Babeş-Bolyai University
Faculty of Mathematics and Computer Science

Techniques of potential theory and geometric function theory in the study of some problems in fluid mechanics

Ph.D. Thesis - Summary

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Contents

Introduction

The theory of univalent functions is an important part of complex analysis and is one of the most attractive directions in geometric function theory of one complex variable. The paper of Koebe (1907) plays a key role in the theory of univalent functions and contains a covering result for the class S of normalized univalent functions on the unit disc U of the complex plane. In 1914 Gronwall obtained the *Area theorem*. By using this theorem, Bieberbach [6] proved in 1916 the sharp estimation of the second coefficient a_2 for functions in the class S, namely $|a_2| \leq 2$. Bieberbach [6] also formulated the following well known conjecture: *If $f \in S$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$, then $|a_n| \leq n$, $n \geq 2$. Equality $|a_n| = n$ for $n \geq 2$ holds if and only if f is a rotation of the Koebe function*. Since then there have been obtained many partial results and there were stated other fundamental conjectures in order to prove the Bieberbach conjecture. The first important step in this direction was obtained by Loewner [66] in 1923, who proved that $|a_3| \leq 3$. In 1936 Robertson proposed a stronger conjecture related to the odd functions in the class S. Next, we mention Milin's conjecture which implies Robertson's conjecture, and thus Bieberbach's conjecture. Finally, Bieberbach's conjecture was successfully solved by L. de Branges [8] in 1985, by using the Loewner method and some fundamental results in the theory of special functions.

Henry Shelby Hele-Shaw (1854-1941) defined the Hele-Shaw cell as an investigation instrument for studying the two dimensional flow of a viscous incompressible fluid between two flat transparent plates that are separated by a very small distance. The Hele-Shaw work was continued by P. Ya. Polubarinova-Kochina [87], and L.A. Galin [34]. They provided a conformal formulation of the Hele-Shaw problem in the case of zero surface tension, by applying the Riemann Mapping Theorem (see Theorem 1.1.6) from a canonical domain (in general the unit disc) onto a phase domain. Other important contributions in the Hele-Shaw field were given by Yu. P. Vinogradov and P. Kufarev (see, e.g., [38]) who proved the existence and uniqueness of the solution for the Polubarinova-Galin equation. An interesting approach was provided by M. Reissig and L. von Wolfersdorf [95] in 1993. Saffman and Taylor [99] in 1958 formulated the first stable exact solution for the ill-posed problem.

The Hele-Shaw flow problem has multiple applications in different fields of natural sciences and engineering, such as: physics, material science, medicine, biology, etc. For example, the Hele-Shaw problem represents a mathematical model for a number of physical situations, as: tumor growths that have the structure of a porous medium, oil distillation, glasses manufacturing.

The theory of univalent functions has provided a powerful tool in the study of various problems concerning the time evolution of the free boundary of a viscous fluid for planar flows in Hele-Show cells under injection. We mention that the evolution in time of starlike domains in the case of zero surface tension was studied by Hohlov, Prokhorov and Vasil'ev [42]. The case of nonzero surface tension remains also valid and can be consulted in [114] (see also [92]). Gustafsson, Prokhorov and Vasil'ev [37] proved that in the case of zero surface tension, the blow-up time for starlike dynamics is ∞. The case of strongly starlike functions of order $\alpha \in (0, 1]$ in the case of zero surface tension was studied by Gustafsson, Prokhorov and Vasil'ev [37], while the case of nonzero surface tension was studied in [114] (see also [38]). In addition, V. M. Entov and P.I. Etingov [24] obtained the invariance in time of some geometric properties of the free boundary for the outer case (unbounded domains with bounded complements). They proved that in the case of zero-surface tension, if the initial domain has a convex complement then the family of domains occupied by the fluid at different moments of time has the same property as long as the solution of the Hele-Shaw problem exists (see also [38]).

This thesis is divided into two parts. The main purpose of the first part of this thesis is to present applications of the theory of univalent functions in the study of Hele-Shaw flow problems, concerning the invariance in time of geometric properties of free boundaries, in both cases of zero and nonzero surface tension, respectively. The second part of this thesis deals with applications of layer potential theory for the Stokes and Brinkman systems in the study of related boundary value problems on Lipschitz domains in Euclidean setting or in compact Riemannian manifolds, with boundary data in L^p or Sobolev spaces.

Part I is related to univalent functions and Hele-Shaw flows problems.

- **Chapter 1** gives definitions, notions and fundamental results concerning univalent functions and Hele-Shaw flow problems, which will be used in the next chapter. All of these results are presented without proofs. The first section presents basic ideas and results in the theory of univalent functions, while the second section deals with the study of some subclasses of univalent functions on the unit disc. Most of these subclasses have analytic and geometric characterizations. Also, there are mentioned various classes of univalent functions on the unit disc: the class S of normalized and univalent functions, the subclass S^* of S consisting of starlike functions with respect to the origin, the subclass K of S consisting of convex functions, etc. This chapter does not contain original results. However, the notion of Φ-likeness on the exterior of the unit disc was recently introduced by P. Curt and D. Fericean [15] (see Definition 1.2.28). Also, the notion of strongly Φ-likeness of order α was introduced by P. Curt, D. Fericean and T. Groşan [16] (see Definition 1.2.29). In the third section we present one of the most important techniques in the theory of univalent functions based on Loewner chains and the Loewner differential equation. The last section deals with the Hele-Shaw flow problem, some practical applications of the Hele-Shaw model in different fields of science and engineering, as well as the Polubarinova-Galin equation for both cases of bounded domains and unbounded domains with bounded complement, in the presence and in the absence of the surface tension. Note that the Polubarinova-Galin equation is an analog of the Loewner differential equation.

- **Chapter 2** contains original results obtained in [15], [16] and [27], related to the invariance in time of Φ-likeness and strongly Φ-likeness of order $\alpha \in (0, 1]$ properties in the case of bounded domains, as well as the case of unbounded domains with bounded complement. We discuss both zero and non-zero surface tension models. Our results in this chapter generalize various results due to Hohlov, Prokhorov and Vasil'ev [42], Vasil'ev and Markina [114], Vasil'ev [112, 113], Gustafsson and Vasil'ev [38], Kornev and Vasil'ev [62].

  The first section of this chapter is based on the original results due to P. Curt and D. Fericean [15], which refer to the time evolution of the boundary of a fluid in the Hele-Shaw flow problem. By applying methods from the theory of univalent functions, we show the invariance in time of Φ-likeness property (a geometric property which includes starlikeness and spirallikeness). The main results presented in Section 2.1 are Theorem 2.1.1, Corollary 2.1.2, Theorem 2.1.4, Corollary 2.1.7, Theorem 2.1.8. Note that Theorem 2.1.1 is a generalization of [42, Theorem 1] (see also Theorem 1.4.3) to the case of Φ-like functions, under the assumption of zero surface tension. Theorem 2.1.4 is a generalization of [114, Theorem 1] (see also Theorem 1.4.4) to the case of Φ-like functions, under the assumption of nonzero small surface tension. In addition, Theorem 2.1.8 is a generalization of [113, Theorem 3], while Theorem 2.1.10 is a generalization of [114, Theorem 3.1] (see also Theorem 1.4.4).

Section 2.2 contains original results obtained by D. Fericean [27], and by P. Curt, D. Fericean and T. Groşan [16]. We prove that the property of strongly Φ-likeness of order $\alpha \in (0,1]$ (a geometric property which includes strongly starlikeness of order α and strongly spirallikeness of order α, respectively) remains invariant in time in two cases: the inner problem and the outer problem, in the absence of the surface tension (see [16]). The case when the surface tension is nonzero but sufficiently small is also treated in Section 2.2 (see [27]). The main results presented in Section 2.2 are Theorem 2.2.1, Corollary 2.2.3, Theorem 2.2.4, Corollary 2.2.6, Theorem 2.2.8 and Theorem 2.2.10.

Section 2.3 of this chapter contains some examples related to the evolution in time of a fluid domain under the assumption of zero surface tension. The examples are related to the solution of the free boundary in the case of injection by considering polynomial functions of degrees 4 and 5. We also provide for those polynomial functions some numerical results obtained by using the programs Matlab and Mathematica. Note that the case of the polynomial function of degree 2 was considered by Polubarinova-Kochina [87] and Galin [34]. They obtained the solution of the free boundary problem in the suction case. The case of polynomials of degree 3 was studied by Huntingford [45]. The case of polynomials of degree 4 was studied in [16] and the polynomials of degree 5 in [27]. Certain numerical results are provided in the injection case for starlike and convex domains (see [16] and [27]).

The layer potential techniques have been successfully used in the analysis of boundary value problems for elliptic equations on Lipschitz domains. Among many valuable contributions in this field we refer to those related to the Stokes and Brinkman equations. Fabes, Kenig and Verchota [26] used a layer potential method to treat the L^2-Dirichlet problem for the Stokes system in Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. Fischer, Hsiao and Wendland [32] used singular perturbation methods and layer potential methods in order to study exterior three-dimensional slow viscous flow problems. Russo [98] presented well-posedness results for boundary value problems associated to the Stokes system on Lipschitz domains in Euclidean setting and in various function spaces. Mitrea and Wright [79] used the layer potential theory to show the well-posedness of the main boundary value problems (Dirichlet, Neumann, Regularity and transmission problems) associated to the Stokes system on Lipschitz domains in Euclidean setting and with boundary data in various function spaces, such as Hardy, Sobolev and Besov spaces. Kohr, Lanza De Cristoforis and Wendland [54] have used a layer potential analysis and the Leray-Schauder degree theory to show an existence result for a nonlinear Neumann-transmission problem for the Stokes and Brinkman systems on bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, with data in L^p, Sobolev, or Besov spaces.

Various boundary value problems for elliptic operators on smooth or even Lipschitz domains in compact Riemannian manifolds have been studied by using layer potential theory. Mitrea, Mitrea and Qiang Shi [73] showed the well-posedness of transmission problems for the Laplace-Beltrami equation on Lipschitz domains in compact Riemannian manifolds, and properties of related singular integral operators on non-smooth manifolds. Recently, Hofmann, Mitrea and Taylor [41] studied elliptic boundary value problems on the class of (two-sided) NTA domains (in the sense of Jerison and Kenig [46]) with Ahlfors regular boundaries and small mean oscillations of the unit normals, in Euclidean setting but also on compact Riemannian manifolds, by using layer potential methods. Kohr, Pintea and Wendland [57] developed a layer potential analysis for a certain type of pseudodifferential matrix operators on Lipschitz domains in compact Riemannian manifolds and used this analysis to treat related boundary value problems.

Part II contains four chapters (Chapters 3-6) and deals with the applications of the layer potential theory in the study of boundary value problems for the Stokes and Brinkman systems on

Lipschitz domains in Euclidean setting or in compact Riemannian manifolds, and with boundary data in L^p or Sobolev spaces.

- **Chapter 3** gives definitions, notions and results that will be used in the elaboration of the next chapters, and focuses on the main properties of the layer potential operators associated to the Stokes and Brinkman equations on Lipschitz domains in Euclidean setting or in compact Riemannian manifolds. This chapter does not contains original results of the author of this thesis. Section 3.1 contains the definition of a Lipschitz domain in $\mathbb{R}^n$ and related Sobolev spaces associated to Lipschitz domains in $\mathbb{R}^n$. These spaces will play a significant role all along this work. In addition, in this section we also present the nontangential trace and conormal derivative operators associated to the Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, as well as the Green formulas related to them. Note that the Stokes equation is a linear form of the Navier-Stokes equation and describes the flow of viscous incompressible fluid with vanishing Reynolds number (for further details we refer to [59]). Also the Brinkman equation describes the flow in porous media and has a similar form as the Stokes equation, except a zero order term. Section 3.2 starts with the definition of a compact operator and their properties. This section represents an introduction to the pseudodifferential operators on $\mathbb{R}^n$ with special attention to elliptic operators and elliptic systems in the sense of Agmon-Douglis-Nirenberg on $\mathbb{R}^n$. In Section 3.3 we describe the class of pseudodifferential operators on compact Riemann manifolds and useful properties of such operators. We also refer to elliptic pseudodifferential operators and elliptic systems in the sense of Agmon-Douglis-Nirenberg on compact Riemannian manifolds. The fourth section is devoted to Fredholm operators and their main properties on Banach spaces, while Section 3.5 contains main results of layer potential theory for the Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$. One of the main results refers to the compactness property of the complementary layer potential operators. It has been obtained by Kohr, Lanza de Cristoforis and Wendland in [54]. We also present invertibility results of related layer potential operators on Lipschitz domains in $\mathbb{R}^n$, obtained by Mitrea and Wright [79]. In the last section we present properties of layer potentials associated with a pseudodifferential Brinkman operator on Lipschitz domains in compact Riemannian manifolds.

- **Chapter 4** contains original results of the author of this thesis concerning the study of a boundary value problem of Dirichlet-transmission type for the Stokes and Brinkman equations on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, with boundary data in L^p or Sobolev spaces. These results have been recently obtained by D. Fericean and W.L. Wendland [31]. The chapter is structured on four sections.

  In the first section we formulate the Dirichlet-transmission problem (4.1.1), while in the second section we get the uniqueness result for this problem. In the next section we obtain an existence result for the Dirichlet-transmission problem. In order to show this result, we use the layer potential theory for both, Brinkman and Stokes equations, and hence a layer potential method that reduces the problem to a uniquely solvable matrix type equation. In the last section we analyze two special cases. The first case refers to an exterior three-dimensional Stokes flow past a porous body that contains a solid core, when the corresponding permeability is large. Asymptotic results for the inner velocity field of the flow inside the porous body, as well as for the force exerted on the porous body, are also obtained. The second case refers to a similar Stokes flow problem but under the hypothesis of low permeability. The novelty of our study is provided by the fact that the transmission conditions in (4.1.1) are expressed in terms of a parameter $\mu \in (0, 1]$ and the given boundary data are chosen in various function spaces, such as Sobolev or L^p spaces, with p near 2. For $n = 3$ and $\mu = 1$, this boundary value problem describes an exterior Stokes flow past a porous particle that contains a solid core, all involved domains being Lipschitz. A similar problem, but in a particular situation, has been analyzed in [103]. The main results presented in this chapter are Theorem 4.2.1, which provides the uniqueness result for the Dirichlet-transmission problem (4.1.1), Theorem 4.3.1 and Theorem 4.3.2, which show existence results for the Dirichlet-transmission problem in Sobolev spaces or L^p spaces, in each of the cases $\mu \in (0,1)$ and $\mu = 1$, respectively, as well as existence and uniqueness results for some boundary value problems that appear in the asymptotic analysis of a special case presented in Section 4.4.

Robin-transmission problems for pseudodifferential Brinkman operators on Lipschitz domains in compact Riemannian manifolds have been studied by Kohr, Pintea and Wendland [57], by using the layer potential theory. Russo and Tartaglione [97] analyzed the Robin problem for the Navier-Stokes equations in an exterior domain $\Omega \subseteq \mathbb{R}^3$ of class C^1. They showed that if the boundary datum belongs to the space $L^q(\partial\Omega)$, $q > \frac{3}{2}$, then the problem has a solution which converges to an assigned constant vector at infinity and takes the boundary value on $\partial\Omega$ in the sense of nontangential convergence. Angot [4] used an asymptotic analysis to show the well-posedness of a Stokes/Brinkman problem with Ochoa-Tapia and Whitaker interface conditions for coupled fluid-porous viscous flows. Alazmi and Vafai [3] analyzed different types of interfacial conditions between a porous medium and a fluid, including the Ochoa-Tapia and Whitaker conditions (5.0.1).

- **Chapter 5** is devoted to the study of some boundary value problems of Robin-transmission type for the Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, when the given boundary data belong to some Sobolev or L^p spaces. This chapter is based on the original results obtained by D. Fericean et al. in [30], [29] and it is structured in two sections. In the first section we use a layer potential method in order to show an existence result for an interface boundary value problem of Robin-transmission type for the Stokes and Brinkman systems on Lipschitz domains in Euclidean setting, when the given boundary data belong to some Sobolev or L^p spaces. The problem is formulated in three adjacent Lipschitz domains, with assigned conditions at infinity and prescribed transmission conditions at the interfaces between these domains. One of them is a Robin-transmission condition, which is formulated in terms of a non-negative matrix multiplication operator P with L^∞ coefficients. The importance of our study is provided by the fact that for some choice of this operator we get the stress jump interface conditions (5.1.13) due to Ochoa-Tapia and Whitaker [84], [85], which are the physical relevant transmission conditions that appear on a fluid-porous interface when the porous medium is governed by the Brinkman equation (see e.g., [90] for details). Indeed, as a particular case, we consider the boundary value problem that describes the exterior Stokes flow of a viscous incompressible fluid past two porous spheres, one of them being embedded into another one, when the shear stress jump conditions (5.1.13) are imposed at the fluid-porous interface. The solution of this problem is determined explicitly together with the streamlines of the flow. The main results presented in Section 5.1 are Theorem 5.1.1, which gives an existence result for the interface problem of Robin-transmission type (5.1.2), when the given boundary datum belong to the L^2- Sobolev space $\mathcal{X}_\nu$ given by (5.1.3), and Theorem 5.1.2, which yields an existence result of the interface problem (5.1.9), when the given boundary datum belong to the L^p-space $\mathfrak{X}_{\nu;p}$ given by (5.1.8), with $p \in \left(\max\left\{1, \frac{2(n-1)}{n+1} - \varepsilon\right\}, 2+\varepsilon\right)$, $n \geq 3$, and some $\varepsilon > 0$.

  The second section of this chapter is devoted to a layer potential analysis for a boundary value problem with Dirichlet and Robin-transmission conditions for Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. In particular, we consider the boundary value problem that describes the exterior Stokes flow of a viscous incompressible fluid past a porous sphere with a solid core inside, when the shear stress jump conditions due to Ochoa-Tapia and Whitaker [84], [85] are imposed at the fluid-porous interface.

The analysis of boundary value problems for the Stokes and Navier-Stokes equations on manifolds has a main role, due to several practical applications of these problems. Among many valuable contributions in this field we mention that Ebin and Marsden [23] studied the fluid flows on surfaces, and Temam and Ziane [109] analyzed the Navier-Stokes equations on thin spherical domains. The analysis of the boundary value problems on compact surfaces, in particular, on the sphere $\mathfrak{S}^2$, is motivated by the flow of viscous incompressible fluids which pass through porous soil or porous rock on the Earth. Kohr, Pintea and Wendland [56, 57] used layer potential methods to study boundary value problems for pseudodifferential Brinkman operators on Lipschitz domains in compact Riemannian manifolds, with given boundary data in L^p, or Sobolev spaces.

- **Chapter 6** is based on the original results of the author of this thesis presented in [28] and is devoted to a layer potential analysis for a boundary value problem of Neumann type associated to the Brinkman system on Lipschitz domains in compact Riemannian manifolds, when the boundary datum belongs to some Sobolev spaces. This chapter is structured on three sections. In the first section we formulate a boundary value problem of Neumann type, (6.1.1), while in the second section we get the uniqueness result for this problem. In the third section we obtain the existence result for the Neumann problem. The main properties are included in Theorem 6.2.1, which gives the uniqueness of the solution to the boundary value problem (6.1.1), and Theorem 6.3.1, which is devoted to the existence and uniqueness of the solution (up to a constant pressure) to the boundary value problem (6.1.1), when the boundary datum belong to arbitrary L^2-Sobolev spaces.

The original results presented in this thesis are based on the following papers:

- P. Curt, **D. Fericean**, *A special class of univalent functions in Hele-Shaw flow problems*, Abstract and Applied Analysis (**ISI**), Volume 2011, Article ID 948236, 10 pages; doi:10.1155/2011/948236.

- P. Curt, **D. Fericean**, T. Groşan, *Φ-like functions in two dimensional free boundary problems*, Mathematica (Cluj), **53 (76)** (2011), 121-130.

- **D. Fericean**, *Strongly Φ-like functions of order α in two-dimensional free boundary problems*, Appl. Math. Comput. (**ISI**), **218** (2012), 7856-7863.

- **D. Fericean**, *Layer potential analysis of a Neumann problem for the Brinkman system*, Mathematica (Cluj), to appear.

- **D. Fericean**, *Boundary value problems with Dirichlet and Robin-transmission conditions. Well-posedness results*, in preparation.

- **D. Fericean**, T. Groşan, M. Kohr, W.L. Wendland, *Interface boundary value problems of Robin-transmission type for the Stokes and Brinkman systems on n-dimensional Lipschitz domains: applications*, Math. Meth. Appl. Sci. (**ISI**), to appear.

- **D. Fericean**, W.L. Wendland, *Layer potential analysis for a Dirichlet-transmission problem in Lipschitz domains in $\mathbb{R}^n$*, submitted.

Keywords

Hele-Shaw flow, free boundary problem, Polubarinova-Galin equation, univalent function, Φ-like function, Lipschitz domain, Stokes system, Brinkman system, Fredholm operator, potential theory, compact Riemannian manifold, transmission problem.

Part I

Univalent functions and Hele-Shaw flow properties

Chapter 1

General results concerning univalent functions and Hele-Shaw flow problems

In this chapter we are going to give definitions, notions and fundamental results concerning univalent functions and Hele-Shaw flow problems, which will be used in the next chapter.

The first section gives basic results concerning holomorphic and univalent functions, while the second section contains certain results regarding some particular subclasses of univalent functions that can be characterized by interesting geometric and analytic conditions. There are presented some general results in the theory of univalent functions, such as sufficient conditions of univalence for holomorphic functions on domains in $\mathbb{C}$ and some examples of univalent functions. A key role is played by the well known Riemann mapping theorem concerning the conformal equivalence of simply connected domains in $\mathbb{C}$. Also, there are mentioned various classes of univalent functions on the unit disc: the class S of normalized and univalent functions, the class S^* of normalized starlike functions on the unit disc U, the class K of normalized convex functions on U, the class $\mathcal{C}$ of normalized close-to-convex functions, the class M_α of α convex functions, the class $\hat{S}_\gamma$ of spirallike functions of type γ and the class of Φ-like functions. In the third section there are presented fundamental results in the theory of Loewner chains and the Loewner differential equation. The last section deals with the Hele-Shaw problem, the Stokes-Leibenzon model, and the Polubarinova-Galin equation. All of these results are presented without proofs. This chapter does not contain original results. However the notion of Φ-likeness on the exterior of the unit disc was recently introduced by P. Curt and D. Fericean [15] (see Definition 1.2.28). Also, the notion of strongly Φ-likeness of order α was introduced by P. Curt, D. Fericean and T. Groşan [16] (see Definition 1.2.29).

We mention that the main sources used in the preparation of this chapter are [22], [35], [37], [38], [39], [53], [81], [88] and [114].

1.1 Preliminaries

This section presents some basic ideas and results in the theory of univalent functions. These results will be useful in the next sections. For more details see [22], [35], [39], [53], and [88], basic sources used in the preparation of this section.

Notations

Let us give some notations which will be used in next chapters.

- $\mathbb{C}$ denotes the complex plane;
- $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane;

- $\mathcal{H}(D)$ denotes the set of holomorphic functions defined on an open set $D \subseteq \mathbb{C}$ with values in $\mathbb{C}$;
- $\mathcal{H}_u(D)$ represents the class of univalent functions from D;
- $U = U(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disc;
- $U^- = \{z : |z| > 1\}$ denotes the exterior of the unit disc;
- $U_r = \{z \in \mathbb{C} : |z| < r\}$ denotes the disc with the center in the origin and of radius r;
- $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ denotes the disc of center z_0 and radius r.

Definition 1.1.1 **[39]** Let $D \subseteq \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$. We say that f is *univalent* if f is holomorphic and injective on D.
We denote by $\mathcal{H}_u(D)$ the set of univalent functions on D.

The following well known result provides a necessary condition of univalence.

Theorem 1.1.2 **[39]** *Let D be a domain in $\mathbb{C}$ and let $f \in \mathcal{H}_u(D)$. Then $f'(z) \neq 0$ for $z \in D$.*

We remark that the above result provides a necessary but not sufficient condition of global univalence for holomorphic functions. Indeed, the entire function $f(z) = e^z$ is locally univalent on $\mathbb{C}$ (i.e. $f'(z) \neq 0$, $z \in \mathbb{C}$), but f is not univalent on the whole complex plane.

The following results provide simple sufficient conditions of univalence for holomorphic functions. Theorem 1.1.3 was obtained by Alexander, Noshiro, Warschawski and Wolff (see, e.g., [81], [35]).

Theorem 1.1.3 *Let $D \subseteq \mathbb{C}$ be a convex domain and let $f : D \to \mathbb{C}$ be a holomorphic function. If $\operatorname{Re} f'(z) > 0$, $z \in D$, then f is univalent on D.*

The next result due to Ozaki and Kaplan [50] is a generalization of Theorem 1.1.3. If D is a convex domain and $g(z) \equiv z$ in Theorem 1.1.4, one obtains Theorem 1.1.3.

Theorem 1.1.4 (**[50]**) *Let $D \subseteq \mathbb{C}$ be a domain and $f, g \in \mathcal{H}(D)$ be such that $g \in \mathcal{H}_u(D)$ and $g(D)$ is a convex domain. If $\operatorname{Re}\left[\frac{f'(z)}{g'(z)}\right] > 0$, $z \in D$, then f is univalent on D.*

Also, we recall that a locally univalent function is conformal, i.e. it preserves angles and orientation. This leads to the notion of conformal equivalence. Next, we present two fundamental results related to this notion (see [22], [39], [53], [88]).

Definition 1.1.5 (**[39]**) Let D_1 and D_2 be two domains in $\mathbb{C}$. The function $f : D_1 \to D_2$ is a *conformal mapping* of D_1 onto D_2 if f is univalent on D_1 and $f(D_1) = D_2$. In this case the domains D_1 and D_2 are called *conformally equivalent*. If f is a conformal mapping of a domain $D \subseteq \mathbb{C}$ onto itself, then f is called an *automorphism (conformal automorphism)* of D.

One of the most important results in the theory of univalent functions is the *Riemann mapping theorem* concerning the conformal equivalence of simply connected domains in $\mathbb{C}$. For various applications of this fundamental result, see [96].

Theorem 1.1.6 (**[35], [39]**) *Let D be a simply connected domain in $\mathbb{C}$ such that $D \neq \mathbb{C}$. Then D and the unit disc U are conformally equivalent. In addition, if $\eta \in D$ is a given point, then there exists a unique conformal mapping f of D onto U such that $f(\eta) = 0$ and $f'(\eta) > 0$.*

Theorem 1.1.7 (**[22], [89]**) *Let $D \subset \mathbb{C}$ be a simply connected domain bounded by a closed Jordan curve. Also, let $f : D \to U$ be a conformal mapping of D onto U. Then there exists a homeomorphism F of $\overline{D}$ onto $\overline{U}$ such that $F|_D = f$.*

1.2 Subclasses of univalent functions on U

This section is concerned with the study of some subclasses of univalent functions on the unit disc. Most of these subclasses have analytic and geometric characterizations. We refer to the class S of normalized and univalent functions on U, the subclass S^* of S consisting of starlike functions with respect to the origin, the subclass K of S consisting of convex functions, the subclass $\mathcal{C}$ of S consisting close-to-convex functions. Also, we present the class M_α of α-convex functions (Mocanu's functions), the class $\hat{S}_\gamma$ of spirallike functions of type γ, and the class of Φ-like functions on U. To this end, we recall the definitions of these subclasses and some of their basic properties. The motivation of these classes of univalent functions is based on the fact that the last section of this chapter deals with the following problem: to determine the time evolution of the free boundary of a viscous fluid for a planar flow in the Hele - Shaw cell model under injection (see [38]). It is known that the notions of starlikeness, strongly starlikeness of order α, convexity in a given direction, are preserved in time for both inner and outer domains (see [38]).

The main sources used in this section are [22], [35], [80], [81], [88].

1.2.1 The classes S and Σ

Next, we consider the class S of univalent functions f on U that are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ (see [22] and [88]). Therefore,

$$S = \{f \in \mathcal{H}_u(U) : f(0) = f'(0) - 1 = 0\}. \tag{1.2.1}$$

Next, let Σ be the class of univalent functions φ on U^- given by

$$\varphi(z) = z + \alpha_0 + \sum_{n=1}^{\infty} \frac{\alpha_n}{z^n}, \quad |z| > 1,$$

such that these functions have a simple pole at ∞ (see [22] and [88]). The class Σ plays an important role in the study of some properties of the class S (see [22] and [88]).

Remark 1.2.1 (i) If $f \in S$ and $g(\zeta) = \dfrac{1}{f\left(\frac{1}{\zeta}\right)}$, $\zeta \in U^-$, then the function g belongs to the class Σ and $g(\zeta) \neq 0$, $\zeta \in U^-$ (see e.g., [81]).

(ii) If $g \in \Sigma$ and $g(\zeta) \neq 0$, $\zeta \in U^-$, then the function f belongs to the class S, where $f(z) = \dfrac{1}{g\left(\frac{1}{z}\right)}$, $0 < |z| < 1$, and $f(0) = 0$ (see e.g., [81]).

The following result, known as the *Area theorem*, was obtained by Gronwall in 1914, and represents a fundamental result in the study of elementary properties of the classes S and Σ.

Theorem 1.2.2 [39] *If $g \in \Sigma$ is given by $g(z) = z + a_0 + \dfrac{a_1}{z} + ... + \dfrac{a_n}{z^n} + ...$, $|z| > 1$, then* $\sum_{n=1}^{\infty} n|a_n|^2 \leq 1$.

By using the area theorem, Bieberbach [6] proved in 1916 the sharp estimation of the second coefficient a_2 for functions in the class S, namely $|a_2| \leq 2$. This result was used to obtain other classical results related to the class S, such as the covering and distortion theorems for the class S (see [22], [88]). These fundamental results related to class S were obtained by Koebe (1907) and Bieberbach [6] (for details, see [22], [35] and [88]).

Theorem 1.2.3 **[6]** *If $f \in S$ is given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$, then $|a_2| \leq 2$. The equality $|a_2| = 2$ holds if and only if $f = k_\theta$ for some $\theta \in \mathbb{R}$.*

Bieberbach [6] also formulated the following conjecture:

Bieberbach's conjecture: *If $f \in S$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$, then $|a_n| \leq n$, $n \geq 2$. The equality $|a_n| = n$ for $n \geq 2$ holds if and only if f is a rotation of the Koebe function.*

Next, we recall the growth and distortion theorem for the class S. For more details, see [22], [88].

Theorem 1.2.4 (**[6]**; see also **[22] and [88]**) *If $f \in S$ then the following statements are true:*

(i) $\dfrac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \dfrac{|z|}{(1-|z|)^2}$, $z \in U$,

(ii) $\dfrac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \dfrac{1+|z|}{(1-|z|)^3}$, $z \in U$,

(iii) $\dfrac{1-|z|}{1+|z|} \leq \left|\dfrac{zf'(z)}{f(z)}\right| \leq \dfrac{1+|z|}{1-|z|}$, $z \in U$.

Equality holds in each of the above relations for some point $z \neq 0$ if and only if f is a rotation of the Koebe function.

In view of the above result and the Hurwitz theorem for univalent functions, it follows the compactness result related to the class S.

Corollary 1.2.5 **[81]** *The class S is compact.*

1.2.2 The class S^* of starlike functions

In this section we consider the class S^* of normalized starlike functions on the unit disc, and we recall some important results related to this class, such as the estimation of coefficients, growth and distortion results. For details, see [22], [81], [88].

Definition 1.2.6 **[81]** Let $f \in \mathcal{H}(U)$ be such that $f(0) = 0$. The function f is called *starlike* if f is univalent on U, $f(0) = 0$ and $f(U)$ is a starlike domain with respect to the origin.

Note that a domain $\Omega \subseteq \mathbb{C}$ is starlike with respect to $z_0 \in \Omega$ if the closed segment between z_0 and z is contained in Ω, for all $z \in \Omega$.

The following result provides the analytical characterization of starlikeness (see e.g., [22], [35], [81] and [88]):

Theorem 1.2.7 (Analytical characterization of starlikeness) *Let $f \in \mathcal{H}(U)$ be such that $f(0) = 0$. Then f is starlike if and only if $f'(0) \neq 0$ and*

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in U. \tag{1.2.2}$$

Denote by S^* the class of normalized and starlike functions on U. Thus,

$$S^* = \{f : U \to \mathbb{C} : \ f \text{ starlike}, \ f(0) = f'(0) - 1 = 0\}.$$

We have that $S^* \subset S$. Also, the Koebe function and its rotations belong to S^*.

Theorem 1.2.8 (**[66]**) *If the function $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ belongs to class S^* then $|a_n| \leq n$, $n \geq 2$. Equality holds if and only if f is a rotation of Koebe function.*

Definition 1.2.9 [88] Assume that $F(\zeta) = a\zeta + a_0 + \frac{a_{-1}}{\zeta} + \ldots$, $|\zeta| > 1$, where $a \neq 0$. The function F is *starlike* on U^- if F is univalent on U^- and the set $E = \mathbb{C} \setminus F(U^-)$ is starlike with respect to zero.

Remark 1.2.10 [88] Let F be a holomorphic function on $U^- = \{\zeta : |\zeta| > 1\}$ such that $F(\zeta) = a\zeta + a_0 + \frac{a_{-1}}{\zeta} + \ldots$, $|\zeta| > 1$, where $a \neq 0$. Then F is starlike on U^- if and only if (see [88])

$$\mathrm{Re}\left[\frac{\zeta F'(\zeta)}{F(\zeta)}\right] > 0, \quad |\zeta| > 1.$$

Therefore, it is natural to consider the following subclass of starlike functions consisting of strongly starlike functions of order $\alpha \in (0,1]$ (see [81]):

Definition 1.2.11 Let f be a holomorphic function on the unit disc U such that $f(0) = 0$ and $f'(0) \neq 0$, and let $\alpha \in (0,1]$. The function f is called *strongly starlike of order α on U* if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad z \in U. \tag{1.2.3}$$

In this case, $f(U)$ is called a *strongly starlike domain of order α*. Let $\hat{S}^*(\alpha)$ be the class of strongly starlike functions of order α on U.

The notion of strongly starlikeness of order α will be useful in another section.

1.2.3 The class K of convex functions

The notion of convexity was introduced by E. Study (1913). His investigation was continued by T. Gronwall and K. Loewner [66]. This section contains the definition of the class K of normalized convex functions on the unit disc, the Alexander theorem concerning the connection between the classes S^* and K, the estimation of coefficients for functions in the class K, and the growth and distortion theorem related to K. For further details, see [22], [88], [35], [81]).

Definition 1.2.12 [81] Let $f : U \to \mathbb{C}$ be a holomorphic function. The function f is called *convex* if f is univalent on U and $f(U)$ is a convex domain.

The following analytical characterization of convexity on the unit disc is very useful in many applications related to convex functions on U (see [22], [81], [88]):

Theorem 1.2.13 (Analytical characterization of convexity). *Let $f \in \mathcal{H}(U)$. Then the function f is convex if and only if $f'(0) \neq 0$ and*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U.$$

Let K be the subset of S consisting of convex functions. Then $K \subset S^* \subset S$. Also, it is clear that the Koebe function $k : U \to \mathbb{C}$, $k(z) = \frac{z}{(1-z)^2}$, belongs to S^* but is not in K. By using Theorems 1.2.7 and 1.2.13, one obtains the following connection between the classes S^* and K, known as the Alexander duality theorem (see e.g., [81]):

Theorem 1.2.14 [81] *Let $f : U \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $f'(0) = 1$. Then $f \in K$ if and only if $F \in S^*$, where $F(z) = zf'(z)$, $z \in U$.*

Also, we mention the covering result for the class K (see e.g., [81]).

Theorem 1.2.15 *If $f \in K$, then $f(U) \supseteq U\left(0, \frac{1}{2}\right)$.*

The following sharp estimation of coefficients holds for the class K:

Theorem 1.2.16 [66] *If the function $f(z) = z + a_2z^2 + \ldots + a_nz^n + \ldots$ belongs to the class K, then $|a_n| \leq 1$, $n = 2, 3, \ldots$. Equality holds if and only if f has the form*

$$f(z) = \frac{z}{1 + e^{i\theta}z}, \quad z \in U, \quad \theta \in \mathbb{R}.$$

Finally, we mention the convexity in the case of the exterior of the unit disc.

Definition 1.2.17 [88] Assume that $F(\zeta) = a\zeta + a_0 + \frac{a_{-1}}{\zeta} + \ldots$, $|\zeta| > 1$, where $a \neq 0$. The function F is *convex* on U^- if F is univalent on U^- and the set $E = \mathbb{C} \setminus F(U^-)$ is convex.

Remark 1.2.18 [88] Let F be a holomorphic function on $U^- = \{\zeta : |\zeta| > 1\}$ such that $F(\zeta) = a\zeta + a_0 + \frac{a_{-1}}{\zeta} + \ldots$, $|\zeta| > 1$, where $a \neq 0$. Then F is convex on U^- if and only if

$$\text{Re}\left[1 + \frac{\zeta F''(\zeta)}{F'(\zeta)}\right] > 0, \quad |\zeta| > 1.$$

1.2.4 The class $\mathcal{C}$ of close-to-convex functions

The following notion of close-to-convexity was introduced by Kaplan:

Definition 1.2.19 [50] Let $f \in \mathcal{H}(U)$. The function f is called *close-to-convex* if there exists a convex function g on U such that

$$\text{Re}\left[\frac{f'(z)}{g'(z)}\right] > 0, \ z \in U. \tag{1.2.4}$$

From Theorem 1.1.4 it follows that each close-to-convex function is univalent on U.

1.2.5 The class M_α of α-convex functions

The notion of α-convexity was introduced by P.T. Mocanu [80] in 1969. This notion provides a continuous passage between starlikeness and convexity, by the variation of parameter α. We give the definition of α-convexity on the unit disc and some basic properties of the α-convex functions. The main sources used in this subsection are [35], [80], [81].

Definition 1.2.20 [80] Let $\alpha \in \mathbb{R}$ and $f : U \to \mathbb{C}$ be a normalized holomorphic function. The function f is called *α-convex* if

$$\text{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0, \ z \in U. \tag{1.2.5}$$

Let M_α be the class of α-convex functions. Then $M_0 = S^*$ and $M_1 = K$.

Next, we give some basic results concerning α-convex functions. For more details, see [35] and [81]. The first result shows that each α-convex function is starlike, for $\alpha \in \mathbb{R}$.

Theorem 1.2.21 (**[80]**; see also **[81]**) *If $\alpha \in \mathbb{R}$, then $M_\alpha \subseteq S^*$. In addition, $M_\beta \subseteq M_\alpha$, for all $\alpha, \beta \in \mathbb{R}$, $0 \leq \frac{\alpha}{\beta} < 1$.*

For $\alpha \geq 0$, we have the following result concerning the duality between the classes S^* and M_α.

Theorem 1.2.22 (**[80]**; see also **[81]**) *Let $\alpha \geq 0$. Then $f \in M_\alpha$ if and only if the function g defined by*

$$g(z) = f(z)\left[\frac{zf'(z)}{f(z)}\right]^\alpha, \quad z \in U,$$

belongs to S^*. The branch of the power function is chosen such that

$$\left.\left[\frac{zf'(z)}{f(z)}\right]^\alpha\right|_{z=0} = 1.$$

1.2.6 The class $\hat{S}_\gamma$ of spirallike functions of type γ

This section is devoted to another subclass of univalent functions, namely the class of spirallike functions, which was introduced by Špaček in 1933. We give the definition of a spirallike domain of type γ, the definition of a spirallike function of type γ, a necessary and sufficient condition of spirallikeness of type γ in the unit disc, a duality result between the classes of starlike and spirallike functions of type γ, and an example of spirallike function of type γ.

The following notion of spirallikeness of type γ was introduced by Špaček (see [22]).

Definition 1.2.23 **[81]** If $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the logarithmic γ-spiral (or γ - spiral) is a curve given by

$$\omega(t) = \omega_0 e^{-(\cos\gamma - i\sin\gamma)t}, \quad t \in \mathbb{R},$$

where $\omega_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

A domain $D \subset \mathbb{C}$, which contains the origin, is called *spirallike of type* γ, where $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if for each $\omega_0 \in D \setminus \{0\}$, the arc of the γ-spiral joining ω_0 to the origin lies in D.

Definition 1.2.24 **[81]** (1) Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $f \in \mathcal{H}(U)$ be such that $f(0) = 0$. The function f is *spirallike of type* γ if f is univalent on U and the domain $f(U)$ is a spirallike domain of type γ.
(2) Let $f \in \mathcal{H}(U)$ be such that $f(0) = 0$. We say that f is *spirallike* if there exists $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that f is spirallike of type γ.

Let $\gamma \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and let $\hat{S}_\gamma$ be the class of normalized spirallike functions of type γ:

$$\hat{S}_\gamma = \left\{f \in \mathcal{H}(U) : \ f(0) = f'(0) - 1 = 0, \ \mathrm{Re}\left[e^{i\gamma}\frac{zf'(z)}{f(z)}\right] > 0, z \in U\right\}.$$

By using the analytical characterization of spirallikeness, we can give the following result, which provides the connection between the classes S^* and $\hat{S}_\gamma$ (see [35], [81]).

Theorem 1.2.25 *Let $\gamma \in \mathbb{R}$ be such that $-\dfrac{\pi}{2} < \gamma < \dfrac{\pi}{2}$. Also, let $f \in \mathcal{H}(U)$ be a normalized function. Then $f \in \hat{S}_\gamma$ if and only if $g \in S^*$, where $g(z) = z\left[\frac{f(z)}{z}\right]^{1+i\mathrm{tg}\,\gamma}$.*

1.2.7 The class of Φ-like functions

In this section we are concerned with Φ-like functions on the unit disc. We discuss the connection between Φ-likeness and univalence.

The notion of Φ-likeness was introduced by L. Brickman [9] in 1973 as a generalization of starlikeness and spirallikeness. For more details see e.g., [9].

Definition 1.2.26 [9] Let f be a holomorphic function on U such that $f(0) = 0$ and $f'(0) \neq 0$. Let Φ be a holomorphic function on $f(U)$ such that $\Phi(0) = 0$ and $\operatorname{Re}\Phi'(0) > 0$. Then f is *Φ-like on U* (or *Φ-like*) if

$$\operatorname{Re}\left[\frac{zf'(z)}{\Phi(f(z))}\right] > 0, \quad z \in U. \tag{1.2.6}$$

The next result of Brickman (see [9]; see also [35]) shows that any Φ-like function is univalent on U, and conversely any univalent function on U is Φ-like for some Φ.

Theorem 1.2.27 [9] *The following relations hold:*
(i) If f is Φ-like, then $f \in \mathcal{H}_u(U)$.
(ii) If $f \in \mathcal{H}_u(U)$, then there exists a function $\Phi \in \mathcal{H}(f(U))$ such that $\Phi(0) = 0$, $\operatorname{Re}\Phi'(0) > 0$ and f is Φ-like.

The notion of Φ-likeness may be also defined on the exterior of the unit disc, not only on the unit disc. This notion was introduced by P. Curt and D. Fericean [15], as follows:

Definition 1.2.28 [15] Let F be a holomorphic function on $U^- = \{\zeta \mid |\zeta| > 1\}$ such that $F(\zeta) = a\zeta + a_0 + \dfrac{a_{-1}}{\zeta} + \ldots$, where $a \neq 0$. Let $\tilde{\Phi}$ be a holomorphic function on $F(U^-)$ such that $\lim\limits_{\zeta\to\infty} \tilde{\Phi}(\zeta) = \infty$ and $\lim\limits_{\zeta\to\infty} \tilde{\Phi}'(\zeta) > 0$. We say that F is *$\tilde{\Phi}$-like on U^-* if

$$\operatorname{Re}\left[\frac{\zeta F'(\zeta)}{\tilde{\Phi}(F(\zeta))}\right] > 0, \quad \zeta \in U^-. \tag{1.2.7}$$

1.2.8 The class of strongly Φ-like functions of order α

The notion of strongly Φ-likeness of order α was introduced by P. Curt, D. Fericean and T. Groşan in [16], as a generalization of strongly starlikeness and spirallikeness of order α.

Definition 1.2.29 [16] Let f be a holomorphic function on the unit disc U such that $f(0) = 0$ and $f'(0) \neq 0$. Let Φ be an holomorphic function on $f(U)$ such that $\Phi(0) = 0$ and $|\arg\Phi'(0)| < \dfrac{\alpha\pi}{2}$, where $\alpha \in (0, 1]$. We say that f is *strongly Φ-like of order α on U* if

$$\left|\arg\left(\frac{zf'(z)}{\Phi(f(z))}\right)\right| < \frac{\alpha\pi}{2}, \quad z \in U. \tag{1.2.8}$$

In this case, $f(U)$ is called a *strongly Φ-like domain of order α.*

The notion of strongly Φ-likeness of order α may be also defined on the exterior of the unit disc, not only on the unit disc. This notion was introduced by P. Curt, D. Fericean and T. Groşan (see [16]).

Definition 1.2.30 [16] Let F be a holomorphic function on $U^- = \{\zeta : |\zeta| > 1\}$ such that

$$F(\zeta) = a\zeta + a_0 + \frac{a_{-1}}{\zeta} + \ldots, \quad |\zeta| > 1,$$

where $a \neq 0$. Let $\alpha \in (0, 1]$ and let Φ be a holomorphic function on $F(U^-)$ such that

$$\lim_{\zeta\to\infty} \Phi(\zeta) = \infty \text{ and } \lim_{\zeta\to\infty} \Phi'(\zeta) > 0.$$

We say that F is *strongly Φ-like of order α on U^-* if

$$\left|\arg\frac{\zeta F'(\zeta)}{\Phi(F(\zeta))}\right| < \frac{\alpha\pi}{2},\ \forall\, \zeta \in U^-. \tag{1.2.9}$$

1.3 Loewner chains and the Loewner equation. Applications

In this section we present one of the most important methods in the theory of univalent functions based on Loewner chains and the Loewner differential equation. The proof of the Bieberbach conjecture due to L. de Branges [8] involved the Loewner differential equation. This section contains some notions and results concerning Loewner chains and the Loewner differential equation.

Definition 1.3.1 [81] Let $f, g : U \to \mathbb{C}$ be two holomorphic functions. The function f is *subordinate* to g (denoted by $f \prec g$ or $f(z) \prec g(z)$) if there exists a function $w \in \mathcal{H}(U)$ with $w(0) = 0$, $|w(z)| < 1, z \in U$ (thus, w is a Schwarz function), such that $f(z) = g(w(z))$, $z \in U$.

If the function g is univalent we have the following result:

Theorem 1.3.2 [81] *If $f, g \in \mathcal{H}(U)$ and the function g is univalent on U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Next, we give the definition of a Loewner chain (univalent subordination chain) (see [88]; see also [35], [81]):

Definition 1.3.3 (**[88]**) A function $f = f(z,t) : U \times [0,\infty) \to \mathbb{C}$ is called a *univalent subordination chain* (or a *Loewner chain*) if $f(\cdot, t)$ is univalent on U, $f(0,t) = 0$ for $t \geq 0$, and

$$f(\cdot, s) \prec f(\cdot, t), \quad 0 \leq s \leq t < \infty. \tag{1.3.1}$$

The subordination (1.3.1) is equivalent to the existence of a unique family of Schwarz functions $v(z,s,t)$, called the *transition functions* of $f(z,t)$, such that

$$f(z,s) = f(v(z,s,t),t), \quad z \in U,\ 0 \leq s \leq t < \infty.$$

Example 1.3.4 [88] The function $f(z,t) = \dfrac{e^t z}{(1-z)^2}$, $z \in U$, $t \geq 0$, is a Loewner chain.

Next, we present the Loewner differential equation and the connection with Loewner chains. First, we recall the Carathéodory class of functions with positive real part on the unit disc (see [35], [81], [88]). Let

$$\mathcal{P} = \Big\{ p \in \mathcal{H}(U) : p(0) = 1,\ \mathrm{Re}\, p(z) > 0,\ z \in U \Big\}$$

be the Carathéodory class of functions with positive real part on U.

The next result is an important characterization of Lowner chains in terms of the Loewner differential equation. This result was obtained by Pommerenke ([88]; for details and applications, see also [35], [88]).

Theorem 1.3.5 [88] *Let $f = f(z,t) : U \times [0,\infty) \to \mathbb{C}$ be such that $f(0,t) = 0$ and $f'(0,t) = e^t$, $t \geq 0$. Then $f(z,t)$ is a Loewner chain if and only if the following conditions are satisfied:*

(i) There exist $r \in (0,1)$ and $K > 0$ such that $f(\cdot,t) \in \mathcal{H}(U_r)$ for $t \geq 0$, $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in U_r$, and $|f(z,t)| \leq Ke^t$, $z \in U_r$, $t \geq 0$.

(ii) There exists a function $p(z,t)$ such that $p(\cdot,t) \in \mathcal{P}$ for $t \geq 0$, $p(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in U$, and

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \quad \textit{a.e.} \quad t \in [0,\infty),\ \forall\, z \in U_r. \tag{1.3.2}$$

Note that $f'(z,t) = \dfrac{\partial f}{\partial z}(z,t)$. Also, we mention that the equation (1.3.2) is called the *Loewner differential equation* (Loewner-Kufarev equation).

1.4 General results concerning Hele-Shaw flow problems

In this section we present the Hele-Shaw flow problem, some practical applications of the Hele-Shaw model in different fields of science and engineering, as well as the Polubarinova-Galin equation for both cases of bounded domains and unbounded domains with bounded complement, in the presence and in the absence of the surface tension γ. The Polubarinova-Galin equation is an analog of the Loewner differential equation, which was studied in the above section. We mention that the main sources used in this section are [37], [38], [92], [112], [113], [114].

In 1898 Henry Shelby Hele-Shaw defined the Hele-Shaw cell as an investigation instrument for studying the two dimensional flow of a viscous incompressible fluid between two flat transparent plates that are separated by a very small distance. In this model the viscous fluid occupies a phase domain with free boundary and more fluid is injected or removed through a point. The free boundary starts moving due to injection/suction. Hele-Shaw flow problem reduces to determine the evolution in time of the initial domain, occupied by the fluid. For more details see [38], [112], [113], [114].

1.4.1 Bounded domains

We start by presenting the basic notions regarding the bounded case (for details, see [38]). In this case we study the flow of a viscous fluid in a planar Hele-Shaw cell under injection through a source (of constant strength Q, which is negative in the case of injection, i.e. $Q < 0$), located at the origin. We can assume that the strength of the source is constant (otherwise by a suitable change of variable we can reduce to the constant strength source). Suppose that at the initial moment the domain $\Omega(0)$ occupied by the fluid is simply connected and it is bounded by an analytic and smooth curve $\Gamma(0) = \partial\Omega(0)$. By using the Riemann Mapping Theorem (see Theorem 1.1.6), the domain $\Omega(t)$ (occupied by the fluid at the moment t) is conformally equivalent to the unit disc $U = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and hence, it can be described by an unique univalent function $f(\zeta, t)$ of U onto $\Omega(t)$ normalized by $f(0, t) = 0$, $f'(0, t) > 0$. Let $\Gamma(t)$ be the boundary of the domain $\Omega(t)$. The function $f(\zeta, 0) = f_0(\zeta)$ produces a parametrization of $\Gamma(0)$, that is $\Gamma(0) = \{f_0(e^{i\theta}),\ \theta \in [0, 2\pi)\}$. In addition, the moving boundary is parameterized by $\Gamma(t) = \{f(e^{i\theta}, t), \theta \in [0, 2\pi)\}$.

The zero surface tension model

In this case, the equation satisfied by the free boundary $\Gamma(t)$ was first derived by L. A. Galin [34], P. Polubarinova-Kochina ([87]), and is given by:

$$\text{Re}[\dot{f}(\zeta, t)\overline{\zeta f'(\zeta, t)}] = -\frac{Q}{2\pi}, \quad \zeta = e^{i\theta} \in \partial U. \tag{1.4.1}$$

We mention that in the previous equality we have used the notations: $f' = \dfrac{\partial f}{\partial \zeta}$, $\dot{f} = \dfrac{\partial f}{\partial t}$.

Taking into account the Schwarz-Poisson formula, the equation (1.4.1) can be written in the following form, which is an analog of the Loewner-Kufarev equation (see [38, page 18]):

$$\dot{f}(\zeta, t) = -\zeta f'(\zeta, t)\frac{Q}{4\pi^2}\int_0^{2\pi} \frac{1}{|f'(e^{i\theta}t)|^2} \cdot \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\theta,\ \zeta \in U. \tag{1.4.2}$$

If we consider in the relation (1.4.2) the limit of ζ to a point on the unit circle, and we use the Plemelj-Sokhotsky formulas [82], the equation (1.4.2) reduces to the equation (1.4.1) (see [38]).

Definition 1.4.1 [38] A *strong or classical solution* in the interval $[0, T)$ is a function $f(\zeta, t)$, $t \in [0, T)$, that is univalent on a neighborhood of $\overline{U}$ and C^1 with respect to $t \in [0, T)$, where T is called the *blow-up time*.

The nonzero surface tension model

In the case of the problem of injection $(Q < 0)$ of the fluid into a bounded simply connected domain with small surface tension $\gamma > 0$, the equation which describes the evolution of the free boundary, i.e., the Polubarinova-Galin equation [114] has the form (see also [38], [87]):

$$\text{Re}[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}] = -\frac{Q}{2\pi} + \gamma H\left[i\frac{\partial\kappa}{\partial\theta}(e^{i\theta},t)\right](\theta), \quad \zeta = e^{i\theta}, \tag{1.4.3}$$

where κ is the curvature of the boundary and it is defined by

$$\kappa(e^{i\theta},t) = \frac{1}{|f'(e^{i\theta},t)|}\text{Re}\left(1 + \frac{e^{i\theta}f''(e^{i\theta},t)}{f'(e^{i\theta},t)}\right), \quad \theta \in [0,2\pi), \tag{1.4.4}$$

and the Hilbert transform in (1.4.3) is given by (see [38])

$$H[\Phi](\theta) := \frac{1}{\pi}p.v.\int_0^{2\pi} \frac{\Phi(e^{i\theta'})}{1 - e^{i(\theta-\theta')}}d\theta', \tag{1.4.5}$$

where the symbol $p.v.$ means the principal value.

Remark 1.4.2 An important fact is related to the analysis of the situation when $\gamma \to 0$. The solution in the limiting case $\gamma \to 0$ is not always the corresponding zero-surface tension solution (see e.g., [105], [113]). In view of this idea, it is important to treat both zero and non zero surface tension cases.

1.4.2 Unbounded domains with bounded complement

Next, denote by $\Omega(t)$ the domain occupied by the fluid at the moment t and $\Gamma(t) = \partial\Omega(t)$. By using the Riemann mapping theorem, the domain $\Omega(t)$ can be described by an univalent function $F(\zeta,t)$ from the exterior of the unit disk $U^- = \{\zeta : |\zeta| > 1\}$ onto $\Omega(t)$, $F(\zeta,t) = a\zeta + a_0 + \frac{a_1}{\zeta} + \ldots$, $|\zeta| > 1$, where $a > 0$ (see also the definition of the class Σ from Section 1.2.1).

• The Polubarinova-Galin equation satisfied by the free boundary is (see [112], [114]):

$$\text{Re}[\dot{F}(\zeta,t)\overline{\zeta F'(\zeta,t)}] = \frac{Q}{2\pi}, \quad \zeta = e^{i\theta}, \tag{1.4.6}$$

for the zero tension surface model.

The existence and uniqueness of the solution of the Polubarinova-Galin equation (1.4.6) was studied by J. Escher and G. Simonett (see [38]).

• In the case of small surface tension model (i.e., for sufficiently small surface tension γ), the Polubarinova-Galin equation has the following form (see e.g., [38]):

$$\text{Re}[\dot{F}(\zeta,t)\overline{\zeta F'(\zeta,t)}] = \frac{Q}{2\pi} - \gamma H\left[i\frac{\partial\kappa}{\partial t}(e^{i\theta},t)\right](\theta), \quad \zeta = e^{i\theta}. \tag{1.4.7}$$

The existence and uniqueness of the solution of the Polubarinova-Galin equation (1.4.7) was studied by M. Kimura in [52].

1.4.3 Invariance in time of some special domains

Starlike domains

The case of starlike domains was studied by Hohlov, Prokhorov and Vasil'ev in [42] in the case of zero surface tension. We suppose that the initial function f_0 is analytic in $\overline{U}$. Then the following result holds (see also [112]):

Theorem 1.4.3 [42] *Let $Q < 0$ and $f_0 \in S^*$ be analytic and univalent in a neighborhood of $\overline{U}$. Then any domain $\Omega(t)$ remains starlike ($f(\cdot, t) \in S^*$) as long as the solution of the Polubarinova-Galin equation exists.*

Next, we present the case of starlike domains in the presence of a small surface tension γ. The following result holds (see [114, Theorem 1]; see also [92, Theorem 3.1]).

Theorem 1.4.4 [114] *Let $Q < 0$ and the surface tension γ be sufficiently small. If the initial domain $\Omega(0)$ is starlike, then there exists some $t(\gamma) \leq T$, such that the family of domains $\Omega(t)$ (i.e. the family of univalent functions $f(\zeta, t)$) preserves this property during the time $t \in [0, t(\gamma)]$, where T is the blow-up time.*

Strongly starlike domains of order α

The case of strongly starlike domains of order α was studied by Gustafsson, Prokhorov, Vasil'ev in [37] (see also [38, p 79]). They proved the following result (see [37]):

Theorem 1.4.5 [37] *Let $f_0 \in \hat{S}^*(\alpha)$, $\alpha \in (0, 1]$, be analytic and univalent in a neighborhood of $\overline{U}$. Then the strong solution $f(\zeta, t)$ of the Polubarinova-Galin equation $(1.4.1)$ determines an univalent subordination chain of strongly starlike functions of order $\alpha(t)$, where $\alpha(t)$ is a strictly decreasing function of t during the life time, and $\alpha(0) = \alpha$.*

Next, let us present the following result related to strongly starlike domains of order $\alpha \in (0, 1]$ in the presence of the surface tension γ (see [38, Theorem 4.3.4]):

Theorem 1.4.6 [38] *Let $Q < 0$ and the surface tension γ be sufficiently small. If the initial domain $\Omega(0)$ is strongly starlike of order α, then there exists $t(\gamma) \leq T$, such that the family of domains $\Omega(t)$ (i.e. the family of univalent functions $f(\zeta, t)$) preserves this property for each $t \in [0, t(\gamma)]$, where T is the blow-up time.*

Convex domains

For the outer case (unbounded domains with bounded complements), the first results in studying the invariance in time of some geometric properties of the free boundary were obtained by V. M. Entov and P.I. Etingov in [24]. They proved that in the case of zero-surface tension, if the initial domain has a convex complement then the family of domains occupied by the fluid at different moments of time has the same property as long as the solution of the Hele-Shaw problem exists (see also [38]).

Chapter 2

Invariant geometric properties in Hele-Shaw flow problems

This chapter contains original results which refer to the invariance in time of Φ-likeness and strongly Φ-likeness of order α properties in the case of bounded domains, as well as the case of unbounded domains with bounded complement. We discuss both zero and nonzero surface tension models. Certain particular cases related to starlikeness, strongly starlikeness of order α, spirallikeness, are also presented. An interesting situation is related to the case $\gamma \to 0$, where γ is the surface tension. Note that the solution in the case $\gamma \to 0$ need not be the corresponding zero surface tension solution (see [105]). This fact is justified by some numerical results obtained in [93] (see also [38], [113]). These arguments motivate our choice of studying the invariance in time of Φ-likeness and strongly Φ-likeness properties of order α in both cases of zero and nonzero surface tension models, respectively.

2.1 Special classes of univalent functions in Hele-Shaw flow problems

The results in this section have been recently obtained by P. Curt and D. Fericean [15].

2.1.1 The inner problem

In this section we obtain the invariance in time of Φ-likeness property for the inner problem.

We study the flow of a viscous fluid in a planar Hele-Shaw cell under injection through a source of constant strength $Q < 0$, located at the origin. Suppose that at the initial moment $t = 0$, the domain $\Omega(0)$ occupied by the fluid is simply connected, contains the origin, and it is bounded by an analytic and smooth curve $\Gamma(0) = \partial\Omega(0)$. By using the Riemann Mapping Theorem (see Theorem 1.1.6), the domain $\Omega(t)$ (occupied by the fluid at the moment t) is conformally equivalent to the unit disc $U = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and hence, it can be described by an unique univalent function $f(\zeta, t)$ of U onto $\Omega(t)$, normalized by $f(0, t) = 0$, $f'(0, t) > 0$. Let $\Gamma(t)$ be the boundary of the domain $\Omega(t)$. The function $f(\zeta, 0) = f_0(\zeta)$ produces a parametrization of $\Gamma(0)$, that is $\Gamma(0) = \{f_0(e^{i\theta}),\ \theta \in [0, 2\pi)\}$. In addition, the moving boundary is parameterized by $\Gamma(t) = \{f(e^{i\theta}, t), \theta \in [0, 2\pi)\}$.

Starting with an initial bounded domain $\Omega(0)$ which is Φ-like, we prove that at each moment $t \in [0, T)$ the domain $\Omega(t)$ is Φ-like (both for zero and nonzero surface tension models). The results presented in this section are due to P. Curt and D. Fericean [15].

• The following result is a generalization of [42, Theorem 1] (see also Theorem 1.4.3) to the case of Φ-like functions, in the case of zero surface tension.

Theorem 2.1.1 [15]. *Let $Q < 0$ and f_0 be a function which is Φ-like on U and univalent on $\overline{U}$.*

Let $f(\zeta,t)$ be the classical solution of the Polubarinova-Galin equation (1.4.1) *with the initial condition $f(\zeta,0)=f_0(\zeta)$. Also let $\Omega=\bigcup\limits_{0\leq t<T}\Omega(t)=\bigcup\limits_{0\leq t<T} f(U,t)$, where T is the blow-up time. If Φ is holomorphic on $\overline{\Omega}$ and satisfies the condition*

$$\text{Re}\Phi'(w)>0,\ \forall\, w\in\overline{\Omega}, \tag{2.1.1}$$

then $f(\zeta,t)$ is Φ-like for $t\in[0,T)$.

Corollary 2.1.2 [15] *Let $Q<0$ and let f_0 be a function which is spirallike of type $\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ on U and univalent on $\overline{U}$. Then the classical solution of the Polubarinova-Galin equation* (1.4.1) *with the initial condition $f(\zeta,0)=f_0(\zeta)$ is spirallike of type α for $t\in[0,T)$, where T is the blow-up time.*

Remark 2.1.3 [15] According to Theorem 2.1.1, we deduce that if the initial domain $\Omega(0)=f(U,0)$ is Φ-like, than any domain $\Omega(t)=f(U,t)$ remains Φ-like for $t\in[0,T)$, where T is the *blow-up time*.

• The following result is a generalization of [114, Theorem 1] (see also Theorem 1.4.4) to the case of Φ-like functions, under the assumption of nonzero small surface tension.

Theorem 2.1.4 [15]. *Let $Q<0$ and the surface tension γ be sufficiently small. If f_0 is a function which is Φ-like on U and univalent on $\overline{U}$, then there exists $t(\gamma)\leq T$ such that the classical solution $f(\zeta,t)$ of the equation* (1.4.3) *with the initial condition $f(\zeta,0)=f_0(\zeta)$ is Φ-like for $t\in[0,t(\gamma))$, where T is the blow-up time, $\Omega=\bigcup\limits_{0\leq t<t(\gamma)}\Omega(t)=\bigcup\limits_{0\leq t<t(\gamma)} f(U,t)$ and Φ is a holomorphic function on $\overline{\Omega}$ which satisfies the condition* (2.1.1).

Remark 2.1.5 [15] Let $Q<0$ and let f_0 be a function that is Φ-like on U and univalent on $\overline{U}$. If f_0 satisfies the condition (1.2.6) for each $\zeta\in\overline{U}$, then there exists a surface tension γ (which depends on f_0) sufficiently small and $t(\gamma)\leq T$ such that the classical solution $f(\zeta,t)$ of the equation (1.4.3) with the initial condition $f(\zeta,0)=f_0(\zeta)$ is Φ-like for $t\in[0,t(\gamma))$, where T is the blow-up time.

Remark 2.1.6 [15] According to Theorem 2.1.4, we deduce that if the initial domain $\Omega(0)=f(U,0)$ is Φ-like, then there exists $t(\gamma)\leq T$ such that the domain $\Omega(t)=f(U,t)$ remains Φ-like for $t\in[0,t(\gamma))$, where T is the blow-up time.

Corollary 2.1.7 [15] *Let $Q<0$ and the surface tension α be sufficiently small. If f_0 is a function which is spirallike of type $\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ on U and univalent on $\overline{U}$, then there exists $t(\gamma)\leq T$ such that the classical solution $f(\zeta,t)$ of the equation* (1.4.3) *with the initial condition $f(\zeta,0)=f_0(\zeta)$ is spirallike of type α for $t\in[0,t(\gamma))$, where T is the blow-up time.*

2.1.2 The outer problem

In this section we obtain the invariance in time of the same geometric property of Φ-likeness in the case of the outer problem (the case of unbounded domains with bounded complement). The case of unbounded domain with bounded complement can be viewed as the dynamics of a contracting bubble in a Hele-Shaw cell since the fluid occupies a neighborhood of infinity and injection (of constant strength $Q<0$) is supposed to take place at infinity.

• Let us first consider the case of zero surface tension. The following result is a generalization of [113, Theorem 3]. The mentioned theorem may be obtained by taking $\tilde{\Phi}(w)\equiv w$ in Theorem 2.1.8 below.

Theorem 2.1.8 [15]. *Let F_0 be a function which $\tilde{\Phi}$-like on U^- and univalent on $\overline{U^-}$. Then the solution $F(\zeta,t)$ of the Polubarinova-Galin equation* (1.4.6) *with the initial condition $F(\zeta,0) = F_0(\zeta)$ is $\tilde{\Phi}$-like for $t \in [0,T)$, where T is the blow-up time, $\Omega = \bigcup_{0\le t<T} \Omega(t) = \bigcup_{0\le t<T} F(U^-,t)$ and $\tilde{\Phi}$ is a holomorphic function on $\overline{\Omega}$ which satisfies the following conditions:*

$$\text{(2.1.2)} \qquad \mathrm{Re}\frac{\tilde{\Phi}(\omega)}{\omega} > 0 \quad \textit{and} \quad \mathrm{Re}\tilde{\Phi}'(\omega) < 2\mathrm{Re}\frac{\tilde{\Phi}(\omega)}{\omega},\ \forall\, \omega \in \overline{\Omega}.$$

Remark 2.1.9 According to Theorem 2.1.8, we deduce that if the initial domain $\Omega(0) = F(U^-,0)$ is Φ-like, then each of the domains $\Omega(t) = F(U^-,t)$ remains Φ-like for $t \in [0,T)$, where T is the blow-up time.

• Next, we consider the case of nonzero small surface tension. The following result is a generalization of [114, Theorem 3.1] (see also Theorem 1.4.4). The mentioned theorem may be obtained by taking $\tilde{\Phi}(w) \equiv w$ in Theorem 2.1.10 below.

Theorem 2.1.10 [15]. *Let $Q < 0$ and let the surface tension γ be sufficiently small. If F_0 is a function which is $\tilde{\Phi}$-like on U^- and univalent on $\overline{U^-}$, then there exists $t(\gamma) \le T$ such that the solution $F(\zeta,t)$ of the equation* (1.4.7) *with the initial condition $F(\zeta,0) = F_0(\zeta)$ is $\tilde{\Phi}$-like for $t \in [0,t(\gamma))$, where T is the blow-up time, $\Omega = \bigcup_{0\le t<t(\gamma)} \Omega(t) = \bigcup_{0\le t<t(\gamma)} F(U^-,t)$ and $\tilde{\Phi}$ is a holomorphic function on $\overline{\Omega}$ which satisfies the conditions* (2.1.2).

Remark 2.1.11 [15] According to Theorem 2.1.10, we deduce that if the initial domain $\Omega(0) = F(U^-,0)$ is Φ-like, then there exists $t(\gamma) \le T$ such that each domain $\Omega(t) = F(U^-,t)$ remains Φ-like for $t \in [0,t(\gamma))$, where T is the blow-up time.

Remark 2.1.12 [15] Let $Q < 0$ and let F_0 be a function that is $\tilde{\Phi}$-like on U^- and univalent on $\overline{U^-}$. If F_0 satisfies the condition (1.2.7) for each $\zeta \in \overline{U^-}$, then there exist a surface tension γ (which depends on F_0) sufficiently small and $t(\gamma) \le T$ such that the classical solution $F(\zeta,t)$ of the equation (1.4.7) with the initial condition $F(\zeta,0) = F_0(\zeta)$ is $\tilde{\Phi}$-like for $t \in [0,t(\gamma))$, where T is the blow-up time.

2.2 Strongly Φ-like functions of order α in two-dimensional free boundary problems

This section contains original results obtained by D. Fericean [27], and by P. Curt, D. Fericean and T. Groşan (see [16]). We show that the property of strongly Φ-likeness of order $\alpha \in (0,1]$ (a geometric property which includes strongly starlikeness of order α and strongly spirallikeness of order α, respectively) remains invariant in time in two cases: the inner problem and the outer problem, in the absence of the surface tension (see [16]). The case when the surface tension is nonzero but sufficiently small will be also treated (see [27]).

2.2.1 The inner problem

Next, we present certain results concerning the evolution in time of strongly Φ-likeness property of order $\alpha \in (0,1]$ for the inner problem (bounded domains) in the absence of surface tension. Let us consider the flow of a viscous fluid in a planar Hele-Shaw cell under injection through a source of constant strength $Q < 0$ that is located at the origin. The following result is a generalization of [112, Theorem 1] (see e.g., [38, Theorem 4.3.2]) for the case of strongly Φ-like functions of order α.

Theorem 2.2.1 [16] *Let $\alpha \in (0,1]$, $Q < 0$ and let f_0 be a strongly Φ-like function of order α on U and univalent on $\overline{U}$. Let $f(\zeta, t)$ be the classical solution of the Polubarinova-Galin equation* $(1.4.1)$ *with the initial condition $f(\zeta, 0) = f_0(\zeta)$. Also let $\Omega = \bigcup\limits_{0 \le t < T} \Omega(t) = \bigcup\limits_{0 \le t < T} f(U, t)$ where T is the blow-up time. If Φ is holomorphic on $\overline{\Omega}$ and satisfies the condition $|\arg \Phi'(w)| < \dfrac{\alpha\pi}{2}$, $\forall\, w \in \overline{\Omega}$, then $f(\zeta, t)$ is strongly Φ-like of order α for $t \in [0, T)$.*

Remark 2.2.2 [16] According to Theorem 2.2.1 we deduce that if the initial domain $\Omega(0) = f(U, 0)$ is strongly Φ-like of order α, than any domain $\Omega(t) = f(U, t)$ remains strongly Φ-like of order α, for $t \in [0, T)$, where T is the blow-up time.

Corollary 2.2.3 [16] *Let $Q < 0$ and let f_0 be a strongly spirallike function of type β and order α on U and univalent on $\overline{U}$, where $\alpha \in (0, 1]$ and $\beta \in \left(-\frac{\alpha\pi}{2}, \frac{\alpha\pi}{2}\right)$. Then the classical solution of the Polubarinova-Galin equation* $(1.4.1)$ *with the initial condition $f(\zeta, 0) = f_0(\zeta)$ is strongly spirallike of type β and order α for $t \in [0, T)$, where T is the blow-up time.*

The next result due to D. Fericean [27] is a generalization of [114, Theorem 3.1] (see also [38, Theorem 4.3.4]) to the case of strongly Φ-like of order α functions, in the presence of a small surface tension.

Theorem 2.2.4 [27] *Let $\alpha \in (0, 1]$, $Q < 0$, and let the surface tension γ be sufficiently small. Also let f_0 be a strongly Φ-like function of order α on U and univalent on $\overline{U}$. Then there exists $t(\gamma) \le T$ such that the classical solution $f(\zeta, t)$ of the Polubarinova-Galin equation* $(1.4.3)$ *with the initial condition $f(\zeta, 0) = f_0(\zeta)$ is strongly Φ-like of order α for $t \in [0, t(\gamma))$, where T is the blow-up time, $\Omega = \bigcup\limits_{0 \le t < t(\gamma)} \Omega(t) = \bigcup\limits_{0 \le t < t(\gamma)} f(U, t)$, and Φ is a holomorphic function on $\overline{\Omega}$, which satisfies the condition $|\arg \Phi'(w)| < \dfrac{\alpha\pi}{2}$, $\forall\, w \in \overline{\Omega}$.*

Remark 2.2.5 [27] According to Theorem 2.2.4, we deduce that if the initial domain $\Omega(0) = f(U, 0)$ is strongly Φ-like of order α, then there exists $t(\gamma) \le T$ such that the domain $\Omega(t) = f(U, t)$ remains strongly Φ-like of order α for $t \in [0, t(\gamma))$, where T is the blow-up time.

Corollary 2.2.6 [27] *Let $Q < 0$ and let the surface tension γ be sufficiently small. Also let f_0 be a strongly spirallike function of type β and order α on U and univalent on $\overline{U}$, where $\alpha \in (0, 1]$ and $\beta \in \left(-\frac{\alpha\pi}{2}, \frac{\alpha\pi}{2}\right)$. Then there exists $t(\gamma) \le T$ such that the classical solution of the Polubarinova-Galin equation* $(1.4.3)$ *with the initial condition $f(\zeta, 0) = f_0(\zeta)$ is strongly spirallike of type β and order α for $t \in [0, t(\gamma))$, where T is the blow-up time.*

Remark 2.2.7 [27] According to Corollary 2.2.6, if the initial domain $\Omega(0) = f(U, 0)$ is strongly spirallike of type β and order α, then there exists $t(\gamma) \le T$ such that the family of domains $\Omega(t) = f(U, t)$ remain strongly spirallike of type β and order α for $t \in [0, t(\gamma))$, where T is the blow-up time.

2.2.2 The outer problem

In this section we obtain the invariance in time of the strongly Φ-likeness of order α property for the outer problem (the case of unbounded domains with bounded complement).

Next, we obtain the analog version of Theorem 2.2.1 to the case of unbounded domains with bounded complement, by assuming zero surface tension. This result is a generalization of [62, Theorem 3] (see also [16, Theorem 4.3.5]). The mentioned theorem may be obtained by taking $\Phi(w) \equiv w$ and $\alpha = 1$ in Theorem 2.2.8 below. The case $\alpha = 1$ was considered in [15] (see also Theorem 2.1.8).

Theorem 2.2.8 [16] *Let $\alpha \in (0,1]$ and F_0 be a strongly Φ-like function of order α on U^- and univalent on $\overline{U^-}$. Then the solution $F(\zeta,t)$ of the Polubarinova-Galin equation (1.4.6) with the initial condition $F(\zeta,0) = F_0(\zeta)$ is strongly Φ-like of order α for $t \in [0,T)$, where T is the blow-up time, $\Omega = \bigcup_{0 \le t < T} \Omega(t) = \bigcup_{0 \le t < T} F(U^-,t)$ and the function Φ is a holomorphic function on $\overline{\Omega}$ which satisfies the following conditions: $\left|\arg \frac{\Phi(w)}{w}\right| < \frac{\alpha\pi}{2}$, $\forall\, w \in \overline{\Omega}$, and $\left|\arg\left(2\frac{\Phi(w)}{w} - \Phi'(w)\right)\right| < \frac{\alpha\pi}{2}$, $\forall\, w \in \overline{\Omega}$.*

Remark 2.2.9 [16] According to Theorem 2.2.8, we deduce that if the initial domain $\Omega(0) = F(U^-,0)$ is strongly Φ-like of order α, then each of the domains $\Omega(t) = F(U^-,t)$ remains strongly Φ-like of order α, for $t \in [0,T)$, where T is the blow-up time.

Next, we present the analog version of Theorem 2.2.4 in the case of unbounded domains with bounded complement, by assuming the nonzero small surface tension. This result is a generalization of [113, Theorem 3] (see also [38, Theorem 4.3.5]) to the case of strongly Φ-like functions of order α.

Theorem 2.2.10 [27] *Let $Q < 0$ and let the surface tension γ be sufficiently small. Let $\alpha \in (0,1]$ and let F_0 be a strongly Φ-like function of order α on U^- and univalent on $\overline{U^-}$. Then there exists $t(\gamma) \le T$ such that the solution $F(\zeta,t)$ of the Polubarinova-Galin equation (1.4.3) with the initial condition $F(\zeta,0) = F_0(\zeta)$ is strongly Φ-like of order α for $t \in [0,t(\gamma))$, where T is the blow-up time, $\Omega(t) = F(U^-,t)$, $\Omega = \bigcup_{0 \le t < t(\gamma)} \Omega(t)$, and Φ is assumed to be a holomorphic function on $\overline{\Omega}$ such that $\left|\arg \frac{\Phi(w)}{w}\right| < \frac{\alpha\pi}{2}$, $\forall\, w \in \overline{\Omega}$ and $\left|\arg\left(2\frac{\Phi(w)}{w} - \Phi'(w)\right)\right| < \frac{\alpha\pi}{2}$, $\forall\, w \in \overline{\Omega}$.*

Remark 2.2.11 [27] Under the assumptions of Theorem 2.2.10, if the initial domain $\Omega(0) = F(U^-,0)$ is strongly Φ-like of order α, then there exists $t(\gamma) \le T$ such that the family of domains $\Omega(t) = F(U^-,t)$ remain strongly Φ-like of order α for $t \in [0,t(\gamma))$, where T is the blow-up time.

2.3 Numerical results

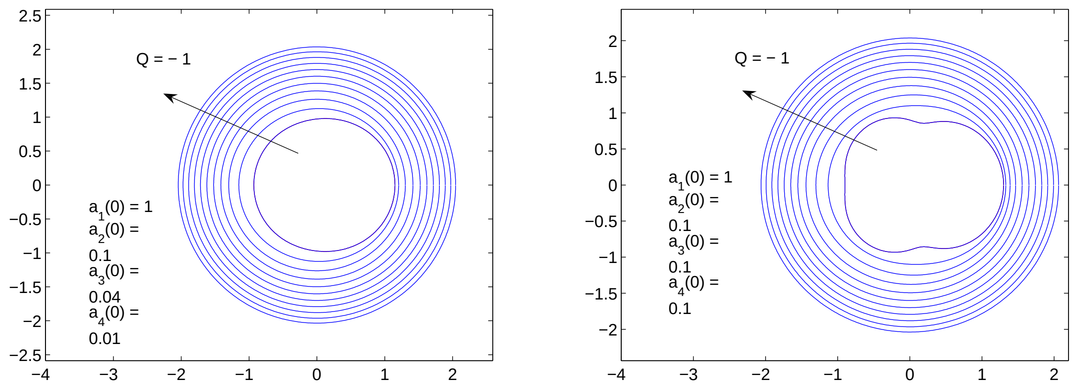
In this section we present some examples of evolution in time of a fluid domain under the assumption of zero surface tension. These numerical results were obtained in [16] and [27]. The case of the polynomial function of degree 2 was considered by Polubarinova-Kochina [87] and Galin [34]. They obtained the solution of the free boundary problem in the suction case. The case of the polynomials of degree 3 was studied by Huntingford [45].

To this aim, we study the case of polynomials of degree 4 in [16] and the polynomials of degree 5 in [27] and we provide some numerical results for the injection case for starlike and convex domains. We consider the polynomial function of degree 4

$$F(\zeta,t) = a_1(t)\zeta + a_2(t)\zeta^2 + a_3(t)\zeta^3 + a_4(t)\zeta^4.$$

It has to satisfy the Polubarinova-Galin equation (1.4.1) that leads to a system of differential equations obtained using Mathematica, which is solved starting from an initial domain $F(U,0)$ given by $F(\zeta,0) = a_1(0)\zeta + a_2(0)\zeta^2 + a_3(0)\zeta^3 + a_4(0)\zeta^4$.

The system obtained using Mathematica was solved numerically by using Matlab for two different initial domains, a convex domain and a starlike domain, respectively. We have also considered a negative value for Q (fluid injection). In the injection case the domain takes a disk shape



(a) (b)

Figure 2.1: Convex initial domain, injection and starlike initial domain, injection.

after some time. In the Figures 2.1 (a) and (b) there are presented the domains variations. We remark that the coefficients $a_k(0)$, $k = 1, \dots, 4$, have been chosen such that $\sum_{k=2}^{4} k|a_k(0)| \leq 1$, which yields that the initial domain $F(U,0)$ is starlike (see e.g., [35]) and the coefficients $a_k(0)$, $k = 1, \dots, 4$, have been chosen such that $\sum_{k=2}^{4} k^2|a_k(0)| \leq 1$, which yields that the initial domain $F(U,0)$ is convex (see e.g., [35]). Next, let us consider the polynomial function of degree 5

$$F(\zeta,t) = a_1(t)\zeta + a_2(t)\zeta^2 + a_3(t)\zeta^3 + a_4(t)\zeta^4 + a_5(t)\zeta^5.$$

Requiring that the above function be a solution of the Polubarinova-Galin equation (1.4.1), and using Mathematica, we obtain a similar system of first order differential equations with the system obtained in the case of polynomial of degree 4. Thus, we start from an initial domain $F(U,0)$, where $F(\zeta,0) = a_1(0)\zeta + a_2(0)\zeta^2 + a_3(0)\zeta^3 + a_4(0)\zeta^4 + a_5(0)\zeta^5$. We have also considered a negative value for Q (fluid injection). In the injection case the domain take a disk shape after some time. We remark that we impose the follwoing condition $\sum_{k=2}^{5} k|a_k| \leq |a_1|$, which yields that the initial domain $F(U,0)$ is starlike (see e.g., [35]).

Part II

Layer potential theory for Stokes and Brinkman systems on Lipschitz domains. Applications

Chapter 3

Layer potential theory for Stokes and Brinkman equations on Lipschitz domains

In this chapter we present the main properties of the layer potential operators associated to the Stokes and Brinkman equations on Lipschitz domains in Euclidean setting or in compact Riemannian manifolds. We introduce the fundamental solutions for the Stokes and Brinkman equations, and we define the associated layer potentials on Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, and also in compact Riemannian manifolds. One of the main related results is the compactness property of the complementary layer potential operators. Useful invertibility results are also presented (see, e.g., [54], [57], [59], [79] for details). In addition, we present definitions, notions and results which will be used in the elaboration of the next chapters. The main sources used in the preparation of this chapter are [19], [44], [54], [55], [56], [57], [59], [78], [79], [111].

The chapter is organized as follows. The first section contains the definition of a Lipschitz domain in $\mathbb{R}^n$ and related Sobolev spaces associated to Lipschitz domains, which will be used all along this thesis. Second section is an introduction to the pseudodifferential operators on $\mathbb{R}^n$ with special attention to elliptic operators on $\mathbb{R}^n$ and elliptic systems in the sense of Agmon-Douglis-Nirenberg on $\mathbb{R}^n$. The next section presents the main properties related to pseudodifferential operators on compact Riemannian manifolds. The fourth section is devoted to Fredholm operators and their main properties on Banach spaces. The fifth section contains the layer potential theory for the Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$. We present the fundamental solutions for both Brinkman and Stokes systems, as well as boundedness properties of the corresponding layer potential operators. One of the main connected results refers to the compactness property of the complementary layer potential operators. Note that a complementary layer potential operator for the Stokes and Brinkman systems is the difference between the corresponding layer potential operator for the Brinkman system and that associated to the Stokes system. This compactness property will be very useful in the treatment of some boundary value problems of transmission type that will be analyzed in the next chapters. In the second section we introduce the pseudodifferential Brinkman operator on a compact boundaryless Riemannian manifold, by following the ideas from [56, 57]. This is an invertible operator and two entries of its inverse determine the corresponding fundamental solution. Note that the pseudodifferential Brinkman operator can be interpreted as an extension of the differential Brinkman operator from the Euclidean setting to compact Riemannian manifolds. Next, we present main results of the layer potential theory for pseudodifferential Brinkman operators on Lipschitz domains in compact Riemannian manifold, which include the fundamental solution for the Brinkman operator and the compactness property of the associated complementary layer potential operators. This chapter does not contains original result of the author of this chapter.

3.1 Lipschitz domains in $\mathbb{R}^n$ and related Sobolev spaces

This section contains the definition of a Lipschitz domain in $\mathbb{R}^n$ and describes some special Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$, which play a significant role all along this work.

3.1.1 Lipschitz domains in $\mathbb{R}^n$

Definition 3.1.1 Let X be a metric space. A function $f : X \to \mathbb{C}$ is called *Lipschitz* if there exists a constant $c > 0$ such that $|f(x) - f(y)| \leq c\,\mathrm{dist}(\mathrm{x}, \mathrm{y}),\ \forall\, x, y \in X$.

Definition 3.1.2 (e.g., **[51], [70], [79]**) A open set $\mathfrak{D} \subset \mathbb{R}^n$ $(n \geq 2)$ is *a bounded Lipschitz domain* if there exists a constant $c > 0$ and a family of hyperplanes $\Xi_i,\ i = 1, ..., m$, a choice of the unit normal $\mathbf{n}_i$ to Ξ_i, and a Lipschitz function $\varphi_i : \Xi_i \to \mathbb{R}$ with the Lipschitz constant c, i.e., $|\varphi_i(x) - \varphi_i(y)| < c|x - y|$ for all $x, y \in Z_i$, such that

(i) For each i, in the system of coordinates determined by $(\Xi_i, \mathbf{n}_i)$, there is an open, vertical, double truncated, circular cylinder Z_i such that $\{Z_i\}_{i=1}^m$ is an open cover of the boundary $\partial\mathfrak{D}$

(ii) If $\mathfrak{D}_i$ is the domain situated above the graph of the function φ_i, then, by working again in the system of coordinates determined by $(\Xi_i, \mathbf{n}_i)$ in $\mathbb{R}^n$, and denoting by ξZ_i the concentric dilatation of $\mathcal{Z}_i$ by a factor $\xi > 0$, one has for each i,

$$\mathfrak{D} \cap 2(c+1)\mathcal{Z}_i = \mathfrak{D}_i \cap 2(c+1)\mathcal{Z}_i,$$
$$\partial\mathfrak{D} \cap 2(c+1)\mathcal{Z}_i = \partial\mathfrak{D}_i \cap 2(c+1)\mathcal{Z}_i.$$

The pair $(\mathcal{Z}_i, \varphi_i)$ is called a *coordinate chart of* $\mathfrak{D}$, and $\partial\mathfrak{D}_i$ is the graph of φ_i in the system of coordinates induced by $\mathcal{Z}_i$.

Let $\mathfrak{D} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We use the notations $\mathfrak{D}_- := \mathfrak{D}$, $\mathfrak{D}_+ := \mathbb{R}^n \setminus \overline{\mathfrak{D}}$. For a fixed parameter $k_0 = k_0(\partial\mathfrak{D}) > 1$, sufficiently large, define the *nontangential approach regions* $\gamma_\pm(\mathbf{x})$, $\mathbf{x} \in \partial\mathfrak{D}$, as (see e.g., [79, p. 27]) $\gamma_\pm(\mathbf{x}) := \{\mathbf{y} \in \mathfrak{D}_\pm : \mathrm{dist}(\mathbf{x}, \mathbf{y}) < k_0\ \mathrm{dist}(\mathbf{y}, \partial\mathfrak{D})\}$, and, for arbitrary $u : \mathfrak{D}_\pm \to \mathbb{R}$, the nontangential maximal function $\mathcal{N}_{k_0}(u)$ by $\mathcal{N}_{k_0}(u)(\mathbf{x}) := \sup\{|u(\mathbf{y})| : \mathbf{y} \in \gamma_\pm(\mathbf{x})\},\ \mathbf{x} \in \partial\mathfrak{D}$, where $\mathcal{N} := \mathcal{N}_{k_0}$ is called the *nontangential maximal operator*.

Remark 3.1.3 If in Definition 3.1.2 the functions φ_i are chosen in the class C^1, then the domain $\mathfrak{D}$ is of class C^1.

3.1.2 Function spaces on $\mathbb{R}^n$

In this section we present some notations and conventions which will be used in the next chapters. First, denote by $\mathbb{Z}$ the set of integers, and by $\mathbb{N}$ the set of positive integers. All along this work, we consider the space $\mathbb{R}^n$ equipped with the norm $|x| := \sqrt{x_1^2 + ... + x_n^2},\ x = (x_1, ..., x_n) \in \mathbb{R}^n$ and with the canonical orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}_j := (\delta_{1j}, \ldots, \delta_{nj}),\ 1 \leq j \leq n$. The notation $\langle\cdot,\cdot\rangle$ refers to the duality pairing between two dual spaces X and X^*. In addition, the same notation is sometimes used for the inner product in some Hilbert spaces, including $\mathbb{R}^n$.

Next, let Ω be an open set and denote by $C^0(\Omega)$ be the space of continuous real valued functions on Ω. The space of r times continuously differentiable real valued functions on Ω is denoted by $C^r(\Omega),\ r \in \mathbb{N}$, and $C^\infty(\Omega) := \bigcap_{r\in\mathbb{N}} C^r(\Omega)$.

The space of functions $\varphi \in C^r(\Omega)$ with compact support is denoted by $C_0^r(\Omega)$, and by $\mathcal{D}'(\Omega)$ we denote the space of distributions on Ω, i.e., the dual of $C_0^\infty(\Omega)$ equipped with the inductive limit topology. Denote also by $\mathrm{supp}(u)$ the support of $u \in \mathcal{D}'(\Omega)$ in Ω defined as the set of points without any open neighborhood in which u vanishes.

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of smooth rapidly decreasing functions. This set is called the *Schwartz space*. Its dual space $\mathcal{S}'(\mathbb{R}^n)$ is called the *space of tempered distributions* in $\mathbb{R}^n$ (see e.g., [117]). In addition, by $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ we denote the Fourier transform defined on the space of tempered distributions, and by $\mathcal{F}^{-1}$ its inverse (we refer to Section 4.1 for more details). Also, as usual, by $\Delta = \partial_1^2 + \cdots + \partial_n^2$ we denote the Laplacian, where all partial derivatives are considered in the sense of distributions.

3.1.3 Review of Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$

Next, we consider a bounded Lipschitz domain $\mathfrak{D} := \mathfrak{D}_- \subset \mathbb{R}^n, n \geq 3,$, i.e., its boundary $\Gamma := \partial\mathfrak{D}$ is locally the graph of a Lipschitz function, and let $\mathfrak{D}_+ := \mathbb{R}^n \setminus \overline{\mathfrak{D}}$. Also let $\mathbf{n}_\Gamma$ be the outward unit normal to Γ, which is defined a.e., with respect to the surface element $d\sigma$, on Γ. For $p \in (1, \infty)$ denote by $L^p(\mathbb{R}^n)$ the Lebesgue space of measurable, p-th power integrable functions on $\mathbb{R}^n$, and, for $p \in (1, \infty)$ and $s \in \mathbb{R}$, denote by $L_s^p(\mathbb{R}^n, \mathbb{R}) := L_s^p(\mathbb{R}^n)$ the Sobolev (Bessel potential) space with smoothness s in $\mathbb{R}^n$, defined by (see e.g., [44], [75])

$$\begin{aligned} L_s^p(\mathbb{R}^n) : &= \left\{(I - \Delta)^{-s/2}\, g :\ g \in L^p(\mathbb{R}^n)\right\} \\ &= \left\{\mathcal{F}^{-1}\left(1 + |\zeta|^2\right)^{-s/2} \mathcal{F}g :\ g \in L^p(\mathbb{R}^n)\right\}. \end{aligned} \tag{3.1.1}$$

Note that the space defined in (3.1.1) is equipped with the following norm $||f||_{L^p(\mathbb{R}^n)} := ||\mathcal{F}^{-1}\left(1 + |\zeta|^2\right)^{-s/2} \mathcal{F}f||_{L^p(\mathbb{R}^n)}$. When the smoothness index is a natural number, i.e., $s = r \in \mathbb{N}$, the classical Sobolev space can be defined by (see e.g., [44])

$$L_r^p(\mathbb{R}^n) := \Big\{f \in L^p(\mathbb{R}^n) :\ ||f||_{L^p(\mathbb{R}^n)} := \sum_{|\gamma| \leq r} ||\partial^\gamma f||_{L^p(\mathbb{R}^n)},\ r \in \mathbb{N} \setminus \{0\},\ 1 < p < \infty\Big\}.$$

For $k \geq 2$, let

$$L_s^p(\mathbb{R}^n, \mathbb{R}^k) := \{u = (u_1, \cdots, u_k) : \mathbb{R}^n \to \mathbb{R}^k : u_j \in L_s^p(\mathbb{R}^n),\ j = 1, \ldots, k\}. \tag{3.1.2}$$

For $p \in (1, \infty)$ and $s \geq 0$, define the L^p-Sobolev spaces of functions with smoothness s in $\mathfrak{D}_\pm$

$$L_s^p(\mathfrak{D}_\pm) := \{f|_{\mathfrak{D}_\pm} : f \in L_s^p(\mathbb{R}^n)\},\ \tilde{L}_s^p(\mathfrak{D}_\pm) := \{f \in L_s^p(\mathbb{R}^n) : \mathrm{supp} f \subseteq \overline{\mathfrak{D}}_\pm\}, \tag{3.1.3}$$

where $\mathrm{supp}f$ is the support of the function f, i.e., the closure of the set of all points where f does not vanish. For $p \in (1, \infty)$ these spaces are Banach spaces. Moreover, for $p = 2$ they are Hilbert spaces. Further, denote by[1] $L_{-s}^p(\mathfrak{D}_\pm) = \left(\tilde{L}_s^q(\mathfrak{D}_\pm)\right)^*$ the dual of the space $\tilde{L}_s^q(\mathfrak{D}_\pm)$, where $q \in (1, \infty)$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. In addition, $L_s^p(\mathfrak{D}_\pm, \mathbb{R}^k)$ and $\tilde{L}_s^p(\mathfrak{D}_\pm, \mathbb{R}^k)$ are the Sobolev spaces of vector functions $\mathbf{u} : \mathfrak{D}_\pm \to \mathbb{R}^k$ having their components in $L_s^p(\mathfrak{D}_\pm)$ and $\tilde{L}_s^p(\mathfrak{D}_\pm)$, respectively, and $L_{-s}^p(\mathfrak{D}_\pm, \mathbb{R}^k) := (\tilde{L}_s^q(\mathfrak{D}_\pm, \mathbb{R}^k))^*$. As customary, for $p = 2$ we use the usual notations $L_s^2(\mathbb{R}^n) := H^s(\mathbb{R}^n)$, $L_s^2(\mathbb{R}^n, \mathbb{R}^k) := H^s(\mathbb{R}^n, \mathbb{R}^k)$, $L_s^2(\mathfrak{D}_\pm) := H^2(\mathfrak{D}_\pm)$. For a complete description o these spaces we refer to [44], [67], [118].

The Sobolev spaces $L_s^p(\mathbb{R}^{n-1})$ with $1 < p < \infty$ and $0 \leq s \leq 1$ are stable with respect to the composition by Lipschitz diffeomorphisms. In addition, they are invariant under the pointwise multiplication by Lipschitz maps.These properties lead to the natural definition of Sobolev spaces

[1] If X is a given Banach space, X^* denotes its dual space.

on Lipschitz boundaries. Thus, if $\Omega \subset \mathbb{R}^n$ is a *graph Lipschitz domain*, i.e., an unbounded region in $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$, then for any $1 < p < \infty$ and $0 \leq s \leq 1$ (see e.g., [44], [75], [107])

$$f \in L^p_s(\partial\Omega) \Leftrightarrow f(\cdot, \varphi(\cdot)) \in L^p_s(\mathbb{R}^{n-1}),$$
$$g \in L^p_{-s}(\partial\Omega) \Leftrightarrow g(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in L^p_{-s}(\mathbb{R}^{n-1}). \tag{3.1.4}$$

Based on a well-known partition of unity argument, these properties can be extended to the case of bounded Lipschitz domains. Therefore, keeping the same notations as above, i.e., if $\mathfrak{D} \subset \mathbb{R}^n$ is a Lipschitz domain with boundary Γ, then (see e.g., [75], [79])

$$L^p_1(\Gamma) := \{f \in L^p(\Gamma) : \nabla_{\tan} f \in L^p(\Gamma)\}, \ 1 < p < \infty,$$
$$L^p_{-s}(\Gamma) = \left(L^q_s(\Gamma)\right)^*, \ 1 < p < \infty, \ 0 \leq s \leq 1, \tag{3.1.5}$$

where $\nabla_{\tan}$ is the tangential gradient on Γ, and $\frac{1}{p} + \frac{1}{q} = 1$.

For further purposes, consider the spaces[2] [54]

$$\begin{aligned} L^p_{s+\frac{1}{p}}(\mathfrak{D}_-, \mathcal{L}_{\mathrm{St}}) := & \left\{(\mathbf{u}, \pi) \in L^p_{s+\frac{1}{p}}(\mathfrak{D}_-, \mathbb{R}^n) \times L^p_{s+\frac{1}{p}-1}(\mathfrak{D}_-) : \right. \\ & \left. \mathcal{L}_{\mathrm{St}}(\mathbf{u}, \pi) = \mathbf{0}, \operatorname{div} \mathbf{u} = 0 \text{ in } \mathfrak{D}_- \right\}, \\ L^p_{s+\frac{1}{p}}(\mathfrak{D}_+, \mathcal{L}_{\mathrm{St}}) := & \left\{(\mathbf{u}, \pi) \in L^p_{s+\frac{1}{p},\mathrm{loc}}(\overline{\mathfrak{D}}_+, \mathbb{R}^n) \times L^p_{s+\frac{1}{p}-1,\mathrm{loc}}(\overline{\mathfrak{D}}_+) : \right. \\ & \left. \mathcal{L}_{\mathrm{St}}(\mathbf{u}, \pi) = \mathbf{0}, \operatorname{div} \mathbf{u} = 0 \text{ in } \mathfrak{D}_+ \right\}, \end{aligned} \tag{3.1.6}$$

where $\mathcal{L}_{\mathrm{St}}(\mathbf{u}, \pi) := -\Delta\mathbf{u} + \nabla\pi$. In particular, for $p = 2$ and $\beta := s - \frac{1}{2} \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, one obtains

$$H^{1+\beta}(\mathfrak{D}_-, \mathcal{L}_{\mathrm{St}}) := L^2_{1+\beta}(\mathfrak{D}_-, \mathcal{L}_{\mathrm{St}}), \ H^{1+\beta}(\mathfrak{D}_+, \mathcal{L}_{\mathrm{St}}) := L^2_{1+\beta}(\mathfrak{D}_+, \mathcal{L}_{\mathrm{St}}).$$

3.1.4 The nontangential trace and conormal derivative operators on Lipschitz domains in $\mathbb{R}^n$

For a fixed constant $k_0 = k_0(\Gamma) > 1$, sufficiently large, define the *non-tangential maximal function* $\mathcal{N}u$ by (see e.g., [79, (2.3)-(2.6)]): $\mathcal{N}(u)(\mathbf{x}) := \sup\{|u(\mathbf{y})| : \mathbf{y} \in \gamma_\pm(\mathbf{x})\}, \ \mathbf{x} \in \Gamma$, for arbitrary $u : \mathfrak{D}_+ \to \mathbb{R}$, where $\gamma_\pm(\mathbf{x}) := \{\mathbf{y} \in \mathfrak{D}_\pm : \operatorname{dist}(\mathbf{x}, \mathbf{y}) < k_0 \operatorname{dist}(\mathbf{y}, \Gamma)\}, \ \mathbf{x} \in \Gamma$, are non-tangential approach regions lying in $\mathfrak{D}_+ = \mathbb{R}^n \setminus \overline{\mathfrak{D}}$ and $\mathfrak{D}_-$, respectively. In addition, the *non-tangential boundary trace operators* $\mathrm{Tr}^\pm$ on Γ are defined in terms of nontangential boundary limits, as $(\mathrm{Tr}^\pm u)(\mathbf{x}) := \lim\limits_{\gamma_\pm(\mathbf{x}) \ni \mathbf{y} \to \mathbf{x}} u(\mathbf{y}), \ \mathbf{x} \in \Gamma$. In particular, denoting by $\cdot|_\Gamma$ the usual restriction to the boundary Γ,

$$\mathrm{Tr}^\pm v = v|_\Gamma, \ \forall\, v \in C^\infty(\overline{\mathfrak{D}}_\pm). \tag{3.1.7}$$

Lemma 3.1.4 (**[2], [14], [44], [79], [83]**) *Let $\mathfrak{D}_- := \mathfrak{D} \subset \mathbb{R}^n$ be a bounded Lipschitz domain with the boundary Γ and let $\mathfrak{D}_+ := \mathbb{R}^n \setminus \overline{\mathfrak{D}}$. Then the following statements hold:*

(a) For any $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$ there exists a linear and bounded operator $\mathrm{Tr}^- : H^s(\mathfrak{D}_-) \to H^{s-\frac{1}{2}}(\Gamma)$ whose action is compatible to that of the restriction to the boundary in (3.1.7). *It is onto and has a right inverse $\mathcal{Z}^- : H^{s-\frac{1}{2}}(\Gamma) \to H^s(\mathfrak{D}_-), \ \mathrm{Tr}^-(\mathcal{Z}^-\phi) = \phi, \ \forall\, \phi \in H^{s-\frac{1}{2}}(\Gamma)$. For $s > \frac{3}{2}$, the operator $\mathrm{Tr}^- : H^s(\mathfrak{D}_-) \to H^1(\Gamma)$ is also linear and bounded.*

[2] By definition $F \in L^p_{s+\frac{1}{p},\mathrm{loc}}(\overline{\mathfrak{D}}_+, \mathbb{R}^n)$ if and only if $F \in L^p_{s+\frac{1}{p}}(B \cap \mathfrak{D}_+)$ for any open ball $B \subseteq \mathbb{R}^n$ with $B \cap \mathfrak{D}_+ \neq \emptyset$.

(b) If $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$*, there exists an operator* $\mathrm{Tr}^+ : H^s_{\mathrm{loc}}(\overline{\mathfrak{D}}_+) \to H^{s-\frac{1}{2}}(\Gamma)$*, which is linear and continuous, with a compatible action to that in* (3.1.7)*. It is onto and has a right inverse*[3] $\mathscr{Z}^+ : H^{s-\frac{1}{2}}(\Gamma) \to H^s_0(\overline{\mathfrak{D}}_+)$.

Let $s \in [0,1]$ be given. In view of the fact that $\mathbf{n}_\Gamma \in L^\infty(\Gamma, \mathbb{R}^n)$, the functional $\nu_\Gamma \in H^{-s}(\Gamma, \mathbb{R}^n) := \left(H^s(\Gamma, \mathbb{R}^n)\right)^*$ defined by $\langle \nu_\Gamma, \mathbf{w}\rangle_{\partial\mathcal{D}} := \int_{\partial\mathfrak{D}} \langle \mathbf{n}_\Gamma, \mathbf{w}\rangle d\sigma,\ \forall\ \mathbf{w} \in H^s(\Gamma, \mathbb{R}^n)$, is well-defined, linear and bounded, and defines the outward unit conormal ν_Γ to Γ. Then one has the following result due to Mitrea and Wright [79, Theorem 10.10] for the Stokes system on general Sobolev or Besov spaces (see also [56, Lemma 2.2] for the extension to the Brinkman system on Lipschitz domains in compact Riemannian manifolds)[4]:

Lemma 3.1.5 [79] *Let* $\mathfrak{D}_- := \mathfrak{D} \subset \mathbb{R}^n$ *be a bounded Lipschitz domain with the boundary* Γ*. Then for any* $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ *the* inner conormal derivative operator $\partial^-_{\nu_\Gamma} : H^{1+\beta}(\mathfrak{D}, \mathcal{L}_{\mathrm{St}}) \to H^{-\frac{1}{2}+\beta}(\Gamma, \mathbb{R}^n)$ *given by*

(3.1.8)
$$\left\langle \partial^-_{\nu_\Gamma}(\mathbf{u}, \pi), \mathbf{\Psi} \right\rangle_\Gamma := 2\int_{\mathfrak{D}} E_{jk}(\mathbf{u})E_{jk}(\mathscr{Z}^-\mathbf{\Psi})d\mathbf{x} - \int_{\mathfrak{D}} \pi \mathrm{div}(\mathscr{Z}^-\mathbf{\Psi})d\mathbf{x},\ \forall\ \mathbf{\Psi} \in H^{\frac{1}{2}+\beta}(\Gamma, \mathbb{R}^n)$$

is well-defined, bounded and linear, where $E_{jk}(\mathbf{u}) := \dfrac{1}{2}\left(\dfrac{\partial u_j}{\partial x_k} + \dfrac{\partial u_k}{\partial x_j}\right)$*. In addition, for any* $(\mathbf{u}, \pi) \in H^{1+\beta}(\mathfrak{D}, \mathcal{L}_{\mathrm{St}})$ *and* $\mathbf{w} \in H^{1-\beta}(\mathfrak{D}, \mathbb{R}^n)$*, one has the Green formula*

$$2\int_{\mathfrak{D}} E_{jk}(\mathbf{u})E_{jk}(\mathbf{w})d\mathbf{x} = \int_{\mathfrak{D}} \pi\ \mathrm{div}\ \mathbf{w}d\mathbf{x} - \left\langle \partial^-_{\nu_\Gamma}(\mathbf{u}, \pi), \mathrm{Tr}^-\mathbf{w} \right\rangle_\Gamma. \tag{3.1.9}$$

Remark 3.1.6 For any $(\mathbf{u}, \pi) \in H^1_{\mathrm{loc}}(\overline{\mathfrak{D}}_+, \mathbb{R}^n) \times L^2_{\mathrm{loc}}(\overline{\mathfrak{D}}_+)$ satisfying the Stokes system and the growth conditions at infinity: $\nabla^k \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{2-n-k}),\ \pi(\mathbf{x}) = O(|\mathbf{x}|^{1-n})$ as $|\mathbf{x}| \to \infty$, $k = 0, 1$ and $n \geq 3$, or $\nabla^k \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-1-k}),\ \pi(\mathbf{x}) = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$, $k = 0, 1$ and $n = 2$, one has the Green formula (see e.g., [79])

$$2\int_{\mathfrak{D}_+} E_{jk}(\mathbf{u})E_{jk}(\mathbf{u})d\mathbf{x} = \int_{\mathfrak{D}_+} \pi \mathrm{div}\mathbf{u}d\mathbf{x} - \langle \partial^+_{\nu_\Gamma}(\mathbf{u}, \pi), \mathrm{Tr}^+\mathbf{u}\rangle_\Gamma. \tag{3.1.10}$$

3.2 Pseudodifferential operators on $\mathbb{R}^n$

In this section we present the main properties of pseudodifferential operators on $\mathbb{R}^n$. The basic sources used in the preparation of this section are [44, Chapter 6], [47], [117, Chapter 7]. Among other important sources in this field, we mention the books by Taylor [106] and Wong [118].

3.2.1 Compact operators

3.2.2 Main properties of pseudodifferential operators on $\mathbb{R}^n$

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of infinitely differentiable functions $u \in C^\infty(\mathbb{R}^n)$ such that, for all multi-indices α and β, one has

$$\sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta u)(x)| < \infty. \tag{3.2.1}$$

[3] If $p \in (1, \infty)$ and $s \in (0, 1)$, $L^p_{s,0}(X)$ is the set of all elements $f \in L^p_s(\mathbb{R}^n)$ with compact support in $X \subseteq \mathbb{R}^n$.

[4] All along this work, one uses the Einstein repeated-index summation convention.

This set[5] is called the *Schwartz space*. The family of semi-norms $|\cdot|_{k;\mathcal{S}}$, $k = 0, 1, 2, \cdots$, defined on $\mathcal{S}$ by

$$|u|_{k;\mathcal{S}} := \max_{|\alpha|+|\beta|\le k} \sup_{x\in\mathbb{R}^n} |x^\alpha(D^\beta u)(x)|, \tag{3.2.2}$$

leads to a topology on $\mathcal{S}(\mathbb{R}^n)$, making it a Frechét space (see e.g., [117, p. 233]).

Let us mention that the Fourier transformation $\mathcal{F}$ is a well-defined map on the space $\mathcal{S}(\mathbb{R}^n)$. For a function $u \in \mathcal{S}(\mathbb{R}^n)$ one has

$$(\mathcal{F}u)(\zeta) := \int_{\mathbb{R}^n} e^{-ix\cdot\zeta}u(x)dx, \tag{3.2.3}$$

where $x\cdot\zeta := \sum_{k=1}^n x_k\zeta_k$, for $x = (x_1,\ldots,x_n)$, $\zeta = (\zeta_1,\ldots,\zeta_n) \in \mathbb{R}^n$, and $i^2 = -1$. For brevity, we use the notation $\hat{u}$ instead of $\mathcal{F}u$.

In fact, the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear, topological isomorphism. Hence there exists the inverse $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and is given by

$$u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\zeta\cdot x}\hat{u}(\zeta)d\zeta. \tag{3.2.4}$$

Note that the linear topological isomorphism $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends to the dual space $\mathcal{S}'(\mathbb{R}^n)$, $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Denoting by $\langle\cdot,\cdot\rangle$ the duality pairing between the spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, one has $\langle\mathcal{F}u, \Psi\rangle = \langle u, \mathcal{F}\Psi\rangle$, for all $u \in \mathcal{S}(\mathbb{R}^n)$, $\Psi \in \mathcal{S}'(\mathbb{R}^n)$ (see e.g., [117, p. 233]).

The classes S^m

Let us consider a differential operator of order $m \in \mathbb{N}$,

$$T(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha, \ \ x \in \mathbb{R}^n, \tag{3.2.5}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_k = i^{-1}\dfrac{\partial}{\partial x_k}$ and $|\alpha| := \alpha_1+...\alpha_n$ is the order of D^α. Assume that $a_\alpha \in C^\infty(\mathbb{R}^n)$. In terms of the Fourier transformation one has (see [117, p. 236])

$$T(x,D)u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\zeta}T(x,\zeta)\hat{u}(\zeta)d\zeta, \tag{3.2.6}$$

where $\zeta^\alpha := \zeta_1^{\alpha_1}...\zeta_n^{\alpha_n}$. The polynomial $T(x,\zeta) := \sum_{|\alpha|\le m} a_\alpha(x)\zeta^\alpha$ is called the (total) *symbol* of $T(x,D)$, and the *principal symbol* $\sigma_m(T)$ of T is $(\sigma_m(T))(x,\zeta) := \sum_{|\alpha|=m} a_\alpha(x)\zeta^\alpha$ (see also [117, p. 236]). If the coefficients a_α of the differential operator (3.2.5) are functions of class C^∞, with bounded derivatives of any order, i.e., $a_\alpha \in C_b^\infty(\mathbb{R}^n)$, then the symbol $T(x,\zeta)$ belongs to the class S^m, $m \in \mathbb{N}$, defined below.

Definition 3.2.1 [117] Let $m \in \mathbb{R}$. Then the set $S^m = S^m(\mathbb{R}^n\times\mathbb{R}^n)$ consisting of all functions $T \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ such that, for all multi-indices α, β,

$$|D_\zeta^\alpha D_x^\beta T(x,\zeta)| \le C_{\alpha,\beta}(1+|\zeta|)^{m-|\alpha|}, \ \forall\, x,\zeta \in \mathbb{R}^n, \tag{3.2.7}$$

is called *the space of symbols of order* m.

One uses the notations $S^{-\infty} := \bigcap S^m$, $S^\infty := \bigcup S^m$.

Next, we use the simplified notation $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$. Let us mention the following useful result:

[5] The set $C_0^\infty(\mathbb{R}^n)$ of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports is included in $\mathcal{S}(\mathbb{R}^n)$.

Proposition 3.2.2 [117] *Let $T \in S^m$ be given. If $u \in \mathcal{S}$, then*

$$T(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\zeta} T(x,\zeta)\hat{u}(\zeta)d\zeta, \tag{3.2.8}$$

is a function $T(x,D)u(x) \in \mathcal{S}$. In addition, the map $T(x,D) : \mathcal{S} \to \mathcal{S}$ is continuous. The bilinear map $S^m \times \mathcal{S} \ni (T,u) \mapsto T(x,D)u \in \mathcal{S}$ is continuous as well.

Remark 3.2.3 The operator $T(x,D)$ given by (3.2.8) is called *pseudodifferential operator* $(p.d.o.)$ *of order m on* $\mathbb{R}^n$. The set of all pseudodifferential operators of order m on $\mathbb{R}^n$ is denoted by $OPS^m(\mathbb{R}^n)$. For a detailed description of this class we refer to [44, Chapter 6], [47], [78], [117, Chapter 7], [118].

Definition 3.2.4 [117] Let $m \in \mathbb{R}$. Then a function $A(x,\zeta)$ belongs to $J^m = J^m(\mathbb{R}^n \times \mathbb{R}^n)$ if the following conditions are satisfied:

(i) $A(x,\zeta)$ is (positively) homogenous of order m, i.e., $A(x, c\zeta) = c^m A(x,\zeta),\ \forall\, c > 0,\ \zeta \neq 0,$

(ii) $A \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$

(iii) Any derivative $D_\zeta^\alpha D_x^\beta A(x,\zeta)$ is bounded on $\mathbb{R}^n \times \mathfrak{S}^{n-1}$, i.e., $A \in C_b^\infty(\mathbb{R}^n \times \mathfrak{S}^{n-1})$, where $\mathfrak{S}^{n-1} := \{\zeta \in \mathbb{R}^n : |\zeta| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

Definition 3.2.5 [117] Let $T(x,\zeta) \in S^m$ be given. Then T is a (-1) *classical symbol* if there are some homogeneous functions $H_m \in J^m$, a symbol $T_{m-1} \in S^{m-1}$, and a cut-off function χ, such that T admits the representation $T(x,\zeta) = \chi(\zeta)H_m(x,\zeta) + T_{m-1}(x,\zeta)$. Note that the existence of the above representation does not depend on the choice of the cut-off function χ (cf. e.g., [117, p. 258]).

The set of (-1) classical symbols is denoted by $Cl^{-1}S^m$, and a symbol $T \in Cl^{-1}S^m$ has the *principal part* given by $\pi T(x,\zeta) := H_m(x,\zeta),\ H_m \in J^m$. Let us mention that the Sobolev space $L_s^2(\mathbb{R}^n)$, $s \in \mathbb{R}$, is defined by (see e.g., [117, p. 260])

$$L_s^2(\mathbb{R}^n) = \left\{ u \in L_{loc}^2(\mathbb{R}^n) : ||u||_s^2 := \int_{\mathbb{R}^n} |\hat{u}(\zeta)|^2(1+|\zeta|^2)^s d\zeta < \infty \right\}. \tag{3.2.9}$$

A main result related to pseudodifferential operators yields that any pseudodifferential operator $T(x,D) \in OPS^m$ extends to a continuous operator on any Sobolev space:

Theorem 3.2.6 (**[43], [117]**) *Let $T \in S^m$ be a given symbol. Then the associated pseudodifferential operator $T(x,D)$ extends to a continuous operator, denoted in the same way,*

$$T(x,D) : L_s^2(\mathbb{R}^n) \to L_{s-m}^2(\mathbb{R}^n),\ \ \forall\, s \in \mathbb{R}.$$

3.2.3 Elliptic operators on $\mathbb{R}^n$

The set of all pseudodifferential operators contains a class of operators that appear in many applications devoted to boundary value problems for partial differential equations, and have parametrices (i.e., inverses modulo compact operators), which are also pseudodifferential operators. This is the class of elliptic operators. The main sources used in the preparation of this section are [44], [117]. First, we give the definition of a parametrix:

Definition 3.2.7 [117] Let $T = T(x,D) \in OPS^m(\mathbb{R}^n)$ be a given pseudodifferential operator. If there exists an operator $B \in OPS^{-m}(\mathbb{R}^n)$ such that

$$TB - I \in OPS^{-\infty}(\mathbb{R}^n),\ BT - I \in OPS^{-\infty}(\mathbb{R}^n), \tag{3.2.10}$$

then B is called a *parametrix* of T.

Note that $OPS^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} OPS^m(\mathbb{R}^n)$, and $A \in OPS^{-\infty}(\mathbb{R}^n)$ if and only if A is a smoothing operator (see e.g., [44, Theorem 6.1.10]).

Definition 3.2.8 [117] An operator $T \in OPS^m(\mathbb{R}^n)$ is called *elliptic of order m* if there exists an operator $B \in OPS^{-m}(\mathbb{R}^n)$ such that $TB - I \in OPS^{-1}(\mathbb{R}^n),\ BT - I \in OPS^{-1}(\mathbb{R}^n)$.

Theorem 3.2.9 (**[43], [44], [106], [117]**) *Let $T \in OPS^m(\mathbb{R}^n)$. Then the operator T is elliptic of order m if and only if it admits a parametrix, i.e., an operator $B \in OPS^{-m}(\mathbb{R}^n)$ satisfying the condition* (3.2.10).

Example 3.2.10 The Laplacian $\triangle := \sum_{j=1}^{n} D_{x_j}^2$ and any zero order perturbation of it, $\triangle + \lambda\mathbb{I}$, $\lambda > 0$, have the principal symbol $\sum_{j=1}^{n} \zeta_j^2$. Therefore, they are elliptic operators. In addition, if $g_{ij}(x)$ a Riemannian metric on $\mathbb{R}^n$ with the inverse $g^{ij}(x)$, then the variable coefficient Laplacian $\triangle := \sum_{j=1}^{n} g^{ij} D_{x_i} D_{x_j} + A$, where A is a first order differential operator, has the principal symbol $\sum_{j=1}^{n} g^{ij}\zeta_i\zeta_j$, and hence is elliptic.

3.2.4 Elliptic operators on domains in $\mathbb{R}^n$

3.2.5 Elliptic systems in the sense of Agmon-Douglis-Nirenberg on $\mathbb{R}^n$

In this subsection we give the definition of elliptic systems in the sense of Agmon-Douglis-Nirenberg on Lipschitz domains in $\mathbb{R}^n$, as well as some of their basic properties. They appear in many applications devoted to elliptic boundary value problems on Lipschitz domains. The main sources used in the preparation of this part are [44], [117].

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $T(x, D) = (T_{jk}(x, D))_{j,k=1,\dots,p},\quad x \in \Omega$ be a matrix of pseudodifferential operators $T_{jk}(x, D)$ with the symbols $T^{jk} = T^{jk}(x, \zeta)$. Assume that there exist numbers $s_j, t_k \in \mathbb{R}$, $j, k = 1, \dots, p$, such that these symbols satisfy the condition $T^{jk} \in \mathbf{S}^{s_j+t_k}(\Omega \times \mathbb{R}^n)$. In particular, we consider the *Agmon-Douglis-Nirenberg system* of partial differential equations (see e.g., [44, p. 328])

$$\sum_{k=1}^{p} \sum_{|\beta|=0}^{s_j+t_k} T_\beta^{jk}(x) D^\beta u_k(x) = f_j(x),\ j = 1, \dots, p, \tag{3.2.11}$$

with the matrix of differential operators $T = (T_{jk})_{j,k=1,\dots,p}$ given by $T_{jk} := \sum_{|\beta|=0}^{s_j+t_k} T_\beta^{jk}(x) D^\beta$, and the corresponding symbols $T^{jk}(x) \in \mathbf{S}^{s_j+t_k}(\Omega \times \mathbb{R}^n)$. Assume that $s_j \le 0$.

Definition 3.2.11 (**[44], [117]**) The matrix $(T^{jk}(x, \zeta))_{j,k=1,\dots,p}$, where

$$T^{jk}(x, \zeta) := \sum_{|\beta|=0}^{s_j+t_k} T_\beta^{jk}(x) i^{|\beta|} \zeta^\beta, \tag{3.2.12}$$

is called the *symbol matrix* of the system (3.2.11), and the *principal part* is defined by $T_{s_j+t_k}^{jk}(x, \zeta) := \sum_{|\beta|=s_j+t_k} T_\beta^{jk}(x) i^{|\beta|} \zeta^\beta$, where $T_{s_j+t_k}^{jk}(x, \zeta)$ is chosen equal to zero if $T^{jk}(x, \zeta)$ has the order $\le s_j + t_k$.

Definition 3.2.12 (**[44], [117]**) Let $(T_{jk})_{j,k=1,\dots p}$ be a matrix of pseudodifferential operators.

Then

- The *characteristic determinant* $H(x, \zeta)$ is given by $H(x, \zeta) := \det[(\tilde{T}_{s_j+t_k}^{jk}(x, \zeta))]_{p\times p}$, where $\tilde{T}_{s_j+t_k}^{jk}(x, \zeta) := |\zeta|^{s_j+t_k} T_{s_j+t_k}^{jk}\left(x, \dfrac{\zeta}{|\zeta|}\right)$.
- The system (T_{jk}) is *elliptic in the sense of Agmon-Douglis -Nirenberg* if $H(x, \zeta) \ne 0$, $\forall\, x \in \Omega,\ \zeta \in \mathbb{R}^n \setminus \{0\}$.

Remark 3.2.13 Having in view the Definition 3.2.12 *it follows that the Definition* 3.2.7 *of a parametrix* $Q_0 = (Q_{jk})_{j,k=1,\dots,p}$ *for the operator* $T = (T_{jk})_{j,k=1,\dots,p}$, *with* $T_{jk} \in OPS^{s_j+t_k}(\Omega)$, *and the existence of a parametrix, are valid as well for Agmon-Douglis-Nirenberg elliptic systems.*

3.3 Pseudodifferential operators on compact Riemannian manifolds

In this section we present the class of pseudodifferential operators on compact Riemann manifolds, as well as some useful properties related to such operators. The main sources used for the preparation of this part are [44, Chapter 8], [47], [117].

3.3.1 General results related to pseudodifferential operators on compact Riemannian manifolds

A topological space M with the property that any point of M has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ is called a *locally Euclidean space*. In addition, a pair (U, q) with $U \subset M$ an open set and q homeomorphism of U onto a open set in $\mathbb{R}^n$ is called a *coordinate map*. If at least two coordinate maps are involved, one uses the notations $q : U_q \to V_q$, where V_q is the range of q. Next, we recall the definition of C^s structure of a locally Euclidean space.

Definition 3.3.1 [117] If $s \in \mathbb{N}$, or $s = \infty$, then a C^s *structure on a locally Euclidean space* M is a family $\mathfrak{F}$ of coordinate maps $q : U_q \to V_q$ such that the following statements hold:

(i) The domains U_q cover M, i.e., $M = \bigcup_{q\in\mathfrak{F}} U_q$.

(ii) Let $q_1, q_2 \in \mathfrak{F}$ such that $U_{q_1} \cap U_{q_2} \neq \emptyset$. Then the overlap map $q_2 \circ {q_1}^{-1} : q_1 (U_{q_1} \cap U_{q_2}) \to q_2 (U_{q_1} \cap U_{q_2})$ is of class C^s.

(iii) If q_0 is a coordinate map such that $q_0 \circ q^{-1}$ and $q \circ q_0^{-1}$ are of class C^s, for any $q \in \mathfrak{F}$, then $q_0 \in \mathfrak{F}$, i.e., the family $\mathfrak{F}$ is maximal with respect to (ii).

It is shown that a C^s structure can be defined by an arbitrary family $\mathfrak{C}$ that satisfies only the first two previous conditions (i) and (ii) (see also [117, p. 114]). Such a family $\mathfrak{C}$ is called a C^s *atlas*.

Definition 3.3.2 [117] A pair $(M, \mathfrak{F})$ is called a C^s *manifold*, if M is a locally Euclidean Hausdorff, second countable space[6] and $\mathfrak{F}$ is a C^s structure on M.

Example 3.3.3 The Euclidean space $\mathbb{R}^n$ is a manifold with an atlas given by a single chart ($\mathbb{R}^n$, $\mathbb{I}$), which provides the *standard* C^∞ *structure on* $\mathbb{R}^n$.

Next, denote by $TM = \bigcup_{p\in M} T_pM$ the *tangent bundle*, where T_pM is the tangent space at the point $p \in M$, and by $T^*M = \bigcup_{p\in M} T_p^*M$ the dual space, i.e., the *cotangent bundle*. Then:

Definition 3.3.4 [117] A manifold M with a Riemannian metric g on the tangent bundle TM is called a *Riemannian manifold*.

Let (M, g) be a compact boundaryless Riemannian manifold of dimension $p \geq 2$ equipped with a smooth Riemannian metric tensor[7] $g := \sum_{j,k=1}^{p} g_{jk} dx_j \otimes dx_k =: g_{jk} dx_j \otimes dx_k$, and let (g^{jk}) be the inverse of (g_{jk}). Let us mention that the volume element on M is given by $\mathrm{dVol} = \sqrt{g} dx_1 \dots dx_p$, where $g := \det(g_{jk})$. Next, we define the following inner product on on the space of one forms $\Lambda^1 TM$ (see e.g., [78], [117]): $\langle dx_j, dx_k\rangle = g^{jk}, \quad \langle X, Y\rangle = X_j g^{jk} Y_k$,

[6] This means that there is a countable basis for the topology of M.

[7] All along this work we use the repeated index summation rule.

where the vector field $X = X^k\partial_k \in TM$ is identified with the one form $X_r dx_r = X^k g_{kr} dx_r$, $X_r = g_{kr}X^k$, and the notation $\langle\cdot,\cdot\rangle$ is used for the inner product.

Let X be an open set in M. Let us denote by $C_0^\infty(X) = C^\infty(X,\mathbb{C})$ the space of C^∞ functions on X with compact support. By using the fact that M is compact, one finds that $C_0^\infty(M) = C^\infty(M)$. Next, one defines the Sobolev scale of complex-valued functions defined on the compact boundaryless Riemannian manifold M.

Definition 3.3.5 [117] Assume that M is a compact boundaryless Riemannian manifold and let $s \geq 0$ be given. Then the *Sobolev space* $L_s^2(M)$ is the set of functions $u : M \to \mathbb{C}$ such that $(\varphi \circ u) \circ q^{-1} \in L_s^2(\mathbb{R}^n)$ for each coordinate map $q : U \to V$ and any $\varphi \in C_0^\infty(U)$. The topology on $L_s^2(M)$ is the weakest topology with respect to which the semi-norms $u \mapsto ||(\varphi \cdot u) \circ q^{-1}||_s$ are continuous.

Now, let $q : U \to V$ be some charts for M. Denote by $q^* f = f \circ q$ the *pull-back* of $f \in C^\infty(V)$ and $q^{*-1}h = h \circ q^{-1}$ the *push-forward* of $h \in C^\infty(U)$.

Definition 3.3.6 [117] Let $T : C^\infty(M) \to C^\infty(M)$ be a linear operator. T is called a *pseudodifferential operator of order* $m \in \mathbb{R}$ *on* M if the push-forward operator $Q(\varphi T\psi, q) := q^{*-1}(\varphi T\psi)q^*$ is a pseudodifferential operator of order m on $\mathbb{R}^n$, i.e., $Q(\varphi T\psi, q) \in OPS^m(\mathbb{R}^n)$, for any coordinate patch (U, q) on M, and for all $\varphi, \psi \in C_0^\infty(U)$.

The space of all pseudodifferential operators of order m on M is denoted by $OPS^m(M)$. In addition, as in the Euclidean case, denote by $\sigma_m(T)(x,\zeta)$ the *principal symbol* of $\varphi T\psi$. Also, let $OC^\infty(M)$ be the set of all integral operators on M with kernels in $C^\infty(M \times M)$.

Theorem 3.3.7 [117] *If $T \in OPS^m(M)$, then the mapping $T : L_s^2(M) \to L_{s-m}^2(M)$ is continuous, for any $s \in \mathbb{R}$.*

Definition 3.3.8 [117] An operator $T \in OPS^m(M)$ is called a (-1) *classical pseudodifferential operator on* M if for any chart (U, q) on M and for all $\varphi, \psi \in C_0^\infty(M)$, $(\varphi T\psi)_q \in OPS^m(\mathbb{R}^n)$, i.e., the push-forward operator $(\varphi T\psi)_q$ is (-1) classic on $\mathbb{R}^n$.

The set of all classical pseudodifferential operators of order m on M is denoted by $OPS_{\rm cl}^m(M)$. As in the Euclidean case, a classical pseudodifferential operator T on M is *elliptic* if its principal symbol does not vanish (for more details, see e.g., [117, p. 307], [47]). In addition, it is possible to construct a parametrix for an elliptic operator $T \in OPS_{\rm cl}^m(M)$, i.e., an operator $T' \in OPS_{\rm cl}^{-m}(M)$ such that $TT' = \mathbb{I} - R$, where $R \in OPS_{\rm cl}^{-1}(M)$.

Theorem 3.3.9 [47] *If T is an elliptic operator of order m on a compact Riemannian manifold M, then, for any $s \in \mathbb{R}$, $T : H^s(M) \to H^{s-m}(M)$ is a Fredholm operator and its index depends only on the principal symbol of T.*

Finally, note that by Rellich's compactness lemma (see e.g., [47, Theorem 11.6]), the inclusion $i : H^t(M) \to H^s(M)$ is compact, for any $t > s$. Therefore:

Corollary 3.3.10 [47] *If T is a pseudodifferential operator of negative order, then T is compact on any Sobolev space.*

3.3.2 Elliptic systems of pseudodifferential operators on compact Riemannian manifolds

3.3.3 Elliptic systems of Agmon-Douglis-Nirenberg type on compact Riemannian manifolds

An important system of pseudodifferential operators on compact Riemannian manifolds is the Agmon-Douglis-Nirenberg elliptic system. Next, we present this notion and related properties, such as the existence of a parametrix, by following [44], [117, p. 334].

Let $s_1, \ldots, s_p, t_1, \ldots, t_p$ be any real numbers. Consider a matrix type operator $T = (T_{ij})_{p\times p}$ such that each entry $T_{ij} \in OPS^{m_{ij}}_{\rm cl}(M)$ is (-1) classic pseudodifferential operator on M of order $m_{ij} \leq s_i + t_j$. The Agmon-Douglis-Nirenberg *principal part* is the matrix $\pi_{_D}(T(x,\zeta)) := (t_{ij}(x,\zeta))_{p\times p}$, with the entries $t_{ij}(x,\zeta) = \pi_{s_i+t_j} T_{ij}$ if the order of the operator T_{ij} is $s_i + t_j$, and 0 otherwise. Denote by $s := (s_1, \ldots, s_p)$ and $t := (t_1, \ldots, t_p)$. The class of all matrix operators as above is denoted by $OClS^{s+t}(M, p\times p)$ (for more details we refer to [117, Chapter 8]).

The matrix type operator T is called *Agmon-Douglis-Nirenberg elliptic* if there exist real numbers $s_1, \ldots, s_p, t_1, \ldots, t_p$, such that $T \in OCIS^{s+t}(M, p\times p)$ and

$$\det\left(\pi_{_D}(T(x,\zeta))\right) \neq 0, \ \forall\, (x,\zeta) \in T^*(M)\setminus\{0\} \tag{3.3.1}$$

(see [117, p. 334]). Then we have the following property:

Theorem 3.3.11 [117] *Let $T = (T_{ij}) \in OCIS^{s+t}(M, p\times p)$ be an Agmon-Douglis-Nirenberg matrix of pseudodifferential operators with ADN numbers $s_1, \ldots, s_p, t_1, \ldots, t_p$. If T is Agmon-Douglis-Nirenberg elliptic then $T : L^2_{\ell+t_1}(M)\oplus\ldots\oplus L^2_{\ell+t_p}(M) \to L^2_{\ell-s_1}(M)\oplus\ldots\oplus L^2_{\ell-s_p}(M)$ is Fredholm for any $\ell\in\mathbb{R}$.*

3.4 Fredholm operators

In this subsection we present the notion of Fredholm operator, as well as related properties. There are excellent sources devoted to this subject, such as [79, p. 205] and [117]. These are frequently used in the preparation of this part.

Let X and Y be Banach spaces, and let $\mathcal{L}(X, Y)$ be the set of continuous, linear maps $T : X \to Y$. Also let X^* be the dual of X, i.e., $X^* := \{f : X \to \mathbb{R} : f \text{ liniar and continuous}\}$. For $f \in X^*$, we use the notation $f(x) := \langle f, x\rangle_X$. Then, for $F \in \mathcal{L}(X, Y)$, one defines the *dual map* $F^* : Y^* \to X^*$ by $\langle F^*y^*, x\rangle_X = \langle y^*, Fx\rangle_Y$. Note that $||F^*|| = ||F||$ and $F^* \in \mathcal{L}(Y^*, X^*)$.

Definition 3.4.1 (**[79], [117]**) Let $F \in \mathcal{L}(X, Y)$. Then F is a *Fredholm operator* if it satisfies the following conditions:

(i) The kernel of F, $\mathrm{Ker}(F) := \{x \in X : Fx = 0\}$, is finite dimensional

(ii) The range of F, $FX := \{y \in Y : \exists\, x \in X \text{ such that } Fx = y\}$, is closed in Y

(iii) The cokernel of F, $\mathrm{Coker}(F) := Y/FX$, is finite dimensional.

The number

$$\mathrm{ind}(F) := \dim(\mathrm{Ker}(F)) - \dim(\mathrm{Coker}(F)) < \infty \tag{3.4.1}$$

is called the *index* of the Fredholm operator F.

Let $\Phi(X, Y) := \{F \in \mathcal{L}(X, Y) : F \text{ Fredholm}\}$ be the set of Fredholm operators from X to Y.

Definition 3.4.2 [79] Let X and Y be Banach spaces. Consider the sets

$$\Phi_+(X, Y) := \Big\{F \in \mathcal{L}(X, Y) : F \text{ has closed range and a finite dimensional kernel}\Big\}, \tag{3.4.2}$$

$$\Phi_-(X, Y) := \{F \in \mathcal{L}(X, Y) : F \text{ has closed range and finite dimensional cokernel}\}\,. \tag{3.4.3}$$

Therefore, the set of semi-Fredholm operators is $\Phi_-(X, Y) \cup \Phi_+(X, Y)$, and the index function $\mathrm{ind} : \Phi(X, Y) \to \mathbb{Z}$, given by (3.4.1), can be extended to the set of all semi-Fredholm operators as (see [79, Definition 11.34])

$$\mathrm{ind} : \Phi_-(X, Y)\cup\Phi_+(X, Y) \to \mathbb{Z}\cap\{\pm\infty\},\ \mathrm{ind(F)} := \dim(\mathrm{Ker(F)}) - \dim(\mathrm{Coker(F)}). \tag{3.4.4}$$

Note that if $F \in \mathcal{L}(X,Y)$, then FX has finite codimension in $Y \iff \dim(Y/FX) < \infty$, i.e., the dimension of the space Y/FX is the *codimension* of FX in Y.

Next, we present the main properties related to Fredholm operators, which are very useful in the applications of layer potential theory to elliptic boundary value problems. These properties can be find in [79] and [117].

Theorem 3.4.3 (**[79], [117]**) *Let X and Y be Banach spaces and let $F \in \mathcal{L}(X,Y)$.*

(i) Let $F \in \Phi_\pm(X,Y)$, $S \in \Phi_\pm(Y,Z)$. Then $SF \in \Phi_\pm(X,Z)$, $\mathrm{ind}(SF) = \mathrm{ind}(S) + \mathrm{ind}(F)$.

(ii) $F \in \Phi_+(X,Y)$ if and only if F is bounded from below modulo compact operators. Therefore, there exist a Banach space Z, a compact operator $K : X \to Z$ and a constant $C > 0$ such that $||x||_X \le C||Fx||_Y + ||Kx||_Z$, $\forall\, x \in X$.

(iiii) $F \in \Phi(X,Y)$ if and only if there exist some operators $S_1, S_2 \in \mathcal{L}(X,Y)$, $K_1 \in \mathcal{K}(Y,Y)$ and $K_2 \in \mathcal{K}(X,X)$ such that $FS_1 = I_Y + K_1$, $S_2F = I_X + K_2$, where $\mathcal{K}(X,X)$ is the set of compact operators from X to X, and $I_X : X \to X$ is the identity operator on X.

Lemma 3.4.4 **[79]** *Let X, Y, Z and W be Banach spaces. Assume that the following diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array} \tag{3.4.5}$$

is commutative, where all arrows represent linear and bounded operators. If three of them are Fredholm operators then the fourth one is Fredholm operator as well.

Lemma 3.4.5 **[79]** *Let X_j, Y_j, $j = 1,2$, be Banach spaces such that the inclusions $X_1 \hookrightarrow X_2$ and $Y_1 \hookrightarrow Y_2$ are continuous. Assume that the inclusion $Y_1 \hookrightarrow Y_2$ has dense range. Let $F \in \Phi(X_1,Y_1) \cap \Phi(X_2,Y_2)$ with the property* $\mathrm{ind}(\mathrm{F}:\mathrm{X}_1 \to \mathrm{Y}_1) = \mathrm{ind}(\mathrm{F}:\mathrm{X}_2 \to \mathrm{Y}_2)$. *Then* $\mathrm{Ker}(\mathrm{F}:\mathrm{X}_1 \to \mathrm{Y}_1) = \mathrm{Ker}(\mathrm{F}:\mathrm{X}_2 \to \mathrm{Y}_2)$.

The next result is devoted to the stability of the index and Fredholm property (see [12] for the version on Banach spaces. The extension to quasi-Banach spaces has been obtained by Kalton, Mayboroda and Mitrea [48], by using the results in [49]; see also [79, Theorem 11.43]).

Theorem 3.4.6 **[12]** *Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples of Banach spaces. Assume that $X_0 + X_1$ and $Y_0 + Y_1$ are analytically convex. Let $F : X_j \to Y_j$, $j = 0,1$ be a bounded, linear operator. Let*[8] *$X_\theta := [X_0, X_1]_\theta$ and $Y_\theta := [Y_0, Y_1]_\theta$, $\theta \in (0,1)$. Then:*

- *F induces a bounded linear operator $F_\theta : X_\theta \to Y_\theta$, $\forall\, \theta \in (0,1)$. In addition, one has the interpolation inequality*

$$||F_\theta||_{\mathcal{L}(X_\theta,Y_\theta)} \le ||F||^{1-\theta}_{\mathcal{L}(X_0,X_0)} ||F||^{\theta}_{\mathcal{L}(X_1,X_1)}, \quad \theta \in (0,1). \tag{3.4.6}$$

- *If there exists $\theta_0 \in (0,1)$ such that $F_{\theta_0} : X_{\theta_0} \to Y_{\theta_0}$ is an isomorphism, then there exists $\varepsilon > 0$ such that $F_\theta : X_\theta \to Y_\theta$ is isomorphism as well, for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$.*

Let us mention that the Sobolev space $L^p_s(X,\mathbb{R}^n)$, $s \in (0,1)$, $p \in (1,\infty)$ can be obtained from the complex interpolation of the spaces $L^p_1(X,\mathbb{R}^n)$ and $L^p(X,\mathbb{R}^n)$ (see e.g., [110]):

$$L^p_s(X,\mathbb{R}^n) = \left[L^p_1(X,\mathbb{R}^n), L^p(X,\mathbb{R}^n)\right]_s .$$

In addition, the space $L^p_1(X,\mathbb{R}^n)$ is densely imbedded into the space $L^p_s(X,\mathbb{R}^n)$, for any $s \in (0,1)$ and $p \in (1,\infty)$ (see also [17], [110]).

The following result was obtained by Kalton and Mitrea [49] (see also [79, Theorem 11.45]).

[8] Note that the brackets $[\cdot,\cdot]_\theta$ belong to the complex interpolation method (see [110] for details).

Theorem 3.4.7 [49] *Under the hypothesis of Theorem* 3.4.6, *assume that the space* $Y_0 \cap Y_1$ *is dense in each*[9] Y_θ, $\theta \in (0,1)$. *If there exists* $\theta_0 \in (0,1)$ *such that* F_{θ_0} *is Fredholm, then there exists* $\varepsilon > 0$ *such that* F_θ *is Fredholm for any* $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$, *and the index is constant, i.e.,* $\mathrm{ind}(F_\theta) = \mathrm{ind}(F_{\theta_0})$, *for any* $\theta \in (0,1)$.

In addition, we have the following useful results:

Lemma 3.4.8 [79] *Let* $F : X \to Y$ *be a Fredholm operator with index zero. Then* F *is invertible if and only if* F *is injective.*

Theorem 3.4.9 [117] *Let* X, Y *be Banach spaces. If* $F : X \to Y$ *is a Fredholm operator and* $K : X \to Y$ *is compact then* $F + K : X \to Y$ *is also a Fredholm operator of the same index, i.e.,*

$$\mathrm{ind}(F+K) = \mathrm{ind}(F). \tag{3.4.7}$$

3.5 Layer potential theory for Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$

This section contains fundamental results and the main properties concerning layer potential theory for Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$. The main sources used in the preparation of this section are [54], [59], [79], [111].

3.5.1 The fundamental solution for Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$

This section is devoted to the fundamental solutions for Stokes and Brinkman systems. The main sources used in the preparation of this section are [54], [59], [111].

The fundamental solution of the Brinkman system

If $\chi > 0$ is a given constant, denote by $\mathcal{G}^{\chi^2}(\mathbf{x},\mathbf{y})$ and $\Pi^{\chi^2}(\mathbf{x},\mathbf{y})$ the fundamental tensor and the fundamental pressure vector, respectively, for the Brinkman system in $\mathbb{R}^n$, $n \geq 2$, i.e.,

$$\left(-\triangle_{\mathbf{x}} + \chi^2\mathbb{I}\right)\mathcal{G}^{\chi^2}(\mathbf{x},\mathbf{y}) + \nabla_{\mathbf{x}}\Pi^{\chi^2}(\mathbf{x},\mathbf{y}) = \mathrm{Dirac}_{\mathbf{y}}(\mathbf{x}),\ \ \mathrm{div}_{\mathbf{x}}\mathcal{G}^{\chi^2}(\mathbf{x},\mathbf{y}) = 0,\ \mathbf{x},\mathbf{y} \in \mathbb{R}^n \tag{3.5.1}$$

in the distributional sense, where $\mathrm{Dirac}_{\mathbf{y}}$ is the Dirac distribution with mass at $\mathbf{y}$. Note that the subscript $\mathbf{x}$ added to the above operators shows the action of these operators with respect to $\mathbf{x}$.

The components of $\mathcal{G}^{\chi^2}(\mathbf{x},\mathbf{y})$ and those of $\Pi^{\chi^2}(\mathbf{x},\mathbf{y})$ are given by (see e.g., [59, Chapter 2], [111, pp. 58-60])

$$\mathcal{G}_{jk}^{\chi^2}(\mathbf{x},\mathbf{y}) = \frac{1}{\omega_n}\left\{\frac{\delta_{jk}}{|\mathbf{x}-\mathbf{y}|^{n-2}}A_1(\chi|\mathbf{x}-\mathbf{y}|) + \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^n}A_2(\chi|\mathbf{x}-\mathbf{y}|)\right\}, \tag{3.5.2}$$

$$\Pi_j^{\chi^2}(\mathbf{x},\mathbf{y}) = \frac{1}{\omega_n}\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^n}, \tag{3.5.3}$$

where δ_{jk} is the Kronecker symbol, i.e, $\delta_{jk} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k, \end{cases}$ ω_n is the surface measure of the unit sphere $\mathfrak{S}^{n-1}$ in $\mathbb{R}^n$, $n \geq 2$, and, with the notation $\mathbf{z} := \mathbf{x} - \mathbf{y} = (z_1, \ldots, z_n)$,

$$\begin{aligned} A_1(z) &:= \frac{\left(\frac{z}{2}\right)^{\frac{n}{2}-1}K_{\frac{n}{2}-1}(z)}{\Gamma\left(\frac{n}{2}\right)} + 2\frac{\left(\frac{z}{2}\right)^{\frac{n}{2}}K_{\frac{n}{2}}(z)}{\Gamma\left(\frac{n}{2}\right)z^2} - \frac{1}{z^2}, \\ A_2(z) &:= \frac{n}{z^2} - 4\frac{\left(\frac{z}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(z)}{\Gamma\left(\frac{n}{2}\right)z^2}, \end{aligned} \tag{3.5.4}$$

[9] This condition is always satisfied in the case of inner complex interpolation spaces.

Also, K_ℓ is the Bessel function of the second kind and order $\ell \geq 0$, and Γ is the Gamma function (see e.g., [1]).

The corresponding stress and pressure tensors $\mathbf{S}^{\chi^2}(\mathbf{x},\mathbf{y})$ and $\mathbf{\Lambda}^{\chi^2}(\mathbf{x},\mathbf{y})$ have the components (see e.g., [59, Chapter 2], [111, pp. 58-60]):

$$
\begin{aligned}
S_{ijk}^{\chi^2}(\mathbf{x},\mathbf{y}) &= -\Pi_j^{\chi^2}(\mathbf{x},\mathbf{y})\delta_{ik} + \frac{\partial \mathcal{G}_{ij}^{\chi^2}(\mathbf{x},\mathbf{y})}{\partial x_k} + \frac{\partial \mathcal{G}_{kj}^{\chi^2}(\mathbf{x},\mathbf{y})}{\partial x_i} \\
&= -\frac{1}{\omega_n}\left\{\delta_{jk}\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^n}D_1(\chi|\mathbf{x}-\mathbf{y}|) + \frac{\delta_{ik}x_j+\delta_{jk}x_i}{|\mathbf{x}|^n}D_2(\chi|\mathbf{x}-\mathbf{y}|)\right. \\
&\qquad \left. + \frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^{n+2}}D_3(\chi|\mathbf{x}-\mathbf{y}|)\right\},
\end{aligned} \tag{3.5.5}
$$

$$
\begin{aligned}
\Lambda_{jk}^{\chi^2}(\mathbf{x},\mathbf{y})n_k(\mathbf{y}) &= -\Xi^{\chi^2}(\mathbf{z})n_k(\mathbf{y}) + \left(\frac{\partial \Pi_j}{\partial y_k} + \frac{\partial \Pi_k}{\partial y_j}\right)(\mathbf{y})n_k(\mathbf{y}) \\
&= \frac{1}{2\pi}\left\{-(\chi^2|\mathbf{z}|^2\ln|\mathbf{z}|+2)\frac{n_j(\mathbf{y})}{|\mathbf{z}|^2} + 4\frac{z_j\mathbf{z}\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{z}|^4}\right\}, \quad n=2 \\
\Lambda_{jk}^{\chi^2}(\mathbf{x},\mathbf{y})n_k(\mathbf{y}) &= -\frac{1}{\omega_n}\left\{2n\frac{z_j\mathbf{z}\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{z}|^{n+2}} + \chi^2 n_j(\mathbf{y})\frac{|\mathbf{z}|^{2-n}}{n-2} - 2\frac{n_j(\mathbf{y})}{|\mathbf{z}|^n}\right\}, \ n\geq 3,
\end{aligned} \tag{3.5.6}
$$

where

$$
\begin{aligned}
D_1(z) &:= 8\frac{\left(\frac{z}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(z)}{\Gamma\left(\frac{n}{2}\right)z^2} - \frac{2n}{z^2} + 1 \\
D_2(z) &:= 8\frac{\left(\frac{z}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(z)}{\Gamma\left(\frac{n}{2}\right)z^2} + 2\frac{\left(\frac{z}{2}\right)^{\frac{n}{2}}K_{\frac{n}{2}}(z)}{\Gamma\left(\frac{n}{2}\right)} - \frac{2n}{z^2} \\
D_3(z) &:= -16\frac{\left(\frac{z}{2}\right)^{\frac{n}{2}+2}K_{\frac{n}{2}+2}(z)}{\Gamma\left(\frac{n}{2}\right)z^2} + \frac{2n(n+2)}{z^2}.
\end{aligned} \tag{3.5.7}
$$

Note that

$$
\left(-\triangle_{\mathbf{x}} + \chi^2\mathbb{I}\right)S_{jk\ell}^{\chi^2}(\mathbf{y},\mathbf{x}) + \frac{\partial \Lambda_{j\ell}^{\chi^2}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0, \quad \frac{\partial S_{jk\ell}^{\chi^2}(\mathbf{y},\mathbf{x})}{\partial x_k} = 0 \ \text{ for } \mathbf{x}\neq\mathbf{y}. \tag{3.5.8}
$$

The fundamental solution of the Stokes system on Lipschitz domains in $\mathbb{R}^n$

The components of the fundamental Stokes tensor $\mathcal{G}(\mathbf{x},\mathbf{y})$ and those of the associated pressure vector $\mathbf{\Pi}(\mathbf{x},\mathbf{y})$, which determine the fundamental solution $(\mathcal{G},\mathbf{\Pi})$ of the Stokes system in $\mathbb{R}^n$, $n\geq 2$, are given by (see e.g., [59, Chapter 2], [64], [111, pp. 38,39])

$$
\begin{aligned}
\mathcal{G}_{jk}(\mathbf{x},\mathbf{y}) &= \frac{1}{2\omega_n}\left\{\frac{\delta_{kj}}{(n-2)|\mathbf{x}-\mathbf{y}|^{n-2}} + \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^n}\right\}, \\
\Pi_j(\mathbf{x},\mathbf{y}) &= \frac{1}{\omega_n}\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^n},
\end{aligned} \qquad n\geq 3, \tag{3.5.9}
$$

$$
\begin{aligned}
\mathcal{G}_{jk}(\mathbf{x}-\mathbf{y}) &= \frac{1}{4\pi}\left\{\frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2} - \delta_{kj}\ln(|\mathbf{x}-\mathbf{y}|)\right\}, \\
\Pi_j(\mathbf{x}-\mathbf{y}) &= \frac{1}{2\pi}\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^2},
\end{aligned} \qquad n=2. \tag{3.5.10}
$$

The corresponding stress and pressure tensors $\mathbf{S}(\mathbf{x},\mathbf{y})$ and $\boldsymbol{\Lambda}(\mathbf{x},\mathbf{y})$ have the components (see e.g., [59, Chapter 3], [111, p. 132])

$$S_{ijk}(\mathbf{x},\mathbf{y}) = -\frac{n}{\omega_n}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^{n+2}},\ \ n\geq 2, \tag{3.5.11}$$

$$\Lambda_{ik}(\mathbf{x},\mathbf{y}) = -\frac{2}{\omega_n}\left(-\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|^n}+n\frac{(x_i-y_i)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^{n+2}}\right),\ \ n\geq 2. \tag{3.5.12}$$

3.5.2 Layer potential operators for the Stokes and Brinkman equations on Lipschitz domains in $\mathbb{R}^n$

Let us now refer to the Brinkman system

$$\mathrm{div}\mathbf{u}=0,\ \ (\triangle-\chi^2\mathbb{I})\mathbf{u}-\nabla\pi=\mathbf{0}, \tag{3.5.13}$$

which describes the flow of a viscous incompressible fluid in a porous medium. The first equation in (3.5.13) is the continuity equation and the second one is the Brinkman equation. Also, note that the constant $\chi>0$ is related to the physical properties of the involved porous medium, i.e., if a is a characteristic length of the domain occupied by the porous medium with permeability κ, then $\chi=\frac{a}{\sqrt{\kappa}}$.

As in the previous sections, by $\mathfrak{D}\subset\mathbb{R}^n,\ n\geq 2$, we denote a bounded Lipschitz domain with boundary $\Gamma:=\partial\mathfrak{D}$ and $\mathfrak{D}_+:=\mathbb{R}^n\setminus\overline{\mathfrak{D}}$. Let us now define the *single* and *double layer potentials*, $\mathbf{V}_{\chi^2;\Gamma}\mathbf{g},\ \mathbf{W}_{\chi^2;\Gamma}\mathbf{h}:\mathbb{R}^n\setminus\Gamma\to\mathbb{R}^n$, associated with this system and having the densities $\mathbf{g}$ and $\mathbf{h}$, as (see e.g., [59, Chapter 3])

$$\big(\mathbf{V}_{\chi^2;\Gamma}\mathbf{g}\big)(\mathbf{x}):=\big\langle\mathcal{G}^{\chi^2}(\mathbf{x},\cdot),\mathbf{g}\big\rangle_\Gamma\ \ \big(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}\big)_k(\mathbf{x}):=\int_\Gamma S^{\chi^2}_{jk\ell}(\mathbf{y},\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})d\Gamma(\mathbf{y}),\ \ \mathbf{x}\in\mathbb{R}^n\setminus\Gamma. \tag{3.5.14}$$

Also let $P^s_{\chi^2;\Gamma}\mathbf{g},\ P^d_{\chi^2;\Gamma}\mathbf{h}:\mathbb{R}^n\setminus\Gamma\to\mathbb{R}$ be the functions given by

$$\big(P^s_{\chi^2;\Gamma}\mathbf{g}\big)(\mathbf{x}):=\big\langle\Pi^{\chi^2}(\mathbf{x},\cdot),\mathbf{g}\big\rangle_\Gamma\ \ \big(P^d_{\chi^2;\Gamma}\mathbf{h}\big)(\mathbf{x}):=\int_\Gamma \Lambda^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y})n_\ell(\mathbf{y})h_j(\mathbf{y})d\Gamma(\mathbf{y}),\ \ \mathbf{x}\in\mathbb{R}^n\setminus\Gamma. \tag{3.5.15}$$

Note that $(\mathbf{g},\mathbf{h})$ are chosen in either of the following spaces:

$$\begin{aligned}&(i)\ \ H^{-\frac{1}{2}+\beta}(\Gamma,\mathbb{R}^n)\times H^{\frac{1}{2}+\beta}_{\mathbf{n}_\Gamma}(\Gamma,\mathbb{R}^n),\ \beta\in\left(-\frac{1}{2},\frac{1}{2}\right)\\ &(ii)\ \ L^p(\Gamma,\mathbb{R}^n)\times L^p_{\mathbf{n}_\Gamma}(\Gamma,\mathbb{R}^n),\ p\in(2-\epsilon,2+\epsilon)\text{ for some }\epsilon:=\epsilon(\Gamma)>0.\end{aligned} \tag{3.5.16}$$

The pairs $(\mathbf{V}_{\chi^2;\Gamma}\mathbf{g},P^s_{\chi^2;\Gamma}\mathbf{g})$ and $(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h},P^d_{\chi^2;\Gamma}\mathbf{h})$ satisfy the Brinkman system in $\mathbb{R}^n\setminus\Gamma$, i.e.,

$$\left.\begin{aligned}&(\triangle-\chi^2\mathbb{I})\mathbf{V}_{\chi^2;\Gamma}\mathbf{g}-\nabla P^s_{\chi^2;\Gamma}\mathbf{g}=\mathbf{0},\ \ \mathrm{div}\mathbf{V}_{\chi^2;\Gamma}\mathbf{g}=0\\ &(\triangle-\chi^2\mathbb{I})\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}-\nabla P^d_{\chi^2;\Gamma}\mathbf{h}=\mathbf{0},\ \ \mathrm{div}\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}=0\end{aligned}\right.\ \ \text{in }\mathbb{R}^n\setminus\Gamma. \tag{3.5.17}$$

Note that in each of the above cases, there exists the principal value (or the boundary version) of the double-layer potential $\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}$ almost everywhere on Γ, and is given by:

$$(\mathbf{K}_{\chi^2;\Gamma}\mathbf{h})_k(\mathbf{x}):=\text{p.v.}\int_\Gamma S^{\chi^2}_{jk\ell}(\mathbf{y},\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})d\Gamma(\mathbf{y}),\ \ a.e.\ \mathbf{x}\in\Gamma, \tag{3.5.18}$$

where p.v. means the principal value of a singular integral[10]. The corresponding boundary traces are $(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h})^\pm:=\mathrm{Tr}^\pm(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h})$.

[10]Note that $\text{p.v.}\int_\Gamma S^{\chi^2}_{jk\ell}(\mathbf{y},\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})d\Gamma(\mathbf{y}):=\lim_{\epsilon\to 0}\int_{\Gamma\setminus\Gamma_\epsilon} S^{\chi^2}_{jk\ell}(\mathbf{y},\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})d\Gamma(\mathbf{y})$, where Γ_ϵ is the portion of Γ located inside the ball in $\mathbb{R}^n$ of radius ϵ and center $\mathbf{x}\in\Gamma$.

Theorem 3.5.1 (**[14], [26], [44], [54], [79]**) *Let $\mathfrak{D} \subset \mathbb{R}^n$ be a bounded Lipschitz domain $(n \geq 2)$ with boundary Γ, and let $\chi \geq 0$, $p \in (1, \infty)$, $r \in [0, 1]$, $\mathbf{g} \in L^p_{r-1}(\Gamma, \mathbb{R}^n)$ and $\mathbf{h} \in L^p_r(\Gamma, \mathbb{R}^n)$. Then a.e. on Γ:*

$$\mathrm{Tr}^+\big(\mathbf{V}_{\chi^2;\Gamma}\mathbf{g}\big) = \mathrm{Tr}^-\big(\mathbf{V}_{\chi^2;\Gamma}\mathbf{g}\big) := \mathcal{V}_{\chi^2;\Gamma}\mathbf{g}, \tag{3.5.19}$$

$$\big(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}\big)^{\pm} = \left(\pm\frac{1}{2}\mathbb{I} + \mathbf{K}_{\chi^2;\Gamma}\right)\mathbf{h}, \tag{3.5.20}$$

$$\partial^{\pm}_{\nu_\Gamma}\left(\mathbf{V}_{\chi^2;\Gamma}\mathbf{g}, P^s_{\chi^2;\Gamma}\mathbf{g}\right) = \left(\mp\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{\chi^2;\Gamma}\right)\mathbf{g}, \tag{3.5.21}$$

$$\partial^{+}_{\nu_\Gamma}\left(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}, P^d_{\chi^2;\Gamma}\mathbf{h}\right) = \partial^{-}_{\nu_\Gamma}\left(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}, P^d_{\chi^2;\Gamma}\mathbf{h}\right) := \mathbf{D}_{\chi^2;\Gamma}\mathbf{h}. \tag{3.5.22}$$

In addition, the following layer potential operators are well-defined and continuous:

$$\begin{aligned}
&\mathbf{V}_{\chi^2;\Gamma} : L^p_{r-1}(\Gamma, \mathbb{R}^n) \to L^p_{\frac{1}{p}+r}(\mathfrak{D}_-, \mathbb{R}^n),\ \mathbf{W}_{\chi^2;\Gamma} : L^p_r(\Gamma, \mathbb{R}^n) \to L^p_{\frac{1}{p}+r}(\mathfrak{D}_-, \mathbb{R}^n),\\
&\mathbf{V}_{\chi^2;\Gamma} : L^p_{r-1}(\Gamma, \mathbb{R}^n) \to L^p_{\frac{1}{p}+r,\mathrm{loc}}(\overline{\mathfrak{D}}_+, \mathbb{R}^n),\ \mathbf{W}_{\chi^2;\Gamma} : L^p_r(\Gamma, \mathbb{R}^n) \to L^p_{\frac{1}{p}+r,\mathrm{loc}}(\overline{\mathfrak{D}}_+, \mathbb{R}^n),\\
&\mathcal{V}_{\chi^2;\Gamma} : L^p_{r-1}(\Gamma, \mathbb{R}^n) \to L^p_r(\Gamma, \mathbb{R}^n),\ \mathbf{K}_{\chi^2;\Gamma} : L^p_r(\Gamma, \mathbb{R}^n) \to L^p_r(\Gamma, \mathbb{R}^n),\\
&\mathbf{K}^*_{\chi^2;\Gamma} : L^p_{r-1}(\Gamma, \mathbb{R}^n) \to L^p_{r-1}(\Gamma, \mathbb{R}^n),\ \mathbf{D}_{\chi^2;\Gamma} : L^p_r(\Gamma, \mathbb{R}^n) \to L^p_{r-1}(\Gamma, \mathbb{R}^n).
\end{aligned}$$

Note that the superscript + (respectively, -) in the above formulas applies for the limiting value of a field evaluated from the external side (respectively, the internal side) of Γ.

The result below has been obtained in [79] in the case $\chi = 0$ (see also [59], [111] for $\chi > 0$).

Theorem 3.5.2 (e.g., **[59], [79], [111]**) *If $\mathfrak{D} \subset \mathbb{R}^n$, is a bounded Lipschitz domain with the boundary Γ, $n \geq 2$, then for any $p \in (1, \infty)$ and $r \in [0, 1]$, one has*

$$\mathrm{Ker}\left(\mathcal{V}_{\chi^2;\Gamma} : L^p_{r-1}(\Gamma, \mathbb{R}^n) \to L^p_r(\Gamma, \mathbb{R}^n)\right) = \mathbb{R}\mathbf{n}_\Gamma,\ \mathbb{R}\mathbf{n}_\Gamma := \{c\mathbf{n}_\Gamma : c \in \mathbb{R}\}. \tag{3.5.23}$$

The layer potentials $\mathbf{V}_{\chi^2;\Gamma}\mathbf{g}$, $\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}$, $P^s_{\chi^2;\Gamma}\mathbf{g}$, $P^d_{\chi^2;\Gamma}\mathbf{h}$ have the following behavior at infinity for $\chi > 0$ (see e.g, [54, p. 1067], [59, Chapter 3]):

$$\begin{aligned}
&\big(\mathbf{V}_{\chi^2;\Gamma}\mathbf{g}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-n}),\ \big(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty, n \geq 2\\
&\big(P^s_{\chi^2;\Gamma}\mathbf{g}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}),\ \big(P^d_{\chi^2;\Gamma}\mathbf{h}\big)(\mathbf{x}) = \mathcal{O}(\ln|\mathbf{x}|) \text{ as } |\mathbf{x}| \to \infty, n = 2\\
&\big(P^s_{\chi^2;\Gamma}\mathbf{g}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-n}),\ \big(P^d_{\chi^2;\Gamma}\mathbf{h}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{2-n}) \text{ as } |\mathbf{x}| \to \infty, n \geq 3.
\end{aligned} \tag{3.5.24}$$

In addition, if $\langle \mathbf{h}, \mathbf{n}_\Gamma \rangle_\Gamma = 0$, one has:

$$\big(\mathbf{W}_{\chi^2;\Gamma}\mathbf{h}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-n}),\ \big(P^d_{\chi^2;\Gamma}\mathbf{h}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-n}), \text{as } |\mathbf{x}| \to \infty. \tag{3.5.25}$$

For $\chi = 0$, one has the asymptotic formulas

$$\begin{aligned}
&(\mathbf{V}_\Gamma\mathbf{g})(\mathbf{x}) = \mathcal{O}(\ln|\mathbf{x}|),\ \big(P^s_\Gamma\mathbf{h}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty, n = 2\\
&(\mathbf{V}_\Gamma\mathbf{g})(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{2-n}),\ \big(P^s_\Gamma\mathbf{h}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty, n \geq 3\\
&(\mathbf{W}_\Gamma\mathbf{h})(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-n}),\ \big(P^d_\Gamma\mathbf{h}\big)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-n}) \text{ as } |\mathbf{x}| \to \infty, n \geq 2.
\end{aligned} \tag{3.5.26}$$

3.5.3 Compactness of the complementary layer potential operators on Lipschitz domains in $\mathbb{R}^n$

Next, we mention the compactness result of the complementary layer potential operators associated to the Stokes and Brinkman systems on Sobolev spaces $\{L^p_s(\Gamma, \mathbb{R}^n)\}$, $\{L^p_{-s}(\Gamma, \mathbb{R}^n)\}$, $s \in (0, 1)$, as well as on the spaces $\{L^p(\Gamma, \mathbb{R}^n)\}$, $\{L^p_1(\Gamma, \mathbb{R}^n)\}$, $p \in (1, \infty)$. Note that by a complementary layer potential operator we mean the difference between a layer potential operator for the Brinkman system and the corresponding layer potential operator for the Stokes system. Let us mention that Cwikel [17] showed that the compactness property is extrapolated to complex interpolation scales of Banach spaces. The following compactness result has been obtained recently by Kohr, Lanza de Cristoforis and Wendland [54].

Theorem 3.5.3 [54] *If $\mathfrak{D} \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded Lipschitz domain with boundary Γ, $\lambda > 0$ is a given constant, then for any $p \in (1,\infty)$ the following operators are compact:*

$$\begin{array}{l} \mathcal{V}_{\chi^2,0;\Gamma} : L^p_{s-1}(\Gamma,\mathbb{R}^n) \to L^p_s(\Gamma,\mathbb{R}^n), \\ \mathbf{K}_{\chi^2,0;\Gamma} : L^p_s(\Gamma,\mathbb{R}^n) \to L^p_s(\Gamma,\mathbb{R}^n), \\ \mathbf{K}^*_{\chi^2,0;\Gamma} : L^p_{s-1}(\Gamma,\mathbb{R}^n) \to L^p_{s-1}(\Gamma,\mathbb{R}^n), \\ \mathbf{D}_{\chi^2,0;\Gamma} : L^p_s(\Gamma,\mathbb{R}^n) \to L^p_{s-1}(\Gamma,\mathbb{R}^n) \end{array} \quad \forall\, s \in (0,1) \tag{3.5.27}$$

and

$$\begin{array}{l} \mathbf{K}_{\chi^2,0;\Gamma} : L^p_1(\Gamma,\mathbb{R}^n) \to L^p_1(\Gamma,\mathbb{R}^n),\ \mathbf{K}_{\chi^2,0;\Gamma} : L^p(\Gamma,\mathbb{R}^n) \to L^p(\Gamma,\mathbb{R}^n), \\ \mathcal{V}_{\chi^2,0;\Gamma} : L^p(\Gamma,\mathbb{R}^n) \to L^p_1(\Gamma,\mathbb{R}^n),\ \mathcal{V}_{\chi^2,0;\Gamma} : L^p_{-1}(\Gamma,\mathbb{R}^n) \to L^p(\Gamma,\mathbb{R}^n), \\ \mathbf{K}^*_{\chi^2,0;\Gamma} : L^p(\Gamma,\mathbb{R}^n) \to L^p(\Gamma,\mathbb{R}^n),\ \mathbf{K}^*_{\chi^2,0;\Gamma} : L^p_{-1}(\Gamma,\mathbb{R}^n) \to L^p_{-1}(\Gamma,\mathbb{R}^n), \\ \mathbf{D}_{\chi^2,0;\Gamma} : L^p_1(\Gamma,\mathbb{R}^n) \to L^p(\Gamma,\mathbb{R}^n),\ \mathbf{D}_{\chi^2,0;\Gamma} : L^p(\Gamma,\mathbb{R}^n) \to L^p_{-1}(\Gamma,\mathbb{R}^n). \end{array} \tag{3.5.28}$$

3.5.4 Invertibility results for related layer potential operators on Lipschitz domains in $\mathbb{R}^n$

The following result has been obtained by Mitrea and Wright [79, Theorems 9.3, 10.13]:

Theorem 3.5.4 [79] *Let $\mathfrak{D} \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded Lipschitz domain with connected boundary $\partial\mathfrak{D}$. Then there exists $\varepsilon = \varepsilon(\partial\mathfrak{D}) > 0$ such that, for any $\gamma \in \mathbb{R} \setminus \left[-\frac{1}{2},\frac{1}{2}\right]$ one has the properties:*

(i) *The operators*

$$\gamma\mathbb{I} + \mathbf{K}_{\partial\mathfrak{D}} : L^p_1(\partial\mathfrak{D},\mathbb{R}^n) \to L^p_1(\partial\mathfrak{D},\mathbb{R}^n),\ \gamma\mathbb{I} + \mathbf{K}^*_{\partial\mathfrak{D}} : L^p(\partial\mathfrak{D},\mathbb{R}^n) \to L^p(\partial\mathfrak{D},\mathbb{R}^n),$$
$$\pm\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{\partial\mathfrak{D}} : L^p(\partial\mathfrak{D},\mathbb{R}^n)/\mathbb{R}\nu \to L^p(\partial\mathfrak{D},\mathbb{R}^n)/\mathbb{R}\nu,$$

are invertible for any $p \in \left(\max\left\{1, \frac{2(n-1)}{n+1} - \varepsilon\right\}, 2+\varepsilon\right)$.

(ii) *For any $s \in (0,1)$ and $p \in (2-\varepsilon, 2+\varepsilon)$, the following operators are invertible:*

$$\gamma\mathbb{I} + \mathbf{K}_{\partial\mathfrak{D}} : L^p_s(\partial\mathfrak{D},\mathbb{R}^n) \to L^p_s(\partial\mathfrak{D},\mathbb{R}^n),\ \gamma\mathbb{I} + \mathbf{K}^*_{\partial\mathfrak{D}} : L^p_{s-1}(\partial\mathfrak{D}) \to L^p_{s-1}(\partial\mathfrak{D},\mathbb{R}^n),$$
$$\pm\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{\partial\mathfrak{D}} : L^p_{s-1}(\partial\mathfrak{D},\mathbb{R}^n)/\mathbb{R}\nu \to L^p_{s-1}(\partial\mathfrak{D},\mathbb{R}^n)/\mathbb{R}\nu.$$

3.6 Layer potential theory for pseudodifferential Brinkman operators on Lipschitz domains in compact Riemannian manifolds

In this section we present main results of the layer potential theory for pseudodifferential Brinkman operators on Lipschitz domains in compact Riemannian manifold, which include the invertibility of the Brinkman operator, the fundamental solution for the Brinkman operator and the compactness property of the complementary layer potential operators. These results have been recently obtained by Kohr, Pintea and Wendland in [55]-[58]. Note that the pseudodifferential Brinkman operators are variable coefficient operators that extend the differential Brinkman operator from the Euclidean setting to the case of compact Riemannian manifolds. The main sources used in the preparation of this section are [19], [55], [56], [57].

3.6.1 Pseudodifferential Brinkman operators on compact Riemannian manifolds

We consider a compact boundaryless manifold (M, g) of dimension $m \geq 2$ equipped with a smooth Riemannian metric tensor $g = \sum_{j,k=1}^{m} g_{jk} dx_j \otimes dx_k =: g_{jk} dx_j \otimes dx_k$, and let (g^{jk}) be the inverse of (g_{jk}). Let us mention that the volume element on M is given by $\mathrm{dVol} = \sqrt{g} dx_1 \ldots dx_m$, where $g := \det(g_{jk})$. Recall that the tangent bundle is denoted by $TM = \bigcup_{p \in M} T_p M$ and the cotangent bundle by $T^*M = \bigcup_{p \in M} T_p^* M$. Let $\mathcal{X}(M) = C^\infty(M, TM)$ denote the $C^\infty(M)$-module of smooth vector fields on M. In a natural way we can identify T^*M with TM and $\Lambda^1 TM$ with $\mathcal{X}(M)$. Next, we define following inner product on $\Lambda^1 TM$ (see e.g., [117]):

$$\langle dx_j, dx_k \rangle = g^{jk}, \quad \langle X, Y \rangle = X_j g^{jk} Y_k, \tag{3.6.1}$$

where the vector field $X = X^k \partial_k \in TM$ is identified with the one form $X_r dx_r = X^k g_{kr} dx_r$, $X_r = g_{kr} X^k$, and the notation $\langle \cdot, \cdot \rangle$ is used for the inner product. Consequently, the gradient operator $\mathrm{grad} : C^\infty(M) \to \mathcal{X}(M)$ is identified with the *exterior derivative operator* $d : C^\infty(M) \to C^\infty(M, \Lambda^1 TM)$, $d = \partial_j dx_j$. In addition, $-\mathrm{div} : \mathcal{X}(M) \to C^\infty(M)$ is identified with the *exterior co-derivative operator* $\delta : C^\infty(M, \Lambda^1 TM) \to C^\infty(M)$, $\delta = d^*$. As usual, by ∇ we denote the Levi-Civita connection on M (for details we refer to [107, Chapter 2]).

Next, assume that $X \in \mathcal{X}(M)$. The symmetric part of the tensor field

$$\nabla X : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M, TM \otimes TM), \; (\nabla X)(Y, Z) = \langle \nabla_Y X, Z \rangle,$$

is called the *deformation* of X and is denoted by $\mathrm{Def}\ X$ (see e.g., [20], [107]). Hence

$$(\mathrm{Def}\ X)(Y, Z) = \frac{1}{2} \{ \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle \}, \quad \forall\, Y, Z \in \mathcal{X}(M). \tag{3.6.2}$$

Definition 3.6.1 (e.g., **[107]**) A vector field $X \in \mathcal{X}(M)$ such that $\mathrm{Def}\ X = 0$ on M, is called a *Killing field*.

All along this work, we assume that [19, 78]

(3.6.3) The manifold M has no nontrivial Killing fields.

In fact, if $\Omega \subset M$ is a given Lipschitz domain, then M may be deformed away from $\overline{\Omega}$ such that the condition (3.6.3) is satisfied (we refer to [78, p. 959] for more details about such manifolds). Note that the Killing fields in $\mathbb{R}^n$ are the usual rigid body motion fields.

Let us now consider the second-order partial differential operator [19, 78]

$$\mathfrak{L} : \mathcal{X}(M) \to \mathcal{X}(M), \;\; \mathfrak{L} := 2\mathrm{Def}^* \mathrm{Def} = -\triangle + \mathrm{d}\delta - 2\mathrm{Ric}, \tag{3.6.4}$$

where Def^* is the adjoint of Def, $\triangle := -(\mathrm{d}\delta + \delta d)$ is the Hodge Laplacian and Ric is the Ricci tensor. Note that $\mathfrak{L}$ is the *natural operator for the Stokes system on an arbitrary Riemannian manifold* (cf. [23]).

For a complete description of differential operators on Riemannian manifolds we refer to [20], [65], [107, Chapter 2].

Further, recall that OPS^{ℓ}_{cl} is the class of classical pseudodifferential operators of order ℓ on M (see Definition 3.3.8). Let $P \in OPS^0_{\mathrm{cl}}(\Lambda^1 TM, \Lambda^1 TM)$ be a self-adjoint and non-negative operator with respect to the $L^2(M, \Lambda^1 TM)$ - inner product $\langle \cdot, \cdot \rangle$, i.e.,

$$\langle Pu, w \rangle = \langle u, Pw \rangle, \;\; \langle Pu, u \rangle \geq 0 \;\text{ for all }\; u, w \in L^2(M, \Lambda^1 TM). \tag{3.6.5}$$

Then the *pseudodifferential Brinkman operator* on M is given by (cf. [56])

$$B_P := \begin{pmatrix} \mathfrak{L} + P & d \\ \delta & 0 \end{pmatrix} : C^\infty(M, \Lambda^1 TM) \times C^\infty(M) \to C^\infty(M, \Lambda^1 TM) \times C^\infty(M). \tag{3.6.6}$$

All along this work we consider the pseudodifferential operator P of the form $P = \lambda^2\mathbb{I}$, where $\lambda \neq 0$ is a constant. Thus, the operator (3.6.6) becomes

$$B_\lambda := \begin{pmatrix} \mathfrak{L} + \lambda^2\mathbb{I} & d \\ \delta & 0 \end{pmatrix} : C^\infty(M, \Lambda^1TM) \times C^\infty(M) \to C^\infty(M, \Lambda^1TM) \times C^\infty(M). \tag{3.6.7}$$

3.6.2 Sobolev spaces on Lipschitz domains in compact Riemannian manifolds

Let $\Omega_+ := \Omega \subset M$ be a Lipschitz domain (i.e., the boundary of the open and connected set Ω can be described in appropriate local coordinates by means of graphs of Lipschitz functions) and assume that $\Omega_- := M \setminus \overline{\Omega}$ is connected. Therefore, the sets $\Omega_\pm$ are both Lipschitz domains.

For a fixed constant $\kappa = \kappa(\partial\Omega) > 0$ denote by $\gamma_\pm(x) := \{y \in \Omega_\pm : |x - y| < (1 + \kappa)\mathrm{dist}(y, \partial\Omega)\}$, $x \in \partial\Omega$ the non-tangential approach regions lying in Ω_+ and Ω_-, respectively. As in the Euclidean case one can define the trace operator on a compact Riemannian manifold. More precisely, by using the same notations as in the Euclidean setting, let $\mathrm{Tr}^\pm$ be the non-tangential boundary trace operators on $\partial\Omega$ given by (see e.g., [76]) $(\mathrm{Tr}^\pm u)(x) := \lim\limits_{\gamma_\pm(x)\ni y\to x} u(y)$, $x \in \partial\Omega$.

Next, for $s \geq 0$, consider the Sobolev spaces of functions

$$H^s(\Omega_\pm) := \{f|_{\Omega_\pm} : f \in H^s(M)\}, \ \ \tilde{H}^s(\Omega_\pm) := \{f \in H^s(M) : \mathrm{supp} f \subseteq \overline{\Omega}_\pm\},$$

and denote by $H^{-s}(\Omega_\pm)$ the dual of the space $\tilde{H}^s(\Omega_\pm)$ with respect to the $L^2(\Omega_\pm)$-duality, i.e., $H^{-s}(\Omega_\pm) = \big(\tilde{H}^s(\Omega_\pm)\big)^*$. In addition, consider the Sobolev spaces of one forms (see e.g., [77]):

$$\begin{aligned} &H^s(\Omega_\pm, \Lambda^1TM|_{\Omega_\pm}) := H^s(\Omega_\pm) \otimes \Lambda^1TM|_{\Omega_\pm}, \\ &\tilde{H}^s(\Omega_\pm, \Lambda^1TM|_{\Omega_\pm}) := \tilde{H}^s(\Omega_\pm) \otimes \Lambda^1TM|_{\Omega_\pm}, \\ &H^{-s}(\Omega_\pm, \Lambda^1TM) := (\tilde{H}^s(\Omega_\pm, \Lambda^1TM))^*. \end{aligned} \tag{3.6.8}$$

Therefore, $H^s(\Omega_\pm, \Lambda^1TM|_{\Omega_\pm})$ is the set of all one forms having their coefficients in $H^s(\Omega_\pm)$.

Further, assume that $\lambda \geq 0$ is a given constant. Then for any $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, consider the spaces

$$\tilde{H}^{-1+\beta}(\Omega_\pm, \Lambda^1TM) := \{\mathbf{f} \in H^{-1+\beta}(M, \Lambda^1TM) : \mathrm{supp}\ \mathbf{f} \subseteq \overline{\Omega}_\pm\}, \tag{3.6.9}$$

$$\begin{aligned} H^{1+\beta}(\Omega_\pm, \mathcal{L}_\lambda) := \big\{(\mathbf{u}, \pi, \mathbf{f}) :& \mathbf{u} \in H^{1+\beta}(\Omega_\pm, \Lambda^1TM), \pi \in H^\beta(\Omega_\pm), \mathbf{f} \in \tilde{H}^{-1+\beta}(\Omega_\pm, \Lambda^1TM) \\ &\text{such that } \mathcal{L}_\lambda(\mathbf{u}, \pi) = \mathbf{f}|_{\Omega_\pm},\ \delta\mathbf{u} = 0 \text{ in } \Omega_\pm\big\}, \end{aligned} \tag{3.6.10}$$

where $\mathcal{L}_\lambda(\mathbf{u}, \pi) := \mathfrak{L}\mathbf{u} + \lambda^2\mathbf{u} + d\pi$.

3.6.3 The nontangential trace and conormal derivative operators on compact Riemannian manifolds

Lemma 3.6.2 (**[19], [78]**) *For every $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, the restriction to the boundary,*

$$C^\infty(\overline{\Omega}_\pm, \Lambda^1TM) \to C^0(\partial\Omega_\pm, \Lambda^1TM),\ u \mapsto u|_{\partial\Omega_\pm},$$

extends to a linear and bounded operator $\mathrm{Tr}^\pm : H^s(\Omega_\pm, \Lambda^1TM) \to H^{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1TM)$, which is onto and has a bounded right inverse $\mathcal{Z}^\pm : H^{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1TM) \to H^s(\Omega_\pm, \Lambda^1TM)$. For $s > \frac{3}{2}$, the operator $\mathrm{Tr}^\pm : H^s(\Omega_\pm, \Lambda^1TM) \to H^1(\partial\Omega_\pm, \Lambda^1TM)$ is also bounded.

The conormal derivative operator for the Brinkman system on Lipschitz domains in Riemannian manifolds has been introduced by Kohr, Pintea and Wendland in [56, Lemma 2.2], as an extension to this setting of the notion of conormal derivative operator for the Stokes system on Euclidean setting[11] due to Mitrea and Wright [79, Theorem 10.10] (see also [19, 55, 56, 78]):

[11]For $s \in [0, 1]$ and some $X \subseteq M$, $\langle\cdot,\cdot\rangle_X := {}_{H^s(X,\Lambda^1TM)}\langle\cdot,\cdot\rangle_{(H^s(X,\Lambda^1TM))^*}$ denotes the pairing between two dual Sobolev spaces $H^s(X, \Lambda^1TM)$ and $\big(H^s(X, \Lambda^1TM)\big)^*$.

Lemma 3.6.3 **[56]** *Let* $\lambda \geq 0$ *be a given constant. Then for any* $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ *the* conormal derivative operator

$$\partial_\nu^\pm : H^{1+\beta}(\Omega_\pm, \mathcal{L}_\lambda) \to H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM), \tag{3.6.11}$$

$$\pm\langle \partial_\nu^\pm(\mathbf{u},\pi,\mathbf{f}), \Phi\rangle_{\partial\Omega} := 2\int_{\Omega_\pm} \langle \mathrm{Def}\ \mathbf{u}, \mathrm{Def}\ (\mathcal{Z}^\pm\Phi)\rangle \mathrm{dVol} + \lambda^2 \int_{\Omega_\pm} \langle \mathbf{u}, \mathcal{Z}^\pm\Phi\rangle \mathrm{dVol}$$
$$+ \int_{\Omega_\pm} \langle \pi, \delta(\mathcal{Z}^\pm\Phi)\rangle \mathrm{dVol} - \langle \mathbf{f}, \mathcal{Z}^\pm\Phi\rangle_{\Omega_\pm},\ \forall\, \Phi \in H^{\frac{1}{2}-\beta}(\partial\Omega, \Lambda^1 TM), \tag{3.6.12}$$

is well defined and bounded. Also, the following Green formula holds:

$$\pm\langle \partial_\nu^\pm(\mathbf{u},\pi,\mathbf{f}), \mathrm{Tr}^\pm\, \mathbf{v}\rangle_{\partial\Omega} - 2\int_{\Omega_\pm} \langle \mathrm{Def}\ \mathbf{u}, \mathrm{Def}\ \mathbf{v}\rangle \mathrm{dVol} - \lambda^2 \int_{\Omega_\pm} \langle \mathbf{u}, \mathbf{v}\rangle \mathrm{dVol}$$
$$= \int_{\Omega_\pm} \langle \pi, \delta\mathbf{v}\rangle \mathrm{dVol} - \langle \mathbf{f}, \mathbf{v}\rangle_{\Omega_\pm} \tag{3.6.13}$$

for all $(\mathbf{u}, \pi, \mathbf{f}) \in H^{1+\beta}(\Omega_\pm, \mathcal{L}_\lambda)$ *and* $\mathbf{v} \in H^{1-\beta}(\Omega_\pm, \Lambda^1 TM)$.

3.6.4 The invertibility of the Brinkman operator on Lipschitz domains in compact Riemannian manifolds

The Brinkman operator (3.6.7) is elliptic in the sense of Agmon-Douglis-Nirenberg (see [55]) and, in view of Theorem 3.3.11, extends to a Fredholm operator with index zero

$$B_\lambda : H^1(M, \Lambda^1 TM) \times L^2(M) \to H^{-1}(M, \Lambda^1 TM) \times L^2(M).$$

The kernel of this operator is the set $\{0\} \times \mathbb{R}$, and its range is $H^{-1}(M, \Lambda^1 TM) \times L^2_*(M)$, where

$$L^2_*(M) := \{q \in L^2(M) : \langle q, 1\rangle = 0\}.$$

In addition, the restriction of the Brinkman operator to $H^1(M, \Lambda^1 TM) \times L^2_*(M)$, denoted by B^0_λ, is invertible (for more details see [55]). Next, let us refer to the second order differential operator

$$\mathfrak{L}_\lambda = 2\mathrm{Def}^*\mathrm{Def} + \lambda^2\mathbb{I} : H^1(M, \Lambda^1 TM) \to H^{-1}(M, \Lambda^1 TM) \tag{3.6.14}$$

which is Fredholm with index zero and injective (due to the assumption (3.6.3)), and hence invertible (see [55, Lemma 5.8] for further details).

Lemma 3.6.4 (**[55], [56]**) *Let* M *be a boundaryless compact Riemannian manifold and let* $\lambda \geq 0$ *be a given constant. Then the operators*

$$\Upsilon_\lambda : L^2_*(M) \to L^2_*(M),\ \ \Upsilon_\lambda := \delta L_\lambda^{-1} d, \tag{3.6.15}$$

$$B^0_\lambda : H^1(M, \Lambda^1 TM) \times L^2_*(M) \to H^{-1}(M, \Lambda^1 TM) \times L^2_*(M), \tag{3.6.16}$$

are invertible. In addition, the inverse of B^0_λ *is the operator*

$$\begin{aligned} &B^0_\lambda : H^{-1}(M, \Lambda^1 TM) \times L^2_*(M) \to H^1(M, \Lambda^1 TM) \times L^2_*(M), \\ &(B^0_\lambda)^{-1} := \begin{pmatrix} \mathfrak{A}_\lambda & \mathfrak{B}_\lambda \\ \mathfrak{C}_\lambda & \mathfrak{D}_\lambda \end{pmatrix}, \end{aligned} \tag{3.6.17}$$

where $\mathfrak{A}_\lambda \in OPS^{-2}_{\mathrm{cl}}$, $\mathfrak{B}_\lambda \in OPS^{-1}_{\mathrm{cl}}$, $\mathfrak{C}_\lambda \in OPS^{-1}_{\mathrm{cl}}$, $\mathfrak{D}_\lambda \in OPS^{0}_{\mathrm{cl}}$ *are the pseudodifferential operators defined as*

$$\mathfrak{A}_\lambda := L_\lambda^{-1} - L_\lambda^{-1} d\, \Upsilon_\lambda^{-1}\, \delta L_\lambda^{-1},\ \ \mathfrak{B}_\lambda := L_\lambda^{-1} d \Upsilon_\lambda^{-1}, \tag{3.6.18}$$

$$\mathfrak{C}_\lambda := \Upsilon_\lambda^{-1} \delta L_\lambda^{-1},\ \ \mathfrak{D}_\lambda := -\, \Upsilon_\lambda^{-1}\, . \tag{3.6.19}$$

For $\lambda = 0$, i.e., for the inverse of the operator B_0^0 of the Stokes system, we use the notation

$$(B_0^0)^{-1} := \begin{pmatrix} \mathfrak{A}_0 & \mathfrak{B}_0 \\ \mathfrak{C}_0 & \mathfrak{D}_0 \end{pmatrix}. \tag{3.6.20}$$

3.6.5 The fundamental solution for the Brinkman operator on Lipschitz domains in compact Riemannian manifolds

In view of Lemma 3.6.4, one obtains the following relations M:

$$\mathfrak{L}_\lambda \mathfrak{A}_\lambda + \mathrm{d}\mathfrak{C}_\lambda = \mathbb{I}, \quad \delta \mathfrak{A}_\lambda = 0, \tag{3.6.21}$$

where $\mathbb{I}$ is the identity operator on $H^{-1}(M, \Lambda^1 TM)$. Let us denote by $\mathcal{G}_\lambda(x,y)$ and $\Pi_\lambda(x,y)$ the Schwartz kernels[12] of the pseudodifferential operators $\mathfrak{A}_\lambda$ and $\mathfrak{C}_\lambda$, respectively. In addition, let $\mathcal{G}(x,y)$ and $\Pi(x,y)$ be the Schwartz kernels of $\mathfrak{A}_0$ and $\mathfrak{C}_0$, respectively. By (3.6.21) one then obtains the following equations on M:

$$(\mathfrak{L}_x + \lambda^2 \mathbb{I})\mathcal{G}_\lambda(x,y) + d_x \Pi_\lambda(x,y) = \mathrm{Dirac}_y(x), \quad \delta_x \mathcal{G}_\lambda(x,y) = 0, \tag{3.6.22}$$

where Dirac_y denotes the Dirac distribution with mass at y. Hence the pair $(\mathcal{G}_\lambda(x,y), \Pi_\lambda(x,y))$ is the *fundamental solution of the Brinkman system on* M (for more details we refer to [56, 57]).

3.6.6 Layer potential operators for the Brinkman system on Lipschitz domains in compact Riemannian manifolds

In this section we present the main properties of layer potential operators for the Brinkman system on Lipschitz domains in compact Riemannian manifolds.

For $s \in [0,1]$, $\mathbf{f} \in H^{s-1}(\partial\Omega, \Lambda^1 TM)$ and $\mathbf{h} \in H^s(\partial\Omega, \Lambda^1 TM)$, the *single-layer potential* $\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}$ is the one form given on $M \setminus \partial\Omega$ by

$$(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f})(x) := \langle \mathcal{G}_\lambda(x,\cdot), \mathbf{f} \rangle_{\partial\Omega}, \ \ x \in M \setminus \partial\Omega. \tag{3.6.23}$$

In addition, the corresponding pressure potential has the expression

$$Q_{\lambda;\partial\Omega}\mathbf{f} := \langle \Pi_\lambda(x,\cdot), \mathbf{f} \rangle_{\partial\Omega}, \ \ x \in M \setminus \partial\Omega. \tag{3.6.24}$$

Similarly, the *double-layer potential* is defined at any point $x \in M \setminus \partial\Omega$ by

(3.6.25)

$$(\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h})(x) := \int_{\partial\Omega} \big\langle -2\big[(\mathrm{Def}_y\, \mathcal{G}_\lambda(x,\cdot))\nu_{\partial\Omega}\big](y) + (\Pi_\lambda)^\top(y,x)\nu_{\partial\Omega}(y), \mathbf{h}(y)\big\rangle d\sigma(y),$$

and the corresponding pressure potential

$$(\mathcal{P}_{\lambda;\partial\Omega}\mathbf{h})(x) := \int_{\partial\Omega} \langle -2[(\mathrm{Def}_y\, \Pi_\lambda(x,\cdot))\nu_{\partial\Omega}](y) - E_\lambda(x,y)\nu_{\partial\Omega}(y), \mathbf{h}(y)\rangle d\sigma(y), \tag{3.6.26}$$

where $E_\lambda(x,y)$ is the Schwartz kernel of $(-\mathfrak{D}_\lambda)^\top \in OPS^0_{\mathrm{cl}}(\mathbb{R}, \mathbb{R})$. Note that

$$\left.\begin{array}{l} \delta(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) = 0, \ (L + \lambda^2\mathbb{I})\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f} + dQ_{\lambda;\partial\Omega}\mathbf{f} = 0 \\ \delta\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h} = 0, \ (L + \lambda^2\mathbb{I})\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h} + d\mathcal{P}_{\lambda;\partial\Omega}\mathbf{h} = 0 \end{array}\right. \quad \text{on } M \setminus \partial\Omega. \tag{3.6.27}$$

[12] Note that the Schwartz kernel of a map $T : \mathcal{S} \to \mathcal{S}'$ represents a distribution $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfies the relation $\langle Tu, v \rangle = \langle K, u \otimes v \rangle$, $u, v \in \mathcal{S}$ (see e.g., [118]).

Further, the principal value (or the boundary version) of the double-layer potential $\mathbf{W}_{P;\partial\Omega}\mathbf{h}$ is given at a.e. $x \in \partial\Omega$ by (see e.g., [19])

$$(\mathbf{K}_{\lambda;\partial\Omega}\mathbf{h})(x) := \text{p.v.} \int_{\partial\Omega} \big\langle -2\big[(\mathrm{Def}_y\, \mathcal{G}_\lambda(x,\cdot))\nu_{\partial\Omega}\big](y) + (\Pi_\lambda)^\top(y,x) \otimes \nu_{\partial\Omega}(y), \mathbf{h}(y)\big\rangle d\sigma_y,$$

where the symbol p.v. refers to the principal value of a singular integral. Thus, one has

$$(\mathbf{K}_{\lambda;\partial\Omega}\mathbf{h})(x) = \lim_{\epsilon\to 0} \int_{\{y\in\partial\Omega\,:\, r(x,y)>\varepsilon\}} \big\langle -2\big[(\mathrm{Def}_y\, \mathcal{G}_\lambda(x,\cdot))\nu_{\partial\Omega}\big](y) + (\Pi_\lambda)^\top(y,x) \otimes \nu_{\partial\Omega}(y), \mathbf{h}(y)\big\rangle d\sigma_y, \tag{3.6.28}$$

where $r(x,y)$ is the geodesic distance between the points x and y in M. In addition, one has the following jump relations a.e. on $\partial\Omega$ (see e.g., [19, 55, 57])

$$\begin{aligned}
&\mathrm{Tr}^{\pm}(\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h}) = \Big(\pm\tfrac{1}{2}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega}\Big)\mathbf{h},\\
&\partial_\nu^{\pm}(\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h}, \mathcal{P}_{\lambda;\partial\Omega}\mathbf{h}) := \mathbf{D}^{\pm}_{\lambda;\partial\Omega}\mathbf{h},\ \ \mathbf{D}^{+}_{\lambda;\partial\Omega}\mathbf{h} - \mathbf{D}^{-}_{\lambda;\partial\Omega}\mathbf{h} \in \mathbb{R}\nu_{\partial\Omega}\\
&\mathrm{Tr}^{+}(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) = \mathrm{Tr}^{-}(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) := \mathcal{V}_{\lambda;\partial\Omega}\mathbf{f},\\
&\partial_\nu{}^{\pm}(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}, Q_{\lambda;\partial\Omega}\mathbf{f}) = \mp\tfrac{1}{2}\mathbf{f} + \mathbf{K}^{*}_{\lambda;\partial\Omega}\mathbf{f},
\end{aligned} \tag{3.6.29}$$

where

$$(\mathbf{K}^{*}_{\lambda;\partial\Omega}\mathbf{f})(x) := \text{p.v.}\int_{\partial\Omega} \big\langle -2[\mathrm{Def}_x \mathcal{G}_\lambda(\cdot,y)\nu](x) + \Pi_\lambda(x,y)\otimes\nu(x), \mathbf{f}(y)\big\rangle_y d\sigma(y), \ \ \text{a.e. } x \in \partial\Omega.$$

Theorem 3.6.5 (**[55], [57]**) *Let $\Omega \subset M$ be a Lipschitz domain and $\lambda \geq 0$ be a given constant. Then for any $s \in [0,1]$ and $\mathbf{f} \in H^{-s}(\partial\Omega, \Lambda^1 TM)$, one has*

$$\mathrm{Tr}^{+}(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) = \mathrm{Tr}^{-}(\mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}) = \mathcal{V}_{\lambda;\partial\Omega}\mathbf{f}. \tag{3.6.30}$$

Theorem 3.6.6 (**[55], [57]**) *If $\Omega \subset M$ is a Lipschitz domain, then for any $s \in [0,1]$ the kernel of the single-layer potential operator $\mathcal{V}_{\lambda,\partial\Omega} : H^{-s}(\partial\Omega, \Lambda^1 TM) \to H^{1-s}(\partial\Omega, \Lambda^1 TM)$ is given by*

$$\mathrm{Ker}\,\big(\mathcal{V}_{\lambda,\partial\Omega} : H^{-s}(\partial\Omega, \Lambda^1 TM) \to H^{1-s}(\partial\Omega, \Lambda^1 TM)\big) = \mathbb{R}\nu,\ \mathbb{R}\nu := \{c\nu : c \in \mathbb{R}\}. \tag{3.6.31}$$

In addition, one has the property $\mathbf{V}_{\lambda,\partial\Omega}\nu = 0$ on M.

In the case $\lambda = 0$ one gets the result by Mitrea and Taylor [78, Lemma 6.1].

3.6.7 Compactness of the complementary layer potential operators on Lipschitz domains in compact Riemannian manifolds

Next, we mention the compactness property of the complementary layer potential operators. By a *complementary layer potential operator* we mean the difference between a layer potential operator for the Brinkman system and the corresponding layer potential operator for the Stokes system. Then one has the following compactness results on the scale of boundary Sobolev spaces:

Theorem 3.6.7 **[57]** *Let $\Omega \subset M$ be a Lipschitz domain and let $\lambda > 0$ be a given constant. Then for any $s \in [0,1]$ the following operators are compact:*

- *The complementary single and double-layer potential operators*

$$\begin{aligned}
&\mathcal{V}_{\lambda,0;\partial\Omega} := \mathcal{V}_{\lambda;\partial\Omega} - \mathcal{V}_{\partial\Omega} : H^{s-1}(\partial\Omega, \Lambda^1 TM) \to H^{s}(\partial\Omega, \Lambda^1 TM)\\
&\mathbf{K}_{\lambda,0;\partial\Omega} := \mathbf{K}_{\lambda;\partial\Omega} - \mathbf{K}_{\partial\Omega} : H^{s}(\partial\Omega, \Lambda^1 TM) \to H^{s}(\partial\Omega, \Lambda^1 TM)
\end{aligned} \tag{3.6.32}$$

- *The adjoint of the complementary layer potential operator*

$$\mathbf{K}^*_{\lambda,0;\partial\Omega} := \mathbf{K}^*_{\lambda;\partial\Omega} - \mathbf{K}^*_{\partial\Omega} : H^{s-1}(\partial\Omega, \Lambda^1 TM) \to H^{s-1}(\partial\Omega, \Lambda^1 TM)$$

- *The complementary hypersingular layer potential operator*

$$\mathbf{D}_{\lambda,0;\partial\Omega} := \mathbf{D}_{\lambda;\partial\Omega} - \mathbf{D}_{\partial\Omega} : H^{s}(\partial\Omega, \Lambda^1 TM) \to H^{s-1}(\partial\Omega, \Lambda^1 TM). \tag{3.6.33}$$

3.6.8 Invertibility results for related layer potential operators on Lipschitz domains in compact Riemannian manifolds

The Fredholm and invertibility results below have been recently obtained:

Theorem 3.6.8 [57] *Let $\Omega \subset M$ be a Lipschitz domain and let $\lambda > 0$, $\mu \in [0,1)$ be given constants. Then for any $s \in (0,1)$ the following statements hold:*

(i) *The operators*

$$\tilde{\mathbf{K}}^{\pm}_{\lambda;\partial\Omega;\mu} := \mp\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega} : H^{s}(\partial\Omega, \Lambda^1 TM) \to H^{s}(\partial\Omega, \Lambda^1 TM) \tag{3.6.34}$$

are Fredholm with index zero.

(ii) *The operators*

$$\tilde{\mathbf{K}}^{\pm}_{\lambda;\partial\Omega;\mu} := \mp\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega} : H^{s}_{\nu}(\partial\Omega, \Lambda^1 TM) \to H^{s}_{\nu}(\partial\Omega, \Lambda^1 TM) \tag{3.6.35}$$

are isomorphisms, where

$$H^{s}_{\nu}(\partial\Omega, \Lambda^1 TM) := \{\Phi \in H^{s}(\partial\Omega, \Lambda^1 TM) : \langle \Phi, \nu \rangle_{\partial\Omega} = 0\}. \tag{3.6.36}$$

Remark 3.6.9 An extension of the results in Theorems 3.6.7 and 3.6.8 to a more general case involving a boundaryless compact Riemannian manifold with arbitrary dimension $m \geq 2$ and a pseudodifferential operator $P \in OPS^0_{\rm cl}(M, \Lambda^1 TM)$ of the form $P = \lambda\mathbb{I}$, $\lambda \in C^\infty(M)$, or the m-dimensional unit sphere $\mathfrak{S}^m$ and an arbitrary pseudodifferential operator $P \in OPS^0_{\rm cl}(\mathfrak{S}^m, \Lambda^1 TM)$, has been obtained in [57].

Chapter 4

Dirichlet-transmission problems for Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$

This chapter contains original results of the author concerning the study of a Dirichlet-transmission problem for the Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. The novelty of our study is provided by the fact that the transmission conditions are expressed in terms of a parameter $\mu \in (0,1]$ and the given boundary data are chosen in various function spaces, such as Sobolev, or L^p spaces, with p near 2. We use the results of layer potential theory presented in the previous chapter in order to get existence and uniqueness results for this problem. Our study has been suggested by some particular cases that have important practical applications. For example, by choosing $n = 3$ and $\mu = 1$, this boundary value problem describes an exterior Stokes flow past a porous particle with a solid core inside, all of the involved domains being Lipschitz. Note that a similar problem, but in a more particular situation, i.e., the problem of Stokes flow past a porous sphere that contains a spherical solid core, has been analyzed by e.g., Srivastava and Srivastava [103]. Our study can be considered an extension of their study to a more general situation. In addition, we analyze two special cases. The first case is devoted to a three-dimensional Stokes flow past a porous body with a solid core inside, when the corresponding permeability is large. In the second case we consider a similar Stokes flow problem but under the hypothesis of low permeability. In order to show the existence and uniqueness of the solution to the corresponding boundary value problem, we use the layer potential theory for both, Brinkman and Stokes systems, and hence a layer potential method that reduces the problem to a uniquely solvable system of Fredholm integral equations. The main results presented in this chapter have been recently obtained by Fericean and Wendland in [31]. In order to prepare this chapter we had in view the sources [54], [59], [60], [61].

4.1 Formulation of the problem

Let $\Omega, \mathfrak{D} \subset \mathbb{R}^n, n \geq 3$, be bounded Lipschitz domains such that $\overline{\mathfrak{D}} \subset \Omega$, and let Γ be the boundary of $\mathfrak{D}$. Also, let $\Omega^- := \Omega \setminus \overline{\mathfrak{D}}$, and $\Omega_+ := \mathbb{R}^n \setminus \overline{\Omega}$. Denote by $\partial\Omega$ the boundary of Ω. Assume that $\partial\Omega$ and Γ are connected. Also, assume that $\mu \in (0,1]$ is a given transmission parameter, $\chi > 0$ is a given constant, and $\mathbf{H}$, $\mathbf{F}$, $\mathbf{G}$ are given vector functions in spaces to be specified below. Next, consider a boundary value problem for the Stokes and Brinkman system, with Dirichlet and transmission conditions. This problem requires to find the pairs $((\mathbf{u}_+,\pi_+),(\mathbf{u}_-,\pi_-))$ satisfying

$$\begin{cases} \text{div }\mathbf{u}_+ = 0, \ \ -\nabla\pi_+ + \Delta\mathbf{u}_+ = 0 \text{ in } \Omega_+, \\ \text{div }\mathbf{u}_- = 0, \ \ -\nabla\pi_- + (\Delta - \chi^2\mathbb{I})\mathbf{u}_- = 0 \text{ in } \ \Omega^-, \\ \text{Tr}^+\mathbf{u}_+ - \text{Tr}^-\mathbf{u}_- = \mathbf{H}, \ \ \partial_\nu^+(\mathbf{u}_+,\pi_+) - \mu\partial_\nu^-(\mathbf{u}_-,\pi_-) = \mathbf{F} \ \text{ on } \ \partial\Omega, \\ \text{Tr}_\Gamma^+\mathbf{u}_- = \mathbf{G} \ \text{ on } \ \Gamma, \\ \nabla^k\mathbf{u}_+(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{2-n-k}), \ \ \pi_+(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-n}) \ \text{ as } \ |\mathbf{x}|\to\infty, \ k = 0,1, \end{cases} \tag{4.1.1}$$

where Tr^+, Tr^- are the trace operators acting on $\partial\Omega$, Tr_Γ^+ is the trace operator acting on Γ, $\partial_\nu^+(\mathbf{u}_+,\pi_+)$ and $\partial_\nu^-(\mathbf{u}_-,\pi_-)$ are the conormal derivatives associated to the pairs $(\mathbf{u}_+,\pi_+)$ and $(\mathbf{u}_-,\pi_-)$, respectively, and corresponding to $\partial\Omega$. In the three-dimensional case $(n = 3)$, the boundary value problem (4.1.1) describes the exterior Stokes flow of a viscous incompressible fluid in the presence of a porous particle that contains a solid core $(\mathfrak{D})$. In this case, $\chi := \frac{a}{\sqrt{\kappa}}$, where a is a characteristic length of the porous particle with permeability κ.

We now consider the following appropriate trace and solution spaces:

1. **Sobolev spaces**
(4.1.2)
$$\begin{aligned} &Y := Y_{\nu,1,\beta} := H_{\mathbf{n}_{\partial\Omega}}^{\frac{1}{2}+\beta}(\partial\Omega,\mathbb{R}^n) \times H^{-\frac{1}{2}+\beta}(\partial\Omega,\mathbb{R}^n) \times H_{\mu_\Gamma}^{-\frac{1}{2}+\beta}(\Gamma,\mathbb{R}^n), \\ &Y^{(1)} := Y_{\nu,1,\beta}^{(1)} := H_{\mathbf{n}_{\partial\Omega}}^{\frac{1}{2}+\beta}(\partial\Omega,\mathbb{R}^n) \times H^{-\frac{1}{2}+\beta}(\partial\Omega,\mathbb{R}^n) \times H_{\mathbf{n}_\Gamma}^{\frac{1}{2}+\beta}(\Gamma,\mathbb{R}^n), \\ &Z := \Big(H_{\text{loc}}^{1+\beta}(\overline{\Omega}_+,\mathbb{R}^n) \times H_{\text{loc}}^{\beta}(\overline{\Omega}_+)\Big) \times \left(H^{1+\beta}(\Omega^-,\mathbb{R}^n) \times H^\beta(\Omega^-)\right), \ \beta \in (-\tfrac{1}{2},\tfrac{1}{2}). \end{aligned}$$

2. **L^p spaces**
$$\begin{aligned} &Y := Y_{\nu,1,p} := L_{1,\mathbf{n}_{\partial\Omega}}^p(\partial\Omega,\mathbb{R}^n) \times L^p(\partial\Omega,\mathbb{R}^n) \times L_{\mu_\Gamma}^p(\Gamma,\mathbb{R}^n), \\ &Y^{(1)} := Y_{\nu,1,p}^{(1)} := L_{1,\mathbf{n}_{\partial\Omega}}^p(\partial\Omega,\mathbb{R}^n) \times L^p(\partial\Omega,\mathbb{R}^n) \times L_{1,\mathbf{n}_\Gamma}^p(\Gamma,\mathbb{R}^n), \\ &Z := \Big(C^2(\Omega_+,\mathbb{R}^n) \times C^1(\Omega_+)\Big) \times \Big(C^2(\Omega^-,\mathbb{R}^n) \times C^1(\Omega^-)\Big). \end{aligned} \tag{4.1.3}$$

for $p \in (2-\epsilon, 2+\epsilon)$, $n \geq 2$ and some $\epsilon = \epsilon(\Gamma) > 0$, which will be specified later on. In order to have a meaningful formulated problem in this case, we require the following non-tangential maximal conditions (see e.g., [79, Theorem 4.13])

$$\mathcal{N}_{\partial\Omega}(\nabla\mathbf{u}_\pm),\ \mathcal{N}_{\partial\Omega}(\pi_\pm) \in L^p(\partial\Omega),\ \mathcal{N}_\Gamma(\nabla\mathbf{u}_-),\ \mathcal{N}_\Gamma(\pi_-) \in L^p(\Gamma), \tag{4.1.4}$$

where $\mathcal{N}_{\partial\Omega}$ is the non-tangential maximal operator for $\partial\Omega$, while $\mathcal{N}_\Gamma$ is that corresponding to Γ. In addition, in this case, the conormal derivatives corresponding to $\partial\Omega$ are given by

$$\partial_\nu^\pm(\mathbf{u},\pi) := \Big(-\pi\mathbb{I} + \nabla\mathbf{u} + \nabla^\top\mathbf{u}\Big)\big|_{\partial\Omega^\pm}\mathbf{n}_{\partial\Omega} \text{ a.e. on } \partial\Omega \tag{4.1.5}$$

in the sense of nontangential limit.

The functions μ_Γ and $\mu_{\partial\Omega}$ are chosen such that[1] $\langle\mathbf{n}_\Gamma,\mu_\Gamma\rangle_\Gamma = 1$ and $\langle\mathbf{n}_{\partial\Omega},\mu_{\partial\Omega}\rangle_{\partial\Omega} = 1$. For any $s \in [0,1]$ and $p \in (1,\infty)$ we further define

$$\begin{aligned} &L_{s,\mathbf{n}_{\partial\Omega}}^p(\partial\Omega,\mathbb{R}^n) := \{\mathbf{f} \in L_s^p(\partial\Omega,\mathbb{R}^n) : \langle\mathbf{f},\mathbf{n}_{\partial\Omega}\rangle_{\partial\Omega} = 0\}, \\ &L_{-s,\mu_{\partial\Omega}}^p(\partial\Omega,\mathbb{R}^n) := \{\mathbf{g} \in L_{-s}^p(\partial\Omega,\mathbb{R}^n) : \langle\mathbf{g},\mu_{\partial\Omega}\rangle_{\partial\Omega} = 0\}, \end{aligned} \tag{4.1.6}$$

and the spaces $L_{s,\mathbf{n}_\Gamma}^p(\Gamma,\mathbb{R}^n)$, $L_{-s,\mu_\Gamma}^p(\Gamma,\mathbb{R}^n)$ are defined similarly. In each of the above cases, assume that $(\mathbf{H},\mathbf{F},\mathbf{G}) \in Y^{(1)}$.

[1]The notation ${}_{X^*}\langle\cdot,\cdot\rangle_X := \langle\cdot,\cdot\rangle_\Gamma$ refers to the duality pairing between two dual spaces X^* and X, defined with respect to Γ.

4.2 Uniqueness result for the boundary value problem $(4.1.1)$

Theorem 4.2.1 [31] *The boundary value problem* $(4.1.1)$ *has at most one solution* $((\mathbf{u}_+,\pi_+),(\mathbf{u}_-,\pi_-))$*, in the case of the Sobolev spaces given in* $(4.1.2)$ *with* $\beta \geq 0$*. The same uniqueness result holds for the problem* $(4.1.1)$, $(4.1.4)$ *in the case of* L^p *spaces given in* $(4.1.3)$ *with* $p \geq 2$*.*

4.3 Layer potential formulation of the problem

In either one of the cases (4.1.2), (4.1.3), we will show the existence of solutions to the boundary value problem $(4.1.1)$, by the use of the layer potential representations:

$$(4.3.1)\quad \begin{array}{l}\mathbf{u}_+ = \mathbf{W}_{\partial\Omega}\mathbf{h} + \mathbf{V}_{\partial\Omega}\mathbf{f},\\ \pi_+ = P^d_{\partial\Omega}\mathbf{h} + P^s_{\partial\Omega}\mathbf{f},\end{array} \text{ in } \Omega_+,\ \begin{array}{l}\mathbf{u}_- := \mathbf{W}_{\chi^2,\partial\Omega}\mathbf{h} + \mathbf{V}_{\chi^2,\partial\Omega}\mathbf{f} + \mathbf{V}_{\chi^2,\Gamma}\mathbf{g},\\ \pi_- = P^d_{\chi^2,\partial\Omega}\mathbf{h} + P^s_{\chi^2,\partial\Omega}\mathbf{f} + P^s_{\chi^2,\Gamma}\mathbf{g},\end{array} \text{ in } \Omega^-,$$

with the unknown densities $(\mathbf{h},\mathbf{f},\mathbf{g}) \in Y$. Thus, we determine the unknown vector field $\mathbf{u}_+$ as a combination of a double-layer potential $\mathbf{W}_{\partial\Omega}\mathbf{h}$ and a single-layer potential $\mathbf{V}_{\partial\Omega}\mathbf{f}$, each of them satisfying the Stokes and continuity equations in (4.1.1). Similarly, $\mathbf{u}_-$ is determined as a combination of a double-layer potential $\mathbf{W}_{\chi^2,\partial\Omega}\mathbf{h}$ and two single-layer potentials, $\mathbf{V}_{\chi^2,\partial\Omega}\mathbf{f}$ and $\mathbf{V}_{\chi^2,\Gamma}\mathbf{g}$, each of them satisfying the Brinkman and continuity equations in (4.1.1).

4.3.1 Boundary integral equations due to the layer potential formulation

Next, we refer to the case $\mu \in (0,1)$. In this case, we obtain the matrix type equation

$$(4.3.2)\qquad \mathbb{M}_{\chi^2,0}\,(\mathbf{h},\mathbf{f},\mathbf{g})^\top = (\mathbf{H},\mathbf{F},\mathbf{G})^\top\,, \text{ where } \mathbb{M}_{\chi^2,0} : Y \to Y^{(1)},$$

$$\mathbb{M}_{\chi^2,0} := \begin{pmatrix} \mathbb{I} - \mathbf{K}_{\chi^2,0,\partial\Omega} & -\mathcal{V}_{\chi^2,0,\partial\Omega} & -\mathcal{V}_{\chi^2,\Gamma,\partial\Omega} \\ (1-\mu)\mathbf{D}_{\partial\Omega} - \mu\mathbf{D}_{\chi^2,0,\partial\Omega} & (1-\mu)\mathcal{K}^*_{\mu;\partial\Omega} - \mu\mathbf{K}^*_{\chi^2,0,\partial\Omega} & -\mu\mathbf{K}^*_{\chi^2,\Gamma,\partial\Omega} \\ \mathbf{K}^*_{\chi^2,\partial\Omega,\Gamma} & \mathcal{V}_{\chi^2,\partial\Omega,\Gamma} & \mathcal{V}_\Gamma + \mathcal{V}_{\chi^2,0,\Gamma} \end{pmatrix},$$

and $\mathcal{K}^*_{\mu;\partial\Omega} := -\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}^*_{\partial\Omega}$. By Theorem 3.5.3, the operator $\mathbb{P} : Y \to Y^{(1)}$ is compact. In addition, in view of the invertibility of $-\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}^*_{\partial\Omega} : H^{-\frac{1}{2}+\beta}(\partial\Omega,\mathbb{R}^n) \to H^{-\frac{1}{2}+\beta}(\partial\Omega,\mathbb{R}^n)$ and $\mathcal{V}_\Gamma : H^{-\frac{1}{2}+\beta}_{\mu_\Gamma}(\Gamma,\mathbb{R}^n) \to H^{\frac{1}{2}+\beta}_{\mathbf{n}_\Gamma}(\Gamma,\mathbb{R}^n)$ for any $\beta \in \left(-\frac{1}{2},\frac{1}{2}\right)$ (see [79]), one finds that $\mathbb{M}_0 : Y \to Y^{(1)}$ is invertible, when Y and $Y^{(1)}$ are given by (4.1.2). A similar invertibility result holds in the case (4.1.3).

4.3.2 The invertibility of the operator $\mathbb{M}_{\chi^2,0}$

• First, we prove the invertibility of the operator $\mathbb{M}_{\chi^2,0} : Y_{\nu,1} \to Y^{(1)}_{\nu,1}$ when the spaces $Y_{\nu,1}$ and $Y^{(1)}_{\nu,1}$ are defined in (4.1.2). As we mentioned before, $\mathbb{P} : Y_{\nu,1} \to Y^{(1)}_{\nu,1}$ is compact and $\mathbb{M}_0 : Y_{\nu,1} \to Y^{(1)}_{\nu,1}$ is Fredholm with index zero. Thus, the operator

$$(4.3.3)\qquad \mathbb{M}_{\chi^2,0} := \mathbb{M}_0 + \mathbb{P} : Y_{\nu,1} \to Y^{(1)}_{\nu,1}$$

is Fredholm with index zero, too. It is also injective. Consequently, in the case (4.1.2), we have the following existence and uniqueness result:

$$\forall\,(\mathbf{H},\mathbf{F},\mathbf{G})^\top \in Y^{(1)},\ \ \exists!\,(\mathbf{h},\mathbf{f},\mathbf{g})^\top \in Y \ \text{ such that }\ \mathbb{M}_{\chi^2,0}\,(\mathbf{h},\mathbf{f},\mathbf{g})^\top = (\mathbf{H},\mathbf{F},\mathbf{G})^\top\,.$$

In addition, the layer potential representations (4.3.1) obtained with the densities $(\mathbf{h}, \mathbf{f}, \mathbf{g})^T$ determine the unique solution $((\mathbf{u}_+, \pi_+), (\mathbf{u}_-, \pi_-)) \in Z$ of the problem (4.1.1).

• The operator $\mathbb{M}_{\chi^2,0} : Y \to Y^{(1)}$ is invertible when the spaces Y, $Y^{(1)}$ and Z are given by (4.1.2) with $\beta > 0$, or (4.1.3). In the same case (4.1.3), with $p \geq 2$, the layer potentials (4.3.1) determine the unique solution to the boundary value problem (4.1.1), which satisfies the conditions (4.1.4) and the estimate (see [78, Theorem 3.1], [79]):

$$\begin{aligned} \|\mathcal{N}_{\partial\Omega}(\nabla \mathbf{u}_+)\|_{L^p(\partial\Omega)} + \|\mathcal{N}_{\partial\Omega}(\pi_+)\|_{L^p(\partial\Omega)} + \|\mathcal{N}_{\partial\Omega}(\nabla \mathbf{u}_-)\|_{L^p(\partial\Omega)} + \|\mathcal{N}_{\partial\Omega}(\pi_-)\|_{L^p(\partial\Omega)} \\ + \|\mathcal{N}_{\Gamma}(\nabla \mathbf{u}_-)\|_{L^p(\Gamma)} + \|\mathcal{N}_{\Gamma}(\pi_-)\|_{L^p(\Gamma)} \leq C \, \|(\mathbf{H}, \mathbf{F}, \mathbf{G})\|_Y , \end{aligned} \tag{4.3.4}$$

with some constant $C > 0$ independent of $\mathbf{H}$, $\mathbf{F}$ and $\mathbf{G}$. Consequently, one obtains:

Theorem 4.3.1 [31] *Let $\Omega, \mathfrak{D} \subset \mathbb{R}^n, n \geq 3$, be bounded Lipschitz domains with connected boundaries $\partial\Omega$ and Γ, respectively, such that $\overline{\mathfrak{D}} \subset \Omega$. Also, let $\Omega^- := \Omega \setminus \overline{\mathfrak{D}}$, and $\Omega_+ := \mathbb{R}^n \setminus \overline{\Omega}$. For given $\mu \in (0,1)$ and $\lambda > 0$, consider the Dirichlet-transmission problem* (4.1.1). *Then:*

(a) For any $\beta \in (-\frac{1}{2}, \frac{1}{2})$ and the boundary data $(\mathbf{H}, \mathbf{F}, \mathbf{G})^\top \in Y^{(1)}$, the equation (4.3.2) *has a unique solution $(\mathbf{h}, \mathbf{f}, \mathbf{g})^\top \in Y$, where Y is the space given in* (4.1.2).

(b) For $p > 1$ there exists $\epsilon = \epsilon(\Gamma) > 0$ such that the equation (4.3.2) *has a unique solution $(\mathbf{h}, \mathbf{f}, \mathbf{g})^\top \in Y$, where Y is the space given in* (4.1.3).

The densities $\mathbf{h}, \mathbf{f}, \mathbf{g}$ and the layer potential representations (4.3.1) *determine a solution $\big((\mathbf{u}_+, \pi_+), (\mathbf{u}_-, \pi_-)\big) \in Z$ to the Dirichlet-transmission problem* (4.1.1), *where Z is the space described in* (4.1.2). *The same existence result holds for the problem* (4.1.1), (4.1.4), *where the space Z is described in* (4.1.3). *In the first case* (4.1.2) *with $\beta \geq 0$, the solution is unique. In the case* (4.1.3) *with $p \geq 2$, the solution of the problem* (4.1.1), (4.1.4) *is also unique and satisfies the estimate* (4.3.4).

4.3.3 The case $\mu = 1$

The matrix equation (4.3.2) has in this case the form:

$$\mathbb{M}_{\chi^2,0} \, (\mathbf{h}, \mathbf{f}, \mathbf{g})^\top = (\mathbf{H}, \mathbf{F}, \mathbf{G})^\top , \tag{4.3.5}$$

where

$$\mathbb{M}_{\chi^2,0} := \begin{pmatrix} \mathbb{I} - \mathbf{K}_{\chi^2,0,\partial\Omega} & -\mathcal{V}_{\chi^2,0,\partial\Omega} & -\mathcal{V}_{\chi^2,\Gamma,\partial\Omega} \\ -\mathbf{D}_{\chi^2,0,\partial\Omega} & -\mathbf{K}^*_{\chi^2,0,\partial\Omega} & -\mathbf{K}^*_{\chi^2,\Gamma,\partial\Omega} \\ \mathbf{K}^*_{\chi^2,\partial\Omega,\Gamma} & \mathcal{V}_{\chi^2,\partial\Omega,\Gamma} & \mathcal{V}_{\Gamma} + \mathcal{V}_{\chi^2,0,\Gamma} \end{pmatrix} : Y \to Y^{(1)}. \tag{4.3.6}$$

Theorem 4.3.2 [31] *Let $\Omega, \mathfrak{D} \subset \mathbb{R}^n, n \geq 3$, be bounded Lipschitz domains with connected boundaries $\partial\Omega$ and Γ, respectively, such that $\overline{\mathfrak{D}} \subset \Omega$. Also, let $\Omega^- := \Omega \setminus \overline{\mathfrak{D}}$, and $\Omega_+ := \mathbb{R}^n \setminus \overline{\Omega}$. For given $\lambda > 0$, consider the problem* (4.1.1), *with $\mu = 1$ and the data $(\mathbf{H}, \mathbf{F}, \mathbf{G})^\top \in Y^{(1)}$, where Y and $Y^{(1)}$ are the spaces given in relations* (4.1.2) *or* (4.1.3). *Then:*

(a) For any $\beta \in (-\frac{1}{2}, \frac{1}{2})$ and the boundary data $(\mathbf{H}, \mathbf{F}, \mathbf{G})^\top \in Y^{(1)}$, the equation (4.3.5) *has a unique solution $(\mathbf{h}, \mathbf{f}, \mathbf{g})^\top \in Y$, where Y is the space given in* (4.1.2).

(b) For $p > 1$ there exists $\epsilon = \epsilon(\Gamma) > 0$ such that the equation (4.3.5) *has a unique solution $(\mathbf{h}, \mathbf{f}, \mathbf{g})^\top \in Y$, where Y is the space given in* (4.1.3).

The densities $\mathbf{h}, \mathbf{f}, \mathbf{g}$ and the layer potential representations (4.3.1) *determine a solution $\big((\mathbf{u}_+, \pi_+), (\mathbf{u}_-, \pi_-)\big) \in Z$ to the Dirichlet-transmission problem* (4.1.1) *with $\mu = 1$, where Z is the space described in* (4.1.2). *The same existence result holds for the problem* (4.1.1), (4.1.4), *with $\mu = 1$ and the space Z defined in* (4.1.3). *In the first case* (4.1.2) *with $\beta \geq 0$, the solution is unique. In the case* (4.1.3) *with $p \geq 2$, the solution of the problem* (4.1.1), (4.1.4) *is also unique and satisfies the estimate* (4.3.4).

4.3.4 Uniqueness result in the particular case when Γ is missing and $\chi = 0$

4.4 Stokes flow past a porous body with a solid core inside

In this section we refer again to the boundary value problem (4.1.1) for $n = 3$ and in two special cases. The first case describes a Stokes flow past a porous body with a solid core inside and with a large permeability, and the second one correspond to a similar flow past a porous body with low permeability. Note that the problem analyzed in the first case has been studied by e.g., Srivastava and Srivastava [103], but only for spherical geometry of the involved domains. We treat this problem in a more general setting of Lipschitz domains.

4.4.1 Stokes flow past a porous body with large permeability and a solid core inside

Let us now assume that $\Omega, \mathfrak{D} \subset \mathbb{R}^3$ are bounded Lipschitz domains such that $\overline{\mathfrak{D}} \subset \Omega$, and let Γ be the boundary of $\mathfrak{D}$. Also, let $\Omega^- := \Omega \setminus \overline{\mathfrak{D}}$, and $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega}$. Denote by $\partial\Omega$ the boundary of Ω. Also, for $\mu = 1$ consider the boundary value problem (4.1.1) in $\mathbb{R}^3$, with the far field conditions

$$\nabla^k(\mathbf{u}_+(\mathbf{x}) - \mathbf{U}_\infty) = \mathcal{O}(|\mathbf{x}|^{-1-k}),\ \ \pi_+(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2}) \text{ as } |\mathbf{x}| \to \infty. \tag{4.4.1}$$

This problem describes the Stokes flow of a viscous incompressible fluid past a porous medium in the presence of a fixed solid core $\mathfrak{D}$. In addition, the flow at infinity is uniform with a constant velocity field $\mathbf{U}_\infty$ and a constant pressure p_∞. For simplicity, choose $p_\infty = 0$. By the relations

$$\mathbf{u}_+ = \mathbf{U}_\infty + \mathbf{v}_+,\ \pi_+ = q_+ \text{ in } \Omega_+,\ \mathbf{u}_- = \mathbf{v}_-,\ \pi_- = q_- \text{ in } \Omega^-, \tag{4.4.2}$$

the above mentioned problem reduces to the non-homogenous Dirichlet transmission problem

$$\begin{cases} \operatorname{div} \mathbf{v}_+ = 0,\ -\nabla q_+ + \Delta \mathbf{v}_+ = 0 \text{ in } \Omega_+ \\ \operatorname{div} \mathbf{v}_- = 0,\ -\nabla q_- + (\Delta - \chi^2\mathbb{I})\mathbf{v}_- = 0 \text{ in } \Omega^- \\ \mathrm{Tr}^+\mathbf{v}_+ - \mathrm{Tr}^-\mathbf{v}_- = -\mathbf{U}_\infty \in H_\nu^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3), \\ \partial_\nu^+(\mathbf{v}_+, q_+) - \partial_\nu^-(\mathbf{v}_-, q_-) = 0 \text{ on } \partial\Omega \\ \mathrm{Tr}^+\mathbf{v}_- = 0 \text{ on } \Gamma \\ \nabla^s\mathbf{v}_+(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1-s}),\ q_+(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2}) \text{ as } |\mathbf{x}| \to \infty,\ \ s = 0, 1. \end{cases} \tag{4.4.3}$$

In view of Theorem 4.3.2, it follows that (4.4.3) has a unique solution $((\mathbf{v}_+, q_+), (\mathbf{v}_-, q_-)) \in (H^1_{\mathrm{loc}}(\overline{\Omega}_+, \mathbb{R}^3) \times L^2_{\mathrm{loc}}(\overline{\Omega}_+)) \times (H^1(\Omega^-, \mathbb{R}^3) \times L^2(\Omega^-))$.

Next, assume that the porous particle has large permeability κ, i.e., $\chi \ll 1$, where $\chi := \frac{a}{\sqrt{\kappa}}$, a is a characteristic length of the particle, and let the formal expansions (with respect to small χ):

$$\mathbf{u}_\pm = \mathbf{u}_\pm^{(0)} + \chi\mathbf{u}_\pm^{(1)} + \chi^2\mathbf{u}_\pm^{(2)} + \dots,\ \pi_\pm = \pi_\pm^{(0)} + \chi\pi_\pm^{(1)} + \chi^2\pi_\pm^{(2)} + \dots \tag{4.4.4}$$

By substituting these expansions into the equations and the boundary conditions of (4.4.3), and collecting the k^{th}-order terms, $k = 0, 1, 2$, with respect to (small) χ, we obtain:

$$\begin{cases} \operatorname{div} \mathbf{u}_+^{(k)} = 0, -\nabla\pi_+^{(k)} + \Delta\mathbf{u}_+^{(k)} = \mathbf{0} \text{ in } \Omega_+ \\ \operatorname{div} \mathbf{u}_-^{(k)} = 0, -\nabla\pi_-^{(k)} + \Delta\mathbf{u}_-^{(k)} = \mathbf{u}^{(k)} \text{ in } \Omega^- \\ \mathrm{Tr}^+\mathbf{u}_+^{(k)} = \mathrm{Tr}^-\mathbf{u}_-^{(k)} \text{ on } \partial\Omega \\ \partial_\nu^+(\mathbf{u}_+^{(k)}, \pi_+^{(k)}) = \partial_\nu^-(\mathbf{u}_-^{(0)}, \pi_-^{(k)}) \text{ on } \partial\Omega \\ \mathrm{Tr}^+\mathbf{u}_-^{(k)} = 0 \text{ on } \Gamma \\ \nabla^s(\mathbf{u}_+^{(k)} - \mathbf{U}_\infty)(\mathbf{x}) = O(|\mathbf{x}|^{-1-s}) \text{ as } |\mathbf{x}| \to \infty,\ s = 0, 1 \end{cases},\ k = 0, 1, 2, \tag{4.4.5}$$

where $\mathbf{u}^{(k)} = \begin{cases} \mathbf{0}, & k = 0, 1 \\ \mathbf{u}_-^{(0)}, & k = 2. \end{cases}$ We now refer to the case $k = 0$. By Theorem 4.2.1, the 0^{th} order boundary value problem (4.4.5) has at most one solution. To show the existence of this solution, we consider the representations

$$\begin{aligned} \mathbf{u}_+^{(0)} &= \mathbf{U}_\infty - \mathbf{V}_\Gamma \mathbf{f},\ \pi_+^{(0)} = -Q_\Gamma^s \mathbf{f} \text{ in } \Omega_+, \\ \mathbf{u}_-^{(0)} &= \mathbf{U}_\infty - \mathbf{V}_\Gamma \mathbf{f},\ \pi_-^{(0)} = -Q_\Gamma^s \mathbf{f} \text{ in } \Omega^-, \end{aligned} \tag{4.4.6}$$

where $\mathbf{f} \in H_{\mu_\Gamma}^{-\frac{1}{2}}(\Gamma, \mathbb{R}^3)$ is an unknown density and $\mu_\Gamma \in H^{\frac{1}{2}}(\Gamma, \mathbb{R}^3)$ is chosen such that $\langle \mu_\Gamma, \mathbf{n}_\Gamma \rangle_\Gamma = 1$. These representations satisfy the equations, the transmission and far field conditions in (4.4.5), corresponding to $k = 0$. By imposing the Dirichlet condition on Γ, and going non-tangentially to the boundary in (4.4.6), one obtains the following equation with unknown $\mathbf{f}$:

$$\mathcal{V}_\Gamma \mathbf{f} = \mathbf{U}_\infty \text{ on } \Gamma. \tag{4.4.7}$$

Since the single-layer potential operator $\mathcal{V}_\Gamma : H_{\mu_\Gamma}^{-\frac{1}{2}}(\Gamma, \mathbb{R}^3) \to H_{\mathbf{n}_\Gamma}^{\frac{1}{2}}(\Gamma, \mathbb{R}^3)$ is invertible (see Theorem 3.5.2; see also [78, Theorem 6.1], [79]) and $\mathbf{U}_\infty \in H_{\mathbf{n}_\Gamma}^{\frac{1}{2}}(\Gamma, \mathbb{R}^3)$, we conclude the equation (4.4.7) has a unique solution $\mathbf{f} \in H_{\mu_\Gamma}^{-\frac{1}{2}}(\Gamma, \mathbb{R}^3)$. Thus, the representations (4.4.6) provide the unique solution $((\mathbf{u}_+^{(0)}, \pi_+^{(0)}), (\mathbf{u}_-^{(0)}, \pi_-^{(0)}))$ of the problem (4.4.5) for $k = 0$. Similarly,

$$((\mathbf{u}_+^{(1)}, \pi_+^{(1)}), (\mathbf{u}_-^{(1)}, \pi_-^{(1)})) = ((0, 0), (0, 0)). \tag{4.4.8}$$

Next, we refer to the boundary value problem (4.4.5) for $k = 2$, and show that it is also uniquely solvable. Note that the representations (4.4.6) yield the non-homogeneous system

$$\text{div } \mathbf{u}_-^{(2)} = 0,\ -\nabla \pi_-^{(2)} + \Delta \mathbf{u}_-^{(2)} = \mathbf{U}_\infty - \mathbf{V}_\Gamma \mathbf{f} \ \text{ in } \ \Omega^-, \tag{4.4.9}$$

where $\mathbf{f} \in H_{\mu_\Gamma}^{-\frac{1}{2}}(\Gamma, \mathbb{R}^3)$ is the unique solution of the equation (4.4.7). We then determine the corresponding solution $((\mathbf{u}_+^{(2)}, \pi_+^{(2)}), (\mathbf{u}_-^{(2)}, \pi_-^{(2)}))$ in the form

$$\begin{aligned} \mathbf{u}_+^{(2)} &= -\int_{\Omega^-} \mathcal{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma \mathbf{f})(\mathbf{y}) d\mathbf{y} + \mathbf{u}_{+,0}^{(2)}(\mathbf{x}),\ \mathbf{x} \in \Omega_+ \\ \pi_+^{(2)} &= -\int_{\Omega^-} \Pi(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma \mathbf{f})(\mathbf{y}) d\mathbf{y} + \pi_{+,0}^{(2)}(\mathbf{x}),\ \mathbf{x} \in \Omega_+, \end{aligned} \tag{4.4.10}$$

and

$$\begin{aligned} \mathbf{u}_-^{(2)} &= -\int_{\Omega^-} \mathcal{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma \mathbf{f})(\mathbf{y}) d\mathbf{y} + \mathbf{u}_{-,0}^{(2)}(\mathbf{x}),\ \mathbf{x} \in \Omega^- \\ \pi_-^{(2)} &= -\int_{\Omega^-} \Pi(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma \mathbf{f})(\mathbf{y}) d\mathbf{y} + \pi_{-,0}^{(2)}(\mathbf{x}),\ \mathbf{x} \in \Omega^-, \end{aligned} \tag{4.4.11}$$

where the integrals over Ω^- are Newtonian potentials. Therefore, we get

$$\begin{cases} \text{div } \mathbf{u}_{+,0}^{(2)} = 0, -\nabla \pi_{+,0}^{(2)} + \Delta \mathbf{u}_{+,0}^{(2)} = \mathbf{0} \ \text{ in } \ \Omega_+ \\ \text{div } \mathbf{u}_{-,0}^{(2)} = 0, -\nabla \pi_{-,0}^{(2)} + \Delta \mathbf{u}_{-,0}^{(2)} = \mathbf{0} \ \text{ in } \ \Omega^- \\ \text{Tr}^+ \mathbf{u}_{+,0}^{(2)} = \text{Tr}^- \mathbf{u}_{-,0}^{(2)} \text{ on } \partial\Omega \\ \partial_\nu^+ (\mathbf{u}_{+,0}^{(2)}, \pi_{+,0}^{(2)}) = \partial_\nu^- (\mathbf{u}_{-,0}^{(2)}, \pi_{-,0}^{(2)}) \text{ on } \partial\Omega \\ \text{Tr}^+ \mathbf{u}_{-,0}^{(2)} = \int_{\Omega^-} \mathcal{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma \mathbf{f})(\mathbf{y}) d\mathbf{y} \text{ on } \Gamma \\ \nabla^s \mathbf{u}_{+,0}^{(2)}(\mathbf{x}) = 0 \text{ as } |\mathbf{x}| \to \infty,\ s = 0, 1. \end{cases} \tag{4.4.12}$$

In order to analyze the problem (4.4.12), we next consider the auxiliary Dirichlet problem:

$$\text{(4.4.13)} \qquad \begin{cases} \operatorname{div} \mathbf{u}_0 = 0, -\nabla\pi_0 + \Delta\mathbf{u}_0 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\mathfrak{D}} \\ \mathrm{Tr}^+\mathbf{u}_0 = \int_{\Omega^-} \mathcal{G}(\cdot,\mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f})(\mathbf{y})d\mathbf{y} \text{ on } \Gamma \\ \nabla^s\mathbf{u}_0(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1-s}) \text{ as } |\mathbf{x}| \to \infty,\ s = 0, 1. \end{cases}$$

By the uniqueness of the solution to the exterior Dirichlet problem for the Stokes system (see e.g., [79, Theorem 10.15]), one finds that there exists a unique solution $(\mathbf{u}_0, \pi_0) \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \mathfrak{D}, \mathbb{R}^3) \times L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \mathfrak{D})$ to the boundary value problem (4.4.13). This solution is given by the layer potentials

$$\text{(4.4.14)} \qquad \mathbf{u}_0 = \mathbf{V}_\Gamma(\mathcal{V}_\Gamma^{-1}\mathcal{N}_{\Omega^-}),\ \ \pi_0 = Q^s_\Gamma(\mathcal{V}_\Gamma^{-1}\mathcal{N}_{\Omega^-}) \text{ in } \mathbb{R}^3 \setminus \overline{\mathfrak{D}}.$$

The pairs $(\mathbf{u}^{(2)}_{+,0}, \pi^{(2)}_{+,0}) \in (H^1_{\mathrm{loc}}(\overline{\Omega}_+, \mathbb{R}^3) \times L^2_{\mathrm{loc}}(\overline{\Omega}_+))$, $(\mathbf{u}^{(2)}_{-,0}, \pi^{(2)}_{-,0}) \in (H^1(\Omega^-, \mathbb{R}^3) \times L^2(\Omega^-))$,

$$\text{(4.4.15)} \qquad \begin{array}{l} \mathbf{u}^{(2)}_{+,0} := \mathbf{u}_0|_{\Omega_+}, \pi^{(2)}_{+,0} := \pi_0|_{\Omega_+} \text{ in } \Omega_+ \\ \mathbf{u}^{(2)}_{-,0} := \mathbf{u}_0|_{\Omega^-}, \pi^{(2)}_{-,0} := \pi_0|_{\Omega^-} \text{ in } \Omega^- \end{array}$$

determine the unique solution of the boundary value problem (4.4.12).

By using again Theorem 4.2.1, we conclude that the unique solution of the boundary value problem (4.4.5) corresponding to $k = 2$ is given by (4.4.10), (4.4.11), (4.4.14) and (4.4.15), i.e.,

$$\text{(4.4.16)} \qquad \begin{aligned} \mathbf{u}^{(2)}_+ &= -\int_{\Omega^-} \mathcal{G}(\mathbf{x},\mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f})(\mathbf{y})d\mathbf{y} + \mathbf{V}_\Gamma(\mathcal{V}_\Gamma^{-1}\mathcal{N}_{\Omega^-}),\ \mathbf{x} \in \Omega_+ \\ \pi^{(2)}_+ &= -\int_{\Omega^-} \Pi(\mathbf{x},\mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f})(\mathbf{y})d\mathbf{y} + Q^s_\Gamma(\mathcal{V}_\Gamma^{-1}\mathcal{N}_{\Omega^-}),\ \ \mathbf{x} \in \Omega_+, \\ \mathbf{u}^{(2)}_- &= -\int_{\Omega^-} \mathcal{G}(\mathbf{x},\mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f})(\mathbf{y})d\mathbf{y} + \mathbf{V}_\Gamma(\mathcal{V}_\Gamma^{-1}\mathcal{N}_{\Omega^-}),\ \mathbf{x} \in \Omega^- \\ \pi^{(2)}_- &= -\int_{\Omega^-} \Pi(\mathbf{x},\mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f})(\mathbf{y})d\mathbf{y} + Q^s_\Gamma(\mathcal{V}_\Gamma^{-1}\mathcal{N}_{\Omega^-}),\ \ \mathbf{x} \in \Omega^- \end{aligned}$$

where $\mathbf{f} \in H^{-\frac{1}{2}}_{\mu_\Gamma}(\Gamma, \mathbb{R}^3)$ is the unique solution of the equation (4.4.7). Also, we get

$$\text{(4.4.17)} \qquad \big(\mathbf{u}^{(2l+1)}_+, \pi^{(2l+1)}_+\big) = (\mathbf{0}, 0) \text{ in } \Omega_+,\ \big(\mathbf{u}^{(2l+1)}_-, \pi^{(2l+1)}_-\big) = (\mathbf{0}, 0) \text{ in } \Omega^-,\ \forall\, l \geq 0.$$

Now, from (4.4.6), (4.4.8), (4.4.16) and (4.4.17) we obtain the following expansion of the inner velocity field $\mathbf{u}_-$ with respect to small χ, up to the order $\mathcal{O}(\chi^4)$:

(4.4.18)

$$\mathbf{u}_- = (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f}) - \chi^2 \int_{\Omega^-} \mathcal{G}(\cdot,\mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f})(\mathbf{y})d\mathbf{y} + \chi^2\mathbf{V}_\Gamma(\mathcal{V}_\Gamma^{-1}\mathcal{N}_{\Omega^-}) + \mathcal{O}(\chi^4).$$

4.4.2 The force exerted by the Stokes flow on the porous particle

By using the expansion (4.4.18), one obtains the following asymptotic formula for the non-dimensional force $\mathbf{F}$ exerted by the Stokes flow on the porous particle:

$$\begin{aligned} \mathbf{F} = \int_{\partial\Omega} \partial^+_\nu(\mathbf{u}_+, \pi_+)d\sigma &= \chi^2 \int_{\Omega^-} (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f})d\mathbf{x} - \chi^4 \int_{\Omega^-}\int_{\Omega^-} \mathcal{G}(\mathbf{x},\mathbf{y}) \cdot (\mathbf{U}_\infty - \mathbf{V}_\Gamma\mathbf{f})(\mathbf{y})d\mathbf{x}d\mathbf{y} \\ &+ \chi^4 \int_{\Omega^-} \mathbf{V}_\Gamma(\mathcal{V}_\Gamma^{-1}\mathcal{N}_{\Omega^-})d\mathbf{x} + \int_\Gamma \partial^+_\nu(\mathbf{u}_-, \pi_-)d\Gamma + \mathcal{O}(\chi^6). \end{aligned} \qquad \text{(4.4.19)}$$

In the absence of the solid core, i.e., when $\Omega^- = \Omega := \Omega_-$, the formulas (4.4.18) and (4.4.19) become (see [60, (139),(147)]):

$$\mathbf{u}_- = \mathbf{U}_\infty - \chi^2 \int_{\Omega_-} \mathcal{G}(\cdot, \mathbf{y}) \cdot \mathbf{U}_\infty d\mathbf{y} + \mathcal{O}(\chi^4),$$

$$F_k = \chi^2 |\Omega_-| U_{\infty,k} - \chi^4 U_{\infty,k} \int_{\partial\Omega} \int_{\partial\Omega} \left(-\delta_{jk} n_l(\mathbf{x}) n_l(\mathbf{y}) + n_k(\mathbf{x}) n_j(\mathbf{y})\right) r d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) + \mathcal{O}\left(\chi^6\right),$$

where $|\Omega_-|$ is the volume of Ω_-.

4.4.3 Stokes flow past a porous body with low permeability and a solid core inside

Next, assume that $\chi \gg 1$, i.e., $\frac{1}{\chi^2} = \frac{\kappa}{a^2} \ll 1$, where κ is the permeability of the porous body with the characteristic length a. Also, consider $\mu = 1$ and the following equivalent form the boundary value problem (4.1.1):

(4.4.20)
$$\begin{cases} \operatorname{div} \mathbf{u}_+ = 0, \ -\nabla \pi_+ + \Delta \mathbf{u}_+ = 0 \text{ in } \Omega_+ \\ \operatorname{div} \mathbf{u}_- = 0, \ -\frac{1}{\chi^2} \nabla \pi_- + (\frac{1}{\chi^2} \Delta - \mathbb{I}) \mathbf{u}_- = 0 \text{ in } \Omega^- \\ \operatorname{Tr}^+ \mathbf{u}_+ - \operatorname{Tr}^- \mathbf{u}_- = 0, \ \partial_\nu^+ (\mathbf{u}_+, \pi_+) - \partial_\nu^- (\mathbf{u}_-, \pi_-) = 0 \text{ on } \partial\Omega \\ \operatorname{Tr}^+ \mathbf{u}_+ = 0 \text{ on } \Gamma \\ \nabla^s (\mathbf{u}_+ - \mathbf{U}_\infty)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1-s}), \ \pi_+(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2}) \text{ as } |\mathbf{x}| \to \infty. \end{cases}$$

To this problem we associate the formal expansions (with respect to small χ^{-2}):

(4.4.21)
$$\mathbf{u}_\pm = \tilde{\mathbf{u}}_\pm^{(0)} + \frac{1}{\chi^2} \tilde{\mathbf{u}}_\pm^{(1)} + \frac{1}{\chi^4} \tilde{\mathbf{u}}_\pm^{(2)} + \ldots, \ \pi_\pm = \tilde{\pi}_\pm^{(0)} + \frac{1}{\chi^2} \tilde{\pi}_\pm^{(1)} + \frac{1}{\chi^4} \tilde{\pi}_\pm^{(2)} + \ldots$$

Substituting them the equations and conditions of the boundary value problem (4.4.20), and collecting the i^{th} order terms ($i = 0, 1, 2$), with respect to χ^{-2} we obtain:
(4.4.22)
$$\begin{cases} \tilde{\mathbf{u}}_-^{(0)} = 0 \ \text{ in } \ \Omega^- \\ -\nabla \tilde{\pi}_-^{(0)} + \Delta \tilde{\mathbf{u}}_-^{(0)} = \tilde{\mathbf{u}}_-^{(1)} \ \text{ in } \ \Omega^- \\ \operatorname{div} \tilde{\mathbf{u}}_-^{(1)} = 0 \text{ in } \Omega^-, \end{cases} \quad \begin{cases} -\nabla \pi_+^{(j)} + \Delta \tilde{\mathbf{u}}_+^{(j)} = 0 \ \text{ in } \ \Omega_+ \\ \operatorname{div} \tilde{\mathbf{u}}_+^{(j)} = 0 \ \text{ in } \ \Omega_+, \ j \geq 0 \\ \tilde{\mathbf{u}}_+^{(0)}(\mathbf{x}) \to \mathbf{U}_\infty \text{ and } \tilde{\mathbf{u}}_+^{(j)}(\mathbf{x}) \to \mathbf{0} \ |\mathbf{x}| \to \infty, j = 1, 2, \ldots \end{cases}$$

Thus, we have obtained a singular perturbation problem. Now, taking into account the first relation in (4.4.22), one finds that, at the leading order, the velocity field of the inner flow $\mathbf{u}_-^0$ is equal to zero in Ω^-, and hence, at this order, the exterior Stokes flow past the stationary porous body with a solid core inside may be viewed as the Stokes flow past only a stationary solid body with the same geometry (as $\Omega^- \cup \overline{\mathfrak{D}}$).

Chapter 5

Robin-transmission problems for Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$

This chapter is devoted to boundary value problems of Robin-transmission type for the Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$ $(n \geq 3)$, when the given boundary datum belong to some Sobolev spaces. The main sources used in the preparation of this chapter are [29], [30], [36], [57].

The study of Robin-transmission problems for Stokes and Brinkman systems is motivated by the fact that the boundary conditions, which should be imposed at the fluid-porous interface between a homogeneous porous medium governed by the Brinkman equation and a viscous fluid, require a jump of shear stress and the continuity of the velocity and normal stress. This jump condition, which is a Robin-transmission type condition, has been derived by Ochoa-Tapia and Whitaker [84], [85], by using the volume averaging techniques. It has been constructed to join the Darcy law with the Brinkman equation (i.e., the zero order perturbation of the Stokes equation), and replaces the usual stress continuity condition at the fluid-porous interface (for other physical details we refer to [90]). The Ochoa-Tapia and Whitaker conditions at a porous-fluid interface Σ have the form [84], [85] (see also [4])

$$\left(\mu\nabla\mathbf{v}^f \cdot \mathbf{n} - \frac{\mu}{\phi}\nabla\mathbf{v}^p \cdot \mathbf{n}\right)\Big|_\Sigma \cdot \tau = \frac{\mu\beta}{\sqrt{\kappa}}\mathbf{v}_\Sigma \cdot \tau,\ \mathbf{v}^f = \mathbf{v}^p = \mathbf{v}_\Sigma, \tag{5.0.1}$$

where the superscripts f and p refer to the fluid and porous region, respectively, μ is the viscosity, κ the permeability and ϕ is a physical parameter of the porous region, and β is a dimensionless parameter of order one. Also, $\mathbf{n}$ is the unit normal vector to Σ and τ is any unit tangent vector from a local basis on Σ. Note that Angot [4] used an asymptotic analysis to show the well-posedness of a Stokes/Brinkman problem with Ochoa-Tapia and Whitaker interface conditions for coupled fluid-porous viscous flows. Alazmi and Vafai [3] analyzed different types of interfacial conditions between a porous medium and a fluid, including the Ochoa-Tapia and Whitaker conditions.

5.1 Interface problems of Robin-transmission type for the Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$

This section contains original results obtained by D. Fericean, T. Groşan, M. Kohr and W. L. Wendland [30]. We use a layer potential method in order to show an existence result for an interface boundary value problem of Robin-transmission type for the Stokes and Brinkman systems on Lipschitz domains in Euclidean setting, when the given boundary data belong to some Sobolev or

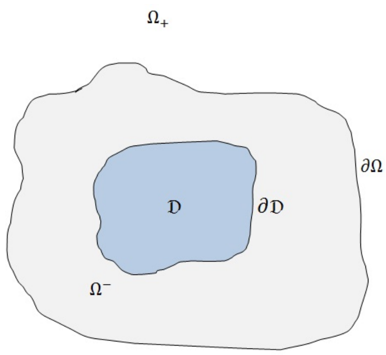


Figure 5.1: Geometry of the problem: domains and boundaries.

L^p spaces. The problem is formulated in three adjacent Lipschitz domains, with assigned conditions at infinity and prescribed transmission conditions at the interfaces between these domains. One of them is a Robin-transmission condition, which is formulated in terms of a non-negative matrix multiplication operator P with L^∞ coefficients. In particular, we consider the boundary value problem that describes the exterior Stokes flow of a viscous incompressible fluid past two porous spheres, one of them being embedded into another one, when the stress jump conditions due to Ochoa-Tapia and Whitaker [84], [85] are imposed at the fluid/porous interface. The solution of this problem is determined explicitly together with the streamlines of the flow for various values of the parameters ζ, λ_1 and λ_2.

5.1.1 Formulation of the problem

Let $\mathfrak{D}$, $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be bounded Lipschitz domains with connected boundaries, such that $\overline{\mathfrak{D}} \subset \Omega$. Let $\Omega^- := \Omega \setminus \overline{\mathfrak{D}}$, $\Omega_+ := \mathbb{R}^n \setminus \overline{\Omega}$, and let $\mathbf{n}_{\partial\Omega}$ and $\mathbf{n}_{\partial\mathfrak{D}}$ be the outward unit normals to $\partial\Omega$ and $\partial\mathfrak{D}$, respectively. Let $P \in L^\infty(\partial\Omega, \mathbb{R}^n \otimes \mathbb{R}^n)$ define a matrix multiplication operator (of type $n \times n$ with $L^\infty(\partial\Omega)$-coefficients) satisfying the non-negativity condition

$$\langle P\mathbf{v}, \mathbf{v}\rangle_{\partial\Omega} \geq 0,\ \forall\, \mathbf{v} \in L^2(\partial\Omega, \mathbb{R}^n). \tag{5.1.1}$$

Recall that $\langle \cdot, \cdot \rangle_{\partial\Omega}$ is the L^2 inner product on $\partial\Omega$, i.e., $\langle \mathbf{u}, \mathbf{v}\rangle_{\partial\Omega} = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{v} d\sigma$.

Let $\alpha \in (0, 1]$, λ_1, $\lambda_2 > 0$ and $\mathbf{U}_\infty \in \mathbb{R}^n$ be given constants. We next consider a boundary value problem of Robin-transmission type for the Stokes and Brinkman systems. This problem requires to find the pairs $((\mathbf{u}, \tilde{\pi}), (\mathbf{u}_-, \tilde{\pi}_-), (\mathbf{u}_+, \tilde{\pi}_+)) \in Z$ such that

$$\begin{cases} \operatorname{div} \mathbf{u} = 0,\ -\nabla\tilde{\pi} + (\triangle - \lambda_1^2 \mathbb{I})\mathbf{u} = \mathbf{0} \text{ in } \mathfrak{D}, \\ \operatorname{div} \mathbf{u}_- = 0,\ -\nabla\tilde{\pi}_- + (\triangle - \lambda_2^2 \mathbb{I})\mathbf{u}_- = \mathbf{0} \text{ in } \Omega^-, \\ \operatorname{div} \mathbf{u}_+ = 0,\ -\nabla\tilde{\pi}_+ + \triangle\mathbf{u}_+ = \mathbf{0} \text{ in } \Omega_+, \\ \operatorname{Tr}^+_{\partial\mathfrak{D}}\mathbf{u}_- - \operatorname{Tr}^-_{\partial\mathfrak{D}}\mathbf{u} = \mathbf{H} \in L^2_{s;\nu_{\partial\mathfrak{D}}}(\partial\mathfrak{D}, \mathbb{R}^n) \text{ on } \partial\mathfrak{D}, \\ \partial^+_{\nu_{\partial\mathfrak{D}}}(\mathbf{u}_-, \tilde{\pi}_-) - \alpha\partial^-_{\nu_{\partial\mathfrak{D}}}(\mathbf{u}, \tilde{\pi}) = \mathbf{F} \in L^2_{s-1}(\partial\mathfrak{D}, \mathbb{R}^n) \text{ on } \partial\mathfrak{D}, \\ \operatorname{Tr}^+_{\partial\Omega}\mathbf{u}_+ - \operatorname{Tr}^-_{\partial\Omega}\mathbf{u}_- = \mathbf{U} \in L^2_{s;\nu_{\partial\Omega}}(\partial\Omega, \mathbb{R}^n) \text{ on } \partial\Omega, \\ \partial^+_{\nu_{\partial\Omega}}(\mathbf{u}_+, \tilde{\pi}_+) - \partial^-_{\nu_{\partial\Omega}}(\mathbf{u}_-, \tilde{\pi}_-) - P\operatorname{Tr}^+_{\partial\Omega}\mathbf{u}_+ = \mathbf{G} \in L^2_{s-1}(\partial\Omega, \mathbb{R}^n) \text{ on } \partial\Omega, \\ \nabla^k(\mathbf{u}_+ - \mathbf{U}_\infty)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{2-n-k}),\ \tilde{\pi}_+(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty,\ k = 0, 1, \end{cases} \tag{5.1.2}$$

where $\operatorname{Tr}^\pm_{\partial\Omega}$ and $\operatorname{Tr}^\pm_{\partial\mathfrak{D}}$ are the non-tangential trace operators for $\partial\Omega$ and $\partial\mathfrak{D}$, respectively. We assume that $\mathbf{U}$ and $\mathbf{H}$ satisfy the conditions $\langle \mathbf{U}, \nu_{\partial\Omega}\rangle_{\partial\Omega} = 0$, $\langle \mathbf{H}, \nu_{\partial\mathfrak{D}}\rangle_{\partial\mathfrak{D}} = 0$ (which are trivial in the physical relevant case: $n = 3$, $\mathbf{U} = \mathbf{0}$ and $\mathbf{H} = \mathbf{0}$). Hence, we assume that $(\mathbf{H}, \mathbf{F}, \mathbf{U}, \mathbf{G})^\top \in$

$\mathcal{X}_\nu$, and define the trace and solution spaces $\mathcal{X}_\nu$ and Z as

(5.1.3)

$$\begin{aligned}\mathcal{X}_\nu &= L^2_{s;\nu_{\partial\mathfrak{D}}}(\partial\mathfrak{D},\mathbb{R}^n)\times L^2_{s-1}(\partial\mathfrak{D},\mathbb{R}^n)\times L^2_{s;\nu_{\partial\Omega}}(\partial\mathfrak{D},\mathbb{R}^n)\times L^2_{s-1}(\partial\Omega,\mathbb{R}^n),\\ Z &= \Big(L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\Big)\times\Big(L^2_{s+\frac{1}{2}}(\Omega^-,\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\Omega^-)\Big)\times\\ &\qquad \Big(L^2_{s+\frac{1}{2},\mathrm{loc}}(\overline{\Omega}_+,\mathbb{R}^n)\times L^p_{s-\frac{1}{2},\mathrm{loc}}(\overline{\Omega}_+)\Big),\end{aligned} \quad s\in(0,1).$$

5.1.2 Layer potentials for the Robin-transmission problem (5.1.2)

In order to show the existence of solutions to the boundary value problem of Robin-transmission type (5.1.2), we now consider the following layer potential representations:

(5.1.4)
$$\begin{aligned}&\mathbf{u}=\mathbf{W}_{\lambda_1;\partial\mathfrak{D}}\mathbf{g}+\mathbf{V}_{\lambda_1;\partial\mathfrak{D}}\mathbf{r},\ \tilde{\pi}=P^d_{\lambda_1;\partial\mathfrak{D}}\mathbf{g}+P^s_{\lambda_1;\partial\mathfrak{D}}\mathbf{r}\ \text{ in } \mathfrak{D},\\ &\mathbf{u}_-=\mathbf{W}_{\lambda_2;\partial\Omega}\mathbf{h}+\mathbf{V}_{\lambda_2;\partial\Omega}\mathbf{f}+\mathbf{W}_{\lambda_2;\partial\mathfrak{D}}\mathbf{g}+\mathbf{V}_{\lambda_2;\partial\mathfrak{D}}\mathbf{r}\ \text{ and}\\ &\tilde{\pi}_-=P^d_{\lambda_2;\partial\Omega}\mathbf{h}+P^s_{\lambda_2;\partial\Omega}\mathbf{f}+P^d_{\lambda_2;\partial\mathfrak{D}}\mathbf{g}+P^s_{\lambda_2;\partial\Omega}\mathbf{r}\ \text{ in } \Omega^-,\\ &\mathbf{u}_+=\mathbf{U}_\infty+\mathbf{W}_{\partial\Omega}\mathbf{h}+\mathbf{V}_{\partial\Omega}\mathbf{f},\ \tilde{\pi}_+=P^d_{\partial\Omega}\mathbf{h}+P^s_{\partial\Omega}\mathbf{f}\ \text{ in } \Omega_+,\end{aligned}$$

where $(\mathbf{g},\mathbf{r},\mathbf{h},\mathbf{f})^\top\in\mathcal{X}_\nu$ are unknown densities. These layer potentials satisfy the equations and the far field conditions in (5.1.2) for any choice of the densities $(\mathbf{g},\mathbf{r},\mathbf{h},\mathbf{f})^\top\in\mathcal{X}_\nu$.

The boundary conditions lead to the matrix equation

(5.1.5)
$$\mathfrak{R}_{\lambda_1,\lambda_2;\alpha}\Phi=\mathcal{B},$$

with given data $\mathcal{B}:=(\mathbf{H},-\mathbf{F},\mathbf{U}-\mathbf{U}_\infty,-\mathbf{G})^\top\in\mathcal{X}_\nu$ and the unknown $\Phi:=(\mathbf{g},\mathbf{r},\mathbf{h},\mathbf{f})^\top\in\mathcal{X}_\nu$. Also the operator $\mathfrak{R}_{\lambda_1,\lambda_2;\alpha}:\mathcal{X}_\nu\to\mathcal{X}_\nu$ is given by

(5.1.6)
$$\begin{pmatrix}\mathbb{I}+\mathbf{K}_{\lambda_2;\lambda_1;\partial\mathfrak{D}} & \mathcal{V}_{\lambda_2;\lambda_1;\partial\mathfrak{D}} & \mathbf{K}_{\lambda_2;\partial\Omega;\partial\mathfrak{D}} & \mathcal{V}_{\lambda_2;\partial\Omega;\partial\mathfrak{D}}\\ -\mathbf{D}_{\lambda_2;\partial\mathfrak{D}}+\alpha\mathbf{D}_{\lambda_1;\partial\mathfrak{D}} & \mathcal{K}^*_{\lambda_2;\lambda_1;\alpha;\partial\mathfrak{D}} & \mathbf{D}_{\lambda_2;\partial\Omega;\partial\mathfrak{D}} & \mathbf{K}^*_{\lambda_2;\partial\Omega;\partial\mathfrak{D}}\\ -\mathbf{K}_{\lambda_1;\partial\mathfrak{D};\partial\Omega} & -\mathcal{V}_{\lambda_1;\partial\mathfrak{D};\partial\Omega} & \mathbb{I}-\mathbf{K}_{\lambda_1;0;\partial\Omega} & -\mathcal{V}_{\lambda_2;0;\partial\Omega}\\ \mathbf{D}_{\lambda_2;\partial\mathfrak{D};\partial\Omega} & \mathbf{K}^*_{\lambda_2;\partial\mathfrak{D};\partial\Omega} & \mathbf{D}_{\lambda_2;0;\partial\Omega}+P\Big(\frac{1}{2}\mathbb{I}+\mathbf{K}_{\partial\Omega}\Big) & \mathbb{I}+\mathbf{K}^*_{\lambda_2;0;\partial\Omega}+P\mathcal{V}_{\partial\Omega}\end{pmatrix}.$$

Theorem 5.1.1 [30] *Assume that $\mathfrak{D},\Omega\subset\mathbb{R}^n$ $(n\geq 3)$ are Lipschitz domains with connected boundaries $\partial\mathfrak{D}$ and $\partial\Omega$, respectively, such that $\overline{\mathfrak{D}}\subset\Omega$. Let $\Omega^-:=\Omega\setminus\overline{\mathfrak{D}}$ and $\Omega_+:=\mathbb{R}^n\setminus\overline{\Omega}$. Also let $P\in L^\infty(\partial\Omega,\mathbb{R}^n\otimes\mathbb{R}^n)$ be a matrix multiplication operator satisfying the non-negativity condition* (5.1.1). *Let $\alpha\in(0,1]$ and $\lambda_1,\lambda_2>0$ be given constants. Then for any $s\in(0,1)$ the equation* (5.1.5) *has a unique solution $(\mathbf{g},\mathbf{r},\mathbf{h},\mathbf{f})^\top\in\mathcal{X}_\nu$ and the layer potentials* (5.1.4) *determine a solution $((\mathbf{u},\tilde{\pi}),(\mathbf{u}_-,\tilde{\pi}_-),(\mathbf{u}_+,\tilde{\pi}_+))\in Z$ to the interface problem of Robin-transmission type* (5.1.2)*, where $\mathcal{X}_\nu$ and Z are the spaces given in* (5.1.3). *For $s\in[\frac{1}{2},1)$ the solution is unique, and for any $\Omega_R:=B_R\cap\Omega_+$, where $B_R\subseteq\mathbb{R}^n$ is an arbitrary open ball such that $\overline{\Omega}\subset B_R$, there exists a constant $C>0$ such that*[1]

$$\begin{aligned}&\|(\mathbf{u},\tilde{\pi})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_{\lambda_1})}+\|(\mathbf{u}_-,\tilde{\pi}_-)\|_{L^2_{s+\frac{1}{2}}(\Omega^-,\mathcal{L}_{\lambda_2})}+\|(\mathbf{u}_+,\tilde{\pi}_+))\|_{L^2_{s+\frac{1}{2}}(\Omega_R,\mathcal{L}_{\mathrm{St}})}\\ &\qquad\leq C\|(\mathbf{H},-\mathbf{F},\mathbf{U}-\mathbf{U}_\infty,-\mathbf{G})\|_{\mathcal{X}_\nu}.\end{aligned} \tag{5.1.7}$$

5.1.3 The Robin-transmission problem with boundary data in L^p spaces

Let $p\in\left(\max\left\{1,\frac{2(n-1)}{n+1}-\varepsilon\right\},2+\varepsilon\right)$, $n\geq 3$, and $\varepsilon=\varepsilon(\partial\mathfrak{D})>0$ be as in Theorem 3.5.4. Next, we consider the interface problem of Robin-transmission type (5.1.2) with boundary data

[1] Recall that the space $L^2_{s+\frac{1}{2}}(\Omega_+,\mathcal{L}_{\mathrm{St}})$ is defined as in (3.1.6). The spaces $L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_{\lambda_1})$ and $L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_{\lambda_1})$ can be defined similarly, by replacing the operator $\mathcal{L}_{\mathrm{St}}$ by $\mathcal{L}_{\lambda_i}$, where $\mathcal{L}_{\lambda_i}(\mathbf{u},\pi):=\nabla\pi-(\triangle-\lambda_i^2\mathbb{I})\mathbf{u}=0$, $i=1,2$.

$(\mathbf{H}, -\mathbf{F}, \mathbf{U} - \mathbf{U}_\infty, -\mathbf{G})^\top \in \mathfrak{X}_{\nu;p}$, where $\mathfrak{X}_{\nu;p}$ is the boundary space

$$\mathfrak{X}_{\nu;p} := L^p_{1;\nu_{\partial\mathfrak{D}}}(\partial\mathfrak{D}, \mathbb{R}^n) \times L^p(\partial\mathfrak{D}, \mathbb{R}^n) \times L^p_{1;\nu_{\partial\Omega}}(\partial\mathfrak{D}, \mathbb{R}^n) \times L^p(\partial\Omega, \mathbb{R}^n). \tag{5.1.8}$$

Also, denote by $\mathcal{M}_{\partial\Omega}$ and $\mathcal{M}_{\partial\mathfrak{D}}$ the non-tangential maximal operators corresponding to $\partial\Omega$ and $\partial\mathfrak{D}$, respectively. In addition, in this case, consider

$$\partial^{\pm}_{\nu_{\partial\Omega}}(\mathbf{v}, q) := \big(-q\mathbb{I} + \big(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\big)\big)\big|_{\partial\Omega_\pm}\mathbf{n}_{\partial\Omega} \text{ a.e. on } \partial\Omega \text{ in the sense of nontangential limit.}$$

Note that if $p \in (1,\infty)$ and $(\mathbf{v}, q)$ satisfies the Stokes system in the Lipschitz domain Ω, such that $\mathcal{M}_{\partial\Omega}(\nabla\mathbf{v}),\ \mathcal{M}_{\partial\Omega}q \in L^p(\partial\Omega)$, then $\mathrm{Tr}^-\mathbf{v} \in L^p_1(\partial\Omega, \mathbb{R}^n)$ and $\partial^-_{\nu_{\partial\Omega}}(\mathbf{v}, q) \in L^p(\partial\Omega, \mathbb{R}^n)$ (see [79, Theorem 4.13]). A similar result can be also obtained in the case of the Brinkman system.

Theorem 5.1.2 [30] *Under the hypothesis of Theorem* 5.1.1 *there exists* $\varepsilon = \varepsilon(\partial\mathfrak{D}) > 0$ *such that for any* $p \in \left(\max\left\{1, \frac{2(n-1)}{n+1} - \varepsilon\right\}, 2+\varepsilon\right)$, *the equation* (5.1.5) *has a unique solution* $(\mathbf{g}, \mathbf{r}, \mathbf{h}, \mathbf{f})^\top \in \mathcal{X}_{\nu;p}$, *and the layer potentials* (5.1.4) *determine a solution* $((\mathbf{u}, \tilde{\pi}), (\mathbf{u}_-, \tilde{\pi}_-), (\mathbf{u}_+, \tilde{\pi}_+)) \in \big(C^2(\mathfrak{D}, \mathbb{R}^n) \times C^1(\mathfrak{D})\big) \times \big(C^2(\Omega^-, \mathbb{R}^n) \times C^1(\Omega^-)\big) \times \big(C^2(\Omega_+, \mathbb{R}^n) \times C^1(\Omega_+)\big)$ *to the interface problem of Robin-transmission type*

$$\begin{cases} \mathrm{div}\,\mathbf{u} = 0,\ -\nabla\tilde{\pi} + (\triangle - \lambda_1^2\mathbb{I})\mathbf{u} = \mathbf{0} \textit{ in } \mathfrak{D}, \\ \mathrm{div}\,\mathbf{u}_- = 0,\ -\nabla\tilde{\pi}_- + (\triangle - \lambda_2^2\mathbb{I})\mathbf{u}_- = \mathbf{0} \textit{ in } \Omega^-, \\ \mathrm{div}\,\mathbf{u}_+ = 0,\ -\nabla\tilde{\pi}_+ + \triangle\mathbf{u}_+ = \mathbf{0} \textit{ in } \Omega_+, \\ \mathcal{M}_{\partial\Omega}(\nabla\mathbf{u}_\pm), \mathcal{M}_{\partial\Omega}(\tilde{\pi}_\pm) \in L^p(\partial\Omega), \\ \mathcal{M}_{\partial\mathfrak{D}}(\nabla\mathbf{u}_-), \mathcal{M}_{\partial\mathfrak{D}}(\nabla\mathbf{u}), \mathcal{M}_{\partial\mathfrak{D}}(\tilde{\pi}_-), \mathcal{M}_{\partial\mathfrak{D}}(\tilde{\pi}) \in L^p(\partial\mathfrak{D}), \\ \mathrm{Tr}^+_{\partial\mathfrak{D}}\mathbf{u}_- - \mathrm{Tr}^-_{\partial\mathfrak{D}}\mathbf{u} = \mathbf{H} \in L^p_{1;\nu_{\partial\mathfrak{D}}}(\partial\mathfrak{D}, \mathbb{R}^n) \textit{ on } \partial\mathfrak{D}, \\ \partial^+_{\nu_{\partial\mathfrak{D}}}(\mathbf{u}_-, \tilde{\pi}_-) - \alpha\partial^-_{\nu_{\partial\mathfrak{D}}}(\mathbf{u}, \tilde{\pi}) = \mathbf{F} \in L^p(\partial\mathfrak{D}, \mathbb{R}^n) \textit{ on } \partial\mathfrak{D}, \\ \mathrm{Tr}^+_{\partial\Omega}\mathbf{u}_+ - \mathrm{Tr}^-_{\partial\Omega}\mathbf{u}_- = \mathbf{U} \in L^p_{1;\nu_{\partial\Omega}}(\partial\Omega, \mathbb{R}^n) \textit{ on } \partial\Omega, \\ \partial^+_{\nu_{\partial\Omega}}(\mathbf{u}_+, \tilde{\pi}_+) - \partial^-_{\nu_{\partial\Omega}}(\mathbf{u}_-, \tilde{\pi}_-) - P\mathrm{Tr}^+_{\partial\Omega}\mathbf{u}_+ = \mathbf{G} \in L^p(\partial\Omega, \mathbb{R}^n) \textit{ on } \partial\Omega, \\ \nabla^k(\mathbf{u}_+ - \mathbf{U}_\infty)(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{2-n-k}),\ \tilde{\pi}_+(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{1-n}) \textit{ as } |\mathbf{x}| \to \infty,\ k = 0, 1, \end{cases} \tag{5.1.9}$$

where $\mathfrak{X}_{\nu;p}$ *is the space given in* (5.1.8). *For any* $p \in [2, 2+\varepsilon)$ *the solution is unique, and there exists a constant* $C > 0$ *such that the following estimate holds:*

$$\begin{aligned} &\|\mathcal{M}_{\partial\mathcal{D}}(\nabla\mathbf{u})\|_{L^p(\partial\mathfrak{D})} + \|\mathcal{M}_{\partial\mathfrak{D}}(\tilde{\pi})\|_{L^p(\partial\mathfrak{D})} + \|\mathcal{M}_{\partial\Omega}(\nabla\mathbf{u}_\pm)\|_{L^p(\partial\Omega)} + \|\mathcal{M}_{\partial\Omega}(\tilde{\pi}_\pm)\|_{L^p(\partial\Omega)} \\ &+ \|\mathcal{M}_{\partial\mathfrak{D}}(\nabla\mathbf{u}_-)\|_{L^p(\partial\mathfrak{D})} + \|\mathcal{M}_{\partial\mathfrak{D}}(\tilde{\pi}_-)\|_{L^p(\partial\mathfrak{D})} \le C\|(\mathbf{H}, -\mathbf{F}, \mathbf{U} - \mathbf{U}_\infty, -\mathbf{G})\|_{\mathcal{X}_{\nu;p}}. \end{aligned} \tag{5.1.10}$$

5.1.4 Stokes flow past two concentric porous spheres

Next, we assume that $\Omega, \mathfrak{D} \subset \mathbb{R}^3$ are two concentric spheres, such that $\overline{\mathfrak{D}} \subset \Omega$, and consider a matrix multiplication operator P of the form

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix},\quad \zeta \in (0,\infty) \text{ is a given constant,} \tag{5.1.11}$$

with respect to a spherical coordinate system $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ having the origin at the center of the spheres and the Ox_1 axis in the direction of $\mathbf{U}_\infty$. Also, choose $\alpha = 1, \mathbf{H} = \mathbf{0}, \mathbf{F} = \mathbf{0}, \mathbf{U} = \mathbf{0}\ \mathbf{G} = \mathbf{0}$ and $U_\infty = 1$ in (5.1.9). Then the interface problem (5.1.9) describes the exterior Stokes flow of a viscous incompressible fluid past two concentric porous spheres, with stress jump conditions for the tangential stresses and the continuity of the velocity components and normal stresses on the porous-fluid interface $\partial\Omega$, and with the continuity of the velocity and stress fields on the interface $\partial\mathfrak{D}$ between the porous media. We determine the solution of this problem by making use of the geometry of the involved flow domains. In view of (5.1.11), the transmission conditions in (5.1.2), corresponding to the porous-fluid interface $\partial\Omega$, take the form

$$(5.1.12)\qquad \begin{array}{l} (\mathbf{u}_+)_r = (\mathbf{u}_-)_r,\ (\mathbf{u}_+)_\theta = (\mathbf{u}_-)_\theta,\ (\mathbf{u}_+)_\phi = (\mathbf{u}_-)_\phi, \\ T_{rr}(\mathbf{u}_+, \tilde{\pi}_+) = T_{rr}(\mathbf{u}_-, \tilde{\pi}_-), \\ T_{r\theta}(\mathbf{u}_+, \tilde{\pi}_+) - T_{r\theta}(\mathbf{u}_-, \tilde{\pi}_-) = \zeta(\mathbf{u}_+)_\theta, \\ T_{r\phi}(\mathbf{u}_+, \tilde{\pi}_+) - T_{r\phi}(\mathbf{u}_-, \tilde{\pi}_-) = \zeta(\mathbf{u}_+)_\phi, \end{array}$$

where $(\mathbf{u}_\pm)|_{\partial\Omega} := ((\mathbf{u}_\pm)_r, (\mathbf{u}_\pm)_\theta, (\mathbf{u}_\pm)_\phi)$ are the velocity fields on $\partial\Omega$. In addition, the conormal derivatives (or stress fields) $\partial^\pm_{\nu_{\partial\Omega}}(\mathbf{u}_\pm, \tilde{\pi}_\pm) := \left(-\tilde{\pi}_\pm\mathbb{I} + \left(\nabla\mathbf{u}_\pm + (\nabla\mathbf{u}_\pm)^\top\right)\right)|_{\partial\Omega}\mathbf{n}_{\partial\Omega}$ have the components $(t_r(\mathbf{u}_\pm, \tilde{\pi}_\pm), t_\theta(\mathbf{u}_\pm, \tilde{\pi}_\pm), t_\phi(\mathbf{u}_\pm, \tilde{\pi}_\pm)) = (T_{rr}(\mathbf{u}_\pm, \tilde{\pi}_\pm), T_{r\theta}(\mathbf{u}_\pm, \tilde{\pi}_\pm), T_{r\phi}(\mathbf{u}_\pm, \tilde{\pi}_\pm))$ with respect to the spherical coordinate system $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$. Therefore, the Robin-transmission conditions in (5.1.12) reduce to the continuity of the normal stress, $T_{rr}(\mathbf{u}_+, \tilde{\pi}_+) = T_{rr}(\mathbf{u}_-, \tilde{\pi}_-)$, and the jump boundary conditions (5.0.1) for the shear stress on the fluid/porous interface $\partial\Omega$, due to Ochoa-Tapia and Whitaker [84], [85]:

$$(5.1.13)\qquad \frac{\partial(\mathbf{u}_+)_\theta}{\partial r} - \frac{\partial(\mathbf{u}_-)_\theta}{\partial r} = \zeta(\mathbf{u}_+)_\theta,\ \frac{\partial(\mathbf{u}_+)_\phi}{\partial r} - \frac{\partial(\mathbf{u}_-)_\phi}{\partial r} = \zeta(\mathbf{u}_+)_\phi,$$

with the jump coefficient $\zeta > 0$. Note that $\zeta = 0$ corresponds to the continuity of the stress field across $\partial\Omega$ (for applications in this case see, e.g., [36]).

Now, we assume that the concentric porous spheres $\mathfrak{D}$ and Ω have the nondimensional radii $r = 1$ (corresponding to $\partial\mathfrak{D}$) and $r = R > 1$ (corresponding to $\partial\Omega$), respectively. The axisymmetric flow configuration implies that $\frac{\partial}{\partial\phi} \equiv 0$, $(\mathbf{u}_\pm)_\phi = 0$ and $u_\phi = 0$. Consequently, the second condition in (5.1.13) is identically satisfied. On the other hand, the Stokes and Brinkman equations in (5.1.2) can be written in spherical coordinates as (see e.g., [36])

$$(5.1.14)\qquad \begin{array}{l} \dfrac{\partial q}{\partial r} = \chi^2 v_r - \Big\{\dfrac{\partial^2 v_r}{\partial r^2} + 2\dfrac{\partial v_r}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v_r}{\partial\theta^2} \\ \qquad\qquad + \dfrac{\cot\theta}{r^2}\dfrac{\partial v_r}{\partial\theta} - \dfrac{2v_r}{r^2} - \dfrac{2}{r^2}\dfrac{\partial v_\theta}{\partial\theta} - 2\dfrac{v_\theta\cot\theta}{r^2}\Big\}, \\ -\dfrac{1}{r}\dfrac{\partial q}{\partial\theta} = \chi^2 v_\theta - \Big\{\dfrac{\partial^2 v_\theta}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial v_\theta}{\partial r} \\ \qquad\qquad + \dfrac{1}{r^2}\dfrac{\partial^2 v_\theta}{\partial\theta^2} + \dfrac{\cot\theta}{r^2}\dfrac{\partial v_\theta}{\partial\theta} + \dfrac{2}{r^2}\dfrac{\partial v_r}{\partial\theta} - v_\theta\dfrac{\operatorname{cosec}^2\theta}{r^2}\Big\}, \end{array}$$

where v_r and v_θ are the spherical coordinates of $\mathbf{v}$, and

$$(5.1.15)\qquad (\mathbf{v}, q) := \left\{\begin{array}{ll} (\mathbf{u}, \tilde{\pi}) & \text{in } \mathfrak{D} \\ (\mathbf{u}_-, \tilde{\pi}_-) & \text{in } \Omega^- \\ (\mathbf{u}_+, \tilde{\pi}_+) & \text{in } \Omega_+, \end{array}\right. \quad \chi := \left\{\begin{array}{ll} \lambda_1 & \text{in } \mathfrak{D} \\ \lambda_2 & \text{in } \Omega^- \\ 0 & \text{in } \Omega_+. \end{array}\right.$$

Also, $\lambda_1 = a/\sqrt{\kappa_0}$ and $\lambda_2 = a/\sqrt{\kappa_-}$ are the parameters of the porous media, with the permeability κ_0 in $\mathfrak{D}$ and κ_- in Ω_-, respectively, and a is a characteristic length (e.g., the dimensional radius of $\mathfrak{D}$). In addition, the boundary conditions in (5.1.2) take the form

(5.1.16)

$$\left\{\begin{array}{l} (\mathbf{u}_+)_r = (\mathbf{u}_-)_r,\ (\mathbf{u}_+)_\theta = (\mathbf{u}_-)_\theta, \\ T_{rr}(\mathbf{u}_+, \tilde{\pi}_+) = T_{rr}(\mathbf{u}_-, \tilde{\pi}_-),\ \text{for } r = R, \\ T_{r\theta}(\mathbf{u}_+, \tilde{\pi}_+) - T_{r\theta}(\mathbf{u}_-, \tilde{\pi}_-) = \zeta u_\theta, \end{array}\right. \quad \left\{\begin{array}{l} (\mathbf{u}_-)_r = u_r,\ (\mathbf{u}_-)_\theta = u_\theta, \\ T_{rr}(\mathbf{u}_-, \tilde{\pi}_-) = T_{rr}(\mathbf{u}, \tilde{\pi}), \quad \text{for } r = 1. \\ T_{r\theta}(\mathbf{u}_-, \tilde{\pi}_-) = T_{r\theta}(\mathbf{u}, \tilde{\pi}), \end{array}\right.$$

In order to have satisfied the continuity equation $\operatorname{div}\mathbf{v} = 0$, we now consider the stream functions

ψ, ψ_- and ψ_+ given by the relations (see, e.g., [59, p. 13])

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta},\ u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\ \text{ in } \mathfrak{D},$$
$$(5.1.17)\qquad (\mathbf{u}_-)_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi_-}{\partial\theta},\ (\mathbf{u}_-)_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi_-}{\partial r}\ \text{ in } \Omega^-,$$
$$(\mathbf{u}_+)_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi_+}{\partial\theta},\ (\mathbf{u}_+)_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi_+}{\partial r}\ \text{ in } \Omega_+.$$

In addition, in view of the far field conditions

$$(5.1.18)\qquad (\mathbf{u}_+)_r \to \cos\theta,\ (\mathbf{u}_+)_\theta \to -\sin\theta\ \text{ as } r\to\infty,$$

and the relations (5.1.17), we get the following asymptotic behavior of the stream function ψ_+ at infinity with respect to the leading order term in r: $\psi_+(r,\theta) \approx \frac{r^2}{2}\sin^2\theta$. According to this behavior we determine the functions $\psi_\pm$ and ψ in the form

$$(5.1.19)\qquad \psi_\pm = f_\pm(r)\sin^2\theta\ \text{ and }\ \psi = f(r)\sin^2\theta.$$

Now, from the equations (5.1.14)-(5.1.15) and the relations (5.1.17) and (5.1.19), we obtain the following ordinary differential equation:

$$(5.1.20)\qquad g^{(iv)} - \frac{4}{r^2}g'' + \frac{8}{r^3}g' - \frac{8}{r^4}g - \zeta\left(g'' - \frac{2}{r^2}g\right) = 0.$$

Using appropriate transformations, the differential equation (5.1.20) can be reduced to a Bessel type equation and its solution in each of the domains Ω_+, Ω_- and $\mathfrak{D}$ is given by

$$f_+(r) = A_+ r + B_+ r^4 + D_+ r^2 + \frac{C_+}{r},$$
$$(5.1.21)\qquad f_-(r) = \frac{C_-}{r} + D_- r^2 + A_-\frac{\sqrt{r}}{\lambda_2^2}I_{\frac{3}{2}}(\lambda_2 r) + B_-\frac{\sqrt{r}}{\lambda_2^2}K_{\frac{3}{2}}(\lambda_2 r),$$
$$f(r) = \frac{C}{r} + Dr^2 + A\frac{\sqrt{r}}{\lambda_1^2}I_{\frac{3}{2}}(\lambda_1 r) + B\frac{\sqrt{r}}{\lambda_1^2}K_{\frac{3}{2}}(\lambda_1 r),$$

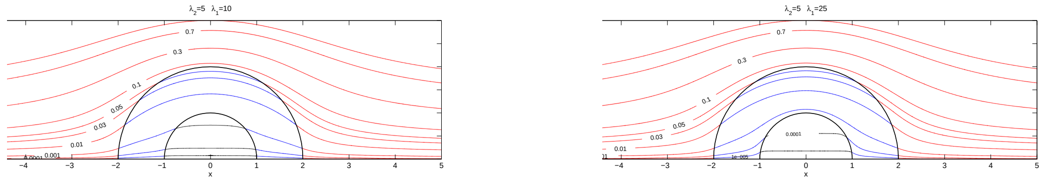
where D, $B_\pm, C_\pm, D_\pm$ are additional unknown real constants. Let us mention that the formulas (5.1.21) have the same form as the general solutions of the Stokes and Brinkman equations in spherical domains given by Zlatanovski [119, (19a)-(19b)]. The (nondimensional) force due to the Stokes flow on the external sphere is

$$(5.1.22)\qquad F|_{r=R} = \int_0^\pi \left(T_{rr}(\mathbf{u}_+,\tilde{\pi}_+)\cos\theta - T_{r\theta}(\mathbf{u}_+,\tilde{\pi}_+)\sin\theta\right)|_{r=R}\sin\theta d\theta = -\frac{4}{R^2}A_+.$$

In order to determine the unknown constants A_+, C_+, A_-, B_-, C_-, D_-, A and D we use the interface conditions (5.1.16), which, in view (5.1.17), take the form

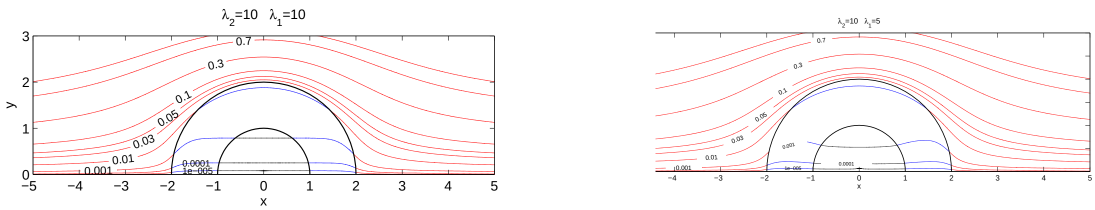
$$(5.1.23)\qquad \begin{cases} f_+(R) = f_-(R) \\ f'_+(R) = f'_-(R) \\ f''_-(R) - f''_+(R) = \zeta f'_-(R) \\ f'''_-(R) - f'''_+(R) = -\lambda_2^2 f'_-(R), \end{cases} \quad \begin{cases} f_-(1) = f(1) \\ f'_-(1) = f'(1) \\ f''_-(1) = f''(1) \\ f'''(1) - f'''_-(1) = -\lambda_2^2 f'_-(1) + \lambda_1^2 f'(1). \end{cases}$$

The system (5.1.23) has been solved for several values of the involved parameters by using the symbolic software Mathematica. In addition, by using the expression (5.1.19) of the stream functions, we have obtained the streamlines of the flow for various values of the involved parameters ζ, λ_1 and λ_2. Figures 5.2 (a) and (b) correspond to the case $\lambda_1 > \lambda_2$, Figure 5.3 (a) is related to the case $\lambda_1 = \lambda_2$ and Figure 5.3 (b) yields the streamlines in the case $\lambda_1 < \lambda_2$. We conclude that:



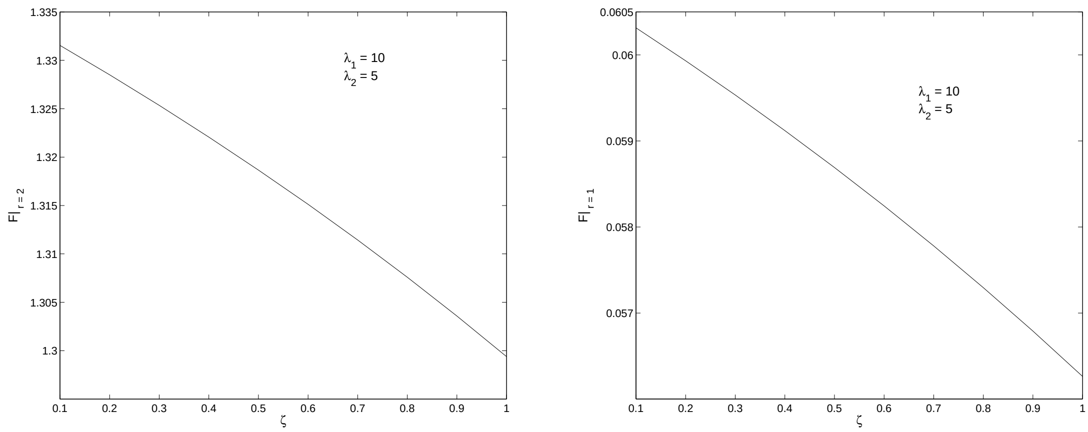
(a) (b)

Figure 5.2: Streamlines for $\lambda_1 = 10$, $\lambda_2 = 5$ and $\lambda_1 = 25$, $\lambda_2 = 5$.



(a) (b)

Figure 5.3: Streamlines for $\lambda_1 = 10$, $\lambda_2 = 10$ and $\lambda_1 = 5$, $\lambda_2 = 10$.



(a) (b)

Figure 5.4: Variation of drag force with ζ.

(a) In the case $\lambda_1 > \lambda_2$ the streamlines of the flow in the domain Ω_- (between the porous spheres) have a similar structure to that of the flow past a solid sphere.

(b) When $\lambda_1 = \lambda_2$, the presence of the smaller porous sphere does not perturb the flow inside the larger porous sphere, since in this case both spheres have the same physical properties.

(c) In the case $\lambda_1 < \lambda_2$ the streamlines are bent down inside the smaller porous sphere, due to the physical properties of the porous spheres.

In addition, the variation of dimensionless drag force on the inner and outer spheres, with respect to ζ, λ_2 and λ_1, respectively, and for $r = 1$ and $R = 2$, is shown in Figures 5.4.

5.2 Boundary value problems with Dirichlet and Robin-transmission conditions for the Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$

This section contains original results of D. Fericean [29], and deals with the layer potential analysis for a boundary value problem with Dirichlet and Robin-transmission conditions for Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. In particular, we refer to a Stokes flow problem past a porous medium with a solid sphere inside, when the stress jump conditions (5.1.13) are imposed on the interface between the fluid and the porous medium. In this special case, we obtain both well-posedness and numerical results. Note that the boundary value problem treated in this section is similar with the boundary value problem (4.1.1). The difference between them is provided by the involved transmission conditions. In the particular $3D$ case due to a spherical porous medium that contains a fixed spherical solid core, the transmission conditions on the porous-fluid interface require the jump of shear stress and the continuity of the velocity and normal stress instead of the usual velocity and stress continuity conditions at the fluid-porous interface that appear in (4.1.1). These shear stress jump conditions [84], [85] are the physical conditions that should be imposed on a fluid-porous interface when the flow inside the porous medium is governed by the Brinkman equation.

Chapter 6

Layer potential analysis of a Neumann problem for the Brinkman system on Lipschitz domains in compact Riemannian manifolds

This chapter is based on the original results of the author of this thesis presented in [28] and is devoted to a layer potential analysis for a boundary value problem of Neumann type associated to the Brinkman system on Lipschitz domains in compact Riemannian manifolds, when the boundary datum belongs to some Sobolev spaces. By using a layer potential method, one obtains the existence and uniqueness result (up to a constant pressure) for this problem. The main sources used in the preparation of this chapter are [41], [56], [73].

6.1 Formulation of the problem

Let $\Omega \subset M$ be a Lipschitz domain on a compact boundaryless Riemannian manifold M, $\dim(M) \geq 2$, and let $G \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$. All along this chapter assume that $\lambda > 0$ is a given constant. For $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ consider the following boundary value problem of Neumann type for the Brinkman system:

$$(6.1.1) \qquad \begin{cases} (\mathfrak{L} + \lambda^2 \mathbb{I})\mathbf{u} + \mathrm{d}\pi = 0, \ \delta\mathbf{u} = 0 \text{ in } \Omega \\ \\ [\partial_\nu^+(\mathbf{u}, \pi)] = [G] \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu. \end{cases}$$

Note that the condition $[\partial_\nu^+(\mathbf{u}, \pi)] = [G]$ is equivalent with $\partial_\nu^+(\mathbf{u}, \pi) - G \in \mathbb{R}\nu$ on $\partial\Omega$.

6.2 Uniqueness result for the Neumann problem (6.1.1)

Theorem 6.2.1 **[28]** *Let $\Omega \subset M$ be a Lipschitz domain on a compact boundaryless Riemannian manifold M, $dim(M) \geq 2$, and let $\lambda > 0$, $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $G \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ be given. Then the boundary value problem of Neumann type* $(6.1.1)$ *has at most one solution* $(\mathbf{u}, \pi) \in H^{1+\beta}(\Omega, \Lambda^1 TM) \times H^\beta(\Omega)$ (*up to a constant pressure*).

6.3 Layer potential formulation of the problem

Next, we show the existence of a solution $(\mathbf{u},\pi)\in H^{1+\beta}(\Omega,\Lambda^1TM)\times H^{\beta}(\Omega)$ of the Neumann problem $(6.1.1)$, by means of the layer potential theory. To this aim, we use the invertibility property of the operators

$$\pm\frac{1}{2}\mathbb{I}+\mathbf{K}_{\lambda,\partial\Omega}:H_{\nu}^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)\to H_{\nu}^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM),$$

for any $\beta\in(-\frac{1}{2},\frac{1}{2})$ as follows from Theorem 3.6.8 (see also [57, Lemma 5.4]). We consider the following double-layer potential and its associated pressure potential:

$$\mathbf{u}=\mathbf{W}_{\lambda,\partial\Omega}\mathbf{h},\pi=\mathcal{P}_{\lambda,\partial\Omega}\mathbf{h}\text{ in }\Omega, \tag{6.3.1}$$

with the density $\mathbf{h}\in H_{\nu}^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)$ in the form

$$\mathbf{h}:=\left(\frac{1}{2}\mathbb{I}+\mathbf{K}_{\lambda,\partial\Omega}\right)^{-1}\left(-\frac{1}{2}\mathbb{I}+\mathbf{K}_{\lambda,\partial\Omega}\right)^{-1}\mathcal{V}_{\lambda,\partial\Omega}G \tag{6.3.2}$$

and with $G\in H_{\nu}^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)$ given. Therefore, $(\mathbf{u},\pi)\in H^{1+\beta}(\Omega,\Lambda^1TM)\times H^{\beta}(\Omega)$. Finally, by using the property (3.6.31), we conclude that $[\partial_{\nu}^{+}(\mathbf{u},\pi)]=[G]$. Consequently, the pair $(\mathbf{u},\pi)$ given by $(6.3.1)$, $(6.3.2)$ is a solution of the Neumann problem $(6.1.1)$, in the space $H^{1+\beta}(\Omega,\Lambda^1TM)\times H^{\beta}(\Omega)$. In view of Theorem 6.2.1, this is the unique solution (up to a constant pressure) of the Neumann problem (6.1.1). The boundedness properties of the layer potentials (6.3.1) and those of the operators in (6.3.2) imply that there exists a constant $C>0$ such that

$$\|\mathbf{u}\|_{H^{1+\beta}(\Omega,\Lambda^1TM)}+\|\pi\|_{H^{\beta}(\Omega)}\leq C\|\,[G]\,\|_{H^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)/\mathbb{R}\nu}. \tag{6.3.3}$$

Theorem 6.3.1 [28] *Let $\Omega\subset M$ be a Lipschitz domain on a compact boundaryless Riemannian manifold M, $dim(M)\geq 2$, and let $\lambda>0$, $\beta\in\left(-\frac{1}{2},\frac{1}{2}\right)$ and $G\in H^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)$ be given. Then the layer potentials* (6.3.1) *with the density $\mathbf{h}\in H_{\nu}^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)$ given by* (6.3.2) *determine the unique solution $(\mathbf{u},\pi)\in H^{1+\beta}(\Omega,\Lambda^1TM)\times H^{\beta}(\Omega)$ (up to a constant pressure) of the boundary value problem of Neumann type* (6.1.1)*, which satisfies the estimate* (6.3.3).

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