BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Matrices of von Neumann regular type

PhD Thesis Summary

Scientific adviser
Prof. Dr. Septimiu Crivei

PhD Student
Iulia-Elena Chiru

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Contents

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Introduction

The concept of a (von Neumann) regular ring was introduced by von Neumann in his famous work [70] as an algebraic tool for studying certain lattices, which were useful in the coordinatization of projective geometry. Since then, there have been many applications of them, in different mathematical branches, such as algebra, functional analysis, differential equations, statistics, probabilities or cryptography. An exhaustive presentation of von Neumann regular rings was first given in Goodearl's monograph [38].

Recall that a ring R is called *von Neumann regular* if every element $a \in R$ is *von Neumann regular*, i.e., there is an element $b \in R$, called an *inner inverse* or *generalized inverse* of a, such that $a = aba$. The definition can be naturally extended to matrices over a ring R as follows: an $m \times n$-matrix A is called *von Neumann regular* if there is an $n \times m$-matrix B such that $A = ABA$, and in this case B is called an *inner inverse* or *generalized inverse* of A. The inner inverse of a matrix has immediate applications in solving linear systems $Ax = b$, when A is singular or rectangular. If the system has a solution y and A is von Neumann regular with an inner inverse B, then $x = Bb$ is a solution, as $Ax = ABb = ABAy = Ay = b$ (e.g., see [12]).

A classical theorem of von Neumann states that every matrix over a von Neumann regular ring is still von Neumann regular and, in particular, every matrix over a field is von Neumann regular. The problem of characterizing von Neumann regular matrices and their generalized inverses over commutative rings was raised by Bhaskara Rao [13] and, in this direction we mention, among many other papers, the works of Bapat, Bhaskara Rao and Prasad [10], Prasad [65], Lam and Swan [49], and the monographs by Ben-Israel and Greville [12] and Bhaskara Rao [14], which contain several important characterizations. This problem has useful applications to control theory, systems theory in polynomial matrices as well as operator algebras (e.g. see [14]).

Von Neumann regularity has a categorical generalization given by Dăscălescu, Năstăsescu, Tudorache and Dăuş [33] as follows: for two objects M and N of an arbitrary category, N is called M-regular if every morphism $f : M \to N$ is regular in the sense that there is a morphism $g : N \to M$ such that $f = fgf$. Regular morphisms in the category of modules have been considered by Kasch and Mader [43]. Recently, regular objects and morphisms in abelian categories have also been studied by Crivei and Kör [32] and Crivei, Koşan and Yildirim [31].

These raised our interest in a further investigation of von Neumann regularity for matrices and its related notions of strong regularity and outer regularity, which will be all studied throughout the present work. The thesis is structured in four chapters and an appendix, which will be described in what follows.

Our interest in the first chapter has been to find a practical criterion for checking von

Neumann regularity of matrices over certain commutative rings as well as for counting them in some finite cases. Our approach can be related to the work of Lam and Swan [49], who gave a characterization of von Neumann regular square matrices over commutative rings in terms of their associated determinantal ideals. In Theorem 1.2.1 we prove that if A is a non-zero $m \times n$-matrix with determinantal rank t over a commutative ring such that A has an invertible $t \times t$-submatrix A', then A is von Neumann regular, and an inner inverse of A can be constructed by using the inverse of A'. Conversely, in Theorem 1.3.2, under certain conditions on the ring R, if A is a non-zero von Neumann regular $m \times n$-matrix with determinantal rank t over a local commutative ring R, then A has an invertible $t \times t$-submatrix. In Theorem 1.3.3, we deduce that, under certain conditions, an $m \times n$-matrix over a local commutative ring is von Neumann regular if and only if its determinantal and McCoy ranks coincide. Based on the above results, we establish Theorem 1.4.2, which gives an intrinsic characterization of a non-zero $m \times n$-matrix A with determinantal rank $\rho(A) = t$ over a local commutative ring to be von Neumann regular, namely A must have an invertible $t \times t$-submatrix. The existence of an invertible $t \times t$-submatrix of A is equivalent to the existence of a unit in the t^{th} compound matrix $C_t(A)$ of A, which consists of all $t \times t$-minors of A.

We also derive consequences to arbitrary commutative rings and products of local commutative rings. In Theorem 1.5.1 we determine the number of von Neumann regular $m \times n$-matrices over a local finite ring R with maximal ideal M such that $|R/M| = |F_q| = q$ as $\sum_{t=0}^{\min(m,n)} |M|^{t(m+n-t)} r(m,n,q,t)$, where $r(m,n,q,t)$ is the number of $m \times n$-matrices over a field F_q having determinantal rank t. As applications, we count von Neumann regular $m \times n$-matrices over rings of residue classes $\mathbb{Z}_l$ and over group algebras $F_q[\mathbb{Z}_l]$ (Corollary 1.5.3), where F_q is a field with q elements whose characteristic divides l. Finally, we discuss von Neumann regular matrices over formal triangular matrix rings and we give the characterization Theorem 1.6.1.

An important subclass of von Neumann regular rings [70] consists of strongly regular rings, which were introduced by Arens and Kaplansky [6] and have been studied in ring theory. This is the topic of the second chapter of the thesis. A ring R with identity is called *strongly regular* if for every $a \in R$ there is $b \in R$ such that $a = a^2b$, and this definition turns out to be left-right symmetric. Restricting it to elements, $a \in R$ is called *strongly regular* if there is $b \in R$ such that $a = a^2b = ba^2$, and in this case b is called a *strong inner* (or *strong generalized) inverse* of a. For general properties of strongly regular rings we refer to [38]. The above definitions may also be given for matrices over some ring R: an $n \times n$-matrix A is called *strongly regular* if there is an $n \times n$-matrix B such that $A = A^2B = BA^2$. Note that matrices over fields need not be strongly regular, as it is the case with von Neumann regularity, so the problem of characterizing strongly regular matrices does already make sense over fields.

We point out that the definitions of von Neumann regular and strongly regular matrices are not intrinsic, but depend on the existence of other matrices with the required properties. Practical applications might involve checking von Neumann regularity or strong regularity of matrices of large size, and this becomes a time-consuming computational problem. Hence it is of interest to have intrinsic characterizations of such properties. In this direction, a natural question is whether a strongly regular matrix with determinantal rank t over a local commutative ring may also be characterized in terms of some property of its associated t^{th} compound matrix.

In the second chapter we answer this in the affirmative. We first establish a more general result over an arbitrary commutative ring R by using the reduced Cayley-Hamilton Theorem (Theorem 2.1.1). For a non-zero matrix $A \in M_n(R)$ with determinantal rank t, we prove that if A is strongly regular, then the trace $\mathrm{Tr}(C_t(A))$ of its t^{th} compound matrix does not belong to the radical of R, while if $\mathrm{Tr}(C_t(A))$ is a unit of R, then A is strongly regular (Theorem 2.1.3). In particular, this implies that a non-zero matrix A with determinantal rank t over a local commutative ring R is strongly regular if and only if the sum of its diagonal $t \times t$-minors is a unit in R, or equivalently, the trace of its t^{th} compound matrix is a unit in R. Moreover, in this case we construct a strong inner inverse of A as $B = -c_t^{-1}(A^{t-1} + c_1 A^{t-2} + \cdots + c_{t-1} I_n)$, where $c_k = (-1)^k \mathrm{Tr}(C_k(A))$ for every $k \in \{1, \ldots, n\}$ (Theorem 2.1.5).

This result has consequences to direct products of local commutative rings and group algebras (e.g., see Corollary 2.2.3). It allows us in Theorem 2.3.1 to count strongly regular matrices over local finite rings. In particular, the number of strongly regular $n \times n$-matrices of determinantal rank t over R is given by $\frac{|GL_n(R)|}{|GL_{n-t}(R)|}$. We also deduce counting results over rings of residue classes $\mathbb{Z}_l$ and over some group algebras $F_q[G]$ (Theorem 2.4.1), where F_q is a field with q elements and G is a group with l elements. Then we deal with strong inner inverses and strong reflexive inverses in arbitrary rings, and we show in Theorem 2.5.2 that if a and b are strongly regular elements of a semiprime ring R having disjoint sets of inner inverses $S(a)$ and $S(b)$ respectively, then $S(a) \subseteq S(b)$ if and only if $b^2 = ab = ba$. Finally, we characterize strongly regular matrices over formal triangular matrix rings (Theorem 2.6.1).

As a natural extension of our research, in chapter three we continued with the study of outer inverses. An element b in a ring R is called an *outer inverse* of $a \in R$ if $bab = b$, which definition can also be applied to matrices: an $n \times m$-matrix B is called an *outer inverse* of an $m \times n$-matrix A if $BAB = B$. If A is a von Neumann regular $m \times n$-matrix with inner inverse $n \times m$-matrix B, then it is well known and easy to see that BAB is an outer inverse of A. Clearly, if A is non-zero von Neumann regular, then it has a non-zero outer inverse. In the first two chapters of our thesis, we have established some intrinsic characterizations of (strongly) von Neumann regular matrices over commutative rings as well as some related counting results. Now we consider the more general class of matrices having non-zero outer inverses over arbitrary rings, and we look for some intrinsic descriptions of such matrices.

We first analyze matrices having non-zero outer inverse in the general case of an arbitrary ring. Thus, we prove that the existence of an entry having a non-zero outer inverse ensures that the matrix A has a non-zero outer inverse, which in turn implies that A must have an entry outside the Jacobson radical of the ring (Theorem 3.1.1). We show that these conditions are equivalent and they provide a constructive criterion for matrices having a non-zero outer inverse in the case of a large class of rings, namely the class of semiperfect rings. Examples of semiperfect rings include local rings, one-sided artinian rings, semiprimary rings and one-sided perfect rings. Our construction successively considers local rings, direct products of local rings (Theorem 3.2.2), and finally, semiperfect rings (Theorem 3.3.1). During the process, we show a result of possible independent interest, namely that elements having a non-zero outer inverse lift modulo one-sided ideals of exchange rings (Theorem 3.1.4 and Corollary 3.1.5).

We also count matrices having a non-zero outer inverse over finite semiperfect rings and finite commutative rings (Proposition 3.4.1), and we give several applications to rings of residue classes, products of Galois rings, quaternion rings over rings of residue classes and finite group algebras. Such results may also have applications to cryptography, by describing and counting the elements of the key space of some cryptosystems, in a similar way as for von Neumann regular matrices, e.g., see the key exchange protocol and the public key encryption with keyword search scheme from [56]. Finally, we characterize matrices having outer inverses over formal triangular matrix rings (Theorem 3.5.1).

Having studied the concepts of von Neumann regular matrices, strongly regular matrices and matrices having a non-zero outer inverse, in our fourth and last chapter of our thesis we derive applications to some generalizations of these. One such generalization is called a *von Neumann local* matrix, which is a matrix $A \in M_n(R)$ such that A or $I_n - A$ is von Neumann regular. This is inspired by the corresponding ring-theoretic concept introduced by Contessa [29]. In this direction, in Theorem 4.1.3, we deduce that A is von Neumann local, with $\rho(A) = t$ and $\rho(I_n - A) = s$, if and only if $C_t(A)$ or $C_s(I_n - A)$ is von Neumann regular. We also upscale such results to direct products of arbitrary rings (or local commutative rings) (e.g., Theorem 4.1.6).

A specialization of the notion of von Neumann local matrix is that of *strongly von Neumann local* matrix, which is an $n \times n$-matrix A such that A or $I_n - A$ is strongly regular. We show in Corollary 4.2.2 that there is a rich supply of such matrices, since for every $A \in M_n(R)$ over a commutative ring R with $\rho(A) = t$, $C_t(A)$ is strongly von Neumann local. A characterization of these matrices over a commutative ring R is given in Theorem 4.2.3 by $C_t(A)$ or $C_s(I_n - A)$ being strongly regular, where $t = \rho(A)$ amd $s = \rho(I_n - A)$. Generalizing further the concept of von Neumann local matrix, we consider the notion of *outer von Neumann local* matrix, which is a matrix $A \in M_n(R)$ such that A or $I_n - A$ has a non-zero outer inverse. We prove in Theorem 4.3.1 that every $A \in M_n(R)$ over an arbitrary semiperfect ring R is outer von Neumann local. When R is an arbitrary local ring, then $A \in M_n(R)$ is outer von Neumann local if and only if A or $I_n - A$ has one of the following properties: it has an invertible entry, or it has an entry with a non-zero outer inverse, or it does not have elements in the Jacobson radical of R (Theorem 4.3.3).

Finally, as we have all these intrinsic characterizations for von Neumann regular, strongly regular and outer regular matrices, developing algorithms for checking those properties for some matrices may be useful. Thus, in the appendix of our thesis, we illustrate some efficient algorithms for matrices over rings of residue classes, along with their implementations in Python and some relevant higher order examples computed with them.

Except for the cited results, all other results in our thesis are our original work and included in our papers [15, 23, 24, 25, 26, 27]. Five of the papers have been published in the journals *Linear and Multilinear Algebra, Linear Algebra and Its Applications, Electronic Journal of Linear Algebra* and *Mathematica*. Also, the main results of the thesis were presented in three conferences.

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Chapter 1

Von Neumann regular matrices

We give a constructive sufficient condition for a matrix over a commutative ring to be von Neumann regular, and we show that it is also necessary over certain local rings. Specifically, under some hypothesis on the ring R, we prove that a matrix A over a local commutative ring R is von Neumann regular if and only if A has an invertible $\rho(A) \times \rho(A)$-submatrix, where $\rho(A)$ is the determinantal rank of A. We deduce consequences to (products of local) commutative rings, and we determine the number of von Neumann regular matrices over some finite rings of residue classes and group algebras. We also discuss von Neumann regular matrices over formal triangular matrix rings. Except for the cited results, all other results are original and are mostly included in our papers [15, 24].

1.1 Preliminaries

Let us recall some terminology on (von Neumann regular) matrices over commutative rings following some classical sources, such as [12, 14, 18, 38].

Throughout the chapter $m, n \geq 2$ will be two integers, and R will be a commutative ring with identity. We denote by $M_{m,n}(R)$ the set of all $m \times n$-matrices over R, and by $M_n(R)$ the set of all $n \times n$-matrices over R. Let $A \in M_{m,n}(R)$. Given subsets $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}$ with $i_1 < \cdots < i_k$ and $J = \{j_1, \dots, j_l\} \subseteq \{1, \dots, n\}$ with $j_1 < \cdots < j_l$, we denote by $A_{I,J}$ the submatrix of A whose rows and columns are indexed by the sets I and J respectively. For each $k \in \{1, \dots, \min(m,n)\}$, the *kth compound matrix* of A is defined as the matrix $C_k(A) \in M_{m',n'}(R)$, where $m' = \binom{m}{k}$ and $n' = \binom{n}{k}$, consisting of the $k \times k$-minors of A, where for every $I' = \{i'_1, \dots, i'_k\}$ with $i'_1 < \cdots < i'_k$ and $J' = \{j'_1, \dots, j'_k\}$ with $j'_1 < \cdots < j'_k$, the (I', J') entry of $C_k(A)$ is $\det(A_{I',J'})$.

For each $k \in \{1, \dots, r = \min(m,n)\}$, $\mathcal{D}_k(A)$ will denote the ideal of R generated by all $k \times k$-minors of A (i.e., all entries of the compound matrix $C_k(A)$), and will be called the *kth determinantal ideal* of A. We have the following ascending chain of ideals in R: $\mathcal{D}_r(A) \subseteq \mathcal{D}_{r-1}(A) \subseteq \cdots \subseteq \mathcal{D}_2(A) \subseteq \mathcal{D}_1(A) \subseteq \mathcal{D}_0(A) = R$.

The *McCoy rank* of A, denoted by $\mathrm{rk}(A)$, is defined as the largest $k \in \{0, \dots, r\}$ for which the determinantal ideal $\mathcal{D}_k(A)$ is faithful, that is: $\mathrm{rk}(A) = \max\{k \in \{0, \dots, r\} \mid \mathrm{Ann}_R(\mathcal{D}_k(A)) = (0)\}$. In other words, the McCoy rank of $A = (a_{ij}) \in M_{m,n}(R)$ is zero if there is a non-zero $c \in R$ such that $ca_{ij} = 0$ for every $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ and, otherwise, the greatest positive integer k with the property that if $c \in R$ is such that $c\det(A') = 0$ for every $k \times k$-submatrix A' of A, then $c = 0$.

The *determinantal rank* of a non-zero $A \in M_{m,n}(R)$, denoted by $\rho(A)$, will be the maximal order of a submatrix of A with non-zero determinant. The determinantal rank of the zero matrix will be zero. For every $A \in M_{m,n}(R)$, one has $\mathrm{rk}(A) \leq \rho(A)$, but in general the determinantal rank and the McCoy rank of a matrix are different.

A ring R is called *local* if it has a unique maximal right ideal. We denote by $\mathrm{rad}(R)$ the Jacobson radical of R, that is, the intersection of its maximal right ideals, and by $U(R)$ the set of units of R.

1.2 Sufficient conditions

We begin with a sufficient condition for an $m \times n$-matrix over a commutative ring to be von Neumann regular, which is of practical interest.

Theorem 1.2.1. *Let $A \in M_{m,n}(R)$ be a non-zero matrix with $\rho(A) = t$. If A has a submatrix $A_{I,J} \in U(M_t(R))$ for some $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$, then A is von Neumann regular. Moreover, an inner inverse of A is the matrix $B \in M_{n,m}(R)$, where $B_{J,I} = A_{I,J}^{-1}$ and the other entries of B are zero.*

We illustrate Theorem 1.2.1 as follows.

Example 1.2.2. The matrix $A = \begin{pmatrix} 0 & 2 & 2 & 6 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 0 \end{pmatrix} \in M_{3,4}(\mathbb{Z})$ has $\rho(A) = \mathrm{rk}(A) = 2$. Let $I = \{2, 3\}$ and $J = \{1, 3\}$. Then $\det(A_{I,J}) = -1$ is invertible in $\mathbb{Z}$. Hence A is von Neumann regular by Theorem 1.2.1, and an inner inverse B of A may be constructed by taking $B_{J,I} = A_{I,J}^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$, and then $B = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix} \in M_{4,3}(\mathbb{Z})$.

One may also use the following extension of [14, Theorem 5.3] in order to give an immediate alternative proof of Theorem 1.2.1.

Theorem 1.2.3. *Let $A = (a_{ij}) \in M_{m,n}(R)$ be such that $\rho(A) = t$. Denote $\Delta = \{(I, J) \mid I \subseteq \{1, \dots, m\}, J \subseteq \{1, \dots, n\}, |I| = |J| = t\}$. Consider the statements:*

(o) *There exists $(I, J) \in \Delta$ such that $\det(A_{I,J}) \in U(R)$.*

(i) *There is a family $(c_{J,I})_{(I,J)\in\Delta}$ of elements of R such that $\sum_{(I,J)\in\Delta} \det(A_{I,J})c_{J,I} = 1$.*

(ii) *There is a family $(c_{J,J})_{(I,J)\in\Delta}$ of elements of R such that $\left(\sum_{(I,J)\in\Delta} \det(A_{I,J})c_{J,I}\right) a_{kl} = a_{kl}$ for every $k \in \{1, \dots, m\}$ and $l \in \{1, \dots, n\}$.*

(iii) *A is von Neumann regular.*

(iv) *$C_t(A)$ is von Neumann regular.*

(v) *There exists a family* $(c_{J,I})_{(I,J)\in\Delta}$ *of elements of* R *such that* $\det(A_{K,L})\left(\sum_{(I,J)\in\Delta}\det(A_{I,J})c_{J,I}\right) = \det(A_{K,L})$ *for every* $(K,L)\in\Delta$.

(vi) *There exists a family* $(c_{J,I})_{(I,J)\in\Delta}$ *of elements of* R *such that* $\sum_{(I,J)\in\Delta}\det(A_{I,J})c_{J,I}$ *is a non-zero idempotent.*

Then (o) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi)*. If* R *is a local ring, then all statements are equivalent.*

In the case of certain rings we may deduce another sufficient condition for a matrix to be von Neumann regular. Recall that a ring R has *finite uniform dimension* if there is a finite direct sum of uniform ideals of R which is essential in R. Also, recall that a ring R is called *morphic* if $R/Ra \cong \mathrm{Ann}_R(a)$ for every $a \in R$, or equivalently, for every $a \in R$, there is $b \in R$ such that $Ra = \mathrm{Ann}_R(b)$ and $Rb = \mathrm{Ann}_R(a)$ [59].

Corollary 1.2.4. *Let* R *be a local ring that satisfies one of the following conditions:*

(i) *Every element of* R *is either a unit or nilpotent.*

(ii) R *has finite uniform dimension and every element of* R *is either a unit or a zero divisor.*

(iii) R *is a morphic ring.*

Let $A \in M_{m,n}(R)$ *be such that* $\rho(A) = \mathrm{rk}(A)$*. Then* A *is von Neumann regular.*

1.3 A necessary condition

The converse of Theorem 1.2.1 does not hold in general, as we may see next.

Example 1.3.1. Following [49, Example 3.3 (C)], consider $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 2 \\ -2 & -8 & -4 \end{pmatrix} \in M_3(\mathbb{Z})$, which has $\rho(A) = 2$. Then A is von Neumann regular, but it does not have any invertible 2×2-submatrix.

In what follows we are interested in finding some conditions under which the converse of Theorem 1.2.1 is true, in the same time having in mind to obtain some characterization of von Neumann regular matrices with some significant computational consequences.

Theorem 1.3.2. *Let* R *be local, and let* $A \in M_{m,n}(R)$ *be a non-zero matrix with* $\rho(A) = t$*. If* A *is von Neumann regular, then* A *has an invertible* $t\times t$*-submatrix.*

It is well known that $\mathrm{rk}(A) \le \rho(A)$ for every $A \in M_{m,n}(R)$, and the equality holds when R is a field. Theorem 1.3.2 gives another instance of the equality of the two ranks.

Corollary 1.3.3. *Let* R *be local, and let* $A \in M_{m,n}(R)$ *be von Neumann regular. Then* $\rho(A) = \mathrm{rk}(A)$.

1.4 Characterizations

First, let us note that [61, Chapter 4, Theorem 18] of Northcott (in the language of finite free modules), [14, Theorem 7.19] of Bhaskara Rao and [49, Theorem 3.2] of Lam and Swan provide necessary and sufficient conditions for a matrix over a commutative ring to be von Neumann regular. We state their result as follows and we give an alternative proof for one of the implications.

Theorem 1.4.1. *A matrix* $A \in M_{m,n}(R)$ *is von Neumann regular if and only if for each* $k \in \{1, \ldots, \min(m,n)\}$, $\mathcal{D}_k(A)$ *is generated by an idempotent.*

Theorems 1.2.1 and 1.3.2 and their Corollaries 1.2.4 and 1.3.3 yield the following characterization of von Neumann regular $m \times n$-matrices over a local commutative ring.

Theorem 1.4.2. *Let* R *be local, and let* $A \in M_{m,n}(R)$. *Then the following are equivalent:*

(1) A *is von Neumann regular.*

(2) A *is either zero or* A *has an invertible* $\rho(A) \times \rho(A)$*-submatrix.*

If R *satisfies (i), (ii) or (iii) from Corollary 1.2.4, then they are further equivalent to:*

(3) $\rho(A) = \mathrm{rk}(A)$.

Recall that $A \in M_{m,n}(R)$ will be denoted by $A_\mathfrak{p}$ when viewed over the localization $R_\mathfrak{p}$ of R at a prime ideal $\mathfrak{p}$. We have the following consequence of Theorem 1.4.2.

Theorem 1.4.3. *Let* $A \in M_{m,n}(R)$. *Consider the statements:*

(1) A *is von Neumann regular.*

(2) For every prime (maximal) ideal $\mathfrak{p}$ *of* R, $A_\mathfrak{p} \in M_{m,n}(R_\mathfrak{p})$ *is either zero or has an invertible* $\rho(A_\mathfrak{p}) \times \rho(A_\mathfrak{p})$*-submatrix.*

Then (1) $\Longrightarrow$ (2).

If $R_\mathfrak{p}$ *satisfies (i), (ii) or (iii) from Corollary 1.2.4 for every prime (maximal) ideal* $\mathfrak{p}$ *of* R, *then* (2) *is further equivalent to:*

(3) For every prime (maximal) ideal $\mathfrak{p}$ *of* R, $\rho(A_\mathfrak{p}) = \mathrm{rk}(A_\mathfrak{p})$.

Theorem 1.4.2 may be extended to $m \times n$-matrices over products of local commutative rings. It is well known and easy to see that von Neumann regularity is well behaved with respect to direct products.

Corollary 1.4.4. *Let* $R = \prod_{k \in K} R_k$ *be a direct product of local commutative rings, and let* $0_{m,n} \neq A \in M_{m,n}(R)$. *For every* $k \in K$, *denote by* $h_k : M_{m,n}(R) \to M_{m,n}(R_k)$ *the canonical projection. Then the following are equivalent:*

(1) A *is von Neumann regular.*

(2) For every $k \in K$, *either* $h_k(A) = 0_{m,n}$ *or* $h_k(A)$ *has an invertible* $t_k \times t_k$*-submatrix, where* $t_k = \rho(h_k(A))$.

If R_k satisfies (i), (ii) or (iii) from Corollary 1.2.4 for every $k \in K$, then they are further equivalent to:

(3) For every $k \in K$, $\rho(h_k(A)) = \mathrm{rk}(h_k(A))$.

Recall that a ring R is called *semiperfect* if $R/\mathrm{rad}(R)$ is semisimple artinian and idempotents lift modulo $\mathrm{rad}(R)$. A commutative ring is semiperfect if and only if it is a finite direct product of commutative local rings [46, Theorem 23.11]. Hence the above corollary is applicable to any commutative semiperfect ring. In particular, it applies to any finite commutative ring, because every such ring is a finite direct product of local finite commutative rings (e.g., see [52, Theorem (VI.2)], hence it is semiperfect.

Theorem 1.4.5. *Let $R = \prod_{k \in K} R_k$ be a direct product of local commutative rings. Let $A \in M_{m,n}(R)$ with $\rho(A) = t$. Then A is von Neumann regular if and only if $C_t(A)$ is von Neumann regular.*

1.5 Counting von Neumann regular matrices

Von Neumann regular elements in rings of residue classes were characterized by Morgado [54], and their number has been determined by Alkam, Osba [3] and Tóth [69]. Also, Castillo-Ramirez and Gadouleau have determined the number of von Neumann regular elements of certain group algebras in their study of von Neumann cellular automata [19]. As applications of Theorem 1.4.2 we generalize the cited results to matrices over such rings, and we determine the numbers of von Neumann regular $m \times n$-matrices over rings of residue classes $\mathbb{Z}_l$ and over group algebras $F_q[\mathbb{Z}_l]$, where F_q is a field such that its characteristic $\mathrm{char}(F_q)$ divides l. In fact, we prove a more general result and we deduce these numbers as particular cases.

Denote by $r(m, n, q, t)$ the number of $m \times n$-matrices over a field F_q with q elements having (determinantal) rank $t \in \{0, \dots, \min(m, n)\}$. Then $r(m, n, q, 0) = 1$ and for every $t \in \{1, \dots, \min(m, n)\}$ we have:

$$r(m,n,q,t) = \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{t-1})(q^n-1)(q^n-q)\cdots(q^n-q^{t-1})}{(q^t-1)(q^t-q)\cdots(q^t-q^{t-1})}$$

by [55, 1.7].

Theorem 1.5.1. *Let R be a local finite ring with maximal ideal M such that $|R/M| = |F_q| = q$. Then the number of von Neumann regular $m \times n$-matrices over R is*

$$V(M_{m,n}(R)) = \sum_{t=0}^{\min(m,n)} |M|^{t(m+n-t)} r(m,n,q,t).$$

Having in mind that von Neumann regularity behaves well with respect to direct products, we have at once the following corollary, which is furthermore applicable to commutative semiperfect rings.

Corollary 1.5.2. *Let $R = \prod_{k=1}^{s} R_k$ be a direct product of local commutative finite rings R_k with maximal ideals M_k such that $|R_k/M_k| = |F_{q_k}| = q_k$ for every $k \in K$. Then the number of* *von Neumann regular $m \times n$-matrices over R is*

$$V(M_{m,n}(R)) = \prod_{k=1}^{s} \sum_{t=0}^{\min(m,n)} |M_k|^{t(m+n-t)} r(m,n,q_k,t).$$

Now consider group algebras $F_q[G]$, where F_q is a field with q elements and $G = \mathbb{Z}_l$.

Corollary 1.5.3. *Let $l \geq 2$ be an integer and let F_q be a finite field with q elements such that $\operatorname{char}(F_q)$ divides l. Let $x^l - 1 = p_1(x)^{r_1} \cdots p_s(x)^{r_s}$ for some distinct irreducible polynomials $p_1(x), \ldots, p_s(x) \in F_q[x]$ with degrees $d_1, \ldots, d_s$ respectively, and positive integers $r_1, \ldots, r_s$. Then the number of von Neumann regular $m \times n$-matrices over the group algebra $F_q[\mathbb{Z}_l]$ is*

$$V(M_{m,n}(F_q[\mathbb{Z}_l])) = \prod_{k=1}^{s} \sum_{t=0}^{\min(m,n)} q_k^{(r_k-1)t(m+n-t)} r(m,n,q_k,t),$$

where for every $k \in \{1, \ldots, s\}$ we have $q_k = q^{d_k}$.

In the end of this section we deal with the following question: *if a matrix over a finite ring is von Neumann regular, then how many inner inverses and how many reflexive inverses does it have?* Recall that a matrix is von Neumann regular if and only if it has an inner inverse if and only if it has a reflexive inverse. But we shall see that, for a given von Neumann regular matrix, the number of its inner inverses might be different of the number of its reflexive inverses.

Let us denote by $I(A)$ and $\operatorname{Ref}(A)$ the sets of all inner inverses and all reflexive inverses of a von Neumann regular matrix A respectively.

Theorem 1.5.4. *Let R be a finite local ring with maximal ideal M such that $|R/M| = |F_q| = q$. Consider the natural ring homomorphism $p : R \to R/M$ and the induced R-module homomorphism $h : M_{m,n}(R) \to M_{m,n}(R/M)$, $h((a_{ij})) = (a_{ij} + M)$. Let $A \in M_{m,n}(R)$ be a von Neumann regular matrix with $\rho(A) = t$.*

Then the number of inner inverses of A is $|I(A)| = |M|^{mn-t^2} \cdot |I(h(A))| = |R|^{mn-t^2}$, and the number of reflexive inverses of A is $|\operatorname{Ref}(A)| = |M|^{t(m+n-2t)} \cdot |\operatorname{Ref}(h(A))| = |R|^{t(m+n-2t)}$.

Corollary 1.5.5. *Let $R = \prod_{k=1}^{s} R_k$ be a direct product of local commutative finite rings. Let $A \in M_{m,n}(R)$ be a von Neumann regular matrix with $\rho(A) = t$. Then the number of inner inverses of A is $|I(A)| = \prod_{k=1}^{s} |R_k|^{mn-t^2}$ and the number of reflexive inverses of A is $|\operatorname{Ref}(A)| = \prod_{k=1}^{s} |R_k|^{t(m+n-2t)}$.*

1.6 Von Neumann regular formal triangular matrix rings

Let us see the behaviour of von Neumann regularity with respect to formal triangular matrix rings. In the introduction of the paper [39] it is claimed that the authors will prove such a result, but it does not appear in the main part of that paper, and we have not been able to find it somewhere else in the literature. For arbitrary rings R and S and an R-S-bimodule M, the formal triangular matrix ring is defined by

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & x \\ 0 & s \end{pmatrix} \middle| r \in R, x \in M, s \in S \right\},$$

which is a ring with respect to the usual addition and multiplication of matrices.

Theorem 1.6.1. *Let R and S be arbitrary rings and let M be an R-S-bimodule. Then $\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \in \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is von Neumann regular if and only if $a_1 \in R$ and $a_2 \in S$ are von Neumann regular and $x \in a_1M + Ma_2$.*

Using Theorem 1.6.1 we may immediately count von Neumann regular elements of formal triangular matrix rings.

Theorem 1.6.2. *Let R and S be finite arbitrary rings, and let M be a finite R-S-bimodule. Let $\mathrm{vnr}(R) = \{r_1, \ldots, r_k\}$ and $\mathrm{vnr}(S) = \{s_1, \ldots, s_l\}$ be the sets of von Neumann regular elements of R and S respectively. Then the number of von Neumann regular matrices in $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is $\sum_{(i,j)\in\{1,\ldots,k\}\times\{1,\ldots,l\}} |r_iM + Ms_j|$.*

Chapter 2

Strongly regular matrices

We prove a necessary condition and a sufficient condition for an $n \times n$-matrix A with determinantal rank $\rho(A) = t$ over an arbitrary commutative ring to be (von Neumann) strongly regular in terms of the trace of its t^{th} compound matrix $C_t(A)$. In particular, a non-zero $n \times n$-matrix A with $\rho(A) = t$ over a local commutative ring R is strongly regular if and only if $\text{Tr}(C_t(A))$ is a unit in R, and in this case we construct a strong inner inverse of A. We derive applications to direct products of local commutative rings and group algebras. We count strongly regular matrices over some finite rings of residue classes and group algebras. We also discuss strong inner inverses and strong reflexive inverses in arbitrary rings as well as strong regular matrices over formal triangular matrix rings. Except for the cited results, all other results are original and are mostly included in our papers [23, 27].

2.1 Characterizations

For general terminology on matrices over commutative rings the reader is referred to the classical sources [12, 14, 18, 38]. We only recall some concepts which are essential in our work. Throughout the chapter $n \geq 2$ will be an integer, and R will be a commutative ring with identity. Also, $GL_n(R)$ is the group of all $n \times n$-matrices whose determinants are units of R. The characteristic polynomial of A will be $p_A(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$, where $c_k = (-1)^k \text{Tr}(C_k(A))$ for every $k \in \{1, \dots, n\}$.

We start with a reduced version of the Cayley-Hamilton Theorem for matrices over commutative rings, which will be useful for proving our characterization of non-zero strongly regular matrices. In the case of matrices over the field $\mathbb{C}$ of complex numbers it was first proved by Segercrantz [68], and then by Hwang [42, Theorem 1]. The latter proof clearly holds for matrices over an arbitrary field. One of the proofs of the Cayley-Hamilton Theorem over a commutative ring R uses its reduction to the case of a field, or even to $\mathbb{C}$ (e.g., see [53, p. 32] and [28, Theorem 3.4] and the details therein). For the reduced Cayley-Hamilton Theorem we use the same idea, which is more generally applicable to universal identities, and we sketch it in what follows.

Theorem 2.1.1. *Let $A \in M_n(R)$ with $\rho(A) = t$. Then $A^{t+1} + c_1A^t + \cdots + c_tA = 0_n$.*

Remark 2.1.2. Theorem 2.1.1 is relevant for $t < n - 1$, since for $t \geq n - 1$ the above identity can be obtained from the Cayley-Hamilton Theorem.

Now we are ready to present one of the main results of the chapter.

Theorem 2.1.3. *Let $A \in M_n(R)$ be a non-zero matrix with $\rho(A) = t$.*
(i) If A is strongly regular, then $c_t \notin \mathrm{rad}(R)$.
(ii) If $c_t \in U(R)$, then A is strongly regular, and a strong inner inverse of A is

$$B = -c_t^{-1}(A^{t-1} + c_1 A^{t-2} + \cdots + c_{t-1} I_n).$$

Let us give a first illustrating example.

Example 2.1.4. (1) Let $A = \begin{pmatrix} 7 & 2 & 8 \\ 6 & 5 & 3 \\ 0 & 10 & 6 \end{pmatrix} \in M_3(\mathbb{Z}_{12})$. By Theorem 2.1.3, A is strongly regular, and a strong inner inverse of A is $B = -c_2^{-1}(A + c_1 I_3) = -5(A - \mathrm{Tr}(A) I_3) = 7(A - 6I_3) = \begin{pmatrix} 7 & 2 & 8 \\ 6 & 5 & 9 \\ 0 & 10 & 0 \end{pmatrix}$.

(2) Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_4)$. By Theorems 1.4.2 and 2.1.3, A is von Neumann regular, but not strongly regular.

In the case of a local commutative ring we give a characterization theorem of (non-zero) strongly regular matrices.

Theorem 2.1.5. *Let R be local and let $A \in M_n(R)$ be a non-zero matrix with $\rho(A) = t$. Then A is strongly regular if and only if $c_t \in U(R)$. In this case, a strong inner inverse of A is $B = -c_t^{-1}(A^{t-1} + c_1 A^{t-2} + \cdots + c_{t-1} I_n)$.*

2.2 Transfer of strong regular property

Let us first see how strong regularity of a matrix of determinantal rank t compares with strong regularity of its t^{th} compound matrix.

Theorem 2.2.1. *Let $A \in M_n(R)$ with $\rho(A) = t$. Consider the statements:*

(i) A is strongly regular.

(ii) $C_t(A)$ is strongly regular.

Then (i) $\Longrightarrow$ (ii). If R is a local ring, then (i) $\Longleftrightarrow$ (ii).

Our characterization of strongly regular non-zero matrices over local commutative rings from Theorem 2.1.5 may also be applied in conjunction with localizations at prime ideals of arbitrary commutative rings in order to decide whether or not a non-zero matrix over an arbitrary commutative ring is strongly regular. We may immediately deduce the following consequence of Theorem 2.1.5. Recall that a matrix $A \in M_n(R)$ will be denoted by $A_{\mathfrak{p}}$ when viewed over the localization $R_{\mathfrak{p}}$ of R at a prime ideal $\mathfrak{p}$.

Theorem 2.2.2. *Let $A \in M_{m,n}(R)$. If A is strongly regular, then for every prime (maximal) ideal $\mathfrak{p}$ of R, $A_{\mathfrak{p}} \in M_{m,n}(R_{\mathfrak{p}})$ is either zero or it has $c_t \in U(R_{\mathfrak{p}})$, where $t = \rho(A_{\mathfrak{p}})$.*

Theorem 2.1.5 may be extended to $n \times n$-matrices over products of local commutative rings as follows. We obtain the following corollary, which may be applied to any semiperfect commutative ring, and in particular to any finite commutative ring.

Corollary 2.2.3. *Let $R = \prod_{k \in K} R_k$ be a direct product of local commutative rings, and let $0_n \neq A \in M_n(R)$. For every $k \in K$, denote by $h_k : M_n(R) \to M_n(R_k)$ the canonical projection. Then the following are equivalent:*

(1) A is strongly regular.

(2) For every $k \in K$, we have either $h_k(A) = 0_n$ or $\rho(h_k(A)) = t_k \geq 1$ and $c_{t_k} = (-1)^k \mathrm{Tr}(C_{t_k}(h_k(A))) \in U(R_k)$.

We have seen in Theorem 2.2.1 that an $n \times n$-matrix A with $\rho(A) = t$ over a local ring R is strongly regular if and only if so is its compound matrix $C_t(A)$. We may extended this result to matrices over direct products of local rings.

Theorem 2.2.4. *Let $R = \prod_{k \in K} R_k$ be a direct product of local commutative rings. Let $A = (a_{ij}) \in M_n(R)$ with $\rho(A) = t$. Then A is strongly regular if and only if $C_t(A)$ is strongly regular.*

Recall that any commutative semiperfect ring is a finite direct product of commutative local rings. Now we have the following corollary.

Corollary 2.2.5. *Let R be a commutative semiperfect ring, and let $A \in M_n(R)$ with $\rho(A) = t$. Then A is strongly regular if and only if $C_t(A)$ is strongly regular.*

2.3 Counting strongly regular matrices

In this section we count strongly regular matrices over finite rings, deduce some related formulas, and give an application to rings of residue classes. Apart from their direct interest, such results may also have implications to the theory of cellular automata [19] or to cryptography, where von Neumann regular matrices may be used in some key exchange protocols and public key encryptions with keyword search scheme [56]. Similarly, strongly regular matrices may serve as key space of some cryptosystems, and determining its size is an important problem.

Now let us count the strongly regular matrices over a local commutative finite ring. As usual, F_q denotes a field with q elements.

Theorem 2.3.1. *Let R be a local finite ring with maximal ideal M such that $|R/M| = |F_q| = q$. Then the number of strongly regular $n \times n$-matrices of determinantal rank t over R is*

$$VS_t(M_n(R)) = \frac{|GL_n(R)|}{|GL_{n-t}(R)|} = |M|^{t(2n-t)} q^{t(n-t)} (q^n - 1)(q^n - q) \cdots (q^n - q^{t-1}).$$

Hence the number of strongly regular $n \times n$-matrices over R is $VS(M_n(R)) = \sum_{i=0}^{n} VS_t(M_n(R))$.

In the next result we relate the numbers $VS_t(M_n(R))$ of strongly regular and $V_t(M_n(R))$ of von Neumann regular $n \times n$-matrices of rank t over a local finite commutative ring with other relevant numbers. Also, $r(n, n, q, t)$ denotes the number of $n \times n$-matrices of rank t over a field F_q with q elements. For every $k \in \{0, \dots, n\}$, denote by

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{t-1})}{(q^t-1)(q^t-q)\cdots(q^t-q^{t-1})}$$

the Gaussian binomial coefficient, which counts the number of k-dimensional subspaces of an n-dimensional vector space over F_q. It is well known that $\binom{n}{k}_q = \binom{n}{n-k}_q$.

Proposition 2.3.2. *Let R be a local finite ring with maximal ideal M such that $|R/M| = |F_q| = q$, and let $t \in \{0, \dots, n\}$. Then:*

(1) $VS_t(M_n(R)) = |M|^{t(2n-t)} \prod_{k=0}^{t-1} VS_1(M_{n-k}(F_q))$.

(2) $VS_t(M_n(R)) = |M|^{t(2n-t)} q^{t(n-t)} \binom{n}{t}_q^{-1} r(n, n, q, t) = q^{t(n-t)} \binom{n}{t}_q^{-1} V_t(M_n(R))$.

Since von Neumann strong regularity behaves well with respect to direct products, we have at once the following corollary.

Corollary 2.3.3. *Let $R = \prod_{k=1}^s R_k$ be a direct product of local commutative finite rings R_k with maximal ideals M_k such that $|R_k/M_k| = |F_{q_k}| = q_k$ for every $k \in K$. Then the number of strongly regular $n \times n$-matrices over R is*

$$VS(M_n(R)) = \prod_{k=1}^{s} \sum_{t=0}^{n} |M_k|^{t(2n-t)} q_k^{t(n-t)} (q_k^n - 1)(q_k^n - q_k) \cdots (q_k^n - q_k^{t-1}).$$

We finally discuss the following question: *if a matrix over a finite ring is strongly regular, then how many strong inner inverses does it have?*

Let us denote by $S(A)$ the set of all strong inner inverses of a strongly regular matrix A.

Theorem 2.3.4. *Let R be a finite local ring with maximal ideal M such that $|R/M| = |F_q| = q$. Consider the natural ring homomorphism $p : R \to R/M$ and the induced R-module homomorphism $h : M_n(R) \to M_n(R/M)$, $h((a_{ij})) = (a_{ij} + M)$. Let $A \in M_n(R)$ be a strongly regular matrix. Then the number of strong inner inverses of A is $|S(A)| = |M|^{(n-t)^2} \cdot |S(h(A))| = |R|^{(n-t)^2}$.*

Corollary 2.3.5. *Let $R = \prod_{k=1}^s R_k$ be a direct product of local commutative finite rings. Let $A \in M_n(R)$ be a strongly regular matrix with $\rho(A) = t$. Then the number of strong inner inverses of A is $|S(A)| = \prod_{k=1}^s |R_k|^{(n-t)^2}$.*

2.4 Counting strongly regular matrices over group algebras

Consider a semisimple group algebra $F_q[G]$, where F_q is a field with q elements and G is a group with l elements. By Maschke's Theorem, the group algebra $F_q[G]$ is semisimple if and only if $\mathrm{char}(F_q)$ does not divide l. In this case, the Wedderburn-Artin Theorem yields an isomorphism of F_q-algebras: $F_q[G] \cong \bigoplus_{k=1}^s M_{n_k}(D_k)$ for some positive integers $n_1, \dots, n_s$ and finite fields $D_1, \dots, D_s$ [64, Theorem 3.4.9]. Note that $|G| = \sum_{k=1}^s n_k^2 d_k$, where $d_k = [D_k : F_q]$

is the degree of the field extension D_k over F_q for every $k \in \{1, \dots, s\}$. In what follows we use the notation $q_k = |D_k| = q^{d_k}$ and $m_k = nn_k$ for every $k \in \{1, \dots, s\}$.

We have seen that a matrix over a field need not be strongly regular, although it is always von Neumann regular. In this section we are interested in counting strongly regular elements of $F_q[G]$ as well as strongly regular $n \times n$-matrices over $F_q[G]$, sometimes even if $F_q[G]$ is not semisimple.

Theorem 2.4.1. *Let G be a group with l elements and let F_q be a field with q elements such that* $\mathrm{char}(F_q)$ *does not divide l. Consider an isomorphism $F_q[G] \cong \bigoplus_{k=1}^{s} M_{n_k}(D_k)$ of F_q-algebras with the above notation. Then:*

(1) the number of strongly regular elements of $F_q[G]$ is

$$VS(F_q[G]) = \prod_{k=1}^{s} \sum_{t=0}^{n_k} q_k^{t(n_k-t)}(q_k^{n_k} - 1)(q_k^{n_k} - q_k) \cdots (q_k^{n_k} - q_k^{t-1}).$$

(2) the number of strongly regular $n \times n$-matrices over $F_q[G]$ is

$$VS(M_n(F_q[G])) = \prod_{k=1}^{s} \sum_{t=0}^{m_k} q_k^{t(m_k-t)}(q_k^{m_k} - 1)(q_k^{m_k} - q_k) \cdots (q_k^{m_k} - q_k^{t-1}).$$

Next we consider a semisimple group algebra $F_q[G]$, where F_q is a field with q elements and G is an *abelian* group with l elements. Then in the above Wedderburn decomposition $F_q[G] \cong \bigoplus_{k=1}^{s} M_{n_k}(D_k)$ all n_k's are one, and all fields D_k are field extensions of F_q by some primitive roots of unity. More precisely, the Perlis-Walker Theorem [64, Theorem 3.5.4] yields an isomorphism of F_q-algebras: $F_q[G] \cong \bigoplus_{d|l} a_d F_q(\zeta_d)$, where ζ_d is a primitive root of unity of order d, $e_d = [F_q(\zeta_d) : F_q]$, n_d is the number of elements of order d of G, $a_d = \frac{n_d}{e_d}$, and $a_d F_q(\zeta_d)$ denotes the direct sum of a_d different fields all of which are isomorphic to the field extension $F_q(\zeta_d)$ of F_q. Note that $|F_q(\zeta_d)| = q^{e_d}$. In what follows we will use this notation.

Since all summands from the Perlis-Walker decomposition of the commutative algebra $F_q[G]$ are fields, all elements of $F_q[G]$ are von Neumann (strongly) regular. So we only count strongly regular $n \times n$-matrices over $F_q[G]$.

Theorem 2.4.2. *Let G be an abelian group with l elements and let F_q be a field with q elements such that* $\mathrm{char}(F_q)$ *does not divide l. Consider an isomorphism $F_q[G] \cong \bigoplus_{d|l} a_d F_q(\zeta_d)$ of F_q-algebras. Then the number of strongly regular $n \times n$-matrices over $F_q[G]$ is*

$$VS(M_n(F_q[G])) = \prod_{d|l} \left(\sum_{t=0}^{n} q^{e_d t(n-t)}(q^{e_d n} - 1)(q^{e_d n} - q^{e_d}) \cdots (q^{e_d n} - q^{e_d(t-1)}) \right)^{a_d}.$$

Finally, we consider a group algebra $F_q[G]$, where F_q is a field with q elements and G is a *cyclic* group with l elements, that is, $G \cong \mathbb{Z}_l$. Unlike the other cases, we will be able to give a formula for counting strongly regular $n \times n$-matrices over $F_q[G]$ even if the group algebra $F_q[G]$ is not semisimple.

Theorem 2.4.3. *Let $l \geq 2$ be an integer and let F_q be a finite field with q elements. Write $x^l - 1 = p_1(x)^{r_1} \cdots p_s(x)^{r_s}$ for some distinct irreducible polynomials $p_1(x), \dots, p_s(x) \in F_q[x]$*

with degrees $d_1, \ldots, d_s$ respectively, and positive integers $r_1, \ldots, r_s$. Then the number of strongly regular $n \times n$-matrices over the group algebra $F_q[\mathbb{Z}_l]$ is

$$VS(M_n(F_q[\mathbb{Z}_l])) = \prod_{k=1}^{s} \sum_{t=0}^{n} q_k^{(r_k-1)t(2n-t)} q_k^{t(n-t)} (q_k^n - 1)(q_k^n - q_k) \cdots (q_k^n - q_k^{t-1}),$$

where $q_k = q^{d_k}$ for every $k \in \{1, \ldots, s\}$.

When G is a finite cyclic group and the group algebra $F_q[G]$ is semisimple, we have a simplification of Theorem 2.4.3 as well as an alternative point of view using field extensions of F_q by some primitive roots of unity, which matches Theorem 2.4.2.

Corollary 2.4.4. *Let $l \geq 2$ be an integer and let F_q be a field with q elements such that $\mathrm{char}(F_q)$ does not divide l. Consider isomorphisms of F_q-algebras $F_q[\mathbb{Z}_l] \cong F_q[x]/(x^l - 1) \cong \bigoplus_{d|l} a_d F_q(\zeta_d)$, where $x^l - 1 = p_1(x) \cdots p_s(x)$ for some distinct irreducible polynomials $p_1(x), \ldots, p_s(x) \in F_q[x]$ with degrees $d_1, \ldots, d_s$ respectively, ζ_d is a primitive root of unity of order d, $e_d = [F_q(\zeta_d) : F_q]$, $\phi(d)$ is Euler's totient function, $a_d = \frac{\phi(d)}{e_d}$,and $a_d F_q(\zeta_d)$ denotes the direct sum of a_d different fields all of which are isomorphic to the field extension $F_q(\zeta_d)$ of F_q. Then the number of strongly regular $n \times n$-matrices over the group algebra $F_q[\mathbb{Z}_l]$ is*

$$\begin{aligned} VS(M_n(F_q[\mathbb{Z}_l])) &= \prod_{k=1}^{s} \sum_{t=0}^{n} q^{d_k t(n-t)} (q^{d_k n} - 1)(q^{d_k n} - q^{d_k}) \cdots (q^{d_k n} - q^{d_k(t-1)}) \\ &= \prod_{d|l} \left(\sum_{t=0}^{n} q^{e_d t(n-t)} (q^{e_d n} - 1)(q^{e_d n} - q^{e_d}) \cdots (q^{e_d n} - q^{e_d(t-1)}) \right)^{a_d}. \end{aligned}$$

2.5 Strong inner and strong reflexive inverses

Throughout this section, the ring R is not necessarily commutative. Hence the following results are applicable to matrix rings.

Recall that an element $a \in R$ is called *strongly regular* if $a \in a^2R \cap Ra^2$ (e.g., see [8, 58]). As noted by Azumaya [8, Lemma 1], if a is strongly regular with $a = a^2u = va^2$ for some $u, v \in R$, then there is a (unique) element $w \in R$ such that $a = a^2w = wa^2$ and $aw = wa$, namely $w = au^2 = v^2a$. Hence $a \in R$ is strongly regular if and only if there is $w \in R$ such that $a = a^2w = wa^2$ and $aw = wa$, and in this case w is called a *strong inner* (or *strong generalized) inverse* of a. An element $u \in R$ is called a *strong reflexive inverse* of $a \in R$ if u is a strong inner inverse of a and a is a strong inner inverse of u. Note also that R is strongly regular if and only if each element of R has a unique reflexive inverse [67, Proposition 3.4].

We denote by $S(a)$ the set of strong inner inverses, and by $\mathrm{SRef}(a)$ the set of strong reflexive inverses of $a \in R$.

Khurana, Lam and Nielsen studied the lifting property of von Neumann regular elements [44, Theorem 4.2]. We discuss the same problem in the case of strongly regular elements.

Theorem 2.5.1. *Let I be an ideal of a ring R and let $x \in R$ be strongly regular modulo I. Then the following are equivalent:*

(1) The element x lifts strongly regularly modulo I.

(2) The element x lifts strongly regularly modulo I and if y is a strong inner inverse of x modulo I, then $y + I$ lifts to a strong inner inverse of any given strongly regular lift of $x + I$.

(3) If x, y are strong reflexive inverses modulo I, then they lift modulo I to strongly reflexive inverses. (In fact, $y + I$ lifts to a strong reflexive inverse of any given strongly regular lift of $x + I$).

An interesting problem on generalized inverses of ring elements is to relate elements in terms of their sets of generalized inverses. In this direction we mention the work of Alahmadi, Jain and Leroy [2], and Lee [50, 51], who studied inner and reflexive inverses in semiprime rings. In what follows we consider the case of strong inner inverses and strong reflexive inverses in semiprime rings.

Recall that R is called a *semiprime* ring if for every $a \in R$ such that $aRa = 0$, one has $a = 0$. For instance, every von Neumann regular ring is semiprime. Also, every reduced ring (i.e., a ring with no non-zero nilpotent elements) is semiprime.

Theorem 2.5.2. *Let R be a semiprime ring, and let $a, b \in R$ be strongly regular elements such that $S(a) \cap S(b) \neq \emptyset$. Then $S(a) \subseteq S(b)$ if and only if $b^2 = ab = ba$.*

Corollary 2.5.3. *Let R be a semiprime ring, and let $a, b \in R$ be strongly regular elements such that $S(a) \cap S(b) \neq \emptyset$. Then $S(a) = S(b)$ if and only if $a = b$.*

Theorem 2.5.4. *Let R be a semiprime ring, and let $a, b \in R$ be strongly regular elements. Then $\mathrm{SRef}(a) \cap \mathrm{SRef}(b) \neq \emptyset$ if and only if $a = b$.*

2.6 Strongly regular formal triangular matrix rings

In this section we study how strong regularity behaves with respect to formal triangular matrix rings.

Theorem 2.6.1. *Let R and S be arbitrary rings and let M be an R-S-bimodule. Then $\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \in \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is strongly regular if and only if $a_1 \in R$ and $a_2 \in S$ are strongly regular and $x \in a_1M + Ma_2$.*

Using Theorem 2.6.1 we may immediately count strongly regular elements of formal triangular matrix rings.

Theorem 2.6.2. *Let R and S be finite arbitrary rings, and let M be a finite R-S-bimodule. Let $\mathrm{svnr}(R) = \{r_1, \ldots, r_k\}$ and $\mathrm{svnr}(S) = \{s_1, \ldots, s_l\}$ be the sets of strongly regular elements of R and S respectively. Then the number of strongly regular matrices in $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is $\sum_{(i,j) \in \{1,\ldots,k\} \times \{1,\ldots,l\}} |r_iM + Ms_j|$.*

Chapter 3

Matrices having a non-zero outer inverse

It is well known that every non-zero von Neumann regular $m \times n$-matrix A over an arbitrary ring has a non-zero outer inverse $n \times m$-matrix B in the sense that $B = BAB$. Generalizing previous work on von Neumann regular matrices, in this chapter the matrices having non-zero outer inverses over semiperfect rings are characterized as the matrices having some entry outside the Jacobson radical of R. Such matrices over finite semiperfect rings and finite commutative rings are counted, and several applications are given. We also discuss outer inverses of formal triangular matrix rings. Except for the cited results, all other results are original and are mostly included in our paper [25].

3.1 Arbitrary rings

Throughout the chapter $m, n \geq 1$ will be integers, and R will be a ring with identity.

We analyze matrices having a non-zero outer inverse by considering first the general case of an arbitrary ring. We have already seen that every non-zero von Neumann regular matrix has a non-zero outer inverse. Now we may relate the existence of a non-zero outer inverse of a matrix with two other conditions as follows.

Theorem 3.1.1. *Let $A = (a_{ij}) \in M_{m,n}(R)$. Consider the following statements:*

(i) There exists some a_{ij} having a non-zero outer inverse.

(ii) A has a non-zero outer inverse.

(iii) $A \notin M_{m,n}(\mathrm{rad}(R))$.

Then (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii).

In the next subsections we will study when the statements of Theorem 3.1.1 are equivalent.

Example 3.1.2. The matrix $A = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in M_3(\mathbb{Z}_4)$ has a non-zero outer inverse by Theorem 3.1.1, because A has an invertible entry. For instance, since the (3,1)-entry of A is 1, a non-zero outer inverse of A is $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{Z}_4)$ by Theorem 3.1.1. But A is not von Neumann regular.

The following concept will be useful.

Definition 3.1.3. Let I be a one-sided ideal of R. We say that *elements having a non-zero outer inverse lift modulo* I, if whenever $x + I \in R/I$ has a non-zero outer inverse in R/I, there is an element $a \in R$ having a non-zero outer inverse in R such that $x - a \in I$.

Recall that a one-sided ideal I of R is called *strongly lifting* if whenever $x^2 - x \in I$ for some $x \in R$ (i.e., x is idempotent modulo I), there is an idempotent $e \in xR$ such that $e - x \in I$ [60]. Note that this property is left-right symmetric, and the Jacobson radical J of a ring is a strongly lifting ideal provided idempotents lift modulo J.

We extend [44, Theorem 4.9] from von Neumann regular elements to elements having a non-zero outer inverse.

Theorem 3.1.4. *Let I be a strongly lifting right ideal of R. Then elements having a non-zero outer inverse lift modulo I.*

Recall that a ring R is called an *exchange ring* (or a *suitable ring*) if there is an idempotent $e \in R$ such that $e - x \in (x^2 - x)R$, and this concept is left-right symmetric [57].

We have some useful consequence of Theorem 3.1.4 in the case of exchange rings and, in particular, von Neumann regular rings, π-regular rings or semiperfect rings, the latter being the case of interest for us.

Corollary 3.1.5. *Let I be a one-sided ideal of an exchange ring R. Then elements having a non-zero outer inverse lift modulo I.*

3.2 Transfer of non-zero outer inverses

We consider a first case when all statements of Theorem 3.1.1 are equivalent.

Theorem 3.2.1. *Let R be local and let $A = (a_{ij}) \in M_{m,n}(R)$. Then the following are equivalent:*

(i) There exists some $a_{ij} \in U(R)$.

(ii) There exists some a_{ij} having a non-zero outer inverse.

(iii) A has a non-zero outer inverse.

(iv) $A \notin M_{m,n}(\mathrm{rad}(R))$.

In this case, a non-zero outer inverse of A is the matrix $B \in M_{n,m}(R)$ having all entries zero, except for the entry (j, i), which is a (non-zero outer) inverse of a_{ij}.

Theorem 3.2.1 may be extended to $m \times n$-matrices over direct products of local rings as follows. Unlike the cases of strong inner inverses or inner inverses, $A \in M_{m,n}(R)$ has a non-zero outer inverse if and only if its projection $h_k(A)$ has a non-zero outer inverse for some $k \in K$. Now we obtain the following result.

Theorem 3.2.2. *Let s be a positive integer, let $R = R_1 \times \cdots \times R_s$ be a direct product of local rings, and let $A = (a_{ij}) \in M_{m,n}(R)$. Then the following are equivalent:*

(i) There exists some a_{ij} having a non-zero outer inverse.

(ii) A has a non-zero outer inverse.

(iii) $A \notin M_{m,n}(\mathrm{rad}(R))$.

In this case, a non-zero outer inverse of A is the matrix $B \in M_{n,m}(R)$ having all entries zero, except for the entry (j, i), which is a non-zero outer inverse of a_{ij}.

3.3 Semiperfect rings

Having prepared the necessary tools, in this section we extend our results to semiperfect rings. If R is a commutative semiperfect ring, then it is well known that it is a finite direct product of commutative local rings, hence Theorem 3.2.2 is directly applicable. Let us see that a similar result is valid for arbitrary semiperfect rings as well.

Theorem 3.3.1. *Let R be a semiperfect ring. Then the following are equivalent for $A = (a_{ij}) \in M_{m,n}(R)$:*

(i) There exists some a_{ij} having a non-zero outer inverse.

(ii) A has a non-zero outer inverse.

(iii) $A \notin M_{m,n}(\mathrm{rad}(R))$.

In this case, a non-zero outer inverse of A is the matrix $B \in M_{n,m}(R)$ having all entries zero, except for the entry (j, i), which is a non-zero outer inverse of a_{ij}.

In general Theorem 3.3.1 does not hold for semilocal rings, as the following example shows. Recall that a ring R is called *semilocal* if $R/\mathrm{rad}(R)$ is semisimple artinian. It is clear by the definitions that every semiperfect ring is semilocal.

Example 3.3.2. Consider the localizations $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(q)}$ of the ring of integers to some distinct primes p and q. Then the ring $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ is a semilocal ring with two maximal ideals (p) and (q) generated by p and q respectively, because

$$R/\mathrm{rad}(R) = R/(pq) \cong R/(p) \times R/(q).$$

But R is not semiperfect, because idempotents do not lift modulo $\mathrm{rad}(R)$. Note that $x = \frac{a}{b} \in R$ has a non-zero outer inverse if and only if $x \in U(R)$ if and only if a and pq are relatively prime.

Now let us choose $p = 3$ and $q = 5$, hence $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$. Let $A = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} \in M_2(R)$. No entry of A has a non-zero outer inverse, but A is invertible, and thus it has a non-zero outer inverse, namely $A^{-1} = \frac{1}{2}\begin{pmatrix} 9 & -5 \\ -5 & 3 \end{pmatrix} \in M_2(R)$. Also, $A' = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ and $A' \notin M_2(\mathrm{rad}(R)) = M_2(15R)$, but it is easily seen that A' has no non-zero outer inverse.

Now we may see how the property having a non-zero outer inverse transfers from a matrix of determinantal rank t to its t^{th} compound matrix.

Theorem 3.3.3. *Let $A = (a_{ij}) \in M_{m,n}(R)$ with $\rho(A) = t$. Consider the statements:*

(i) A has a non-zero outer inverse.

(ii) $C_t(A)$ has a non-zero outer inverse.

Then $(i) \Longrightarrow (ii)$. If R is a semiperfect ring, then $(i) \Longleftrightarrow (ii)$.

Since every commutative semiperfect ring is a finite direct product of commutative local rings, we immediately have the following corollary.

Corollary 3.3.4. *Let R be a commutative semiperfect ring, and let $A \in M_{m,n}(R)$ with $\rho(A) = t$. Then A has a non-zero outer inverse if and only if $C_t(A)$ has a non-zero outer inverse.*

3.4 Counting matrices having non-zero outer inverses

For a finite semiperfect ring, Theorem 3.3.1 allows us to easily determine the number of $m \times n$-matrices over R having a non-zero outer inverse, which will be denoted by $VO(M_{m,n}(R))$.

Proposition 3.4.1. *Let R be commutative semiperfect, say $R = R_1 \times \cdots \times R_s$ for some local finite commutative rings. Then:*

$$VO(M_{m,n}(R)) = \prod_{k=1}^{s} |M_{m,n}(R_k)| - \prod_{k=1}^{s} |M_{m,n}(\mathrm{rad}(R_k))|.$$

Next we count matrices having a non-zero outer inverse over some finite group algebras $F_q[G]$, where F_q is a field with q elements and G is a group with l elements.

Proposition 3.4.2. *Let G be a group with l elements, and let F_q be a field with q elements such that $\mathrm{char}(F_q)$ does not divide l. Then:*

(i) $VO(M_{m,n}(F_q[G])) = \prod_{k=1}^{s} q_k^{mnn_k^2} - 1$.

(ii) If G is abelian, then $VO(M_{m,n}(F_q[G])) = \prod_{k=1}^{s} q^{mne_d a_d} - 1$.

Finally, we consider the more interesting case of a *cyclic* group G with l elements, that is, $G \cong \mathbb{Z}_l$. This time we have a formula for the number of $m \times n$-matrices over $F_q[G]$ having a non-zero outer inverse even if the group algebra $F_q[G]$ is not semisimple.

Proposition 3.4.3. *Let $l \geq 2$ be an integer and let F_q be a finite field with q elements. Write $x^l - 1 = p_1(x)^{r_1} \cdots p_s(x)^{r_s}$ for some distinct irreducible polynomials $p_1(x), \ldots, p_s(x) \in F_q[x]$ with degrees $d_1, \ldots, d_s$ respectively, and positive integers $r_1, \ldots, r_s$. Then:*

$$VO(M_{m,n}(F_q[\mathbb{Z}_l])) = \prod_{k=1}^{s} q_k^{r_k mn} - \prod_{k=1}^{s} q_k^{(r_k - 1)mn},$$

where $q_k = q^{d_k}$ for every $k \in \{1, \ldots, s\}$.

3.5 Outer inverses of formal triangular matrix rings

In this section we see how the property of having an outer inverse behaves with respect to formal triangular matrix rings.

Theorem 3.5.1. *Let R and S be arbitrary rings and let M be an R-S-bimodule. Then $\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \in \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ has an outer inverse $\begin{pmatrix} b_1 & y \\ 0 & b_2 \end{pmatrix} \in \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ if and only if $a_1 \in R$ has outer inverse $b_1 \in R$ and $a_2 \in S$ has outer inverse $b_2 \in S$. Moreover, $\begin{pmatrix} b_1 & y \\ 0 & b_2 \end{pmatrix}$ is non-zero if and only if at least one of b_1 and b_2 is non-zero.*

Using Theorem 3.5.1 we may immediately count matrices having a non-zero outer inverse in formal triangular matrix rings.

Theorem 3.5.2. *Let R and S be finite arbitrary rings, and let M be a finite R-S-bimodule. Let $\operatorname{ovnr}(R)$ and $\operatorname{ovnr}(S)$ be the sets of elements of R and S having a non-zero outer inverse respectively. Then the number of matrices in $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ having a non-zero outer inverse is*

$$|R| \cdot |S| \cdot |M| - (|R| - |\operatorname{ovnr}(R)|) \cdot (|S| - |\operatorname{ovnr}(S)|) \cdot |M|.$$

Chapter 4

Applications

We use our previous results on von Neumann regular matrices, strongly regular matrices and matrices having a non-zero outer inverse to derive applications to some generalizations of these concepts, called von Neumann local, strongly von Neumann local and outer von Neumann local matrices. Among other properties, we show that the t^{th} compound matrix of every matrix of determinantal rank t over a commutative local ring is strongly von Neumann local, and every matrix over an arbitrary semiperfect ring is outer von Neumann local. Except for the cited results, all other results are original and are mostly included in our paper [26].

4.1 Von Neumann local matrices

Contessa [29] introduced *von Neumann local* rings as the rings R with the property that a or $1-a$ is von Neumann regular for every $a \in R$. Clearly, every von Neumann regular ring and every local ring is von Neumann local. Also, every von Neumann local ring is an exchange ring. Von Neumann local rings have been also studied by Abu Osba, Henriksen and Alkam [1], and Anderson and Badawi [4], which specialized their definition to elements. Thus, an element $a \in R$ is called *von Neumann local* if a or $1-a$ is von Neumann regular. Clearly, every von Neumann regular element is von Neumann local.

In particular, a matrix $A \in M_n(R)$ is von Neumann local if A or $I_n - A$ is von Neumann regular. We show that there is a rich supply of von Neumann local matrices.

Theorem 4.1.1. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) \leq 1$. Then A is von Neumann local.*

Corollary 4.1.2. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) = t$. Then $C_t(A)$ is von Neumann local.*

Using our characterizations of von Neumann regular matrices, we may immediately deduce corresponding characterizations of von Neumann local matrices.

Theorem 4.1.3. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) = t$ and $\rho(I_n - A) = s$. Then the following are equivalent:*

(1) A is von Neumann local.

(2) $C_t(A)$ *or* $C_s(I_n - A)$ *is von Neumann regular.*

(3) $A = 0_n$ *or* A *has an invertible* $t \times t$*-submatrix or* $A = I_n$ *or* $I_n - A$ *has an invertible* $s \times s$*-submatrix.*

Next we characterize von Neumann local 2×2-matrices over commutative local rings only in terms of determinants.

Theorem 4.1.4. *Let* R *be a commutative local ring, and let* $A \in M_2(R)$*. Then the following are equivalent:*

(1) A *is von Neumann local.*

(2) $\det(A) \in U(R) \cup \{0\}$ *or* $\det(I_2 - A) \in U(R) \cup \{0\}$.

If R *is not a field, then they are further equivalent to:*

(3) $\det(A) \notin \operatorname{rad}(R) \setminus \{0\}$ *or* $\det(I_2 - A) \notin \operatorname{rad}(R) \setminus \{0\}$.

(4) $\det(A) \in U(R) \cup \{0\}$ *or* $1 - \operatorname{Tr}(A) \in U(R)$.

Example 4.1.5. The matrix $A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{Z}_4)$ is von Neumann local (since $I_2 - A$ is invertible), but not von Neumann regular by Theorem 1.4.2.

Unlike the case of von Neumann regularity, the property of being von Neumann local is not well behaved with respect to direct products (e.g., see [1, p.2644]). By [1, Theorem 3.1], a direct product $R = \prod_{k\in K} R_k$ is von Neumann local if and only if there is $l \in K$ such that R_l is von Neumann local and R_k is von Neumann regular for every $k \in K \setminus \{l\}$. Next we state an element-wise version of this result, whose commutative version was given by Anderson and Badawi [4, Theorem 5.1]. Let us denote by $\operatorname{vnr}(R)$ (respectively $\operatorname{vnl}(R)$) the set of von Neumann regular (respectively von Neumann local) elements of a ring R.

Theorem 4.1.6. *Let* $R = \prod_{k\in K} R_k$ *be a direct product of arbitrary rings. Then* $\operatorname{vnl}(R) = \prod_{k\in K} \operatorname{vnl}(R_k)$ *if and only if* $\operatorname{vnl}(R_k) = \operatorname{vnr}(R_k)$ *for all but at most one* $k \in K$*. In particular,* R *is a von Neumann local ring if and only if there is at most one* $k \in K$ *such that* R_k *is not von Neumann regular, but* R_k *is von Neumann local.*

4.2 Strongly von Neumann local matrices

We consider a specialization of the notion of von Neumann local element of a ring. Thus, an element $a \in R$ is called *strongly von Neumann local* if a or $1-a$ is strongly regular. Clearly, every strongly regular element is strongly von Neumann local, and every strongly von Neumann local element is von Neumann local. In particular, a matrix $A \in M_n(R)$ is strongly von Neumann local if A or $I_n - A$ is strongly regular. Note that our concept of strongly von Neumann local element is different of the one with the same name from [1].

Next we improve some results from von Neumann local matrices.

Theorem 4.2.1. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) \leq 1$. Then A is strongly von Neumann local.*

Corollary 4.2.2. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) = t$. Then $C_t(A)$ is strongly von Neumann local.*

Our previous results on strongly regular matrices may be immediately applied to derive corresponding properties of strongly von Neumann local matrices.

Theorem 4.2.3. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) = t$ and $\rho(I_n - A) = s$. Then the following are equivalent:*

(1) A is strongly von Neumann local.

(2) $C_t(A)$ or $C_s(I_n - A)$ is strongly regular.

(3) $A = 0_n$ or $c_t \in U(R)$ or $A = I_n$ or $d_s \in U(R)$, where $c_t = (-1)^t \mathrm{Tr}(C_t(A))$ and $d_s = (-1)^s \mathrm{Tr}(C_s(I_n - A))$.

Next we show that von Neumann local and strongly von Neumann local matrices coincide in this case of 2×2-matrices over commutative local rings..

Theorem 4.2.4. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A is strongly von Neumann local if and only if A is von Neumann local.*

Example 4.2.5. By Theorem 4.2.4, in order to find an example of a von Neumann local matrix which is not strongly von Neumann local over a commutative local ring we need to look for some matrix having a larger size than 2×2. Let us take

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix} \in M_3(\mathbb{Z}_4).$$

Since $\rho(A) = 2$ and A has an invertible 2×2-submatrix, A is von Neumann regular by Theorem 1.4.2, and consequently, A is von Neumann local. Since the sum of diagonal 2×2-submatrices of A is 0, A is not strongly regular by Theorem 2.1.5. Now consider

$$I_3 - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 2 \end{pmatrix}.$$

Since $\rho(I_3 - A) = 3$ and $\det(I_3 - A) = 2 \notin U(\mathbb{Z}_4)$, $I_3 - A$ is not strongly regular by Theorem 2.1.5, and consequently, A is not strongly von Neumann local.

We have already seen that in general the property of being von Neumann local is not well behaved with respect to direct products, and we have given Theorem 4.1.6. Now we deal with the similar problem for strongly von Neumann local elements in an arbitrary ring R. Let us denote by $\mathrm{svnr}(R)$ (respectively $\mathrm{svnl}(R)$) the set of strongly regular (respectively strongly von Neumann local) elements of R.

Theorem 4.2.6. *Let $R = \prod_{k \in K} R_k$ be a direct product of arbitrary rings. Then $\mathrm{svnl}(R) = \prod_{k \in K} \mathrm{svnl}(R_k)$ if and only if $\mathrm{svnl}(R_k) = \mathrm{svnr}(R_k)$ for all but at most one $k \in K$. In particular, R is a strongly von Neumann local ring if and only if there is at most one $k \in K$ such that R_k is not strongly regular, but R_k is strongly von Neumann local.*

4.3 Outer von Neumann local matrices

Generalizing von Neumann local elements of a ring, an element $a \in R$ is called *outer von Neumann local* if a or $1-a$ has a non-zero outer inverse. An element having a non-zero outer inverse will also be called an *outer von Neumann regular* element. Clearly, every outer von Neumann regular is outer von Neumann local, and every von Neumann local element is outer von Neumann local. In particular, a matrix $A \in M_n(R)$ is outer von Neumann local if A or $I_n - A$ is outer von Neumann regular.

In Theorem 3.3.1 we have given a characterization of matrices having a non-zero outer inverse over semiperfect rings. We may use it to obtain the following result.

Theorem 4.3.1. *Let R be a semiperfect ring. Then every matrix $A = (a_{ij}) \in M_n(R)$ is outer von Neumann local.*

Example 4.3.2. (1) Let us first give an example of an outer von Neumann local matrix over a non-semiperfect ring. Consider the semilocal ring $R = \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$ (which is not semiperfect), and the matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$. Direct calculations show that A does not have a non-zero outer inverse. But we have $B(I_2 - A)B = B$ for $B = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \in M_2(R)$, hence $I_2 - A = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ has a non-zero outer inverse, and consequently, A is outer von Neumann local.

(2) The above matrix A also gives an example of an outer von Neumann local matrix which is not von Neumann local. Indeed, direct calculations show that neither A nor $I_2 - A$ is von Neumann regular (or note that the first determinantal ideals of A and $I_2 - A$ are not generated by an idempotent and use Theorem 1.4.1). Hence A is not von Neumann local.

Theorem 4.3.3. *Let R be an arbitrary local ring and let $A \in M_n(R)$. Then the following are equivalent:*

(1) A is outer von Nemann local.

(2) A or $I_n - A$ has an invertible entry.

(3) A or $I_n - A$ has an entry with a non-zero outer inverse.

(4) $A \notin M_n(\mathrm{rad}(R))$ or $I_n - A \notin M_n(\mathrm{rad}(R))$.

Finally, let us see how the property of being outer von Neumann local behaves with respect to direct products. Let us denote by $\mathrm{ovnr}(R)$ (respectively $\mathrm{ovnl}(R)$) the set of outer von Neumann regular (respectively outer von Neumann local) elements of a ring R.

Theorem 4.3.4. *Let $R = \prod_{k \in K} R_k$ be a direct product of arbitrary rings. Then $(a_k)_{k\in K} \in \mathrm{ovnl}(R)$ if and only if there is $j \in K$ such that $a_j \in \mathrm{ovnl}(R_j)$.*

Appendix

The definitions of a von Neumann regular, strongly regular or outer regular matrix need finding a suitable matrix B satisfying some property related to A. In this way, the verifications of strong regularity, von Neumann regularity or outer regularity by definitions are time-consuming, the critical part being the enumeration of possible candidates for B. In contrast, our results offer intrinsic characterizations of these three notions, which can be developed into much more efficient algorithms. We illustrate them for matrices over rings of residue classes.

The algorithms for strongly von Neumann local and von Neumann local matrices over rings of residue classes are easily derived from these three basic ones. We have seen that every matrix over a ring of residue classes is outer von Neumann local.

By using our algorithms, higher order examples may be easily obtained.

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