BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Rickart-type objects in abelian categories

PhD Thesis Summary

Scientific adviser
Prof. Dr. Septimiu Crivei

PhD Student
Simona-Maria Radu

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Keywords

- CS-Rickart object
- strongly CS-Rickart object
- CS-Baer object
- strongly CS-Baer object
- abelian category
- duality principle
- fully faithful functor
- adjoint functors
- module category
- endomorphism ring

Introduction

The purpose of this thesis has been to introduce and study the notion of CS-Rickart object in abelian categories as a common generalization of Rickart object and extending object. We have also considered a relevant particularization, namely CS-Baer object, which unifies the theories of Baer objects and extending objects. All these concepts had been previously considered in module categories, but we have developed new specific techniques in order to approach them categorically. The main advantage of working at the level of generality of abelian categories is that the dual results follow automatically by the duality principle. In this way we give a unified approach of dual notions, which have been treated separately in the literature for module categories. In subsidiary, one also has consequences in particular abelian categories, other than module categories.

A basic module-theoretic result states that a module M is semisimple if and only if all its submodules are direct summands. One may obtain various generalizations of semisimplicity by considering only some submodules of a given module to be direct summands. For instance, if M is the right R-module R, then R is a (von Neumann) regular ring if and only if every finitely generated submodule of M (i.e., right ideal of R) is a direct summand. Given a module M, one may also define an extension of semisimplicity by using certain submodules related to the endomorphisms of M. As such, a module M is called Rickart if every endomorphism of M has the kernel a direct summand of M [56]. Using a different approach, one may consider another generalization of semisimplicity, namely: a module M is called extending (or CS-module) if all submodules of M are essential in direct summands of M [37].

A natural question is what happens when one restricts the definition of an extending module M only to some submodules related to the endomorphisms of M in the same style in which one obtains the concept of Rickart module from that of semisimple module? This is the way to obtain the so-called CS-Rickart modules, introduced and studied by Abyzov, Nhan and Quynh [1, 2], which are defined as modules M such that every endomorphism of M has the kernel essential in a direct summand of M. We point out that CS-Rickart modules may be viewed as playing the role of extending modules in the world of Rickart modules. The properties of CS-Rickart modules are sometimes similar to those of Rickart modules, but one needs different techniques to obtain them, in the same way as one uses different approaches to study extending modules in contrast to semisimple modules.

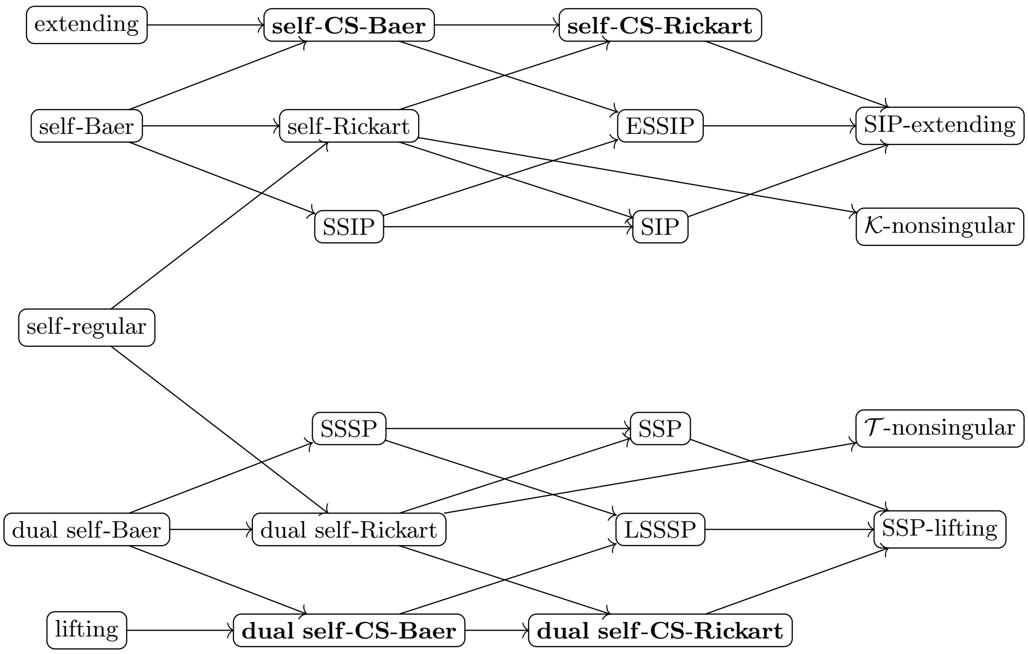
Rickart objects and their duals in abelian categories have been considered and investigated by Crivei, Kör and Olteanu [22, 23]. On one hand, they extend regular objects which have been approached in abelian categories by Dăscălescu, Năstăsescu, Tudorache and Dăuş [34]. Thus, an object is regular if and only if it is both Rickart and dual Rickart. The main interest in their study stems from the work of von Neumann [77] on regular rings, and Fieldhouse [40] and Zelmanowitz [81] on certain concepts of regular modules. The main idea of the approach from [22] was to split the study of regular objects into two directions, one of Rickart objects and the other of dual Rickart objects. Since they are dual concepts, it is enough to study one of them and use the duality principle in abelian categories.

On the other hand, Rickart objects and their duals extend to abelian categories Rickart modules and their duals, which have been approached by Lee, Rizvi and Roman [56, 57], and in particular, Baer and dual Baer modules studied by Rizvi and Roman [69, 70] and Keskin Tütüncü and Tribak [53] respectively. The origin of (dual) Baer modules and (dual) Rickart modules can be found in the work of Kaplansky [50] on Baer rings and Maeda [58] on Rickart rings respectively. Examples of Baer rings include von Neumann regular right self-injective rings, von Neumann algebras, endomorphism rings of semisimple modules, while examples of Rickart rings include Baer rings, von Neumann regular rings, right hereditary rings, endomorphism rings of arbitrary direct sums of copies of a right hereditary ring. The general theory of (dual) Rickart objects in abelian categories may be efficiently applied both to the study of regular objects and to the study of (dual) Baer objects in abelian categories, as underlined in [22, 23].

Returning to module theory, extending and lifting modules have been fruitful topics of research for the last decades, due to their significant applications to general ring and module theory. Examples of extending modules include uniform modules and injective modules, while examples of lifting modules include hollow modules and projective modules over perfect rings. The reader is referred to the monographs [13, 37] for further information on extending and lifting modules. Recently, CS-Rickart and dual CS-Rickart modules have been introduced and investigated by Abyzov, Nhan and Quynh [1, 2], Tribak has considered dual CS-Rickart modules in case of Dedekind domains [75], and Nhan has studied CS-Baer modules (under the name of essentially Baer modules) [63].

The above notions have some strong versions, obtained by replacing direct summands by fully invariant direct summands in their definitions. E.g., see the work of Al-Saadi and Ibrahiem on (dual) strongly Rickart modules [3, 4], Ebrahimi Atani, Khoramdel and Dolati Pish Hesari [39] on strongly extending modules, Wang [78] on strongly lifting modules, Crivei and Olteanu on (dual) strongly Rickart objects in abelian categories [24, 25].

Motivated by all the above, we introduce and study (dual) CS-Rickart and (dual) CS-Baer objects in abelian categories as well as their strong versions. To this end, we shall take advantage of some techniques used for (dual) Rickart and (dual) Baer objects, and we shall develop some new ones, inspired by the theory of extending and lifting modules. We depict in the following diagram the main concepts of the thesis and the way in which they relate to each other.



In what follows we briefly present the structure description and the main contributions of the thesis. All results are original, except for those explicitly cited or recalled. Our results and their proofs will be mainly presented for CS-Rickart and CS-Baer objects, the dual ones following automatically by the duality principle in abelian categories. Versions for strong CS-Rickart and strong CS-Baer objects are not mentioned explicitly in the introduction, but have been given throughout the thesis.

In Chapter 1 we introduce and study CS-Rickart objects in abelian categories. We begin with a section on preliminaries, where we recall some needed notions and notation, which will be used throughout the thesis. In Section 2 we define relative CS-Rickart objects and self-CS-Rickart objects in abelian categories, and illustrate and delimit our concepts. We show that an object M is strongly self-CS-Rickart if and only if M is weak duo and self-CS-Rickartif and only if M is self-CS-Rickart and the ring $\mathrm{End}_{\mathcal{A}}(M)$ is abelian (Corollary 1.2.3, Proposition 1.2.4). Also, for any object M, we prove that M-Rickart objects are precisely the M-$\mathcal{K}$-nonsingular M-CS-Rickart objects (Theorem 1.2.8). Section 3 shows that the class of CS-Rickart objects is well behaved with respect to direct summands (Theorem 1.3.1). We prove that every self-CS-Rickart object is SIP-extending, a suitable property on direct summands which generalizes the summand intersection property (SIP) (Corollary 1.3.6). In Section 4, for objects M and $N_1, \dots, N_n$ of an abelian category $\mathcal{A}$, we prove a necessary and sufficient condition for $\bigoplus_{i=1}^n N_i$ to be M-CS-Rickart, namely N_i is M-CS-Rickart for every $i \in \{1, \dots, n\}$ (Theorem 1.4.2). In general, coproducts of self-CS-Rickart objects need not be self-CS-Rickart, but we have the following result. Consider a direct sum decomposition $M = \bigoplus_{i \in I} M_i$ in an abelian category $\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(M_i, M_j) = 0$ for every distinct $i, j \in I$. Then a necessary and sufficent condition for M to be self-CS-Rickart is that M_i is self-CS-Rickart for each $i \in I$ (Theorem 1.4.7). Section 5 deals with classes all of whose objects are self-CS-Rickart. For instance, if there are enough injectives in abelian category $\mathcal{A}$ and $\mathcal{C}$ is a class of objects of $\mathcal{A}$ which is closed under binary direct sums and contains all injective objects of $\mathcal{A}$, then we prove that every object of $\mathcal{C}$ is extending if and only if every object of $\mathcal{C}$ is self-CS-Rickart if and only if every object of $\mathcal{C}$ is SIP-extending (Theorem 1.5.1). We also deduce characterizations of right perfect and weakly (semi)hereditary rings in terms of self-CS-Rickart or dual self-CS-Rickart properties (Corollaries 1.5.4, 1.5.7).

In Chapter 2 we investigate how CS-Rickart properties transfer through functors between abelian categories. In Section 1 we show that a functor between abelian categories need not preserve or reflect CS-Rickart properties. Nevertheless, if we consider fully faithful functors or adjoint pairs with some reasonable additional properties, we are able to obtain transfer results on CS-Rickart properties. In Section 1 we consider fully faithful functors between abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact fully faithful covariant functor between abelian categories, and let M and N be objects of $\mathcal{A}$. If $\mathrm{Im}(F)$ is closed under subobjects or quotients, and N is M-CS-Rickart, we prove that $F(N)$ is $F(M)$-CS-Rickart. Also, if $\mathrm{Im}(F)$ is closed under direct summands and $F(N)$ is $F(M)$-CS-Rickart, then N is M-CS-Rickart (Theorem 2.1.3). In Section 2 we consider an adjoint pair (L, R), where $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are covariant functors between abelian categories such that L is exact and $M, N \in \mathrm{Stat}(R) = \{B \in \mathcal{B} \mid \varepsilon_B \text{ is an isomorphism}\}$. We prove that if N is M-CS-Rickart in $\mathcal{B}$, then $R(N)$ is $R(M)$-CS-Rickart in $\mathcal{A}$. Also, we show that if R reflects zero objects, in particular if R is faithful, and $R(N)$ is $R(M)$-CS-Rickart in $\mathcal{A}$, then N is M-CS-Rickart in $\mathcal{B}$ (Theorem 2.2.1). As a consequence, we deduce that if L is exact and R is fully faithful, then N is M-CS-Rickart in $\mathcal{B}$ if and only if $R(N)$ is $R(M)$-CS-Rickart in $\mathcal{A}$ (Theorem 2.2.2). Our theorems are widely applicable, and we illustrate them in several relevant contexts in Section 3. We discuss applications to Giraud and co-Giraud subcategories (Corollary 2.3.1), functor categories, localizing and colocalizing subcategories, adjoint triples of functors (Corollary 2.3.6), Frobenius functors and recollements between abelian categories. We present consequences to arbitrary Grothendieck categories, but also to the particular cases of (graded) module and comodule categories. In Section 4 we give properties on the transfer of CS-Rickart property to endomorphism rings of (graded) modules and comodules. Among the results, for a right R-module M with $S = \mathrm{End}_R(M)$, we prove that if M is im-local-retractable and S is a self-CS-Rickart right S-module, then M is a self-CS-Rickart right R-module (Corollary 2.4.2). Also, if M is a self-CS-Rickart right R-module which is flat as a left S-module, then S is a self-CS-Rickart right S-module (Corollary 2.4.3).

In Chapter 3 we introduce and study CS-Baer objects in an abelian category $\mathcal{A}$ with AB3*. In Section 1 we introduce these concepts, and we illustrate them by a series of examples. We show that an object M is strongly self-CS-Baer if and only if M is weak duo and self-CS-Baer if and only if M is self-CS-Baer and $\mathrm{End}_{\mathcal{A}}(M)$ is abelian (Corollary 3.1.3). Sections 2 and 3 are dedicated to the comparison of relative CS-Baer objects with their most important particularizations, namely relative Baer and extending objects. For objects M and N, we show that N is M-Baer if and only if N is M-CS-Baer and M-$\mathcal{K}$-nonsingular (Theorem 3.2.2). We consider some stronger forms of M-$\mathcal{K}$-nonsingularity and M-$\mathcal{K}$-cononsingularity, called $\mathcal{E}$-M-$\mathcal{K}$-nonsingularity and $\mathcal{E}$-M-$\mathcal{K}$-cononsingularity respectively. We show that if N is M-CS-Baer and $\mathcal{E}$-M-$\mathcal{K}$-cononsingular, then M is extending (Corollary 3.3.4). We also prove that if an object M is extending and $\mathcal{E}$-M-$\mathcal{K}$-nonsingular, then M is self-CS-Baer and $\mathcal{E}$-M-$\mathcal{K}$-cononsingular (Corollary 3.3.6). We characterize dual self-CS-Baer rings as lifting rings, or equivalently, semiperfect rings (Proposition 3.3.7). Sections 4 and 5 study the relationship between relative CS-Baer objects and two relevant generalizations, namely objects with certain summand intersection properties called ESSIP and SSIP-extending, and relative CS-Rickart objects. We prove that every self-CS-Baer object is ESSIP, and if A and B are objects such that $A \oplus B$ is ESSIP, then B is A-CS-Baer (Corollary 3.4.4, Lemma 3.4.5). Also, if M is self-CS-Rickart and SSIP-extending, then M is self-CS-Baer, while the converse holds if $\mathrm{Soc}(M)$ is an essential subobject of M (Theorem 3.5.3, Corollary 3.5.4). In particular, if M is a finitely cogenerated right R-module or R is a right semiartinian ring, then M is self-CS-Baer if and only if M is self-CS-Rickart and SSIP-extending. Also, if M is a finitely generated right R-module or R is a right max ring, then M is dual self-CS-Baer if and only if M is dual self-CS-Rickart and SSSP-lifting (Corollary 3.5.5). Section 6 studies products and coproducts of relative CS-Baer objects. The class of relative CS-Baer objects is closed under direct summands (Corollary 3.6.2), but in general not under coproducts. Nevertheless, we show that a necessary and sufficient condition for $\bigoplus_{i=1}^{n} N_i$ to be M-CS-Baer is that each N_i is M-CS-Baer (Theorem 3.6.4). Given a direct sum decomposition $M = \bigoplus_{i=1}^{n} M_i$ such that $\mathrm{Hom}_{\mathcal{A}}(M_i, M_j) = 0$ for every distinct $i, j \in \{1, \ldots, n\}$, we prove that M is self-CS-Baer if and only if each M_i is self-CS-Baer (Theorem 3.6.6). In Section 7 we determine the structure of dual self-CS-Baer modules over Dedekind domains (Theorems 3.7.1, 3.7.2, 3.7.3). In Section 8 we give some results (such as Theorem 3.8.1) on classes all of whose objects are self-CS-Baer, having a similar flavour as the corresponding ones for the self-CS-Rickart property. In Section 9 we study when relative CS-Baer properties transfer through functors between abelian categories. Our main results refer to fully faithful functors and adjoint functors between abelian categories, under the same reasonable assumptions as in the case of the CS-Rickart property (Theorems 3.9.1, 3.9.3). In Section 10 we study the transfer of CS-Baer property to endomorphism rings of (graded) modules and comodules. Among the results, for a right R-module M with $S = \mathrm{End}_R(M)$, we prove that if M is im-local-retractable and S is a self-CS-Baer right S-module, then M is a self-CS-Baer right R-module (Corollary 3.10.3). Also, if M is a self-CS-Baer right R-module which is finitely generated projective as a left S-module, then S is a self-CS-Baer right S-module (Corollary 3.10.4).

This thesis relies on our results from the papers [18, 26, 27, 28] and the preprints [29, 30]. The papers have been published or accepted for publication in *Journal of Algebra and its Applications*, *Bulletin of the Belgian Mathematical Society "Simon Stevin"*, *Quaestiones Mathematicae* and *Communications in Algebra*. Additionally, parts of this thesis have been communicated in several international conferences.

In this summary we omit stating dual results in abelian categories.

Chapter 1

CS-Rickart objects in abelian categories

We introduce and study the concept of CS-Rickart object in abelian categories as a common generalization of Rickart object and extending object. We establish several characterizations of CS-Rickart objects, we study direct summands and (co)products of such objects and we analyze classes all of whose objects are CS-Rickart. Except for the cited results, all other results are original, and are included in our papers [26] and [28].

1.1 Preliminaries

We begin by setting some general notation and terminology, which will be used throughout the thesis.

Let $\mathcal{A}$ be an abelian category. For every morphism $f : M \to N$ in $\mathcal{A}$ we denote by $\mathrm{ker}(f) : \mathrm{Ker}(f) \to M$, $\mathrm{coker}(f) : N \to \mathrm{Coker}(f)$, $\mathrm{coim}(f) : M \to \mathrm{Coim}(f)$ and $\mathrm{im}(f) : \mathrm{Im}(f) \to N$ the kernel, the cokernel, the coimage and the image of f respectively. Note that $\mathrm{Coim}(f) \cong \mathrm{Im}(f)$, because $\mathcal{A}$ is abelian. For a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we also write $C = B/A$. A morphism $f : A \to B$ is called a *section* if there is a morphism $f' : B \to A$ such that $f'f = 1_A$, and a *retraction* if there is a morphism $f' : B \to A$ such that $ff' = 1_B$.

An abelian category is called *AB3* (*AB3**) if it has arbitrary coproducts (products). Note that AB3 (AB3*) abelian categories have arbitrary sums (intersections). The category $\mathrm{Mod}(R)$ of right modules over a unitary ring R and the category ${}^C\mathcal{M}$ of left comodules over a coalgebra C over a field (see [33, Corollary 2.2.8]) are typical examples of Grothendieck categories, hence also AB3 and AB3* abelian categories. For further information on abelian categories the reader is referred to [41, 72].

1.2 Relative CS-Rickart objects

Now we introduce the main concepts of the thesis, namely (strongly) relative CS-Rickart objects, which generalize both (strongly) relative Rickart objects and (strongly) extending objects in abelian categories. We also give their duals.

Definition 1.2.1. Let M and N be objects of an abelian category $\mathcal{A}$. Then N is called:

(1) *(strongly) M-CS-Rickart* if for every morphism $f : M \to N$ there are an essential monomorphism $e : \mathrm{Ker}(f) \to L$ and a (fully invariant) section $s : L \to M$ in $\mathcal{A}$ such that $\ker(f) = se$. Equivalently, N is M-CS-Rickart if and only if for every morphism $f : M \to N$, $\mathrm{Ker}(f)$ is essential in a (fully invariant) direct summand of M.

(2) *dual (strongly) M-CS-Rickart* if for every morphism $f : M \to N$ there are a (fully coinvariant) retraction $r : N \to P$ and a superfluous epimorphism $t : P \to \mathrm{Coker}(f)$ in $\mathcal{A}$ such that $\mathrm{coker}(f) = tr$. Equivalently, N is dual M-CS-Rickart if and only if for every morphism $f : M \to N$, $\mathrm{Im}(f)$ lies above a (fully invariant) direct summand of N.

(3) *(strongly) self-CS-Rickart* if N is (strongly) N-CS-Rickart.

(4) *dual (strongly) self-CS-Rickart* if N is dual (strongly) N-CS-Rickart.

Next let us see how relative CS-Rickart and strongly relative CS-Rickart objects further relate to each other.

Proposition 1.2.2. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that every direct summand of M is isomorphic to a subobject of N. Then N is strongly M-CS-Rickart if and only if N is M-CS-Rickart and M is weak duo.*

Corollary 1.2.3. *Let M be an object of an abelian category $\mathcal{A}$. Then M is strongly self-CS-Rickart if and only if M is self-CS-Rickart and weak duo.*

Proposition 1.2.4. *Let M be an object of an abelian category $\mathcal{A}$. Then M is strongly self-CS-Rickart if and only if M is self-CS-Rickart and* $\mathrm{End}_{\mathcal{A}}(M)$ *is abelian.*

Now we consider some examples in module categories.

Example 1.2.5. Consider the matrix ring $R = \begin{pmatrix} K & M \\ 0 & D \end{pmatrix}$, where K is a field, D is an integral domain which is not a field and contains K, the Jacobson radical $\mathrm{Rad}(D) = 0$, and M is a torsion-free D-module. For instance, one may consider $R = \begin{pmatrix} K & K[X] \\ 0 & K[X] \end{pmatrix}$.

Then R is a self-Rickart right R-module, and thus it is a self-CS-Rickart right R-module, but it is not a strongly self-CS-Rickart right R-module, because R is not abelian. Also, R is not a dual (strongly) self-CS-Rickart right R-module.

Example 1.2.6. Let A be a ring, and let G be a subgroup of the group $\mathrm{Aut}(A)$ of ring automorphisms of A. The skew group ring is given by $A * G = \bigoplus_{g\in G} Ag$ with addition given componentwise and multiplication defined by $(ag)(bh) = ab^{g^{-1}}gh \in Agh$ for every $a, b \in A$ and $g, h \in G$. If $G = \{g_1, g_2, \dots, g_n\}$, $a \in A$ and $\beta = a_1g_1 + a_2g_2 + \ldots + a_ng_n \in A * G$ with $a_i \in A$, define

$$a \cdot \beta = a^{g_1}a_1^{g_1} + a^{g_2}a_2^{g_2} + \ldots + a^{g_n}a_n^{g_n}.$$

Then A is a right $A * G$-module.

Following [56, Example 3.13], consider the ring $A = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. If $g \in \operatorname{Aut}(A)$ is the conjugation by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it follows that $G = \{1, g\}$ is a subgroup of $\operatorname{Aut}(A)$. Consider the skew group ring $R = A * G$ and the right R-module $M = A$. Then the right R-module M is self-CS-Rickart, and its endomorphism ring is abelian. Hence the right R-module M is strongly self-CS-Rickart by Proposition 1.2.4. But it is not strongly self-Rickart [56, Example 3.13].

Example 1.2.7. If M is a uniserial object (i.e., its subobject lattice is a chain) of an abelian category, then M is clearly strongly self-CS-Rickart and dual strongly self-CS-Rickart, but neither (strongly) self-Rickart, nor dual (strongly) self-Rickart. We give some example in comodule categories, which are known to be Grothendieck [33, Corollary 2.2.8]. Let C be a vector space with basis $\{c_n \mid n \in \mathbb{N}\}$ over a field K. Then C is a coalgebra over K with comultiplication and counit defined for every $n \in \mathbb{N}$ by $\Delta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i}$ and $\varepsilon(c_n) = \delta_{0n}$ (the Kronecker symbol) respectively. This is a coalgebra called the divided power coalgebra [33, Example 1.1.4], for which the dual algebra C^* is isomorphic to the algebra $K[[X]]$ of formal power series and the category of C-comodules is isomorphic to the category of torsion $K[[X]]$-modules [33, Examples 1.3.8, 3.2.7]. Since C is a uniserial (left and right) C-comodule [31, Example 1.4], C is a strongly self-CS-Rickart and dual strongly self-CS-Rickart C-comodule.

Other examples in abelian categories can be obtained by using the transfer of the (strong) CS-Rickart property via functors developed in Chapter 2.

The following theorem, which generalizes [1, Lemmas 6,7], further relates relative Rickart and relative CS-Rickart objects.

Theorem 1.2.8. *Let M and N be objects of an abelian category $\mathcal{A}$. Then N is (strongly) M-CS-Rickart and N is M-$\mathcal{K}$-nonsingular if and only if N is (strongly) M-Rickart.*

Theorem 1.2.8 yields the following corollary in module categories.

Corollary 1.2.9. *Every non-singular (strongly) self-CS-Rickart right R-module is (strongly) self-Rickart and every non-cosingular dual (strongly) self-CS-Rickart right R-module is dual (strongly) self-Rickart.*

1.3 Direct summands of relative CS-Rickart objects

As in the case of (strongly) relative Rickart objects and (strongly) extending objects, we see that (strongly) relative CS-Rickart objects are well behaved with respect to direct summands.

Theorem 1.3.1. *Let $r : M \to M'$ be an epimorphism and $s : N' \to N$ a monomorphism in an abelian category $\mathcal{A}$. If r is a retraction and N is (strongly) M-CS-Rickart, then N' is (strongly) M'-CS-Rickart.*

The following consequence of Theorem 1.3.1 generalizes [1, Lemma 1] from the category of modules to abelian categories.

Corollary 1.3.2. *Let M and N be objects of an abelian category $\mathcal{A}$, M' a direct summand of M and N' a direct summand of N. If N is (strongly) M-CS-Rickart, then N' is (strongly) M'-CS-Rickart.*

In case of self-CS-Rickart objects, we need the following generalizations of SIP (SSIP) property, inspired by the corresponding module-theoretic notions [2, 51].

Definition 1.3.3. An object M of an abelian category $\mathcal{A}$ is called *SSIP-extending* (*SIP-extending*) if for any family of (two) subobjects of M which are essential in direct summands of M, their intersection is essential in a direct summand of M.

Next we consider some strict versions of SSIP-extending and SIP-extending objects in abelian categories.

Definition 1.3.4. An object M of an abelian category $\mathcal{A}$ is called *strictly SSIP-extending* (*strictly SIP-extending*) if for any family of (two) subobjects of M which are essential in direct summands of M, their intersection is essential in a fully invariant direct summand of M.

Proposition 1.3.5. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that every direct summand of M is isomorphic to a subobject of N, and N is (strongly) M-CS-Rickart. Then M is (strictly) SIP-extending.*

The following consequence generalizes [1, Propositions 1,2].

Corollary 1.3.6. *Let $\mathcal{A}$ be an abelian category. Every (strongly) self-CS-Rickart object of $\mathcal{A}$ is (strictly) SIP-extending.*

The following lemma, which generalizes [2, Proposition 3.7], will be useful.

Lemma 1.3.7. *Let A and B be objects of an abelian category $\mathcal{A}$. If $A \oplus B$ is (strictly) SIP-extending, then B is (strongly) A-CS-Rickart.*

Corollary 1.3.8. *Let M be an object of an abelian category $\mathcal{A}$. If $M \oplus M$ is (strictly) SIP-extending, then M is (strongly) self-CS-Rickart.*

1.4 Coproducts of relative CS-Rickart objects

Now we analyze the behaviour of relative (strongly) CS-Rickart and, in particular, (strongly) self-CS-Rickart objects with respect to coproducts. We begin with the study of finite coproducts of (strongly) relative CS-Rickart objects.

Theorem 1.4.1. *Let $\mathcal{A}$ be an abelian category. Let M, N_1 and N_2 be objects of $\mathcal{A}$ such that N_1 and N_2 are (strongly) M-CS-Rickart. Then $N_1 \oplus N_2$ is (strongly) M-CS-Rickart.*

Now we may immediately deduce our main theorem on finite coproducts involving relative CS-Rickart objects.

Theorem 1.4.2. *Let $\mathcal{A}$ be an abelian category. Let M and $N_1, \dots, N_n$ be objects of $\mathcal{A}$. Then $\bigoplus_{i=1}^n N_i$ is (strongly) M-CS-Rickart if and only if N_i is (strongly) M-CS-Rickart for every $i \in \{1, \dots, n\}$.*

We illustrate the above result with an application to module categories. Following again [37, 74] and using the notation preceding Corollary 3.2.4, for any right R-module M, denote by $Z_2(M)$ and $\overline{Z}^2(M)$ the submodules of M determined by the equalities $Z_2(M)/Z(M) = Z(M/Z(M))$ and $\overline{Z}^2(M) = \overline{Z}(\overline{Z}(M))$.

Corollary 1.4.3. *Let M be a right R-module. Then:*

(1) $Z_2(M)$ and $M/Z_2(M)$ are (strongly) M-CS-Rickart if and only if $Z_2(M)$ is a direct summand of M and M is (strongly) self-CS-Rickart.

(2) M is dual (strongly) $\overline{Z}^2(M)$-CS-Rickart and dual (strongly) $M/\overline{Z}^2(M)$-CS-Rickart if and only if $\overline{Z}^2(M)$ is a direct summand of M and M is dual (strongly) self-CS-Rickart.

Under some finiteness conditions we have the following result.

Corollary 1.4.4. *Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ has coproducts, let M be a finitely generated object of $\mathcal{A}$, and let $(N_i)_{i\in I}$ be a family of objects of $\mathcal{A}$. Then $\bigoplus_{i\in I} N_i$ is (strongly) M-CS-Rickart if and only if N_i is (strongly) M-CS-Rickart for every $i \in I$.*

We also have the next theorem on products of relative CS-Rickart objects.

Theorem 1.4.5. *Let $\mathcal{A}$ be an abelian category. Let M be an (a strictly) SSIP-extending object of $\mathcal{A}$, and let $(N_i)_{i\in I}$ be a family of objects of $\mathcal{A}$ having a product. Then $\prod_{i\in I} N_i$ is (strongly) M-CS-Rickart if and only if N_i is (strongly) M-CS-Rickart for every $i \in I$.*

In general the coproduct of two (strongly) self-CS-Rickart objects is not (strongly) self-CS-Rickart, as we may see in the following example.

Example 1.4.6. Consider the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and the right R-modules $M_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$. Then M_1 and M_2 are self-Rickart, hence also self-CS-Rickart right R-modules [56, Example 2.9]. Since $\operatorname{End}(M_1) \cong \mathbb{Z} \cong \operatorname{End}(M_2)$, M_1 and M_2 are strongly self-CS-Rickart by Proposition 1.2.4. But we have seen that $R = M_1 \oplus M_2$ is not a self-CS-Rickart right R-module. Thus $R = M_1 \oplus M_2$ is not strongly self-CS-Rickart again by Proposition 1.2.4.

Nevertheless, we have the following results.

Theorem 1.4.7. *Let $\mathcal{A}$ be an abelian category, and let $M = \bigoplus_{i\in I} M_i$ be a direct sum decomposition in $\mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(M_i, M_j) = 0$ for every $i, j \in I$ with $i \neq j$. Then M is self-CS-Rickart if and only if M_i is self-CS-Rickart for each $i \in I$.*

Theorem 1.4.8. *Let $\mathcal{A}$ be an abelian category, and let $M = \bigoplus_{i\in I} M_i$ be a direct sum decomposition in $\mathcal{A}$.*

(1) Then M is strongly self-CS-Rickart if and only if M_i is strongly self-CS-Rickart for each $i \in I$ and $\mathrm{Hom}_{\mathcal{A}}(M_i, M_j) = 0$ for every $i, j \in I$ with $i \neq j$.

(2) Assume that I is finite. Then M is dual strongly self-CS-Rickart if and only if M_i is dual strongly self-CS-Rickart for each $i \in I$ and $\mathrm{Hom}_{\mathcal{A}}(M_i, M_j) = 0$ for every $i, j \in I$ with $i \neq j$.

We end this section with some results on the structure of (dual) strongly self-CS-Rickart modules over a Dedekind domain.

Corollary 1.4.9. *Let R be a Dedekind domain with quotient field K, and let M be a non-zero R-module.*

(i) Assume that M is torsion. The following are equivalent:

  (1) M is strongly self-CS-Rickart.

  (2) M is dual strongly self-CS-Rickart.

  (3) M is weak duo.

  (4) $M \cong \bigoplus_{i \in I} M_i$, where for each $i \in I$, either $M_i \cong E(R/P_i)$ or $M_i \cong R/P_i^{n_i}$ for some distinct maximal ideals P_i of R and positive integers n_i.

(ii) Assume that M is finitely generated.

  (1) The following are equivalent:

    (a) M is strongly self-CS-Rickart.

    (b) M is weak duo.

    (c) $M \cong J$ for some ideal J of R or $M \cong \bigoplus_{i=1}^{k} R/P_i^{n_i}$ for some distinct maximal ideals $P_1, \dots, P_k$ of R and positive integers $n_1, \dots, n_k$.

  (2) The following are equivalent:

    (a) M is dual strongly self-CS-Rickart.

    (b) $M \cong \bigoplus_{i=1}^{k} R/P_i^{n_i}$ for some distinct maximal ideals P_i of R and positive integers $n_1, \dots, n_k$.

(iii) Assume that M is injective. Then the following are equivalent:

  (1) M is strongly self-CS-Rickart.

  (2) M is dual strongly self-CS-Rickart.

  (3) $M \cong K$ or $M \cong \bigoplus_{i \in I} E(R/P_i)$ for some distinct maximal ideals P_i of R.

1.5 Classes all of whose objects are self-CS-Rickart

In this section we obtain several characterizations of classes all of whose objects are (dual) (strongly) self-CS-Rickart, mainly in connection with (weak duo) injective, (weak duo) projective, (strongly) extending and (strongly) lifting objects.

Theorem 1.5.1. *Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ has enough injectives. Let $\mathcal{C}$ be a class of objects of $\mathcal{A}$ which is closed under binary direct sums and contains all injective objects of $\mathcal{A}$. Then the following are equivalent:*

- *(i) Every object of $\mathcal{C}$ is (strongly) extending.*
- *(ii) Every object of $\mathcal{C}$ is (strongly) self-CS-Rickart.*
- *(iii) Every object of $\mathcal{C}$ is (strictly) SIP-extending.*

We deduce a number of corollaries of Theorem 1.5.1.

Corollary 1.5.2. *Let $\mathcal{A}$ be an abelian category. The following are equivalent:*

- *(i) Every object of $\mathcal{A}$ has an injective envelope.*
- *(ii) $\mathcal{A}$ has enough injectives and every injective object of $\mathcal{A}$ is extending.*
- *(iii) $\mathcal{A}$ has enough injectives and every injective object of $\mathcal{A}$ is self-CS-Rickart.*
- *(iv) $\mathcal{A}$ has enough injectives and every injective object of $\mathcal{A}$ is SIP-extending.*

Note that every object of a Grothendieck category $\mathcal{A}$ has an injective envelope, so every injective object of $\mathcal{A}$ is extending, and consequently self-CS-Rickart, by Corollary 1.5.2.

As consequences of Theorem 1.5.1 and Corollary 1.5.2 for module categories we obtain the following partially known results (see [1, Lemmas 5, 11]).

Corollary 1.5.3. *The following are equivalent for a unitary ring R with Jacobson radical $J(R)$:*

- *(i) Every right R-module is extending.*
- *(ii) Every right R-module is self-CS-Rickart.*
- *(iii) Every right R-module is SIP-extending.*
- *(iv) Every right R-module is lifting.*
- *(v) Every right R-module is dual self-CS-Rickart.*
- *(vi) Every right R-module is SSP-lifting.*
- *(vii) R is a left and right artinian serial ring with $(J(R))^2 = 0$.*

Corollary 1.5.4. *The following are equivalent for a unitary ring R:*

- *(i) R is right perfect.*
- *(ii) Every projective right R-module is lifting.*
- *(iii) Every projective right R-module is dual self-CS-Rickart.*
- *(iv) Every projective right R-module is SSP-lifting.*

Corollary 1.5.5. *The following are equivalent for a coalgebra C over a field:*

- (i) *C is right perfect.*
- (ii) *C is right semiperfect and every projective right C-comodule is lifting.*
- (iii) *C is right semiperfect and every projective right C-comodule is dual self-CS-Rickart.*
- (iv) *C is right semiperfect and every projective right C-comodule is SSP-lifting.*

Theorem 1.5.6. *Let $\mathcal{A}$ be a Grothendieck category.*

- (1) *Assume that $\mathcal{A}$ has a family of finitely generated projective generators. Then the following are equivalent:*
  - (i) *Every (finitely generated) projective object of $\mathcal{A}$ is weakly (semi)hereditary.*
  - (ii) *Every (finitely generated) projective object of $\mathcal{A}$ is self-CS-Rickart.*
  - (iii) *Every (finitely generated) projective object of $\mathcal{A}$ is SIP-extending.*
- (2) *Assume that $\mathcal{A}$ is locally finitely generated. Then the following are equivalent:*
  - (i) *Every (finitely cogenerated) injective object of $\mathcal{A}$ is weakly (semi)cohereditary.*
  - (ii) *Every (finitely cogenerated) injective object of $\mathcal{A}$ is dual self-CS-Rickart.*
  - (iii) *Every (finitely cogenerated) injective object of $\mathcal{A}$ is SSP-lifting.*

The following corollary extends a part of [1, Theorem 8]. Also, compare it with [2, Theorems 3.11, 3.12] for nonsingular right (semi)hereditary rings. Note that a right weakly hereditary ring is the same as a right Σ-extending ring (or right co-H-ring) [37, Corollary 11.13].

Corollary 1.5.7. *Let R be a unitary ring.*

- (1) *The following are equivalent:*
  - (i) *R is right weakly (semi)hereditary.*
  - (ii) *Every (finitely generated) projective right R-module is weakly (semi)hereditary.*
  - (iii) *Every (finitely generated) projective right R-module is self-CS-Rickart.*
  - (iv) *Every (finitely generated) projective right R-module is SIP-extending.*
- (2) *The following are equivalent:*
  - (i) *Every (finitely cogenerated) injective right R-module is weakly (semi)cohereditary.*
  - (ii) *Every (finitely cogenerated) injective right R-module is dual self-CS-Rickart.*
  - (iii) *Every (finitely cogenerated) injective right R-module is SSP-lifting.*

Corollary 1.5.8. *Let C be a coalgebra over a field.*

- (1) *Assume that C is left and right semiperfect. Then the following are equivalent:*

  - *(i) Every (finitely generated) projective right C-comodule is weakly (semi)hereditary.*
  - *(ii) Every (finitely generated) projective right C-comodule is self-CS-Rickart.*
  - *(iii) Every (finitely generated) projective right C-comodule is SIP-extending.*

(2) The following are equivalent:

  - *(i) Every (finitely cogenerated) injective right C-comodule is weakly (semi)cohereditary.*
  - *(ii) Every (finitely cogenerated) injective right C-comodule is dual self-CS-Rickart.*
  - *(iii) Every (finitely cogenerated) injective right C-comodule is SSP-lifting.*

We continue with some results on classes all of whose objects are (dual) strongly self-CS-Rickart.

Theorem 1.5.9. *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category.*

(1) The following are equivalent:

  - *(i) Every finitely cogenerated object of $\mathcal{A}$ is weak duo semisimple.*
  - *(ii) Every finitely cogenerated object of $\mathcal{A}$ is strongly self-CS-Rickart.*
  - *(iii) Every finitely cogenerated object of $\mathcal{A}$ is weak duo and every finitely cogenerated injective object of $\mathcal{A}$ is strongly self-CS-Rickart.*

(2) The following are equivalent:

  - *(i) Every finitely generated object of $\mathcal{A}$ is weak duo regular.*
  - *(ii) Every finitely generated object of $\mathcal{A}$ is dual strongly self-CS-Rickart.*
  - *(iii) Every finitely generated object of $\mathcal{A}$ is weak duo and every finitely generated projective object of $\mathcal{A}$ is dual strongly self-CS-Rickart.*

Theorem 1.5.10. *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category.*

(1) The following are equivalent:

  - *(i) Every (finitely generated) subobject of a projective object of $\mathcal{A}$ is weak duo projective.*
  - *(ii) Every (finitely generated) projective object of $\mathcal{A}$ is strongly self-CS-Rickart.*

(2) The following are equivalent:

  - *(i) Every (finitely cogenerated) factor object of an injective object of $\mathcal{A}$ is weak duo injective.*
  - *(ii) Every (finitely cogenerated) injective object of $\mathcal{A}$ is dual strongly self-CS-Rickart.*

Chapter 2

Transfer of CS-Rickart property via functors

In the present chapter we study the transfer of CS-Rickart property via functors between abelian categories. We consider fully faithful functors and adjoint pairs of functors between abelian categories, under some reasonable assumptions. We discuss applications to Giraud and co-Giraud subcategories, functor categories, localizing and colocalizing subcategories, adjoint triples of functors, Frobenius functors and recollements between abelian categories. We derive consequences for endomorphism rings of (graded) modules and comodules. Except for the cited results, all other results are original, and are included in our papers [27, 29].

2.1 Transfer via fully faithful functors

In general, a functor between abelian categories need not preserve or reflect (strongly) CS-Rickart properties, as we may see in the following examples.

Example 2.1.1. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$ and the forgetful covariant functor $F : \mathrm{Mod}(R) \to \mathrm{Mod}(\mathbb{Z})$ between module categories. Then F is an exact faithful functor which is not full. Note that $\mathbb{Z}_2$ and $\mathbb{Z}_{16}$ are right self-injective rings [54, Corollary 3.13]. Then $R = \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$ is also a right self-injective ring [54, Corollary 3.11B]. Hence the right R-module R is extending [54, Corollary 6.80], and so it is self-CS-Rickart. It is also strongly self-CS-Rickart by Proposition 1.2.4. But the $\mathbb{Z}$-module $F(R) = \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$ is not (strongly) self-CS-Rickart.

Example 2.1.2. For every $\mathbb{Z}$-module M, denote by $t(M)$ the largest torsion submodule of M and by $d(M)$ the largest divisible (i.e., injective) submodule of M. Let $\mathcal{A}$ be the category of $\mathbb{Z}$-modules, and let $\mathcal{B}$ be the category of torsion $\mathbb{Z}$-modules. Note that both $\mathcal{A}$ and $\mathcal{B}$ are abelian categories. Consider the covariant functor $F : \mathcal{A} \to \mathcal{B}$ defined by $F(M) = t(d(M))$ on objects M of $\mathcal{A}$ and accordingly on homomorphisms. Then F is a left exact functor. Also, F is a full, but not faithful functor [21, Example 4.1]. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{p^\infty}$ for some prime p. Then $F(M) = \mathbb{Z}_{p^\infty}$ is a torsion injective $\mathbb{Z}$-module, hence it is an injective object in the category $\mathcal{B}$. Moreover, it is well known that every object of $\mathcal{B}$ has an injective envelope, namely the injective envelope of a torsion $\mathbb{Z}$-module A is $t(E(A))$, where $E(A)$ is the injective envelope of A in $\mathcal{A}$. Then the injective object $F(M)$ of $\mathcal{B}$ is a self-CS-Rickart object in $\mathcal{B}$ by Corollary 1.5.2. It is also strongly self-CS-Rickart. By Corollary 1.3.2, the $\mathbb{Z}$-module M is not self-CS-Rickart, because its direct summand $\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$ is not self-CS-Rickart. Therefore the $\mathbb{Z}$-module M is not strongly self-CS-Rickart.

We continue with a result on preservation and reflection of relative (strongly) CS-Rickart properties and their duals via fully faithful functors under some suitable conditions. For a covariant functor $F : \mathcal{A} \to \mathcal{B}$, we denote by $\mathrm{Im}(F)$ the essential image of F, which consists of all objects B of $\mathcal{B}$ such that $B \cong F(A)$ for some object A of $\mathcal{A}$.

Theorem 2.1.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful covariant functor between abelian categories. Let M and N be objects of $\mathcal{A}$.*

(i) Assume that $\mathrm{Im}(F)$ *is closed under subobjects or quotients. If F is left exact and N is (strongly) M-CS-Rickart, then $F(N)$ is (strongly) $F(M)$-CS-Rickart.*

(ii) Assume that $\mathrm{Im}(F)$ *is closed under direct summands. If F is left exact and $F(N)$ is (strongly) $F(M)$-CS-Rickart, then N is (strongly) M-CS-Rickart.*

The following result is immediate.

Corollary 2.1.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be an equivalence of abelian categories, and let M, N be objects of $\mathcal{A}$. Then N is (dual) (strongly) M-CS-Rickart if and only if $F(N)$ is (dual) (strongly) $F(M)$-CS-Rickart.*

Theorem 2.1.3 may be used to produce new examples of (strongly) relative CS-Rickart objects and their duals. We illustrate this in the following example.

Example 2.1.5. Let $\mathcal{T}$ be a hereditary torsion class (i.e., a class closed under subobjects, coproducts, quotients and extensions) of an abelian category $\mathcal{A}$. Then $\mathcal{T}$ is an abelian category. Let M and N be objects of $\mathcal{T}$. By using Theorem 2.1.3 for the embedding functor $F : \mathcal{T} \to \mathcal{A}$, it follows that N is (strongly) M-CS-Rickart in $\mathcal{T}$ if and only if $F(N)$ is (strongly) $F(M)$-CS-Rickart in $\mathcal{A}$. For instance, consider the abelian group $G = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^n}$ for some distinct primes p and q and positive integer n. Then G is a strongly self-CS-Rickart abelian group, hence G is strongly self-CS-Rickart in $\mathcal{T}$ as well.

2.2 Transfer via adjoint functors

Next we show that (dual) relative CS-Rickart properties transfer via adjoint functors under reasonable conditions. Let (L, R) be an adjoint pair of covariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ between abelian categories. Denote by $\varepsilon : LR \to 1_{\mathcal{B}}$ and $\eta : 1_{\mathcal{A}} \to RL$ the counit and the unit of adjunction respectively. Following [11], denote by $\mathrm{Stat}(R)$ the full subcategory of $\mathcal{B}$ consisting of *R-static* objects, that is, objects B of $\mathcal{B}$ such that $\varepsilon_B : LR(B) \to B$ is an isomorphism. Also, denote by $\mathrm{Adst}(R)$ the full subcategory of $\mathcal{A}$ consisting of *R-adstatic* objects, that is, objects A of $\mathcal{A}$ such that $\eta_A : A \to RL(A)$ is an isomorphism. Note that R is fully faithful if and only if every object of $\mathcal{B}$ is R-static, while L is fully faithful if and only if every object of A is R-adstatic.

Theorem 2.2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let (L, R) be an adjoint pair of covariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$. Assume that L is exact, and let $M, N \in \mathrm{Stat}(R)$.*

- *(i) If N is (strongly) M-CS-Rickart in $\mathcal{B}$, then $R(N)$ is (strongly) $R(M)$-CS-Rickart in $\mathcal{A}$.*
- *(ii) If R reflects zero objects, in particular if R is faithful, and $R(N)$ is (strongly) $R(M)$-CS-Rickart in $\mathcal{A}$, then N is (strongly) M-CS-Rickart in $\mathcal{B}$.*

The main consequence of Theorem 2.2.1 is the following result.

Theorem 2.2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let (L, R) be an adjoint pair of covariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$. Assume that L is exact and R is fully faithful. Let M and N be objects of $\mathcal{B}$. Then N is (strongly) M-CS-Rickart in $\mathcal{B}$ if and only if $R(N)$ is (strongly) $R(M)$-CS-Rickart in $\mathcal{A}$.*

2.3 Applications

We give several applications of our theorems, showing that their hypotheses hold in a large number of relevant situations.

Giraud and co-Giraud subcategories

Let $\mathcal{A}$ be an abelian category and let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is called a *reflective* (*coreflective*) subcategory of $\mathcal{A}$ if the inclusion functor $i : \mathcal{C} \to \mathcal{A}$ has a left (right) adjoint. In any of the two cases, i is fully faithful. If $\mathcal{C}$ is a reflective (coreflective) subcategory of $\mathcal{A}$ and the left (right) adjoint of the inclusion functor i preserves kernels (cokernels), then the subcategory $\mathcal{C}$ is called *Giraud* (*co-Giraud*). In this case, the left (right) adjoint of i is exact.

Corollary 2.3.1. *Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ a full subcategory of $\mathcal{A}$ and $i : \mathcal{C} \to \mathcal{A}$ the inclusion functor. Assume that $\mathcal{C}$ is a Giraud subcategory of $\mathcal{A}$. Let M and N be objects of $\mathcal{C}$. Then N is (strongly) M-CS-Rickart in $\mathcal{C}$ if and only if $i(N)$ is (strongly) $i(M)$-CS-Rickart in $\mathcal{A}$.*

For Grothendieck categories we have the following corollary.

Corollary 2.3.2. *Let $\mathcal{A}$ be a Grothendieck category with a generator U with $R = \mathrm{End}_{\mathcal{A}}(U)$. Let $S = \mathrm{Hom}_{\mathcal{A}}(U, -) : \mathcal{A} \to \mathrm{Mod}(R)$ and let $T : \mathrm{Mod}(R) \to \mathcal{A}$ be a left adjoint of S. Let M and N be objects of $\mathcal{A}$. Then N is an (a strongly) M-CS-Rickart object of $\mathcal{A}$ if and only if $S(N)$ is an (a strongly) $S(M)$-CS-Rickart right R-module.*

Following [33, Section 2.2], let C be a coalgebra over a field k, and let ${}^{C}\mathcal{M}$ be the (Grothendieck) category of left C-comodules. Left C-comodules will be identified with rational right C^*-modules, where $C^* = \mathrm{Hom}_k(C, k)$. Let $i : {}^{C}\mathcal{M} \to \mathrm{Mod}(C^*)$ be the inclusion functor, and let $\mathrm{Rat} : \mathrm{Mod}(C^*) \to {}^{C}\mathcal{M}$ be the functor which associates to every right C^*-module its rational C^*-submodule. Then (i, Rat) is an adjoint pair, hence ${}^{C}\mathcal{M}$ is a coreflective subcategory of $\mathrm{Mod}(C^*)$. If C is a right semiperfect coalgebra, then the functor Rat is exact [33, Corollary 3.2.12], hence ${}^C\mathcal{M}$ is a co-Giraud subcategory of $\mathrm{Mod}(C^*)$. Then Corollary 2.3.1 yields the following consequence.

Corollary 2.3.3. *Let C be a right semiperfect coalgebra over a field. Let M and N be left C-comodules. Then N is a dual (strongly) M-CS-Rickart left C-comodule if and only if $i(N)$ is a dual (strongly) $i(M)$-CS-Rickart right C^*-module.*

Functor categories

We recall some facts on functor categories associated to module categories, following [7, 45]. Let $\mathrm{Mod}(R)$ and $\mathrm{Mod}(R^{\mathrm{op}})$ be the categories of right R-modules and left R-modules respectively. Also, let $\mathrm{mod}(R)$ be the category of finitely presented right R-modules. Let $(\mathrm{mod}(R), \mathrm{Ab})$ be the category of covariant functors from $\mathrm{mod}(R)$ to the category Ab of abelian groups, and $((\mathrm{mod}(R))^{\mathrm{op}}, \mathrm{Ab})$ the category of contravariant functors from $\mathrm{mod}(R)$ to Ab. It is well known that both $(\mathrm{mod}(R), \mathrm{Ab})$ and $((\mathrm{mod}(R))^{\mathrm{op}}, \mathrm{Ab})$ are Grothendieck categories.

Corollary 2.3.4. *(1) Let M and N be right R-modules. Then N is a (strongly) M-CS-Rickart right R-module if and only if $\mathrm{Hom}_R(-, N)$ is a (strongly) $\mathrm{Hom}_R(-, M)$-CS-Rickart object in $((\mathrm{mod}(R))^{\mathrm{op}}, \mathrm{Ab})$.*

(2) Let M and N be left R-modules. Then N is a dual (strongly) M-CS-Rickart left R-module if and only if $- \otimes_R N$ is a dual (strongly) $- \otimes_R M$-CS-Rickart object of $(\mathrm{mod}(R), \mathrm{Ab})$.

Localizing and colocalizing subcategories

Now we have the following consequence of Theorem 2.2.2 (or Corollary 2.3.1).

Corollary 2.3.5. *Let $\mathcal{A}$ be a locally small abelian category, let $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$, and let $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ be the quotient functor. Let M and N be objects of $\mathcal{A}/\mathcal{C}$. Assume that $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$ with section functor $S : \mathcal{A}/\mathcal{C} \to \mathcal{A}$. Then N is (strongly) M-CS-Rickart in $\mathcal{A}/\mathcal{C}$ if and only if $S(N)$ is (strongly) $S(M)$-CS-Rickart in $\mathcal{A}$.*

Adjoint triples

Recall that an *adjoint triple* of functors is a triple (L, F, R) of covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $L, R : \mathcal{B} \to \mathcal{A}$ such that (L, F) and (F, R) are adjoint pairs of functors. Then F is an exact functor. It is known that L is fully faithful if and only if so is R [38, Lemma 1.3]. Now Theorem 2.2.2 yields the following consequence.

Corollary 2.3.6. *Let (L, F, R) be an adjoint triple of covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $L, R : \mathcal{B} \to \mathcal{A}$ between abelian categories.*

(i) Let M and N be objects of $\mathcal{A}$, and assume that F is fully faithful. If L is exact, then N is (strongly) M-CS-Rickart in $\mathcal{A}$ if and only if $F(N)$ is (strongly) $F(M)$-CS-Rickart in $\mathcal{B}$.

(ii) Let M and N be objects of $\mathcal{B}$, and assume that L or R is fully faithful. Then N is (strongly) M-CS-Rickart in $\mathcal{B}$ if and only if $R(N)$ is (strongly) $R(M)$-CS-Rickart in $\mathcal{A}$.

Now let A, C be rings and let ${}_CB_A$ be a bimodule. Let $R = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ be the formal triangular matrix ring constructed from A, B, C. Following [44, Chapter 4, Section A, Exercises 19 and 20], consider the covariant functors:

$$J_{23} : \mathrm{Mod}(C) \to \mathrm{Mod}(R), \quad J_{23}(N) = \left(N\otimes_C B \ {}^{0}_{N} \right),$$

$$J_3 : \mathrm{Mod}(C) \to \mathrm{Mod}(R), \quad J_3(N) = J_{23}(N)/\left(N\otimes_C B \ {}^{0}_{0} \right),$$

$$P_3 : \mathrm{Mod}(R) \to \mathrm{Mod}(C), \quad P_3(M) = M \left(\begin{smallmatrix} 0 & 0 \\ 0 & C \end{smallmatrix} \right).$$

Then (J_{23}, P_3, J_3) is an adjoint triple [44, Chapter 4, Section A, Exercise 22].

Corollary 2.3.7. *Consider the ring $R = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$, where A, C are rings and ${}_CB_A$ is a bimodule. Let M and N be right C-modules. Then:*

(1) N is a (strongly) M-CS-Rickart right C-module if and only if $J_3(N)$ is a (strongly) $J_3(M)$-CS-Rickart right R-module.

(2) N is a dual (strongly) M-CS-Rickart right C-module if and only if $J_{23}(N)$ is a dual (strongly) $J_{23}(M)$-CS-Rickart right R-module.

Following [62], we recall some notation and terminology on graded modules. In what follows G will denote a group with identity element e, and R will be a G-graded ring. For a G-graded ring $R = \bigoplus_{\sigma\in G} R_\sigma$, denote by $\mathrm{gr}(R)$ the (Grothendieck) category which has as objects the G-graded unital right R-modules and as morphisms the morphisms of G-graded unital right R-modules. We consider the following functors:

1. The *induced functor* $\mathrm{Ind} : \mathrm{Mod}(R_e) \to \mathrm{gr}(R)$ defined as follows: for a right R_e-module N, $\mathrm{Ind}(N)$ is the graded right R-module $M = N \otimes_{R_e} R$, where the gradation of $M = \bigoplus_{\sigma\in G} M_\sigma$ is given by $M_\sigma = N_\sigma \otimes_{R_e} R$ for every $\sigma \in G$.
2. The *coinduced functor* $\mathrm{Coind} : \mathrm{Mod}(R_e) \to \mathrm{gr}(R)$ defined as follows: for a right R_e-module N, $\mathrm{Coind}(N)$ is the graded right R-module $M^* = \bigoplus_{\sigma\in G} M'_\sigma$, where

$$M'_\sigma = \{f \in \mathrm{Hom}_{R_e}(R, N) \mid f(R_{\sigma'}) = 0 \text{ for every } \sigma' \neq \sigma^{-1}\}.$$

Corollary 2.3.8. *Let R be a G-graded ring, and let M and N be right R_e-modules. Then:*

(1) N is an (a strongly) M-CS-Rickart right R_e-module if and only if $\mathrm{Coind}(N)$ *is a (strongly)* $\mathrm{Coind}(M)$*-CS-Rickart graded right R-module.*

(2) N is a dual (strongly) M-CS-Rickart right R_e-module if and only if $\mathrm{Ind}(N)$ *is a dual (strongly)* $\mathrm{Ind}(M)$*-CS-Rickart graded right R-module.*

Frobenius functors

Recall that a covariant functor $F : \mathcal{A} \to \mathcal{B}$ is called *Frobenius* if there is a covariant functor $G : \mathcal{B} \to \mathcal{A}$ such that (F, G, F) (or (G, F, G)) is an adjoint triple [10]. Then F and G are exact functors, and Corollary 2.3.6 yields the following consequence.

Corollary 2.3.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful Frobenius functor between abelian categories. Let M and N be objects of $\mathcal{A}$. Then N is (dual) (strongly) M-CS-Rickart in $\mathcal{A}$ if and only if $F(N)$ is (dual) (strongly) $F(M)$-CS-Rickart in $\mathcal{B}$.*

We illustrate the above result with some situations from graded module and comodule categories.

Corollary 2.3.10. *Let R be a G-graded ring, let $U : \mathrm{gr}(R) \to \mathrm{Mod}(R)$ be the forgetful functor and let M, N be objects of $\mathrm{gr}(R)$.*

(1) Assume that G is finite. Then N is a (strongly) M-CS-Rickart graded right R-module if and only if $U(N)$ is a (strongly) $U(M)$-CS-Rickart right R-module.

(2) N is a dual (strongly) M-CS-Rickart graded right R-module if and only if $U(N)$ is a dual (strongly) $U(M)$-CS-Rickart right R-module.

Corollary 2.3.11. *Let C be a finite-dimensional coalgebra over a field. Let $i : {}^{C}\mathcal{M} \to \mathrm{Mod}(C^*)$ be the inclusion functor. Let M and N be left C-comodules. Then N is a (dual) (strongly) M-CS-Rickart left C-comodule if and only if $i(N)$ is a (dual) (strongly) $i(M)$-CS-Rickart right c^*-module.*

Recollements

Now let us recall the concept of recollement of abelian categories, following [68]. For an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, we denote by $\mathrm{Ker}(F)$ the kernel of F, which consists of all objects A of $\mathcal{A}$ such that $F(A) = 0$. A *recollement* of abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, denoted by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, is a diagram of functors:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "i"] & \mathcal{B} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & \mathcal{C} \arrow[l, bend right, "l"'] \arrow[l, bend left, "r"]
\end{tikzcd}$$

which satisfy the following conditions: (i) (l, e, r) is an adjoint triple; (ii) (q, i, p) is an adjoint triple; (iii) i, l and r are fully faithful; (iv) $\mathrm{Im}(i) = \mathrm{Ker}(e)$.

Now Corollary 2.3.6 yields the following consequence.

Corollary 2.3.12. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of abelian categories as above.*

(i) Let M and N be objects of $\mathcal{A}$. If q is exact, then N is (strongly) M-CS-Rickart in $\mathcal{A}$ if and only if $i(N)$ is (strongly) $i(M)$-CS-Rickart in $\mathcal{B}$.

(ii) Let M and N be objects of $\mathcal{C}$. Then N is (strongly) M-CS-Rickart in $\mathcal{C}$ if and only if $r(N)$ is (strongly) $r(M)$-CS-Rickart in $\mathcal{B}$.

Example 2.3.13. Following [56, Example 3.13], consider the ring $A = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$, and the subgroup $G = \{1, g\}$ of $\mathrm{Aut}(A)$, where g is the conjugation by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Consider the skew group ring $A * G$ and the ring A^G of elements of A fixed by the automorphisms of G. Note that A is a left and right $A * G$-module as well as a left and right A^G-module. Following [68, Example 2.9], there are bimodule homomorphisms $\phi : A \otimes_{A^G} A \to A * G$ and $\psi : A \otimes_{A*G} A \to A^G$, which give a Morita ring $\Lambda = \Lambda_{(\phi,\psi)} = \begin{pmatrix} A^G & A \\ A & A * G \end{pmatrix}$. Furthermore, by using the idempotent $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of Λ, this yields the following recollement of module categories (see [68, Definition 2.1]):

$$\begin{tikzcd}
\mathrm{Mod}(A^G/\mathrm{Im}(\psi)) \arrow[r, "i"] & \mathrm{Mod}(\Lambda) \arrow[l, bend right, "-\otimes_{\Lambda}(A^G/\mathrm{Im}(\psi))"'] \arrow[l, bend left, "\mathrm{Hom}_{\Lambda}(A^G/\mathrm{Im}(\psi),-)"] \arrow[r, "(-)e"] & \mathrm{Mod}(A * G) \arrow[l, bend right, "-\otimes_{A*G} e\Lambda"'] \arrow[l, bend left, "\mathrm{Hom}_{A*G}(\Lambda e,-)"]
\end{tikzcd}$$

It is known that A is a self-CS-Rickart right $A * G$-module by Example 1.2.6. Then $\mathrm{Hom}_{A*G}(\Lambda e, A)$ is a self-CS-Rickart right Λ-module by Corollary 2.3.12.

2.4 Endomorphism rings of self-CS-Rickart objects

In this section we give some applications to endomorphism rings of (graded) modules and comodules. We start with a specific result for modules, for which we need to introduce the following concepts.

Definition 2.4.1. A right R-module M is called:

(1) *im-local-retractable* if for every monomorphism $k : K \to M$ and for every $x \in K$, there exists a homomorphism $h : M \to K$ such that $x \in \mathrm{Im}(hk)$.

(2) *im-local-coretractable* if for every epimorphism $c : M \to C$ and for every $z \in C$, there exists a homomorphism $h : C \to M$ such that $z \in \mathrm{Im}(ch)$.

Corollary 2.4.2. *Let M be a right R-module, and let $S = \mathrm{End}_R(M)$.*

(1) If M is im-local-retractable and S is a (strongly) self-CS-Rickart right S-module, then M is a (strongly) self-CS-Rickart right R-module.

(2) If M is im-local-coretractable and S is a dual (strongly) self-CS-Rickart left S-module, then M is a dual (strongly) self-CS-Rickart right R-module.

Following [47], a covariant functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called *faithfully exact* provided the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact if and only if the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact. Let ${}_SM_R$ be a bimodule. The right R-module M is called *faithfully projective* if the functor $\mathrm{Hom}_R(M, -) : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is faithfully exact. The right R-module M is called *faithfully injective* if the functor $\mathrm{Hom}_R(-, M) : \mathrm{Mod}(R) \to \mathrm{Mod}(S^{\mathrm{op}})$ is faithfully exact. The left S-module M is called *faithfully flat* if the functor $- \otimes_S M : \mathrm{Mod}(S) \to \mathrm{Mod}(R)$ is faithfully exact.

Corollary 2.4.3. *Let M be a right R-module and $S = \mathrm{End}_R(M)$.*

(1) Assume that M is a flat left S-module.

  (i) If M is a (strongly) self-CS-Rickart right R-module, then S is a (strongly) self-CS-Rickart right S-module.

  (ii) If M is a faithfully projective right R-module and S is a (strongly) self-CS-Rickart right S-module, then M is a (strongly) self-CS-Rickart right R-module.

(2) Assume that M is a projective right R-module.

  (i) If S is a (strongly) dual self-CS-Rickart right S-module, then M is a dual (strongly) self-CS-Rickart right R-module.

  (ii) If M is a faithfully flat left S-module and M is a dual (strongly) self-CS-Rickart right R-module, then S is a dual (strongly) self-CS-Rickart right S-module.

(3) Assume that M is an injective left S-module.

  (i) If M is a dual (strongly) self-CS-Rickart right R-module, then S is a (strongly) self-CS-Rickart left S-module.

  (ii) If M is a faithfully injective right R-module and S is a (strongly) self-CS-Rickart left S-module, then M is a dual (strongly) self-CS-Rickart right R-module.

Chapter 3

CS-Baer objects in abelian categories

We introduce and study CS-Baer objects in abelian categories, which form a subclass of CS-Rickart objects, and generalize both Baer and extending objects. We show that the theory of CS-Rickart objects may be applied in the study of CS-Baer objects. We investigate CS-Baer objects in relationship with Baer objects, extending objects, objects having certain summand intersection properties and CS-Rickart objects. We also study (co)products of CS-Baer objects, and we determine the complete structure of dual self-CS-Baer modules over Dedekind domains. Finally, we discuss classes all of whose objects are self-CS-Baer, the transfer of CS-Baer property via functors, and give applications to endomorphism rings. Except for the cited results, all other results are original, and are included in our papers [18] and [30].

3.1 Relative CS-Baer objects

We introduce the main notions of this chapter, namely (strongly) CS-Baer objects and their duals in abelian categories.

Definition 3.1.1. Let M and N be objects of an abelian category $\mathcal{A}$.

(1) If $\mathcal{A}$ is AB3*, then N is called:

  (i) *(strongly) M-CS-Baer* if for every family $(f_i)_{i\in I}$ of morphisms $f_i : M \to N$, $\bigcap\limits_{i\in I} \operatorname{Ker}(f_i)$ is essential in a (fully invariant) direct summand of M.

  (ii) *(strongly) self-CS-Baer* if N is (strongly) N-CS-Baer.

(2) If $\mathcal{A}$ is AB3, then N is called:

  (i) *dual (strongly) M-CS-Baer* if for every family $(f_i)_{i\in I}$ of morphisms $f_i : M \to N$, $\sum\limits_{i\in I} \operatorname{Im}(f_i)$ lies above a (fully invariant) direct summand of N.

  (ii) *dual (strongly) self-CS-Baer* if N is dual (strongly) N-CS-Baer.

Relative CS-Baer objects are related to strongly relative CS-Baer objects as follows.

Proposition 3.1.2. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*, and every direct summand of M is isomorphic to a subobject of N. Then N is strongly M-CS-Baer if and only if N is M-CS-Baer and M is weak duo.*

The following corollary generalizes the module-theoretic result [63, Theorem 2], and often will be implicitly used throughout the thesis.

Corollary 3.1.3. *Let M be an object of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. Then the following are equivalent:*

(i) M is strongly self-CS-Baer.

(ii) M is self-CS-Baer and weak duo.

(iii) M is self-CS-Baer and $\mathrm{End}_{\mathcal{A}}(M)$ is abelian.

Next we give some examples and non-examples of (dual) relative CS-Baer and (dual) strongly relative CS-Baer objects. Further examples in connection with the other concepts of the paper will be given in the corresponding sections later on.

Example 3.1.4. Now we consider some example in module categories.

As in Example 1.2.5, consider the matrix ring $R = \begin{pmatrix} K & M \\ 0 & D \end{pmatrix}$, where K is a field, D is an integral domain which is not a field and contains K, the Jacobson radical $\mathrm{Rad}(D) = 0$, and M is a torsion-free D-module. A particular example can be $R = \begin{pmatrix} K & K[X] \\ 0 & K[X] \end{pmatrix}$. Then R is a self-CS-Baer right R-module. It is not a strongly self-CS-Baer right R-module, because $\mathrm{End}_R(R) \cong R$ is not abelian. Note also that the right R-module R is not dual (strongly) self-CS-Baer, because it is not dual (strongly) self-CS-Rickart Exemple 1.2.5.

Example 3.1.5. Every uniserial object of an abelian category is clearly strongly self-CS-Baer and dual strongly self-CS-Baer, but neither (strongly) self-Baer, nor dual (strongly) self-Baer. In particular, the divided power coalgebra C (see Example 1.2.7) is a strongly self-CS-Baer and dual strongly self-CS-Baer C-comodule.

3.2 Relative CS-Baer versus relative Baer objects

Every relative Baer object of an abelian category is clearly relative CS-Baer, but the converse does not hold in general, as the following example show.

Example 3.2.1. Let $n > 1$ be an integer and let p be a prime. Let $M_1 = \mathbb{Z}_{p^n}$ and $M_2 = \mathbb{Z}_{p^n} \oplus \mathbb{Z}_{p^{n+1}}$. Then M_1 and M_2 are extending and weak duo (see [60, Proposition A.12] and [65, Theorem 3.10]), and hence they are strongly self-CS-Baer by Corollary 3.1.3. However, M_1 and M_2 are not strongly self-Baer by [25, Corollary 6.7].

Theorem 3.2.2. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. Then N is (strongly) M-CS-Baer and N is M-$\mathcal{K}$-nonsingular if and only if N is (strongly) M-Baer.*

The following corollary generalizes the module-theoretic result [63, Theorem 1].

Corollary 3.2.3. *Let M be an object of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. Then M is (strongly) self-CS-Baer and M-$\mathcal{K}$-nonsingular if and only if M is (strongly) self-Baer.*

Corollary 3.2.4. *Every non-singular (strongly) self-CS-Baer right R-module is (strongly) self-Baer and every non-cosingular dual (strongly) self-CS-Baer right R-module is dual (strongly) self-Baer.*

3.3 Relative CS-Baer versus extending objects

In this section we are interested in relating the (dual) relative CS-Baer property with the extending (lifting) property. Note that every (dual) self-Rickart object and every extending (lifting) object of an abelian category $\mathcal{A}$ is (dual) self-CS-Rickart by definitions.

Example 3.3.1. (i) We have seen that the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ is self-CS-Baer, but not weak duo. It is neither extending (see [60, p. 19]), nor strongly extending.

(ii) The $\mathbb{Z}$-module $\mathbb{Q}$ is dual strongly self-CS-Baer, but it is not (strongly) lifting. Note that for any $0 \neq f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Q})$, $\mathrm{Im}(f) = \mathbb{Q}$ is indecomposable.

For objects M and N of an abelian category $\mathcal{A}$, denote $U = \mathrm{Hom}_{\mathcal{A}}(M, N)$.

For every subobject X of M and every subobject Z of U, we denote:

$$l_U(X) = \{f \in U \mid X \subseteq \mathrm{Ker}(f)\}, \quad r_M(Z) = \bigcap_{f \in Z} \mathrm{Ker}(f).$$

For every subobject Y of N and every subobject Z of U, we denote:

$$l'_U(Y) = \{f \in U \mid \mathrm{Im}(f) \subseteq Y\}, \quad r'_N(Z) = \sum_{f \in Z} \mathrm{Im}(f).$$

Definition 3.3.2. Let M and N be objects of an abelian category $\mathcal{A}$. Then N is called:

(i) *$\mathcal{E}$-M-$\mathcal{K}$-nonsingular* if for any morphism $f : M \to N$ in $\mathcal{A}$, $\mathrm{Ker}(f)$ essential in a direct summand of M implies $f = 0$.

(ii) *$\mathcal{E}$-M-$\mathcal{K}$-cononsingular* if for any subobjects X and Y of M such that $X \subseteq Y$, $l_U(X) = l_U(Y)$ implies that X is essential in Y.

Theorem 3.3.3. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. If N is (strongly) M-CS-Baer and $\mathcal{E}$-M-$\mathcal{K}$-cononsingular, then M is (strongly) extending.*

Corollary 3.3.4. *Let M be an object of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. If M is (strongly) self-CS-Baer and $\mathcal{E}$-M-$\mathcal{K}$-cononsingular, then M is (strongly) extending.*

Theorem 3.3.5. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3* and any direct summand of M is isomorphic to a subobject of N. If M is (strongly) extending and N is $\mathcal{E}$-M-$\mathcal{K}$-nonsingular, then N is (strongly) M-CS-Baer and $\mathcal{E}$-M-$\mathcal{K}$-cononsingular.*

Corollary 3.3.6. *Let M be an object of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. If M is (strongly) extending and $\mathcal{E}$-M-$\mathcal{K}$-nonsingular, then M is (strongly) self-CS-Baer and $\mathcal{E}$-M-$\mathcal{K}$-cononsingular.*

We continue with the following module-theoretic result, which also shows that the dual (strong) self-CS-Baer property is left-right symmetric for rings.

Proposition 3.3.7. *Let R be a unitary ring. Then the following are equivalent:*

(i) R is a dual (strongly) self-CS-Baer right R-module.

(ii) R is a (strongly) lifting right R-module.

(iii) R is an (abelian) semiperfect ring.

We also give a related characterization for dual (strongly) self-CS-Rickart rings, which shows that the dual (strong) self-CS-Rickart property is left-right symmetric for rings (see [75, Proposition 2.12] for a different proof).

Proposition 3.3.8. *Let R be a unitary ring. Then R is a dual (strongly) self-CS-Rickart right R-module if and only if R is an (abelian) semiregular ring.*

3.4 Relative CS-Baer versus ESSIP objects

We shall see in the next section that CS-Baer and CS-Rickart objects may be related by means of some conditions involving direct summands. In order to get there we need now to prepare the setting.

In the study of (strongly) self-CS-Rickart objects, it is useful to consider the following concepts generalizing SIP (SSIP).

Definition 3.4.1. An object M of an abelian category $\mathcal{A}$ with AB3* is called:

(1) *SIP-extending* (*SSIP-extending*) if for any two (family of) subobjects of M which are essential in direct summands of M, their intersection is essential in a direct summand of M.

(2) *strictly SIP-extending* (*strictly SSIP-extending*) if for any two (family of) subobjects of M which are essential in direct summands of M, their intersection is essential in a fully invariant direct summand of M.

(3) *ESIP* (*ESSIP*) if for any two (family of) direct summands of M, their intersection is essential in a direct summand of M.

(4) *strictly ESIP* (*strictly ESSIP*) if for any two (family of) direct summands of M, their intersection is essential in a fully invariant direct summand of M.

Lemma 3.4.2. *Let M be an object of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3* and the socle* $\mathrm{Soc}(M)$ *of M is essential in M. Then M is (strictly) SSIP-extending if and only if M is (strictly) ESSIP.*

Theorem 3.4.3. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3* and any direct summand of M is isomorphic to a subobject of N. If N is (strongly) M-CS-Baer, then M is (strictly) ESSIP.*

Corollary 3.4.4. *Assume that $\mathcal{A}$ is an AB3* abelian category. Then every (strongly) self-CS-Baer object is (strictly) ESSIP.*

In general the converse of the above corollary does not hold. Nevertheless, we have the following property.

Lemma 3.4.5. *Let M and N be objects of an abelian category $\mathcal{A}$. If $\mathcal{A}$ is AB3* and $M \oplus N$ is (strictly) ESSIP, then N is (strongly) M-CS-Baer.*

Corollary 3.4.6. *Let M be an object of an abelian category $\mathcal{A}$. If $\mathcal{A}$ is AB3* and $M \oplus M$ is (strictly) ESSIP, then M is (strongly) self-CS-Baer.*

3.5 Relative CS-Baer versus relative CS-Rickart objects

Every relative CS-Baer object is clearly relative CS-Rickart. The converse does not hold in general, as we may see in the following example.

Example 3.5.1. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}^{(\mathbb{R})}$. By [56, Remark 2.28], M is not self-Baer, but it is self-Rickart, and hence M is self-CS-Rickart. Since $\mathbb{Z}$ is non-singular, then so is M [44, Proposition 1.22]. Now Corollary 3.2.4 allows us to deduce that M is not self-CS-Baer.

The following proposition shows that one may use the theory of relative CS-Rickart objects in order to develop the theory of relative CS-Baer objects in abelian categories. We illustrate its applications several times throughout the thesis.

Proposition 3.5.2. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. Then N is (strongly) M-CS-Baer if and only if N^I is (strongly) M-CS-Rickart for any set I.*

Next we show how the notions of (strictly) SSIP-extending and (strictly) ESSIP objects are related to that of (strongly) CS-Baer object.

Theorem 3.5.3. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. If N is (strongly) M-CS-Rickart and M is (strictly) SSIP-extending, then N is (strongly) M-CS-Baer. The converse holds if any direct summand of M is isomorphic to a subobject of N, and* $\mathrm{Soc}(M)$ *is an essential subobject of M.*

The following corollary generalizes the module-theoretic result [63, Theorem 3].

Corollary 3.5.4. *Let M and N be objects of an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3*. If M is (strongly) self-CS-Rickart and (strictly) SSIP-extending, then M is (strongly) self-CS-Baer. The converse holds if* $\mathrm{Soc}(M)$ *is an essential subobject of M.*

We present some illustrations of Corollary 3.5.4 in module and comodule categories.

Corollary 3.5.5. *Let R be a unitary ring, and let M be a right R-module.*

(1) Assume that M is finitely cogenerated or R is a right semiartinian ring. Then M is (strongly) self-CS-Baer if and only if M is (strongly) self-CS-Rickart and (strictly) SSIP-extending.

(2) Assume that M is finitely generated or R is a right max ring. Then M is dual (strongly) self-CS-Baer if and only if M is dual (strongly) self-CS-Rickart and (strictly) SSSP-lifting.

Corollary 3.5.6. *Let C be a coalgebra over a field, and let M be a left C-comodule. Then:*

(1) M is (strongly) self-CS-Baer if and only if M is (strongly) self-CS-Rickart and (strictly) SSIP-extending.

(2) If C is right semiperfect, then M is dual (strongly) self-CS-Baer if and only if M is dual (strongly) self-CS-Rickart and (strictly) SSSP-lifting.

3.6 Coproducts of relative CS-Baer objects

Now we analyze the behaviour of (strongly) relative CS-Baer objects with respect to direct summands and (co)products.

Corollary 3.6.1. *Let $r : M \to M'$ be an epimorphism and $s : N' \to N$ a monomorphism in an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ is AB3* and r is a retraction. If N is (strongly) M-CS-Baer, then N' is (strongly) M'-CS-Baer.*

The following corollary generalizes the module-theoretic result [63, Theorem 4].

Corollary 3.6.2. *Let M and N be objects of an abelian category $\mathcal{A}$, M' a direct summand of M and N' a direct summand of N. Assume that $\mathcal{A}$ is AB3*. If N is (strongly) M-CS-Baer, then N' is (strongly) M'-CS-Baer.*

Example 3.6.3. Consider the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and the right R-modules $M_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$. Since M_1 and M_2 are self-Baer right R-modules, M_1 and M_2 are self-CS-Baer. But we have seen in Example 1.4.6 that $R = M_1 \oplus M_2$ is not a self-CS-Rickart right R-module, hence R is not a self-CS-Baer right R-module.

Theorem 3.6.4. *Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ is AB3*. Let $M, N_1, \ldots, N_n$ be objects of $\mathcal{A}$. Then $\bigoplus_{i=1}^{n} N_i$ is (strongly) M-CS-Baer if and only if N_i is (strongly) M-CS-Baer for any $i \in \{1, \ldots, n\}$.*

Theorem 3.6.5. *Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ is AB3*. Let M be a (strictly) SSIP-extending object of $\mathcal{A}$, and let $(N_i)_{i\in I}$ be a family of objects of $\mathcal{A}$. Then $\prod_{i\in I} N_i$ is (strongly) M-CS-Baer if and only if N_i is (strongly) M-CS-Baer for every $i \in I$.*

Next we study the behaviour of the (strong) self-CS-Baer property with respect to direct sum decompositions.

Theorem 3.6.6. *Let $\mathcal{A}$ be an abelian category, and let $M = \bigoplus_{i\in I} M_i$ be a direct sum decomposition in $\mathcal{A}$ for some finite set I.*

(i) If $\mathrm{Hom}_{\mathcal{A}}(M_i, M_j) = 0$ for every $i, j \in I$ with $i \neq j$, then M is self-CS-Baer if and only if M_i is self-CS-Baer for each $i \in I$.

(ii) M is strongly self-CS-Baer if and only if M_i is strongly self-CS-Baer for each $i \in I$ and $\mathrm{Hom}_{\mathcal{A}}(M_i, M_j) = 0$ for every $i, j \in I$ with $i \neq j$.

In case of module categories, we may add a condition that allows us to deal with (possibly) infinite direct sum decompositions as follows.

Theorem 3.6.7. *Let R be a unitary ring, and let $M = \bigoplus_{i\in I} M_i$ be a direct sum decomposition of a right R-module M into submodules M_i such that for every submodule L of M, $L = \bigoplus_{i\in I}(L\cap M_i)$. Then M is (strongly) self-CS-Baer if and only if M_i is (strongly) self-CS-Baer for each $i \in I$.*

3.7 Dual self-CS-Baer modules over Dedekind domains

The aim of this section is to determine the structure of dual (strongly) self-CS-Baer modules over Dedekind domains. The next theorem shows that we can reduce the problem to the case of modules over discrete valuation rings.

Let M be a module over a Dedekind domain R. We denote by $T(M)$ the torsion submodule of M, i.e., $T(M) = \{x \in M \mid \mathrm{Ann}_R(x) \neq 0\}$. Let $\mathbf{P}$ denote the set of non-zero prime ideals of R. For any $\mathfrak{p} \in \mathbf{P}$, the $\mathfrak{p}$-primary component of M will be denoted by $T_{\mathfrak{p}}(M)$, that is, $T_{\mathfrak{p}}(M) = \{x \in M \mid \mathfrak{p}^n x = 0 \text{ for some integer } n \geq 0\}$.

Theorem 3.7.1. *Let R be a non-local Dedekind domain with quotient field K, and let M be an R-module. Then the following are equivalent:*

(i) M is a dual self-CS-Baer module;

(ii) $M = T(M) \oplus L$ such that $T(M)$ is a dual self-CS-Baer module and $L \cong K^{(I)}$ for some index set I;

(iii) $M = \left(\bigoplus_{\mathfrak{p}\in\mathbf{P}} T_{\mathfrak{p}}(M)\right) \oplus L$ such that each $T_p(M)$ ($\mathfrak{p} \in \mathbf{P}$) is a dual self-CS-Baer $R_{\mathfrak{p}}$-module and $L \cong K^{(I)}$ for some index set I.

Now our purpose is to describe the structure of dual self-CS-Baer modules over discrete valuation rings. In the remainder of this section we assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m}$, quotient field K and $Q = K/R$.

For natural numbers n_1, n_2, …, n_s, let $B(n_1, n_2, \ldots, n_s)$ denote the direct sum of arbitrarily many copies of $R/\mathfrak{m}^{n_1}$, $R/\mathfrak{m}^{n_2}$, …, $R/\mathfrak{m}^{n_s}$.

Theorem 3.7.2. *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$, quotient field K and $Q = K/R$. Let I_1 and I_2 be two index sets and let a, b, c and n be non-negative integers. Then an R-module M is dual self-CS-Baer if and only if M is isomorphic to one of the following modules: (i) $K^a \oplus Q^b \oplus R^c$ with $a \leq 1$ if R is incomplete, or; (ii) $K^{(I_1)} \oplus Q^{(I_2)} \oplus B(n)$, or; (iii) $K^{(I_1)} \oplus B(n, n+1)$.*

Let R be a Dedekind domain with quotient field K, and let $\mathfrak{p}$ be a non-zero prime ideal of R. By $B_{\mathfrak{p}}(n_1, \ldots, n_s)$ we denote the direct sum of arbitrarily many copies of $R/\mathfrak{p}^{n_1}$, $R/\mathfrak{p}^{n_2}$, …, $R/\mathfrak{p}^{n_s}$ for some non-negative integers n_1, n_2, …, n_s. We will denote by $R(\mathfrak{p}^\infty)$ the $\mathfrak{p}$-primary component of the torsion R-module K/R. Combining Theorems 3.7.1 and 3.7.2, we obtain the following structure result.

Theorem 3.7.3. *Let R be a non-local Dedekind domain with quotient field K and let M be an R-module. Then the following are equivalent:*

(i) M is a dual self-CS-Baer module;

(ii) $M = (\bigoplus_{\mathfrak{p}\in\mathbf{P}} T_{\mathfrak{p}}(M)) \oplus L$ such that $L \cong K^{(\Lambda)}$ for some index set Λ and for every non-zero prime ideal $\mathfrak{p}$ of R, $T_{\mathfrak{p}}(M) \cong R(\mathfrak{p}^\infty)^{(\Lambda)} \oplus B_{\mathfrak{p}}(n)$ or $T_{\mathfrak{p}}(M) \cong B_{\mathfrak{p}}(n, n+1)$, where Λ is an index set and n is a non-negative integer.

Next, we exhibit some examples of dual self-CS-Rickart modules which are not dual self-CS-Baer.

Example 3.7.4. Let R be a discrete valuation ring with quotient field K. Comparing [75, Theorem 3.14] with Theorem 3.7.2, we obtain many examples of dual self-CS-Rickart R-modules which are not dual self-CS-Baer. For example, for any positive integer n, the R-modules $K^{(\mathbb{N})} \oplus R^n$, $(K/R)^{(\mathbb{N})} \oplus R^n$ and $K^{(\mathbb{N})} \oplus (K/R)^{(\mathbb{N})} \oplus R^n$ are dual self-CS-Rickart, but they are not dual self-CS-Baer.

Combining Theorems 3.6.6 and 3.6.7, Theorems 3.7.2 and 3.7.3, we get the following corollary.

Corollary 3.7.5. *Let R be a Dedekind domain with quotient field K and let M be an R-module.*

(i) If R is a discrete valuation ring, then M is dual strongly self-CS-Baer if and only if $M \cong R$ or $M \cong K$ or $M \cong K/R$ or $M \cong R/\mathfrak{m}^n$ or $M \cong K \oplus R/\mathfrak{m}^n$ for some non-negative integer n.

(ii) If R is not local, then M is dual strongly self-CS-Baer if and only if one of the following conditions holds:

(a) $M = \left(\bigoplus_{\mathfrak{p}\in\mathbf{P}} T_{\mathfrak{p}}(M)\right) \oplus L$ with $L \cong K$ and for every non-zero prime ideal $\mathfrak{p}$ of R, there exists a non-negative integer n_p depending on $\mathfrak{p}$ such that $T_{\mathfrak{p}}(M) \cong R/\mathfrak{p}^{n_p}$.

(b) $M = \bigoplus_{\mathfrak{p}\in\mathbf{P}} T_{\mathfrak{p}}(M)$ such that for every non-zero prime ideal $\mathfrak{p}$ of R, $T_{\mathfrak{p}}(M) \cong R(\mathfrak{p}^\infty)$ or $T_{\mathfrak{p}}(M) \cong R/\mathfrak{p}^{n_p}$ for some non-negative integer n_p depending on $\mathfrak{p}$.

Example 3.7.6. Let R be a Dedekind domain with quotient field K. Comparing Theorem 3.7.3 and Corollary 3.7.5, we see that $K^{(\mathbb{N})}$ is dual self-CS-Baer, but not dual strongly self-CS-Baer.

3.8 Classes all of whose objects are self-CS-Baer

In this section we give some results on classes all of whose objects are (strongly) self-CS-Baer.

Theorem 3.8.1. *Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ has AB3* and enough injectives. Let $\mathcal{C}$ be a class of objects of $\mathcal{A}$ which is closed under binary direct sums and contains all injective objects of $\mathcal{A}$. Then the following are equivalent:*

(i) Every object of $\mathcal{C}$ is (strongly) self-CS-Baer.

(ii) Every object of $\mathcal{C}$ is (strictly) ESSIP.

(iii) Every object of $\mathcal{C}$ is (strongly) self-CS-Rickart.

(iv) Every object of $\mathcal{C}$ is (strictly) SIP-extending.

(v) Every object of $\mathcal{C}$ is (strongly) extending.

Corollary 3.8.2. *Let $\mathcal{A}$ be an abelian category. If $\mathcal{A}$ has AB3*, then the following are equivalent:*

(i) Every object of $\mathcal{A}$ has an injective envelope.

(ii) $\mathcal{A}$ has enough injectives and every injective object of $\mathcal{A}$ is self-CS-Baer.

(iii) $\mathcal{A}$ has enough injectives and every injective object of $\mathcal{A}$ is ESSIP.

(iv) $\mathcal{A}$ has enough injectives and every injective object of $\mathcal{A}$ is self-CS-Rickart.

(v) $\mathcal{A}$ has enough injectives and every injective object of $\mathcal{A}$ is SIP-extending.

(vi) $\mathcal{A}$ has enough injectives and every injective object of $\mathcal{A}$ is extending.

We illustrate these results in module categories as follows.

Corollary 3.8.3. *The following are equivalent for a unitary ring R with Jacobson radical $J(R)$:*

(i) Every right R-module is self-CS-Baer.

(ii) Every right R-module is ESSIP.

(iii) Every right R-module is self-CS-Rickart.

(iv) Every right R-module is SIP-extending.

(v) Every right R-module is extending.

(vi) Every right R-module is dual self-CS-Baer.

(vii) Every right R-module is LSSSP.

(viii) Every right R-module is dual self-CS-Rickart.

(ix) Every right R-module is SSP-lifting.

(x) Every right R-module is lifting.

(xi) R is a left and right artinian serial ring with $(J(R))^2 = 0$.

Corollary 3.8.4. *The following are equivalent for a unitary ring R:*

(i) R is right perfect.

(ii) Every projective right R-module is dual self-CS-Baer.

(iii) Every projective right R-module is LSSSP.

(iv) Every projective right R-module is dual self-CS-Rickart.

(v) Every projective right R-module is SSP-lifting.

(vi) Every projective right R-module is lifting.

Theorem 3.8.5. *Let* $\mathcal{A}$ *be a Grothendieck category.*

(1) Assume that $\mathcal{A}$ *has enough projectives, and the class of projective objects is closed under products. Then the following are equivalent:*

  (i) Every projective object of $\mathcal{A}$ *is weakly hereditary.*
  (ii) Every projective object of $\mathcal{A}$ *is self-CS-Baer.*
  (iii) Every projective object of $\mathcal{A}$ *is ESSIP.*
  (iv) Every projective object of $\mathcal{A}$ *is self-CS-Rickart.*
  (v) Every projective object of $\mathcal{A}$ *is SIP-extending.*

(2) Assume that the class of injective objects is closed under coproducts. Then the following are equivalent:

  (i) Every injective object of $\mathcal{A}$ *is weakly cohereditary.*
  (ii) Every injective object of $\mathcal{A}$ *is dual self-CS-Baer.*
  (iii) Every injective object of $\mathcal{A}$ *is LSSSP.*
  (iv) Every injective object of $\mathcal{A}$ *is dual self-CS-Rickart.*
  (v) Every injective object of $\mathcal{A}$ *is SSP-lifting.*

Corollary 3.8.6. *Let R be a unitary ring.*

(1) Assume that R is right perfect left coherent. Then the following are equivalent:

  (i) Every projective right R-module is weakly hereditary.
  (ii) Every projective right R-module is self-CS-Baer.
  (iii) Every projective right R-module is ESSIP.
  (iv) Every projective right R-module is self-CS-Rickart.
  (v) Every projective right R-module is SIP-extending.

(2) Assume that R is right noetherian. Then the following are equivalent:

- *(i) Every injective right R-module is weakly cohereditary.*
- *(ii) Every injective right R-module is dual self-CS-Baer.*
- *(iii) Every injective right R-module is LSSSP.*
- *(iv) Every injective right R-module is dual self-CS-Rickart.*
- *(v) Every injective right R-module is SSP-lifting.*

3.9 Transfer of CS-Baer property via functors

In this section we study when relative CS-Baer properties transfer via functors between abelian categories. One sees that this is not always the case, by reviewing the same examples from Chapter 2 from the point of view of relative CS-Baer properties.

As in the case of CS-Rickart property, now we consider fully faithful functors and adjoint pairs of functors between abelian categories.

Theorem 3.9.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful covariant functor between abelian categories. Let M and N be objects of $\mathcal{A}$.*

- *(i) Assume that $\mathcal{A}$ and $\mathcal{B}$ have AB3* and* $\mathrm{Im}(F)$ *is closed under subobjects or quotients. If F is left exact and preserves products, and N is (strongly) M-CS-Baer, then $F(N)$ is (strongly) $F(M)$-CS-Baer.*
- *(ii) Assume that $\mathcal{A}$ and $\mathcal{B}$ have AB3 and* $\mathrm{Im}(F)$ *is closed under direct summands. If F is left exact and preserves products, and $F(N)$ is (strongly) $F(M)$-CS-Baer, then N is (strongly) M-CS-Baer.*

Corollary 3.9.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be an equivalence of abelian categories, and let M, N be objects of $\mathcal{A}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ have AB3*. Then N is (strongly) M-CS-Baer if and only if $F(N)$ is (strongly) $F(M)$-CS-Baer.*

For adjoint pairs of functors we also have the following theorem. We have seen in Proposition 3.5.2 that if $\mathcal{A}$ is an AB3* abelian category, then N is (strongly) M-CS-Baer if and only if N^I is (strongly) M-CS-Rickart for any set I. This characterization allows us to deduce properties of CS-Baer objects from the corresponding properties of CS-Rickart objects, and we will make use of it.

Theorem 3.9.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let (L, R) be an adjoint pair of covariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ have AB3*, L is exact, and let M, N be objects of $\mathcal{B}$ such that $M, N^I \in$* $\mathrm{Stat}(R)$ *for every set I.*

- *(i) If N is (strongly) M-CS-Baer in $\mathcal{B}$, then $R(N)$ is (strongly) $R(M)$-CS-Baer in $\mathcal{A}$.*
- *(ii) If R reflects zero objects, in particular if R is faithful, and $R(N)$ is (strongly) $R(M)$-CS-Baer in $\mathcal{A}$, then N is (strongly) M-CS-Baer in $\mathcal{B}$.*

As an immediate consequence of Theorem 3.9.3 we have the following result.

Theorem 3.9.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let (L, R) be an adjoint pair of covariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ have AB3*, L is exact and R is fully faithful. Let M and N be objects of $\mathcal{B}$. Then N is (strongly) M-CS-Baer in $\mathcal{B}$ if and only if $R(N)$ is (strongly) $R(M)$-CS-Baer in $\mathcal{A}$.*

Corollary 3.9.5. *Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ a full subcategory of $\mathcal{A}$ and $i : \mathcal{C} \to \mathcal{A}$ the inclusion functor. Assume that $\mathcal{A}$ has AB3* and $\mathcal{C}$ is a Giraud subcategory of $\mathcal{A}$. Let M and N be objects of $\mathcal{C}$. Then N is (strongly) M-CS-Baer in $\mathcal{C}$ if and only if $i(N)$ is (strongly) $i(M)$-CS-Baer in $\mathcal{A}$.*

Corollary 3.9.6. *Let $\mathcal{A}$ be a locally small abelian category, let $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$, and let $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ be the quotient functor. Let M and N be objects of $\mathcal{A}/\mathcal{C}$. Assume that $\mathcal{A}$ has AB3* and $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$ with section functor $S : \mathcal{A}/\mathcal{C} \to \mathcal{A}$. Then N is (strongly) M-CS-Baer in $\mathcal{A}/\mathcal{C}$ if and only if $S(N)$ is (strongly) $S(M)$-CS-Baer in $\mathcal{A}$.*

Corollary 3.9.7. *Let (L, F, R) be an adjoint triple of covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $L, R : \mathcal{B} \to \mathcal{A}$ between abelian categories. Assume that $\mathcal{A}$ and $\mathcal{B}$ have AB3*.*

(i) Let M and N be objects of $\mathcal{A}$, and assume that F is fully faithful. If L is exact, then N is (strongly) M-CS-Baer in $\mathcal{A}$ if and only if $F(N)$ is (strongly) $F(M)$-CS-Baer in $\mathcal{B}$.

(ii) Let M and N be objects of $\mathcal{B}$, and assume that L or R is fully faithful. Then N is (strongly) M-CS-Baer in $\mathcal{B}$ if and only if $R(N)$ is (strongly) $R(M)$-CS-Baer in $\mathcal{A}$.

Corollary 3.9.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful Frobenius functor between abelian categories. Let M and N be objects of $\mathcal{A}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ have AB3*. Then N is (strongly) M-CS-Baer in $\mathcal{A}$ if and only if $F(N)$ is (strongly) $F(M)$-CS-Baer in $\mathcal{B}$.*

Corollary 3.9.9. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of abelian categories given by the following diagram of functors:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "i"] \arrow[r, <-, bend left=50, "q"] \arrow[r, <-, bend right=50, "p"'] & \mathcal{B} \arrow[r, "e"] \arrow[r, <-, bend left=50, "l"] \arrow[r, <-, bend right=50, "r"'] & \mathcal{C}
\end{tikzcd}$$

(i) Let M and N be objects of $\mathcal{A}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ have AB3. If q is exact, then N is (strongly) M-CS-Baer in $\mathcal{A}$ if and only if $i(N)$ is (strongly) $i(M)$-CS-Baer in $\mathcal{B}$.*

(ii) Let M and N be objects of $\mathcal{C}$. Assume that $\mathcal{B}$ and $\mathcal{C}$ have AB3. Then N is (strongly) M-CS-Baer in $\mathcal{C}$ if and only if $r(N)$ is (strongly) $r(M)$-CS-Baer in $\mathcal{B}$.*

3.10 Endomorphism rings of self-CS-Baer objects

In this section we give some applications to endomorphism rings of objects, mainly in module categories. We begin with a useful generalization of Corollary 2.4.2 on (strongly) relative CS-Rickart modules.

Theorem 3.10.1. *Let M be a right R-module with $S = \mathrm{End}_R(M)$, and let A, B be right R-modules.*

(1) If A is im-local-retractable, $A \in \mathrm{Stat}(\mathrm{Hom}_R(M, -))$ and $\mathrm{Hom}_R(M, B)$ is a (strongly) $\mathrm{Hom}_R(M, A)$-CS-Rickart right S-module, then B is a (strongly) A-CS-Rickart right R-module.

(2) If B is im-local-coretractable, $B \in \mathrm{Refl}(\mathrm{Hom}_R(-, M))$ and $\mathrm{Hom}_R(A, M)$ is a dual (strongly) $\mathrm{Hom}_R(B, M)$-CS-Rickart left S-module, then B is a dual (strongly) A-CS-Rickart right R-module.

Corollary 3.10.2. *Let M be a right R-module with $S = \mathrm{End}_R(M)$, and let A, B be right R-modules.*

(1) If A is im-local-retractable, $A \in \mathrm{Stat}(\mathrm{Hom}_R(M, -))$ and $\mathrm{Hom}_R(M, B)$ is a (strongly) $\mathrm{Hom}_R(M, A)$-CS-Baer right S-module, then B is a (strongly) A-CS-Baer right R-module.

(2) If B is im-local-coretractable, $B \in \mathrm{Refl}(\mathrm{Hom}_R(-, M))$ and $\mathrm{Hom}_R(A, M)$ is a dual (strongly) $\mathrm{Hom}_R(B, M)$-CS-Baer left S-module, then B is a dual (strongly) A-CS-Baer right R-module.

We immediately have the following corollary on endomorphism rings.

Corollary 3.10.3. *Let M be a right R-module, and let $S = \mathrm{End}_R(M)$.*

(1) If M is im-local-retractable and S is a (strongly) self-CS-Baer right S-module, then M is a (strongly) self-CS-Baer right R-module.

(2) If M is im-local-coretractable and S is a dual (strongly) self-CS-Baer left S-module, then M is a dual (strongly) self-CS-Baer right R-module.

Next we present some conditions under which the converses of the above results hold.

Corollary 3.10.4. *Let M be a right R-module and $S = \mathrm{End}_R(M)$.*

(1) Assume that M is a finitely generated projective left S-module.

  (i) If M is a (strongly) self-CS-Baer right R-module, then S is a (strongly) self-CS-Baer right S-module.

  (ii) If M is a faithfully projective right R-module and S is a (strongly) self-CS-Baer right S-module, then M is a (strongly) self-CS-Baer right R-module.

(2) Assume that M is a finitely generated projective right R-module.

  (i) If S is a (strongly) dual self-CS-Baer right S-module, then M is a dual (strongly) self-CS-Baer right R-module.

  (ii) If M is a faithfully flat left S-module and M is a dual (strongly) self-CS-Baer right R-module, then S is a dual (strongly) self-CS-Baer right S-module.

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