiconvex iff for every $y \in Y$, the lower level set $f^{-1}(y - C)$ is convex.*

3° *f is explicitly C-quasiconvex iff for any points $y \in Y$, $x \in f^{-1}(y - C)$, $x' \in f^{-1}(y - \operatorname{int} C)$ and $t \in (0, 1)$ we have $(1-t)x+tx' \in f^{-1}(y - \operatorname{int} C)$.*

1.2.3 Semistrictly C-quasiconvex vector functions

In this last subsection, we introduce our new concept of semistrict quasiconvexity for vector functions (as defined in Definition 1.2.7) and explore its relationships with other established concepts for vector functions, including quasiconvexity, explicit quasiconvexity, and semistrict quasiconvexity in the sense of Flores-Bazán. To this aim, let us consider that $C \subseteq Y$ is a proper solid convex cone. Under these assumptions, it is well-known that $C - C = Y$ (i.e., C generates the space), $\operatorname{bd} C \neq \emptyset$ (since $0 \in C \setminus \operatorname{int} C \subseteq \operatorname{bd} C$) and $C \setminus (-C) \neq \emptyset$ (otherwise, $C \subseteq -C$ would entail $Y = C - C \subseteq -C - C = -C \neq Y$, a contradiction).

We now introduce a new concept of semistrict quasiconvexity for vector functions.

Definition 1.2.7. *We say that $f : D \to Y$ is semistrictly C-quasiconvex if for any $y \in Y$, $x, x' \in D$ and $t \in (0, 1)$ we have*

$$f(x) \in y - \operatorname{bd} C \quad \text{and} \quad f(x') \in y - \operatorname{int} C \quad \Longrightarrow \quad f((1-t)x + tx') \in y - \operatorname{int} C.$$

Remark 2. *Similarly to the characterization of explicitly C-quasiconvex functions given by Proposition 1.2.6 (3°), it can be easily seen that a function $f : D \to Y$ is semistrictly C-quasiconvex if and only if for any points $y \in Y$, $x \in f^{-1}(y - \operatorname{bd} C)$, $x' \in f^{-1}(y - \operatorname{int} C)$ and $t \in (0, 1)$ we have $(1 - t)x + tx' \in f^{-1}(y - \operatorname{int} C)$.*

Remark 3. *Let $C' \subseteq Y$ be a convex cone such that $\operatorname{int} C' = \operatorname{int} C$. Since C is solid, it follows that $\operatorname{cl} C' = \operatorname{cl}(\operatorname{int} C') = \operatorname{cl}(\operatorname{int} C) = \operatorname{cl} C$, hence $\operatorname{bd} C' = \operatorname{bd} C$. Therefore, f is semistrictly C-quasiconvex if and only if f is semistrictly C'-quasiconvex.*

Subsequently, we investigate the relationship between semistrict C-quasiconvexity and other concepts of generalized convexity.

Lemma 1.2.8. *If $f : D \to Y$ is both semistrictly C-quasiconvex and C-quasiconvex, then f is explicitly C-quasiconvex.*

Lemma 1.2.9. *If C is closed, then every explicitly C-quasiconvex function $f : D \to Y$ is both C-quasiconvex and semistrictly C-quasiconvex.*

The following result gives a new characterization of explicitly C-quasiconvex vector functions.

Theorem 1.2.10. *If C is closed, then for any function $f : D \to Y$ the following assertions are equivalent:*

1° *f is explicitly C-quasiconvex.*

2° *f is both C-quasiconvex and semistrictly C-quasiconvex.*

In what follows, given a function $f : D \to Y$, it will be convenient to introduce the notation

$$\Theta_f(x, x', y) = \{t \in [0,1] \mid (1-t)x + tx' \notin f^{-1}(y - C)\}$$

for any points $x, x' \in D$ and $y \in Y$.

Remark 4. *Let $f : D \to Y$ be a function and let $y \in Y$. If $x, x' \in f^{-1}(y-C)$, then*

$$\Theta_f(x, x', y) = \{t \in (0,1) \mid (1-t)x + tx' \notin f^{-1}(y - C)\}.$$

Remark 5. *For any function $f : D \to Y$, the following assertions are equivalent:*

1° *f is C-quasiconvex.*

2° *For any $y \in Y$ and $x, x' \in f^{-1}(y - C)$ we have $\Theta_f(x, x', y) = \emptyset$.*

Lemma 1.2.11. *Assume that C is closed and $f : D \to Y$ is semistrictly C-quasiconvex. Then, for any $y \in Y$ and $x, x' \in f^{-1}(y-C)$, the set $\Theta_f(x, x', y)$ is either empty or reduces to a singleton.*

Definition 1.2.12. *We say that a function $f : D \to Y$ is C-lower semicontinuous along line segments if for any points $x, x' \in D$ the function*

$$t \in [0,1] \longmapsto f((1-t)x + tx') \in Y$$

has closed C-lower level sets, i.e.,

$$\{t \in [0,1] \mid f((1-t)x + tx') \in y - C\} \tag{1.6}$$

is closed for any $y \in Y$.

Remark 6. *Let $f : D \to Y$ be a function and let $x, x' \in D$. For any $y \in Y$ the level set* (1.6) *can be represented as*

$$\{t \in [0,1] \mid f((1-t)x + tx') \in y - C\} = [0,1] \setminus \Theta_f(x, x', y).$$

Theorem 1.2.13. *Assume that C is closed. If $f : D \to Y$ is semistrictly C-quasiconvex and C-lower semicontinuous along line segments, then f is C-quasiconvex.*

We conclude this section by establishing a relationship between semistrict C-quasiconvexity and another concept of generalized convexity, that has been introduced by Flores-Bazán in [23] (see also Flores-Bazán and Vera [25], and Flores-Bazán and Hernández [24]).

Definition 1.2.14. *Let $P \subseteq Y$ be a nonempty set. We say that a vector-valued function $f : D \to Y$ is* semistrictly $\langle P \rangle$-quasiconvex *in the sense of Flores-Bazán if for any $x, x' \in D$ and $t \in (0,1)$,*

$$f(x') \in f(x) - P \quad \Longrightarrow \quad f((1-t)x + tx') \in f(x) - P.$$

Theorem 1.2.15. *Assume that C is a closed convex cone and consider $P \in \{\operatorname{int} C,\, C \setminus (-C),\, Y \setminus (-C)\}$. Then, every semistrictly C-quasiconvex function $f : D \to Y$ is semistrictly $\langle P \rangle$-quasiconvex in the sense of Flores-Bazán.*

1.3 Characterization of semistrictly/explicitly cone-quasiconvex vector functions by means of nonlinear scalarization functions

The last section contains the central result that characterizes semistrictly C-quasiconvex vector functions through the use of nonlinear scalarization functions. We also present some known results concerning the characterization of C-convex (respectively, semistrictly $\langle \operatorname{int} C \rangle$-quasiconvex, semistrictly $\langle C \rangle$-quasiconvex, explicitly C-quasiconvex) vector functions by means of the nonlinear scalarization functions σ_a. Finally, we establish some connections between our new concept of semistrict C-quasiconvexity and well-known notions such as componentwise semistrict quasiconvexity and componentwise explicit quasiconvexity by examining the specific context of the finite-dimensional real Euclidean space $Y = \mathbb{R}^m$, which is partially ordered by the standard cone $C = \mathbb{R}^m_+$.

To this aim, we assume that $C \subseteq Y$ is a proper solid closed convex cone. Given a point $e \in \operatorname{int} C$, we denote by $\sigma : Y \to \mathbb{R}$ the nonlinear scalarization function in the sense of Gerstewitz (Tammer) [29], which is defined for all $y \in Y$ by

$$\sigma(y) \;:=\; \min\{s \in \mathbb{R} \mid y \in se - C\}.$$

One can also associate to each point $a \in Y$ a function $\sigma_a : Y \to \mathbb{R}$, defined for all $y \in Y$ by

$$\sigma_a(y) = \sigma(y - a),$$

the initial function σ being recovered as σ_0. These functions have been widely applied in vector and set optimization and are recognized by various names in the field of mathematical economics (see, e.g., Göpfert *et al.* [32], Khan, Tammer and Zălinescu [45], Luc [53], Tammer and Weidner [30], Tammer and Zălinescu [83] and the references therein). According to Göpfert *et al.* [32, Th. 2.3.1], the following properties hold:

$$\{y \in Y \mid \sigma(y) \leq 0\} = -C, \tag{1.7}$$

$$\{y \in Y \mid \sigma(y) = 0\} = -\operatorname{bd} C, \tag{1.8}$$

$$\{y \in Y \mid \sigma(y) < 0\} = -\operatorname{int} C. \tag{1.9}$$

Furthermore, for any $\lambda \in \mathbb{R}$ and $y \in Y$ we have

$$\sigma(y - \lambda e) = \sigma(y) - \lambda. \tag{1.10}$$

A characterization of explicitly C-quasiconvex vector functions by means of the nonlinear scalarization functions σ_a is given in the following theorem.

Theorem 1.3.1. *A vector function $f : D \to Y$ is explicitly C-quasiconvex if and only if for every $a \in Y$, the scalar function $\sigma_a \circ f : D \to \mathbb{R}$ is explicitly quasiconvex.*

Corollary 1.3.2. *If $f_1, \dots, f_n$ are lower semicontinuous along line segments, then the following assertions are equivalent:*

1° *f is semistrictly $\mathbb{R}^n_+$-quasiconvex.*

2° *f is explicitly $\mathbb{R}^n_+$-quasiconvex.*

3° *f is componentwise explicitly quasiconvex.*

4° *f is componentwise semistricty quasiconvex.*

Chapter 2
Properties of fractional functions

The property of preserving convexity of sets by direct and inverse images is significant in optimization, particularly in convex optimization because it simplifies problem formulations, enables the use of efficient algorithms, and plays a fundamental role in convex duality theory.

This property is closely related to convex functions. However, other classes of functions such as ratios between a convex function and a concave one (particularly, a quadratic function and an affine one, or two affine functions), have also proven significant in scalar optimization. Additionally, the study of vector-valued functions with fractional-type scalar components has garnered substantial attention in the domain of vector optimization. Works by researchers such as Cambini and Martein [14], Göpfert *et al.* [32], Schaible [73], and Stancu-Minasian [79] have explored this area comprehensively, among others referenced therein. While set-valued optimization remains a vital field of study (see, e.g., Khan, Tammer and Zălinescu [45]), there has been a limited introduction of concepts related to fractional-type set-valued functions in the literature. Noteworthy contributions include works like Bhatia and Mehra [8], as well as recent papers such as the one by Das and Nahak [18]. Within this chapter we give some characterizations of two special classes of fractional-type set-valued functions in terms of convexity-preserving properties of sets by direct and inverse images.

The chapter is divided into four sections. **Section 2.1** is dedicated to introducing the general framework and necessary tools of set-valued and convex analysis.

Section 2.2 investigates the concept of set-valued affine functions, as defined by Tan [84]. In particular, we demonstrate that the inverse of such a function is affine as well—a distinction from other concepts of affine set-valued functions, as illustrated by Kuroiwa *et al.* [48, Ex. 2]. Additionally, we extend the classical results of Rothblum [70] from finite-dimensional Euclidean spaces to real linear spaces. Ultimately, we introduce a class of set-valued ratios of affine functions and present a series of convexity-preserving properties.

In **Section 2.3**, we delve into an alternative concept of affine set-valued functions introduced in the litreture by Gorokhovik [33, 34] and deliver a set of results that will hold significant importance for the subsequent section.

Finally, in the last **Section 2.4**, we present a series of results related to convexity-preserving properties of sets through direct and inverse images for general set-valued maps. In addition, we investigate the scenario involving set-valued ratios of affine functions, wherein we extend and generalize certain convexity-preserving findings presented in the preceding sections. The results of this part of the thesis were published in the papers Orzan and Popovici [60, 61] and Orzan [59].

2.1 General framework and preliminaries

In the opening section of this chapter, we will cover some basic definitions and results of set-valued and convex analysis, that will prove helpful as we progress further. To this aim, let us consider that X and Y are two real linear spaces. As usual in set-valued analysis (see, e.g., Aubin and Frankowska [1]), for any set-valued function $F : X \to \mathcal{P}(Y)$ we denote by

$$\operatorname{dom} F = \{x \in X \mid F(x) \neq \emptyset\}$$

the domain of F. We say that F is proper if $\operatorname{dom} F \neq \emptyset$. The (direct) image of a set $A \subseteq X$ by F is defined by

$$F(A) = \bigcup_{x \in A} F(x).$$

There are different manners to define the inverse image of a set $B \subseteq Y$ by a set-valued map F, two of them being currently used in set-valued analysis (Aubin and Frankowska [1]), namely:

$$F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}, \tag{1.11}$$

$$F^{+1}(B) = \{x \in X \mid F(x) \subseteq B\}. \tag{1.12}$$

The set $F^{-1}(B)$ is called the inverse image of B by F and $F^{+1}(B)$ is called the core of B by F (also known as the lower inverse image and the upper inverse image of B by F, respectively). They are related by (see, e.g., Kassay and Rădulescu [44, Sec. 1.3])

$$F^{+1}(B) = X \setminus F^{-1}(Y \setminus B). \tag{1.13}$$

2.2 Affine multifunctions in the sense of Tan

The generalization of the affinity concept to set-valued functions has been approached in various manners within the literature. Noteworthy examples include the works of Deutsch and Singer [19], Nikodem and Popa [57], Tan [84], Gorokhovik [33], Gorokhovik and Zabreiko [34] and the references cited therein. Among these approaches, one particular instantiation, drawing inspiration from Tan [84], will serve as the main tool in this section and is presented in what follows.

Definition 2.2.1. A set-valued function $G : X \to \mathcal{P}(Y)$ is said to be affine (in the sense of Tan) if

$$G((1-t)x^1 + tx^2) = (1-t)G(x^1) + tG(x^2) \tag{1.14}$$

for all $x_1, x_2 \in \operatorname{dom} G$ and $t \in \mathbb{R}$.

Proposition 2.2.2. *For any set-valued function $G : X \to \mathcal{P}(Y)$ the following assertions are equivalent:*

1° *G is affine.*

2° *For all $x^1, x^2 \in \operatorname{dom} G$ and $t \in \mathbb{R}$ we have*

$$(1-t)G(x^1) + tG(x^2) \subseteq G((1-t)x^1 + tx^2).$$

Theorem 2.2.3. *Let $G : X \to \mathcal{P}(Y)$ be a set-valued affine function. Then the inverse of G, i.e., the set-valued function $G^{-1} : Y \to \mathcal{P}(X)$, is affine.*

Corollary 2.2.4. *If $g : E \to Y$ is a vector-valued affine function, defined on a nonempty affine set $E \subseteq X$, then the set-valued function $g^{-1} : Y \to \mathcal{P}(X)$ is affine.*

2.2.1 Vector-valued ratios of affine functions

We begin this subsection by extending the notion of vector-valued ratios of affine functions, originally introduced by Rothblum [70] within finite-dimensional Euclidean spaces, to the framework of general real linear spaces.

Definition 2.2.5. A vector-valued function $f : D \to Y$, defined on a nonempty convex set $D \subseteq X$, is said to be a ratio of affine functions if there exist a vector-valued affine function $g : X \to Y$ and a real-valued affine function $h : X \to \mathbb{R}$, such that

$$D \subseteq \{x \in X \mid h(x) > 0\}$$

and

$$f(x) = \frac{g(x)}{h(x)},\ \forall\, x \in D.$$

The following propositions extend to the framework of general real linear spaces some results obtained within $\mathbb{R}^n$ by Rothblum (see [70, Props. 1, 2 and 3] along with subsequent remarks). Their proofs are ommited, since they follow the main lines in Rothblum [70].

Proposition 2.2.6. *Given a vector-valued function $f : D \to Y$ defined on a nonempty convex set $D \subseteq X$, the following assertions are equivalent:*

1° $\operatorname{conv} f(S) \subseteq f(\operatorname{conv} S)$ *for every set* $S \subseteq D$.

2° $f(A)$ *is convex for every convex set* $A \subseteq D$, *i.e.,* f *preserves the convexity of sets by direct images.*

Proposition 2.2.7. *Given a vector-valued function $f : D \to Y$ defined on a nonempty convex set $D \subseteq X$, the following assertions are equivalent:*

1° $f(\operatorname{conv} S) \subseteq \operatorname{conv} f(S)$ *for every set* $S \subseteq D$.

2° $f^{-1}(B)$ *is convex for every convex set* $B \subseteq Y$, *i.e., function* f *preserves the convexity of sets by inverse images.*

Proposition 2.2.8. *Let $D \subseteq X$ be a nonempty convex set. If $f : D \to Y$ is a vector-valued ratio of affine functions, then*

$$\operatorname{conv} f(S) = f(\operatorname{conv} S) \ \textit{ for every set } \ S \subseteq D.$$

Therefore f preserves the convexity of sets by direct and inverse images.

2.2.2 Set-valued ratios of affine functions

In this subsection we introduce a class of set-valued ratios of affine functions, by slightly modifying the one proposed by Orzan in [59, Def. 3.3]. We show that these fractional-type functions preserve the convexity of sets through direct and inverse images, a property that, as we have already stated, is very significant in optimization, especially in the context of convex optimization.

Definition 2.2.9. Let $F : X \to \mathcal{P}(Y)$ be a set-valued function with the domain nonempty and convex. We say that F is a set-valued ratio of affine functions (in the sense of Tan) if there exist a proper set-valued affine function

$G : X \to \mathcal{P}(Y)$ and a real-valued affine function $h : X \to \mathbb{R}$, other than the null functional, such that

$$\operatorname{dom} F \subseteq \{x \in X \mid h(x) > 0\} \cap \operatorname{dom} G$$

and

$$F(x) = \begin{cases} \dfrac{G(x)}{h(x)} & \text{if } x \in \operatorname{dom} F \\ \emptyset & \text{if } x \in X \setminus \operatorname{dom} F. \end{cases} \tag{1.15}$$

Theorem 2.2.10. *Let $F : X \to \mathcal{P}(Y)$ be a set-valued ratio of affine functions defined by* (1.15)*. Then there exist a vector-valued ratio of affine functions $f : \operatorname{dom} F \to Y$ and a linear subspace $M \subseteq Y$, such that*

$$F(x) = \begin{cases} f(x) + M & \text{if } x \in \operatorname{dom} F \\ \emptyset & \text{if } x \in X \setminus \operatorname{dom} F. \end{cases} \tag{1.16}$$

Theorem 2.2.11. *Consider a set-valued function $F : X \to \mathcal{P}(Y)$ of type* (1.16)*, where $\operatorname{dom} F \subseteq X$ and $M \subseteq Y$ are nonempty convex sets, while $f : \operatorname{dom} F \to Y$ is a vector-valued function that preserves the convexity of sets by direct images. Then, for every convex set $A \subseteq X$, the set $F(A)$ is convex, i.e., F preserves the convexity of sets by direct images.*

Theorem 2.2.12. *Consider a set-valued function $F : X \to \mathcal{P}(Y)$ of type* (1.16)*, where $\operatorname{dom} F \subseteq X$ and $M \subseteq Y$ are nonempty convex sets, while $f : \operatorname{dom} F \to Y$ is a vector-valued function that preserves the convexity of sets by inverse images. Then, for every convex set $B \subseteq Y$, the sets $F^{-1}(B)$ and $F^{+1}(B) \cap \operatorname{dom} F$ are convex, i.e., F preserves the convexity of sets by lower inverse images as well as by upper inverse images in the sense of Berge.*

Corollary 2.2.13. *If $F : X \to \mathcal{P}(Y)$ is a set-valued ratio of affine functions defined by* (1.15)*, then the following assertions hold:*

1° *F preserves the convexity of sets by direct images.*

2° *F preserves the convexity of sets by lower inverse images.*

3° *F preserves the convexity of sets by upper inverse images in the sense of Berge.*

2.3 Affine multifunctions in the sense of Gorokhovik

In the subsequent section of this chapter, we will investigate another type of affinity for set-valued functions, introduced by V. Gorokhovik [33], which encompasses the previously mentioned concept of affinity by Tan. To this aim, we will also revisit the classical notions of convexity and concavity for set-valued functions.

Definition 2.3.1. A set-valued function $G : X \to \mathcal{P}(Y)$ is said to be
(i) *convex* if

$$(1-t)G(x^1) + tG(x^2) \subseteq G((1-t)x^1 + tx^2)$$

for any $x^1, x^2 \in \operatorname{dom} G$ and $t \in [0,1]$;
(ii) *concave* if $\operatorname{dom} G$ is a convex subset of X and

$$G((1-t)x^1 + tx^2) \subseteq (1-t)G(x^1) + tG(x^2)$$

for any $x^1, x^2 \in \operatorname{dom} G$ and $t \in [0,1]$;
(iii) *affine* (in the sense of Gorokhovik) if

$$G((1-t)x^1 + tx^2) = (1-t)G(x^1) + tG(x^2)$$

for any $x^1, x^2 \in \operatorname{dom} G$ and $t \in [0,1]$.

Let $K \subseteq X$ be a cone (i.e. a set with $0_X \in K = \mathbb{R}_+ \cdot K$).

Definition 2.3.2 (Nikodem [56])**.** *Let $F : X \to \mathcal{P}_0(Y)$ be a set-valued function. Then F is called K-quasiconcave if it satisfies the condition*

$$F(x^1) \subseteq A + K \text{ and } F(x^2) \subseteq A + K \Rightarrow F((1-t)x^1 + tx^2) \subseteq A + K$$

for any convex set $A \subseteq Y$, $x^1, x^2 \in X$ and $t \in [0,1]$.

Remark 7. *If $K = \{0_X\}$, then F is quasiconvex.*

The following notions and results will be used in the subsequent part. For any two points $a, b \in \mathbb{R}, a, b > 0$ we consider the ratio of affine functions $\sigma_{a,b} : [0,1] \to [0,1]$ defined by

$$\sigma_{a,b}(t) = \frac{tb}{(1-t)a + tb}. \tag{1.17}$$

Remark 8. *It is easy to see that the function $\sigma_{a,b}$ is well-defined and has the following properties:*

1° $\sigma_{a,b}$ *is strictly increasing;*

2° $\sigma_{a,b}$ *is convex if $a > b$, $\sigma_{a,b}$ is concave if $a < b$ and $\sigma_{a,b}(t) = t$ for any $t \in [0, 1]$ if $a = b$;*

3° $\sigma_{a,b}$ *is bijective and its inverse $\sigma_{a,b}^{-1}$ is $\sigma_{b,a}$.*

2.3.1 Set-valued ratios of affine functions

In our way to explore convexity preserving properties based on Gorokhovik's concept of affinity for set-valued maps, we will deliver a similar definition to Definition 2.2.9. This will facilitate our work and, at the same time, will set the stage for the upcoming results.

Definition 2.3.3. Let $F : X \to \mathcal{P}(Y)$ be a set-valued function with $\operatorname{dom} F$ a convex set. We say that F is a *set-valued ratio of affine functions* (in the sense of Gorokhovik) if there exists a proper affine set-valued function $G : X \to \mathcal{P}(Y)$ and a real-valued affine function $h : X \to \mathbb{R}$, other than the null functional, such that

$$\operatorname{dom} F \subseteq \{x \in X \mid h(x) > 0\} \cap \operatorname{dom} G$$

and

$$F(x) = \begin{cases} \dfrac{G(x)}{h(x)} & \text{if } x \in \operatorname{dom} F \\ \emptyset & \text{if } x \in X \setminus \operatorname{dom} F. \end{cases}$$

Lemma 2.3.4. *Let F be the ratio between a set-valued function G and a positive affine function h. The next statements are true:*

1° *if G is convex, then*

$$(1 - \sigma_{h(x^1),h(x^2)}(t))F(x^1) + \sigma_{h(x^1),h(x^2)}(t)F(x^2) \subseteq F((1 - t)x^1 + tx^2)$$

for any points $x^1, x^2 \in \operatorname{dom} F$ and $t \in [0, 1]$;

2° *if G is concave, then*

$$F((1 - t)x^1 + tx^2) \subseteq (1 - \sigma_{h(x^1),h(x^2)}(t))F(x^1) + \sigma_{h(x^1),h(x^2)}(t)F(x^2)$$

for any points $x^1, x^2 \in \operatorname{dom} F$ and $t \in [0, 1]$;

3° *if G is affine, then*

$$F((1-t)x^1 + tx^2) = (1 - \sigma_{h(x^1),h(x^2)}(t))F(x^1) + \sigma_{h(x^1),h(x^2)}(t)F(x^2)$$

for any points $x^1, x^2 \in \operatorname{dom} F$ and $t \in [0,1]$.

Proposition 2.3.5. *Let F be the ratio between a convex set-valued function G and a positive affine function h. Then*

1° *for any point $x \in X$, the set $F(x)$ is convex;*

2° *for any point $y \in Y$, the set $F^{-1}(y)$ is convex.*

2.4 Convexity-preserving properties

We conclude this chapter by a two-part section. In the first subsection we deliver some convexity-preserving results for general set-valued functions. The second subsection deals with set-valued ratios of affine functions (in the sense of Gorokhovik) and generalizes some of the results obtained in the previous sections.

2.4.1 The case of general set-valued functions

Proposition 2.4.1. *For any set-valued map $F : X \to \mathcal{P}(Y)$, the following assertions are equivalent:*

1° $\operatorname{conv} F(A) \subseteq F(\operatorname{conv} A)$ *for any set $A \subseteq X$.*

2° *For any convex set $C \subseteq X$, the set $F(C)$ is convex.*

3° $\operatorname{conv} F(\{x^1, x^2\}) \subseteq F(\operatorname{conv} \{x^1, x^2\})$ *for any $x^1, x^2 \in X$.*

4° $\operatorname{conv} F(\{x^1, x^2\}) \subseteq F(\operatorname{conv} \{x^1, x^2\})$ *for any $x^1, x^2 \in \operatorname{dom} F$.*

Corollary 2.4.2. *For any set-valued map $F : X \to \mathcal{P}(Y)$ the following assertions are equivalent:*

1° $\operatorname{conv} F^{-1}(A) \subseteq F^{-1}(\operatorname{conv} A)$ *for any set $A \subseteq Y$.*

2° *For any convex set $C \subseteq Y$, the set $F^{-1}(C)$ is convex, i.e., F is quasi-convex in the sense of Nikodem.*

3° $\operatorname{conv} F^{-1}(\{y^1, y^2\}) \subseteq F^{-1}(\operatorname{conv} \{y^1, y^2\})$ *for any $y^1, y^2 \in Y$.*

4° $\operatorname{conv} F^{-1}(\{y^1, y^2\}) \subseteq F^{-1}(\operatorname{conv} \{y^1, y^2\})$ *for any $y^1, y^2 \in \operatorname{dom} F^{-1}$.*

Proposition 2.4.3. *For any set-valued function $F : X \to \mathcal{P}(Y)$ such that $\operatorname{dom} F$ is convex, the following assertions are equivalent:*

1° $F(\operatorname{conv} A) \subseteq \operatorname{conv} F(A)$ *for any set* $A \subseteq \operatorname{dom} F$.

2° *For any convex set $C \subseteq Y$, the set $F^{+1}(C) \cap \operatorname{dom} F$ is convex.*

3° $F(\operatorname{conv}\{x^1, x^2\}) \subseteq \operatorname{conv} F(\{x^1, x^2\})$ *for any* $x^1, x^2 \in \operatorname{dom} F$.

2.4.2 The case of set-valued ratios of affine functions

Theorem 2.4.4. *Let $F : X \to \mathcal{P}(Y)$ be the ratio of a convex set-valued function and a positive affine function. Then for any convex set $C \subseteq X$, the set $F(C)$ is convex.*

Corollary 2.4.5. *If $F : X \to \mathcal{P}(Y)$ is a set-valued ratio of affine functions, then for any convex set $C \subseteq X$, the set $F(C)$ is convex.*

Theorem 2.4.6. *Let $F : X \to \mathcal{P}(Y)$ be a set-valued ratio of affine functions as given by Definition 2.3.3. Then for any convex set C from Y, the set $F^{+1}(C) \cap \operatorname{dom} F$ is convex.*

Theorem 2.4.7. *Let $F : X \to \mathcal{P}(Y)$ be the ratio of a concave set-valued map and a positive affine function with $\operatorname{dom} F \subseteq X$ nonempty and convex. Then for any convex set $C \subseteq Y$, the set $F^{-1}(C)$ is convex.*

Corollary 2.4.8. *Let $F : X \to \mathcal{P}(Y)$ be a ratio of affine set-valued functions. Then for any convex set $B \in \mathcal{P}(Y)$, the set $F^{-1}(B)$ is convex.*

Corollary 2.4.9. *Let $F : X \to \mathcal{P}(Y)$ be an affine set-valued function. Then for any convex set $B \in \mathcal{P}(Y)$, the set $F^{-1}(B)$ is convex.*

Chapter 3
Dinkelbach type approximation algorithms for fractional problems

Fractional optimization problems have gained significant attention from researchers due to their relevance in modeling various real-world processes. Numerous authors have dedicated substantial efforts to studying this subject, such as Avriel et al. [2], Cambini and Martein [14], Stancu-Minasian [79], Elbenani and Ferland [21] and the references therein. Examples of fractional optimization problems can also be identified in mathematics in multiple publications among which we mention the papers of Crouzeix [15, 16], Hadjisavvas [26], Schaible [71, 72], Rodenas [69], Shi [77], Boţ et al.[10, 11], Tammer [81]. In this chapter of the thesis we establish some approximation versions of the classical Dinkelbach algorithm for nonlinear fractional optimization problems in Banach spaces.

The chapter is structured into five sections. The first one, **Section 3.1**, introduces our general framework and the original Dinkelbach algorithm.

Section 3.2 presents the case in which at any step of the algorithm, the generated point desired to be a minimizer can only be determined with a given error $\varepsilon > 0$ and provides a sufficient condition under which the functional A/B reaches the minimum value.

In **Section 3.3** we deliver the Dinkelbach algorithm with errors decreasing to zero and show that under the same Palais-Smale compactness condition from the previous section, the algorithm is convergent to the pair $(\lambda^*, \widetilde{x})$, in other words, the solution of our fractional optimization problem. **Section 3.4** puts forward the Dinkelbach-Ekeland approximation algorithm for our fractional problem.

Finally, the last **Section 3.5** offers sufficient conditions that make the achievement of the Palais-Smale requirement possible and grants our final result concerning the convergence of the Dinkelbach-Ekeland algorithm.

The results of this chapter were published in the paper Orzan and Precup [62].

3.1 Fractional optimization problems and the Dinkelbach algorithm

A fractional optimization problem can be expressed in its general form as follows

$$\frac{A(x)}{B(x)} \longrightarrow \min_{x \in D},$$

where A and B are two given functionals defined on a nonempty set D and $B(x) \neq 0$ for all $x \in D$. Solving the problem means to obtain a pair (λ^*, x^*), where λ^* is the minimal value of A/B and x^* is a minimum point, i.e.,

$$\lambda^* = \frac{A(x^*)}{B(x^*)} = \min_D \frac{A}{B}.$$

In addition to direct minimization methods, there exist specific techniques that transform the problem into an optimization problem involving a non-ratio functional. One such technique, introduced by Dinkelbach [20], is based on the parametric problem

$$A(x) - \lambda B(x) \longrightarrow \min_{x \in D}.$$

The method involves generating a sequence (λ_k) of parameter values that ultimately leads to the minimum of the ratio A/B. Alongside this sequence, another sequence (x_k) is generated to converge to the desired minimum point. In order to enhance our understanding of this method and facilitate comparison with the forthcoming results, we provide a brief overview of Dinkelbach's algorithm and outline the main stages of its convergence proof. Assuming that D is a nonempty arbitrary set, $A : D \to \mathbb{R}$ is bounded from bellow, and $B : D \to \mathbb{R}$ has the property that

$$0 < c \leq B(x) \leq C \quad \text{for all} \quad x \in D,$$

for some real numbers $c, C \in \mathbb{R}$, the Dinkelbach algorithm reads as follows:

Algorithm 3.1.1 (Dinkelbach's algorithm).

Step 0 **(initialization)** Take any point $x_0 \in D$, calculate $\lambda_0 = \frac{A(x_0)}{B(x_0)}$ and set $k = 1$.

Step k (cycle step for $k \geq 1$) Find a point $x_k \in D$ such that

$$A(x_k) - \lambda_{k-1} B(x_k) = m_k := \min_D (A - \lambda_{k-1} B),$$

calculate

$$\lambda_k = \frac{A(x_k)}{B(x_k)}$$

and perform Step $k+1$.

Theorem 3.1.2. *Dinkelbach's algorithm is convergent and if $\lambda^* = \lim \lambda_k$ and $x^* \in D$ is a point such that*

$$A(x^*) - \lambda^* B(x^*) = \min_D (A - \lambda^* B),$$

then

$$\lambda^* = \frac{A(x^*)}{B(x^*)} = \min_D \frac{A}{B}. \tag{1.18}$$

We conclude this section by recalling the Ekeland variational principle and some of its consequences (see, e.g., Ekeland [22], Frigon [27]), which will prove to be of great importance for our work. We also recall the Palais-Smale (compactness) condition, denoted for simplicity by (PS).

Theorem 3.1.3 (Ekeland)**.** *Let (X, d) be a complete metric space and let $E : X \to \mathbb{R}$ be a lower semicontinuous function bounded from below. Then given $\varepsilon > 0$ and $u_0 \in X$, there exists a point $u \in X$ such that*

$$E(u) \leq E(v) + \varepsilon d(u, v) \quad \text{for all } v \in X$$

and

$$E(u) \leq E(u_0) - \varepsilon d(u, u_0).$$

Corollary 3.1.4. *Let $(X, |\cdot|)$ be a Banach space and $E : X \to \mathbb{R}$ a C^1 functional bounded from below. Then for every $\varepsilon > 0$, there exists an element $u \in X$ such that*

$$E(u) \leq \inf_X E + \varepsilon, \quad |E'(u)| \leq \varepsilon$$

A C^1 functional E defined on a Banach space is said to satisfy the *(PS) condition* if any sequence (x_k) with

$$E(x_k) \to l \quad (l \in \mathbb{R}) \quad \text{and} \quad E'(x_k) \to 0$$

has a convergent subsequence.

Theorem 3.1.5. *Let $(X, |\cdot|)$ be a Banach space and $E : X \to \mathbb{R}$ be a C^1 functional bounded from below that satisfies the (PS) condition. Then there exists a point $x^* \in X$ with*

$$E(x^*) = \inf_X E \quad \text{and} \quad E'(x^*) = 0.$$

3.2 Dinkelbach algorithm with fixed error

In this section, we are going to discuss the case when Dinkelbach's algorithm incorporates a specific predetermined error threshold $\varepsilon > 0$. Consequently, Algorithm 4.3.1 is modified as follows:

Algorithm 3.2.1 (Dinkelbach algorithm with fixed error).

Step 0 **(initialization)** Take any point $x_0 \in D$, calculate $\lambda_0 = \frac{A(x_0)}{B(x_0)}$ and set $k = 1$.

Step k **(cycle step for** $k \geq 1$) Find a point $x_k \in D$ such that

$$A(x_k) - \lambda_{k-1} B(x_k) \leq \inf_D (A - \lambda_{k-1} B) + \varepsilon, \tag{1.19}$$

calculate

$$\lambda_k = \min \left\{ \frac{A(x_k)}{B(x_k)},\ \lambda_{k-1} \right\}$$

and perform Step $k+1$.

Theorem 3.2.2. *If* $\widetilde{\lambda} = \lim \lambda_k$ *and* $\widetilde{x} \in D$ *is such that*

$$A(\widetilde{x}) - \widetilde{\lambda} B(\widetilde{x}) \leq \inf_D \left(A - \widetilde{\lambda} B \right) + \varepsilon,$$

then

$$\inf_D \frac{A}{B} \leq \widetilde{\lambda} \leq \inf_D \frac{A}{B} + \frac{\varepsilon}{c} \tag{1.20}$$

and

$$\widetilde{\lambda} - \frac{\varepsilon}{c} \leq \frac{A(\widetilde{x})}{B(\widetilde{x})} \leq \widetilde{\lambda} + \frac{\varepsilon}{c}. \tag{1.21}$$

According to Theorem 4.4.2, $\widetilde{\lambda}$ differs from $\inf_D A/B$ with at most ε/c and $\widetilde{x}$ is a $2\varepsilon/c$ approximation minimum point of $\inf_D A/B$.

Next we attempt to see the way in which the functional A/B reaches the value $\widetilde{\lambda}$. To this aim we need a topological structure on D, the continuity of A and B, and a compactness condition.

Theorem 3.2.3. *Assume, in addition, that D is a metric space, A and B are continuous on D and the following compactness condition holds:*

(C) *any sequence (y_k) of elements of D for which the sequence $(A(y_k)/B(y_k))$ is convergent has a convergent subsequence in D.*

Then there exists a point $\widetilde{x} \in \overline{\{x_k\}}$ (either a term or an accumulation point of the sequence (x_k) generated by the approximation Dinkerbach's algorithm) such that $\widetilde{\lambda} = A(\widetilde{x}) / B(\widetilde{x})$.

3.3 Dinkelbach algorithm with errors decreasing to zero

In the previous section we studied Dinkelbach's algorithm with a fixed error $\varepsilon > 0$. In what follows, we establish that if the error ε_k, accepted at any step, decreases to zero, then the algorithm converges to $(\lambda^*, \widetilde{x})$, where λ^* is the infimum of A/B and that is reached under the compactness assumption stated in 3.2.3. The modified algorithm reads as follows:

Algorithm 3.3.1 (Dinkelbach algorithm with errors decreasing to zero)**.**

Step 0 **(initialization)** Take any point $x_0 \in D$, calculate $\lambda_0 = \frac{A(x_0)}{B(x_0)}$ and set $k = 1$.

Step k **(cycle step for** $k \geq 1$) Find a point $x_k \in D$ such that

$$A\left(x_k\right) - \lambda_{k-1} B\left(x_k\right) \leq \inf_D \left(A - \lambda_{k-1} B\right) + \varepsilon_k, \tag{1.22}$$

calculate

$$\lambda_k = \min\left\{\frac{A\left(x_k\right)}{B\left(x_k\right)},\ \lambda_{k-1}\right\}$$

and perform Step $k+1$.

Theorem 3.3.2. *If* $\widetilde{\lambda} = \lim \lambda_k$ *and* $\widetilde{x} \in D$ *is a point such that*

$$A\left(\widetilde{x}\right) - \widetilde{\lambda} B\left(\widetilde{x}\right) \leq \inf_D \left(A - \widetilde{\lambda} B\right) + \varepsilon,$$

then we have

$$\widetilde{\lambda} = \lambda^* = \inf_D \frac{A}{B} \tag{1.23}$$

and

$$\frac{A\left(\widetilde{x}\right)}{B\left(\widetilde{x}\right)} \leq \inf_D \frac{A}{B} + \frac{\varepsilon}{c}.$$

We note that the result in Theorem 3.2.3 remains valid in this context as well. More precisely we have:

Theorem 3.3.3. *Under the assumptions of Theorem* 3.2.3*, if* (λ_k) *and* (x_k) *are the sequences given by Dinkelbach algorithm with error decreasing to zero and* $\lambda^* = \lim \lambda_k$*, then there exists a point* $x^* \in \overline{\{x_k\}}$ *(either a term or an accumulation point of the sequence* (x_k)*) such that*

$$\lambda^* = \frac{A\left(x^*\right)}{B\left(x^*\right)} = \min_D \frac{A}{B}.$$

In addition, we have that

$$\inf_D \left(A - \lambda_k B\right) \to \inf_D \left(A - \lambda^* B\right) = 0 \quad \text{as } k \to \infty. \tag{1.24}$$

3.4 Dinkelbach-Ekeland approximation algorithm

Our next goal is to fulfil the Palais-Smale compactness condition (assumption **(C)** from Theorem 3.2.3) for noncompact sets D. We get closer to this desire if we can ensure additional properties on the sequence (x_k). In order to achive this, we will present a modified version of Dinkelbach's algorithm by making use of Ekeland's variational principle to generate the points of the sequence (x_k). To this aim, let $(X, |\cdot|)$ be a Banach space and $A, B : X \to \mathbb{R}$ two C^1 functionals such that A is bounded from below and B satisfies

$$0 < c \leq B(x) \leq C \quad \text{for all} \quad x \in X.$$

Let us also consider that (ε_k) is a decreasing sequence of positive real numbers. Then the Dinkelbach-Ekeland algorithm description follows below:

Algorithm 3.4.1 (Dinkelbach-Ekeland algorithm)**.**

Step 0 **(initialization)** Take any point $x_0 \in X$, calculate $\lambda_0 = \dfrac{A(x_0)}{B(x_0)}$ and set $k = 1$.

Step k **(cycle step for** $k \geq 1$) By using Ekeland's principle, we find a point $x_k \in X$ such that

$$A(x_k) - \lambda_{k-1}B(x_k) \leq \inf_X (A - \lambda_{k-1}B) + \varepsilon_k, \tag{1.25}$$

$$|A'(x_k) - \lambda_{k-1}B'(x_k)| \leq \varepsilon_k$$

then calculate

$$\lambda_k = \min\left\{\frac{A(x_k)}{B(x_k)},\ \lambda_{k-1}\right\}$$

and perform Step $k+1$.

Theorem 3.4.2. *Under the above assumptions, if* (λ_k) *and* (x_k) *are the sequences given by the algorithm and* $\widetilde{\lambda} = \lim \lambda_k$, *then*

$$A(x_k) - \widetilde{\lambda}B(x_k) \to \inf_X \left(A - \widetilde{\lambda}B\right) \quad \text{and} \quad A'(x_k) - \widetilde{\lambda}B'(x_k) \to 0. \tag{1.26}$$

If in addition the functional $A - \widetilde{\lambda}B$ *satisfies the (PS) condition, then there exists a point* $x^* \in \overline{\{x_k\}}$ *(either a term or an accumulation point of the sequence* (x_k)*) such that*

$$\widetilde{\lambda} = \lambda^* = \frac{A(x^*)}{B(x^*)} = \min_X \frac{A}{B}.$$

3.5 Sufficient conditions for the (PS) requirement

In the final section of this chapter, we deliver some sufficient conditions under which a functional F satisfies the (PS) condition. They require some topological properties on F' that are well-known in nonlinear analysis.

Lemma 3.5.1. *Let $(X, |\cdot|)$ be a Hilbert space and $F : X \to \mathbb{R}$ a C^1 functional having the following two properties:*

(i) *any sequence (x_k) for which $(F(x_k))$ converges is bounded;*

(ii) *the operator $N := I - F'$ is completely continuous.*

Then F satisfies the (PS) condition.

A stronger topological condition on F' guarantees that any sequence (x_k) satisfying $F'(x_k) \to 0$ is entirely convergent (F satisfies strongly the (PS) condition).

Lemma 3.5.2. *Let $(X, |\cdot|)$ be a Hilbert space and $F : X \to \mathbb{R}$ a C^1 functional such that*

(ii*) *the operator $N = I - F'$ is a contraction on X.*

Then F strongly satisfies the (PS) condition.

Returning to functionals of the form $A - \lambda B$, as in our Dinkelbach-Ekeland algorithm, the following result can be established.

Proposition 3.5.3. *Let $(X, |\cdot|)$ be a Hilbert space, $A, B : X \to \mathbb{R}$ be C^1 functionals such there exist the constants c, C with $0 < c \leq B \leq C$. Then the following statements hold:*

(a) *If A is coercive and the operators $I - A'$ and B' are completely continuous, then for each $\lambda \in \mathbb{R}$, the functional $A - \lambda B$ satisfies the (PS) condition.*

(b) *If the operators $I - A'$ and B' are L_A- respectively L_B-Lipschitz continuous with $L_A < 1$, then the functional $A - \lambda B$ satisfies strongly the (PS) condition for every λ with $|\lambda| < (1 - L_A)/L_B$.*

As far as the Dinkelbach-Ekeland algorithm is concerned, we have the following final result on its convergence.

Theorem 3.5.4. *Let $(X, |\cdot|)$ be a Hilbert space, the conditions of Theorem* 3.4.2 *hold, (λ_k) and (x_k) be the sequences given by Algorithm* 3.4.1, *and* $\widetilde{\lambda} = \lim \lambda_k$.

(a) *If A is coercive and the operators $I - A'$ and B' are completely continuous, then there exists a point $x^* \in \overline{\{x_k\}}$ (either a term or an accumulation point of the sequence (x_k)) such that*

$$\widetilde{\lambda} = \lambda^* = \frac{A(x^*)}{B(x^*)} = \min_X \frac{A}{B}. \tag{1.27}$$

(b) *If the operators $I - A'$ and B' are L_A- respectively L_B-Lipschitz continuous with $L_A < 1$ and*

$$\max\{\lambda_0, -\lambda_{-1}\} < \frac{1 - L_A}{L_B}, \tag{1.28}$$

where $\lambda_{-1} = \inf_X A/B$, then the sequence (x_k) converges to some x^* and (1.27) *holds.*

Chapter 4
Componentwise Dinkelbach algorithms for fractional problems

The demand for computational algorithms to address fractional optimization problems arises from a wide array of practical scenarios. These contexts frequently involve situations where the objective revolves around optimizing a performance metric that is expressed as a ratio, as evidenced in the research conducted by Shen and Yu [75, 76], Elbenani and Ferland [21], and the references therein. Consequently, these problems carry substantial relevance across various fields, encompassing economics, industrial planning, medical strategy development and related domains. Similar challenges are also encountered in diverse mathematical domains, such as graph theory and game theory, as evidenced by the research of Stancu-Minasian [79], Stancu-Minasian and Tigan [80] and the references therein. In this section of the thesis, we present a Dinkelbach-type algorithm designed to compute partial minimizers for fractional optimization problems.

The chapter is divided into four sections. **Section 4.1** introduces our general framework and provides the background necessary for the reader to familiarize themselves with fractional optimization problems where the objective function involved (a ratio of two functions) is defined on the Cartesian product of two real normed spaces, namely X and Y. The section also clearly defines the objective of this part of the thesis: determining the so-called partial minimizers. These are points in $X \times Y$ with the property that one of their variables minimizes the objective function when the other variable is held constant.

In **Section 4.2** we demonstrate some results concerning the relationship between global minimizers and partial minimizers (Proposition 4.2.1) in real normed spaces. Additionally, we investigate the distinct case when $A(x, y)$ and $B(x, y)$ have separate variables and show that in this framework the partial minimizer coincides with the global minimizer (Proposition 4.2.2).

Another analogous result with the first one, related to the context of finite-dimensional Euclidean spaces, is also presented towards the conclusion of this section.

In **Section 4.3**, we commence by introducing the original Dinkelbach algorithm (Algorithm 4.3.1), which is applied to objective functions depending on two variables. Subsequently, we present our componentwise variant of the algorithm (Algorithm 4.3.2) and provide an in-depth exploration of the associated drawbacks. Furthermore, we address practical measures to effectively alleviate these limitations.

Ultimately, the last **Section 4.4** is devoted to the convergence of our algorithm. The section demonstrates that introducing additional assumptions concerning the spaces and functions involved – such as Lipschitz-type continuity, partial Fréchet differentiability, and coercivity – provides a foundation for determining adequate conditions for Algorithm 4.3.2 to converge towards a partial minimizer.

The results of this part of the thesis were published in the paper Günther, Orzan and Precup [36].

4.1 The optimization problem

In the first section of this chapter, we will describe the general optimization problem of interest in order to clearly delineate the distinction between the standard problem of finding minimizers for fractional functionals and our specific framework. With the aim of accomplishing this, let us consider throughout the entire chapter that $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ are two real normed spaces, $D_1 \subseteq X$, $D_2 \subseteq Y$ are nonempty sets and $A, B : D_1 \times D_2 \to \mathbb{R}$ are functions such that A is bounded from below and there exist $c, C \in \mathbb{R}$ such that

$$0 < c \leq B(x, y) \leq C \quad \text{for all} \quad (x, y) \in D_1 \times D_2.$$

The general fractional optimization problem we are interested in is given by

$$\frac{A(x, y)}{B(x, y)} \longrightarrow \min_{(x,y) \in D_1 \times D_2}. \tag{P}$$

In problem (P), one is usually searching for points $(x^*, y^*) \in D_1 \times D_2$ such that

$$\frac{A(x^*, y^*)}{B(x^*, y^*)} = \min_{(x,y) \in D_1 \times D_2} \frac{A(x, y)}{B(x, y)},$$

which means that (x^*, y^*) is a global minimizer. In contrast to this standard approach, we switch our focus on the problem of finding points (x^*, y^*) of

$D_1 \times D_2$ with the property that

$$\frac{A(x^*, y^*)}{B(x^*, y^*)} = \min_{x \in D_1} \frac{A(x, y^*)}{B(x, y^*)}, \tag{1.29}$$

$$\frac{A(x^*, y^*)}{B(x^*, y^*)} = \min_{y \in D_2} \frac{A(x^*, y\ )}{B(x^*, y\ )}. \tag{1.30}$$

Points satisfying (1.29) and (1.30) are called partial minimizers for problem (P).

When dealing with fractional optimization problems, one famous method for computing global minimizers is the Dinkelbach algorithm (see Dinkelbach [20] and the references Crouzeix and Ferland [15, 16], Orzan and Precup [62], Ródenas, López and Verastegui [69], Shi [77] and Tammer [81]).

In order to obtain partial minimizers, we use a modified version of the Dinkelbach algorithm, which relies on solving at each iteration step two parametric problems of the following type

$$A(x, y) - \lambda B(x, y) \to \min_{x \in D_1}, \tag{$P_y(\lambda)$}$$

$$A(x, y) - \lambda B(x, y) \to \min_{y \in D_2}, \tag{$P_x(\lambda)$}$$

where $\lambda \in \mathbb{R}$ is a parameter. Our algorithmic procedure generates three sequences $(\lambda_k), (x_k)$ and (y_k), where x_k is a solution of the problem $(P_{y_{k-1}}(\lambda_{k-1}))$, y_k is a solution of $(P_{x_k}(\lambda_{k-1}))$ and

$$\lambda_k = \frac{A(x_k, y_k)}{B(x_k, y_k)}.$$

Under some appropriate conditions on the fractional optimization problem, the convergence of the sequences is ensured and we obtain that the point (x^*, y^*), where x^* and y^* are the limits of (x_k) and (y_k), is a partial minimizer of $\frac{A}{B}$, while the limit of (λ_k) equals the value $\dfrac{A(x^*, y^*)}{B(x^*, y^*)}$.

4.2 Relationships between global minimizers and partial minimizers

In the subsequent section, we outline the connections between global minimizers, partial minimizers, and critical points. Specifically, we prove that under certain conditions on the objective function, partial minimizers coincide with global minimizers. Furthermore, we investigate the scenario in which functions A and B involved in the fractional problem have seperate variables. Eventually, we deliver a result in the same spirit as the previous ones for the particular framework of finite dimensional Euclidean spaces.

Proposition 4.2.1. *Let X and Y be two real normed spaces and function $f : X \times Y \to \mathbb{R}$. Then:*

1° *Any global minimizer of f is a partial minimizer, but the converse implication is not generally true, even in the case when the function is convex.*

2° *If f is a C^1 Fréchet differentiable function, then any partial minimizer is a critical point.*

3° *If f is convex, then (x, y) is a global minimizer if and only if the origin belongs to the subdifferential of f at (x, y), i.e., $0 \in \partial f(x, y)$. In particular, if f is convex and C^1 Fréchet differentiable, then any partial minimizer is a global minimizer.*

The next result comes as a sufficient condition under which the partial minimizer points are global minimizers, for a fractional function $\dfrac{A}{B}$.

Proposition 4.2.2. *If $A, B : D_1 \times D_2 \to \mathbb{R}$ have separate variables in the sense that*

$$A(x, y) = A_1(x) + A_2(y), \quad B(x, y) = B_1(x) + B_2(y)$$

then any partial minimizer is a global minimizer of the ratio $A(x, y)/B(x, y)$.

Remark 9. *In the particular case of finite dimensional Euclidean spaces, the following relationships can be established between global minimizers, partial minimizers and critical points for our fractional optimization problem.*

Proposition 4.2.3. *Assume that $D_1 \subseteq \mathbb{R}^q$, $D_2 \subseteq \mathbb{R}^p$ are nonempty, open and convex sets, $A, B : D_1 \times D_2 \to \mathbb{R}$ are two Fréchet differentiable functions, B is positive. Consider the following assertions:*

1° *(x^*, y^*) is a global minimizer of $\frac{A}{B}$.*

2° *(x^*, y^*) is a partial minimizer of $\frac{A}{B}$.*

3° *(x^*, y^*) is a critical point of $\frac{A}{B}$, i.e., $\nabla \frac{A}{B}(x^*, y^*) = 0$.*

Then $1° \implies 2° \implies 3°$. If at least one of the following assumptions holds true:

(i) A is convex and B is affine,

(ii) A is non-negative and convex, and B is concave,

then $1° \iff 2° \iff 3°$.

4.3 The componentwise Dinkelbach algorithm

In this section, we will present the componentwise Dinkelbach algorithm, designed to compute partial minimizers for fractional optimization problems. We begin by restating the original Dinkelbach algorithm, which is applied to objective functions of the form $\dfrac{A}{B}$ and depends on two variables. This review will provide us with a clearer comprehension of our subsequent componentwise algorithm. We also contextualize some of the weaknesses inherent to our algorithm and offer practical suggestions to mitigate these limitations.

Algorithm 4.3.1 (Dinkelbach's algorithm).

Step 0 **(initialization)** Take any point $(x_0, y_0) \in D_1 \times D_2$, calculate

$$\lambda_0 := \frac{A(x_0, y_0)}{B(x_0, y_0)}$$

and set $k = 1$.

Step k **(cycle step for** $k \geq 1$**)** Find a point $(x_k, y_k) \in D_1 \times D_2$ such that

$$A(x_k, y_k) - \lambda_{k-1} B(x_k, y_k) = \min_{D_1 \times D_2} (A - \lambda_{k-1} B),$$

calculate

$$\lambda_k = \frac{A(x_k, y_k)}{B(x_k, y_k)}$$

and perform Step $k + 1$.

Remark 10. *It is known that, under certain assumptions on the objective function, both sequences* (x_k) *and* (y_k) *generated by the Dinkelbach algorithm are convergent to* x^* *and* y^**, and the point* (x^*, y^*) *is a global minimizer of* A/B *on* $D_1 \times D_2$ *(see Dinkelbach [20]).*

We are going to give now our modified version of the Dinkelbach algorithm for computing partial minimizers of fractional optimization problems.

Algorithm 4.3.2 (Componentwise Dinkelbach algorithm).

Step 0 **(initialization)** Take any point $(x_0, y_0) \in D_1 \times D_2$, calculate

$$\lambda_0 := \frac{A(x_0, y_0)}{B(x_0, y_0)}$$

and set $k = 1$.

Step k (cycle step) Find a point $x_k \in D_1$ such that

$$\min_{x\in D_1} [A(\cdot, y_{k-1}) - \lambda_{k-1}B(\cdot, y_{k-1})] = A(x_k, y_{k-1}) - \lambda_{k-1}B(x_k, y_{k-1}), \tag{1.31}$$

then find $y_k \in D_2$ such that

$$\min_{y\in D_2} [A(x_k, \cdot) - \lambda_{k-1}B(x_k, \cdot)] = A(x_k, y_k) - \lambda_{k-1}B(x_k, y_k), \tag{1.32}$$

next calculate

$$\lambda_k := \frac{A(x_k, y_k)}{B(x_k, y_k)} \tag{1.33}$$

and perform Step $k + 1$.

Proposition 4.3.3. *The minimization problems given by* (1.31) *and* (1.32) *have solutions if any of the following conditions is fulfilled:*

(a) D_1, D_2 are compact sets, A is is lower semicontinuous (l.s.c.) in each variable and B is continuous in each variable.

(b) $D_1 = X$ and $D_2 = Y$ are finite-dimensional normed spaces, A is l.s.c. in each variable, coercive in x (i.e., $\lim_{|x|_X\to\infty} A(x, y) = \infty$ for every $y \in Y$), coercive in y (i.e., $\lim_{|y|_Y\to\infty} A(x, y) = \infty$ for every $x \in X$) and B is continuous in each variable.

(c) $D_1 = X$, $D_2 = Y$ are reflexive Banach spaces, A is both l.s.c. and convex in each variable, coercive in x, coercive in y and nonnegative, whereas B is both upper semicontinuous and concave in each variable.

4.4 Convergence of the componentwise Dinkelbach algorithm

The final part of this chapter offers a series of results concerning the convergence of our algorithm. It is shown that further assumptions on the involved spaces and functions, such as Lipschitz-type continuity, partial Fréchet differentiability, and coercivity, enable us to establish sufficient conditions for the convergence of Algorithm 4.3.2 to a partial minimizer.

The next theorem demonstrates the convergence of the sequence (λ_k) given by our Algorithm 4.3.2.

Theorem 4.4.1. *The sequence (λ_k) given by Algorithm 4.3.2 is nonincreasing and convergent to a real number λ^*.*

Theorem 4.4.2. *Assume that* D_1, D_2 *are closed subsets of the normed spaces* X *and* Y, *respectively, the operators* A *and* B *are continuous, and*

$$x_k \to x^*, \qquad y_k \to y^*. \tag{1.34}$$

Then the following assertions hold true:

(a) One has

$$\lambda^* = \frac{A(x^*, y^*)}{B(x^*, y^*)}.$$

(b) Denoting

$$m_{k,1} := A(x_k, y_{k-1}) - \lambda_{k-1} B(x_k, y_{k-1}), \quad m_{k,2} := A(x_k, y_k) - \lambda_{k-1} B(x_k, y_k),$$

one has

$$\lim_{k\to\infty} m_{k,1} = \lim_{k\to\infty} m_{k,2} = 0.$$

(c) (x^*, y^*) *is a partial minimizer on* $D_1 \times D_2$ *of* $A(x, y)/B(x, y)$.

In our way to ensure that condition (1.34) takes place, we first proceed to guarantee the boundedness of one of the sequences (x_k) and (y_k). To this aim, we are going to assume that $D_1 = X$, $D_2 = Y$ are entire normed spaces and the following hypothesis holds true:

(H1) The functional $A(x, y)$ is coercive in x uniformly w.r.t. y, i.e., for any $M > 0$, there is $l_M > 0$ such that $A(x, y) \geq M$ for $|x|_X \geq l_M$ and all $y \in Y$.

Proposition 4.4.3. *Under assumption (H1), the sequence* (x_k) *is bounded.*

In the following part of the section we proceed with the essential step of ensuring the convergence of the sequences $(x_k), (y_k)$, which will conclude the convergence of our Algorithm 4.3.2. In order to achieve this, we consider that X and Y are Hilbert spaces, $D_1 = X$, $D_2 = Y$ and A, B are C^1 functionals with their partial (Fréchet) derivatives (see, e.g., Precup [66]) satisfying some Lipschitz continuity or monotonicity conditions related to their derivatives.

We denote by A'_x, A'_y, B'_x and B'_y the partial Fréchet derivatives of A and B, and we assume that

(H2) The derivatives B'_x and B'_y are bounded on $X \times Y$.

Now we define the operators

$$N_{11}(x,y) := c_1 x - A'_x(x,y), \qquad N_{12}(x,y) := c_1 y - A'_y(x,y),$$
$$N_{21}(x,y) := c_2 x - B'_x(x,y), \qquad N_{22}(x,y) := c_2 y - B'_y(x,y),$$

where

$$c_1 \geq \max\{1, \lambda_0\} + \lambda_0 c_2 \quad \text{and} \quad c_2 > 0. \tag{1.35}$$

Furthermore we also assume that

(H3) There exist some positive constants $a_{ij}, b_{ij} \in \mathbb{R}$, $i, j \in \{1,2\}$, such that

$$\begin{aligned} |N_{11}(x,y) - N_{11}(\bar{x},\bar{y})|_X &\leq a_{11}|x-\bar{x}|_X + a_{12}|y-\bar{y}|_Y, \\ |N_{12}(x,y) - N_{12}(\bar{x},\bar{y})|_Y &\leq a_{21}|x-\bar{x}|_X + a_{22}|y-\bar{y}|_Y, \\ |N_{21}(x,y) - N_{21}(\bar{x},\bar{y})|_X &\leq b_{11}|x-\bar{x}|_X + b_{12}|y-\bar{y}|_Y, \\ |N_{22}(x,y) - N_{22}(\bar{x},\bar{y})|_Y &\leq b_{21}|x-\bar{x}|_X + b_{22}|y-\bar{y}|_Y \end{aligned}$$

for all $x, \bar{x} \in X$ and $y, \bar{y} \in Y$.

We denote

$$c_{11} := a_{11} + b_{11},\ c_{12} := a_{12} + b_{12},\ c_{21} := a_{21} + b_{21},\ c_{22} := a_{22} + b_{22},$$
$$a := \frac{c_{12}c_{21}}{(1-c_{11})(1-c_{22})}$$

and we assume that

(H4) $c_{11} < 1,\ c_{22} < 1$ and $a < 1$.

We now present a sufficient condition for the convergence of the sequences (x_k) and (y_k), reminiscent of Banach's fixed point theorem.

Theorem 4.4.4. *Under the assumptions (H1)-(H4), the sequences (x_k) and (y_k) are convergent.*

Remark 11. *Taking into account Theorem 4.4.2, under the assumptions of Theorem 4.4.4, if x^* and y^* are the limits of the sequences (x_k) and (y_k) given by the componentwise Dinkelbach algorithm, then (x^*, y^*) is a partial minimizer for $A(x,y)/B(x,y)$.*

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