BABEŞ-BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

NEW RESULTS IN GEOMETRIC FUNCTION THEORY OF ONE COMPLEX VARIABLE

Ph.D. Thesis - Summary

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Keywords

analytic function, univalent function, starlike function, gamma-starlike function of order alpha, convex function, close-to-convex function, meromorphic function, functions with varying arguments, bi-univalent functions, m-fold symmetric functions, composition of functions, convolution, extreme points, differential subordination, Sălăgean derivative, Ruscheweyh derivative, fractional operator, fractional differintegral operator, Sălăgean integral operator, Bernardi integral operator, integro-differential operator, coefficient estimates, Fekete-Szegő inequalities, Hankel determinant, arithmetic mean, geometric mean, harmonic mean, Chebyshev polynomials, Poisson distribution series.

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Contents

Introduction

The geometric function theory is a branch of complex analysis, which studies the geometric properties of analytic functions. The first significant works in the geometric function theory of one complex variable appeared at the beginning of the XXth century. P. Koebe with the work published in the year 1907 ([68]), attracted the attention of the researchers on the study of univalent functions. Further it was obtained an important result, the area theorem, by T. Gronwall ([47]), and by L. Bieberbach ([16, 17]). L. Bieberbach also obtained a bound for the absolute value of the coefficient a_2 of a univalent function ([17]) and stated the famous hypothesis regarding in general the coefficient a_n, where $n = 2, 3, \ldots$ (see Conjecture 1.1.1), which was just proved in the year 1984, by L. De Branges ([30]). L. Bieberbach in [17] also obtained a sharp bound for the absolute value of the expression $a_3 - a_2^2$ for a univalent function.

After the publication of these results, the research directions have become more and more varied. Several famous mathematicians studied and made important contributions in this field. In Romania stood out with their remarkable results romanian mathematicians, like G. Călugăreanu and P. T. Mocanu. G. Călugăreanu in [21, 22] obtained necessary and sufficient conditions for univalency of an holomorphic function in a disk centered in origin. P. T. Mocanu in [88] established a relation between starlike and convex functions, introducing alpha-convex functions. A revolutionary method, the differential subordinations method (or the admissible functions method) was obtained by S. S. Miller and P. T. Mocanu in [83, 84]. Using this method a lot of previously known results could be proved more easily, and many new results were subsequently obtained.

Among numerous treatises and monographs devoted to complex analysis, respectively to geometric function theory of one or more complex variables, we mention those of P. T. Mocanu, G. Ş. Sălăgean and T. Bulboacă [92], S. S. Miller and P. T. Mocanu [85], I. Graham and G. Kohr [46], C. Pommerenke [108], P. L. Duren [32], G. Kohr and P. T. Mocanu [70], P. Hamburg, P. T. Mocanu and N. Negoescu [50], P. T. Mocanu, D. Breaz, G. I. Oros and Gh. Oros [91], G. Ş. Sălăgean [118], G. Kohr [69],

G. Kohr and P. Liczberski [71], P. Curt [29], T. Bulboacă [18], A. W. Goodman [42], L. V. Ahlfors [2], D. J. Hallenbeck and T. H. MacGregor [49].

In this thesis, the researches that are part of the Cluj-Napoca school of mathematics on the geometric function theory of one complex variable are continued and some results are obtained that extend other results obtained by mathematicians from our country and from other countries. Classes of analytic, meromorphic, respectively bi-univalent functions are studied, some of them being defined using operators. Results related to differential subordinations are also obtained. The thesis is structured in four chapters.

In the first chapter, both notions and basic results from the geometric function theory are presented, as well as special classes of functions, the differential subordinations method, respectively differential and integral operators. I have tried to make a presentation in a unitary form, and at the same time to highlight the definitions and results used in the following chapters.
Thus, in Section 1.1, basic notions related to univalent and meromorphic functions are presented. Bieberbach's conjecture regarding the estimation of the coefficients of a univalent function, respectively de Brange's theorem, the Fekete-Szegő inequality for univalent functions, an univalence criterion for an analytic function, the property regarding the existing bijection relation between the class of univalent functions and a subclass of the class of meromorphic functions are stated.
In Section 1.2, different classes of functions are presented, such as the Carathéodory class, the class of Schwarz functions, the class of starlike, convex, close-to-convex, alpha-convex, gamma-starlike, starlike of order α, convex of order α, gamma-starlike of order alpha, strongly starlike of order γ, strongly convex of order γ, δ-uniformly convex, δ-uniformly starlike functions, as well as other classes of functions associated with starlikeness and convexity. Properties are also stated for some of these classes, such as coefficient estimates, Fekete-Szegő inequalities, conditions for an analytic function to belong to a certain class.
In Section 1.3 the differential subordinations method (or the admissible functions method) is described.
In Section 1.4, the classes of starlike and convex meromorphic functions are presented. Necessary and sufficient conditions for starlikeness and convexity of meromorphic functions are stated.
In Section 1.5, operators are presented, such as the Sălăgean differential operator $\mathcal{D}^n$, the Sălăgean integral operator $\mathcal{I}^n$, the Ruscheweyh operator $\mathcal{R}^\lambda$, the fractional differ-integral operator Ω_z^λ, the fractional operator $\mathbb{D}_\lambda^{\nu,n}$, the Al-Oboudi differential operator

$\mathcal{D}_\delta^n$, the generalized Sălăgean integral operator $\mathcal{I}_\delta^n$ defined by J. Patel, the Bernardi integral operator $\mathcal{L}_c$.
With the exception of Remark 1.5.7, Remark 1.5.9 and Remark 1.5.11, the chapter does not contain original results. The mentioned remarks can be found in the work [129].

In the second chapter, results are obtained on analytic or meromorphic functions using operators.
In Section 2.1, different results are obtained using the fractional operator $\mathbb{D}_\lambda^{\nu,n}$, which is the composition of the fractional differintegral operator Ω_z^λ, the Sălăgean operator $\mathcal{D}^n$ and the Ruscheweyh operator $\mathcal{R}^\nu$.
In Subsection 2.1.1 a class of analytic functions defined by this operator is introduced. Inclusion relations between different subclasses of the class are obtained, conditions for belonging to the class of the convolution of two analytic functions are also obtained, the convexity of the class is proved, extreme points of the class and other properties of the class are obtained. With the exception of Theorem 2.1.1, the results in this subsection are original and can be found in the paper [129], paper published in the Mediterranean Journal of Mathematics, an ISI rated journal with the impact factor 1.305.
In the other subsections, differential subordinations are investigated, geometric properties of analytic functions, coefficients bounds and Fekete-Szegő inequalities for classes of analytic functions are obtained. All the results in these subsections are obtained using the fractional operator $\mathbb{D}_\lambda^{\nu,n}$. With the exception of Lemma 2.1.1, Corollary 2.1.5, Corollary 2.1.6, Corollary 2.1.7, Corollary 2.1.8, Corollary 2.1.9 and Corollary 2.1.10, the results in these subsections are original and can be found in the papers [39, 130, 131].
Most of the results in this section are generalizations of results previously obtained by other mathematicians.
In Section 2.2, a class of meromorphic functions defined by using a fractional operator defined in a similar way to the operator $\mathbb{D}_\lambda^{\nu,n}$ is introduced. Inclusion relations between some subclasses of the class and conditions for belonging to the class of some integral operators are obtained. The obtained results are generalizations of some results obtained by other mathematicians. With the exception of Lemma 2.2.1, the results in this section are original and can be found in the work [37].
In Section 2.3 a new operator is defined, which generalizes several operators introduced by other mathematicians. The new operator is defined using the operators $\Omega_z^\lambda, \mathcal{D}^n, \mathcal{R}^\nu$. A class of analytic functions is introduced using the new operator and

the properties of this class are obtained. Using this operator differential subordinations are also investigated. The results in this section are original and are presented in the work [133].
In Section 2.4, inclusion relations are obtained between the δ-uniformly convex, δ-uniformly starlike function classes, the classes $\mathcal{S}_\lambda^*, \mathcal{C}_\lambda$ and the class $US(n,\alpha)$, class defined by using the Sălăgean differential operator $\mathcal{D}^n$. These inclusion relations are associated with Poisson distribution series. With the exception of the Theorem 2.4.1, the results in this section are original and can be found in the work [36].
In Section 2.5, the generalized Sălăgean integro-differential operator is introduced, using the Al-Oboudi differential operator $\mathcal{D}_\delta^n$ and the generalized Sălăgean integral operator $\mathcal{I}_\delta^n$. This operator generalizes the operator introduced by Á. O. Páll-Szabó in the work [104]. Differential subordinations are investigated and known results are generalized. The results in this section are original and are presented in the paper [38].

In the third chapter, various results on some classes of analytic functions associated with starlikeness and convexity are obtained.
In Section 3.1, various differential subordinations involving arithmetic, geometric, respectively harmonic means of the expressions $p(z)$ and $p(z)+\frac{zp'(z)}{p(z)}$ are generalized. A starlikeness criterion and a strongly starlikeness of order θ criterion for analytic functions are obtained. With the exception of the Lemma 3.1.1, the results in this section are original and can be found in the work [35], work published in Mathematica Slovaca, an ISI rated journal with the impact factor 0.996. In the work [73] the authors generalized the results from this section.
In Section 3.2, after the presentation of the Chebyshev polynomials, coefficient estimates, respectively a Fekete-Szegő inequality for a class of analytic functions that satisfy a subordination condition associated to Chebyshev polynomials are obtained. The obtained results are generalizations of known results. With the exception of Corollary 3.2.1, Corollary 3.2.2 and Corollary 3.2.3, the results in this section are original and can be found in the paper [132]. The authors of the works [54], [124] generalized the results from this section.
In Section 3.3, bi-univalent functions, m-fold symmetric functions are presented. Coefficient estimates and the Fekete-Szegő inequality are obtained for a new subclass of m-fold symmetric bi-univalent functions which satisfy subordination conditions. The obtained results generalize other known results. With the exception of the Corollary 3.3.9, the results in this section are original and can be found in the work [41].
In Section 3.4, a bound for the second Hankel determinant for gamma-starlike functions of order alpha is obtained, in the case $0 \leq \gamma \leq 1$. The obtained result extends

the bounds for the second Hankel determinant for other classes of functions. Some of these results were obtained by other mathematicians. The results in this section are original and are presented in the work [40].

In the last chapter, a new operator obtained by the convolution of the Sălăgean operator $\mathcal{D}^n$ and the Ruscheweyh operator $\mathcal{R}^n$ is defined and a class of analytic functions with varying arguments defined by this operator is introduced. The image properties of this class through the Bernardi operator are also studied. The results from this chapter are original and they are presented in the work [106].

The bibliography of this Ph.D. thesis contains a number of 147 titles, 13 of which are signed by the author, 4 being in collaboration, and 2 being published în ISI rated journals with impact factor.

The original results presented in the thesis, are contained in the following papers:

1. **E. Szatmari**, *On a class of analytic functions defined by a fractional operator*, Mediterr. J. Math., 15:158, (2018). WoS, Impact factor: 1,305

2. **E. Gavriş**, *Differential subordinations and Pythagorean means*, Math. Slovaca, 70 (2020). No. 5, 1135-1140. WoS, Impact factor: 0,996

3. **E. Szatmari**, Ş. Altınkaya, *Coefficient estimates and Fekete-Szegő inequality for a class of analytic functions satisfying subordinate condition associated with Chebyshev polynomials*, Acta Univ. Sapientiae Math., 11 (2) (2019), 430-436. WoS

4. **E. Szatmari**, *Differential subordinations obtained by using a fractional operator*, Stud. Univ. Babeş-Bolyai Math., 63 (4) (2018), 475-482. WoS

5. **E. Gavriş**, Ş. Altınkaya, *Coefficient estimates and Fekete-Szegő inequalities for a new subclass of m-fold symmetric bi-univalent functions satisfying subordinate conditions*, Int. J. Nonlinear Anal. Appl., 14 (2023) 1, 3145-3154 (electronic). WoS

6. **E. Gavriş**, *The second Hankel determinant for gamma-starlike functions of order alpha*, submitted.

7. **E. Gavriş**, *On a class of meromorphic functions defined by using a fractional operator*, Mathematica (Cluj), 63 (86), No. 1, 2021, 77-84. Scopus

8. **E. Gavriş**, *Inclusion relations of analytic functions associated with Poisson distribution series and Sălăgean operator*, An. Univ. Oradea Fasc. Mat., 17 (2) 2020, 47-53. BDI

9. Á.O. Páll-Szabó, O. Engel, **E. Szatmari**, *Certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative*, Acta Univ. Apulensis Math. Inform., 51 (2017), 61-74. BDI

10. **E. Gavriş**, *Differential subordinations obtained by using generalized Sălăgean integro-differential operator*, Acta Univ. Apulensis Math. Inform., 71 (2022), 127-136. BDI

11. **E. Gavriş**, *Coefficient bounds and Fekete-Szegő problem for some classes of analytic functions defined by using a fractional operator*, submitted.

12. **E. Szatmari**, *Some properties of analytic functions obtained by using a fractional operator*, Asia Pacific Journal of Mathematics, 5 (2) (2018), 151-172. Scopus

13. **E. Szatmari**, Á.O. Páll-Szabó, *Differential subordination results obtained by using a new operator*, General Mathematics, 25 (1-2) (2017), 119-131. BDI

A part of the original results, proved in the thesis, were presented at the following international conferences:

1. 13th International Symposium on Geometric Function Theory and Applications, August 3-6, 2017, Arad. The title of the presentation: Some results using a fractional operator

2. 6th International Conference on Mathematics and Informatics, September 7-9, 2017, Târgu Mureş. The title of the presentation: Coefficient bounds and Fekete-Szegő problem for some classes of analytic functions defined by using a fractional operator

3. 3rd International Conference on Mathematics and Computer Science, June 14-16, 2018, Braşov. The title of the presentation: Coefficient estimates and Fekete-Szegő inequalities for a new subclass of m-fold symmetric bi-univalent functions satisfying subordinate conditions

4. XIX Conference on Analytic Functions and Related Topics, June 25-29, 2018, Rzeszów, Poland. The title of the presentation: Differential subordinations and Pythagorean means

5. 16th International Conference on Applied Mathematics and Computer Science, July 3-6, 2019, Cluj-Napoca. The title of the presentation: The second Hankel determinant for gamma-starlike functions of order alpha

6. 7th International Conference on Mathematics and Informatics, September 2-4, 2019, Târgu Mureş. The title of the presentation: Differential subordinations obtained by using generalized Sălăgean integro-differential operator

Chapter 1

Definitions and preliminary results

We begin with some notions and results from the geometric function theory. We present first the class of univalent functions, the class of meromorphic functions, the Carathéodory class, Schwarz functions class, the notion of subordination. We present various classes of univalent functions, such as starlike functions, convex functions, close-to-convex functions and classes of functions related to starlikeness and convexity. In the next section, the differential subordinations method is described. Subclasses of meromorphic functions are also presented. The final section is dedicated to some differential and integral operators.

1.1 Basic definitions and results from the theory of univalent functions

We denote the complex plane by $\mathbb{C}$, and the open disk of center $z_0 \in \mathbb{C}$ and radius $r > 0$ by $\mathcal{U}(z_0, r)$,

$$\mathcal{U}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

We denote the open disk $\mathcal{U}(0, r)$ by $\mathcal{U}_r$, and the unit disk $\mathcal{U}_1$ by $\mathcal{U}$.

Let $\mathcal{H}(\mathcal{U})$ denote the class of analytic functions in $\mathcal{U}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}^* = \{1, 2, \ldots\}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathcal{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\},$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathcal{U}) : f(z) = z + a_{n+1} z^{n+1} + \ldots\},$$

with $\mathcal{A}_1 = \mathcal{A}$. So, the series expansion of a function $f \in \mathcal{A}$ is of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \quad z \in \mathcal{U}. \tag{1.1}$$

Definition 1.1.1. *[92, p. I] An analytic function in a domain D we say it is univalent in this domain if it is injective in D.*

We will denote by $\mathcal{S}$ the class of univalent functions in $\mathcal{U}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The series expansion of a function $f \in \mathcal{S}$ is of the form (1.1).

Conjecture 1.1.1. ***(Bieberbach's conjecture)*** *[92, Conjecture 1.2.1, p. 6] If the function $f(z) = z + a_2 z^2 + \ldots$ belongs to the class $\mathcal{S}$, then $|a_n| \le n, n = 2, 3, \ldots$.*

Bieberbach's conjecture was proved by L. de Branges in 1984, using Löwner's method.

Theorem 1.1.1. *[17] If the function $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ belongs to the class $\mathcal{S}$, then $|a_3 - a_2^2| \le 1$. The result is sharp.*

One of the most simple univalence criteria is given by the following theorem obtained by K. Noshiro [95], S. Warschawski [145] and J. Wolff [146].

Theorem 1.1.2. *[92, Theorem 4.5.1, p. 86] If the function f is analytic in the convex domain $D \subset \mathbb{C}$ and if there exists a number $\gamma \in \mathbb{R}$ such that*

$$\Re\big[e^{i\gamma} f'(z)\big] > 0, \quad z \in D,$$

then the function f is univalent in D.

We denote by Σ the class of functions φ meromorphic with a simple pole $\xi = \infty$, univalent in $\mathcal{U}^- = \{\xi \in \mathbb{C}_\infty : |\xi| > 1\}$, which have the Laurent series expansion of the form

$$\varphi(\xi) = \xi + \alpha_0 + \frac{\alpha_1}{\xi} + \cdots + \frac{\alpha_n}{\xi^n} + \ldots, \quad |\xi| > 1. \tag{1.2}$$

Let

$$\Sigma_0 = \{\varphi \in \Sigma : \varphi(\xi) \ne 0, \xi \in \mathcal{U}^-\}.$$

Property 1.1.1. *[92, Property 1.1.2, p. 2] There exists a bijection between the classes $\mathcal{S}$ and Σ_0, so class Σ is "wider" than class $\mathcal{S}$.*

Remark 1.1.1. *[92, p. 3] It is observed that if $\varphi \in \Sigma$ and $c \in \mathbb{C} \setminus \varphi(\mathcal{U}^-)$, then the function*

$$f(z) = \frac{1}{\varphi(\frac{1}{z}) - c} = z + (c - \alpha_0) z^2 + \ldots, \quad z \in \mathcal{U}$$

has the property $f \in \mathcal{S}$.

1.2 Classes of functions

1.2.1 The Carathéodory class. The Schwarz functions class. Subordination

Definition 1.2.1. *[92, Definition 3.1.1, p. 35] The Carathéodory class is defined by*

$$\mathcal{P} = \{p \in \mathcal{H}(\mathcal{U}) : p(0) = 1, \Re p(z) > 0, z \in \mathcal{U}\}.$$

Remark 1.2.1. *The series expansion of a function $p \in \mathcal{P}$ is of the form*

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \quad z \in \mathcal{U}. \tag{1.3}$$

Example 1.2.1. *[92, p. 35] The function $p(z) = \frac{1+z}{1-z} \in \mathcal{P}$, because it maps $\mathcal{U}$ onto the right-half plane $\{w : \Re w > 0\}$.*

Theorem 1.2.1. *[80] Let $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$. Then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \nu \leq 0 \\ 2, & 0 \leq \nu \leq 1 \\ 4\nu - 2, & \nu \geq 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, equality holds if and only if $p_1(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then equality holds if and only if $p_1(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. Inequality becomes equality when $\nu = 0$ if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z}, \quad 0 \leq \lambda \leq 1$$

or one of its rotations. While for $\nu = 1$, equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Remark 1.2.2. *[80] It is also possible to obtain a result of the same form as that of Theorem 1.2.1. Thus, for $0 < \nu < 1$ the following inequalities hold:*

$$|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2, \quad 0 < \nu \leq \frac{1}{2}$$

and

$$|c_2 - \nu c_1^2| + (1 - \nu)|c_1|^2 \leq 2, \quad \frac{1}{2} \leq \nu < 1.$$

Result of the same type with Theorem 1.2.1 can be found in [111]:

Lemma 1.2.1. *[111] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \dots$ is a function with positive real part, then*

$$|c_2 - \nu c_1^2| \le 2 \max\{1; |2\nu - 1|\}.$$

The result is sharp for the function $p_1(z) = \frac{1+z^2}{1-z^2}$ or $p_1(z) = \frac{1+z}{1-z}$.

Theorem 1.2.2. *[108] If $p \in \mathcal{P}$ is of the form (1.3), then*

$$|p_n| \le 2, \quad n \in \mathbb{N}^*$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \le 2 - \frac{|p_2|^2}{2}.$$

Theorem 1.2.3. *[79] Let $p \in \mathcal{P}$ be of the form (1.3). Then there exist $x, z \in \mathbb{C}$ with $|x| \le 1$ and $|z| \le 1$ such that*

$$2p_2 = p_1^2 + x(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z.$$

Definition 1.2.2. *[92, Definition 3.1.1, p. 35] The Schwarz functions class is defined by*

$$\mathcal{B} = \{\phi \in \mathcal{H}(\mathcal{U}) : \phi(0) = 0, |\phi(z)| < 1, z \in \mathcal{U}\}.$$

Theorem 1.2.4. *[65] Let the Schwarz function w be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \dots, \quad z \in \mathcal{U}. \tag{1.4}$$

Then

$$|w_1| \le 1, \; |w_2 - t w_1^2| \le 1 + (|t| - 1)|w_1|^2 \le \max\{1, |t|\},$$

where $t \in \mathbb{C}$.

Definition 1.2.3. *[85, p. 4] Let $f, F \in \mathcal{H}(\mathcal{U})$. The function f is said to be subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there exists a Schwarz function w, such that $f(z) = F\big[w(z)\big], z \in \mathcal{U}$.*

Remark 1.2.3. *[85, p.4] If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$.*

1.2.2 Starlike, convex and close-to-convex functions

Definition 1.2.4. *[92, Definition 4.1.1, p. 49] Let the function $f \in \mathcal{H}(\mathcal{U})$ with $f(0) = 0$. We say that the function f is starlike in $\mathcal{U}$ with respect to the origin (or, simply, starlike) if the function f is univalent in $\mathcal{U}$ and $f(\mathcal{U})$ is a starlike domain with respect to the origin.*

Theorem 1.2.5. *[92, Theorem 4.1.2, p. 49] Let the function $f \in \mathcal{H}(\mathcal{U})$ with $f(0) = 0$. Then the function f is starlike if and only if $f'(0) \neq 0$ and*

$$\Re\frac{zf'(z)}{f(z)} > 0, \quad z \in \mathcal{U}.$$

Definition 1.2.5. *[92, Definition 4.1.3, p. 53] We will note with $\mathcal{S}^*$ the class of the functions $f \in \mathcal{A}$ which are starlike (and normalized) in the unit disk, i. e.*

$$\mathcal{S}^* = \left\{f \in \mathcal{A} : \Re\frac{zf'(z)}{f(z)} > 0, z \in \mathcal{U}\right\}.$$

Remark 1.2.4. *[92, Remark 4.1.1, p. 53] If $f \in \mathcal{A}$, using the language of subordinations, we have*

$$f \in \mathcal{S}^* \iff \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}.$$

Definition 1.2.6. *[85, p. 8] A domain D in $\mathbb{C}$ is said to be convex if the line segment joining any two points of D lies entirely in D.*

Definition 1.2.7. *[85, p. 8] The convex hull of a set E in $\mathbb{C}$ is the intersection of all convex sets containing E. This smallest convex set containing E will be denoted by coE.*

Lemma 1.2.2. *[4] If $p(z)$ is analytic in $\mathcal{U}, p(0) = 1$ and $\Re\big(p(z)\big) > \frac{1}{2}, z \in \mathcal{U}$, then for any function F analytic in $\mathcal{U}$, the function $p * F$ takes its values in the convex hull of $F(\mathcal{U})$.*

Definition 1.2.8. *[92, Definition 4.2.1, p. 55] The function $f \in \mathcal{H}(\mathcal{U})$ is called convex function in $\mathcal{U}$ (or, simply, convex) if f is univalent in $\mathcal{U}$ and $f(\mathcal{U})$ is a convex domain.*

Theorem 1.2.6. *[92, Theorem 4.2.1, p. 56] Let the function $f \in \mathcal{H}(\mathcal{U})$. Then the function f is convex if and only if $f'(0) \neq 0$ and*

$$\Re\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in \mathcal{U}.$$

Theorem 1.2.7. ***(Alexander's duality theorem)*** *[9] The function f is convex in $\mathcal{U}$ if and only if the function $F(z) = zf'(z)$ is starlike in $\mathcal{U}$.*

Definition 1.2.9. *[92, Definition 4.2.2, p. 58] We will denote by $\mathcal{K}$ the class of functions $f \in \mathcal{A}$ which are convex (and normalized) in the unit disk $\mathcal{U}$, i. e.*

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \Re \frac{zf''(z)}{f'(z)} + 1 > 0, z \in \mathcal{U} \right\}.$$

Definition 1.2.10. *[92, Definition 4.6.1, p. 90] The function $f \in \mathcal{H}(\mathcal{U})$ is called close-to-convex if there exists a function ϕ convex in $\mathcal{U}$, such that*

$$\Re \frac{f'(z)}{\phi'(z)} > 0, \quad z \in \mathcal{U}.$$

1.2.3 Classes of functions related to starlikeness and convexity

The well-known class of alpha-convex functions, introduced by P. T. Mocanu in 1969 (see [88]) is a transition between starlike and convex functions.

Definition 1.2.11. *[85, p.10] The class of alpha-convex functions is defined by*

$$\mathcal{M}_\alpha = \{ f \in \mathcal{A} : \Re J(\alpha, f; z) > 0 \},$$

where

$$J(\alpha, f; z) \equiv (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right),$$

for $\alpha \in \mathbb{R}$.

Remark 1.2.5. *$\mathcal{M}_0 = \mathcal{S}^*$ and $\mathcal{M}_1 = \mathcal{K}$.*

The notion of gamma-starlike functions was introduced by Z. Lewandowski et al (see [77]) in 1974.

Definition 1.2.12. *[77] The class of γ-starlike functions is defined by*

$$\mathcal{L}_\gamma = \{ f \in \mathcal{A} : \Re \mathcal{L}(\gamma, f; z) > 0 \},$$

where

$$\mathcal{L}(\gamma, f; z) \equiv \left(\frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left(\frac{zf''(z)}{f'(z)} + 1 \right)^{\gamma},$$

for $\gamma \in \mathbb{R}$.

Remark 1.2.6. *$\mathcal{L}_0 = \mathcal{S}^*$ and $\mathcal{L}_1 = \mathcal{K}$.*

Definition 1.2.13. *[113] The class of starlike functions of order* α *contains analytic functions in* $\mathcal{U}$*, satisfying the conditions* $f(0) = 0, f'(0) \neq 0$ *and*

$$\Re \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathcal{U},$$

where $0 \leq \alpha < 1$*.*

Definition 1.2.14. *[113] The class of convex functions of order* α *contains analytic functions in* $\mathcal{U}$*, satisfying the conditions* $f'(0) \neq 0$ *and*

$$\Re \left(\frac{zf''(z)}{f'(z)} + 1 \right) > \alpha, \quad z \in \mathcal{U},$$

where $0 \leq \alpha < 1$*.*

These classes were introduced by M. S. Robertson and are denoted by $\mathcal{S}^*(\alpha)$, respectively $\mathcal{K}(\alpha)$.

Definition 1.2.15. *[40] Let* $f \in \mathcal{A}$ *be given by* (1.1)*, and let* $\gamma \in \mathbb{R}, 0 \leq \alpha < 1$*. Then* $f \in \mathcal{L}_\gamma(\alpha)$*, called the class of gamma-starlike functions of order alpha if and only if*

$$\Re \left[\left(\frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left(\frac{zf''(z)}{f'(z)} + 1 \right)^{\gamma} \right] > \alpha, \quad z \in \mathcal{U}. \tag{1.5}$$

Remark 1.2.7. $\mathcal{L}_0(\alpha) = \mathcal{S}^*(\alpha), \mathcal{L}_1(\alpha) = \mathcal{K}(\alpha)$ *and* $\mathcal{L}_\gamma(0) = \mathcal{L}_\gamma$*.*

The following two classes of functions were studied by P. T. Mocanu and M. Nunokawa (see [89, 90, 96]).

Definition 1.2.16. *Let* $0 < \gamma \leq 1$*. A function* $f \in \mathcal{A}$ *is called strongly starlike of order* γ *if*

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2}\gamma, \quad z \in \mathcal{U}.$$

A function $f \in \mathcal{A}$ *is called strongly convex of order* γ *if*

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2}\gamma, \quad z \in \mathcal{U}.$$

Let $\mathcal{Q}$ be the class of functions $\phi \in \mathcal{P}$ such that $\phi(\mathcal{U})$ is convex and symmetrical with respect to the real axis.

Definition 1.2.17. *[67, 80] For some* $\phi \in \mathcal{Q}$*, let the classes* $\mathcal{S}^*(\phi), \mathcal{K}(\phi)$ *and* $\mathcal{C}(\phi, \psi)$ *be defined, respectively, by*

$$\mathcal{S}^*(\phi) = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \prec \phi(z), z \in \mathcal{U} \right\},$$

$$\mathcal{K}(\phi) = \left\{ f : f \in \mathcal{A}, 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in \mathcal{U} \right\}$$

and

$$\mathcal{C}(\phi, \psi) = \left\{ f : f \in \mathcal{A}, h \in \mathcal{K}(\psi), \frac{f'(z)}{h'(z)} \prec \phi(z), z \in \mathcal{U} \right\}.$$

Remark 1.2.8. *Note that $f \in \mathcal{K}(\phi)$ if and only if $zf' \in \mathcal{S}^*(\phi)$, and $f \in \mathcal{C}(\phi, \psi)$ if and only if $\exists g \in \mathcal{S}^*(\psi)$ such that $\frac{zf'(z)}{g(z)} \prec \phi(z)$ in $\mathcal{U}$.*

Definition 1.2.18. *[62] A function f of the form* (1.1) *is said to be δ-uniformly convex in $\mathcal{U}$, if it satisfies the following condition:*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta\left|\frac{zf''(z)}{f'(z)}\right|, \quad \delta \geq 0.$$

The class of all δ-uniformly convex functions are denoted by $\delta - \mathcal{UCV}$, studied by S. Kanas and A. Wisniowska [62].

Theorem 1.2.8. *[62] Let $f \in \mathcal{A}$. If for some $\delta, 0 \leq \delta < \infty$, the inequality*

$$\sum_{k=2}^{\infty} k(k-1)|a_k| \leq \frac{1}{\delta + 2}$$

holds, then $f \in \delta - \mathcal{UCV}$. The number $\frac{1}{\delta+2}$ can not be increased.

Definition 1.2.19. *[63] A function f of the form* (1.1) *belongs to the class $\delta - \mathcal{ST}$, if it satisfies the following condition*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad \delta \geq 0.$$

This class is also studied by S. Kanas and A. Wisniowska [63].

Theorem 1.2.9. *[63] If for a function of the form* (1.1) *the condition*

$$\sum_{k=2}^{\infty} [k(\delta + 1) - \delta]|a_k| \leq 1$$

holds true for some $\delta, 0 \leq \delta < \infty$, then $f \in \delta - \mathcal{ST}$.
The result is sharp with equality for the function $f(z) = z - \frac{z^k}{k(\delta+1)-\delta}$.

Remark 1.2.9. *For $\delta = 0$ the classes $\delta - \mathcal{UCV}$ and $\delta - \mathcal{ST}$ are reduced to the classes of convex and starlike functions studied by M. S. Robertson and for $\delta = 1$ these classes are reduced to the classes of uniformly convex and uniformly starlike functions studied by A. W. Goodman [43, 44].*

Definition 1.2.20. *[109] The classes* $\mathcal{S}_\lambda^*$ *and* $\mathcal{C}_\lambda$ *are defined by*

$$\mathcal{S}_\lambda^* = \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \lambda, z \in \mathcal{U}, \lambda > 0 \right\}$$

and

$$\mathcal{C}_\lambda = \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)}{f'(z)} \right| < \lambda, z \in \mathcal{U}, \lambda > 0 \right\}.$$

Remark 1.2.10. *We have,*

$$f(z) \in \mathcal{C}_\lambda \iff zf'(z) \in \mathcal{S}_\lambda^*, \lambda > 0.$$

Theorem 1.2.10. *[109] If for a function of the form* (1.1) *the condition*

$$\sum_{k=2}^{\infty} (k + \lambda - 1)|a_k| \leq \lambda$$

holds true for some $\lambda, \lambda > 0$*, then* $f \in \mathcal{S}_\lambda^*$.

1.3 Differential subordinations

Definition 1.3.1. *[85, p. 15] Let* Ω *and* Δ *be any sets in* $\mathbb{C}$*, let* p *be analytic in the unit disk* $\mathcal{U}$ *with* $p(0) = a$ *and let* $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$*. If*

$$\{\psi(p(z), zp'(z), z^2p''(z); z) | z \in \mathcal{U}\} \subset \Omega \implies p(\mathcal{U}) \subset \Delta, \tag{1.6}$$

then ψ *is called admissible function.*

Remark 1.3.1. *[85, p. 15, 16] If* Δ *is a simply connected domain containing the point* a *and* $\Delta \neq \mathbb{C}$*, then there is a conformal mapping* q *of* $\mathcal{U}$ *onto* Δ *such that* $q(0) = a$*. In this case* (1.6) *can be rewritten as*

$$\{\psi(p(z), zp'(z), z^2p''(z); z) | z \in \mathcal{U}\} \subset \Omega \implies p(z) \prec q(z).$$

If Ω *is also a simply connected domain and* $\Omega \neq \mathbb{C}$*, then there is a conformal mapping* h *of* $\mathcal{U}$ *onto* Ω *such that* $h(0) = \psi(a, 0, 0; 0)$*. If in addition, the function* $\psi(p(z), zp'(z), z^2p''(z); z)$ *is analytic in* $\mathcal{U}$*, then* (1.6) *can be rewritten as*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \implies p(z) \prec q(z).$$

Definition 1.3.2. *[85, p. 16] Let* $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ *and let* h *be univalent in* $\mathcal{U}$*. If* p *is analytic in* $\mathcal{U}$ *and satisfies the (second-order) differential subordination*

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \prec h(z), \tag{1.7}$$

then p *is called a solution of the differential subordination. The univalent function* q *is called a dominant of the solutions of the differential subordination, or more simply a dominant, if* $p \prec q$ *for all* p *satisfying* (1.7)*. A dominant* $\tilde{q}$ *that satisfies* $\tilde{q} \prec q$ *for all dominants* q *of* (1.7) *is said to be the best dominant of* (1.7)*. (Note that the best dominant is unique up to a rotation of* $\mathcal{U}$*).*

Definition 1.3.3. *[85, Definition 2.2b, p.21] We denote by* Q *the set of functions* q *that are analytic and injective on* $\overline{\mathcal{U}} \setminus E(q)$*, where*

$$E(q) = \left\{\zeta \in \partial\mathcal{U} : \lim_{z \to \zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ *for* $\zeta \in \partial\mathcal{U} \setminus E(q)$*.*

Lemma 1.3.1. *[85, Lemma 2.2d, p.24] Let* $q \in Q$*, with* $q(0) = a$*, and let* $p(z) = a + a_n z^n + ...$ *be analytic in* $\mathcal{U}$ *with* $p(z) \not\equiv a$ *and* $n \geq 1$*. If* p *is not subordinate to* q*, then there exist points* $z_0 = r_0 e^{i\theta_0} \in \mathcal{U}$ *and* $\zeta_0 \in \partial\mathcal{U} \setminus E(q)$*, and an* $m \geq n \geq 1$ *for which* $p(\mathcal{U}_{r_0}) \subset q(\mathcal{U})$*,*

$$p(z_0) = q(\zeta_0),$$

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),$$

$$\Re\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\Re\left[\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$$

Definition 1.3.4. *[85, Case 1, p.33] Let* k *be a positive integer,* $a \in \mathbb{C}$ *with* $|a| < M, M > 0$*. The class of admissible functions* $\Psi_k[M, a]$*, consists of those functions* $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ *that satisfy the admissibility condition:*

$$\big|\psi(r, s, t; z)\big| \geq M, \quad z \in \mathcal{U},$$

where

$$r = Me^{i\theta},$$

$$s = m\frac{M\big|M - \overline{a}e^{i\theta}\big|^2}{M^2 - |a|^2}e^{i\theta},$$

$$\Re\frac{t}{s} + 1 \geq m\frac{|M - \overline{a}e^{i\theta}|^2}{M^2 - |a|^2},$$

$\theta \in \mathbb{R}$ *and* $m \geq k$*.*

Theorem 1.3.1. *[85, Theorem 2.3 h, (ii), p. 34] Let $p \in \mathcal{H}[a,k]$. If $\psi \in \Psi_k[M,a]$, then*

$$\left|\psi(p(z), zp'(z), z^2p"(z); z)\right| < M \implies \left|p(z)\right| < M.$$

Definition 1.3.5. *[85, Case 2, p. 34] Let k be a positive integer, $a \in \mathbb{C}$ with $\Re a > 0$. The class of admissible functions $\Psi_k[a]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\Re\psi(\rho i, \sigma, \mu + i\nu; z) \leq 0, \quad z \in \mathcal{U},$$

where $\rho, \sigma, \mu, \nu \in \mathbb{R}$,

$$\sigma \leq -\frac{k}{2}\frac{\left|a - i\rho\right|^2}{\Re a}, \sigma + \mu \leq 0.$$

Theorem 1.3.2. *[85, Theorem 2.3 i, (ii), p. 35] Let $p \in \mathcal{H}[a,k]$. If $\psi \in \Psi_k[a]$, then*

$$\Re\psi\big(p(z), zp'(z), z^2p"(z); z\big) > 0 \implies \Re p(z) > 0.$$

Lemma 1.3.2. *[83] Let $\phi(u,v)$ be a complex valued function, $\phi : D \to \mathbb{C}, D \subset \mathbb{C}^2$, and let $u = u_1 + \mathrm{i}u_2, v = v_1 + \mathrm{i}v_2$. Suppose that the function $\phi(u,v)$ satisfies the following conditions:*
(i) $\phi(u,v)$ is continuous in D,
(ii) $(1,0) \in D$ and $\Re(\phi(1,0)) > 0$,
(iii) $\Re(\phi(\mathrm{i}u_2, v_1)) \leq 0$ for all $(\mathrm{i}u_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$.
Let $p(z) = 1 + p_1z + p_2z^2 + \dots$ be regular in $\mathcal{U}$ such that $(p(z), zp'(z)) \in D$ for all $z \in \mathcal{U}$. If

$$\Re(\phi(p(z), zp'(z))) > 0, \quad z \in \mathcal{U},$$

then $\Re(p(z)) > 0, z \in \mathcal{U}$.

Remark 1.3.2. *The function $\phi(u,v)$ is a particular case of an admissible function of the same type as the one in the Definition 1.3.5, and the conclusion is from Theorem 1.3.2.*

Theorem 1.3.3. *[85, Theorem 3.1b, p.71] Let h be convex in $\mathcal{U}$, with $h(0) = a$, $\gamma \neq 0$ and $\Re\gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt.$$

The function q is convex and is the best (a,n)-dominant.

Lemma 1.3.3. *[92, Lemma 13.5.1, p.419] Let q be a convex function in $\mathcal{U}$ and let*

$$h(z) = q(z) + m\alpha zq'(z),$$

where $\alpha > 0$ and m is a positive integer. If $p \in \mathcal{H}[q(0), m]$ and

$$p(z) + \alpha zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z),$$

and this result is sharp.

1.4 Subclassses of meromorphic functions

Let the function φ of the form (1.2) be a meromorphic function in $\mathcal{U}^- = \{\xi \in \mathbb{C}_\infty : |\xi| > 1\}$, with a simple pole $\xi = \infty$. We denote the set $\mathbb{C} \setminus \phi(\mathcal{U}^-)$ by $E(\phi)$.

Definition 1.4.1. *[92, Definition 4.8.1, p. 102] We say that the function φ of the form (1.2) is a starlike function in $\mathcal{U}^-$ if φ is univalent in $\mathcal{U}^-$ and the set $E(\varphi)$ is starlike with respect to the origin.*

Definition 1.4.2. *[92, Definition 4.8.3, p. 103] Let the function $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \ldots, 0 < |z| < 1$, be a meromorphic function in $\mathcal{U}^* = \{z : 0 < |z| < 1\}$. We say that the function g is starlike in $\mathcal{U}^*$ if the function $\varphi(\xi) = g\big(\frac{1}{\xi}\big), \xi \in \mathcal{U}^-$ is starlike in $\mathcal{U}^-$.*

Theorem 1.4.1. *[92, Theorem 4.8.1, p. 103] Let $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \ldots, z \in \mathcal{U}^*$ be a meromorphic function in $\mathcal{U}$ with $g(z) \neq 0, z \in \mathcal{U}^*$. Then the function g is starlike in $\mathcal{U}^*$ if and only if g is univalent in $\mathcal{U}^*$ and*

$$\Re\bigg(-\frac{zg'(z)}{g(z)}\bigg) > 0, \quad z \in \mathcal{U}^*.$$

Definition 1.4.3. *[92, Definition 4.8.4, p. 104] We say that the function φ of the form (1.2) is a convex function in $\mathcal{U}^-$ if φ is univalent in $\mathcal{U}^-$ and the set $E(\varphi)$ is convex.*

Theorem 1.4.2. *[92, Theorem 4.8.2, p. 104] Let* $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \dots, z \in \mathcal{U}^*$ *be a meromorphic function in* $\mathcal{U}$ *with* $g(z) \neq 0, z \in \mathcal{U}^*$. *Then the function* g *is convex in* $\mathcal{U}$ *if and only if* g *is univalent in* $\mathcal{U}^*$ *and*

$$\Re\left(-\left(\frac{zg''(z)}{g'(z)} + 1\right)\right) > 0, \quad z \in \mathcal{U}^*.$$

1.5 Differential and integral operators

A convolution (or a Hadamard product) " $*$ " between two functions $f, g \in \mathcal{A}$ of the form $f(z) = z + \sum\limits_{k=1}^{\infty} a_{k+1} z^{k+1}$ and $g(z) = z + \sum\limits_{k=1}^{\infty} b_{k+1} z^{k+1}$, $z \in \mathcal{U}$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=1}^{\infty} a_{k+1} b_{k+1} z^{k+1}.$$

Definition 1.5.1. *[117] For* $f \in \mathcal{A}$, *the Sălăgean differential operator* $\mathcal{D}^n$ *of order* n, $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, *is defined by*

$$\mathcal{D}^0 f(z) = f(z),$$

$$\mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z),$$

$$\mathcal{D}^n f(z) = \mathcal{D}(\mathcal{D}^{n-1} f(z)), \quad n \in \mathbb{N}^*.$$

Remark 1.5.1. *[117] The series expression of the operator* $\mathcal{D}^n$ *for the function* $f \in \mathcal{A}$ *of the form* (1.1) *is given by*

$$\mathcal{D}^n f(z) = z + \sum_{k=1}^{\infty} (k+1)^n a_{k+1} z^{k+1}, \quad n \in \mathbb{N}.$$

Definition 1.5.2. *[117] For* $f \in \mathcal{A}$, *the Sălăgean integral operator* $\mathcal{I}^n$ *of order* n, $n \in \mathbb{N}$ *is defined by*

$$\mathcal{I}^0 f(z) = f(z),$$

$$\mathcal{I}^1 f(z) = \mathcal{I} f(z) = \int_0^z f(t) t^{-1} dt,$$

$$\mathcal{I}^n f(z) = \mathcal{I}(\mathcal{I}^{n-1} f(z)).$$

Remark 1.5.2. *[117] The series expression of the operator* $\mathcal{I}^n$ *for the function* $f \in \mathcal{A}$ *of the form* (1.1) *is given by*

$$\mathcal{I}^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k, \quad n \in \mathbb{N}.$$

Definition 1.5.3. *[115] The Ruscheweyh operator $\mathcal{R}^\lambda : \mathcal{A} \to \mathcal{A}$, $\lambda \geq -1$, is defined by*

$$\mathcal{R}^\lambda f(z) = \frac{z}{(1-z)^{1+\lambda}} * f(z), \quad z \in \mathcal{U},$$

and for $\lambda \in \mathbb{N}$ this operator is defined by

$$\mathcal{R}^\lambda f(z) = \frac{z(z^{\lambda-1} f(z))^{(\lambda)}}{\lambda!}, \quad z \in \mathcal{U}.$$

Remark 1.5.3. *[115] The series expression of Ruscheweyh operator for $f \in \mathcal{A}$ of the form* (1.1) *is given by*

$$\mathcal{R}^\lambda f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(k+1+\lambda)}{\Gamma(\lambda+1)\Gamma(k+1)} a_{k+1} z^{k+1}, \quad \lambda > -1, z \in \mathcal{U},$$

and Γ is the familiar gamma function.

The following operators were defined by S. Owa [101].

Definition 1.5.4. *[101] The fractional integral operator $D_z^{-\mu}$ of order $\mu, \mu > 0$, for the function $f \in \mathcal{A}$ is defined by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt, \quad z \in \mathcal{U},$$

where the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$.

Also, the fractional derivative operator D_z^λ is defined of order $\lambda, \lambda \geq 0$, for the function $f \in \mathcal{A}$ by

$$D_z^\lambda f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\lambda} dt, & 0 \leq \lambda < 1 \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z), & n \leq \lambda < n+1 \end{cases}, \quad n \in \mathbb{N},$$

where the multiplicity of $(z-t)^{-\lambda}$ is understood similarly.

The following operator was defined by S. Owa and H. M. Srivastava [102].

Definition 1.5.5. *[102] The fractional differintegral operator $\Omega_z^\lambda : \mathcal{A} \to \mathcal{A}, -\infty < \lambda < 2$, is defined by*

$$\Omega_z^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), \quad z \in \mathcal{U},$$

where $D_z^\lambda f(z)$ is the fractional integral of order λ, $-\infty < \lambda < 0$, and a fractional derivative of order $\lambda, 0 \leq \lambda < 2$.

Remark 1.5.4. *[102] The series expression of the operator Ω_z^λ for the function $f \in \mathcal{A}$ of the form* (1.1) *is given by*

$$\Omega_z^\lambda f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(k+2)}{\Gamma(k+2-\lambda)} a_{k+1} z^{k+1}, \quad -\infty < \lambda < 2, z \in \mathcal{U}.$$

P. Sharma, R. K. Raina and G. Ş. Sălăgean [120] defined the operator $\mathbb{D}_\lambda^{\nu,n}$ as:

Definition 1.5.6. *[120] The fractional operator $\mathbb{D}_\lambda^{\nu,n} : \mathcal{A} \to \mathcal{A}$ for $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}$ is the composition of fractional differintegral operator, the Sălăgean operator and the Ruscheweyh operator.*

Remark 1.5.5. *[120]*

$$\mathbb{D}_\lambda^{\nu,n} f(z) = \mathcal{R}^\nu \mathcal{D}^n \Omega_z^\lambda f(z) = \begin{cases} \mathcal{D}^n \Omega_z^\lambda f(z), & \nu = 0 \\ (1 - \frac{1}{\nu})\mathbb{D}_\lambda^{\nu-1,n} f(z) + \frac{1}{\nu} z(\mathbb{D}_\lambda^{\nu-1,n} f(z))', & \nu \neq 0 \end{cases}. \tag{1.8}$$

Remark 1.5.6. *[120] The series expression of $\mathbb{D}_\lambda^{\nu,n} f(z)$ for $f \in \mathcal{A}$ of the form* (1.1) *is given by*

$$\mathbb{D}_\lambda^{\nu,n} f(z) = z + \sum_{k=1}^{\infty} \frac{(\nu+1)_k}{(2-\lambda)_k} (k+1)^{n+1} a_{k+1} z^{k+1}, \tag{1.9}$$

$-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}, z \in \mathcal{U}$, *where the symbol $(\gamma)_k$ denotes the usual Pochhammer symbol, for $\gamma \in \mathbb{C}$, defined by*

$$(\gamma)_k = \begin{cases} 1, & k = 0 \\ \gamma(\gamma+1)...(\gamma+k-1), & k \in \mathbb{N}^* \end{cases} = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \quad \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

Remark 1.5.7. *[129] Using the definition of the operator $\mathbb{D}_\lambda^{\nu,n}$, respectively the relation* (1.9)*, we observe that*

$$\mathbb{D}_\lambda^{\nu,n} f(z) = \mathcal{R}^\nu \mathcal{D}^n \Omega_z^\lambda f(z) = \begin{cases} \mathcal{R}^\nu \Omega_z^\lambda f(z), & n = 0 \\ z(\mathbb{D}_\lambda^{\nu,n-1} f(z))', & n \neq 0 \end{cases} \tag{1.10}$$

and

$$\mathbb{D}_\lambda^{\nu,n} f(z) = \mathcal{R}^\nu \mathcal{D}^n \Omega_z^\lambda f(z) = \begin{cases} \mathcal{R}^\nu \mathcal{D}^n f(z), & \lambda = 0 \\ \frac{1-\lambda}{2-\lambda}\mathbb{D}_{\lambda-1}^{\nu,n} f(z) + \frac{1}{2-\lambda} z(\mathbb{D}_{\lambda-1}^{\nu,n} f(z))', & \lambda \neq 0 \end{cases}. \tag{1.11}$$

Remark 1.5.8. *[120] The fractional operator $\mathbb{D}_0^{\nu,0}$ is precisely the Ruscheweyh derivative operator $\mathcal{R}^\nu$ of order $\nu, \nu > -1$, and $\mathbb{D}_\lambda^{0,0}$ is the fractional differintegral operator Ω_z^λ of order $\lambda, -\infty < \lambda < 2$, while $\mathbb{D}_0^{0,n} = \mathcal{D}^n$ and $\mathbb{D}_\lambda^{1-\lambda,n} = \mathcal{D}^{n+1}$ are the Sălăgean operators, respectively, of order n and n+1, $n \in \mathbb{N}$.*

Remark 1.5.9. *[129] The fractional operator* $\mathbb{D}_1^{1,n}$ *is the Sălăgean operator* $\mathcal{D}^{n+2}$.

Remark 1.5.10. *[120] The operator* $\mathbb{D}_\lambda^{\nu,n}$ *satisfies the following identity:*

$$\mathbb{D}_\lambda^{\nu+1,n}f(z) = \frac{\nu}{\nu+1}\mathbb{D}_\lambda^{\nu,n}f(z) + \frac{1}{\nu+1}z(\mathbb{D}_\lambda^{\nu,n}f(z))', \tag{1.12}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}$.

Remark 1.5.11. *[129] Making use of* (1.10) *and* (1.11)*, we obtain that the operator* $\mathbb{D}_\lambda^{\nu,n}$ *satisfies the following identities:*

$$\mathbb{D}_\lambda^{\nu,n+1}f(z) = z(\mathbb{D}_\lambda^{\nu,n}f(z))', \tag{1.13}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}$,

and

$$\mathbb{D}_{\lambda+1}^{\nu,n}f(z) = -\frac{\lambda}{1-\lambda}\mathbb{D}_\lambda^{\nu,n}f(z) + \frac{1}{1-\lambda}z(\mathbb{D}_\lambda^{\nu,n}f(z))', \tag{1.14}$$

where $-\infty < \lambda < 1, \nu > -1, n \in \mathbb{N}$.

Definition 1.5.7. *[4] For a function* $f \in \mathcal{A}, \delta \geq 0$ *and* $n \in \mathbb{N}$*, the Al-Oboudi differential operator* $\mathcal{D}_\delta^n f$ *is defined by*

$$\mathcal{D}_\delta^0 f(z) = f(z),$$

$$\mathcal{D}_\delta^1 f(z) = (1-\delta)f(z) + \delta z f'(z) = \mathcal{D}_\delta f(z),$$

$$\mathcal{D}_\delta^n f(z) = \mathcal{D}_\delta\big(\mathcal{D}_\delta^{n-1}f(z)\big), \quad z \in \mathcal{U}.$$

Remark 1.5.12. $\mathcal{D}_\delta^n$ *is a linear operator and for* $f \in \mathcal{A}$,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we have

$$\mathcal{D}_\delta^n f(z) = z + \sum_{k=2}^{\infty}\big[1+(k-1)\delta\big]^n a_k z^k, \quad z \in \mathcal{U} \tag{1.15}$$

and

$$\mathcal{D}_\delta^{n+1}f(z) = (1-\delta)\mathcal{D}_\delta^n f(z) + \delta z\big(\mathcal{D}_\delta^n f(z)\big)', \quad z \in \mathcal{U}. \tag{1.16}$$

For $\delta = 1$*, we obtain the Sălăgean differential operator (see Definition* 1.5.1*).*

Remark 1.5.13. *Differentiating* (1.16)*, we obtain*

$$\big(\mathcal{D}_\delta^{n+1}f(z)\big)' = \big(\mathcal{D}_\delta^n f(z)\big)' + \delta z\big(\mathcal{D}_\delta^n f(z)\big)'', \quad z \in \mathcal{U}. \tag{1.17}$$

Definition 1.5.8. *[103] For a function $f \in \mathcal{A}, \delta > 0$ and $n \in \mathbb{N}$, the operator $\mathcal{I}_\delta^n f$ is defined by*

$$\mathcal{I}_\delta^0 f(z) = f(z),$$
$$\mathcal{I}_\delta^1 f(z) = \frac{1}{\delta} z^{1-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-2} f(t)dt = \mathcal{I}_\delta f(z),$$
$$\mathcal{I}_\delta^n f(z) = \mathcal{I}_\delta\big(\mathcal{I}_\delta^{n-1} f(z)\big), \quad z \in \mathcal{U}.$$

Remark 1.5.14. *If $f \in \mathcal{A}$ and $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, then*

$$\mathcal{I}_\delta^n f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1}{1+(k-1)\delta}\right]^n a_k z^k, \quad z \in \mathcal{U} \tag{1.18}$$

and

$$\delta z\big(\mathcal{I}_\delta^n f(z)\big)' = \mathcal{I}_\delta^{n-1} f(z) - (1-\delta)\mathcal{I}_\delta^n f(z), \quad z \in \mathcal{U}. \tag{1.19}$$

For $\delta = 1$, we obtain the Sălăgean integral operator (see Definition 1.5.2*).*

Remark 1.5.15. *Using* (1.19)*, we have*

$$\big(\mathcal{I}_\delta^n f(z)\big)' = \big(\mathcal{I}_\delta^{n+1} f(z)\big)' + \delta z\big(\mathcal{I}_\delta^{n+1} f(z)\big)'', \quad z \in \mathcal{U}. \tag{1.20}$$

Definition 1.5.9. *[92, p. 384] We define the Bernardi integral operator $\mathcal{L}_c : \mathcal{A} \to \mathcal{A}$*

$$\mathcal{L}_c f(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt, \quad c > -1.$$

Chapter 2

New results on analytic or meromorphic functions obtained by using some operators

The results in this chapter are obtained by using some operators.

In Section 2.1, we obtain various results using the fractional operator $\mathbb{D}_{\lambda}^{\nu,n}$. A class of analytic functions defined by this operator is introduced. Inclusion relations, convolution property, extreme points of the class and other results are given. Differential subordinations are investigated and geometric properties of analytic functions are obtained. In the last subsection, coefficient bounds and Fekete-Szegő inequalities are obtained for some classes of analytic functions, involving the fractional operator $\mathbb{D}_{\lambda}^{\nu,n}$.

In Section 2.2, a class of meromorphic functions defined by using a fractional operator is introduced. Some inclusion relations and other properties of the class are investigated.

In Section 2.3, a new operator is defined. A certain subclass of analytic functions is also introduced using the new operator, and some properties of this class are obtained. Some differential subordinations using the new operator are also investigated.

In Section 2.4, we obtain some inclusion relations between the classes of δ-uniformly convex functions, δ-uniformly starlike functions, respectively the class $\mathcal{US}(n,\alpha)$, defined in a similar way as the another two, using the Sălăgean operator.

In Section 2.5, we introduce a generalized Sălăgean integro-differential operator, using the Al-Oboudi differential operator $\mathcal{D}_{\delta}^{n}$ and the generalized Sălăgean integral operator $\mathcal{I}_{\delta}^{n}$. Differential subordinations are investigated and some previously known results are generalized.

2.1 The fractional operator $\mathbb{D}_\lambda^{\nu,n}$

In this section, some results using the fractional operator $\mathbb{D}_\lambda^{\nu,n}$ defined by (1.8) and (1.9) are obtained.

2.1.1 On a class of analytic functions defined by the operator $\mathbb{D}_\lambda^{\nu,n}$

Definition 2.1.1. *[129] Let $f \in \mathcal{A}$. We say that the function f is in the class $\mathbb{R}_\lambda^{\nu,n}(\alpha)$, where $0 \leq \alpha < 1, -\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}$, if f satisfies the condition*

$$\Re\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)' > \alpha, \quad z \in \mathcal{U}. \tag{2.1}$$

In our investigation, we shall need the following definition and theorem:

Definition 2.1.2. *[4] A sequence $a_0, a_1, \ldots, a_n, \ldots$ of nonnegative numbers is called a convex null sequence if $a_n \to 0$ as $n \to \infty$ and $a_0 - a_1 \geq a_1 - a_2 \geq \cdots \geq a_n - a_{n+1} \geq \cdots \geq 0$.*

The following theorem due to L. Fejér [34] is also used by F. M. Al-Oboudi [4].

Theorem 2.1.1. *[34] Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function $p(z) = \frac{c_0}{2} + \sum\limits_{k=1}^{\infty} c_k z^k, z \in \mathcal{U}$ is analytic and $\Re p(z) > 0$ in $\mathcal{U}$.*

Theorem 2.1.2. *[129] $\mathbb{R}_\lambda^{\nu+1,n}(\alpha) \subset \mathbb{R}_\lambda^{\nu,n}(\alpha)$.*

Remark 2.1.1. *[129] Theorem 2.1.2 can be expressed in the following form:*

$$\Re\left((\mathbb{D}_\lambda^{\nu,n} f(z))' + \frac{1}{\nu+1} z(\mathbb{D}_\lambda^{\nu,n} f(z))''\right) > \alpha \implies \Re(\mathbb{D}_\lambda^{\nu,n} f(z))' > \alpha.$$

If we take $\nu = 1 - \lambda$ in Theorem 2.1.2, we obtain the following result.

Corollary 2.1.1. *[129] If $f \in \mathcal{A}, -\infty < \lambda < 2, n \in \mathbb{N}$ and $0 \leq \alpha < 1$, then*

$$\Re\left[(\mathcal{D}^n f(z))' + \frac{4-\lambda}{2-\lambda} z(\mathcal{D}^n f(z))'' + \frac{1}{2-\lambda} z^2 (D^n f(z))'''\right] > \alpha$$
$$\implies \Re[(\mathcal{D}^n f(z))' + z(\mathcal{D}^n f(z))''] > \alpha,$$

where $z \in \mathcal{U}$ and $\mathcal{D}^n$ is the Sălăgean operator defined in the Definition 1.5.1.

Example 2.1.1. *[129] Taking $n = 0$ in Corollary 2.1.1, we have:*

$$\Re\left(f'(z) + \frac{4-\lambda}{2-\lambda} zf''(z) + \frac{1}{2-\lambda} z^2 f'''(z)\right) > \alpha \implies \Re(f'(z) + zf''(z)) > \alpha, \quad z \in \mathcal{U}.$$

Theorem 2.1.3. *[129]* $\mathbb{R}_\lambda^{\nu,n+1}(\alpha) \subset \mathbb{R}_\lambda^{\nu,n}(\alpha)$.

Remark 2.1.2. *[129] Theorem 2.1.3 can be expressed in the following form:*

$$\Re\big((\mathbb{D}_\lambda^{\nu,n} f(z))' + z(\mathbb{D}_\lambda^{\nu,n} f(z))''\big) > \alpha \implies \Re(\mathbb{D}_\lambda^{\nu,n} f(z))' > \alpha.$$

Theorem 2.1.4. *[129]* $\mathbb{R}_{\lambda+1}^{\nu,n}(\alpha) \subset \mathbb{R}_\lambda^{\nu,n}(\alpha)$, *for* $-\infty < \lambda < 1$.

Remark 2.1.3. *[129] Theorem 2.1.4 can be expressed in the following form:*

$$\Re\left((\mathbb{D}_\lambda^{\nu,n} f(z))' + \frac{1}{1-\lambda} z(\mathbb{D}_\lambda^{\nu,n} f(z))''\right) > \alpha \implies \Re(\mathbb{D}_\lambda^{\nu,n} f(z))' > \alpha.$$

Theorem 2.1.5. *[129] Let* $f \in \mathbb{R}_\lambda^{\nu,n}(\alpha)$ *and* $g \in \mathcal{K}$, *where* $\mathcal{K}$ *denotes the class of convex functions. Then* $f * g \in \mathbb{R}_\lambda^{\nu,n}(\alpha)$.

Theorem 2.1.6. *[129] The set* $\mathbb{R}_\lambda^{\nu,n}(\alpha)$ *is convex.*

Theorem 2.1.7. *[129] The extreme points of* $\mathbb{R}_\lambda^{\nu,n}(\alpha)$ *are*

$$f_x(z) = z + 2(1-\alpha)\sum_{k=1}^{\infty} \frac{(2-\lambda)_k}{(k+1)^{n+2}(\nu+1)_k} x^k z^{k+1}, \quad |x| = 1, z \in \mathcal{U}. \tag{2.2}$$

Corollary 2.1.2. *[129] Let* $f \in \mathbb{R}_\lambda^{\nu,n}(\alpha)$. *Then*

$$|a_{k+1}| \leq \frac{2(1-\alpha)(2-\lambda)_k}{(k+1)^{n+2}(\nu+1)_k}, \quad k \geq 1.$$

The result is sharp.

Corollary 2.1.3. *[129] Let* $f \in \mathbb{R}_\lambda^{\nu,n}(\alpha)$. *Then*

$$|f(z)| \leq r + \sum_{k=1}^{\infty} \frac{2(1-\alpha)(2-\lambda)_k}{(k+1)^{n+2}(\nu+1)_k} r^{k+1}, \quad |z| = r,$$

$$|f'(z)| \leq 1 + \sum_{k=1}^{\infty} \frac{2(1-\alpha)(2-\lambda)_k}{(k+1)^{n+1}(\nu+1)_k} r^{k}, \quad |z| = r.$$

The result is sharp.

Theorem 2.1.8. *[129] Let* $f \in \mathbb{R}_\lambda^{\nu+1,n}(\alpha)$. *Then* $f \in \mathbb{R}_\lambda^{\nu,n}(\beta)$, *where*

$$\beta = 2\alpha - 1 + 2(1-\alpha)(\nu+1)\int_0^1 \frac{t^\nu}{t+1} dt.$$

If we take $\lambda = 0, \nu = 1$ in Theorem 2.1.8, we obtain the following result.

Corollary 2.1.4. *[129] If $f \in \mathcal{A}, n \in \mathbb{N}$, then*

$$\Re\left[(\mathcal{D}^n f(z))' + 2z(\mathcal{D}^n f(z))'' + \frac{1}{2}z^2(\mathcal{D}^n f(z))'''\right] > \alpha$$

$$\Longrightarrow \Re[(\mathcal{D}^n f(z))' + z(\mathcal{D}^n f(z))''] > \beta,$$

where $z \in \mathcal{U}$, $\beta = 3 - 2\alpha - 4(1-\alpha)\ln 2$ and $\mathcal{D}^n$ is the Sălăgean operator defined in the Definition 1.5.1.

Remark 2.1.4. *[129] Using the result of Corollary 2.1.4, we obtain $\beta > \alpha$. So, Theorem 2.1.8 gives us a better result than Theorem 2.1.2.*

Example 2.1.2. *[129] If we take $\alpha = \frac{3}{4}$ in Corollary 2.1.4, we obtain:*

$$\Re\left[(\mathcal{D}^n f(z))' + 2z(\mathcal{D}^n f(z))'' + \frac{1}{2}z^2(\mathcal{D}^n f(z))'''\right] > \frac{3}{4}$$

$$\Longrightarrow \Re[(\mathcal{D}^n f(z))' + z(\mathcal{D}^n f(z))''] > \frac{3}{2} - \ln 2, \quad z \in \mathcal{U}.$$

Remark 2.1.5. *[129] If we take $\lambda = 1, \nu = 0, n = 0, \alpha = \frac{1}{2}$ in Theorem 2.1.8, we obtain the following result obtained in [28]:*

$$\Re[f'(z) + 3zf''(z) + z^2 f'''(z)] > \frac{1}{2} \Longrightarrow \Re[f'(z) + zf''(z)] > \ln 2, \quad z \in \mathcal{U}.$$

Theorem 2.1.9. *[129] Let $f \in \mathbb{R}_\lambda^{\nu,n+1}(\alpha)$. Then $f \in \mathbb{R}_\lambda^{\nu,n}(\beta)$, where*

$$\beta = 2\alpha - 1 + 2(\alpha - 1)\ln 2.$$

Theorem 2.1.10. *[129] Let $f \in \mathbb{R}_{\lambda+1}^{\nu,n}(\alpha), -\infty < \lambda < 1$. Then $f \in \mathbb{R}_\lambda^{\nu,n}(\beta)$, where*

$$\beta = 2\alpha - 1 + 2(1-\alpha)(1-\lambda)\int_0^1 \frac{t^{-\lambda}}{t+1}dt.$$

2.1.2 Differential subordinations obtained by using the operator $\mathbb{D}_\lambda^{\nu,n}$

Theorem 2.1.11. *[130] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + \frac{1}{\nu+1}zg'(z), \quad \nu > -1.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_\lambda^{\nu+1,n} f(z)\right)' \prec h(z), \tag{2.3}$$

then

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec g(z)$$

and the result is sharp.

Taking $\lambda = n = 0$ in Theorem 2.1.11 we obtain the following result, which is a particular case of Theorem 2.3 from [97].

Corollary 2.1.5. *[97] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + \frac{1}{\nu+1} z g'(z).$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathcal{R}^{\nu+1} f(z)\right)' \prec h(z),$$

then

$$\left(\mathcal{R}^{\nu} f(z)\right)' \prec g(z)$$

and the result is sharp.

Theorem 2.1.12. *[130] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + \frac{1}{1-\lambda} z g'(z), \quad -\infty < \lambda < 1.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)' \prec h(z), \tag{2.4}$$

then

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec g(z)$$

and the result is sharp.

Theorem 2.1.13. *[130] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + z g'(z).$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_\lambda^{\nu,n+1} f(z)\right)' \prec h(z), \tag{2.5}$$

then

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec g(z)$$

and the result is sharp.

Taking $\lambda = \nu = 0$ in Theorem 2.1.13 we obtain the following result, which is a particular case of Theorem 2 from [98]:

Corollary 2.1.6. *[98] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z).$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathcal{D}^{n+1} f(z)\right)' \prec h(z),$$

then

$$\left(\mathcal{D}^{n} f(z)\right)' \prec g(z)$$

and the result is sharp.

Taking $\lambda = 0$ and $\nu = n$ in Theorem 2.1.12 or in Theorem 2.1.13 we obtain the following result from [7]:

Corollary 2.1.7. *[7] Let g be a convex function such that $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $n \in \mathbb{N}$ and the following differential subordination holds

$$\frac{1}{z}\mathbb{D}_0^{n+1,n+1} f(z) + \frac{n}{n+1} z\left(\mathbb{D}_0^{n,n} f(z)\right)'' \prec h(z), \quad z \in \mathcal{U},$$

then

$$\left(\mathbb{D}_0^{n,n} f(z)\right)' \prec g(z), \quad z \in \mathcal{U}$$

and this result is sharp.

Theorem 2.1.14. *[130] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec h(z), \quad z \in \mathcal{U}, \tag{2.6}$$

then

$$\frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z} \prec g(z)$$

and the result is sharp.

Taking $\lambda = \nu = 0$ in Theorem 2.1.14 we obtain the following result, which is a particular case of Theorem 4 from [98]:

Corollary 2.1.8. *[98] Let g be a convex function such that $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z).$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathcal{D}^n f(z)\right)' \prec h(z), \quad z \in \mathcal{U},$$

then

$$\frac{\mathcal{D}^n f(z)}{z} \prec g(z), \quad z \in \mathcal{U}$$

and this result is sharp.

Taking $\lambda = n = 0$ in Theorem 2.1.14 we obtain the following result which is a particular case of Theorem 2.5 from [97]:

Corollary 2.1.9. *[97] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z).$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathcal{R}^\nu f(z)\right)' \prec h(z), \quad z \in \mathcal{U},$$

then

$$\frac{\mathcal{R}^\nu f(z)}{z} \prec g(z)$$

and the result is sharp.

Taking $\lambda = 0$ and $\nu = n$ in Theorem 2.1.14 we obtain the following result from [7]:

Corollary 2.1.10. *[7] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $n \in \mathbb{N}, f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_0^{n,n} f(z)\right)' \prec h(z), \quad z \in \mathcal{U},$$

then

$$\frac{\mathbb{D}_0^{n,n} f(z)}{z} \prec g(z), \quad z \in \mathcal{U}$$

and this result is sharp.

Theorem 2.1.15. *[130] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \right)' \prec h(z), \quad z \in \mathcal{U}, \tag{2.7}$$

then

$$\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Theorem 2.1.16. *[130] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \right)' \prec h(z), \quad z \in \mathcal{U}, -\infty < \lambda < 1, \tag{2.8}$$

then

$$\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Theorem 2.1.17. *[130] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \right)' \prec h(z), \quad z \in \mathcal{U}, \tag{2.9}$$

then

$$\frac{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), \quad z \in \mathcal{U}$$

and the result is sharp.

2.1.3 Some properties of analytic functions otained by using the operator $\mathbb{D}_\lambda^{\nu,n}$

Theorem 2.1.18. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, \frac{mMe^{i\theta}}{\nu + Me^{i\theta}} + Me^{i\theta}\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu+1,n} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu+1,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right)\right| < M, \quad z \in \mathcal{U} \tag{2.10}$$

implies

$$\left|\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M, \quad z \in \mathcal{U}. \tag{2.11}$$

Remark 2.1.6. *[131] Making use of* (1.13)*, the inequalities* (2.10) *and* (2.11) *from Theorem 2.1.18 become:*

$$\left|\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_\lambda^{\nu+1,n+1} f(z)}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right)\right| < M$$

and

$$\left|\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M.$$

Taking, respectively, $\nu = n = \lambda = 0$ and $\nu = \lambda = 0$, $n = 1$ in Theorem 2.1.18 we obtain the following corollaries.

Corollary 2.1.11. *[131] Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi(Me^{i\theta}, m + Me^{i\theta})\right| \geq M,$$

where $M > 1, m \geq 1, \theta \in \mathbb{R}$, and let $f \in \mathcal{A}$, with $f(z) \neq 0$ and $\mathcal{R}^1 f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{zf'(z)}{f(z)}, \frac{z\big(\mathcal{R}^1 f(z)\big)'}{\mathcal{R}^1 f(z)}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{zf'(z)}{f(z)}\right| < M, \quad z \in \mathcal{U}.$$

Corollary 2.1.12. *[131] Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi(Me^{i\theta}, m + Me^{i\theta})\right| \geq M,$$

where $M > 1, m \geq 1, \theta \in \mathbb{R}$, and let $f \in \mathcal{A}$, with $\mathcal{D}^1 f(z) \neq 0$ and $\mathcal{R}^1\mathcal{D}^1 f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{z\big(\mathcal{D}^1 f(z)\big)'}{\mathcal{D}^1 f(z)}, \frac{z\big(\mathcal{R}^1\mathcal{D}^1 f(z)\big)'}{\mathcal{R}^1\mathcal{D}^1 f(z)}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{z\big(\mathcal{D}^1 f(z)\big)'}{\mathcal{D}^1 f(z)}\right| < M, \quad z \in \mathcal{U}.$$

Theorem 2.1.19. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{\nu + \rho i} + \rho i\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1 + \rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu+1,n} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu+1,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right) > 0, \quad z \in \mathcal{U} \tag{2.12}$$

implies

$$\Re\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0, \quad z \in \mathcal{U}. \tag{2.13}$$

Remark 2.1.7. *[131] The inequalities* (2.12) *and* (2.13) *from Theorem 2.1.19 can be expressed:*

$$\Re\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_\lambda^{\nu+1,n+1} f(z)}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right) > 0$$

and

$$\Re\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0.$$

Taking, respectively, $\nu = n = \lambda = 0$ and $\nu = \lambda = 0, n = 1$ in Theorem 2.1.19, we obtain the following corollaries.

Corollary 2.1.13. *[131] Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{\rho i} + \rho i\right) \leq 0,$$

where $\rho, \sigma \in \mathbb{R}, \sigma \leq -\frac{1}{2}(1+\rho^2)$ and let $f \in \mathcal{A}$ with $f(z) \neq 0$ and $\mathcal{R}^1 f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\left(\frac{zf'(z)}{f(z)}, \frac{z\big(\mathcal{R}^1 f(z)\big)'}{\mathcal{R}^1 f(z)}\right) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{zf'(z)}{f(z)} > 0, \quad z \in \mathcal{U}.$$

Remark 2.1.8. *[131] Corollary 2.1.13 is a starlikeness criterion.*

Corollary 2.1.14. *[131] Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{\rho i} + \rho i\right) \leq 0,$$

where $\rho, \sigma \in \mathbb{R}, \sigma \leq -\frac{1}{2}(1+\rho^2)$ and let $f \in \mathcal{A}$ with $\mathcal{D}^1 f(z) \neq 0$ and $\mathcal{R}^1\mathcal{D}^1 f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\left(\frac{z\big(\mathcal{D}^1 f(z)\big)'}{\mathcal{D}^1 f(z)}, \frac{z\big(\mathcal{R}^1\mathcal{D}^1 f(z)\big)'}{\mathcal{R}^1\mathcal{D}^1 f(z)}\right) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{z\big(\mathcal{D}^1 f(z)\big)'}{\mathcal{D}^1 f(z)} > 0, \quad z \in \mathcal{U}.$$

Remark 2.1.9. *[131] Corollary 2.1.14 is a convexity criterion.*

Taking $\psi(p(z), zp'(z)) = p(z) + \delta\frac{zp'(z)}{p(z)}$ in the proof of Theorem 2.1.18 and Theorem 2.1.19, we obtain the following corollaries.

Corollary 2.1.15. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, M > 1, \delta \in \mathbb{R}$ and let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu+1,n} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.*
The inequality

$$\left|(1-\delta)\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} + \delta\frac{\mathbb{D}_\lambda^{\nu,n+2} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M, \quad z \in \mathcal{U}.$$

Corollary 2.1.16. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, \delta \in \mathbb{R}$ and let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu+1,n} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.*
The inequality

$$\Re\left[(1-\delta)\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} + \delta\frac{\mathbb{D}_\lambda^{\nu,n+2} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right] > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0, \quad z \in \mathcal{U}.$$

Theorem 2.1.20. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\Big(Me^{i\theta}, m + Me^{i\theta}\Big)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu,n+1} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu,n+1} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right)\right| < M, \quad z \in \mathcal{U} \tag{2.14}$$

implies

$$\left|\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M, \quad z \in \mathcal{U}. \tag{2.15}$$

Remark 2.1.10. *[131] The inequalities (2.14) and (2.15) from Theorem 2.1.20 can be expressed:*

$$\left|\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_\lambda^{\nu,n+2} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right)\right| < M$$

and

$$\left|\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M.$$

Theorem 2.1.21. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{\rho i} + \rho i\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu,n+1} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu,n+1} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right) > 0, \quad z \in \mathcal{U} \tag{2.16}$$

implies

$$\Re\frac{z\big(\mathbb{D}_\lambda^{\nu,n}f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n}f(z)}>0,\quad z\in\mathcal{U}. \tag{2.17}$$

Remark 2.1.11. *[131] The inequalities* (2.16) *and* (2.17) *from Theorem 2.1.21 can be expressed:*

$$\Re\phi\bigg(\frac{\mathbb{D}_\lambda^{\nu,n+1}f(z)}{\mathbb{D}_\lambda^{\nu,n}f(z)},\frac{\mathbb{D}_\lambda^{\nu,n+2}f(z)}{\mathbb{D}_\lambda^{\nu,n+1}f(z)}\bigg)>0$$

and

$$\Re\frac{\mathbb{D}_\lambda^{\nu,n+1}f(z)}{\mathbb{D}_\lambda^{\nu,n}f(z)}>0.$$

Theorem 2.1.22. *[131] Let $\nu>-1, n\in\mathbb{N}, -\infty<\lambda<1, M>1, m\geq 1, \theta\in\mathbb{R}$ and let $\phi:\mathbb{C}^2\to\mathbb{C}$ be an admissible function that satisfies the condition*

$$\bigg|\phi\bigg(Me^{i\theta},\frac{mMe^{i\theta}}{-\lambda+Me^{i\theta}}+Me^{i\theta}\bigg)\bigg|\geq M.$$

Also, let $f\in\mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n}f(z)\neq 0$ and $\mathbb{D}_{\lambda+1}^{\nu,n}f(z)\neq 0$ for $z\in\mathcal{U}\setminus\{0\}$.
The inequality

$$\bigg|\phi\bigg(\frac{z\big(\mathbb{D}_\lambda^{\nu,n}f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n}f(z)},\frac{z\big(\mathbb{D}_{\lambda+1}^{\nu,n}f(z)\big)'}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\bigg)\bigg|<M,\quad z\in\mathcal{U} \tag{2.18}$$

implies

$$\bigg|\frac{z\big(\mathbb{D}_\lambda^{\nu,n}f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n}f(z)}\bigg|<M,\quad z\in\mathcal{U}. \tag{2.19}$$

Remark 2.1.12. *[131] The inequalities* (2.18) *and* (2.19) *from Theorem 2.1.22 can be expressed:*

$$\bigg|\phi\bigg(\frac{\mathbb{D}_\lambda^{\nu,n+1}f(z)}{\mathbb{D}_\lambda^{\nu,n}f(z)},\frac{\mathbb{D}_{\lambda+1}^{\nu,n+1}f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\bigg)\bigg|<M$$

and

$$\bigg|\frac{\mathbb{D}_\lambda^{\nu,n+1}f(z)}{\mathbb{D}_\lambda^{\nu,n}f(z)}\bigg|<M.$$

Theorem 2.1.23. *[131] Let $\nu>-1, n\in\mathbb{N}, -\infty<\lambda<1, \rho,\sigma\in\mathbb{R}$ and let $\phi:\mathbb{C}^2\to\mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\bigg(\rho i,\frac{\sigma}{-\lambda+\rho i}+\rho i\bigg)\leq 0,$$

where $\sigma\leq-\frac{1}{2}(1+\rho^2)$. Also, let $f\in\mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n}f(z)\neq 0$ and $\mathbb{D}_{\lambda+1}^{\nu,n}f(z)\neq 0$ for $z\in\mathcal{U}\setminus\{0\}$.
The inequality

$$\Re\phi\bigg(\frac{z\big(\mathbb{D}_\lambda^{\nu,n}f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n}f(z)},\frac{z\big(\mathbb{D}_{\lambda+1}^{\nu,n}f(z)\big)'}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\bigg)>0,\quad z\in\mathcal{U} \tag{2.20}$$

implies

$$\Re\frac{z\big(\mathbb{D}_\lambda^{\nu,n}f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n}f(z)} > 0, \quad z\in\mathcal{U}. \tag{2.21}$$

Remark 2.1.13. *[131] The inequalities* (2.20) *and* (2.21) *from Theorem 2.1.23 can be expressed:*

$$\Re\phi\bigg(\frac{\mathbb{D}_\lambda^{\nu,n+1}f(z)}{\mathbb{D}_\lambda^{\nu,n}f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n+1}f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\bigg) > 0$$

and

$$\Re\frac{\mathbb{D}_\lambda^{\nu,n+1}f(z)}{\mathbb{D}_\lambda^{\nu,n}f(z)} > 0.$$

Theorem 2.1.24. *[131] Let $\nu > -1, n\in\mathbb{N}, -\infty < \lambda < 2, M > 0, m \geq 1, \theta\in\mathbb{R}$ and let $\phi : \mathbb{C}^3 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\bigg|\phi\bigg(Me^{i\theta}, mMe^{i\theta}, mMe^{i\theta} + L\bigg)\bigg| \geq M,$$

where $\Re(Le^{-i\theta}) \geq (m-1)mM$. Also, let $f\in\mathcal{A}$.
The inequality

$$\bigg|\phi\bigg(\mathbb{D}_\lambda^{\nu,n}f(z), \mathbb{D}_\lambda^{\nu,n+1}f(z), \mathbb{D}_\lambda^{\nu,n+2}f(z)\bigg)\bigg| < M, \quad z\in\mathcal{U}$$

implies

$$\big|\mathbb{D}_\lambda^{\nu,n}f(z)\big| < M, \quad z\in\mathcal{U}.$$

Theorem 2.1.25. *[131] Let $\nu > -1, n\in\mathbb{N}, -\infty < \lambda < 1, M > 0, m \geq 1, \theta\in\mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\bigg|\phi\bigg(Me^{i\theta}, \frac{-\lambda+m}{1-\lambda}Me^{i\theta}\bigg)\bigg| \geq M.$$

Also, let $f\in\mathcal{A}$.
The inequality

$$\bigg|\phi\bigg(\mathbb{D}_\lambda^{\nu,n}f(z), \mathbb{D}_{\lambda+1}^{\nu,n}f(z)\bigg)\bigg| < M, \quad z\in\mathcal{U}$$

implies

$$\big|\mathbb{D}_\lambda^{\nu,n}f(z)\big| < M, \quad z\in\mathcal{U}.$$

Theorem 2.1.26. *[131] Let $\nu > -1, n\in\mathbb{N}, -\infty < \lambda < 1, M > 1, m \geq 1, \theta\in\mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\bigg|\phi\bigg(Me^{i\theta}, Me^{i\theta}\bigg(1 + \frac{m}{(\nu+1)Me^{i\theta} - \nu - \lambda}\bigg)\bigg)\bigg| \geq M.$$

Also, let $f \in \mathcal{A}$ *with* $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ *and* $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \neq 0$ *for* $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right| < M, \quad z \in \mathcal{U}.$$

Theorem 2.1.27. *[131] Let* $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 1, \rho, \sigma \in \mathbb{R}$ *and let* $\phi : \mathbb{C}^2 \to \mathbb{C}$ *be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i + \frac{\sigma}{(\nu+1)\rho i - \nu - \lambda}\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. *Also, let* $f \in \mathcal{A}$ *with* $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ *and* $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \neq 0$ *for* $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} > 0, \quad z \in \mathcal{U}.$$

Theorem 2.1.28. *[131] Let* $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 0, M > 1, m \geq 1, \theta \in \mathbb{R}$ *and let* $\phi : \mathbb{C}^2 \to \mathbb{C}$ *be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, \frac{1}{\lambda}\big(1 - (1-\lambda)Me^{i\theta} - m\big)\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ *with* $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ *and* $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \neq 0$ *for* $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+2}^{\nu,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right| < M, \quad z \in \mathcal{U}.$$

Theorem 2.1.29. *[131] Let* $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 0, \rho, \sigma \in \mathbb{R}$ *and let* $\phi : \mathbb{C}^2 \to \mathbb{C}$ *be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{1}{\lambda}\left(1 - (1-\lambda)\rho i - \frac{\sigma}{\rho i}\right)\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. *Also, let* $f \in \mathcal{A}$ *with* $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ *and* $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \neq 0$ *for* $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+2}^{\nu,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} > 0, \quad z \in \mathcal{U}.$$

Theorem 2.1.30. *[131] Let* $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, M > 1, m \geq 1, \theta \in \mathbb{R}$ *and let* $\phi : \mathbb{C}^3 \to \mathbb{C}$ *be an admissible function that satisfies the condition*

$$\left|\phi\Big(Me^{i\theta}, (1+m)Me^{i\theta}, (1+3m)Me^{i\theta} + L\Big)\right| \geq M,$$

where $\Re(Le^{-i\theta}) \geq (m-1)mM$. *Also, let* $f \in \mathcal{A}$.
The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z}, \frac{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{z}, \frac{\mathbb{D}_{\lambda}^{\nu,n+2} f(z)}{z}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z}\right| < M, \quad z \in \mathcal{U}.$$

Theorem 2.1.31. *[131] Let* $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, \rho, \sigma, \mu, \nu \in \mathbb{R}$, *and let* $\phi : \mathbb{C}^3 \to \mathbb{C}$ *be an admissible function that satisfies the condition*

$$\Re\phi\big(\rho i, \rho i + \sigma, \rho i + 3\sigma + \mu + i\nu\big) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2), \sigma + \mu \leq 0$. *Also, let* $f \in \mathcal{A}$.
The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z}, \frac{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{z}, \frac{\mathbb{D}_{\lambda}^{\nu,n+2} f(z)}{z}\right) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z} > 0, \quad z \in \mathcal{U}.$$

Theorem 2.1.32. *[131] Let* $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 1, M > 1, m \geq 1, \theta \in \mathbb{R}$ *and let* $\phi : \mathbb{C}^2 \to \mathbb{C}$ *be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, \left(1 + \frac{m}{1-\lambda}\right)Me^{i\theta}\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$.
The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{z}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z}\right| < M, \quad z \in \mathcal{U}.$$

Theorem 2.1.33. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 1, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i + \frac{\sigma}{1-\lambda}\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$.
The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{z}\right) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z} > 0, \quad z \in \mathcal{U}.$$

Theorem 2.1.34. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, Me^{i\theta}\left(1 + \frac{m}{(\nu+1)Me^{i\theta} - \nu}\right)\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_{\lambda}^{\nu,n+1} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda}^{\nu+1,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right| < M, \quad z \in \mathcal{U}.$$

Theorem 2.1.35. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 2, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i + \frac{\sigma}{(\nu+1)\rho i - \nu}\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_{\lambda}^{\nu,n+1} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda}^{\nu+1,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}\right) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{\mathbb{D}_{\lambda}^{\nu+1,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)} > 0, \quad z \in \mathcal{U}.$$

Theorem 2.1.36. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 1, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, Me^{i\theta}\left(1 + \frac{m}{(1-\lambda)Me^{i\theta} + \nu + \lambda}\right)\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n}f(z) \neq 0$ and $\mathbb{D}_{\lambda}^{\nu+1,n}f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu+1,n}f(z)}{\mathbb{D}_{\lambda}^{\nu+1,n}f(z)}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}\right| < M, \quad z \in \mathcal{U}.$$

Theorem 2.1.37. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 1, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i + \frac{\sigma}{(1-\lambda)\rho i + \nu + \lambda}\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n}f(z) \neq 0$ and $\mathbb{D}_{\lambda}^{\nu+1,n}f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu+1,n}f(z)}{\mathbb{D}_{\lambda}^{\nu+1,n}f(z)}\right) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)} > 0, \quad z \in \mathcal{U}.$$

Theorem 2.1.38. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 1, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, Me^{i\theta}\left(1 + \frac{m}{(1-\lambda)Me^{i\theta} + \lambda}\right)\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n}f(z) \neq 0$ and $\mathbb{D}_{\lambda}^{\nu,n+1}f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n+1}f(z)}{\mathbb{D}_{\lambda}^{\nu,n+1}f(z)}\right)\right| < M, \quad z \in \mathcal{U}$$

implies

$$\left|\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}\right| < M, \quad z \in \mathcal{U}.$$

Theorem 2.1.39. *[131] Let $\nu > -1, n \in \mathbb{N}, -\infty < \lambda < 1, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\bigg(\rho i, \rho i + \frac{\sigma}{(1-\lambda)\rho i + \lambda}\bigg) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu,n+1} f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$.
The inequality

$$\Re\phi\bigg(\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\bigg) > 0, \quad z \in \mathcal{U}$$

implies

$$\Re\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0, \quad z \in \mathcal{U}.$$

2.1.4 Coefficient bounds and Fekete-Szegő problem for some classes of analytic functions defined by using the operator $\mathbb{D}_\lambda^{\nu,n}$

H. M. Srivastava, P. Sharma, R. K. Raina [126] introduced the following function classes by using the linear operator $\mathbb{D}_\lambda^{\nu,n}$ defined by (1.8) for some $\eta, 0 \leq \eta < 1$ and $\gamma \geq 0$ and for some $\phi \in \mathcal{Q}$ (see page 14) as:

$$\mathcal{S}_\lambda^{\nu,n}(\eta, [\phi]) = \bigg\{ f : f \in \mathcal{A}, \frac{1}{1-\eta}\bigg(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)} - \eta\bigg) \prec \phi(z) \bigg\},$$

$$\mathcal{C}_\lambda^{\nu,n}(\eta, [\phi], [\psi]) = \bigg\{ f : f \in \mathcal{A}, \mathbb{D}_\lambda^{\nu,n} g \in \mathcal{S}^*(\psi), \frac{1}{1-\eta}\bigg(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} g(z)} - \eta\bigg) \prec \phi(z) \bigg\},$$

and

$$\mathcal{R}_\lambda^{\nu,n}(\eta, \gamma, [\phi], [\psi]) = \bigg\{ f : f \in \mathcal{A}, \mathbb{D}_\lambda^{\nu,n} g \in \mathcal{S}^*(\psi),$$

$$\frac{1}{1-\eta}\bigg((1-\gamma)\frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} g(z)} + \gamma\frac{\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\big(\mathbb{D}_\lambda^{\nu,n} g(z)\big)'} - \eta\bigg) \prec \phi(z) \bigg\}.$$

Remark 2.1.14. *[39] For $\eta = \lambda = \nu = n = 0$ we obtain*

$$\mathcal{S}_0^{0,0}(0, [\phi]) = \mathcal{S}^*(\phi),$$

$$\mathcal{C}_0^{0,0}(0, [\phi], [\psi]) = \mathcal{C}(\phi, \psi),$$

the two classes being defined in Definition 1.2.17.

We shall use the following lemma to prove some of our results.

Lemma 2.1.1. *[66] Assume that $\eta(z) = e_1 + e_2 z + \ldots$ is analytic in $\mathcal{U}$ with $|\eta(z)| \leq 1$. Then $|e_1|^2 + |e_2| \leq 1$.*

Theorem 2.1.40. *[39] Let $f \in \mathcal{A}$ be of the form (1.1). If the function f is in the class $\mathcal{S}_\lambda^{\nu,n}(\eta, [\phi])$, then*

$$\left|a_{k+1}\right| \leq \frac{(2-\lambda)_k \prod\limits_{j=0}^{k-1} \big(j + 2(1-\eta)\big)}{(1)_k (\nu+1)_k (k+1)^{n+1}}, \quad k \in \mathbb{N}^*.$$

Remark 2.1.15. *[39] For $\lambda = \nu = n = 0, \phi(z) = \dfrac{1+(1-2\alpha)z}{1-z}, 0 \leq \alpha < 1$, the result of the Theorem 2.1.40 was obtained in [113].*

Theorem 2.1.41. *[39] Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$, B_k real numbers, $k = 1, 2, \ldots$ and $B_1 > 0$. If $f(z)$ given by (1.1) is in the class $\mathcal{S}_\lambda^{\nu,n}(\eta, [\phi])$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{(1-\eta)(2-\lambda)(3-\lambda)}{2^2 3^{n+1}(\nu+1)(\nu+2)} \left|2B_2 - \dfrac{B_1^2}{2^{2n}(3-\lambda)(\nu+1)}\gamma_0\right|, & \text{if } \mu \leq \sigma_1 \\ \dfrac{(1-\eta)(2-\lambda)(3-\lambda)}{2 \cdot 3^{n+1}(\nu+1)(\nu+2)} B_1, & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{(1-\eta)(2-\lambda)(3-\lambda)}{2^2 3^{n+1}(\nu+1)(\nu+2)} \left| -2B_2 + \dfrac{B_1^2}{2^{2n}(3-\lambda)(\nu+1)}\gamma_0\right|, & \text{if } \mu \geq \sigma_2. \end{cases}$$

Further, if $\sigma_1 < \mu \leq \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n+1}(3-\lambda)(\nu+1)}{3^{n+1}(1-\eta)(2-\lambda)(\nu+2)B_1} \left|1 - \frac{B_2}{B_1} + \frac{\gamma_0 B_1}{2^{2n+1}(3-\lambda)(\nu+1)}\right| |a_2|^2$$
$$\leq \frac{(1-\eta)(2-\lambda)(3-\lambda)}{2 \cdot 3^{n+1}(\nu+1)(\nu+2)} B_1.$$

If $\sigma_3 \leq \mu < \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n+1}(3-\lambda)(\nu+1)}{3^{n+1}(1-\eta)(2-\lambda)(\nu+2)B_1} \left|1 + \frac{B_2}{B_1} - \frac{\gamma_0 B_1}{2^{2n+1}(3-\lambda)(\nu+1)}\right| |a_2|^2$$
$$\leq \frac{(1-\eta)(2-\lambda)(3-\lambda)}{2 \cdot 3^{n+1}(\nu+1)(\nu+2)} B_1,$$

where

$$\sigma_1 = \frac{2^{2n+1}(3-\lambda)(\nu+1)\big[B_2 - B_1 + B_1^2(1-\eta)\big]}{3^{n+1}(1-\eta)(2-\lambda)(\nu+2)B_1^2},$$
$$\sigma_2 = \frac{2^{2n+1}(3-\lambda)(\nu+1)\big[B_2 + B_1 + B_1^2(1-\eta)\big]}{3^{n+1}(1-\eta)(2-\lambda)(\nu+2)B_1^2},$$
$$\sigma_3 = \frac{2^{2n+1}(3-\lambda)(\nu+1)\big[B_2 + B_1^2(1-\eta)\big]}{3^{n+1}(1-\eta)(2-\lambda)(\nu+2)B_1^2},$$
$$\gamma_0 = (1-\eta)\big[3^{n+1}\mu(2-\lambda)(\nu+2) - 2^{2n+1}(3-\lambda)(\nu+1)\big].$$

These results are sharp.

Remark 2.1.16. *[39] For $\lambda = \nu = n = \eta = 0$, the result of the Theorem 2.1.41 was obtained in [80] and for $\lambda = \nu = 0$, in [45].*

Theorem 2.1.42. *[39] Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$, B_k real numbers, $k = 1, 2, \dots$ and $B_1 > 0$, and let $f(z)$ be in the class $\mathcal{S}_\lambda^{\nu,n}(\eta, [\phi])$. For a complex number μ we have:*

$$|a_3 - \mu a_2^2| \leq \frac{(1-\eta)(2-\lambda)(3-\lambda)B_1}{2 \cdot 3^{n+1}(\nu+1)(\nu+2)} \max\left\{1; \left| -\frac{B_2}{B_1} + \frac{\gamma_0 B_1}{2^{2n+1}(3-\lambda)(\nu+1)} \right|\right\},$$

where

$$\gamma_0 = (1-\eta)\big(3^{n+1}\mu(2-\lambda)(\nu+2) - 2^{2n+1}(3-\lambda)(\nu+1)\big).$$

The result is sharp.

Theorem 2.1.43. *[39] Let $f \in \mathcal{A}$ be of the form (1.1). If the function f is in the class $\mathcal{C}_\lambda^{\nu,n}(\eta, [\phi], [\psi])$, then*

$$\left|a_{k+1}\right| \leq \frac{(2-\lambda)_k\big(1 + k(1-\eta)\big)}{(\nu+1)_k (k+1)^{n+1}}, \quad k \in \mathbb{N}^*.$$

Theorem 2.1.44. *[39] Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ be analytic in $\mathcal{U}$ and let $\psi(z) = 1 + C_1 z + C_2 z^2 + \dots$ be univalent in $\mathcal{U}$, C_k real numbers, $k = 1, 2, \dots$ and $C_1 > 0$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \dots \in \mathcal{C}_\lambda^{\nu,n}(\eta, [\phi], [\psi])$, then*

$$|a_3 - \mu a_2^2| \leq K(\mu, \lambda, \nu, n, C_1, C_2) + L(\mu, \lambda, \nu, n, \eta, B_1, B_2, C_1),$$

where

$$K(\mu, \lambda, \nu, n, C_1, C_2) =$$

$$\begin{cases} \dfrac{(2-\lambda)(3-\lambda)}{2^2 \cdot 3^{n+2}(\nu+1)(\nu+2)} \left| 2C_2 - \dfrac{C_1^2}{2^{2n}(3-\lambda)(\nu+1)}\gamma_0 \right|, & \text{if } \mu \leq \sigma_1 \\[2ex] \dfrac{(2-\lambda)(3-\lambda)}{2 \cdot 3^{n+2}(\nu+1)(\nu+2)} C_1, & \text{if } \sigma_1 \leq \mu \leq \sigma_2\ , \\[2ex] \dfrac{(2-\lambda)(3-\lambda)}{2^2 \cdot 3^{n+2}(\nu+1)(\nu+2)} \left| -2C_2 + \dfrac{C_1^2}{2^{2n}(3-\lambda)(\nu+1)}\gamma_0 \right|, & \text{if } \mu \geq \sigma_2 \end{cases}$$

$$L(\mu, \lambda, \nu, n, \eta, B_1, B_2, C_1) =$$

$$\begin{cases} \dfrac{(1-\eta)(2-\lambda)(3-\lambda)}{3^{n+2}(\nu+1)(\nu+2)}\left|B_2-\alpha_2 B_1^2\right| + \dfrac{|\alpha_1|C_1(2-\lambda)}{2^{n+1}(\nu+1)}|B_1|, \\ \qquad if\ \dfrac{|\alpha_1|C_1(2-\lambda)}{2^{n+1}(\nu+1)}|B_1| \geq \dfrac{2(1-\eta)(2-\lambda)(3-\lambda)}{3^{n+2}(\nu+1)(\nu+2)}\big(|B_1| - \left|B_2-\alpha_2 B_1^2\right|\big) \\ \dfrac{(1-\eta)(2-\lambda)(3-\lambda)|B_1|}{3^{n+2}(\nu+1)(\nu+2)} + \dfrac{3^{n+2}(2-\lambda)(\nu+2)|B_1|^2 C_1^2 |\alpha_1|^2}{2^{2n+4}(1-\eta)(3-\lambda)(\nu+1)\big(|B_1| - \left|B_2-\alpha_2 B_1^2\right|\big)}, \\ \qquad otherwise \end{cases},$$

$$\sigma_1 = \frac{2^{2n+1}(3-\lambda)(\nu+1)(C_2 - C_1 + C_1^2)}{3^{n+1}(2-\lambda)(\nu+2)C_1^2},$$
$$\sigma_2 = \frac{2^{2n+1}(3-\lambda)(\nu+1)(C_2 + C_1 + C_1^2)}{3^{n+1}(2-\lambda)(\nu+2)C_1^2},$$
$$\gamma_0 = \frac{3^{n+2}\mu(2-\lambda)(\nu+2)}{4} - 2^{2n+1}(3-\lambda)(\nu+1),$$
$$\alpha_1 = \frac{2^{n+1}(1-\eta)(3-\lambda)}{3^{n+2}(\nu+2)} - \frac{\mu(1-\eta)(2-\lambda)}{2^{n+2}(\nu+1)},$$
$$\alpha_2 = \frac{3^{n+2}\mu(1-\eta)(2-\lambda)(\nu+2)}{2^{2n+4}(3-\lambda)(\nu+1)}.$$

Remark 2.1.17. *[39] For $\lambda = \nu = n = \eta = 0$, the result of the Theorem 2.1.44 was obtained in [66].*

Theorem 2.1.45. *[39] Let $f \in \mathcal{A}$ be of the form (1.1). If the function f is in the class $\mathcal{R}_\lambda^{\nu,n}(\eta, 0, [\phi], [\psi])$, then*

$$\left|a_{k+1}\right| \leq \frac{(2-\lambda)_k\big(1+k(1-\eta)\big)}{(\nu+1)_k(k+1)^n}, \quad k \in \mathbb{N}^*.$$

Theorem 2.1.46. *[39] Let $f \in \mathcal{A}$ given by (1.1). If the function f is in the class $\mathcal{R}_\lambda^{\nu,n}(\eta, 1, [\phi], [\psi])$, then*

$$\left|a_{k+1}\right| \leq \frac{(2-\lambda)_k\big(3k+3+(1-\eta)k(2k+1)\big)}{3(\nu+1)_k(k+1)^{n+1}}, \quad k \in \mathbb{N}^*.$$

Theorem 2.1.47. *[39] Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ be analytic in $\mathcal{U}$ and let $\psi(z) = 1 + C_1 z + C_2 z^2 + \dots$ be univalent in $\mathcal{U}$, C_k real numbers, $k = 1, 2, \dots$ and $C_1 > 0$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \dots \in \mathcal{R}_\lambda^{\nu,n}(\eta, \gamma, [\phi], [\psi])$, then*

$$|a_3 - \mu a_2^2| \leq M(\mu, \lambda, \nu, n, C_1, C_2) + N(\mu, \lambda, \nu, n, \eta, \gamma, B_1, B_2, C_1),$$

where

$$M(\mu, \lambda, \nu, n, C_1, C_2) =$$

$$\begin{cases} \dfrac{(2-\lambda)(3-\lambda)}{2^2 \cdot 3^{n+1}(\nu+1)(\nu+2)}\left|2C_2 - \dfrac{C_1^2}{2^{2n}(3-\lambda)(\nu+1)}\gamma_0\right|, & \textit{if } \mu \leq \sigma_1 \\ \dfrac{(2-\lambda)(3-\lambda)}{2 \cdot 3^{n+1}(\nu+1)(\nu+2)}C_1, & \textit{if } \sigma_1 \leq \mu \leq \sigma_2\ , \\ \dfrac{(2-\lambda)(3-\lambda)}{2^2 \cdot 3^{n+1}(\nu+1)(\nu+2)}\left|-2C_2 + \dfrac{C_1^2}{2^{2n}(3-\lambda)(\nu+1)}\gamma_0\right|, & \textit{if } \mu \geq \sigma_2 \end{cases}$$

$$N(\mu, \lambda, \nu, n, \eta, \gamma, B_1, B_2, C_1) =$$

$$\begin{cases} \dfrac{(1-\eta)(2-\lambda)(3-\lambda)}{3^{n+1}(1+2\gamma)(\nu+1)(\nu+2)}\left|B_2 - \alpha_2 B_1^2\right| + \dfrac{C_1(2-\lambda)|\alpha_1|}{2^{n+1}(\nu+1)}|B_1|, \\ \quad \textit{if } \dfrac{C_1(2-\lambda)|\alpha_1|}{2^{n+1}(\nu+1)}|B_1| \geq \dfrac{2(1-\eta)(2-\lambda)(3-\lambda)}{3^{n+1}(1+2\gamma)(\nu+1)(\nu+2)}\Big(|B_1| - \left|B_2 - \alpha_2 B_1^2\right|\Big) \\ \\ \dfrac{(1-\eta)(2-\lambda)(3-\lambda)|B_1|}{3^{n+1}(1+2\gamma)(\nu+1)(\nu+2)} + \dfrac{3^{n+1}(1+2\gamma)(2-\lambda)(\nu+2)|B_1|^2 C_1^2 \left|\alpha_1\right|^2}{2^{2n+4}(1-\eta)(3-\lambda)(\nu+1)\big(|B_1| - \left|B_2 - \alpha_2 B_1^2\right|\big)}, \\ \quad \textit{otherwise} \end{cases},$$

$$\sigma_1 = \frac{2^{2n+1}(3-\lambda)(\nu+1)(C_2 - C_1 + C_1^2)}{3^{n+1}(2-\lambda)(\nu+2)C_1^2},$$

$$\sigma_2 = \frac{2^{2n+1}(3-\lambda)(\nu+1)(C_2 + C_1 + C_1^2)}{3^{n+1}(2-\lambda)(\nu+2)C_1^2},$$

$$\gamma_0 = 3^{n+1}\mu(2-\lambda)(\nu+2) - 2^{2n+1}(3-\lambda)(\nu+1),$$

$$\alpha_1 = \frac{2^{n+1}(1-\eta)(1+3\gamma)(3-\lambda)}{3^{n+1}(1+\gamma)(1+2\gamma)(\nu+2)} - \frac{\mu(1-\eta)(2-\lambda)}{2^n(1+\gamma)(\nu+1)},$$

$$\alpha_2 = \frac{3^{n+1}\mu(1-\eta)(1+2\gamma)(2-\lambda)(\nu+2)}{2^{2n+2}(1+\gamma)^2(3-\lambda)(\nu+1)}.$$

2.2 On a class of meromorphic functions defined by using the operator $\mathcal{D}_\lambda^{\nu,n}$ and some integral operators

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

which are analytic in $\mathcal{U}^* = \{z : 0 < |z| < 1\}$.

Motivated by [120], we define the fractional operator $\mathcal{D}_\lambda^{\nu,n} : \Sigma \to \Sigma$, by

$$\mathcal{D}_\lambda^{\nu,n} f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\nu+1)_{k+1}}{(2-\lambda)_{k+1}}(k+2)^{n+1} a_k z^k, \tag{2.22}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}, z \in \mathcal{U}^*$ and the symbol $(\gamma)_k$ denotes the Pochhammer symbol, for $\gamma \in \mathbb{C}$.
We note that the operator

$$\mathcal{D}_0^{0,n} f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} (k+2)^n a_k z^k$$

was introduced and studied in [138].

Remark 2.2.1. *[37] The operator $\mathcal{D}_\lambda^{\nu,n}$ satisfies the following identities:*

$$\mathcal{D}_\lambda^{\nu,n+1} f(z) = 2\mathcal{D}_\lambda^{\nu,n} f(z) + z\big(\mathcal{D}_\lambda^{\nu,n} f(z)\big)', \tag{2.23}$$

$$\mathcal{D}_\lambda^{\nu+1,n} f(z) = \frac{\nu+2}{\nu+1}\mathcal{D}_\lambda^{\nu,n} f(z) + \frac{1}{\nu+1} z\big(\mathcal{D}_\lambda^{\nu,n} f(z)\big)', \tag{2.24}$$

$$\mathcal{D}_{\lambda+1}^{\nu,n} f(z) = \frac{2-\lambda}{1-\lambda}\mathcal{D}_\lambda^{\nu,n} f(z) + \frac{1}{1-\lambda} z\big(\mathcal{D}_\lambda^{\nu,n} f(z)\big)', \tag{2.25}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}$.

Definition 2.2.1. *[37] A function $f \in \Sigma$ is said to be in the class $SD_\lambda^{\nu,n}(\alpha)$ if it satisfies*

$$\Re\left(\frac{\mathcal{D}_\lambda^{\nu,n+1} f(z)}{\mathcal{D}_\lambda^{\nu,n} f(z)} - 2\right) < -\alpha, \quad z \in \mathcal{U}, \tag{2.26}$$

for some $\alpha (0 \le \alpha < 1), -\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}$.

To prove our results, we need the following lemma, known as Jack's lemma. An extension of this is the Jack-Miller-Mocanu Lemma [84, 92].

Lemma 2.2.1. *[51] Let the function w be regular and nonconstant in $|z| < 1$, with $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then we have $z_0 w'(z_0) = k w(z_0)$, where k is a real number and $k \ge 1$.*

To prove our results, we use the methods used in [24, 138].

Theorem 2.2.1. *[37] $SD_\lambda^{\nu,n+1}(\alpha) \subset SD_\lambda^{\nu,n}(\alpha), \quad n \in \mathbb{N}$.*

Remark 2.2.2. *[37] Taking $\lambda = 0$ and $\nu = 0$, we obtain Theorem 2.1 from [138].*

Using Lemma 1.3.2 instead of Lemma 2.2.1 we will obtain an improvement of Theorem 2.2.1.

Theorem 2.2.2. *[37] $SD_\lambda^{\nu,n+1}(\alpha) \subset SD_\lambda^{\nu,n}(\beta)$, for $n \in \mathbb{N}$, where*

$$\beta = \frac{5 + 2\alpha - \sqrt{(3-2\alpha)^2 + 8}}{4}, \tag{2.27}$$

and $\beta \in (\alpha, 1)$.

Remark 2.2.3. *[37] Taking $\lambda = 0$ and $\nu = 0$, we obtain a particular case of Theorem 2.5 from [5].*

Theorem 2.2.3. *[37] $SD_{\lambda}^{\nu+1,n}(\alpha) \subset SD_{\lambda}^{\nu,n}(\alpha), \quad \nu > -1.$*

Theorem 2.2.4. *[37] $SD_{\lambda+1}^{\nu,n}(\alpha) \subset SD_{\lambda}^{\nu,n}(\alpha), \quad -\infty < \lambda < 1.$*

Theorem 2.2.5. *[37] Let $f \in \Sigma$ satisfying the condition*

$$\Re\left(\frac{\mathcal{D}_{\lambda}^{\nu,n+1} f(z)}{\mathcal{D}_{\lambda}^{\nu,n} f(z)} - 2\right) < -\alpha + \frac{1-\alpha}{2(1-\alpha+c)}, \quad z \in \mathcal{U},$$

$$n \in \mathbb{N}, -\infty < \lambda < 2, \nu > -1, c > 0, \tag{2.28}$$

then

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt \in SD_{\lambda}^{\nu,n}(\alpha).$$

Remark 2.2.4. *[37] Taking $\lambda = 0$ and $\nu = 0$, we obtain Theorem 2.2 from [138].*

Theorem 2.2.6. *[37] $f \in SD_{\lambda}^{\nu,n}(\alpha)$ if and only if the integral operator $F \in SD_{\lambda}^{\nu,n+1}(\alpha)$, where $F(z) = \frac{1}{z^2} \int_0^z t f(t) dt$.*

Remark 2.2.5. *[37] Taking $\lambda = 0$ and $\nu = 0$, we obtain Theorem 2.3 from [138].*

Theorem 2.2.7. *[37] $f \in SD_{\lambda}^{\nu,n}(\alpha)$ if and only if the integral operator $F \in SD_{\lambda}^{\nu+1,n}(\alpha)$, where $F(z) = \frac{\nu+1}{z^{\nu+2}} \int_0^z t^{\nu+1} f(t) dt$.*

Theorem 2.2.8. *[37] $f \in SD_{\lambda}^{\nu,n}(\alpha)$ if and only if the integral operator $F \in SD_{\lambda+1}^{\nu,n}(\alpha)$, where $F(z) = \frac{1-\lambda}{z^{2-\lambda}} \int_0^z t^{1-\lambda} f(t) dt$.*

2.3 The operator $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}$

Definition 2.3.1. *[133] Let $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}, \alpha, \beta \geq 0$. Denote by $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}$ the operator given by $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} : \mathcal{A} \to \mathcal{A}$,*

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = (1-\alpha-\beta)\mathcal{R}^{\nu}\mathcal{D}^{n} f(z) + \alpha\mathcal{R}^{\nu}\Omega_z^{\lambda} f(z) + \beta\mathcal{D}^{n}\Omega_z^{\lambda} f(z),$$

for $z \in \mathcal{U}$, where the operators $\mathcal{R}^{\nu}$, $\mathcal{D}^{n}$ and Ω_z^{λ} are defined in Definition 1.5.3, Definition 1.5.1 and Definition 1.5.5, respectively.

Remark 2.3.1. *[133] $\mathcal{R}^\nu \mathcal{D}^n f(z)$ is the composition of the Sălăgean operator and the Ruscheweyh derivative, $\mathcal{R}^\nu \Omega_z^\lambda f(z)$ is the composition of fractional differintegral operator and the Ruscheweyh derivative, and $\mathcal{D}^n \Omega_z^\lambda f(z)$ is the composition of fractional differintegral operator and the Sălăgean operator.*

Remark 2.3.2. *[133] If $f \in \mathcal{A}$, $f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}$, then*

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = z + \sum_{k=1}^{\infty} \Bigg((1-\alpha-\beta)\frac{(\nu+1)_k}{(2)_k}(k+1)^{n+1} + \alpha\frac{(\nu+1)_k}{(2-\lambda)_k}(k+1) + \beta\frac{(1)_k}{(2-\lambda)_k}(k+1)^{n+1} \Bigg) a_{k+1} z^{k+1}, \tag{2.29}$$

for $z \in \mathcal{U}$.

Remark 2.3.3. *[133] $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = (1-\alpha-\beta)\mathbb{D}_0^{\nu,n} f(z) + \alpha \mathbb{D}_\lambda^{\nu,0} f(z) + \beta \mathbb{D}_\lambda^{0,n} f(z)$, for $z \in \mathcal{U}$, where $\mathbb{D}_\lambda^{\nu,n}$ is defined in (1.8).*

Remark 2.3.4. *[133] For $\alpha = 0$ and $\beta = 0$, we obtain $\mathscr{D}_{0,0}^{\lambda,\nu,n} f(z) = \mathcal{R}^\nu \mathcal{D}^n f(z)$, where $z \in \mathcal{U}$.*

For $\alpha = 1$ and $\beta = 0$, we obtain $\mathscr{D}_{1,0}^{\lambda,\nu,n} f(z) = \mathcal{R}^\nu \Omega_z^\lambda f(z)$, where $z \in \mathcal{U}$.

For $\alpha = 0$ and $\beta = 1$, we obtain $\mathscr{D}_{0,1}^{\lambda,\nu,n} f(z) = \mathcal{D}^n \Omega_z^\lambda f(z)$, where $z \in \mathcal{U}$.

For $\beta = 0$ and $\nu = 0$, we obtain $\mathscr{D}_{\alpha,0}^{\lambda,0,n} f(z) = (1-\alpha)\mathcal{D}^n f(z) + \alpha \Omega_z^\lambda f(z)$, where $z \in \mathcal{U}$.

For $\alpha = 0$ and $n = 0$, we obtain $\mathscr{D}_{0,\beta}^{\lambda,\nu,0} f(z) = (1-\beta)\mathcal{R}^\nu f(z) + \beta \Omega_z^\lambda f(z)$, where $z \in \mathcal{U}$.

For $\alpha + \beta = 1$ and $\lambda = 0$, we obtain $\mathscr{D}_{1-\beta,\beta}^{0,\nu,n} f(z) = (1-\beta)\mathcal{R}^\nu f(z) + \beta \mathcal{D}^n f(z)$, where $z \in \mathcal{U}$.

For $\alpha + \beta = 1, \lambda = 0$ and $\nu = n$, we obtain $\mathscr{D}_{1-\beta,\beta}^{0,n,n} f(z) = (1-\beta)\mathcal{R}^n f(z) + \beta \mathcal{D}^n f(z), z \in \mathcal{U}$. This operator was introduced and studied in [6].

For $\alpha = \beta = n = 0$, we obtain $\mathscr{D}_{0,0}^{\lambda,\nu,0} f(z) = \mathcal{R}^\nu f(z)$, and for $\beta = \lambda = n = 0$, we obtain $\mathscr{D}_{\alpha,0}^{0,\nu,0} f(z) = \mathcal{R}^\nu f(z)$, where $z \in \mathcal{U}$.

For $\alpha = \beta = \nu = 0$, we obtain $\mathscr{D}_{0,0}^{\lambda,0,n} f(z) = \mathcal{D}^n f(z)$, and for $\alpha = \lambda = \nu = 0$, we obtain $\mathscr{D}_{0,\beta}^{0,0,n} f(z) = \mathcal{D}^n f(z)$, where $z \in \mathcal{U}$.

For $\alpha = 0$ and $\lambda = \nu = 1$, we obtain $\mathscr{D}_{0,\beta}^{1,1,n} f(z) = \mathcal{D}^{n+1} f(z)$, where $z \in \mathcal{U}$.

For $\alpha = 1$ and $\beta = \nu = 0$, we obtain $\mathscr{D}_{1,0}^{\lambda,0,n} f(z) = \Omega_z^\lambda f(z)$ and for $\alpha = n = 0$ and $\beta = 1$, we obtain $\mathscr{D}_{0,1}^{\lambda,\nu,0} f(z) = \Omega_z^\lambda f(z)$, where $z \in \mathcal{U}$.

For $\lambda = \nu = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{0,0,n} f(z) = (1-\alpha)\mathcal{D}^n f(z) + \alpha f(z)$, where $z \in \mathcal{U}$.

For $\lambda = n = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{0,\nu,0} f(z) = (1-\beta)\mathcal{R}^\nu f(z) + \beta f(z)$, where $z \in \mathcal{U}$.

For $\nu = n = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{\lambda,0,0} f(z) = (1-\alpha-\beta) f(z) + (\alpha+\beta)\Omega_z^{\lambda} f(z)$, where $z \in \mathcal{U}$.

For $\lambda = 0$ and $\nu = 1$, we obtain $\mathscr{D}_{\alpha,\beta}^{0,1,n} f(z) = (1-\alpha-\beta)\mathcal{D}^{n+1} f(z) + \alpha\mathcal{D}^1 f(z) + \beta\mathcal{D}^n f(z)$, where $z \in \mathcal{U}$.

For $\lambda = 1$ and $\nu = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{1,0,n} f(z) = (1-\alpha-\beta)\mathcal{D}^n f(z) + \alpha\mathcal{D}^1 f(z) + \beta\mathcal{D}^{n+1} f(z)$, where $z \in \mathcal{U}$.

For $\lambda = \nu = 1$, we obtain $\mathscr{D}_{\alpha,\beta}^{1,1,n} f(z) = (1-\alpha)\mathcal{D}^{n+1} f(z) + \alpha\mathcal{D}^2 f(z)$, where $z \in \mathcal{U}$.

For $\lambda = \nu = n = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{0,0,0} f(z) = f(z)$, for $\alpha = \beta = \nu = n = 0$, we obtain $\mathscr{D}_{0,0}^{\lambda,0,0} f(z) = f(z)$, for $\alpha = 1$ and $\lambda = \nu = 0$, we obtain $\mathscr{D}_{1,\beta}^{0,0,n} f(z) = f(z)$, and for $\beta = 1$ and $\lambda = n = 0$, we obtain $\mathscr{D}_{\alpha,1}^{0,\nu,0} f(z) = f(z)$, for $z \in \mathcal{U}$.

2.3.1 On a class of analytic functions defined by the operator $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}$

Definition 2.3.2. *[133] Let $f \in \mathcal{A}$. We say that the function f is in the class $\mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$, where $0 \leq \delta \leq 1, \alpha, \beta \geq 0, -\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}$, if f satisfies the condition*

$$\Re(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z))' > \delta, \quad z \in \mathcal{U}. \tag{2.30}$$

Theorem 2.3.1. *[133] Let $f \in \mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$ and $g \in \mathcal{K}$, where $\mathcal{K}$ denotes the class of convex functions. Then $f * g \in \mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$.*

Theorem 2.3.2. *[133] The set $\mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$ is convex.*

2.3.2 Differential subordinations obtained by using the operator $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}$

Theorem 2.3.3. *[133] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)\right)' \prec h(z), \quad z \in \mathcal{U}, \tag{2.31}$$

then

$$\frac{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)}{z} \prec g(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Theorem 2.3.4. *[133] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathscr{D}_{\alpha,\beta}^{\lambda,\nu+1,n}f(z)}{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)}\right)' \prec h(z), \quad z \in \mathcal{U}, \tag{2.32}$$

then

$$\frac{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu+1,n}f(z)}{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)} \prec g(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Theorem 2.3.5. *[133] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)}{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)}\right)' \prec h(z), \quad z \in \mathcal{U}, \tag{2.33}$$

then

$$\frac{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)}{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)} \prec g(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Theorem 2.3.6. *[133] Let g be a convex function, $g(0) = 0$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z) + \mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z) + \alpha\big(\mathcal{D}\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z) - \mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)\big) \prec h(z), \quad z \in \mathcal{U}, \tag{2.34}$$

then

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z) \prec g(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Theorem 2.3.7. *[133] Let* $h(z)=\dfrac{1+(2\delta-1)z}{1+z}$ *be a convex function in* U*, where* $0\le\delta<1$*. If* $f\in\mathcal{A}$ *verifies the differential subordination*

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)+\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)+\alpha\big(\mathcal{D}\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)-\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)\big)\prec h(z),\quad z\in\mathcal{U},\tag{2.35}$$

then

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)\prec g(z),\quad z\in\mathcal{U},$$

where g *is given by* $g(z)=2\delta-1+2(1-\delta)\dfrac{\ln(1+z)}{z},\quad z\in\mathcal{U}.$
The function g *is convex and is the best dominant.*

Theorem 2.3.8. *[133] Let* g *be a convex function,* $g(0)=1$ *and let* h *be a function such that*

$$h(z)=g(z)+zg'(z),\quad z\in\mathcal{U}.$$

If $f\in\mathcal{A}$ *verifies the differential subordination*

$$\frac{1}{z}\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+2}f(z)+\frac{1}{z}\alpha\big(\mathcal{D}^2\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)-\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)\big)\prec h(z),\quad z\in\mathcal{U},\tag{2.36}$$

then

$$\big(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)\big)'\prec g(z),\quad z\in\mathcal{U}$$

and the result is sharp.

Theorem 2.3.9. *[133] Let* $h(z)=\dfrac{1+(2\delta-1)z}{1+z}$ *be a convex function in* $\mathcal{U}$*, where* $0\le\delta<1$*. If* $f\in\mathcal{A}$ *verifies the differential subordination*

$$\frac{1}{z}\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+2}f(z)+\frac{1}{z}\alpha\big(\mathcal{D}^2\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)-\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)\big)\prec h(z),\tag{2.37}$$

then

$$\big(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)\big)'\prec g(z),\quad z\in\mathcal{U},$$

where g *is given by* $g(z)=2\delta-1+2(1-\delta)\dfrac{\ln(1+z)}{z},\quad z\in\mathcal{U}.$
The function g *is convex and is the best dominant.*

2.4 Inclusion relations of analytic functions associated with Poisson distribution series and Sălăgean operator $\mathcal{D}^n$

In this section, results using the Sălăgean differential operator $\mathcal{D}^n$ defined in Definition 1.5.1 are obtained.

Using the Sălăgean operator, Kanas and Yuguchi [64] introduced the class $\mathcal{US}(n,\alpha)$ as

$$\mathcal{US}(n,\alpha)=\left\{f\in\mathcal{A}:\Re\left(\frac{z\big(\mathcal{D}^n f(z)\big)'}{\mathcal{D}^n f(z)}\right)>\alpha\left|\frac{z(\mathcal{D}^n f(z))'}{\mathcal{D}^n f(z)}-1\right|,\alpha\geq 0,z\in\mathcal{U}\right\}.$$

It is easy to see that $\mathcal{US}(1,\alpha)=\alpha-\mathcal{UCV}$ and $\mathcal{US}(0,\alpha)=\alpha-\mathcal{ST}$.

Porwal [110] introduced Poisson distribution series as

$$K(m,z)=z+\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}e^{-m}z^k.$$

We consider the linear operator $I(m):\mathcal{A}\to\mathcal{A}$ (see [122]) defined by

$$I(m)f=K(m,z)*f(z)=z+\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}e^{-m}a_k z^k.$$

We establish some inclusion relations between the classes $\mathcal{US}(n,\alpha)$, $\delta-\mathcal{UCV}$ and $\delta-\mathcal{ST}$.

We need the following result.

Theorem 2.4.1. *[64] If $f\in\mathcal{US}(n,\alpha)$, then*

$$|a_k|\leq\frac{(P_1)_{k-1}}{\Gamma(k)k^n},\quad k\in\mathbb{N}^*-\{1\},$$

where P_1 is the coefficient of z in the function

$$p_k(z)=1+\sum_{k=1}^{\infty}P_k z^k=\frac{z(\mathcal{D}^n f_k(z))'}{\mathcal{D}^n f_k(z)},$$

where $f_k(z)$ is the extremal function for the class $\mathcal{US}(n,\alpha)$, and the symbol $(\beta)_k$ represents the Pochhammer symbol, for $\beta\in\mathbb{C}$.

Theorem 2.4.2. *[36] If $m>0, f\in\mathcal{US}(n,\alpha)$ and the inequality*

$$(1+\delta)\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-2)!}\frac{(P_1)_{k-1}}{\Gamma(k)k^n}+\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}\frac{(P_1)_{k-1}}{\Gamma(k)k^n}\leq e^m \tag{2.38}$$

is satisfied, then $I(m)f\in\delta-\mathcal{ST}$.

Theorem 2.4.3. *[36] If $m>0, f\in\mathcal{US}(n,\alpha)$ and the inequality*

$$(1+\delta)\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-3)!}\frac{(P_1)_{k-1}}{\Gamma(k)k^n}+(3+2\delta)\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-2)!}\frac{(P_1)_{k-1}}{\Gamma(k)k^n}+\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}\frac{(P_1)_{k-1}}{\Gamma(k)k^n}\leq e^m$$

is satisfied, then $I(m)f\in\delta-\mathcal{UCV}$.

Theorem 2.4.4. *[36] If $m > 0, f \in \mathcal{US}(n,\alpha)$ and the inequality*

$$\sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-3)!} \frac{(P_1)_{k-1}}{\Gamma(k)k^n} + 2\sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-2)!} \frac{(P_1)_{k-1}}{\Gamma(k)k^n} \leq \frac{e^m}{\delta+2}$$

is satisfied, then $I(m)f \in \delta - \mathcal{UCV}$.

Theorem 2.4.5. *[36] If $m > 0, f \in \mathcal{US}(n,\alpha)$ and the inequality*

$$\sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-2)!} \frac{(P_1)_{k-1}}{\Gamma(k)k^n} + \lambda\sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} \frac{(P_1)_{k-1}}{\Gamma(k)k^n} \leq \lambda e^m \tag{2.39}$$

is satisfied, then $I(m)f \in \mathcal{S}_\lambda^*$.

Theorem 2.4.6. *[36] If $m > 0, f \in \mathcal{US}(n,\alpha)$ and the inequality*

$$\sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-3)!} \frac{(P_1)_{k-1}}{\Gamma(k)k^n} + (2+\lambda)\sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-2)!} \frac{(P_1)_{k-1}}{\Gamma(k)k^n} + \lambda\sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} \frac{(P_1)_{k-1}}{\Gamma(k)k^n} \leq \lambda e^m$$

is satisfied, then $I(m)f \in \mathcal{C}_\lambda$.

In the followings, we use the integal operator

$$G(m,z) = \int_0^z \frac{I(m)f(t)}{t} dt$$

or equivalently

$$G(m,z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{k!} e^{-m} a_k z^k, \tag{2.40}$$

defined in [122], and we obtain some inclusion relations for $G(m,z)$ belonging to the classes $\delta - \mathcal{UCV}$, $\mathcal{C}_\lambda$ and $\mathcal{US}(n,\alpha)$.

Theorem 2.4.7. *[36] If $m > 0, f \in \mathcal{US}(n,\alpha)$ then $G(m,z)$ defined in (2.40) is in $\delta - \mathcal{UCV}$ if (2.38) is satisfied.*

Theorem 2.4.8. *[36] If $m > 0, f \in \mathcal{US}(n,\alpha)$ and the inequality*

$$\sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-2)!} \frac{(P_1)_{k-1}}{\Gamma(k)k^n} \leq \frac{e^m}{\delta+2}$$

is satisfied, then $G(m,z) \in \delta - \mathcal{UCV}$.

Theorem 2.4.9. *[36] If $m > 0, f \in \mathcal{US}(n,\alpha)$ and the inequality (2.39) is satisfied, then $G(m,z) \in \mathcal{C}_\lambda$.*

2.5 Differential subordinations obtained by using generalized Sălăgean integro-differential operator $\mathcal{L}_{\lambda\delta}^n$

Many recently published works contain studies on integro-differential operators (see [1, 99, 100, 104]). In the paper [38] I generalized the operator from [104].

Definition 2.5.1. *[38] Let $n \in \mathbb{N}, \delta \geq 0$ and $\lambda \geq 0$ with $\delta \neq \frac{\lambda-1}{\lambda}$. For $f \in \mathcal{A}$, let*

$$\mathcal{L}_{\lambda\delta}^n f(z) = \frac{1}{1-\lambda+\lambda\delta}\big[(1-\lambda)\mathcal{D}_\delta^n f(z) + \lambda\delta\mathcal{I}_\delta^n f(z)\big], \quad z \in \mathcal{U}, \tag{2.41}$$

where the differential operator $\mathcal{D}_\delta^n f$ and the integral operator $\mathcal{I}_\delta^n f$ are given by Definition 1.5.7 and Definition 1.5.8, respectively.

Remark 2.5.1. *[38] We have*

$$\mathcal{L}_{0\delta}^n f(z) = \mathcal{D}_\delta^n f(z),$$

$$\mathcal{L}_{1\delta}^n f(z) = \mathcal{I}_\delta^n f(z),$$

$$\mathcal{L}_{\lambda\delta}^0 f(z) = \mathcal{L}_{\lambda 0}^n f(z) = f(z)$$

and

$$\mathcal{L}_{\lambda 1}^n f(z) = (1-\lambda)\mathcal{D}^n f(z) + \lambda\mathcal{I}^n f(z) \quad \textit{(see [104]).}$$

Remark 2.5.2. *[38] For $f \in \mathcal{A}, f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ by using (1.15) and (1.18), we have*

$$\mathcal{L}_{\lambda\delta}^n f(z) = z + \frac{1}{1-\lambda+\lambda\delta}\sum_{k=2}^{\infty}\left[\big(1-\lambda\big)\big(1+(k-1)\delta\big)^n + \frac{\lambda\delta}{\big(1+(k-1)\delta\big)^n}\right]a_k z^k, \quad z \in \mathcal{U}. \tag{2.42}$$

Theorem 2.5.1. *[38] If $0 \leq \alpha < 1, f \in \mathcal{A}_m$, $m \in \{1,2,3,\dots\}$ and*

$$\Re\left[\big(\mathcal{L}_{\lambda\delta}^{n+1} f(z)\big)' + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\delta z\Big(\big(\mathcal{I}_\delta^{n+1} f(z)\big)'' + \big(\mathcal{I}_\delta^n f(z)\big)''\Big)\right] > \alpha, \quad z \in \mathcal{U}, \tag{2.43}$$

then

$$\Re\big(\mathcal{L}_{\lambda\delta}^n f(z)\big)' > \gamma, \quad z \in \mathcal{U},$$

where

$$\gamma = \gamma(\alpha) = 2\alpha - 1 + \frac{2(1-\alpha)}{\delta m}\int_0^1 \frac{t^{\frac{1}{\delta m}-1}}{1+t}dt.$$

Example 2.5.1. *[38] For $m = 1, \lambda = \frac{1}{2}, \delta = \frac{1}{2}, n = 0$ and $\alpha = \frac{1}{2}$ we obtain that the inequality*

$$\Re\left(f'(z) + \frac{zf''(z)}{2}\right) > \frac{1}{2}, \quad z \in \mathcal{U},$$

implies

$$\Re f'(z) > 2 - 2ln2, \quad z \in \mathcal{U}.$$

Theorem 2.5.2. *[38] Let q be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + m\delta zq'(z), m \in \mathbb{N}^*, \delta > 0, \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}_m$ verifies the following subordination

$$\left(\mathcal{L}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\delta z\left(\left(\mathcal{I}_\delta^{n+1} f(z)\right)'' + \left(\mathcal{I}_\delta^{n} f(z)\right)''\right) \prec h(z), \quad z \in \mathcal{U}, \tag{2.44}$$

then

$$\left(\mathcal{L}_{\lambda\delta}^{n} f(z)\right)' \prec q(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Remark 2.5.3. *[38] Taking $m = 1$ and $\delta = 1$, we obtain Theorem 3 from [104].*

Remark 2.5.4. *[38] Taking $\lambda = 0$, we obtain Theorem 2.2 from [23].*

Theorem 2.5.3. *[38] Let q be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + mzq'(z), \quad m \in \mathbb{N}^*, z \in \mathcal{U}.$$

If $f \in \mathcal{A}_m$ verifies the following subordination

$$\left(\mathcal{L}_{\lambda\delta}^{n} f(z)\right)' \prec h(z), \quad z \in \mathcal{U}, \tag{2.45}$$

then

$$\frac{\mathcal{L}_{\lambda\delta}^{n} f(z)}{z} \prec q(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Remark 2.5.5. *[38] Taking $m = 1$ and $\delta = 1$, we obtain Theorem 1 from [104].*

Remark 2.5.6. *[38] Taking $\lambda = 0$, we obtain Theorem 2.3 from [23].*

Theorem 2.5.4. *[38] Let q be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + mzq'(z), \quad m \in \mathbb{N}^*, z \in \mathcal{U}.$$

If $f \in \mathcal{A}_m$ verifies the following subordination

$$\left(\frac{z\mathcal{L}_{\lambda\delta}^{n+1} f(z)}{\mathcal{L}_{\lambda\delta}^{n} f(z)} \right)' \prec h(z), \quad z \in \mathcal{U}, \tag{2.46}$$

then

$$\frac{\mathcal{L}_{\lambda\delta}^{n+1} f(z)}{\mathcal{L}_{\lambda\delta}^{n} f(z)} \prec q(z), \quad z \in \mathcal{U}$$

and the result is sharp.

Remark 2.5.7. *[38] Taking $m = 1$ and $\delta = 1$, we obtain Theorem 2 from [104].*

Chapter 3

New results on some classes of analytic functions related to starlikeness and convexity

In this chapter, we obtain new results related to starlikeness and convexity.

In Section 3.1, several differential subordination results, involving arithmetic, geometric and harmonic means of the expressions $p(z)$ and $p(z)+\frac{zp'(z)}{p(z)}$ are generalized.

In Section 3.2, a new class of analytic functions satisfying subordinate condition associated with Chebyshev polynomials is defined. Coefficient estimates and Fekete-Szegő inequality for this class are given.

In Section 3.3, a new subclass of m-fold symmetric bi-univalent functions is introduced. Estimates of the Taylor-Maclaurin coefficients are obtained. Fekete-Szegő functional problem for functions in this new subclass is also investigated.

In Section 3.4, we give upper bound for the second Hankel determinant for gamma-starlike functions of oder α, for $0 \leq \gamma \leq 1$. This result generalizes previously obtained results for upper bounds of second Hankel determinants for various classes.

3.1 Differential subordinations and Pythagorean means

The three "classic" means, i.e. the arithmetic mean, the geometric mean and the harmonic mean are sometimes called Pythagorean means. All these means were generalized by their weighted forms. A special case of them, is the convex weighted mean.

Hence, for $0 \leq \alpha \leq 1$, we have

$$CWA(x_1, x_2) = \alpha x_1 + (1-\alpha)x_2,$$

$$CWG(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha},$$
$$CWH(x_1, x_2) = \frac{1}{\alpha \frac{1}{x_1} + (1-\alpha)\frac{1}{x_2}} = \frac{x_1 x_2}{\alpha x_2 + (1-\alpha)x_1},$$

where CWA denotes the convex weighted arithmetic mean, CWG represents the convex weighted geometric mean and CWH is the convex weighted harmonic mean. The Pythagorean means are frequently used in geometric function theory.

The well-known class of α-convex functions (see Definition 1.2.11) plays an important role in this direction, being defined using the convex weighted arithmetic mean, and at the same time being a transition between starlike and convex functions.

The class of gamma-starlike functions (see Definition 1.2.12) is defined in a similar manner, using the convex weighted geometric mean.

There are many other works in geometric function theory, related to arithmetic and geometric means (see for example [57–60, 75, 125, 136]).
In geometric function theory, harmonic means are considered in a few works (see for example [26, 27, 137]).

Differential subordinations involving both, convex weighted arithmetic means and convex weighted geometric means were studied in [56], while differential subordinations involving harmonic means in [61], respectively, convex weighted harmonic means, recently, in [55].

In this section, we generalize some of these results.

The following lemma due to Nunokawa [96], is expressed in a different form. This form is used in [56].

Lemma 3.1.1. *[56] Let $p \in \mathcal{H}(\mathcal{U})$ such that $p(0) = 1, p(z) \not\equiv 1$. If $z_0 \in \mathcal{U}$ verifies the equalities*

$$|\arg p(z_0)| = \max\{\arg p(z) : |z| \le |z_0|\} = \theta\frac{\pi}{2},$$
$$p(z_0) = (ix)^{\theta},$$

then

$$|\arg[z_0 p'(z_0)]| = (\theta + 1)\frac{\pi}{2},$$
$$|z_0 p'(z_0)| = \left|\frac{\theta x^{\theta}}{2}\left(x + \frac{1}{x}\right)\right|.$$

Theorem 3.1.1. *[35] Let $p \in \mathcal{H}(\mathcal{U})$ such that $p(0) = 1$. Also, let $\alpha \in [0;1], \beta \in [0;1], \delta \in [1;2], \gamma \in [0;1]$ and $\theta \in (0;1]$.*
If

$$\left|\arg\left(\alpha[p(z)]^{\delta} + (1-\alpha)\frac{[p(z)]^{\gamma}\left[p(z) + \frac{zp'(z)}{p(z)}\right]^{1-\gamma}}{\beta\frac{zp'(z)}{p^2(z)} + 1}\right)\right| < \theta\frac{\pi}{2}, \quad z \in \mathcal{U}, \tag{3.1}$$

then

$$|\arg p(z)| < \theta\frac{\pi}{2}, \quad z \in \mathcal{U}. \tag{3.2}$$

Remark 3.1.1. *[35] For $\beta = 0$, we obtain the result of Theorem 2.4, p. 128 in [56].*

Remark 3.1.2. *[35] For $\alpha = 0$ and $\gamma = 0$, we have a particular case of the Theorem 2.5, p. 1726, obtained in [55].*

Remark 3.1.3. *[35] For $\alpha = 0$, $\gamma = 0$ and $\beta = \frac{1}{2}$, we obtain the result of Theorem 2.7, p.1251 in [61].*

Taking $\theta = 1$ in Theorem 3.1.1, we obtain the following result.

Corollary 3.1.1. *[35] Let $p \in \mathcal{H}(\mathcal{U})$ such that $p(0) = 1$. Also, let $\alpha \in [0;1], \beta \in [0;1], \delta \in [1;2]$ and $\gamma \in [0;1]$.*
If

$$\Re\left(\alpha[p(z)]^\delta + (1-\alpha)\frac{[p(z)]^\gamma\left[p(z) + \frac{zp'(z)}{p(z)}\right]^{1-\gamma}}{\beta\frac{zp'(z)}{p^2(z)} + 1}\right) > 0, \quad z \in \mathcal{U},$$

then

$$\Re p(z) > 0, \quad z \in \mathcal{U}.$$

Considering convenient values for α, β, γ in Corollary 3.1.1, we obtain the followings.

Remark 3.1.4. *[35] For $\beta = 0$, we obtain the result of Theorem 2.3, p. 127 in [56].*

Remark 3.1.5. *[35] For $\alpha = 0$ and $\gamma = 0$, we have a particular case of Theorem 2.4, p. 1724, obtained in [55].*

Remark 3.1.6. *[35] For $\alpha = 0$, $\gamma = 0$ and $\beta = \frac{1}{2}$, we obtain the result of Theorem 2.3, p.1247 [61].*

Remark 3.1.7. *[35] For $\alpha = 0$ and $\beta = 0$, we get a result obtained in [78].*

Remark 3.1.8. *[35] For $\delta = 1$, $\beta = 0$ and $\gamma = 0$, we get a result obtained in [116].*

Setting $p(z) = \frac{zf'(z)}{f(z)}$ in Corollary 3.1.1, we obtain the following result:

Corollary 3.1.2. *[35] Let $f \in \mathcal{A}$ and also let $\alpha \in [0;1], \beta \in [0;1], \delta \in [1;2]$ and $\gamma \in [0;1]$.*
If

$$\Re\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^\delta + (1-\alpha)\frac{\left[\frac{zf'(z)}{f(z)}\right]^{1+\gamma}\left[1 + \frac{zf''(z)}{f'(z)}\right]^{1-\gamma}}{\beta\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\beta)\frac{zf'(z)}{f(z)}}\right) > 0, \quad z \in \mathcal{U},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathcal{U}.$$

Therefore f is starlike in $\mathcal{U}$.

Remark 3.1.9. *[35] If we put $\alpha = 0$ and $\beta = 0$ in Corollary 3.1.2, we obtain the well known result that α-convex functions are starlike. This result has been proved in various ways (see [86–88]).*

Remark 3.1.10. *[35] If we put $\delta = 1$, $\beta = 0$ and $\gamma = 0$ in Corollary 3.1.2, we obtain the well known result that γ-starlike functions are starlike (see [77, 78]).*

Setting $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 3.1.1, we obtain the following result:

Corollary 3.1.3. *[35] Let $f \in \mathcal{A}$ and also let $\alpha \in [0;1], \beta \in [0;1], \delta \in [1;2], \gamma \in [0;1]$ and $\theta \in (0;1]$.*
If

$$\left|\arg\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\frac{\left[\frac{zf'(z)}{f(z)}\right]^{1+\gamma}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\gamma}}{\beta\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\beta)\frac{zf'(z)}{f(z)}}\right)\right| < \theta\frac{\pi}{2}, \quad z \in \mathcal{U},$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \theta\frac{\pi}{2}, \quad z \in \mathcal{U}.$$

Therefore f is strongly starlike of order θ in $\mathcal{U}$.

3.2 Coefficient estimates and Fekete-Szegő inequality for a class of analytic functions satisfying subordinate condition associated with Chebyshev polynomials

Chebyshev polynomials are of four kinds, but the most common are the Chebyshev polynomials of the first kind,

$$T_n(x) = \cos n\theta, \quad x \in [-1, 1],$$

and the second kind,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x \in [-1, 1],$$

where n denotes the polynomial degree and $x = \cos\theta$.
Applications of Chebyshev polynomials for analytic functions can be found in [11, 13, 20, 33].
Let
$$\mathcal{H}(z,t) = \frac{1}{1-2tz+z^2},$$
where $t = \cos\theta, \theta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.
We have
$$\mathcal{H}(z,t) = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\theta}{\sin\theta} z^n = 1 + 2\cos\theta z + (3\cos^2\theta - \sin^2\theta)z^2 + \dots$$
$$= 1 + U_1(t)z + U_2(t)z^2 + \cdots, \quad z \in \mathcal{U}, t \in \left(\frac{1}{2}, 1\right], \tag{3.3}$$
where
$$U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}}, \quad n \in \mathbb{N}^*$$
are the Chebyshev polynomials of second kind.
Furthermore, we know that
$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$
and
$$U_1(t) = 2t, U_2(t) = 4t^2 - 1, \dots.$$

In the followings, we define a new class of analytic functions, being motivated by the following result:

Corollary 3.2.1. *[56] Let $f \in \mathcal{A}$ and also let $\alpha \in [0,1], a \in [0,1], \delta \in [1,2]$ and $\mu \in [0,1]$. If*
$$\Re\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) > a, \quad z \in \mathcal{U},$$
then
$$\Re\left(\frac{zf'(z)}{f(z)}\right) > a, \quad z \in \mathcal{U},$$
so f is starlike of order a in $\mathcal{U}$.

Definition 3.2.1. *[132] We say that $f \in \mathcal{A}$ of the form (1.1) belongs to $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$ if*
$$\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) \prec \mathcal{H}(z,t), \tag{3.4}$$
the power is considered to have principal value, $\alpha \in [0,1], \delta \in [1,2]$ and $\mu \in [0,1]$.

Taking $\alpha = \delta = t = 1$ and $w(z) = z$, we obtain the following example.

Example 3.2.1. *[132] The function* $f(z) = \frac{z}{1-z} e^{\frac{z}{1-z}}$ *with the series expansion* $f(z) = z + 2z^2 + \frac{7}{2}z^3 + \cdots$ *belongs to* $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$.

Remark 3.2.1. *The result of the above example is obtained using the subordination (3.4), in which we take* $\alpha = \delta = t = 1$ *and* $w(z) = z$. *So, the subordination (3.4) reduces to the equality* $\frac{zf'(z)}{f(z)} = \frac{1}{1-2z+z^2}$, *which results from a direct computation using the function from the example.*

Theorem 3.2.1. *[132] Let* $f \in \mathcal{A}$ *of the form (1.1) belong to the class* $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$. *Then*

$$|a_2| \leq \frac{2t}{\alpha\delta + (1-\alpha)(2-\mu)} \tag{3.5}$$

and for $\lambda \in \mathbb{C}$

$$\begin{aligned}\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{\alpha\delta + (1-\alpha)(3-2\mu)} \max\Bigg\{1, \Bigg|2t\Bigg(&\frac{2\lambda\big(\alpha\delta + (1-\alpha)(3-2\mu)\big)}{\big(\alpha\delta + (1-\alpha)(2-\mu)\big)^2} \\ &- \frac{3 + \frac{2(1-\alpha)(1-\mu) - \alpha(\delta^2-\mu^2) - \mu^2}{\alpha\delta + (1-\alpha)(2-\mu)}}{2\big(\alpha\delta + (1-\alpha)(2-\mu)\big)}\Bigg) - \frac{4t^2-1}{2t}\Bigg|\Bigg\}.\end{aligned} \tag{3.6}$$

Taking $\alpha = 1 - \beta, \delta = 1$ and $\mu = 0$ in Theorem 3.2.1, we obtain the following result:

Corollary 3.2.2. *[11] Let* $f \in \mathcal{A}$ *of the form (1.1) satisfying the condition*

$$\left((1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \prec \mathcal{H}(z,t),$$

where $\beta \in [0,1]$. *Then*

$$|a_2| \leq \frac{2t}{1+\beta}$$

and for $\lambda \in \mathbb{C}$

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{1+2\beta} \max\left\{1, \left|2t\left(\frac{2\lambda(1+2\beta)}{(1+\beta)^2} - \frac{1+3\beta}{(1+\beta)^2}\right) - \frac{4t^2-1}{2t}\right|\right\}.$$

Taking $\alpha = 0$ in Theorem 3.2.1, we obtain the following result:

Corollary 3.2.3. *[13] Let* $f \in \mathcal{A}$ *of the form (1.1) satisfying the condition*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\mu} \prec \mathcal{H}(z,t),$$

where $\mu \in [0,1]$. *Then*

$$|a_2| \leq \frac{2t}{2-\mu}$$

and for $\lambda \in \mathbb{C}$

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{3-2\mu} \max\left\{1, \left|2t\left(\frac{2\lambda(3-2\mu)}{(2-\mu)^2} + \frac{\mu^2+5\mu-8}{2(2-\mu)^2}\right) - \frac{4t^2-1}{2t}\right|\right\}.$$

3.3 Coefficient estimates and Fekete-Szegő inequalities for a new subclass of m-fold symmetric bi-univalent functions satisfying subordinate conditions

Every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z, \quad z \in \mathcal{U}$$

and

$$f(f^{-1}(w)) = w \quad |w| < r_0(f); r_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$. The class of bi-univalent functions in $\mathcal{U}$ is denoted by σ.

A function is said to be m-fold symmetric (see [107]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}, \qquad m \in \mathbb{N}^*, \ z \in \mathcal{U}. \tag{3.7}$$

The class of m-fold symmetric univalent functions, which are normalized by the above series expansion (3.7), is denoted by $\mathcal{S}_m$. The functions in the class $\mathcal{S}$ are one fold symmetric. Analogous to the concept of m-fold symmetric univalent functions, is defined the concept of m-fold symmetric bi-univalent functions. Each function f in the class σ generates an m-fold symmetric bi-univalent function for each positive integer m. The normalized form of f is given in (3.7) and f^{-1} is given in the followings.

$$\begin{aligned} g(w) =& w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right]w^{2m+1} \\ &- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \dots, \end{aligned} \tag{3.8}$$

where $f^{-1} = g$. The class of m-fold symmetric bi-univalent functions is denoted by σ_m.
Recently, many authors investigated coefficient estimates and Fekete-Szegő functional problem for subclasses of m-fold symmetric bi-univalent functions ([3,8,12,14,15,19, 31,74,81,93,94,119,127,128,134,139–144]).

Huo Tang *et al.* [134] introduced the following subclasses of m-fold symmetric bi-univalent functions.

Definition 3.3.1. *[134, Definition 1, p.1066] A function $f(z)$, given by* (3.7) *, is said to be in the class $\mathcal{H}_{\sigma,m}(\phi)$, if the following conditions are satisfied:*

$$f \in \sigma_m, \qquad f'(z) \prec \phi(z) \qquad \text{and} \qquad g'(w) \prec \phi(w),$$

where the function $g(w)$ is defined by (3.8).

Definition 3.3.2. *[134, Definition 3, p. 1078] A function $f(z)$, given by* (3.7)*, is said to be in the class $\mathcal{M}_{\sigma,m}(\lambda, \phi)$ if the following conditions are satisfied:*

$$f \in \sigma_m, \qquad (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z)$$

and

$$(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \phi(w),$$

where the function $g(w)$ is defined by (3.8).

Ş. Altınkaya and S. Yalçın [10] introduced the following subclass of bi-univalent functions.

Definition 3.3.3. *[10] A function $f \in \sigma$ is said to be in $\mathcal{S}_\sigma(\lambda, \phi), 0 \leq \lambda \leq 1$, if the following subordinations hold*

$$\frac{zf'(z) + (2\lambda^2 - \lambda)\, z^2 f''(z)}{4(\lambda - \lambda^2)\, z + (2\lambda^2 - \lambda)\, zf'(z) + (2\lambda^2 - 3\lambda + 1)\, f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w) + (2\lambda^2 - \lambda)\, w^2 g''(w)}{4(\lambda - \lambda^2)\, w + (2\lambda^2 - \lambda)\, wg'(w) + (2\lambda^2 - 3\lambda + 1)\, g(w)} \prec \phi(w),$$

where $g = f^{-1}$.

Motivated by the definition of the above subclass of bi-univalent functions, we introduce below a new subclass of m-fold symmetric bi-univalent functions in a similar manner.

Definition 3.3.4. *[41] A function $f \in \sigma_m$ is said to be in the class $\mathcal{S}_{\sigma_m}(\lambda, \phi)$, $0 \le \lambda \le 1$, if the following subordination conditions hold*

$$\frac{zf'(z) + (2\lambda^2 - \lambda) z^2 f''(z)}{4(\lambda - \lambda^2) z + (2\lambda^2 - \lambda) zf'(z) + (2\lambda^2 - 3\lambda + 1) f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w) + (2\lambda^2 - \lambda) w^2 g''(w)}{4(\lambda - \lambda^2) w + (2\lambda^2 - \lambda) wg'(w) + (2\lambda^2 - 3\lambda + 1) g(w)} \prec \phi(w),$$

where $g = f^{-1}$.

Remark 3.3.1. *[41]*

$$\mathcal{S}_{\sigma_m}(0, \phi) = \mathcal{M}_{\sigma,m}(0, \phi),$$
$$\mathcal{S}_{\sigma_m}\left(\frac{1}{2}, \phi\right) = \mathcal{H}_{\sigma,m}(\phi),$$
$$\mathcal{S}_{\sigma_m}(1, \phi) = \mathcal{M}_{\sigma,m}(1, \phi),$$
$$\mathcal{S}_{\sigma_1}(\lambda, \phi) = \mathcal{S}_{\sigma}(\lambda, \phi).$$

In the followings, we introduce a function ϕ used in [134].
ϕ is an analytic function with positive real part in the unit disk $\mathcal{U}$ such that

$$\phi(0) = 1 \qquad \text{and} \qquad \phi'(0) > 0$$

and $\phi(\mathcal{U})$ is symmetric with respect to the real axis. This function has a series expansion of the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \quad B_1 > 0.$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk $\mathcal{U}$ with

$$u(0) = v(0) = 0, \quad \max\{|u(z)|, |v(z)|\} < 1$$

and

$$u(z) = b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \ldots,$$
$$v(w) = c_m w^m + c_{2m} w^{2m} + c_{3m} w^{3m} + \ldots.$$

We have the following inequalities (see [134])

$$|b_m| \le 1, |b_{2m}| \le 1 - |b_m|^2, |c_m| \le 1 \text{ and } |c_{2m}| \le 1 - |c_m|^2. \tag{3.9}$$

By simple computations, are obtained the followings

$$\phi(u(z)) = 1 + B_1 b_m z^m + (B_1 b_{2m} + B_2 b_m^2) z^{2m} + \ldots, \quad |z| < 1, \tag{3.10}$$

and

$$\phi(v(w)) = 1 + B_1 c_m w^m + (B_1 c_{2m} + B_2 c_m^2) w^{2m} + \dots, \quad |w| < 1. \tag{3.11}$$

We begin by finding the estimates on the coefficients $\left|a_{m+1}\right|$ and $\left|a_{2m+1}\right|$ for functions in the class $\mathcal{S}_{\sigma_m}(\lambda, \phi)$.

Theorem 3.3.1. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{S}_{\sigma_m}(\lambda, \phi)$. Then*

$$\left|a_{m+1}\right| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{\left|\big(\beta(m+1) - 2\alpha\gamma\big)B_1^2 - 2\alpha^2 B_2\right| + 2B_1\alpha^2}} \tag{3.12}$$

and

$$\left|a_{2m+1}\right| \leq$$

$$\begin{cases} \dfrac{\left(|\beta(m+1) - \alpha\gamma| + |\alpha\gamma|\right) B_1}{\left|\beta(\beta(m+1) - 2\alpha\gamma)\right|}, \\ \\ \quad \textit{if } |\beta|(m+1)|B_2| \leq \left(|\beta(m+1) - \alpha\gamma| + |\alpha\gamma|\right) B_1 \\ \\ \dfrac{\big(|\beta(m+1) - \alpha\gamma| + |\alpha\gamma|\big)\left|(\beta(m+1) - 2\alpha\gamma)B_1^2 - 2\alpha^2 B_2\right|B_1 + 2\alpha^2|\beta|(m+1)|B_2|B_1}{\left|\beta(\beta(m+1) - 2\alpha\gamma)\right|\big(\left|(\beta(m+1) - 2\alpha\gamma)B_1^2 - 2\alpha^2 B_2\right| + 2B_1\alpha^2\big)}, \\ \\ \quad \textit{if } |\beta|(m+1)|B_2| > \left(|\beta(m+1) - \alpha\gamma| + |\alpha\gamma|\right) B_1 \end{cases}, \tag{3.13}$$

where

$$\alpha = m + 2\lambda^2 m^2 - \lambda m^2 - 4\lambda^2 + 4\lambda,$$

$$\beta = 2(m + 4\lambda^2 m^2 - 2\lambda m^2 - 2\lambda^2 + 2\lambda),$$

$$\gamma = \big(2\lambda - 1\big)\big((m+2)\lambda - 1\big).$$

Taking $m = 1$ in Theorem 3.3.1, we obtain the following corollary.

Corollary 3.3.1. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{S}_{\sigma}(\lambda, \phi)$. Then*

$$\left|a_2\right| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|\big(\beta - \alpha\gamma\big)B_1^2 - \alpha^2 B_2\right| + B_1\alpha^2}}$$

and

$$\left|a_3\right| \leq \begin{cases} \dfrac{\left(|2\beta-\alpha\gamma|+|\alpha\gamma|\right)B_1}{2\left|\beta(\beta-\alpha\gamma)\right|}, \\ \qquad \textit{if } 2|\beta||B_2| \leq \left(|2\beta-\alpha\gamma|+|\alpha\gamma|\right)B_1 \\ \\ \dfrac{\left(|2\beta-\alpha\gamma|+|\alpha\gamma|\right)\left|(\beta-\alpha\gamma)B_1^2-\alpha^2B_2\right|B_1+2\alpha^2|\beta||B_2|B_1}{2\left|\beta(\beta-\alpha\gamma)\right|\left(\left|(\beta-\alpha\gamma)B_1^2-\alpha^2B_2\right|+B_1\alpha^2\right)}, \\ \qquad \textit{if } 2|\beta||B_2| > \left(|2\beta-\alpha\gamma|+|\alpha\gamma|\right)B_1 \end{cases},$$

where

$$\alpha = 1+3\lambda-2\lambda^2,$$

$$\beta = 2(1+2\lambda^2),$$

$$\gamma = (2\lambda-1)(3\lambda-1).$$

Remark 3.3.2. *[41] The estimate for $\left|a_2\right|$ asserted by Corollary 3.3.1 is obtained in Theorem 1 in [10].*

Taking $\lambda = 0$ in Theorem 3.3.1, we obtain the following corollary.

Corollary 3.3.2. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{M}_{\sigma,m}(0,\phi)$. Then*

$$\left|a_{m+1}\right| \leq \frac{B_1\sqrt{B_1}}{m\sqrt{\left|B_1^2-B_2\right|+B_1}}$$

and

$$\left|a_{2m+1}\right| \leq \begin{cases} \dfrac{m+1}{2m^2}B_1, & \textit{if } |B_2| \leq B_1 \\ \\ \dfrac{(m+1)B_1\left(|B_1^2-B_2|+|B_2|\right)}{2m^2\left(|B_1^2-B_2|+B_1\right)}, & \textit{if } |B_2| > B_1 \end{cases}.$$

Remark 3.3.3. *[41] The results of Corollary 3.3.2 are obtained taking $\lambda = 0$ in Theorem 5 in [134].*

Taking $\lambda = \frac{1}{2}$ in Theorem 3.3.1, we obtain the following corollary.

Corollary 3.3.3. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{H}_{\sigma,m}(\phi)$. Then*

$$\left|a_{m+1}\right| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{(m+1)\left(2(m+1)B_1+\left|(2m+1)B_1^2-2(m+1)B_2\right|\right)}}$$

and

$$\left|a_{2m+1}\right| \leq \begin{cases} \dfrac{B_1}{2m+1}, & \text{if } |B_2| \leq B_1 \\ \\ \dfrac{2(m+1)^3|B_2|B_1(2m+1) + \left|(2m+1)B_1^2 - 2(m+1)B_2\right|B_1}{(2m+1)\big(|(2m+1)B_1^2 - 2(m+1)B_2| + 2B_1(m+1)\big)}, & \text{if } |B_2| > B_1 \end{cases}.$$

Remark 3.3.4. *[41] The estimate for $\left|a_{m+1}\right|$ asserted by Corollary 3.3.3 is obtained in Theorem 1 in [134].*

Taking $\lambda = 1$ in Theorem 3.3.1, we obtain the following corollary.

Corollary 3.3.4. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{M}_{\sigma,m}(1,\phi)$. Then*

$$\left|a_{m+1}\right| \leq \frac{B_1\sqrt{B_1}}{m\sqrt{(m+1)\big(\left|B_1^2 - (m+1)B_2\right| + B_1(m+1)\big)}}$$

and

$$\left|a_{2m+1}\right| \leq \begin{cases} \dfrac{B_1}{2m^2}, & \text{if } |B_2| \leq B_1 \\ \\ \dfrac{\left|B_1^2 - (m+1)B_2\right|B_1 + (m+1)|B_2|B_1}{2m^2\big(\left|B_1^2 - (m+1)B_2\right| + B_1(m+1)\big)}, & \text{if } |B_2| > B_1 \end{cases}.$$

Remark 3.3.5. *[41] The results of Corollary 3.3.4 are obtained taking $\lambda = 1$ in Theorem 5 in [134].*

Next we shall solve the Fekete-Szegő problem for functions in the class $\mathcal{S}_{\sigma_m}(\lambda,\phi)$.

Theorem 3.3.2. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{S}_{\sigma_m}(\lambda,\phi)$. Also let $\delta \in \mathbb{R}$. Then*

$$\left|a_{2m+1} - \delta a_{m+1}^2\right| \leq \begin{cases} \dfrac{B_1}{|\beta|}, & \text{for } 0 \leq \left|h(\delta)\right| < \dfrac{1}{2|\beta|} \\ 2B_1\left|h(\delta)\right|, & \text{for } \left|h(\delta)\right| \geq \dfrac{1}{2|\beta|} \end{cases}, \tag{3.14}$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{2\big[\big(\beta(m+1) - 2\alpha\gamma\big)B_1^2 - 2\alpha^2 B_2\big]},$$

$$\alpha = m + 2\lambda^2 m^2 - \lambda m^2 - 4\lambda^2 + 4\lambda,$$

$$\beta = 2(m + 4\lambda^2 m^2 - 2\lambda m^2 - 2\lambda^2 + 2\lambda) \neq 0,$$

$$\gamma = \big(2\lambda - 1\big)\big((m+2)\lambda - 1\big).$$

Taking $m = 1$ in Theorem 3.3.2, we obtain the following corollary.

Corollary 3.3.5. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{S}_\sigma(\lambda, \phi)$. Also let $\delta \in \mathbb{R}$. Then*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \dfrac{B_1}{2(2\lambda^2+1)}, & \text{for } 0 \leq \left|h(\delta)\right| < \dfrac{1}{4(2\lambda^2+1)} \\ 2B_1\left|h(\delta)\right|, & \text{for } \left|h(\delta)\right| \geq \dfrac{1}{4(2\lambda^2+1)} \end{cases},$$

where

$$h(\delta) = \frac{B_1^2(1-\delta)}{2\big[(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)B_1^2 - (1+3\lambda-2\lambda^2)^2 B_2\big]}.$$

Taking $\delta = 1$ and $\delta = 0$ in Theorem 3.3.2, we have the following corollaries:

Corollary 3.3.6. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{S}_{\sigma_m}(\lambda, \phi)$. Then*

$$\left|a_{2m+1} - a_{m+1}^2\right| \leq \begin{cases} \dfrac{B_1}{|\beta|}, & \text{for } 0 \leq \left|h(1)\right| < \dfrac{1}{2|\beta|} \\ 2B_1\left|h(1)\right|, & \text{for } \left|h(1)\right| \geq \dfrac{1}{2|\beta|} \end{cases},$$

where

$$h(1) = \frac{B_1^2(m-1)}{2\big[\big(\beta(m+1) - 2\alpha\gamma\big)B_1^2 - 2\alpha^2 B_2\big]},$$

$$\alpha = m + 2\lambda^2 m^2 - \lambda m^2 - 4\lambda^2 + 4\lambda,$$

$$\beta = 2(m + 4\lambda^2 m^2 - 2\lambda m^2 - 2\lambda^2 + 2\lambda) \neq 0,$$

$$\gamma = \big(2\lambda - 1\big)\big((m+2)\lambda - 1\big).$$

Corollary 3.3.7. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{S}_{\sigma_m}(\lambda, \phi)$. Then*

$$\left|a_{2m+1}\right| \leq$$

$$\begin{cases} \dfrac{B_1}{|\beta|}, \\ \qquad \text{for } \dfrac{B_2}{B_1^2} \in \left(-\infty; -\dfrac{(m+1)(|\beta|-\beta)+2\alpha\gamma}{2\alpha^2}\right) \cup \left(\dfrac{(m+1)(|\beta|+\beta)-2\alpha\gamma}{2\alpha^2}; +\infty\right) \\ \dfrac{B_1^3(m+1)}{\left|(\beta(m+1)-2\alpha\gamma)B_1^2 - 2\alpha^2 B_2\right|}, \\ \qquad \text{for } \dfrac{B_2}{B_1^2} \in \left(-\dfrac{(m+1)(|\beta|-\beta)+2\alpha\gamma}{2\alpha^2}; \dfrac{\beta(m+1)-2\alpha\gamma}{2\alpha^2}\right) \cup \\ \qquad\qquad \left(\dfrac{\beta(m+1)-2\alpha\gamma}{2\alpha^2}; \dfrac{(m+1)(|\beta|+\beta)-2\alpha\gamma}{2\alpha^2}\right) \end{cases},$$

where

$$\alpha = m + 2\lambda^2 m^2 - \lambda m^2 - 4\lambda^2 + 4\lambda,$$
$$\beta = 2(m + 4\lambda^2 m^2 - 2\lambda m^2 - 2\lambda^2 + 2\lambda) \neq 0,$$
$$\gamma = \big(2\lambda - 1\big)\big((m+2)\lambda - 1\big).$$

Taking $\lambda = 0$ in Theorem 3.3.2, we obtain the following corollary.

Corollary 3.3.8. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{M}_{\sigma,m}(0,\phi)$. Also let $\delta \in \mathbb{R}$. Then*

$$\big|a_{2m+1} - \delta a_{m+1}^2\big| \leq \begin{cases} \dfrac{B_1}{2m}, & \text{for } 0 \leq \big|h(\delta)\big| < \dfrac{1}{4m} \\ 2B_1\big|h(\delta)\big|, & \text{for } \big|h(\delta)\big| \geq \dfrac{1}{4m} \end{cases},$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{4m^2\big(B_1^2 - B_2\big)}.$$

Remark 3.3.6. *[41] The result of Corollary 3.3.8 is obtained taking $\lambda = 0$ in Theorem 6 in [134].*

Taking $\lambda = \frac{1}{2}$, Theorem 3.3.2 reduces to the corresponding result of Huo Tang *et al.* [134].

Corollary 3.3.9. *[134, Th. 2, p. 1070] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{H}_{\sigma,m}(\phi)$. Also let $\delta \in \mathbb{R}$. Then*

$$\big|a_{2m+1} - \delta a_{m+1}^2\big| \leq \begin{cases} \dfrac{B_1}{2m+1}, & \text{for } 0 \leq \big|h(\delta)\big| < \dfrac{1}{2(2m+1)} \\ 2B_1\big|h(\delta)\big|, & \text{for } \big|h(\delta)\big| \geq \dfrac{1}{2(2m+1)} \end{cases},$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{2(m+1)\big[(2m+1)B_1^2 - 2(m+1)B_2\big]}.$$

Taking $\lambda = 1$ in Theorem 3.3.2, we obtain the following corollary:

Corollary 3.3.10. *[41] Let the function $f(z)$, given by (3.7), be in the class $\mathcal{M}_{\sigma,m}(1,\phi)$. Also let $\delta \in \mathbb{R}$. Then*

$$\big|a_{2m+1} - \delta a_{m+1}^2\big| \leq \begin{cases} \dfrac{B_1}{2m(2m+1)}, & \text{for } 0 \leq \big|h(\delta)\big| < \dfrac{1}{4m(2m+1)} \\ 2B_1\big|h(\delta)\big|, & \text{for } \big|h(\delta)\big| \geq \dfrac{1}{4m(2m+1)} \end{cases},$$

where

$$h(\delta) = \frac{B_1^2(m+1-2\delta)}{4m^2(m+1)\big[B_1^2 - (m+1)B_2\big]}.$$

Remark 3.3.7. *[41] The result of Corollary 3.3.10 is obtained taking $\lambda = 1$ in Theorem 6 in [134].*

3.4 The second Hankel determinant for gamma-starlike functions of order alpha

The qth Hankel determinant for $f \in \mathcal{A}, q \geq 1, n \geq 1$ is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

We consider

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

There exist many papers concerning upper bounds for $H_2(2)$ for several classes of analytic functions (see for example [25, 52, 53, 72, 76, 82, 112, 123, 135, 147]).

To prove the following theorem, we use the method used in [53].

Theorem 3.4.1. *[40] Let $f \in \mathcal{L}_\gamma(\alpha), 0 \leq \gamma \leq 1, 0 \leq \alpha < 1$, be of the form* (1.1). *Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \begin{cases} \psi(2), & \text{if } \gamma \in [0, \epsilon] \text{ and } \alpha = h(\gamma) \\ & \text{or } \gamma \in (0, \epsilon) \text{ and } \alpha \in \big[0, h(\gamma)\big) \\ & \text{or } \frac{-B}{2A} > 4 \\ \psi\left(\sqrt{\frac{-B}{2A}}\right), & \text{otherwise} \end{cases}, \tag{3.15}$$

where

$\psi : [0, 2] \to \mathbb{R}, \psi(p) = \frac{(1-\alpha)^2 (Ap^4 + Bp^2 + C)}{144(1+\gamma)^4 (1+2\gamma)^2 (1+3\gamma)}$,

$A = |S| - 3(1+\gamma)^2 \big(3(1+\gamma)(7\gamma^2 + 4\gamma + 1) + 2(1-\alpha)\gamma(-7\gamma^2 + 8\gamma + 11)\big)$,

$S = (1-\alpha)^2 (37\gamma^5 + 25\gamma^4 - 45\gamma^3 - 361\gamma^2 - 220\gamma - 12)$

$+6(1-\alpha)(1+\gamma)^2 \gamma(-7\gamma^2 + 8\gamma + 11) + 3(1+\gamma)^3 (7\gamma^2 + 4\gamma + 1)$,

$B = 24\gamma(1+\gamma)^2 \big((1-\alpha)(-7\gamma^2 + 8\gamma + 11) + 6\gamma(1+\gamma)\big)$,

$C = 144(1+\gamma)^4(1+3\gamma)$,

$\epsilon \approx 0.01471$ *is a solution of the equation* $-37\gamma^4 - 253\gamma^3 - 603\gamma^2 - 263\gamma + 4 = 0$,

$h : [0,1] \to \mathbb{R}, h(\gamma) = \frac{5\gamma^5+11\gamma^4-75\gamma^3+181\gamma^2+154\gamma+12-4(\gamma+1)(2\gamma+1)\sqrt{3(\gamma+1)(-7\gamma^5-25\gamma^4+190\gamma^2+55\gamma+3)}}{-37\gamma^5-25\gamma^4+45\gamma^3+361\gamma^2+220\gamma+12}$.

Taking $\gamma = 0$ in Theorem 3.4.1, we have the following result obtained in [25,72,76].

Corollary 3.4.1. *[40] Let* $f \in \mathcal{S}^*(\alpha), 0 \leq \alpha < 1$, *be of the form* (1.1). *Then*

$$\left|a_2a_4 - a_3^2\right| \leq (1-\alpha)^2.$$

Taking $\gamma = 0$ and $\alpha = 0$ in Theorem 3.4.1, we have the following result obtained in [53].

Corollary 3.4.2. *[40] Let* $f \in \mathcal{S}^*$, *be of the form* (1.1). *Then*

$$\left|a_2a_4 - a_3^2\right| \leq 1.$$

Taking $\gamma = 1$ in Theorem 3.4.1, we have the following result obtained in [72].

Corollary 3.4.3. *[40] Let* $f \in \mathcal{K}(\alpha), 0 \leq \alpha < 1$, *be of the form* (1.1). *Then*

$$\left|a_2a_4 - a_3^2\right| \leq (1-\alpha)^2 \frac{36 - 36\alpha + 17\alpha^2}{144(2 - 2\alpha + \alpha^2)}.$$

Taking $\gamma = 1$ and $\alpha = 0$ in Theorem 3.4.1, we have the following result obtained in [53].

Corollary 3.4.4. *[40] Let* $f \in \mathcal{K}$, *be of the form* (1.1). *Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{8}.$$

Taking $\alpha = 0$ in Theorem 3.4.1, we have the following result.

Corollary 3.4.5. *[40] Let* $f \in \mathcal{L}_\gamma, 0 \leq \gamma \leq 1$, *be of the form* (1.1). *Then*

$$\left|a_2a_4 - a_3^2\right| \leq$$

$$\begin{cases} \frac{(16\gamma^4+80\gamma^3+257\gamma^2+142\gamma+9)(1-\gamma)}{9(1+\gamma)^4(1+2\gamma)^2(1+3\gamma)}, & \text{if } 0 \leq \gamma \leq \epsilon \text{ or } \frac{3(1+\gamma)^2(-\gamma^2+14\gamma+11)}{37\gamma^4+253\gamma^3+603\gamma^2+263\gamma-4} > 1 \\ \frac{112\gamma^5+768\gamma^4+2236\gamma^3+1700\gamma^2+372\gamma-4}{(1+2\gamma)^2(1+3\gamma)(37\gamma^4+253\gamma^3+603\gamma^2+263\gamma-4)}, & \text{otherwise} \end{cases},$$

where $\epsilon \approx 0.01471$ *is a solution of the equation* $-37\gamma^4 - 253\gamma^3 - 603\gamma^2 - 263\gamma + 4 = 0$.

Chapter 4

Certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative

In this chapter, a new operator given by the convolution of the Sălăgean operator $\mathcal{D}^n$ and the Ruscheweyh operator $\mathcal{R}^n$ is defined and a class of analytic functions with varying arguments defined by this operator is introduced. The properties of the image of this class through the Bernardi operator are also studied.

Definition 4.1. *[106] Let $n \in \mathbb{N}$. Denote by $\mathcal{DR}^n$ the operator given by the Hadamard product (convolution) of the Sălăgean operator $\mathcal{D}^n$ and the Ruscheweyh operator $\mathcal{R}^n$, $\mathcal{DR}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathcal{DR}^n f(z) = \mathcal{D}^n \left(\frac{z}{1-z} \right) * \mathcal{R}^n f(z), \quad z \in \mathcal{U}.$$

Remark 4.1. *[106] If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathcal{DR}^n f(z) = z + \sum_{k=2}^{\infty} \frac{k^n (n+k-1)!}{n! (k-1)!} a_k z^k, \quad z \in \mathcal{U}.$$

Definition 4.2. *[106] For $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}$ let $P(n, \lambda, A, B)$ denote the subclass of $\mathcal{A}$ which contain functions $f(z)$ of the form* (1.1) *such that*

$$(1-\lambda)(\mathcal{DR}^n f(z))' + \lambda(\mathcal{DR}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}.$$

Definition 4.3. *[121] A function f of the form* (1.1) *is said to be in the class $V(\theta_k)$ if $f \in A$ and $arg(a_k) = \theta_k$,$\forall k \geq 2$. If $\exists \delta \in \mathbb{R}$ such that $\theta_k + (k-1)\delta \equiv \pi (mod\ 2\pi), \forall k \geq 2$ then f is said to be in the class $V(\theta_k, \delta)$. The union of $V(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V.*

Let $VP(n, \lambda, A, B)$ denote the subclass of V consisting of functions $f(z) \in P(n, \lambda, A, B)$.

Theorem 4.1. *[106] Let the function f be of the form* (1.1) *belonging to V. Then $f(z) \in VP(n, \lambda, A, B)$, if and only if*

$$T(f) = \sum_{k=2}^{\infty} k^{n+1} C_k (1+B) |a_k| \leq B - A, \tag{4.1}$$

where

$$C_k = [n + 1 + \lambda (k-1)(n+k+1)] \frac{(n+k-1)!}{(n+1)!\,(k-1)!}.$$

The extremal functions are:

$$f(z) = z + \frac{B-A}{k^{n+1} C_k (1+B)} e^{i\theta_k} z^k, \quad k \geq 2.$$

Corollary 4.1. *[106] Let the function f be of the form* (1.1) *belonging to the class $VP(n, \lambda, A, B)$. Then*

$$|a_k| \leq \frac{B-A}{k^{n+1} C_k (1+B)}, \quad k \geq 2.$$

The result (4.1) *is sharp for the functions*

$$f(z) = z + \frac{B-A}{k^{n+1} C_k (1+B)} e^{i\theta_k} z^k, \quad k \geq 2.$$

Theorem 4.2. *[106] Let the function f be of the form* (1.1) *belonging to the class $VP(n, \lambda, A, B)$. Then*

$$|z| - \frac{B-A}{2^{n+1} C_2 (1+B)} |z|^2 \leq |f(z)| \leq |z| + \frac{B-A}{2^{n+1} C_2 (1+B)} |z|^2.$$

The result is sharp.

Corollary 4.2. *[106] Let the function f be of the form* (1.1) *belonging to the class $VP(n, \lambda, A, B)$. Then $f(z) \in \mathcal{U}(0, r_1)$, where $r_1 = 1 + \dfrac{B-A}{2^{n+1} C_2 (1+B)}$.*

Theorem 4.3. *[106] Let the function f be of the form* (1.1) *belonging to the class $VP(n, \lambda, A, B)$. Then*

$$1 - \frac{B-A}{2^{n} C_2 (1+B)} |z| \leq |f'(z)| \leq 1 + \frac{B-A}{2^{n} C_2 (1+B)} |z|.$$

The result is sharp.

Corollary 4.3. *[106] Let the function f be of the form* (1.1) *belonging to the class $VP(n, \lambda, A, B)$. Then $f'(z) \in \mathcal{U}(0, r_2)$, where $r_2 = 1 + \dfrac{B-A}{2^n C_2\,(1+B)}$.*

Theorem 4.4. *[106] Let the function f be of the form* (1.1) *belonging to the class $VP(n, \lambda, A, B)$, with $arg(a_k) = \theta_k$ where $\theta_k \equiv \pi, \forall k \geq 2$. Define*

$$f_1(z) = z$$

and

$$f_k(z) = z - \frac{B-A}{k^{n+1} C_k\,(1+B)} z^k, \quad k \geq 2; z \in \mathcal{U}.$$

Then $f(z) \in VP(n, \lambda, A, B)$ if and only if $f(z)$ can expressed by

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \text{ where } \mu_k \geq 0 \text{ and } \sum_{k=1}^{\infty} \mu_k = 1.$$

Corollary 4.4. *[106] Let $VP_\pi(n, \lambda, A, B) = VP(n, \lambda, A, B) \cap V(\pi, 0)$. The extreme points of $VP_\pi(n, \lambda, A, B)$ are*

$$f_1(z) = z \quad \text{and} \quad f_k(z) = z - \frac{B-A}{k^{n+1} C_k (1+B)} z^k, \quad k \geq 2, \; z \in \mathcal{U}.$$

If we combine theorem 4.4 with Silverman's theorem 5 from [121] we get the following corollary:

Corollary 4.5. *[106] The closed convex hull of $VP(n, \lambda, A, B)$ is*

$$cl\ co\ VP(n, \lambda, A, B) = \left\{ f \;|f \in \mathcal{A}, \; \sum_{k=2}^{\infty} k^{n+1} C_k\,(1+B)\,|a_k| \leq B - A \right\}.$$

The extreme points of $cl\ co\ VP(n, \lambda, A, B)$ are

$$\boldsymbol{E}(cl\ co\ VP(n, \lambda, A, B)) = \left\{ z + \frac{B-A}{k^{n+1} C_k (1+B)} \xi z^k, \; |\xi| = 1, \; k \geq 2 \right\}.$$

Theorem 4.5. *[106] If $f \in VP(n, \lambda, 2\alpha - 1, B)$ then $\mathcal{L}_c f \in VP(n, \lambda, 2\beta - 1, B)$, where*

$$\beta = \beta(\alpha) = \frac{B + 1 + 2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

The result is sharp.

Theorem 4.6. *[106] If $f \in VP(n, \lambda, A, B)$ then $\mathcal{L}_c f \in VP(n, \lambda, A^*, B)$, where*

$$A^* = \frac{B + A(c+1)}{c+2} > A.$$

The result is sharp.

Theorem 4.7. *[106] If $f \in VP(n, \lambda, A, B)$ then $\mathcal{L}_c f \in VP(n, \lambda, A, B^*)$, where*

$$B^* = \frac{A\,(1+B)\,(c+2) + (B-A)\,(c+1)}{(1+B)\,(c+2) - (B-A)\,(c+1)} < B.$$

The result is sharp.

Conclusions and future research directions

In this thesis, classes of analytic, meromorphic, respectively bi-univalent functions are studied, some of them being defined by using operators. There are also results related to differential subordinations.

In what follows, we will present future research directions that can be approached to expand the original results from the thesis, respectively to obtain new ones. The following type of results could be obtained:

- various results using the operator from Section 2.1 for classes of functions with negative coefficients;
- more results using the operators from Sections 2.2 and 2.5;
- inclusion relations of analytic functions associated with Poisson distribution series, similar to those in Section 2.4, using other differential operators;
- other differential subordinations involving the Pythagorean means;
  For example in the paper [73] the authors extended the results from [35] (Section 3.1 of this thesis).
- coefficient estimates and Fekete-Szegő inequalities for various classes of m-fold symmetric, bi-univalent functions or associated with Chebyshev polynomials;
  The results from Sections 3.2 and 3.3 can also be extended. The authors of the works [54], [124] extended the results of [132] (Section 3.2 of this thesis).
- estimates of the second Hankel determinant for different classes of functions defined by subordination.

Another field where interesting results can be obtained is that of harmonic univalent functions, respectively harmonic meromorphic functions. Classes of harmonic functions can be defined using the operators from the thesis and their properties can be studied.

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