Babeș–Bolyai University Cluj-Napoca
Institute for Doctoral Studies
Doctoral School of Mathematics and Computer Science

Tree representations of path algebras over tame quivers

PhD Thesis Summary

Doctoral advisor:
Prof. dr. Andrei Marcus

PhD student:
Ábel Lőrinczi

Cluj-Napoca
2022

Contents

Keywords

Quivers, representations of quivers, path algebras, indecomposable modules, exceptional modules, tree representations, Auslander–Reiten theory, preprojective modules, preinjective modules, regular modules.

Introduction

The main goal of representation theory of finite dimensional associative algebras is to understand the structure of finite dimensional module categories, in order to classify all indecomposable modules of a given algebra and all morphisms between them, up to isomorphism. In this thesis we consider path algebras over tame quivers and our aim is to study and describe as explicitly as possible the category of indecomposable modules over the path algebra. This category is equivalent to the category of representations of the quiver, which in many situations is easier to study, hence we will mainly focus on describing the latter.

Given a quiver representation, we choose some bases for the vector spaces associated to the vertices and consider the linear maps restricted to these basis elements. We define the coefficient quiver of this representation, in which the vertices are the basis elements and we have an arrow between two vertices if the matrix coefficient corresponding to these two basis elements is non-zero. Coefficient quivers were considered by Crawley-Boevey to deal with matrix problems and representations of quivers (see [5]).

In [27] Ringel proved that every indecomposable exceptional module, i.e. one without self-extensions, has appropriate bases, so that the coefficient quiver of its representation is a tree. This means that these representations, called tree representations, can be exhibited using $0-1$ matrices, such that the number of non-zero entries is $d-1$, where d is the length of the module. One of the steps in the proof involves a choice of basis, which seems to depend on the underlying field and Ringel posed the question whether there exist tree representations that are independent of this choice of basis, hence being field independent. In this thesis we answer this question in the positive, in the case of tame quivers of type $\widetilde{\mathbb{E}}_6$ and $\widetilde{\mathbb{D}}_m$.

Ringel later gave a simpler proof of his result, using covering theory in [29]. He also conjectured in [28] that if d is a positive root of the corresponding Kac-Moody root system, then there is an indecomposable tree module with dimension vector d, and in the wild hereditary case, if d is imaginary, then there should be more than one isomorphism class of tree modules having the same dimension vector.

This conjecture was proven in the case of the n-Kronecker quiver, where $n \geq 3$ by Weist in [40], where he also gave an explicit construction of the coefficient quivers of the indecomposable tree modules. This is an extension of the results presented in his dissertation by Fahr, see [10], where he considers 3-Kronecker representations with dimension vectors (d, e), where $d < e < 2d$. Later, Weist proved the existence of more than one isomorphism classes of indecomposable tree modules for every imaginary Schur root in [41], where he also stated explicit methods on how to construct these tree modules.

In [11] Gabriel gave a full list of indecomposable representations for Dynkin quivers using $0-1$-matrices. All the given representations, except 4 of them, were tree representations. This list of tree representations was completed by Crawley-Boevey in [5]. Regarding the Euclidean case, Mróz gave a full list of the indecomposable tree representations for the quiver $\widetilde{\mathbb{D}}_4$ in [22]. His results were later generalized by the author and Szántó, giving a full list of tree representations for indecomposable preprojective and preinjective modules for the quiver $\widetilde{\mathbb{D}}_m$ over a closed field k, see [19]. We mention that only one of the representations was proven to be indecomposable, all the others were checked by computer for fixed values of n, thus the checking was not complete.

In [14] Kussin and Meltzer described a method to explicitly determine the indecomposable preprojective and preinjective representations of $\widetilde{\mathbb{D}}_m$ and $\widetilde{\mathbb{E}}_6$ over an arbitrary field, but these representations are not tree representations in general. Later, in [13] Kędzierski and Meltzer generalized these results and gave a method for calculating indecomposable preprojective and preinjective representations of $\widetilde{\mathbb{E}}_8$ over any field and all indecomposable representations for algebraically closed fields. However these methods don't result in tree representations in general.

Using a computer generated proof, the author of this these together with Lénárt and Szöllősi managed to describe explicitly, in a field independent manner, all the exceptional tree representations in the case of the canonically oriented $\widetilde{\mathbb{E}}_6$ quiver in [17], thus answering the question raised by Ringel in the positive. We also conjectured that every tree representation of a Euclidean quiver is field independent.

We later gave a complete and general list corresponding to the exceptional modules over the path algebra of the canonically oriented Euclidean quiver $\widetilde{\mathbb{D}}_6$ and a method to obtain tree representations for exceptionals in the canonically oriented general case $\widetilde{\mathbb{D}}_m$ from that list, see [16].

This thesis is split into 4 chapters, having the following structure.

Chapter I contains the basic notions and definitions, along with some well-known results concerning the representation theory or finite dimensional associative algebras, which we will use throughout the remainder of the thesis. Our main references for this chapter were the books [35] and [36].

In Chapter II, based on the article [17] and its appendix [15] we present a complete and general list of tree representations corresponding to the exceptional modules over the path algebra of the canonically oriented Euclidean quiver $\Delta(\widetilde{\mathbb{E}}_6)$. In Definition II.2.1 we introduce the notion of field independency for modules and for short exact sequences. Lemmas II.2.2 and II.2.4 and Proposition II.2.3 constitute the theoretical elements of the techniques used to prove the correctness of the tree representations presented in Section II.3. We then present the outline of the methods used in obtaining field independent tree representations of exceptional modules for the quiver $\Delta(\widetilde{\mathbb{E}}_6)$. The main result of this chapter is Section II.3, where we list field independent tree representations for every indecomposable exceptional module over the path algebra of the quiver $\Delta(\widetilde{\mathbb{E}}_6)$.

In Chapter III, based on the article [16] and its appendix [15], besides giving a complete list of tree representation for the exceptional modules over the path algebra of the canonically oriented quiver $\Delta(\widetilde{\mathbb{D}}_6)$, we also describe a method in Section III.2 for constructing tree representations for $\Delta(\widetilde{\mathbb{D}}_m)$, where $m \geq 4$ using tree representations of $\Delta(\widetilde{\mathbb{D}}_6)$.

Finally, in Chapter IV, based on the article [18] we verify computationally a conjecture on the field independence of tree representations of Euclidean quivers of type $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{D}}_5$ and $\widetilde{\mathbb{E}}_6$, with dimension vector bounded by the minimal radical vector of the quiver. This includes a large class of exceptional representations, in particular all the regular non-homogeneous exceptionals.

Some of the results of this thesis have been presented at various national and international conferences.

Chapter I

Preliminaries

In this chapter we introduce the basic notions, along with some well-known results concerning the representation theory of associative algebras.

I.1 Quivers and modules

I.2 Auslander–Reiten theory

I.3 Finite and infinite representation type quivers

I.4 Extensions of quiver representations

I.5 Tree representations and Schofield sequences

Chapter II

Tree representations of the quiver $\widetilde{\mathbb{E}}_6$

In this chapter we will give a complete and general list of tree representations corresponding to the exceptional modules over the path algebra of the canonically oriented Euclidean quiver $\widetilde{\mathbb{E}}_6$. The proof (involving induction and symbolic computation with block matrices) was partially generated by a purposefully developed computer software and is available on arXiv as an appendix. All the representations listed remain valid over any base field, answering a question raised by Ringel in [27]. The results presented here were published in the article [17] and its appendix [15].

II.1 Basic notions and definitions

Consider the canonically oriented Euclidean quiver of type $\widetilde{\mathbb{E}}_6$, denoted from now on by $\Delta(\widetilde{\mathbb{E}}_6)$, having the following shape:

$$\Delta(\widetilde{\mathbb{E}}_6): \qquad \begin{array}{ccccccccc} & & & & 7 & & & & \\ & & & & \downarrow & & & & \\ & & & & 6 & & & & \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longleftarrow & 4 & \longleftarrow & 5 \end{array}$$

Therefore, we have $\Delta(\widetilde{\mathbb{E}}_6)_0 = \{1, \dots, 7\}$ for the set of vertices and $\Delta(\widetilde{\mathbb{E}}_6)_1 = \{(1,2), (2,3), (4,3), (5,4), (6,3), (7,6)\}$ for the set of arrows.

The Euler and Tits form in this case is

$$\langle x, y \rangle = \sum_{i=1}^{7} x_i y_i - x_1 y_2 - x_2 y_3 - x_4 y_3 - x_5 y_4 - x_6 y3 - x_7 y_6$$

$$q_{\Delta(\widetilde{\mathbb{E}}_6)}(x) = \sum_{i=1}^{7} x_i^2 - x_1 x_2 - x_2 x_3 - x_4 x_3 - x_5 x_4 - x_6 x_3 - x_7 x_6$$

Note that the Tits form is independent of the orientation of the quiver and it is positive semi-definite with radical $\mathbb{Z}\delta$, where $\delta = (1, 2, 3, 2, 1, 2, 1)$.

II.2 Proving the field independent tree module property

In this part we describe the method used to prove the tree module property for every representation given in the lists in Section II.3 both from the theoretical and practical perspective. The method presented here is general (in the sense that it could be applied to any tame quiver), so as stated before, Q denotes an arbitrary tame quiver and k an arbitrary field.

We will use the "field independent" qualifier in relation to representations and short exact sequences in the following precise manner:

Definition II.2.1. Let $M \in \operatorname{mod} kQ$ be an (exceptional) indecomposable module. We say that:

(1) The module M is *field independent (exceptional) indecomposable* if in the corresponding representation $M = (M_i, M_\alpha)$ all the elements in the matrices M_α are either 0 or 1 and for any field k' if we consider a module $M' \in \operatorname{mod} k'Q$ such that $\underline{\dim} M = \underline{\dim} M'$ and every matrix M'_α from the corresponding representation $M' = (M'_i, M'_\alpha)$ is formally the same as M_α (for all arrows α), then M' is also (exceptional) indecomposable in $\operatorname{mod} k'Q$.

(2) The module M has the *field independent tree property* if it is a tree module in $\operatorname{mod} kQ$ and it is also a *field independent* (exceptional) indecomposable module (i.e. if we consider the corresponding representation with formally the same matrices over any other field k', we still get an exceptional indecomposable tree module in $\operatorname{mod} k'Q$).

(3) A short exact sequence of the form

$$0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} X \longrightarrow 0$$

is *field independent* (with $X, Y, Z \in \operatorname{mod} kQ$) if all the elements in the matrices of the representations X, Y and Z are either 0 or 1, all the elements in the matrices f_i and g_i of the embedding $f = (f_i)_{i \in Q_0}$ respectively the projection $g = (g_i)_{i \in Q_0}$ are either 0 or 1 or -1 and in any field k' the sequence $0 \longrightarrow Y' \xrightarrow{f'} Z' \xrightarrow{g'} X' \longrightarrow 0$ is also exact, where $X', Y', Z' \in \operatorname{mod} k'Q$, $f' : Y' \to Z'$, $g' : Z' \to X'$ correspond in order to X, Y, Z, $f : Y \to Z$, $g : Z \to X$ with the respective dimension vectors unchanged and with all matrices (both from the representations and from the morphisms) being formally the same when considering them over k' instead of k.

The following proposition and lemmas constitute the theoretical elements of the technique used to prove the formulas in Section II.3 in a field independent way:

Lemma II.2.2. *For a module $M \in \operatorname{mod} kQ$ we have M is exceptional indecomposable if and only if* $\dim_k \operatorname{End}(M) = 1$ *and* $\underline{\dim} M \neq \delta$.

Proposition II.2.3. *Let $X, Y, X', Y' \in \operatorname{mod} kQ$ be indecomposable modules. If $M \in \operatorname{mod} kQ$ such that*

(a) *there is an exceptional $Z \in \operatorname{mod} kQ$ such that (X, Y) and (X', Y') are Schofield pairs associated to Z,*

(b) *there exist two short exact sequences*

$$0 \longrightarrow Y \longrightarrow M \longrightarrow X \longrightarrow 0$$

and

$$0 \longrightarrow Y' \longrightarrow M \longrightarrow X' \longrightarrow 0,$$

(c) $X \not\cong X'$ *or* $Y \not\cong Y'$,

(d) $\dim_k \operatorname{Ext}^1(X, Y) = \dim_k \operatorname{Ext}^1(X', Y') = 1$

then M *is exceptional indecomposable.*

Lemma II.2.4. *Let* $X, Y, Z \in \operatorname{mod} kQ$ *and* $f = (f_i)_{i\in Q_0}$, $g = (g_i)_{i\in Q_0}$ *families of* k*-linear maps* $f_i : Y_i \longrightarrow Z_i$, $g_i : Z_i \longrightarrow X_i$. *Then there is a short exact sequence* $0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} X \longrightarrow 0$ *if and only if the following conditions hold (we identify the maps* f_i *and* g_i *with their matrices in the canonical basis):*

(a) *the matrices* f_i *(respectively* g_i*) have maximal column (respectively row) ranks,*

(b) $f_{t(\alpha)} Y_\alpha = Z_\alpha f_{s(\alpha)}$ *and* $g_{t(\alpha)} Z_\alpha = X_\alpha g_{s(\alpha)}$, *for all* $\alpha \in Q_1$,

(c) $g_i f_i = 0$, *for all* $i \in Q_0$,

(d) $\underline{\dim} Z = \underline{\dim} X + \underline{\dim} Y$.

Lemma II.2.5. *If* $X, Y \in \operatorname{mod} kQ$ *are indecomposable modules such that* X *is regular and* Y *is preprojective, or* X *is preinjective and* Y *is regular or both of them are preprojectives (or preinjectives) and there is a path in the Auslander–Reiten quiver from the vertex corresponding to* Y *to the vertex corresponding to* X*, then* $\dim_k \operatorname{Ext}^1(X, Y) = -\langle \underline{\dim} X, \underline{\dim} Y \rangle$.

We are now ready to describe the process of proving the formulas from Section II.3.

The process of proving the field independent tree property

Suppose we have formulas defining families of matrices $(M_\alpha^{(n)})_{\alpha\in Q_1}$ depending on some $n \in \mathbb{N}$. The elements of the matrices $M_\alpha^{(n)}$ are either 0 or 1, so they can be considered over an arbitrary field k. We want to prove that the representation of the quiver Q given as $M = M^{(n)} = (M_i^{(n)}, M_\alpha^{(n)})$ has the field independent tree property (where the dimension of each k-space $M_i^{(n)}$ is in accordance with the column and row sizes of the matrices $M_\alpha^{(n)}$, thus the formulas also determine $\underline{\dim} M$). Suppose that $\underline{\dim} M$ is such that it coincides with the dimension vector of an exceptional indecomposable (see Lemma II.2.2). Suppose also that the number of elements equal to 1 in the matrices $M_\alpha^{(n)}$ is exactly $\ell(M) - 1$. So, in order to prove the field independent tree module property, we need only to show that M is field independent indecomposable. We may use one of the following lines of reasoning:

(1) *Prove that* $\dim_k \operatorname{End}(M) = 1$ *in any field* k *and use Lemma II.2.2.* This may be done by writing the matrix A of the homogeneous system of linear equations defining $\operatorname{End}(M)$ and showing that the corank of A is one (i.e. the solution space is one dimensional). In order to compute the rank of A, it must be echelonized (brought to row echelon form) using elementary operations on rows

and/or columns in a "field independent way". This means that every single elementary operation used in the process of echelonizing A must be such that the elements in the resulting matrix are either 0, 1 or -1 and the result is exactly the same if performed in any field k. For example if in the case of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ we perform the elementary row operation $r_2 \leftarrow r_2 - r_1$, then we get $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - r_1} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ if performed in $\mathbb{R}$, or $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - r_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ if performed in $\mathbb{Z}_2$. Hence it has different ranks if considered over different fields. A crucial element of this proof is to ensure something like this never happens, but the result of every single elementary operation performed is formally the same matrix, independently of the field it is considered in.

(2) *Perform an induction on n, making use of Proposition II.2.3.* First prove the formula for the starting values of n using method (1) above (typically for $n = 0$, but the structure of the block matrices depending on n might require to make additional proofs for small values of n). Then suppose the formula gives field independent exceptional indecomposables $M^{(n')} = (M_i^{(n')}, M_\alpha^{(n')})$ for all $n' < n$. Find two pairs of modules (X, Y) and (X', Y') conforming to all requirements of Proposition II.2.3, such that any of these four representations is obtained either using formula $M^{(n')}$ for some $n' < n$ (or some permuted version of it) or some other formulas proved already to give field independent exceptional indecomposables. If the quiver Q presents some symmetries, then a permuted version of the formula $\tilde{M}^{(n')} = (\tilde{M}_i^{(n')}, \tilde{M}_{i\to j}^{(n')})$ may also be used in the induction step, where $(\tilde{M}_i^{(n')})_{i\in Q_0} = (M_{\sigma(i)}^{(n')})_{i\in Q_0}$ and $(\tilde{M}_{i\to j}^{(n')})_{(i\to j)\in Q_1} = (M_{\sigma(i)\to\sigma(j)}^{(n')})_{(i\to j)\in Q_1}$ for some permutation σ. One has to construct here the two field independent short exact sequences of the form $0 \to Y \to M^{(n)} \to X \to 0$ and $0 \to Y' \to M^{(n)} \to X' \to 0$ in order to show their existence. Once the matrices of the morphisms are constructed, Lemma II.2.4 can be used to prove that indeed these form short exact sequences in any field k. We emphasize that conditions (a), (b) and (c) from Lemma II.2.4 must be verified in a "field independent way": the rank of the matrices must be checked using field independent echelonization as explained before, and the result of the matrix arithmetic operations used in (b) and (c) must be formally the same, independently of the underlying field.

(3) *Perform a direct proof, making use of Proposition II.2.3.* Use two pairs of modules (X, Y) and (X', Y') conforming to all requirements of Proposition II.2.3, such that any of these four representations are obtained by some formulas showed already to give field independent exceptional indecomposables, and prove the existence of the two field independent short exact sequences $0 \to Y \to M^{(n)} \to X \to 0$ and $0 \to Y' \to M^{(n)} \to X' \to 0$ by constructing them using Lemma II.2.4 in the "field independent way".

The proof process described is extremely cumbersome, time-consuming and error-prone if performed by a human, therefore we have implemented a proof assistant software to help us in carrying it out. The proof assistant can perform any of the steps (1), (2) or (3) based on some input given in a LaTeX file. The input data consists of the formulas $(M_\alpha^{(n)})_{\alpha\in Q_1}$ defining the representations and the choice for the short exact sequences required in (2) and (3), together with the families of matrices defining the morphisms. All this data must be given in a LaTeX document with a well-defined structure, in order for the proof assistant to be able to parse it and extract the relevant information. The matrices are given either as "usual matrices" (of fixed size, with elements equal to either 1, −1 or 0), or symbolic block-matrices of variable size, depending on the parameter $n \in \mathbb{N}$. Every block-matrix is built using the following three types of blocks: zero block of size $n_1 \times n_2$, the identity block I_n and a block denoted by E_n having ones on the secondary diagonal and zeros everywhere else (note that $E_n^2 = I_n$ in every field). We have used the document processor LyX to edit the input document and export it to LaTeX (in this way ensuring a syntactically correct LaTeX file).

These are the steps performed by the software:

- It reads and stores the data $M^{(n)} = (M_i^{(n)}, M_\alpha^{(n)})$ defining every representation of $M^{(n)}$.
- Computes the total number of elements equal to 1 in the matrices $M_\alpha^{(n)}$ and compares it against $\ell(M^{(n)})$ to ensure their number is exactly $\ell(M^{(n)}) - 1$.
- If instructed to perform along method (1), it computes the matrix A of the homogeneous system of linear equations defining $\mathrm{End}(M^{(n)})$ and shows that it can be brought to echelon form by performing exactly the same elementary operations resulting in exactly the same matrix (formally) if considered in any field. In this way it ensures that the corank of A is one independently of the field. Note that it can perform in this mode only with formulas where n has any given concrete value.
- If instructed (and given sufficient data) it performs all checks required by methods (2) or (3) based on Proposition II.2.3. First it checks in the list provided in [39] to see that both pairs (X, Y) and (X', Y') are Schofield pairs associated to the exceptional indecomposable $Z \in \mathrm{mod}\, kQ$, such that $\underline{\dim} Z = \underline{\dim} M^{(n)}$, then verifies conditions (c) and (d) from Proposition II.2.3. It is ensured that the requirements of Lemma II.2.5 are met and condition (d) is validated. Finally, it ensures the existence of two short exact sequences of the form $0 \longrightarrow Y \xrightarrow{f} M^{(n)} \xrightarrow{g} X \longrightarrow 0$ and $0 \longrightarrow Y' \xrightarrow{f'} M^{(n)} \xrightarrow{g'} X' \longrightarrow 0$ by reading the matrices of the morphisms f, f', g and g' and showing that every elementary operation and block-matrix arithmetic may be performed in a field independent way in order to fulfill every requirement of Lemma II.2.4.

Every single operation performed by the proof assistant software is written to an output LaTeX document (this is the rather lengthy generated appendix, [15]). Everything (including the elementary operations and the details of computing the block matrix sums and products) is output a detailed step-by-step fashion as if written "by hand". In this way one does not have to believe in the correctness of the implementation, because the complete proof is "on paper" and every single step may be crosschecked and verified by a human mathematician.

II.2.1 Notations

The matrices given in Section II.3 are written using blocks of various sizes. The row and column size of blocks are given by expressions of the form $an + b$, where $n \in \mathbb{N}$ is a parameter, a is a given non-negative integer, b is a given integer. Every matrix here is composed either of identity blocks or rectangular zero blocks. We denote the identity block simply by 1 and the zero block by 0. The row and column sizes will be written as “decorations” along the border of the matrix, like in the following example:

$$\begin{array}{c} \\ 2n{+}2 \\ n{+}1 \\ 2n{+}2 \\ n \end{array} \begin{array}{c} \begin{array}{cc} 2n{+}2 & 2n{+}2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

where this matrix is of size $(6n+5)\times(4n+4)$ and is composed of two identity blocks (each having $2n+2$ rows and columns) and six zero blocks with various compatible sizes.

The matrices may be given using arithmetic expressions containing symbolic block-matrices and identifiers referencing other matrices. Possible operations are: addition, direct sum defined as $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and a special kind of “sum” denoted by $\boxplus$ which adds the right hand side matrix into the upper right corner of left hand side matrix. Formally: if $A \in \mathcal{M}_{m,n}(k)$ and $B \in \mathcal{M}_{m',n'}(k)$ are matrices such that $m' \le m$ and $n' \le n$, then $A \boxplus B = A + \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, where $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{m,n}(k)$ is obtained by adding as many zero columns to the left of B and as many zero rows beneath it to make the resulting matrix of the same size as the matrix A. This operation is useful to insert nonzero elements into the upper right part of a matrix obtained by direct sum.

Representations are given as families having similar block-matrices. For example $P(6n+4,5)$ denotes such a family of representations (where $n \in \mathbb{N}$). Sometimes one needs the previous or next value of n when writing matrices in terms of others, therefore we need to substitute n. Substitution is denoted like $P(6n+4,5)[n \mapsto n-1]$, which in this case is the module $P(6n-2,5)$ for any fixed value of n.

For a representation $Z = (Z_i, Z_\alpha)$ we only give the matrices Z_α. For a module Z and an arrow $\alpha \in Q_1$ we denote the matrix Z_α by M^Z_α. In the case when we give all the matrices “by value” a representation will be written like this (with $d_i \in \mathbb{N}$ for $i \in Q_0$ and with the matrices M^Z_α in this specific order):

$$\underline{\dim} Z = (d_1,\ d_2,\ d_3,\ d_4,\ d_5,\ d_6,\ d_7)$$
$$Z = \left(M^Z_{1\to 2},\ M^Z_{2\to 3},\ M^Z_{4\to 3},\ M^Z_{5\to 4},\ M^Z_{6\to 3},\ M^Z_{7\to 6}\right).$$

There is another notation, when writing the matrices with expressions using the operations $\oplus$ and $\boxplus$, referencing other matrices of representations. In this case there are always two other representations Y, X and a specific arrow α' such that the matrices of Z can be given as $M^Z_\alpha = M^Y_\alpha \oplus M^X_\alpha$ for all $\alpha \neq \alpha'$, and $M^Z_{\alpha'} = \left(M^Y_{\alpha'} \oplus M^X_{\alpha'}\right) \boxplus M$ for a matrix M containing exactly one element equal to 1 and all the other elements being zero. Therefore we give the representation Z in the following form (specifying the matrix

M):

$$\begin{aligned}\underline{\dim} Z &= (d_1,\ d_2,\ d_3,\ d_4,\ d_5,\ d_6,\ d_7)\\ M^Z_\alpha &= M^Y_\alpha \oplus M^X_\alpha, \text{ for } \alpha \neq \alpha'\\ M^Z_{\alpha'} &= \left(M^Y_{\alpha'} \oplus M^X_{\alpha'}\right) \boxplus M.\end{aligned}$$

For small values of n we may give some representations concretely (the general formula may work only for $n > 0$ or $n > 1$ in some cases).

II.3 Tree representations of the quiver $\Delta(\widetilde{\mathbb{E}}_6)$

In this section we list the formulas describing the matrices of the representations corresponding to exceptional modules: the preprojective indecomposables (Subsection II.3.1), the preinjective indecomposables (Section II.3.2) and the regular non-homogeneous indecomposables with dimension vector below δ (Subsection II.3.3). For convenience, at the beginning of each of the following subsections, we present a graphical representation of the corresponding part of the Auslander–Reiten quiver. Blue arrows show the existence of a so-called irreducible monomorphism, while red arrows represent irreducible epimorphisms between suitable indecomposable modules (for details see [3]).

In the case of preprojectives and preinjectives the representations can be grouped in families of the form $P(6n+r, i)$ respectively $I(6n+r, i)$, where $i \in \{1, \dots, 7\}$ and $r \in \{0, \dots, 5\}$. Representations belonging to the same family have similar dimension vectors and matrices, depending only on the parameter $n \in \mathbb{N}$. The matrices listed here are rigorously proved to be correct in the appendix to this article ([15]) using the method described in Subsection II.2.

II.3.1 The preprojective indecomposable representations

The preprojective indecomposable modules correspond to the vertices of the preprojective part of the Auslander–Reiten quiver, as shown in Figure II.1.

Due to the symmetry of the quiver $\Delta(\widetilde{\mathbb{E}}_6)$ we give only the families of representations of the form $P(m, 1)$, $P(m, 2)$ and $P(m, 3)$. For all the other representations we can use the permutations $\sigma = (1, 5)(2, 4)$ and $\tau = (1, 7)(2, 6)$ to write them in terms of $P(m, 1)$, $P(m, 2)$ and $P(m, 3)$ in the following way ($m \geq 0$):

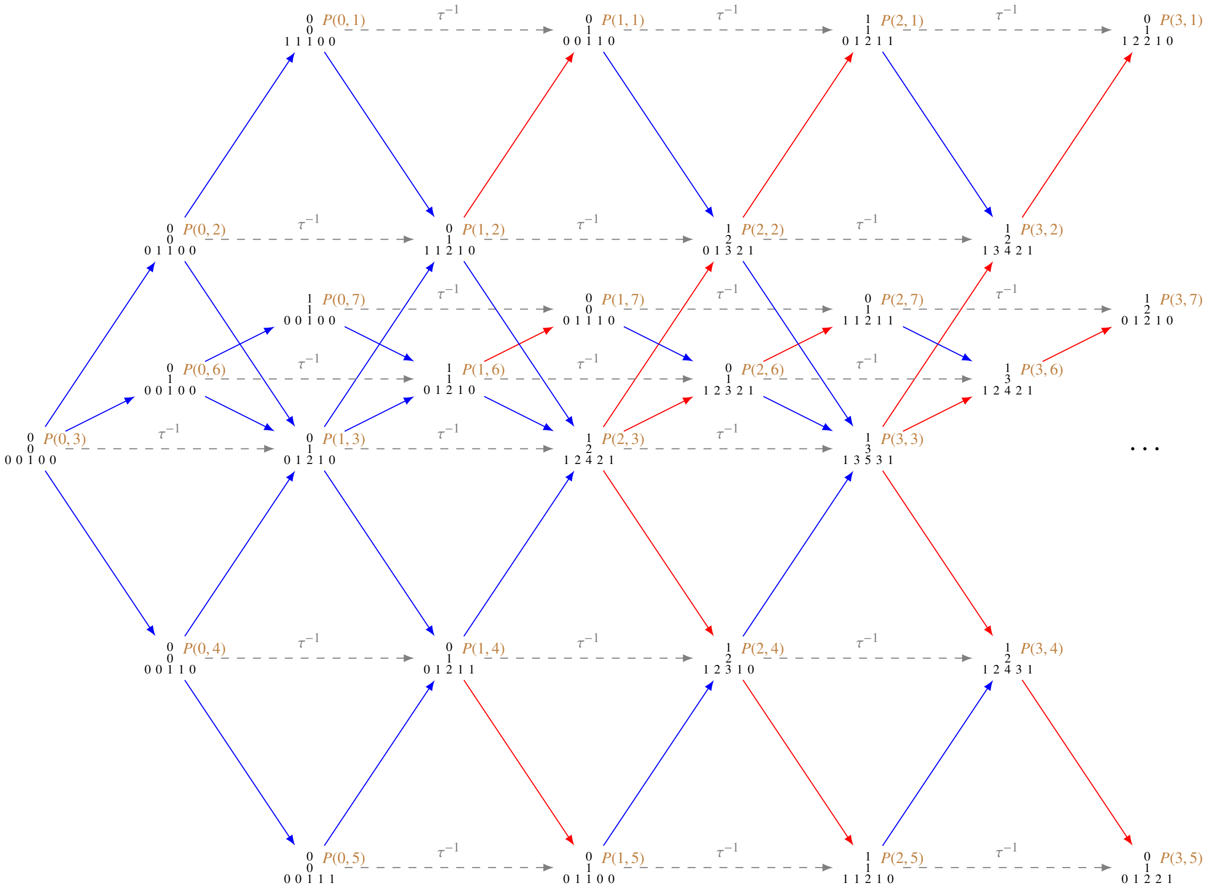
$$\underline{\dim} P(m,5) = (d_{\sigma(i)})_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0} \text{ and } \underline{\dim} P(m,7) = (d_{\tau(i)})_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0},$$

where $\underline{\dim} P(m, 1) = (d_i)_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0}$,

$$\underline{\dim} P(m,4) = (d_{\sigma(i)})_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0} \text{ and } \underline{\dim} P(m,6) = (d_{\tau(i)})_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0},$$

where $\underline{\dim} P(m, 2) = (d_i)_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0}$ for the dimension vectors, respectively

$$P(m,5) = \left(M_{\sigma(i)\to\sigma(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1} \text{ and } P(m,7) = \left(M_{\tau(i)\to\tau(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1},$$



The preprojective part of the Auslander–Reiten quiver $\Delta(\widetilde{\mathbb{E}}_6)$

where $P(m,1) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1}$,

$$P(m,4) = \left(M_{\sigma(i)\to\sigma(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1} \text{ and } P(m,6) = \left(M_{\tau(i)\to\tau(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1},$$

where $P(m,2) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1}$ for the matrices.

In what follows we list the tree representations for preprojective families of the form $P(m,1)$, $P(m,2)$ and $P(m,3)$:

$\underline{\dim} P(6n,1) = (n+1, 2n+1, 3n+1, 2n, n, 2n, n)$

$$P(6n,1) = \left(\begin{smallmatrix} n+1 \\ n \end{smallmatrix} \overset{n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} 2n+1 \\ n \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} 2n \\ n+1 \end{smallmatrix} \overset{2n}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \begin{smallmatrix} n+1 \\ 2n \end{smallmatrix} \overset{2n}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} n \\ n \end{smallmatrix} \overset{n}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \begin{smallmatrix} n+1 \\ 2n \end{smallmatrix} \overset{2n}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} n \\ n \end{smallmatrix} \overset{n}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right)$$

$\underline{\dim} P(6n+1,1) = (n, 2n, 3n+1, 2n+1, n, 2n+1, n)$

$$P(6n+1,1) = \left(\begin{smallmatrix} n \\ n \end{smallmatrix} \overset{n}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \begin{smallmatrix} 2n \\ n+1 \end{smallmatrix} \overset{2n}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \begin{smallmatrix} n+1 \\ 2n \end{smallmatrix} \overset{2n}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} 2n+1 \\ n \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \overset{n}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} n \\ 2n+1 \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} n+1 \\ n \end{smallmatrix} \overset{n}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right)$$

$\underline{\dim} P(6n+2,1) = (n, 2n+1, 3n+2, 2n+1, n+1, 2n+1, n+1)$

$$P(6n+2,1) = \left(\begin{array}{c|c} & n \\ \hline n & 1 \\ n & 1 \\ 1 & 0 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ n+1 & 0 \end{array} + \begin{array}{c|c} & 2n+1 \\ \hline n+1 & 0 \\ 2n+1 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ n+1 & 0 \end{array}, \begin{array}{c|c} & n+1 \\ \hline n+1 & 1 \\ n & 0 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline n+1 & 0 \\ 2n+1 & 1 \end{array}, \begin{array}{c|c} & n+1 \\ \hline n & 0 \\ n+1 & 1 \end{array} \right)$$

$\underline{\dim} P(6n+3,1) = (n+1, 2n+2, 3n+2, 2n+1, n, 2n+1, n)$

$$P(6n+3,1) = \left(\begin{array}{c|c} & n+1 \\ \hline n+1 & 0 \\ n+1 & 1 \end{array}, \begin{array}{c|c} & 2n+2 \\ \hline n & 0 \\ 2n+2 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ n+1 & 0 \end{array}, \begin{array}{c|c} & n \\ \hline n & 1 \\ n+1 & 0 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ n+1 & 0 \end{array} + \begin{array}{c|c} & 2n+1 \\ \hline n+1 & 0 \\ 2n+1 & 1 \end{array}, \begin{array}{c|c} & n \\ \hline n & 1 \\ n & 1 \\ 1 & 0 \end{array} \right)$$

$\underline{\dim} P(6n+4,1) = (n+1, 2n+1, 3n+3, 2n+2, n+1, 2n+2, n+1)$

$$P(6n+4,1) = \left(\begin{array}{c|c} & n+1 \\ \hline n+1 & 1 \\ n & 0 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ n+2 & 0 \end{array}, \begin{array}{c|c} & 2n+2 \\ \hline 2n+2 & 1 \\ n+1 & 0 \end{array} + \begin{array}{c|c} & 2n+2 \\ \hline n+1 & 0 \\ 2n+2 & 1 \end{array}, \begin{array}{c|cc} & n & 1 \\ \hline n & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ n & 1 & 0 \end{array}, \begin{array}{c|c} & 2n+2 \\ \hline n+1 & 0 \\ 2n+2 & 1 \end{array}, \begin{array}{c|c} & n+1 \\ \hline n+1 & 0 \\ n+1 & 1 \end{array} \right)$$

$\underline{\dim} P(6n+5,1) = (n, 2n+2, 3n+3, 2n+2, n+1, 2n+2, n+1)$

$$P(6n+5,1) = \left(\begin{array}{c|c} & n \\ \hline n & 1 \\ 2 & 0 \\ n & 1 \end{array}, \begin{array}{c|c} & 2n+2 \\ \hline 2n+2 & 1 \\ n+1 & 0 \end{array} + \begin{array}{c|c} & 2n+2 \\ \hline n+1 & 0 \\ 2n+2 & 1 \end{array}, \begin{array}{c|c} & 2n+2 \\ \hline 2n+2 & 1 \\ n+1 & 0 \end{array}, \begin{array}{c|c} & n+1 \\ \hline n+1 & 1 \\ n+1 & 0 \end{array}, \begin{array}{c|c} & 2n+2 \\ \hline n+1 & 0 \\ 2n+2 & 1 \end{array}, \begin{array}{c|c} & n+1 \\ \hline n+1 & 0 \\ n+1 & 1 \end{array} \right)$$

$\underline{\dim} P(6n,2) = (2n, 4n+1, 6n+1, 4n, 2n, 4n, 2n)$

$$P(0,2) = \left(0, \begin{bmatrix} 1 \end{bmatrix}, 0, 0, 0, 0\right)$$

$$M_\alpha^{P(6n,2)} = M_\alpha^{P(6n+5,1)[n \mapsto n-1]} \oplus M_\alpha^{P(6n,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n,2)} = \left(M_{7\to 6}^{P(6n+5,1)[n \mapsto n-1]} \oplus M_{7\to 6}^{P(6n,1)}\right) \boxplus \begin{array}{c|cc} & 1 & n-1 \\ \hline n-1 & 0 & 0 \\ 1 & 1 & 0 \end{array}, \quad n > 0$$

$\underline{\dim} P(6n+1,2) = (2n+1, 4n+1, 6n+2, 4n+1, 2n, 4n+1, 2n)$

$$P(1,2) = \left(\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, 0 \right)$$

$$M_\alpha^{P(6n+1,2)} = M_\alpha^{P(6n,1)} \oplus M_\alpha^{P(6n+1,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n+1,2)} = \left(M_{7\to 6}^{P(6n,1)} \oplus M_{7\to 6}^{P(6n+1,1)}\right) \boxplus \begin{array}{c@{}c} & \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{array}{c} n-1 \\ 1 \\ n \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad n > 0$$

$$\underline{\dim} P(6n+2,2) = (2n, 4n+1, 6n+3, 4n+2, 2n+1, 4n+2, 2n+1)$$

$$M_{\alpha}^{P(6n+2,2)} = M_{\alpha}^{P(6n+1,1)} \oplus M_{\alpha}^{P(6n+2,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n+2,2)} = \left(M_{7\to 6}^{P(6n+1,1)} \oplus M_{7\to 6}^{P(6n+2,1)}\right) \boxplus \begin{array}{c@{}c} & \begin{array}{cc} 1 & n \end{array} \\ \begin{array}{c} n \\ 1 \\ n \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\underline{\dim} P(6n+3,2) = (2n+1, 4n+3, 6n+4, 4n+2, 2n+1, 4n+2, 2n+1)$$

$$P(3,2) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$M_{\alpha}^{P(6n+3,2)} = M_{\alpha}^{P(6n+2,1)} \oplus M_{\alpha}^{P(6n+3,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n+3,2)} = \left(M_{7\to 6}^{P(6n+2,1)} \oplus M_{7\to 6}^{P(6n+3,1)}\right) \boxplus \begin{array}{c@{}c} & \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{array}{c} n-1 \\ 1 \\ n+1 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad n > 0$$

$$\underline{\dim} P(6n+4,2) = (2n+2, 4n+3, 6n+5, 4n+3, 2n+1, 4n+3, 2n+1)$$

$$M_{\alpha}^{P(6n+4,2)} = M_{\alpha}^{P(6n+3,1)} \oplus M_{\alpha}^{P(6n+4,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n+4,2)} = \left(M_{7\to 6}^{P(6n+3,1)} \oplus M_{7\to 6}^{P(6n+4,1)}\right) \boxplus \begin{array}{c@{}c} & \begin{array}{cc} n & 1 \end{array} \\ \begin{array}{c} n \\ 1 \\ n \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\underline{\dim} P(6n+5,2) = (2n+1, 4n+3, 6n+6, 4n+4, 2n+2, 4n+4, 2n+2)$$

$$M_{\alpha}^{P(6n+5,2)} = M_{\alpha}^{P(6n+3,5)} \oplus M_{\alpha}^{P(6n,7)[n\mapsto n+1]}, \quad \text{for } \alpha \neq (1 \to 2)$$

$$M_{1\to 2}^{P(6n+5,2)} = \left(M_{1\to 2}^{P(6n+3,5)} \oplus M_{1\to 2}^{P(6n,7)[n\mapsto n+1]}\right) \boxplus \begin{array}{c@{}c} & \begin{array}{cc} 1 & n \end{array} \\ \begin{array}{c} n \\ 1 \\ n \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\underline{\dim} P(6n,3) = (3n, 6n, 9n+1, 6n, 3n, 6n, 3n)$$

$$P(0,3) = (0,0,0,0,0,0)$$

$$M_{\alpha}^{P(6n,3)} = M_{\alpha}^{P(6n+5,2)[n\mapsto n-1]} \oplus M_{\alpha}^{P(6n,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n,3)} = \left(M_{7\to 6}^{P(6n+5,2)[n\mapsto n-1]} \oplus M_{7\to 6}^{P(6n,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} n-1 & 1 \end{array} \\ \begin{array}{c} n-1 \\ 1 \\ 3n \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}, \quad n > 0$$

$\underline{\dim} P(6n+1,3) = (3n, 6n+1, 9n+2, 6n+1, 3n, 6n+1, 3n)$

$$P(1,3) = \left(0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, 0\right)$$

$$M_{\alpha}^{P(6n+1,3)} = M_{\alpha}^{P(6n,2)} \oplus M_{\alpha}^{P(6n+1,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n+1,3)} = \left(M_{7\to 6}^{P(6n,2)} \oplus M_{7\to 6}^{P(6n+1,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{array}{c} 3n-1 \\ 1 \\ n \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad n > 0$$

$\underline{\dim} P(6n+2,3) = (3n+1, 6n+2, 9n+4, 6n+2, 3n+1, 6n+2, 3n+1)$

$$P(2,3) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$M_{\alpha}^{P(6n+2,3)} = M_{\alpha}^{P(6n+1,2)} \oplus M_{\alpha}^{P(6n+2,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n+2,3)} = \left(M_{7\to 6}^{P(6n+1,2)} \oplus M_{7\to 6}^{P(6n+2,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & n \end{array} \\ \begin{array}{c} 3n \\ 1 \\ n \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad n > 0$$

$\underline{\dim} P(6n+3,3) = (3n+1, 6n+3, 9n+5, 6n+3, 3n+1, 6n+3, 3n+1)$

$$P(3,3) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$M_{\alpha}^{P(6n+3,3)} = M_{\alpha}^{P(6n+2,2)} \oplus M_{\alpha}^{P(6n+3,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{P(6n+3,3)} = \left(M_{7\to 6}^{P(6n+2,2)} \oplus M_{7\to 6}^{P(6n+3,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{array}{c} 3n \\ 1 \\ n+1 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad n > 0$$

$\underline{\dim} P(6n+4,3) = (3n+2, 6n+4, 9n+7, 6n+4, 3n+2, 6n+4, 3n+2)$

$$M_{\alpha}^{P(6n+4,3)} = M_{\alpha}^{P(6n+3,2)} \oplus M_{\alpha}^{P(6n+4,1)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M^{P(6n+4,3)}_{7\to6} = \left(M^{P(6n+3,2)}_{7\to6} \oplus M^{P(6n+4,1)}_{7\to6}\right) \boxplus \begin{array}{c} \\ 3n+1 \\ 1 \\ n \end{array} \begin{array}{cc} n & 1 \\ \hline 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}$$

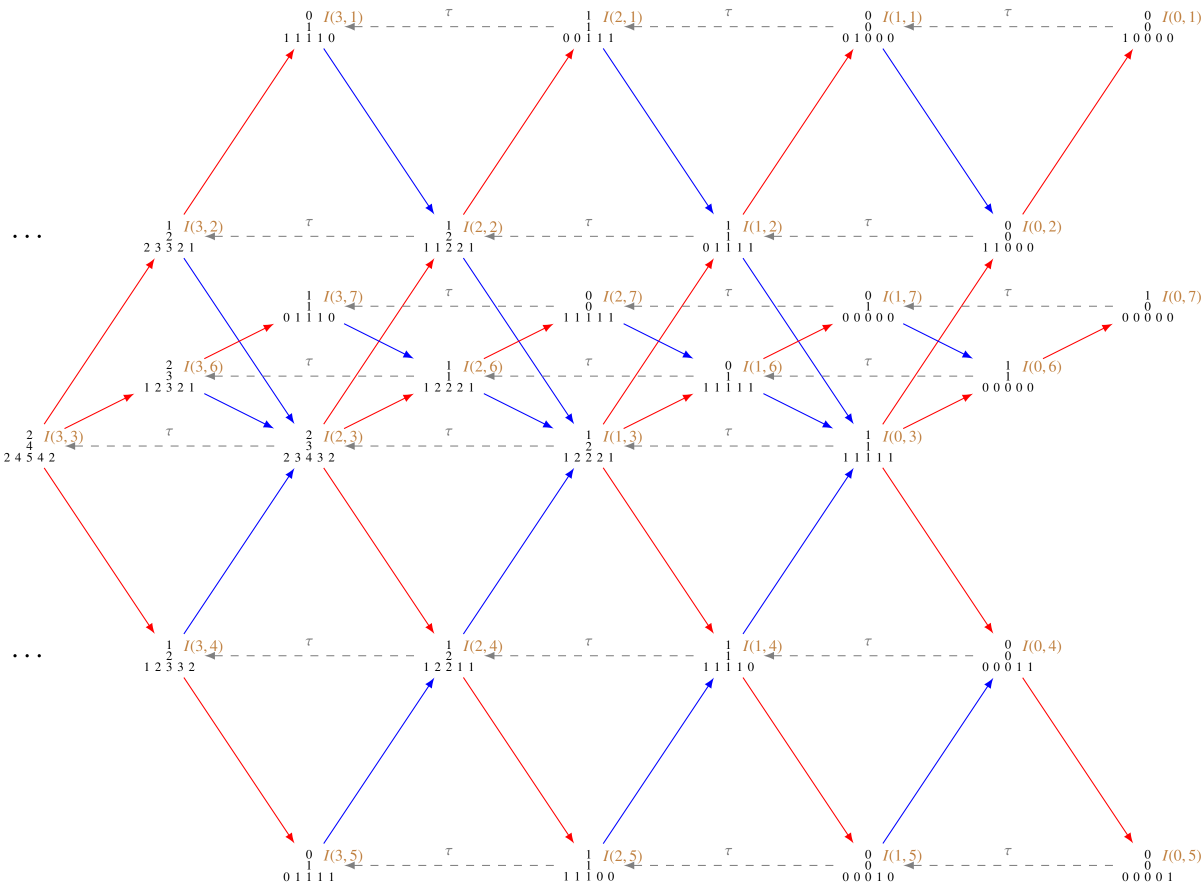
$$\underline{\dim} P(6n+5,3) = (3n+2, 6n+5, 9n+8, 6n+5, 3n+2, 6n+5, 3n+2)$$

$$M^{P(6n+5,3)}_{\alpha} = M^{P(6n+3,5)}_{\alpha} \oplus M^{P(6n+5,4)}_{\alpha}, \quad \text{for } \alpha \neq (2 \to 3)$$

$$M^{P(6n+5,3)}_{2\to3} = \left(M^{P(6n+3,5)}_{2\to3} \oplus M^{P(6n+5,4)}_{2\to3}\right) \boxplus \begin{array}{c} \\ 3n+1 \\ 1 \end{array} \begin{array}{cc} 1 & 4n+3 \\ \hline 0 & 0 \\ 1 & 0 \end{array}$$

II.3.2 The preinjective indecomposable modules

The preinjective indecomposable modules correspond to the vertices of the preinjective part of the Auslander–Reiten quiver, as shown in Figure II.2.



The preinjective part of the Auslander–Reiten quiver

Due to the symmetry of the quiver $\Delta(\widetilde{\mathbb{E}}_6)$ we give only the families of representations of the form $I(m,1)$, $I(m,2)$ and $I(m,3)$. For all the other representations we can use the permutations $\sigma = (1,5)(2,4)$ and $\tau = (1,7)(2,6)$ to write them in terms of $I(m,1)$, $I(m,2)$ and $I(m,3)$ in the following way ($m \geq 0$):

$$\underline{\dim} I(m,5) = (d_{\sigma(i)})_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0} \text{ and } \underline{\dim} I(m,7) = (d_{\tau(i)})_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0},$$

where $\underline{\dim} I(m,1) = (d_i)_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0}$,

$$\underline{\dim} I(m,4) = (d_{\sigma(i)})_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0} \text{ and } \underline{\dim} I(m,6) = (d_{\tau(i)})_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0},$$

where $\underline{\dim} I(m,2) = (d_i)_{i\in\Delta(\widetilde{\mathbb{E}}_6)_0}$ for the dimension vectors, respectively

$$I(m,5) = \left(M_{\sigma(i)\to\sigma(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1} \text{ and } I(m,7) = \left(M_{\tau(i)\to\tau(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1},$$

where $I(m,1) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1}$,

$$I(m,4) = \left(M_{\sigma(i)\to\sigma(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1} \text{ and } I(m,6) = \left(M_{\tau(i)\to\tau(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1},$$

where $I(m,2) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{E}}_6)_1}$ for the matrices.

In what follows we list the tree representations for preprojective families of the form $I(m,1)$, $I(m,2)$ and $I(m,3)$:

$\underline{\dim} I(6n,1) = (n+1, 2n, 3n, 2n, n, 2n, n)$

$I(0,1) = (0,0,0,0,0,0)$

$$I(6n,1) = \left(\begin{array}{c|cc} & n-1 & 2 \\ \hline n-1 & 1 & 0 \\ 2 & 0 & 1 \\ n-1 & 1 & 0 \end{array}, \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ n & 0 \end{array} + \begin{array}{c|c} & 2n \\ \hline n & 0 \\ 2n & 1 \end{array}, \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ n & 0 \end{array}, \begin{array}{c|c} & n \\ \hline n & 1 \\ n & 0 \end{array}, \begin{array}{c|c} & 2n \\ \hline n & 0 \\ 2n & 1 \end{array}, \begin{array}{c|c} & n \\ \hline n & 0 \\ n & 1 \end{array} \right), \quad n > 0$$

$\underline{\dim} I(6n+1,1) = (n, 2n+1, 3n, 2n, n, 2n, n)$

$I(1,1) = (0,0,0,0,0,0)$

$$I(6n+1,1) = \left(\begin{array}{c|c} & n \\ \hline 1 & 0 \\ n & 1 \\ n & 0 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ n-1 & 0 \end{array}, \begin{array}{c|cc} & 1 & 2n-1 \\ \hline 1 & 1 & 0 \\ n & 0 & 0 \\ 2n-1 & 0 & 1 \end{array}, \begin{array}{c|cc} & 1 & n-1 \\ \hline 1 & 1 & 0 \\ n-1 & 0 & 0 \\ 1 & 1 & 0 \\ n-1 & 0 & 1 \end{array}, \right.$$

$$\left. \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ n & 0 \end{array} + \begin{array}{c|cc} & 1 & 2n-1 \\ \hline n+1 & 0 & 0 \\ 2n-1 & 0 & 1 \end{array}, \begin{array}{c|cc} & 1 & n-1 \\ \hline 1 & 0 & 0 \\ 1 & 1 & 0 \\ n-1 & 0 & 1 \\ n-1 & 0 & 1 \end{array} \right), \quad n > 0$$

$\underline{\dim} I(6n+2,1) = (n, 2n, 3n+1, 2n+1, n+1, 2n+1, n+1)$

$$I(6n+2,1) = \left(\begin{smallmatrix} n \\ n \end{smallmatrix} \overset{n}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \begin{smallmatrix} 2n \\ n+1 \end{smallmatrix} \overset{2n}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \begin{smallmatrix} n+1 \\ 2n \end{smallmatrix} \overset{2n}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} 2n+1 \\ n \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} n+1 \\ n \end{smallmatrix} \overset{n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} n \\ 2n+1 \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \overset{n+1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right)$$

$\underline{\dim} I(6n+3,1) = (n+1, 2n+1, 3n+1, 2n+1, n, 2n+1, n)$

$$I(3,1) = \left(\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, 0, \begin{bmatrix} 1 \end{bmatrix}, 0 \right)$$

$$I(6n+3,1) = \left(\begin{smallmatrix} 1 \\ n+1 \\ n-1 \end{smallmatrix} \overset{n+1}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + \begin{smallmatrix} n+2 \\ n-1 \end{smallmatrix} \overset{n-1 \quad 2}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \begin{smallmatrix} 2n+1 \\ n \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \begin{smallmatrix} n+1 \\ 2n \end{smallmatrix} \overset{1 \quad 2n}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}, \right.$$

$$\left. \begin{smallmatrix} 2n+1 \\ n \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} 1 \\ 1 \\ n-1 \\ n \end{smallmatrix} \overset{1 \quad n-1}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}, \begin{smallmatrix} 1 \\ n \\ 2n \end{smallmatrix} \overset{1 \quad 2n}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}, \begin{smallmatrix} n+1 \\ n \end{smallmatrix} \overset{n}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right), \quad n > 0$$

$\underline{\dim} I(6n+4,1) = (n+1, 2n+2, 3n+2, 2n+1, n+1, 2n+1, n+1)$

$$I(6n+4,1) = \left(\begin{smallmatrix} n+1 \\ n+1 \end{smallmatrix} \overset{n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} 2n+2 \\ n \end{smallmatrix} \overset{2n+2}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} n+1 \\ 2n+1 \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \overset{n+1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} 2n+1 \\ n+1 \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \begin{smallmatrix} n+1 \\ 2n+1 \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \begin{smallmatrix} 1 \\ n \\ n \end{smallmatrix} \overset{1 \quad n}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}} \right)$$

$\underline{\dim} I(6n+5,1) = (n, 2n+1, 3n+2, 2n+2, n+1, 2n+2, n+1)$

$$I(5,1) = \left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$I(6n+5,1) = \left(\begin{smallmatrix} 1 \\ n \\ n \end{smallmatrix} \overset{n}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}, \begin{smallmatrix} 2n+1 \\ n+1 \end{smallmatrix} \overset{2n+1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \begin{smallmatrix} n+2 \\ 2n \end{smallmatrix} \overset{1 \quad 2n}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}, \begin{smallmatrix} 2n+2 \\ n \end{smallmatrix} \overset{2n+2}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \begin{smallmatrix} 1 \\ 1 \\ n \\ n \end{smallmatrix} \overset{1 \quad n}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}, \begin{smallmatrix} 1 \\ n \\ 2n+1 \end{smallmatrix} \overset{1 \quad 2n+1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}, \begin{smallmatrix} n+1 \\ n+1 \end{smallmatrix} \overset{n+1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right),$$

$n > 0$

$\underline{\dim} I(6n,2) = (2n+1, 4n+1, 6n, 4n, 2n, 4n, 2n)$

$$I(0,2) = \left(\begin{bmatrix} 1 \end{bmatrix}, 0, 0, 0, 0, 0 \right)$$

$$M_\alpha^{I(6n,2)} = M_\alpha^{I(6n+1,1)} \oplus M_\alpha^{I(6n,1)}, \quad \text{for } \alpha \neq (5 \to 4)$$

$$M^{I(6n,2)}_{5\to4} = \left(M^{I(6n+1,1)}_{5\to4} \oplus M^{I(6n,1)}_{5\to4}\right) \boxplus \begin{matrix} & \begin{matrix}1 & n-1\end{matrix} \\ \begin{matrix}1\\2n-1\end{matrix} & \begin{bmatrix}1&0\\0&0\end{bmatrix}\end{matrix}, \quad n>0$$

$$\underline{\dim} I(6n+1,2) = (2n, 4n+1, 6n+1, 4n+1, 2n+1, 4n+1, 2n+1)$$

$$I(1,2) = \left(0, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right)$$

$$M^{I(6n+1,2)}_{\alpha} = M^{I(6n+3,7)}_{\alpha} \oplus M^{I(6n,5)}_{\alpha}, \quad \text{for } \alpha \neq (1\to2)$$

$$M^{I(6n+1,2)}_{1\to2} = \left(M^{I(6n+3,7)}_{1\to2} \oplus M^{I(6n,5)}_{1\to2}\right) \boxplus \begin{matrix} & \begin{matrix}1 & n-1\end{matrix} \\ \begin{matrix}1\\2n\end{matrix} & \begin{bmatrix}1&0\\0&0\end{bmatrix}\end{matrix}, \quad n>0$$

$$\underline{\dim} I(6n+2,2) = (2n+1, 4n+1, 6n+2, 4n+2, 2n+1, 4n+2, 2n+1)$$

$$I(2,2) = \left(\begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$M^{I(6n+2,2)}_{\alpha} = M^{I(6n+3,1)}_{\alpha} \oplus M^{I(6n+2,1)}_{\alpha}, \quad \text{for } \alpha \neq (2\to3)$$

$$M^{I(6n+2,2)}_{2\to3} = \left(M^{I(6n+3,1)}_{2\to3} \oplus M^{I(6n+2,1)}_{2\to3}\right) \boxplus \begin{matrix} & \begin{matrix}1 & 2n-1\end{matrix} \\ \begin{matrix}1\\1\\3n-1\end{matrix} & \begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix}\end{matrix}, \quad n>0$$

$$\underline{\dim} I(6n+3,2) = (2n+2, 4n+3, 6n+3, 4n+2, 2n+1, 4n+2, 2n+1)$$

$$M^{I(6n+3,2)}_{\alpha} = M^{I(6n+5,5)}_{\alpha} \oplus M^{I(6n+2,7)}_{\alpha}, \quad \text{for } \alpha \neq (5\to4)$$

$$M^{I(6n+3,2)}_{5\to4} = \left(M^{I(6n+5,5)}_{5\to4} \oplus M^{I(6n+2,7)}_{5\to4}\right) \boxplus \begin{matrix} & \begin{matrix}1 & n\end{matrix} \\ \begin{matrix}1\\2n\end{matrix} & \begin{bmatrix}1&0\\0&0\end{bmatrix}\end{matrix}$$

$$\underline{\dim} I(6n+4,2) = (2n+1, 4n+3, 6n+4, 4n+3, 2n+2, 4n+3, 2n+2)$$

$$M^{I(6n+4,2)}_{\alpha} = M^{I(6n+5,1)}_{\alpha} \oplus M^{I(6n+4,1)}_{\alpha}, \quad \text{for } \alpha \neq (5\to4)$$

$$M^{I(6n+4,2)}_{5\to4} = \left(M^{I(6n+5,1)}_{5\to4} \oplus M^{I(6n+4,1)}_{5\to4}\right) \boxplus \begin{matrix} & \begin{matrix}1 & n\end{matrix} \\ \begin{matrix}1\\2n+1\end{matrix} & \begin{bmatrix}1&0\\0&0\end{bmatrix}\end{matrix}$$

$$\underline{\dim} I(6n+5,2) = (2n+2, 4n+3, 6n+5, 4n+4, 2n+2, 4n+4, 2n+2)$$

$$M^{I(6n+5,2)}_{\alpha} = M^{I(6n+1,5)[n\mapsto n+1]}_{\alpha} \oplus M^{I(6n+4,7)}_{\alpha}, \quad \text{for } \alpha \neq (5\to4)$$

$$M^{I(6n+5,2)}_{5\to4} = \left(M^{I(6n+1,5)[n\mapsto n+1]}_{5\to4} \oplus M^{I(6n+4,7)}_{5\to4}\right) \boxplus \begin{matrix} & \begin{matrix}1 & n\end{matrix} \\ \begin{matrix}1\\2n+1\end{matrix} & \begin{bmatrix}1&0\\0&0\end{bmatrix}\end{matrix}$$

$$\underline{\dim} I(6n,3) = (3n+1, 6n+1, 9n+1, 6n+1, 3n+1, 6n+1, 3n+1)$$

$$I(0,3) = \left([1],[1],[1],[1],[1],[1]\right)$$

$$M_\alpha^{I(6n,3)} = M_\alpha^{I(6n+1,6)} \oplus M_\alpha^{I(6n,7)}, \quad \text{for } \alpha \neq (1 \to 2)$$

$$M_{1\to 2}^{I(6n,3)} = \left(M_{1\to 2}^{I(6n+1,6)} \oplus M_{1\to 2}^{I(6n,7)}\right) \boxplus \begin{array}{c|cc} & n-1 & 1 \\ \hline 1 & 0 & 1 \\ 3n+2 & 0 & 0 \end{array}, \quad n > 0$$

$$\underline{\dim} I(6n+1,3) = (3n+1, 6n+2, 9n+2, 6n+2, 3n+1, 6n+2, 3n+1)$$

$$M_\alpha^{I(6n+1,3)} = M_\alpha^{I(6n+3,1)} \oplus M_\alpha^{I(6n+1,2)}, \quad \text{for } \alpha \neq (6 \to 3)$$

$$M_{6\to 3}^{I(6n+1,3)} = \left(M_{6\to 3}^{I(6n+3,1)} \oplus M_{6\to 3}^{I(6n+1,2)}\right) \boxplus \begin{array}{c|cc} & 4n & 1 \\ \hline 1 & 0 & 1 \\ 3n & 0 & 0 \end{array}$$

$$\underline{\dim} I(6n+2,3) = (3n+2, 6n+3, 9n+4, 6n+3, 3n+2, 6n+3, 3n+2)$$

$$I(2,3) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

$$M_\alpha^{I(6n+2,3)} = M_\alpha^{I(6n+3,4)} \oplus M_\alpha^{I(6n+2,5)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{I(6n+2,3)} = \left(M_{7\to 6}^{I(6n+3,4)} \oplus M_{7\to 6}^{I(6n+2,5)}\right) \boxplus \begin{array}{c|cc} & n & 1 \\ \hline 1 & 0 & 1 \\ 4n+1 & 0 & 0 \end{array}, \quad n > 0$$

$$\underline{\dim} I(6n+3,3) = (3n+2, 6n+4, 9n+5, 6n+4, 3n+2, 6n+4, 3n+2)$$

$$M_\alpha^{I(6n+3,3)} = M_\alpha^{I(6n+5,1)} \oplus M_\alpha^{I(6n+3,2)}, \quad \text{for } \alpha \neq (2 \to 3)$$

$$M_{2\to 3}^{I(6n+3,3)} = \left(M_{2\to 3}^{I(6n+5,1)} \oplus M_{2\to 3}^{I(6n+3,2)}\right) \boxplus \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 1 & 0 & 1 \\ 3n+1 & 0 & 0 \end{array}$$

$$\underline{\dim} I(6n+4,3) = (3n+3, 6n+5, 9n+7, 6n+5, 3n+3, 6n+5, 3n+3)$$

$$M_\alpha^{I(6n+4,3)} = M_\alpha^{I(6n+5,6)} \oplus M_\alpha^{I(6n+4,7)}, \quad \text{for } \alpha \neq (2 \to 3)$$

$$M_{2\to 3}^{I(6n+4,3)} = \left(M_{2\to 3}^{I(6n+5,6)} \oplus M_{2\to 3}^{I(6n+4,7)}\right) \boxplus \begin{array}{c|cc} & 1 & 2n \\ \hline 1 & 1 & 0 \\ 7n & 0 & 0 \end{array}$$

$$\underline{\dim} I(6n+5,3) = (3n+3, 6n+6, 9n+8, 6n+6, 3n+3, 6n+6, 3n+3)$$

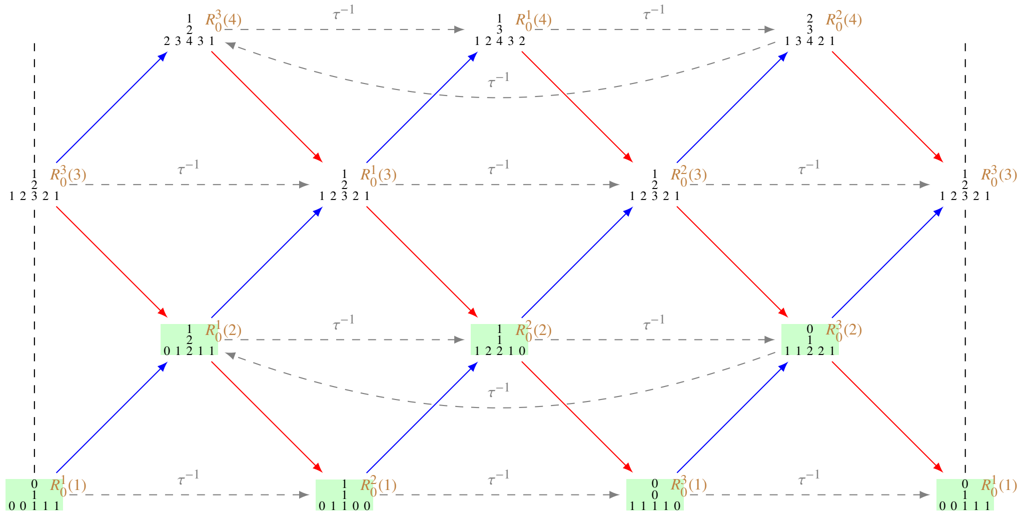
$$M_\alpha^{I(6n+5,3)} = M_\alpha^{I(6n+1,1)[n\mapsto n+1]} \oplus M_\alpha^{I(6n+5,2)}, \quad \text{for } \alpha \neq (7 \to 6)$$

$$M_{7\to 6}^{I(6n+5,3)} = \left(M_{7\to 6}^{I(6n+1,1)[n\mapsto n+1]} \oplus M_{7\to 6}^{I(6n+5,2)}\right) \boxplus \begin{array}{c|cc} & 1 & 2n+1 \\ \hline 1 & 1 & 0 \\ 2n+1 & 0 & 0 \end{array}$$

II.3.3 The exceptional regular modules

There are only a finite number of exceptional regular modules. These are the non-homogeneous indecomposable regulars with dimension vector falling below $\delta = (1,2,3,2,1,2,1)$, marked with green in Figures II.3, II.4 and II.5. Note that $\underline{\dim} R_0^l(3) = \underline{\dim} R_1^l(3) = \underline{\dim} R_\infty^{l'}(2) = \delta$, where $l \in \{1,2,3\}$, $l' \in \{1,2\}$.

Representations of regular simples of $\Delta(\widetilde{\mathbb{E}}_6)$ are also given in [34], we include them here only for the sake of completeness:



The regular non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{E}}_6)}$

$$\underline{\dim} R_0^1(1) = (0,0,1,1,1,1,0),$$
$$R_0^1(1) = \left(0,0,\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\end{bmatrix},0\right)$$

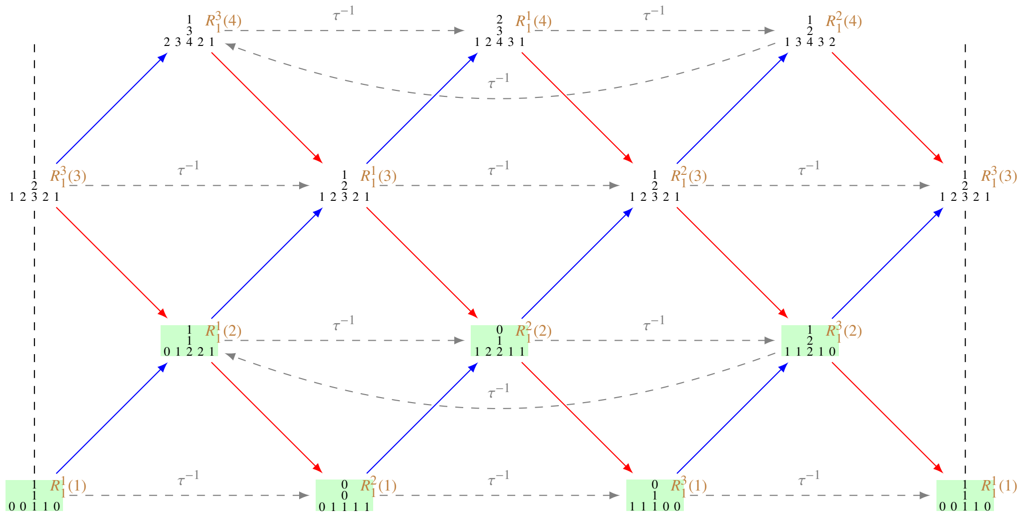
$$\underline{\dim} R_0^2(1) = (0,1,1,0,0,1,1),$$
$$R_0^2(1) = \left(0,\begin{bmatrix}1\end{bmatrix},0,0,\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\end{bmatrix}\right)$$

$$\underline{\dim} R_0^3(1) = (1,1,1,1,0,0,0)$$
$$R_0^3(1) = \left(\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\end{bmatrix},0,0,0\right)$$

$$\underline{\dim} R_0^1(2) = (0,1,2,1,1,2,1),$$
$$R_0^1(2) = \left(0,\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$\underline{\dim} R_0^2(2) = (1,2,2,1,0,1,1),$$
$$R_0^2(2) = \left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, 0, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right)$$

$$\underline{\dim} R_0^3(2) = (1,1,2,2,1,1,0),$$
$$R_0^3(2) = \left(\begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, 0\right)$$



The regular non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{E}}_6)}$

$$\underline{\dim} R_1^1(1) = (0,0,1,1,0,1,1),$$
$$R_1^1(1) = \left(0, 0, \begin{bmatrix}1\end{bmatrix}, 0, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right)$$

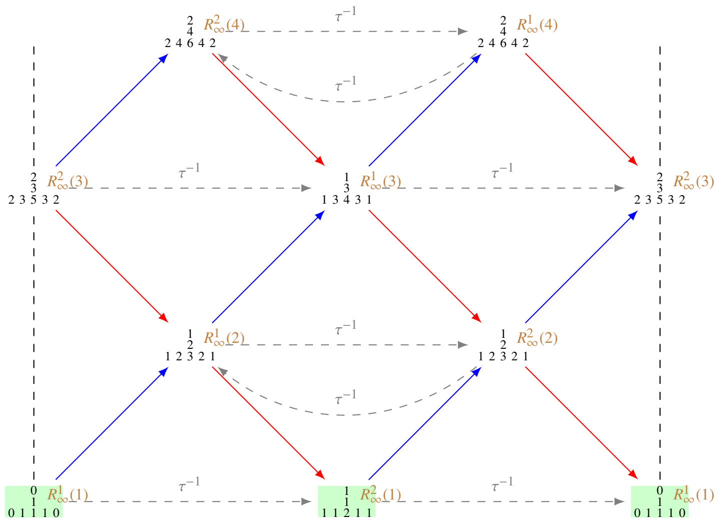
$$\underline{\dim} R_1^2(1) = (0,1,1,1,1,0,0),$$
$$R_1^2(1) = \left(0, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, 0, 0\right)$$

$$\underline{\dim} R_1^3(1) = (1,1,1,0,0,1,0),$$
$$R_1^3(1) = \left(\begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, 0, 0, \begin{bmatrix}1\end{bmatrix}, 0\right)$$

$$\underline{\dim} R_1^1(2) = (0,1,2,2,1,1,1),$$
$$R_1^1(2) = \left(0, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right)$$

$$\underline{\dim} R_1^2(2) = (1,2,2,1,1,1,0),$$
$$R_1^2(2) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, 0 \right)$$

$$\underline{\dim} R_1^3(2) = (1,1,2,1,0,2,1),$$
$$R_1^3(2) = \left(\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$



The regular non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{E}}_6)}$

$$\underline{\dim} R_\infty^1(1) = (0,1,1,1,0,1,0),$$
$$R_\infty^1(1) = \left(0, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, 0, \begin{bmatrix} 1 \end{bmatrix}, 0 \right)$$

$$\underline{\dim} R_\infty^2(1) = (1,1,2,1,1,1,1),$$
$$R_\infty^2(1) = \left(\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix} \right)$$

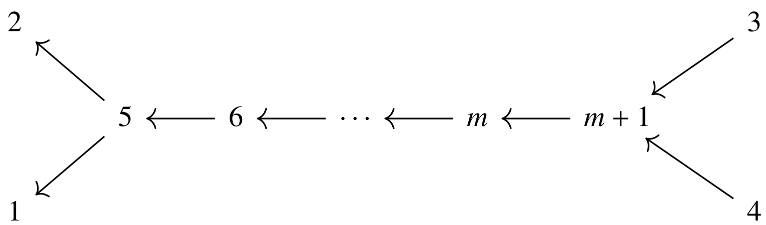
Chapter III

Tree representations of the quiver $\widetilde{\mathbb{D}}_m$

In this chapter we will give a complete and general list of tree representations corresponding to the exceptional modules over the path algebra of the canonically oriented Euclidean quiver $\widetilde{\mathbb{D}}_m$. The proof (involving induction and symbolic computation with block matrices) was partially generated by a purposefully developed computer software and is available on arXiv as an appendix. All the representations listed remain valid over any base field, answering a question raised by Ringel in [27]. The results presented here were published in the article [16] and its appendix [15].

III.1 Basic notions and definitions

Consider the canonically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_m$, denoted from now on by $\Delta(\widetilde{\mathbb{D}}_6)$, having the following shape:



Therefore, we have $\Delta(\widetilde{\mathbb{D}}_m)_0 = \{1, \ldots, m, m+1\}$ and since we have at most one arrow connecting two different vertices, the set of arrows is the following:

$$\Delta(\widetilde{\mathbb{D}}_m)_1 = \{(5 \to 1),\ (5 \to 2),\ (3 \to m+1),\ (4 \to m+1),\ (6 \to 5),\ (7 \to 6),\ \ldots,\ (m+1 \to m)\}.$$

The Tits form in this case is

$$q_{\Delta(\widetilde{\mathbb{D}}_m)}(x) = \frac{1}{4}\left((2x_1 - x_5)^2 + (2x_2 - x_5)^2 + (x_{m+1} - 2x_3)^2 + (x_{m+1} - 2x_4)^2 + 2\sum_{i=5}^{m}(x_i - x_{i+1})^2\right).$$

Note that this is independent of the orientation of the quiver and it is positive semi-definite with radical $\mathbb{Z}\delta$, where $\delta = (1, 1, 1, 1, 2, \ldots, 2)$.

III.2 Constructing tree representations for $\Delta(\widetilde{\mathbb{D}}_m)$ from trees of $\Delta(\widetilde{\mathbb{D}}_6)$

In this section we present an explicit method for solving the following problem: given an exceptional root x in $\Delta(\widetilde{\mathbb{D}}_m)$ where $m \geq 4$, construct an (exceptional) tree representation $M \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_m)$ such that

$\underline{\dim} M = x$. Recall that a positive real root x is exceptional if $\partial x \neq 0$, or if $\partial x = 0$ then $x < \delta$. Throughout this section we denote the identity matrix by I_n (in case $n = 0$ we take I_0 to be the null morphism).

We begin with two lemmas on the form of real roots of the quiver $\Delta(\widetilde{\mathbb{D}}_m)$, where $m \geq 6$.

Lemma III.2.1. *Let x be a real root of the quiver $\Delta(\widetilde{\mathbb{D}}_m)$. Then x has one of the following forms:*

- $x^{(1)} = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}})$, *where* $i = m - 3$;
- $x^{(2)} = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}}, \overbrace{b, \dots, b}^{j \text{ times}})$, *where* $i, j \in \mathbb{N}^*$, $i + j = m - 3$ *and* $a \neq b$;
- $x^{(3)} = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}}, \overbrace{b, \dots, b}^{j \text{ times}}, \overbrace{a, \dots, a}^{k \text{ times}})$, *where* $i, j, k \in \mathbb{N}^*$, $i + j + k = m - 3$ *and* $a \neq b$;
- $x^{(4)} = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}}, \overbrace{b, \dots, b}^{j \text{ times}}, \overbrace{c, \dots, c}^{k \text{ times}})$, *where* $i, j, k \in \mathbb{N}^*$, $i + j + k = m - 3$ *and* a, b, c *are pairwise different.*

Combining these four possibilities for x we get the following (alternative) form:

Lemma III.2.2. *Let x be a real root of the quiver $\Delta(\widetilde{\mathbb{D}}_m)$. Then x has the form* $x = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}}, \overbrace{b, \dots, b}^{j \text{ times}}, \overbrace{c, \dots, c}^{k \text{ times}})$ *with a, b and c not necessarily distinct,* $i, j, k \in \mathbb{N}^*$ *and* $i + j + k = m - 3$.

Let us denote by $\mathfrak{R}_m$ the set of exceptional roots over $\Delta(\widetilde{\mathbb{D}}_m)$.

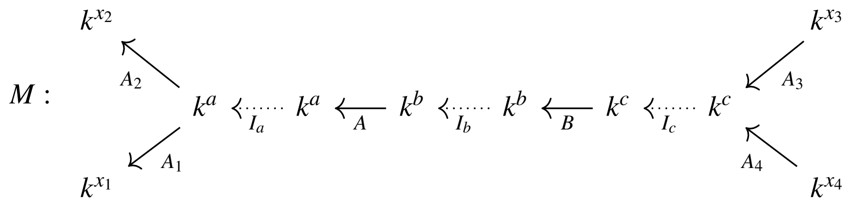
For $m \geq 7$ we introduce $\mathfrak{p}_m : \mathfrak{R}_m \to \mathfrak{R}_6$, where $\mathfrak{p}_m(x) = x'$ with $x \in \mathfrak{R}_m$ constructed according to the following cases (as specified in Lemma III.2.1):

- if $x = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}})$, where $i = m - 3$, then $x' = (x_1, x_2, x_3, x_4, a, a, a)$;
- if $x = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}}, \overbrace{b, \dots, b}^{j \text{ times}})$, where $i, j \in \mathbb{N}^*$, $i + j = m - 3$ and $a \neq b$, then $x' = (x_1, x_2, x_3, x_4, a, a, b)$ in case $i \geq 2$, else $x' = (x_1, x_2, x_3, x_4, a, b, b)$;
- if $x = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}}, \overbrace{b, \dots, b}^{j \text{ times}}, \overbrace{a, \dots, a}^{k \text{ times}})$, where $i, j, k \in \mathbb{N}^*$, $i + j + k = m - 3$ and $a \neq b$, then $x' = (x_1, x_2, x_3, x_4, a, b, a)$;
- if $x = (x_1, x_2, x_3, x_4, \overbrace{a, \dots, a}^{i \text{ times}}, \overbrace{b, \dots, b}^{j \text{ times}}, \overbrace{c, \dots, c}^{k \text{ times}})$, where $i, j, k \in \mathbb{N}^*$, $i + j + k = m - 3$ and a, b, c are pairwise different, then $x' = (x_1, x_2, x_3, x_4, a, b, c)$.

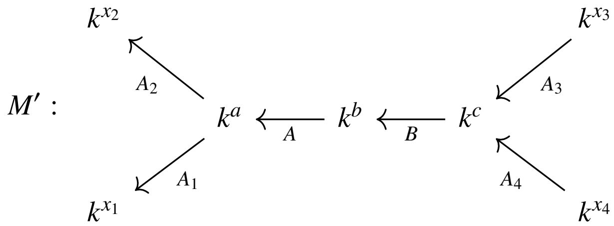
Lemma III.2.3. *For any $m \geq 7$, the previously introduced $\mathfrak{p}_m : \mathfrak{R}_m \to \mathfrak{R}_6$ is a well-defined surjective function. Moreover, defects are also kept (i.e. for all $x \in \mathfrak{R}_m$, $\partial_{k\Delta(\widetilde{\mathbb{D}}_m)} x = \partial_{k\Delta(\widetilde{\mathbb{D}}_6)} \mathfrak{p}_m(x)$).*

In the following drawings the dotted arrows represent zero or more arrows of the form $k^d \xleftarrow[I_d]{} k^d$ ($d \in \{a, b, c\}$), connecting vertices with the same dimension, with suitable identity matrices associated to them. We can state the following:

Lemma III.2.4. *Let $m \geq 7$, $x \in \mathfrak{R}_m$ (as in Lemma III.2.2), $x' = (x_1, x_2, x_3, x_4, a, b, c) \in \mathfrak{R}_6$ such that $\mathfrak{p}_m(x) = x'$ and two representations $M = (M_\alpha, M_i) \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_m)$ and $M' = (M'_\alpha, M'_i) \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_6)$ with $\underline{\dim} M = x$ and $\underline{\dim} M' = x'$ having the following matrices:*



and



Then M is exceptional if and only if M' is exceptional.

In what follows, we are going to construct explicitly a function $T_m : \mathfrak{R}_m \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_m)$ such that $T_m(x)$ with $\underline{\dim} T_m(x) = x$ is a tree representation for any exceptional root x ($m \geq 4$). In this context we treat $\operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_m)$ as a set consisting of only "matrix representations" of Q, where a "matrix representation" is just a collection of matrices of compatible dimensions together with induced vector spaces of the form k^s, encoding a representation of Q.

Constructing tree representations of $\Delta(\widetilde{\mathbb{D}}_6)$

We begin with the $m = 6$ case, since by construction the lists given in Section III.4 define exactly such a function T_6. One can take any exceptional root x over $\Delta(\widetilde{\mathbb{D}}_6)$, identify the corresponding family of representations (based on ∂x and the general forms of the dimension vectors) and apply the right formula for obtaining the matrices of the representations. So we can state the following:

Proposition III.2.5. *For any exceptional root x over $\Delta(\widetilde{\mathbb{D}}_6)$ the listed formulas in Section III.4 define a function $T_6 : \mathfrak{R}_6 \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_6)$ with $T_6(x)$ a tree representation.*

Constructing tree representations of $\Delta(\widetilde{\mathbb{D}}_m)$, for $m \geq 7$

For the $m \geq 7$ case we define $T_m : \mathfrak{R}_m \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_m)$ as follows: for $x \in \mathfrak{R}_m$ let $T_m(x) = M$ where the representation $M \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_m)$ is constructed based on $M' = T_6(\mathfrak{p}_m(x)) \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_6)$; specific matrices of the representation M are $M_{5\to1} = M'_{5\to1}$, $M_{5\to2} = M'_{5\to2}$, $M_{3\to(m+1)} = M'_{3\to7}$, $M_{4\to(m+1)} = M'_{4\to7}$; the other matrices are given based on the possible forms of x (see Lemma III.2.1):

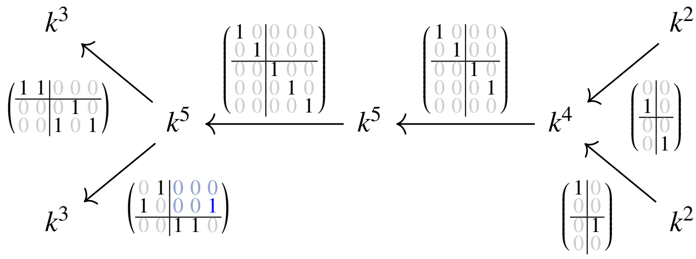
- if $x = (x_1, x_2, x_3, x_4, \overbrace{a, \ldots, a}^{i \text{ times}})$, where $i = m - 3$, then $M_{m\to(m-1)} = M'_{6\to5}$, $M_{(m+1)\to m} = M'_{7\to6}$ and for all other arrows assign identity matrices I_a;

- if $x = (x_1, x_2, x_3, x_4, \overbrace{a, \ldots, a}^{i \text{ times}}, \overbrace{b, \ldots, b}^{j \text{ times}})$, where $i, j \in \mathbb{N}^*$, $i + j = m - 3$ and $a \neq b$, then $M_{(4+i)\to(3+i)} = M'_{6\to5}$, $M_{(4+i+1)\to(4+i)} = M'_{7\to6}$ in case $i \geq 2$, else (if $i = 1$) $M_{6\to5} = M'_{6\to5}$, $M_{7\to6} = M'_{7\to6}$ – for all other arrows assign compatible identity matrices (either I_a or I_b);
- if $x = (x_1, x_2, x_3, x_4, \overbrace{a, \ldots, a}^{i \text{ times}}, \overbrace{b, \ldots, b}^{j \text{ times}}, \overbrace{a, \ldots, a}^{k \text{ times}})$, where $i, j, k \in \mathbb{N}^*$, $i + j + k = m - 3$ and $a \neq b$, then $M_{(4+i+1)\to(4+i)} = M'_{6\to5}$, $M_{(4+i+j+1)\to(4+i+j)} = M'_{7\to6}$ and compatible identity matrices for all other arrows;
- if $x = (x_1, x_2, x_3, x_4, \overbrace{a, \ldots, a}^{i \text{ times}}, \overbrace{b, \ldots, b}^{j \text{ times}}, \overbrace{c, \ldots, c}^{k \text{ times}})$, where $i, j, k \in \mathbb{N}^*$, $i + j + k = m - 3$ and a, b, c are pairwise different, then $M_{(4+i+1)\to(4+i)} = M'_{6\to5}$, $M_{(4+i+j+1)\to(4+i+j)} = M'_{7\to6}$ and compatible identity matrices for all other arrows.

Proposition III.2.6. *For $m \geq 7$ the previously defined function $T_m : \mathfrak{R}_m \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_m)$ gives tree representations for all exceptional roots, i.e. for any $x \in \mathfrak{R}_m$ the representation $T_m(x) \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_m)$ is a tree representation.*

Example III.2.7. Suppose we need a tree representation for the preprojective indecomposable $P(6,7)_{\Delta(\widetilde{\mathbb{D}}_8)} \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_8)$. We have that $\underline{\dim} P(6,7)_{\Delta(\widetilde{\mathbb{D}}_8)} = (3,3,2,2,5,5,5,4,4) \in \mathfrak{R}_8$. We compute its corresponding exceptional root over $\Delta(\widetilde{\mathbb{D}}_6)$: $\mathfrak{p}_8(3,3,2,2,5,5,5,4,4) = (3,3,2,2,5,5,4) \in \mathfrak{R}_6$. Due to Lemma III.2.3 we know that defects are kept by the function $\mathfrak{p}_8$, so we have to search for the corresponding representation among the list of preprojective families in Subsection III.4.1. We identify the family $P(8n+4,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$ with dimension vector of the form $\underline{\dim} P(8n+4,6)_{\Delta(\widetilde{\mathbb{D}}_6)} = (4n+3, 4n+3, 4n+2, 4n+2, 8n+5, 8n+5, 8n+4)$, which for $n = 0$ gives exactly our root. Using the formula given there, we construct the tree representation of $T_6(\mathfrak{p}_8(3,3,2,2,5,5,5,4,4)) = P(4,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$.

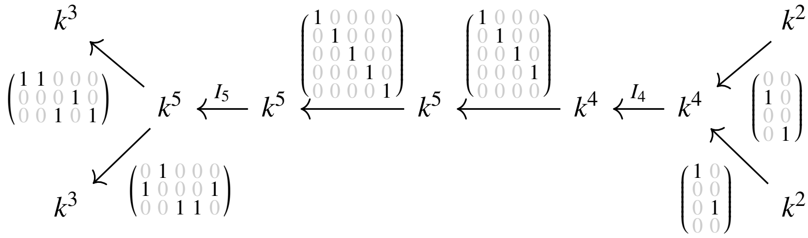
$P(4,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$:



As indicated, this representation may be constructed by first forming a direct sum of $P(4,1)_{\Delta(\widetilde{\mathbb{D}}_6)}$ with $P(6,2)_{\Delta(\widetilde{\mathbb{D}}_6)}$ and then inserting the matrix block $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ into to upper right corner of the matrix associated to the arrow $(5 \to 1)$, thus bringing in an extra element equal to one into the matrix.

Now we are ready to construct our initial representation $T_8(3,3,2,2,5,5,5,4,4) = P(6,7)_{\Delta(\widetilde{\mathbb{D}}_8)}$ using the described method, by taking the matrices associated to the arrows $(5 \to 1)$, $(5 \to 2)$, $(3 \to 7)$, $(4 \to 7)$, $(6 \to 5)$ and $(7 \to 6)$ from $P(4,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$, associating them to the arrows $(5 \to 1)$, $(5 \to 2)$, $(3 \to 9)$, $(4 \to 9)$, $(7 \to 6)$ respectively $(8 \to 7)$ in $P(6,7)_{\Delta(\widetilde{\mathbb{D}}_8)}$ and putting identity matrices on the remaining arrows:

$P(6,7)_{\Delta(\widetilde{\mathbb{D}}_8)}$:



For the $m = 4$ and $m = 5$ cases we first state some analogous lemmas and then the explicit construction for $T_4 : \mathfrak{R}_4 \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_4)$ and $T_5 : \mathfrak{R}_5 \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_5)$.

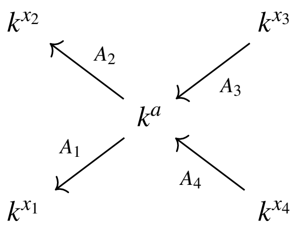
For $m = 4$ we introduce $\mathfrak{i}_4 : \mathfrak{R}_4 \to \mathfrak{R}_6$, where $\mathfrak{i}_4(x_1, x_2, x_3, x_4, a) = (x_1, x_2, x_3, x_4, a, a, a)$ and for $m = 5$ we introduce $\mathfrak{i}_5 : \mathfrak{R}_5 \to \mathfrak{R}_6$, where $\mathfrak{i}_5(x_1, x_2, x_3, x_4, a, b) = (x_1, x_2, x_3, x_4, a, a, b)$.

Lemma III.2.8. *The previously defined* $\mathfrak{i}_4 : \mathfrak{R}_4 \to \mathfrak{R}_6$ *and* $\mathfrak{i}_5 : \mathfrak{R}_5 \to \mathfrak{R}_6$ *are well-defined injective functions. Moreover, defects are also kept (i.e. for all* $x \in \mathfrak{R}_4$, $\partial_{k\Delta(\widetilde{\mathbb{D}}_4)} x = \partial_{k\Delta(\widetilde{\mathbb{D}}_6)} \mathfrak{i}_4(x)$ *and for all* $x \in \mathfrak{R}_5$, $\partial_{k\Delta(\widetilde{\mathbb{D}}_5)} x = \partial_{k\Delta(\widetilde{\mathbb{D}}_6)} \mathfrak{i}_5(x)$*).*

Lemma III.2.9. *The following statements are true:*

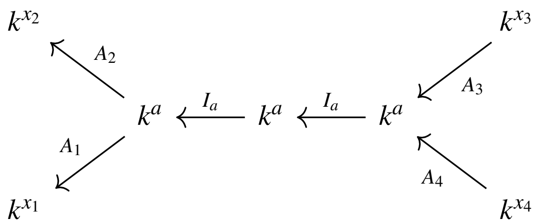
(a) Let $x = (x_1, x_2, x_3, x_4, a) \in \mathfrak{R}_4$, $\mathfrak{i}_4(x) = x' = (x_1, x_2, x_3, x_4, a, a, a) \in \mathfrak{R}_6$ *and two representations* $V \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_4)$ *and* $V' \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_6)$ *such that* $\underline{\dim} V = x$ *and* $\underline{\dim} V' = x'$ *having the following matrices:*

V :



and

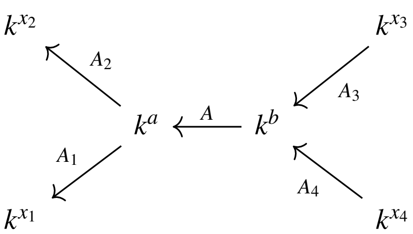
V' :



Then V is exceptional if and only if V′ is exceptional.

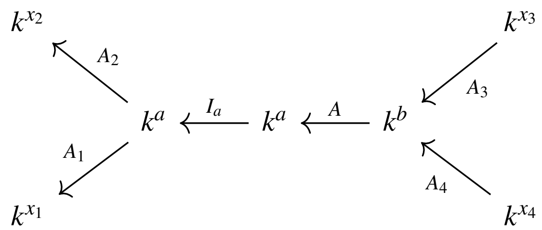
(b) Let $x = (x_1, x_2, x_3, x_4, a, b) \in \mathfrak{R}_5$, $\mathfrak{i}_5(x) = x' = (x_1, x_2, x_3, x_4, a, a, b) \in \mathfrak{R}_6$ *and two representations* $Z \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_5)$ *and* $Z' \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_6)$ *such that* $\underline{\dim} Z = x$ *and* $\underline{\dim} Z' = x'$ *having the following matrices:*

Z :



and

Z' :



Then Z is exceptional if and only if Z' is exceptional.

As one can see, the representations $V', Z' \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_6)$ are somewhat special, in the sense that they must have identity matrices associated to arrows connecting vertices of equal dimension on the central axis of the quiver. Upon inspection of the lists given in Section III.4 it can be seen that all representations were constructed to fulfill this requirement. So we can state:

Lemma III.2.10. *All the (exceptional) tree representations in the case $\Delta(\widetilde{\mathbb{D}}_6)$ (listed in Section III.4) have identity matrices associated to the arrows on the central axis, which connect vertices of equal dimension.*

Constructing tree representations of $\Delta(\widetilde{\mathbb{D}}_4)$ and $\Delta(\widetilde{\mathbb{D}}_5)$

We are now ready to give the functions $T_4 : \mathfrak{R}_4 \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_4)$ and $T_5 : \mathfrak{R}_5 \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_5)$. For any $x \in \mathfrak{R}_4$ let $T_4(x) = V$ be constructed based on $V' = T_6(\mathfrak{i}_4(x)) \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_6)$ in the following way: $V_{5\to1} = V'_{5\to1}$, $V_{5\to2} = V'_{5\to2}$, $V_{3\to5} = V'_{3\to7}$ and $V_{4\to5} = V'_{4\to7}$. Similarly, For any $x \in \mathfrak{R}_5$ let $T_5(x) = Z$ be constructed based on $Z' = T_6(\mathfrak{i}_5(x)) \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_6)$ in the following way: $Z_{5\to1} = Z'_{5\to1}$, $Z_{5\to2} = Z'_{5\to2}$, $Z_{3\to6} = Z'_{3\to7}$, $Z_{4\to6} = Z'_{4\to7}$ and $Z_{6\to5} = Z'_{7\to6}$.

Proposition III.2.11. *Using the previous definitions, the following is true:*

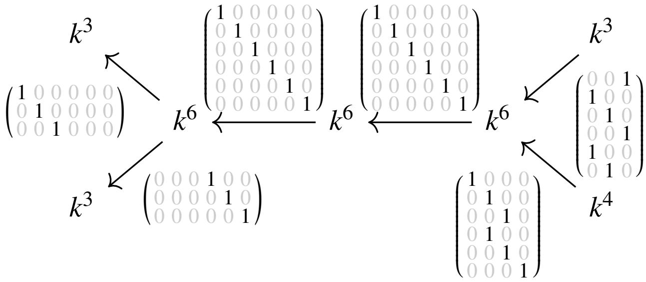
(a) The function $T_4 : \mathfrak{R}_4 \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_4)$ gives tree representations for all exceptional roots, i.e. for any $x \in \mathfrak{R}_4$ the representation $T_4(x) \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_4)$ is a tree representation.

(b) The function $T_5 : \mathfrak{R}_5 \to \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_5)$ gives tree representations for all exceptional roots, i.e. for any $x \in \mathfrak{R}_5$ the representation $T_5(x) \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_5)$ is a tree representation.

Example III.2.12. Suppose we need a tree representation for the preinjective indecomposable $I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)} \in \operatorname{rep} k\Delta(\widetilde{\mathbb{D}}_4)$. We have that $\underline{\dim} I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)} = (3,3,3,4,6) \in \mathfrak{R}_4$. We compute its corresponding exceptional root over $\Delta(\widetilde{\mathbb{D}}_6)$: $\mathfrak{i}_4(3,3,3,4,6) = (3,3,3,4,6,6,6) \in \mathfrak{R}_6$. Due to Lemma III.2.8 we know that defects are kept by the function $\mathfrak{i}_4$, so we have to search for the corresponding representation among the list of preinjective families in Subsection III.4.2. We identify the family $I(8n+4,4)_{\Delta(\widetilde{\mathbb{D}}_6)}$ – obtained from $I(8n+4,3)_{\Delta(\widetilde{\mathbb{D}}_6)}$ via the suitable permutation $\tau = (3,4)$ as explained there – with dimension vector of the form $\underline{\dim} I(8n+4,4)_{\Delta(\widetilde{\mathbb{D}}_6)} = (2n+1, 2n+1, 2n+1, 2n+2, 4n+2, 4n+2, 4n+2)$, which for $n = 1$ gives exactly our root. Using the formula given there, we construct the tree representation of

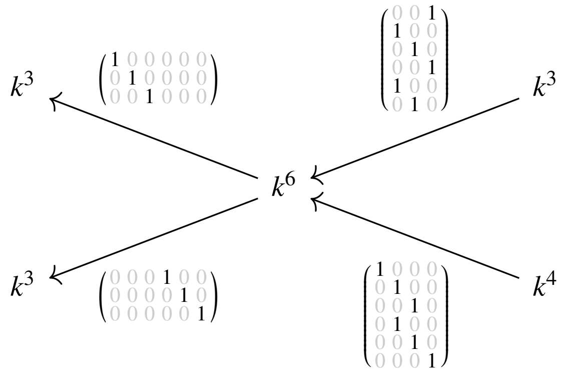
$T_6(\mathrm{i}_4(3,3,3,4,6)) = I(12,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$.

$I(12,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$:



Now we are ready to construct our initial representation $T_4(3,3,3,4,6) = I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)}$ using the described method, by taking the matrices associated to the arrows $(5 \to 1)$, $(5 \to 2)$, $(3 \to 7)$ and $(4 \to 7)$ from $I(12,6)_{\Delta(\widetilde{\mathbb{D}}_6)}$ and associating them to the arrows $(5 \to 1)$, $(5 \to 2)$, $(3 \to 5)$, respectively $(4 \to 5)$ in $I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)}$.

$I(6,4)_{\Delta(\widetilde{\mathbb{D}}_4)}$:



III.3 Proving the field independent tree module property

In this section we give a short overview of the method used to prove the tree module property for every representation given in the lists in Section III.4. The method presented here has been used already in the case of the canonically oriented $\widetilde{\mathbb{E}}_6$ in Chapter II.

Throughout this section we will use the “field independent” qualifier in relation to representations and short exact sequences according to Definition II.2.1.

The technique used to obtain and prove the formulas in Section III.4 (in a field independent way) consists of a mixture of computer experimentation using the computer algebra system GAP [2] followed by a computer aided proof performed by a proof assistant software developed in the purely functional programming language Clean [1], specifically for this purpose. The proof uses our prior knowledge on the existence of certain Schofield sequences (see [39]) and it is based on Proposition II.2.3, proved in Chapter II.

The formulas for the matrices listed in Section III.4 were obtained after extensive experimentation and testing in GAP, working over small finite fields (for details see Remarks 8, 9 and 10 from [39]). Then the “guessed” formulas were introduced into an input LaTeX document which in turn was processed by the proof assistant. The computer aided proof is basically an induction on the dimensions of the representations (detailed in Subsection 1.3 of [15]). For given input data (supposedly) defining short exact sequences (the two different Schofield sequences required by Proposition II.2.3) the proof assistant verifies using Lemma II.2.4 that indeed, two short exact sequences may be constructed using the given matrices (in a field independent way). To complete the proof for the tree module property, it also counts the total number of ones in matrices.

For some more details on the block-matrix arithmetic, rank computation and other steps performed by the proof assistant software we refer to Section II.2 and to [15].

III.4 Tree representations of the quiver $\Delta(\widetilde{\mathbb{D}}_6)$

In this section we list the formulas describing the matrices of the representations corresponding to exceptional modules: the preprojective indecomposables (Subsection III.4.1), the preinjective indecomposables (Subsection III.4.2) and the regular non-homogeneous indecomposables with dimension vector below δ (Subsection III.4.3). For convenience, at the beginning of each of the following subsections, we present a graphical representation of the corresponding part of the Auslander–Reiten quiver. Blue arrows show the existence of a so-called irreducible monomorphism, while red arrows represent irreducible epimorphisms between suitable indecomposable modules (for details see [3]).

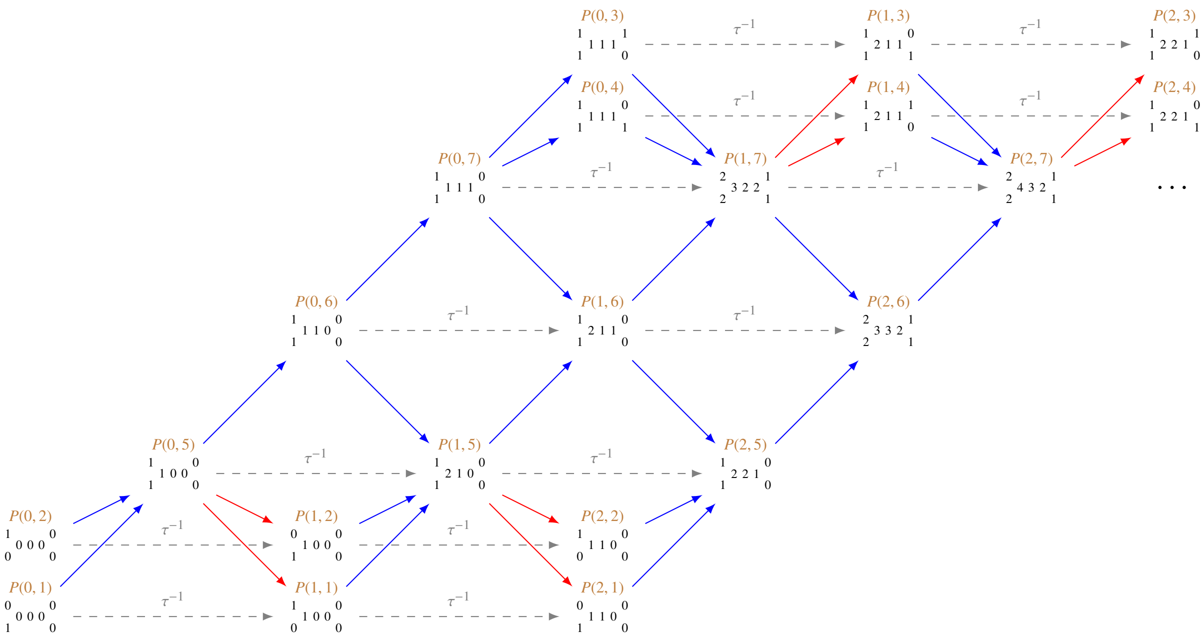
In the case of preprojectives and preinjectives the representations can be grouped in families of the form $P(8n+r, i)$ respectively $I(8n+r, i)$, where $i \in \{1, \dots, 7\}$ and $r \in \{0, \dots, 7\}$. Representations belonging to the same family have similar dimension vectors and matrices, depending only on the parameter $n \in \mathbb{N}$. The matrices listed are written using blocks of various sizes, with the same notation as in Subsection II.2.1. Every matrix is composed either of identity blocks or rectangular zero blocks. We denote the identity block simply by 1 and the zero block by 0. For small values of n we may give some representations concretely, when the general formula only works for $n > 0$. The formulas for the matrices listed here are rigorously proved to be correct – i.e. they give a field independent tree representation of the respective family in the sense of Definition II.2.1 [17]. The Appendix also contains a more detailed presentation of some of the representations from the lists (e.g. matrices written out explicitly for small values of $n = 0, 1, 2, \dots$).

III.4.1 The preprojective indecomposable representations

The preprojective indecomposable modules correspond to the vertices of the preprojective part of the Auslander–Reiten quiver, as shown in Figure III.1 (on the vertices we provide the dimension vectors in the graphical form corresponding to the shape of $\widetilde{\mathbb{D}}_6$).

Due to the symmetry of the quiver $\Delta(\widetilde{\mathbb{D}}_6)$ we give only the families of representations of the form $P(s, 1)$, $P(s, 3)$, $P(s, 5)$, $P(s, 6)$ and $P(s, 7)$. For $P(s, 2)$ and $P(s, 4)$ we can use the permutations $\sigma = (1, 2)$ and $\tau = (3, 4)$ to write them in terms of $P(s, 1)$ and $P(s, 3)$ in the following way ($s \geq 0$):

$$\underline{\dim} P(s, 2) = (d_{\sigma(i)})_{i \in \Delta(\widetilde{\mathbb{D}}_6)_0}, \text{ where } \underline{\dim} P(s, 1) = (d_i)_{i \in \Delta(\widetilde{\mathbb{D}}_6)_0}$$



The preprojective part of the Auslander–Reiten quiver $\Delta(\widetilde{\mathbb{D}}_6)$

$$\underline{\dim} P(s,4) = (d_{\tau(i)})_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}\,, \text{ where } \underline{\dim} P(s,3) = (d_i)_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}$$

for the dimension vectors, respectively

$$P(s,2) = \left(M_{\sigma(i)\to\sigma(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}\,, \text{ where } P(s,1) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}$$

$$P(s,4) = \left(M_{\tau(i)\to\tau(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}\,, \text{ where } P(s,3) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}$$

for the matrices.

In what follows we list the tree representations for preprojective families of the form $P(s,1)$, $P(s,3)$, $P(s,5)$, $P(s,6)$ and $P(s,7)$:

$\underline{\dim} P(8n,1) = (2n+1, 2n, 2n, 2n, 4n, 4n, 4n)$,

$P(0,1) = (0,0,0,0,0,0)$,

$$P(8n,1) = \left(\begin{smallmatrix} & {\scriptstyle 2n-1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 2n-1} \\ {\scriptstyle 1} & 0 & 0 & 1 & 0 \\ {\scriptstyle 2n-1} & 1 & 0 & 0 & 1 \\ {\scriptstyle 1} & 0 & 1 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 2n} & {\scriptstyle 2n} \\ {\scriptstyle 2n} & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 4n} \\ {\scriptstyle 4n} & 1 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 4n} \\ {\scriptstyle 4n} & 1 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 2n} \\ {\scriptstyle 2n} & 0 \\ {\scriptstyle 2n} & 1 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 2n} \\ {\scriptstyle 2n} & 1 \\ {\scriptstyle 2n} & 0 \end{smallmatrix} \right), \quad n > 0;$$

$\underline{\dim} P(8n+1,1) = (2n, 2n+1, 2n, 2n, 4n+1, 4n, 4n)$,

$$P(8n+1,1) = \left(\begin{smallmatrix} & {\scriptstyle 2n} & {\scriptstyle 2n} & {\scriptstyle 1} \\ {\scriptstyle 2n} & 1 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 2n} & {\scriptstyle 1} & {\scriptstyle 2n} \\ {\scriptstyle 1} & 0 & 1 & 0 \\ {\scriptstyle 2n} & 1 & 0 & 1 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 4n} \\ {\scriptstyle 4n} & 1 \\ {\scriptstyle 1} & 0 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 4n} \\ {\scriptstyle 4n} & 1 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 2n} \\ {\scriptstyle 2n} & 0 \\ {\scriptstyle 2n} & 1 \end{smallmatrix}, \begin{smallmatrix} & {\scriptstyle 2n} \\ {\scriptstyle 2n} & 1 \\ {\scriptstyle 2n} & 0 \end{smallmatrix} \right);$$

$\underline{\dim} P(8n+2,1) = (2n+1, 2n, 2n, 2n, 4n+1, 4n+1, 4n)$,

$$P(8n+2,1) = \left(\begin{array}{c|ccc} & 2n & 1 & 2n \\ \hline 1 & 0 & 1 & 0 \\ 2n & 1 & 0 & 1 \end{array}, \begin{array}{c|ccc} & 2n & 2n & 1 \\ \hline 2n & 1 & 1 & 0 \end{array}, \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array}, \begin{array}{c|c} & 4n \\ \hline 4n & 1 \\ 1 & 0 \end{array}, \begin{array}{c|c} & 2n \\ \hline 2n & 0 \\ 2n & 1 \end{array}, \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ 2n & 0 \end{array} \right);$$

$\underline{\dim} P(8n+3,1) = (2n, 2n+1, 2n, 2n, 4n+1, 4n+1, 4n+1)$,

$$P(8n+3,1) = \left(\begin{array}{c|ccc} & 2n & 2n & 1 \\ \hline 2n & 1 & 1 & 0 \end{array}, \begin{array}{c|ccc} & 2n & 1 & 2n \\ \hline 1 & 0 & 1 & 0 \\ 2n & 1 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array}, \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array}, \begin{array}{c|c} & 2n \\ \hline 2n & 0 \\ 2n & 1 \\ 1 & 0 \end{array}, \begin{array}{c|c} & 2n \\ \hline 2n & 1 \\ 2n+1 & 0 \end{array} \right);$$

$\underline{\dim} P(8n+4,1) = (2n+2, 2n+1, 2n+1, 2n+1, 4n+2, 4n+2, 4n+2)$,

$$P(8n+4,1) = \left(\begin{array}{c|cccc} & 2n & 1 & 1 & 2n \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2n & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{array}, \begin{array}{c|cc} & 2n+1 & 2n+1 \\ \hline 2n+1 & 1 & 1 \end{array}, \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array}, \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 0 \\ 2n+1 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+1 & 0 \end{array} \right);$$

$\underline{\dim} P(8n+5,1) = (2n+1, 2n+2, 2n+1, 2n+1, 4n+3, 4n+2, 4n+2)$,

$$P(8n+5,1) = \left(\begin{array}{c|ccc} & 2n+1 & 2n+1 & 1 \\ \hline 2n+1 & 1 & 1 & 0 \end{array}, \begin{array}{c|ccc} & 2n+1 & 1 & 2n+1 \\ \hline 1 & 0 & 1 & 0 \\ 2n+1 & 1 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \\ 1 & 0 \end{array}, \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 0 \\ 2n+1 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+1 & 0 \end{array} \right);$$

$\underline{\dim} P(8n+6,1) = (2n+2, 2n+1, 2n+1, 2n+1, 4n+3, 4n+3, 4n+2)$,

$$P(8n+6,1) = \left(\begin{array}{c|ccc} & 2n+1 & 1 & 2n+1 \\ \hline 1 & 0 & 1 & 0 \\ 2n+1 & 1 & 0 & 1 \end{array}, \begin{array}{c|ccc} & 2n+1 & 2n+1 & 1 \\ \hline 2n+1 & 1 & 1 & 0 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \\ 1 & 0 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 0 \\ 2n+1 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+1 & 0 \end{array} \right);$$

$\underline{\dim} P(8n+7,1) = (2n+1, 2n+2, 2n+1, 2n+1, 4n+3, 4n+3, 4n+3)$,

$$P(8n+7,1) = \left(\begin{array}{c|ccc} & 2n+1 & 2n+1 & 1 \\ \hline 2n+1 & 1 & 1 & 0 \end{array}, \begin{array}{c|ccc} & 2n+1 & 1 & 2n+1 \\ \hline 1 & 0 & 1 & 0 \\ 2n+1 & 1 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 0 \\ 2n+1 & 1 \\ 1 & 0 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+2 & 0 \end{array} \right);$$

$\underline{\dim} P(8n,3) = (2n+1, 2n+1, 2n+1, 2n, 4n+1, 4n+1, 4n+1)$,

$$P(0,3) = \left([1], [1], [1], [1], [1], 0 \right),$$

$$P(8n,3) = \left(\begin{array}{c|ccccc} & 2n-1 & 1 & 2n-1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{array}, \begin{array}{c|cccc} & 2n-1 & 2n-1 & 1 & 2 \\ \hline 2n-1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}, \right.$$

$$\left. \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array}, \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array}, \begin{array}{c|ccc} & 2n-1 & 1 & 1 \\ \hline 2n-1 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}, \begin{array}{c|cc} & 2n-1 & 1 \\ \hline 2n-1 & 1 & 0 \\ 2n-1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right), \quad n > 0;$$

$\underline{\dim} P(8n+1,3) = (2n+1, 2n+1, 2n, 2n+1, 4n+2, 4n+1, 4n+1)$,

$$P(1,3) = \left(\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, 0, \begin{bmatrix}1\end{bmatrix}\right),$$

$$P(8n+1,3) = \left(\begin{array}{c|cccccc|} & 2n-1 & 2n-1 & 1 & 1 & 1 & 1 \\ \hline 2n-1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}, \begin{array}{c|cccccc|} & 2n-1 & 1 & 2n-1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline \end{array}, \right.$$

$$\left. \begin{array}{c|cc|} & 4n & 1 \\ \hline 4n & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ \hline \end{array}, \begin{array}{c|c|} & 4n+1 \\ \hline 4n+1 & 1 \\ \hline \end{array}, \begin{array}{c|cc|} & 2n-1 & 1 \\ \hline 2n-1 & 0 & 0 \\ 2n-1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ \hline \end{array}, \begin{array}{c|ccc|} & 2n-1 & 1 & 1 \\ \hline 2n-1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2n-1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline \end{array} \right), \quad n > 0;$$

$\underline{\dim} P(8n+2,3) = (2n+1, 2n+1, 2n+1, 2n, 4n+2, 4n+2, 4n+1)$,

$$P(2,3) = \left(\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, 0\right),$$

$$P(8n+2,3) = \left(\begin{array}{c|ccccc|} & 2n-1 & 1 & 2n-1 & 1 & 2 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \hline \end{array}, \begin{array}{c|cccccc|} & 2n-1 & 2n-1 & 1 & 1 & 1 & 1 \\ \hline 2n-1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}, \right.$$

$$\left. \begin{array}{c|c|} & 4n+2 \\ \hline 4n+2 & 1 \\ \hline \end{array}, \begin{array}{c|cc|} & 4n & 1 \\ \hline 4n & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ \hline \end{array}, \begin{array}{c|ccc|} & 2n-1 & 1 & 1 \\ \hline 2n-1 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline \end{array}, \begin{array}{c|cc|} & 2n-1 & 1 \\ \hline 2n-1 & 1 & 0 \\ 2n-1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ \hline \end{array} \right), \quad n > 0;$$

$\underline{\dim} P(8n+3,3) = (2n+1, 2n+1, 2n, 2n+1, 4n+2, 4n+2, 4n+2)$,

$$P(3,3) = \left(\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, 0, \begin{bmatrix}1\\0\end{bmatrix}\right),$$

$$P(8n+3,3) = \left(\begin{array}{c|cccccc|} & 2n-1 & 2n-1 & 1 & 1 & 1 & 1 \\ \hline 2n-1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}, \begin{array}{c|cccccc|} & 2n-1 & 1 & 2n-1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline \end{array}, \right.$$

$$\left. \begin{array}{c|c|} & 4n+2 \\ \hline 4n+2 & 1 \\ \hline \end{array}, \begin{array}{c|c|} & 4n+2 \\ \hline 4n+2 & 1 \\ \hline \end{array}, \begin{array}{c|cc|} & 2n-1 & 1 \\ \hline 2n-1 & 0 & 0 \\ 2n-1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \\ \hline \end{array}, \begin{array}{c|ccc|} & 2n-1 & 1 & 1 \\ \hline 2n-1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2n-1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ \hline \end{array} \right), \quad n > 0;$$

$\underline{\dim} P(8n+4,3) = (2n+2, 2n+2, 2n+2, 2n+1, 4n+3, 4n+3, 4n+3)$,

$$P(4,3) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right),$$

$$P(8n+4,3) = \left(\begin{array}{c|ccccccc} & 2n-1 & 1 & 2n-1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}, \begin{array}{c|ccccccc} & 2n-1 & 2n-1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2n-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}, \right.$$

$$\left. \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|cccc} & 2n-1 & 1 & 1 & 1 \\ \hline 2n-1 & 0 & 0 & 0 & 0 \\ 2n-1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|ccc} & 2n-1 & 1 & 1 \\ \hline 2n-1 & 1 & 0 & 0 \\ 2n-1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right), \quad n > 0;$$

$\underline{\dim} P(8n+5,3) = (2n+2, 2n+2, 2n+1, 2n+2, 4n+4, 4n+3, 4n+3)$,

$$P(8n+5,3) = \left(\begin{array}{c|cccc} & 2n+1 & 2n+1 & 1 & 1 \\ \hline 2n+1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|cccc} & 2n+1 & 1 & 2n+1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2n+1 & 1 & 0 & 1 & 0 \end{array}, \right.$$

$$\left. \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 4n+2 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 0 \\ 2n+1 & 1 \\ 1 & 0 \end{array}, \begin{array}{c|cc} & 2n+1 & 1 \\ \hline 2n+1 & 1 & 0 \\ 1 & 0 & 1 \\ 2n & 0 & 0 \\ 1 & 0 & 1 \end{array} \right);$$

$\underline{\dim} P(8n+6,3) = (2n+2, 2n+2, 2n+2, 2n+1, 4n+4, 4n+4, 4n+3)$,

$$P(8n+6,3) = \left(\begin{array}{c|cccc} & 2n+1 & 1 & 2n+1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2n+1 & 1 & 0 & 1 & 0 \end{array}, \begin{array}{c|cccc} & 2n+1 & 2n+1 & 1 & 1 \\ \hline 2n+1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+4 \\ \hline 4n+4 & 1 \end{array}, \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 4n+2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{array}, \begin{array}{c|c} & 2n+2 \\ \hline 2n+1 & 0 \\ 2n+2 & 1 \end{array}, \right.$$

$$\left. \begin{array}{c|c} & 2n+1 \\ \hline 2n+1 & 1 \\ 2n+2 & 0 \end{array} \right);$$

$\underline{\dim} P(8n+7,3) = (2n+2, 2n+2, 2n+1, 2n+2, 4n+4, 4n+4, 4n+4)$,

$$P(8n+7,3)=\left(\begin{array}{cc} & \begin{matrix}2n+1 & 2n+1 & 1 & 1\end{matrix}\\ \begin{matrix}2n+1\\1\end{matrix} & \begin{bmatrix}1&1&0&0\\0&0&0&1\end{bmatrix}\end{array},\begin{array}{cc} & \begin{matrix}2n+1 & 1 & 2n+1 & 1\end{matrix}\\ \begin{matrix}1\\2n+1\end{matrix} & \begin{bmatrix}0&1&0&0\\1&0&1&0\end{bmatrix}\end{array},\begin{array}{cc} & 4n+4\\ 4n+4 & \begin{bmatrix}1\end{bmatrix}\end{array},\begin{array}{cc} & 4n+4\\ 4n+4 & \begin{bmatrix}1\end{bmatrix}\end{array},\begin{array}{cc} & 2n+1\\ \begin{matrix}2n+1\\2n+1\\2\end{matrix} & \begin{bmatrix}0\\1\\0\end{bmatrix}\end{array},\right.$$

$$\left.\begin{array}{cc} & \begin{matrix}2n+1 & 1\end{matrix}\\ \begin{matrix}2n+1\\1\\2n+1\\1\end{matrix} & \begin{bmatrix}1&0\\0&1\\0&0\\0&1\end{bmatrix}\end{array}\right);$$

$$\underline{\dim}P(8n,5)=(4n+1,4n+1,4n,4n,8n+1,8n,8n),$$
$$M_\alpha^{P(8n,5)}=M_\alpha^{P(8n,1)}\oplus M_\alpha^{P(8n+1,1)},\quad \text{for } \alpha\neq(5\to1),$$
$$M_{5\to1}^{P(8n,5)}=\left(M_{5\to1}^{P(8n,1)}\oplus M_{5\to1}^{P(8n+1,1)}\right)\boxplus\begin{array}{cc} & \begin{matrix}4n & 1\end{matrix}\\ \begin{matrix}2n\\1\end{matrix} & \begin{bmatrix}0&0\\0&1\end{bmatrix}\end{array};$$

$$\underline{\dim}P(8n+1,5)=(4n+1,4n+1,4n,4n,8n+2,8n+1,8n),$$
$$M_\alpha^{P(8n+1,5)}=M_\alpha^{P(8n+1,1)}\oplus M_\alpha^{P(8n+2,1)},\quad \text{for } \alpha\neq(5\to2),$$
$$M_{5\to2}^{P(8n+1,5)}=\left(M_{5\to2}^{P(8n+1,1)}\oplus M_{5\to2}^{P(8n+2,1)}\right)\boxplus\begin{array}{cc} & \begin{matrix}4n & 1\end{matrix}\\ \begin{matrix}2n\\1\end{matrix} & \begin{bmatrix}0&0\\0&1\end{bmatrix}\end{array};$$

$$\underline{\dim}P(8n+2,5)=(4n+1,4n+1,4n,4n,8n+2,8n+2,8n+1),$$
$$M_\alpha^{P(8n+2,5)}=M_\alpha^{P(8n+2,1)}\oplus M_\alpha^{P(8n+3,1)},\quad \text{for } \alpha\neq(7\to6),$$
$$M_{7\to6}^{P(8n+2,5)}=\left(M_{7\to6}^{P(8n+2,1)}\oplus M_{7\to6}^{P(8n+3,1)}\right)\boxplus\begin{array}{cc} & \begin{matrix}4n & 1\end{matrix}\\ \begin{matrix}4n\\1\end{matrix} & \begin{bmatrix}0&0\\0&1\end{bmatrix}\end{array};$$

$$\underline{\dim}P(8n+3,5)=(4n+2,4n+2,4n+1,4n+1,8n+3,8n+3,8n+3),$$
$$P(3,5)=\left(\begin{bmatrix}1&1&0\\1&0&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right),$$
$$M_\alpha^{P(8n+3,5)}=M_\alpha^{P(8n,3)}\oplus M_\alpha^{P(8n+3,3)},\quad \text{for } \alpha\neq(5\to2),$$
$$M_{5\to2}^{P(8n+3,5)}=\left(M_{5\to2}^{P(8n,3)}\oplus M_{5\to2}^{P(8n+3,3)}\right)\boxplus\begin{array}{cc} & \begin{matrix}4n-1 & 1 & 2\end{matrix}\\ \begin{matrix}2n-1\\1\\1\end{matrix} & \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\end{array},\quad n>0;$$

$$\underline{\dim}P(8n+4,5)=(4n+3,4n+3,4n+2,4n+2,8n+5,8n+4,8n+4),$$
$$M_\alpha^{P(8n+4,5)}=M_\alpha^{P(8n+4,1)}\oplus M_\alpha^{P(8n+5,1)},\quad \text{for } \alpha\neq(5\to1),$$
$$M_{5\to1}^{P(8n+4,5)}=\left(M_{5\to1}^{P(8n+4,1)}\oplus M_{5\to1}^{P(8n+5,1)}\right)\boxplus\begin{array}{cc} & \begin{matrix}4n+2 & 1\end{matrix}\\ \begin{matrix}2n+1\\1\end{matrix} & \begin{bmatrix}0&0\\0&1\end{bmatrix}\end{array};$$

$$\underline{\dim} P(8n+5,5) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+6, 8n+5, 8n+4),$$
$$M_\alpha^{P(8n+5,5)} = M_\alpha^{P(8n+5,1)} \oplus M_\alpha^{P(8n+6,1)}, \quad \text{for } \alpha \neq (5 \to 2),$$
$$M_{5\to 2}^{P(8n+5,5)} = \left(M_{5\to 2}^{P(8n+5,1)} \oplus M_{5\to 2}^{P(8n+6,1)}\right) \boxplus \begin{array}{rc} & \begin{matrix} 4n+2 & 1 \end{matrix} \\ \begin{matrix} 2n+1 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim} P(8n+6,5) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+6, 8n+6, 8n+5),$$
$$M_\alpha^{P(8n+6,5)} = M_\alpha^{P(8n+6,1)} \oplus M_\alpha^{P(8n+7,1)}, \quad \text{for } \alpha \neq (7 \to 6),$$
$$M_{7\to 6}^{P(8n+6,5)} = \left(M_{7\to 6}^{P(8n+6,1)} \oplus M_{7\to 6}^{P(8n+7,1)}\right) \boxplus \begin{array}{rc} & \begin{matrix} 4n+2 & 1 \end{matrix} \\ \begin{matrix} 4n+2 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim} P(8n+7,5) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+7, 8n+7, 8n+7),$$
$$M_\alpha^{P(8n+7,5)} = M_\alpha^{P(8n+7,2)} \oplus M_\alpha^{P(8n,2)[n\mapsto n+1]}, \quad \text{for } \alpha \neq (5 \to 2),$$
$$M_{5\to 2}^{P(8n+7,5)} = \left(M_{5\to 2}^{P(8n+7,2)} \oplus M_{5\to 2}^{P(8n,2)[n\mapsto n+1]}\right) \boxplus \begin{array}{rc} & \begin{matrix} 4n+3 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n \end{matrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array};$$

$$\underline{\dim} P(8n,6) = (4n+1, 4n+1, 4n, 4n, 8n+1, 8n+1, 8n),$$
$$M_\alpha^{P(8n,6)} = M_\alpha^{P(8n,1)} \oplus M_\alpha^{P(8n+2,2)}, \quad \text{for } \alpha \neq (5 \to 1),$$
$$M_{5\to 1}^{P(8n,6)} = \left(M_{5\to 1}^{P(8n,1)} \oplus M_{5\to 1}^{P(8n+2,2)}\right) \boxplus \begin{array}{rc} & \begin{matrix} 4n & 1 \end{matrix} \\ \begin{matrix} 2n \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim} P(8n+1,6) = (4n+1, 4n+1, 4n, 4n, 8n+2, 8n+1, 8n+1),$$
$$M_\alpha^{P(8n+1,6)} = M_\alpha^{P(8n+1,1)} \oplus M_\alpha^{P(8n+3,2)}, \quad \text{for } \alpha \neq (6 \to 5),$$
$$M_{6\to 5}^{P(8n+1,6)} = \left(M_{6\to 5}^{P(8n+1,1)} \oplus M_{6\to 5}^{P(8n+3,2)}\right) \boxplus \begin{array}{rc} & \begin{matrix} 4n & 1 \end{matrix} \\ \begin{matrix} 4n \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array};$$

$$\underline{\dim} P(8n+2,6) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+3, 8n+3, 8n+2),$$
$$P(2,6) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$
$$M_\alpha^{P(8n+2,6)} = M_\alpha^{P(8n,3)} \oplus M_\alpha^{P(8n+2,4)}, \quad \text{for } \alpha \neq (5 \to 2),$$
$$M_{5\to 2}^{P(8n+2,6)} = \left(M_{5\to 2}^{P(8n,3)} \oplus M_{5\to 2}^{P(8n+2,4)}\right) \boxplus \begin{array}{rc} & \begin{matrix} 4n & 1 & 1 \end{matrix} \\ \begin{matrix} 2n-1 \\ 1 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \quad n > 0;$$

$$\underline{\dim} P(8n+3,6) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+4, 8n+3, 8n+3),$$
$$M_\alpha^{P(8n+3,6)} = M_\alpha^{P(8n+1,4)} \oplus M_\alpha^{P(8n+3,3)}, \quad \text{for } \alpha \neq (6 \to 5),$$

$$M^{P(8n+3,6)}_{6\to5} = \left(M^{P(8n+1,4)}_{6\to5} \oplus M^{P(8n+3,3)}_{6\to5}\right) \boxplus \begin{array}{c|ccc} & 4n & 1 & 1 \\ \hline 4n & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array};$$

$\underline{\dim} P(8n+4,6) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+5, 8n+5, 8n+4),$

$$M^{P(8n+4,6)}_{\alpha} = M^{P(8n+4,1)}_{\alpha} \oplus M^{P(8n+6,2)}_{\alpha}, \quad \text{for } \alpha \neq (5 \to 1),$$

$$M^{P(8n+4,6)}_{5\to1} = \left(M^{P(8n+4,1)}_{5\to1} \oplus M^{P(8n+6,2)}_{5\to1}\right) \boxplus \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 2n+1 & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$\underline{\dim} P(8n+5,6) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+6, 8n+5, 8n+5),$

$$M^{P(8n+5,6)}_{\alpha} = M^{P(8n+5,1)}_{\alpha} \oplus M^{P(8n+7,2)}_{\alpha}, \quad \text{for } \alpha \neq (6 \to 5),$$

$$M^{P(8n+5,6)}_{6\to5} = \left(M^{P(8n+5,1)}_{6\to5} \oplus M^{P(8n+7,2)}_{6\to5}\right) \boxplus \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 4n+2 & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$\underline{\dim} P(8n+6,6) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+7, 8n+7, 8n+6),$

$$M^{P(8n+6,6)}_{\alpha} = M^{P(8n+6,1)}_{\alpha} \oplus M^{P(8n,2)[n\mapsto n+1]}_{\alpha}, \quad \text{for } \alpha \neq (5 \to 2),$$

$$M^{P(8n+6,6)}_{5\to2} = \left(M^{P(8n+6,1)}_{5\to2} \oplus M^{P(8n,2)[n\mapsto n+1]}_{5\to2}\right) \boxplus \begin{array}{c|cc} & 1 & 4n+3 \\ \hline 2n & 0 & 0 \\ 1 & 1 & 0 \end{array};$$

$\underline{\dim} P(8n+7,6) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+8, 8n+7, 8n+7),$

$$M^{P(8n+7,6)}_{\alpha} = M^{P(8n+5,3)}_{\alpha} \oplus M^{P(8n+7,4)}_{\alpha}, \quad \text{for } \alpha \neq (6 \to 5),$$

$$M^{P(8n+7,6)}_{6\to5} = \left(M^{P(8n+5,3)}_{6\to5} \oplus M^{P(8n+7,4)}_{6\to5}\right) \boxplus \begin{array}{c|ccc} & 4n+2 & 1 & 1 \\ \hline 4n+2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array};$$

$\underline{\dim} P(8n,7) = (4n+1, 4n+1, 4n, 4n, 8n+1, 8n+1, 8n+1),$

$$M^{P(8n,7)}_{\alpha} = M^{P(8n,1)}_{\alpha} \oplus M^{P(8n+3,1)}_{\alpha}, \quad \text{for } \alpha \neq (5 \to 1),$$

$$M^{P(8n,7)}_{5\to1} = \left(M^{P(8n,1)}_{5\to1} \oplus M^{P(8n+3,1)}_{5\to1}\right) \boxplus \begin{array}{c|cc} & 4n & 1 \\ \hline 2n & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$\underline{\dim} P(8n+1,7) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+3, 8n+2, 8n+2),$

$$M^{P(8n+1,7)}_{\alpha} = M^{P(8n+1,6)}_{\alpha} \oplus M^{R_1^2(1)}_{\alpha}, \quad \text{for } \alpha \neq (4 \to 7),$$

$$M^{P(8n+1,7)}_{4\to7} = \left(M^{P(8n+1,6)}_{4\to7} \oplus M^{R_1^2(1)}_{4\to7}\right) \boxplus \begin{array}{c|c} & 1 \\ \hline 8n & 0 \\ 1 & 1 \end{array};$$

$\underline{\dim} P(8n+2,7) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+4, 8n+3, 8n+2)$,

$$M_\alpha^{P(8n+2,7)} = M_\alpha^{P(8n+2,5)} \oplus M_\alpha^{R_1^2(2)}, \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to 7}^{P(8n+2,7)} = \left(M_{3\to 7}^{P(8n+2,5)} \oplus M_{3\to 7}^{R_1^2(2)}\right) \boxplus \begin{array}{c|c} & 1 \\ \hline 8n & 0 \\ 1 & 1 \end{array};$$

$\underline{\dim} P(8n+3,7) = (4n+2, 4n+2, 4n+1, 4n+1, 8n+4, 8n+4, 8n+3)$,

$$P(3,7) = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right),$$

$$M_\alpha^{P(8n+3,7)} = M_\alpha^{P(8n+2,3)} \oplus M_\alpha^{P(8n+3,3)}, \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{P(8n+3,7)} = \left(M_{5\to 1}^{P(8n+2,3)} \oplus M_{5\to 1}^{P(8n+3,3)}\right) \boxplus \begin{array}{c|ccc} & 4n & 1 & 1 \\ \hline 2n & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}, \quad n > 0;$$

$\underline{\dim} P(8n+4,7) = (4n+3, 4n+3, 4n+2, 4n+2, 8n+5, 8n+5, 8n+5)$,

$$M_\alpha^{P(8n+4,7)} = M_\alpha^{P(8n+4,1)} \oplus M_\alpha^{P(8n+7,1)}, \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{P(8n+4,7)} = \left(M_{5\to 1}^{P(8n+4,1)} \oplus M_{5\to 1}^{P(8n+7,1)}\right) \boxplus \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 2n+1 & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$\underline{\dim} P(8n+5,7) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+7, 8n+6, 8n+6)$,

$$M_\alpha^{P(8n+5,7)} = M_\alpha^{P(8n+5,6)} \oplus M_\alpha^{R_1^2(1)}, \quad \text{for } \alpha \neq (4 \to 7),$$

$$M_{4\to 7}^{P(8n+5,7)} = \left(M_{4\to 7}^{P(8n+5,6)} \oplus M_{4\to 7}^{R_1^2(1)}\right) \boxplus \begin{array}{c|c} & 1 \\ \hline 8n+4 & 0 \\ 1 & 1 \end{array};$$

$\underline{\dim} P(8n+6,7) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+8, 8n+7, 8n+6)$,

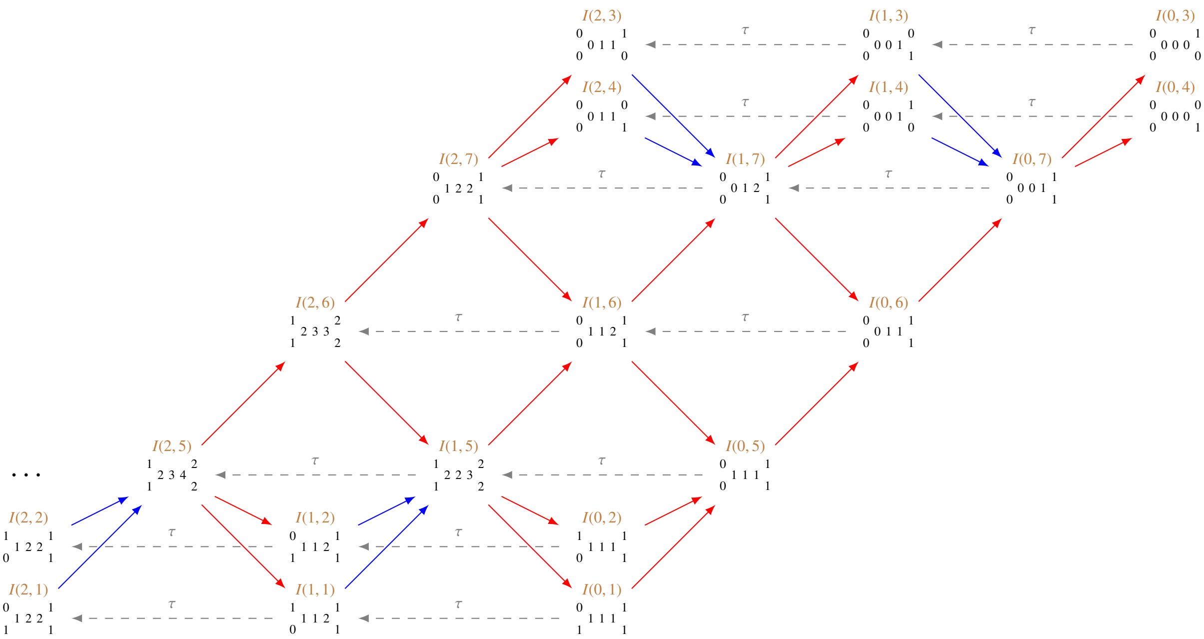
$$M_\alpha^{P(8n+6,7)} = M_\alpha^{P(8n+6,6)} \oplus M_\alpha^{R_1^3(1)}, \quad \text{for } \alpha \neq (5 \to 2),$$

$$M_{5\to 2}^{P(8n+6,7)} = \left(M_{5\to 2}^{P(8n+6,6)} \oplus M_{5\to 2}^{R_1^3(1)}\right) \boxplus \begin{array}{c|c} & 1 \\ \hline 4n+3 & 0 \\ 1 & 1 \end{array};$$

$\underline{\dim} P(8n+7,7) = (4n+4, 4n+4, 4n+3, 4n+3, 8n+8, 8n+8, 8n+7)$,

$$M_\alpha^{P(8n+7,7)} = M_\alpha^{P(8n+7,2)} \oplus M_\alpha^{P(8n+2,2)[n\mapsto n+1]}, \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{P(8n+7,7)} = \left(M_{5\to 1}^{P(8n+7,2)} \oplus M_{5\to 1}^{P(8n+2,2)[n\mapsto n+1]}\right) \boxplus \begin{array}{c|cc} & 1 & 4n+4 \\ \hline 2n+1 & 0 & 0 \\ 1 & 1 & 0 \end{array}.$$



The preinjective part of the Auslander–Reiten quiver of $\Delta(\widetilde{\mathbb{D}}_6)$

III.4.2 The preinjective indecomposable modules

The preinjective indecomposable modules correspond to the vertices of the preinjective part of the Auslander–Reiten quiver, as shown in Figure III.2.

Due to the symmetry of the quiver $\Delta(\widetilde{\mathbb{D}}_6)$ we give only the families of representations of the form $I(s,1)$, $I(s,3)$, $I(s,5)$, $I(s,6)$ and $I(s,7)$. For $I(s,2)$ and $I(s,4)$ we can use the permutations $\sigma = (1,2)$ and $\tau = (3,4)$ to write them in terms of $I(s,1)$ and $I(s,3)$ in the following way ($s \geq 0$):

$$\underline{\dim} I(s,2) = (d_{\sigma(i)})_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}\text{, where } \underline{\dim} I(s,1) = (d_i)_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}$$

$$\underline{\dim} I(s,4) = (d_{\tau(i)})_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}\text{, where } \underline{\dim} I(s,3) = (d_i)_{i\in\Delta(\widetilde{\mathbb{D}}_6)_0}$$

for the dimension vectors, respectively

$$I(s,2) = \left(M_{\sigma(i)\to\sigma(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}\text{, where } I(s,1) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}$$

$$I(s,4) = \left(M_{\tau(i)\to\tau(j)}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}\text{, where } I(s,3) = \left(M_{i\to j}\right)_{(i\to j)\in\Delta(\widetilde{\mathbb{D}}_6)_1}$$

for the matrices.

In what follows we list the tree representations for preinjective families of the form $I(s,1)$, $I(s,3)$, $I(s,5)$, $I(s,6)$ and $I(s,7)$:

$\underline{\dim} I(8n,1) = (2n+1, 2n, 2n+1, 2n+1, 4n+1, 4n+1, 4n+1)$,

$$I(8n,1)=\left(\begin{array}{c|cc} & 2n & 2n+1\\ \hline 2n+1 & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n & 2n+1\\ \hline 2n & 1 & 0\end{array},\ \begin{array}{c|c} & 4n+1\\ \hline 4n+1 & 1\end{array},\ \begin{array}{c|c} & 4n+1\\ \hline 4n+1 & 1\end{array},\ \begin{array}{c|cc} & 2n & 1\\ \hline 2n & 1 & 0\\ 2n & 1 & 0\\ 1 & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n & 1\\ \hline 2n & 1 & 0\\ 1 & 0 & 1\\ 2n & 1 & 0\end{array}\right);$$

$\underline{\dim} I(8n+1,1)=(2n,2n+1,2n+1,2n+1,4n+1,4n+1,4n+2)$,

$$I(1,1)=\left(0,\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}0 & 1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right),$$

$$I(8n+1,1)=\left(\begin{array}{c|cccc} & 1 & 1 & 2n-1 & 2n\\ \hline 1 & 1 & 1 & 0 & 0\\ 2n-1 & 0 & 0 & 1 & 0\end{array},\ \begin{array}{c|ccc} & 1 & 2n & 2n\\ \hline 1 & 1 & 0 & 0\\ 2n & 0 & 0 & 1\end{array},\ \begin{array}{c|c} & 4n+1\\ \hline 4n+1 & 1\end{array},\right.$$

$$\left.\begin{array}{c|cc} & 1 & 4n+1\\ \hline 4n+1 & 0 & 1\end{array},\ \begin{array}{c|ccc} & 1 & 2n-1 & 1\\ \hline 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 2n-1 & 0 & 1 & 0\\ 2n-1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{array},\ \begin{array}{c|cccc} & 1 & 1 & 2n-2 & 1\\ \hline 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ 2n-2 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 1\\ 1 & 0 & 1 & 0 & 0\\ 2n-2 & 0 & 0 & 1 & 0\end{array}\right),\quad n>0;$$

$\underline{\dim} I(8n+2,1)=(2n+1,2n,2n+1,2n+1,4n+1,4n+2,4n+2)$,

$$I(8n+2,1)=\left(\begin{array}{c|cc} & 2n & 2n+1\\ \hline 2n+1 & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n & 2n+1\\ \hline 2n & 1 & 0\end{array},\ \begin{array}{c|cc} & 1 & 4n+1\\ \hline 4n+1 & 0 & 1\end{array},\ \begin{array}{c|c} & 4n+2\\ \hline 4n+2 & 1\end{array},\ \begin{array}{c|c} & 2n+1\\ \hline 2n+1 & 1\\ 2n+1 & 1\end{array},\ \begin{array}{c|cc} & 2n & 1\\ \hline 1 & 0 & 0\\ 2n & 1 & 0\\ 2n & 1 & 0\\ 1 & 0 & 1\end{array}\right);$$

$\underline{\dim} I(8n+3,1)=(2n,2n+1,2n+1,2n+1,4n+2,4n+2,4n+2)$,

$$I(8n+3,1)=\left(\begin{array}{c|ccc} & 1 & 2n & 2n+1\\ \hline 2n & 0 & 1 & 0\end{array},\ \begin{array}{c|cc} & 2n+1 & 2n+1\\ \hline 2n+1 & 0 & 1\end{array},\ \begin{array}{c|c} & 4n+2\\ \hline 4n+2 & 1\end{array},\ \begin{array}{c|c} & 4n+2\\ \hline 4n+2 & 1\end{array},\ \begin{array}{c|c} & 2n+1\\ \hline 2n+1 & 1\\ 2n+1 & 1\end{array},\ \begin{array}{c|cc} & 2n & 1\\ \hline 1 & 0 & 0\\ 2n & 1 & 0\\ 2n & 1 & 0\\ 1 & 0 & 1\end{array}\right);$$

$\underline{\dim} I(8n+4,1)=(2n+2,2n+1,2n+2,2n+2,4n+3,4n+3,4n+3)$,

$$I(8n+4,1)=\left(\begin{array}{c|cc} & 2n+1 & 2n+2\\ \hline 2n+2 & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n+1 & 2n+2\\ \hline 2n+1 & 1 & 0\end{array},\ \begin{array}{c|c} & 4n+3\\ \hline 4n+3 & 1\end{array},\ \begin{array}{c|c} & 4n+3\\ \hline 4n+3 & 1\end{array},\ \begin{array}{c|cc} & 2n+1 & 1\\ \hline 2n+1 & 1 & 0\\ 2n+1 & 1 & 0\\ 1 & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n+1 & 1\\ \hline 2n+1 & 1 & 0\\ 1 & 0 & 1\\ 2n+1 & 1 & 0\end{array}\right);$$

$\underline{\dim} I(8n+5,1)=(2n+1,2n+2,2n+2,2n+2,4n+3,4n+3,4n+4)$,

$$I(8n+5,1)=\left(\begin{array}{c|cc} & 2n+1 & 2n+2 \\ \hline 2n+1 & 1 & 0\end{array},\ \begin{array}{c|cc} & 2n+1 & 2n+2 \\ \hline 2n+2 & 0 & 1\end{array},\ \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1\end{array},\ \begin{array}{c|cc} & 1 & 4n+3 \\ \hline 4n+3 & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n+1 & 1 \\ \hline 1 & 0 & 0 \\ 2n+1 & 1 & 0 \\ 2n+1 & 1 & 0 \\ 1 & 0 & 1\end{array},\ \begin{array}{c|c} & 2n+2 \\ \hline 2n+2 & 1 \\ 2n+2 & 1\end{array}\right);$$

$\underline{\dim} I(8n+6,1)=(2n+2,2n+1,2n+2,2n+2,4n+3,4n+4,4n+4)$,

$$I(8n+6,1)=\left(\begin{array}{c|cc} & 2n+1 & 2n+2 \\ \hline 2n+2 & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n+1 & 2n+2 \\ \hline 2n+1 & 1 & 0\end{array},\ \begin{array}{c|cc} & 1 & 4n+3 \\ \hline 4n+3 & 0 & 1\end{array},\ \begin{array}{c|c} & 4n+4 \\ \hline 4n+4 & 1\end{array},\ \begin{array}{c|cc} & 2n+1 & 1 \\ \hline 1 & 0 & 0 \\ 2n+1 & 1 & 0 \\ 2n+1 & 1 & 0 \\ 1 & 0 & 1\end{array},\ \begin{array}{c|c} & 2n+2 \\ \hline 2n+2 & 1 \\ 2n+2 & 1\end{array}\right);$$

$\underline{\dim} I(8n+7,1)=(2n+1,2n+2,2n+2,2n+2,4n+4,4n+4,4n+4)$,

$$I(8n+7,1)=\left(\begin{array}{c|ccc} & 1 & 2n+1 & 2n+2 \\ \hline 2n+1 & 0 & 1 & 0\end{array},\ \begin{array}{c|cc} & 2n+2 & 2n+2 \\ \hline 2n+2 & 0 & 1\end{array},\ \begin{array}{c|c} & 4n+4 \\ \hline 4n+4 & 1\end{array},\ \begin{array}{c|c} & 4n+4 \\ \hline 4n+4 & 1\end{array},\ \begin{array}{c|cc} & 2n+1 & 1 \\ \hline 1 & 0 & 0 \\ 2n+1 & 1 & 0 \\ 2n+1 & 1 & 0 \\ 1 & 0 & 1\end{array},\ \begin{array}{c|c} & 2n+2 \\ \hline 2n+2 & 1 \\ 2n+2 & 1\end{array}\right);$$

$\underline{\dim} I(8n,3)=(2n,2n,2n+1,2n,4n,4n,4n)$,

$$I(8n,3)=\left(\begin{array}{c|cc} & 2n & 2n \\ \hline 2n & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n & 2n \\ \hline 2n & 1 & 0\end{array},\ \begin{array}{c|c} & 4n \\ \hline 4n & 1\end{array},\ \begin{array}{c|c} & 4n \\ \hline 4n & 1\end{array},\ \begin{array}{c|ccc} & 1 & 2n-1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 2n-1 & 0 & 1 & 0 \\ 2n-1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1\end{array},\ \begin{array}{c|cc} & 2n-1 & 1 \\ \hline 1 & 0 & 1 \\ 2n-1 & 1 & 0 \\ 1 & 0 & 1 \\ 2n-1 & 1 & 0\end{array}\right);$$

$\underline{\dim} I(8n+1,3)=(2n,2n,2n,2n+1,4n,4n,4n+1)$,

$I(1,3)=\left(0,0,0,0,0,[1]\right)$,

$$I(8n+1,3)=\left(\begin{array}{c|ccccc} & 1 & 1 & 1 & 2n-2 & 2n-1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 2n-2 & 0 & 0 & 0 & 1 & 0\end{array},\ \begin{array}{c|cccc} & 1 & 1 & 2n-1 & 2n-1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2n-1 & 0 & 0 & 0 & 1\end{array},\ \begin{array}{c|c} & 4n \\ \hline 4n & 1\end{array},\right.$$

$$\left.\begin{array}{c|cc} & 1 & 4n \\ \hline 4n & 0 & 1\end{array},\ \begin{array}{c|cc} & 1 & 2n-1 \\ \hline 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2n-1 & 0 & 1 \\ 2n-1 & 0 & 1\end{array},\ \begin{array}{c|cccc} & 1 & 1 & 2n-2 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2n-2 & 0 & 0 & 1 & 0 \\ 2n-2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1\end{array}\right),\quad n>0;$$

$\underline{\dim} I(8n+2,3) = (2n, 2n, 2n+1, 2n, 4n, 4n+1, 4n+1)$,

$I(2,3) = \left(0, 0, 0, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, 0\right)$,

$$I(8n+2,3) = \left(\begin{array}{c|cc} & 2n & 2n \\ \hline 2n & 0 & 1 \end{array},\ \begin{array}{c|cc} & 2n & 2n \\ \hline 2n & 1 & 0 \end{array},\ \begin{array}{c|cc} & 1 & 4n \\ \hline 4n & 0 & 1 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|ccc} & 1 & 2n-1 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 2n-1 & 0 & 1 & 0 \\ 2n-1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array},\ \begin{array}{c|cc} & 2n-1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2n-1 & 1 & 0 \\ 1 & 0 & 1 \\ 2n-1 & 1 & 0 \end{array} \right), \quad n > 0;$$

$\underline{\dim} I(8n+3,3) = (2n, 2n, 2n, 2n+1, 4n+1, 4n+1, 4n+1)$,

$I(3,3) = \left(0, 0, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, 0, \begin{bmatrix}1\end{bmatrix}\right)$,

$$I(8n+3,3) = \left(\begin{array}{c|cccc} & 1 & 1 & 2n-1 & 2n \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2n-1 & 0 & 0 & 1 & 0 \end{array},\ \begin{array}{c|cc} & 2n+1 & 2n \\ \hline 2n & 0 & 1 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|c} & 4n+1 \\ \hline 4n+1 & 1 \end{array},\ \begin{array}{c|cc} & 2n-1 & 1 \\ \hline 1 & 0 & 1 \\ 1 & 0 & 0 \\ 2n-1 & 1 & 0 \\ 2n-1 & 1 & 0 \\ 1 & 0 & 1 \end{array},\ \begin{array}{c|cc} & 1 & 2n \\ \hline 1 & 1 & 0 \\ 2n & 0 & 1 \\ 2n & 0 & 1 \end{array} \right), \quad n > 0;$$

$\underline{\dim} I(8n+4,3) = (2n+1, 2n+1, 2n+2, 2n+1, 4n+2, 4n+2, 4n+2)$,

$$I(8n+4,3) = \left(\begin{array}{c|cc} & 2n+1 & 2n+1 \\ \hline 2n+1 & 0 & 1 \end{array},\ \begin{array}{c|cc} & 2n+1 & 2n+1 \\ \hline 2n+1 & 1 & 0 \end{array},\ \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array},\ \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array},\ \begin{array}{c|ccc} & 1 & 2n & 1 \\ \hline 1 & 1 & 0 & 0 \\ 2n & 0 & 1 & 0 \\ 2n & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array},\ \begin{array}{c|cc} & 2n & 1 \\ \hline 1 & 0 & 1 \\ 2n & 1 & 0 \\ 1 & 0 & 1 \\ 2n & 1 & 0 \end{array} \right);$$

$\underline{\dim} I(8n+5,3) = (2n+1, 2n+1, 2n+1, 2n+2, 4n+2, 4n+2, 4n+3)$,

$$I(5,3) = \left(\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 1 \\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}, \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 1\end{bmatrix} \right),$$

$$I(8n+5,3) = \left(\begin{array}{c|ccccc} & 1 & 1 & 1 & 2n-1 & 2n \\ \hline 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2n-1 & 0 & 0 & 0 & 1 & 0 \end{array},\ \begin{array}{c|ccc} & 1 & 2n+1 & 2n \\ \hline 1 & 1 & 0 & 0 \\ 2n & 0 & 0 & 1 \end{array},\ \begin{array}{c|c} & 4n+2 \\ \hline 4n+2 & 1 \end{array},\right.$$

$$\left. \begin{array}{c|cc} & 1 & 4n+2 \\ \hline 4n+2 & 0 & 1 \end{array},\ \begin{array}{c|ccc} & 1 & 2n-1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2n-1 & 0 & 1 & 0 \\ 2n-1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array},\ \begin{array}{c|ccc} & 1 & 1 & 2n \\ \hline 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2n & 0 & 0 & 1 \\ 2n & 0 & 0 & 1 \end{array} \right), \quad n > 0;$$

$\underline{\dim} I(8n+6,3) = (2n+1, 2n+1, 2n+2, 2n+1, 4n+2, 4n+3, 4n+3)$,

$$I(6,3) = \left(\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

$$I(8n+6,3) = \left(\begin{array}{c|cc} & 2n+1 & 2n+1 \\ \hline 2n+1 & 0 & 1 \end{array}, \begin{array}{c|cc} & 2n+1 & 2n+1 \\ \hline 2n+1 & 1 & 0 \end{array}, \begin{array}{c|cc} & 1 & 4n+2 \\ \hline 4n+2 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \right.$$

$$\left. \begin{array}{c|ccc} & 1 & 2n & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2n & 0 & 1 & 0 \\ 2n & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}, \begin{array}{c|cc} & 2n & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2n & 1 & 0 \\ 1 & 0 & 1 \\ 2n & 1 & 0 \end{array} + \begin{array}{c|ccc} & 1 & 1 & 2n-1 \\ \hline 1 & 0 & 1 & 0 \\ 4n+2 & 0 & 0 & 0 \end{array} \right), \quad n > 0;$$

$\underline{\dim} I(8n+7,3) = (2n+1, 2n+1, 2n+1, 2n+2, 4n+3, 4n+3, 4n+3)$,

$$I(8n+7,3) = \left(\begin{array}{c|cccc} & 1 & 1 & 2n & 2n+1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2n & 0 & 0 & 1 & 0 \end{array}, \begin{array}{c|cc} & 2n+2 & 2n+1 \\ \hline 2n+1 & 0 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|c} & 4n+3 \\ \hline 4n+3 & 1 \end{array}, \begin{array}{c|c} & 2n+1 \\ \hline 1 & 0 \\ 2n+1 & 1 \\ 2n+1 & 1 \end{array}, \begin{array}{c|ccc} & 1 & 2n & 1 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2n & 0 & 1 & 0 \\ 2n & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right);$$

$\underline{\dim} I(8n,5) = (4n, 4n, 4n+1, 4n+1, 8n+1, 8n+1, 8n+1)$,

$I(0,5) = \left(0, 0, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}\right)$,

$M_\alpha^{I(8n,5)} = M_\alpha^{I(8n,1)} \oplus M_\alpha^{I(8n+7,1)[n \mapsto n-1]}$, for $\alpha \neq (3 \to 7)$,

$$M_{3\to7}^{I(8n,5)} = \left(M_{3\to7}^{I(8n,1)} \oplus M_{3\to7}^{I(8n+7,1)[n \mapsto n-1]} \right) \boxplus \begin{array}{c|cc} & 2n-1 & 1 \\ \hline 1 & 0 & 1 \\ 4n & 0 & 0 \end{array}, \quad n > 0;$$

$\underline{\dim} I(8n+1,5) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+2, 8n+2, 8n+3)$,

$$I(1,5) = \left(\begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right),$$

$M_\alpha^{I(8n+1,5)} = M_\alpha^{I(8n+4,4)} \oplus M_\alpha^{I(8n+1,4)}$, for $\alpha \neq (3 \to 7)$,

$$M_{3\to7}^{I(8n+1,5)} = \left(M_{3\to7}^{I(8n+4,4)} \oplus M_{3\to7}^{I(8n+1,4)} \right) \boxplus \begin{array}{c|ccc} & 1 & 1 & 2n-1 \\ \hline 1 & 0 & 1 & 0 \\ 4n+1 & 0 & 0 & 0 \end{array}, \quad n > 0;$$

$\underline{\dim} I(8n+2,5) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+2, 8n+3, 8n+4)$,

$M_\alpha^{I(8n+2,5)} = M_\alpha^{I(8n+2,1)} \oplus M_\alpha^{I(8n+1,1)}$, for $\alpha \neq (5 \to 1)$,

$$M_{5\to1}^{I(8n+2,5)} = \left(M_{5\to1}^{I(8n+2,1)} \oplus M_{5\to1}^{I(8n+1,1)} \right) \boxplus \begin{array}{c|cc} & 4n & 1 \\ \hline 2n & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$\underline{\dim} I(8n+3,5) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+3, 8n+4, 8n+4)$,

$M_\alpha^{I(8n+3,5)} = M_\alpha^{I(8n+3,1)} \oplus M_\alpha^{I(8n+2,1)}, \quad \text{for } \alpha \neq (3 \to 7)$,

$$M_{3\to7}^{I(8n+3,5)} = \left(M_{3\to7}^{I(8n+3,1)} \oplus M_{3\to7}^{I(8n+2,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & 2n \end{array} \\ \begin{array}{c} 1 \\ 4n+1 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+4,5) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+5, 8n+5, 8n+5)$,

$M_\alpha^{I(8n+4,5)} = M_\alpha^{I(8n+4,1)} \oplus M_\alpha^{I(8n+3,1)}, \quad \text{for } \alpha \neq (5 \to 1)$,

$$M_{5\to1}^{I(8n+4,5)} = \left(M_{5\to1}^{I(8n+4,1)} \oplus M_{5\to1}^{I(8n+3,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & 4n+1 \end{array} \\ \begin{array}{c} 1 \\ 2n+1 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+5,5) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+6, 8n+6, 8n+7)$,

$M_\alpha^{I(8n+5,5)} = M_\alpha^{I(8n+5,1)} \oplus M_\alpha^{I(8n+4,1)}, \quad \text{for } \alpha \neq (3 \to 7)$,

$$M_{3\to7}^{I(8n+5,5)} = \left(M_{3\to7}^{I(8n+5,1)} \oplus M_{3\to7}^{I(8n+4,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{array}{c} 1 \\ 4n+3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+6,5) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+6, 8n+7, 8n+8)$,

$M_\alpha^{I(8n+6,5)} = M_\alpha^{I(8n+6,1)} \oplus M_\alpha^{I(8n+5,1)}, \quad \text{for } \alpha \neq (4 \to 7)$,

$$M_{4\to7}^{I(8n+6,5)} = \left(M_{4\to7}^{I(8n+6,1)} \oplus M_{4\to7}^{I(8n+5,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{array}{c} 1 \\ 4n+3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+7,5) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+7, 8n+8, 8n+8)$,

$M_\alpha^{I(8n+7,5)} = M_\alpha^{I(8n+7,1)} \oplus M_\alpha^{I(8n+6,1)}, \quad \text{for } \alpha \neq (3 \to 7)$,

$$M_{3\to7}^{I(8n+7,5)} = \left(M_{3\to7}^{I(8n+7,1)} \oplus M_{3\to7}^{I(8n+6,1)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 1 & 2n+1 \end{array} \\ \begin{array}{c} 1 \\ 4n+3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n,6) = (4n, 4n, 4n+1, 4n+1, 8n, 8n+1, 8n+1)$,

$M_\alpha^{I(8n,6)} = M_\alpha^{I(8n+2,4)} \oplus M_\alpha^{I(8n,3)}, \quad \text{for } \alpha \neq (3 \to 7)$,

$$M_{3\to7}^{I(8n,6)} = \left(M_{3\to7}^{I(8n+2,4)} \oplus M_{3\to7}^{I(8n,3)}\right) \boxplus \begin{array}{cc} & \begin{array}{cc} 2n & 1 \end{array} \\ \begin{array}{c} 1 \\ 4n \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+1,6) = (4n, 4n, 4n+1, 4n+1, 8n+1, 8n+1, 8n+2)$,

$$I(1,6) = \left(0, 0, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),$$

$M_\alpha^{I(8n+1,6)} = M_\alpha^{I(8n+1,1)} \oplus M_\alpha^{I(8n+7,2)[n \mapsto n-1]}, \quad \text{for } \alpha \neq (5 \to 2)$,

$$M_{5\to 2}^{I(8n+1,6)} = \left(M_{5\to 2}^{I(8n+1,1)} \oplus M_{5\to 2}^{I(8n+7,2)[n\mapsto n-1]}\right) \boxplus \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 2n} \end{array}\!\!\begin{array}{c} \begin{matrix} {\scriptstyle 1} & {\scriptstyle 4n-1} \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad n > 0;$$

$\underline{\dim} I(8n+2,6) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+2, 8n+3, 8n+3)$,

$$M_\alpha^{I(8n+2,6)} = M_\alpha^{I(8n+2,2)} \oplus M_\alpha^{I(8n,1)}, \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to 7}^{I(8n+2,6)} = \left(M_{3\to 7}^{I(8n+2,2)} \oplus M_{3\to 7}^{I(8n,1)}\right) \boxplus \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 4n+1} \end{array}\!\!\begin{array}{c} \begin{matrix} {\scriptstyle 2n} & {\scriptstyle 1} \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+3,6) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+3, 8n+3, 8n+4)$,

$$I(3,6) = \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \right),$$

$$M_\alpha^{I(8n+3,6)} = M_\alpha^{I(8n+5,4)} \oplus M_\alpha^{I(8n+3,3)}, \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{I(8n+3,6)} = \left(M_{5\to 1}^{I(8n+5,4)} \oplus M_{5\to 1}^{I(8n+3,3)}\right) \boxplus \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 2n} \end{array}\!\!\begin{array}{c} \begin{matrix} {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 4n-1} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \quad n > 0;$$

$\underline{\dim} I(8n+4,6) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+4, 8n+5, 8n+5)$,

$$M_\alpha^{I(8n+4,6)} = M_\alpha^{I(8n+6,3)} \oplus M_\alpha^{I(8n+4,4)}, \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to 7}^{I(8n+4,6)} = \left(M_{3\to 7}^{I(8n+6,3)} \oplus M_{3\to 7}^{I(8n+4,4)}\right) \boxplus \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 4n} \end{array}\!\!\begin{array}{c} \begin{matrix} {\scriptstyle 1} & {\scriptstyle 2n} \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+5,6) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+5, 8n+5, 8n+6)$,

$$M_\alpha^{I(8n+5,6)} = M_\alpha^{I(8n+5,2)} \oplus M_\alpha^{I(8n+3,1)}, \quad \text{for } \alpha \neq (5 \to 1),$$

$$M_{5\to 1}^{I(8n+5,6)} = \left(M_{5\to 1}^{I(8n+5,2)} \oplus M_{5\to 1}^{I(8n+3,1)}\right) \boxplus \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 2n+1} \end{array}\!\!\begin{array}{c} \begin{matrix} {\scriptstyle 1} & {\scriptstyle 4n+1} \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+6,6) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+6, 8n+7, 8n+7)$,

$$M_\alpha^{I(8n+6,6)} = M_\alpha^{I(8n+6,2)} \oplus M_\alpha^{I(8n+4,1)}, \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to 7}^{I(8n+6,6)} = \left(M_{3\to 7}^{I(8n+6,2)} \oplus M_{3\to 7}^{I(8n+4,1)}\right) \boxplus \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 4n+3} \end{array}\!\!\begin{array}{c} \begin{matrix} {\scriptstyle 1} & {\scriptstyle 2n+1} \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n+7,6) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+7, 8n+7, 8n+8)$,

$$M_\alpha^{I(8n+7,6)} = M_\alpha^{I(8n+7,2)} \oplus M_\alpha^{I(8n+5,1)}, \quad \text{for } \alpha \neq (3 \to 7),$$

$$M_{3\to 7}^{I(8n+7,6)} = \left(M_{3\to 7}^{I(8n+7,2)} \oplus M_{3\to 7}^{I(8n+5,1)}\right) \boxplus \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 4n+3} \end{array}\!\!\begin{array}{c} \begin{matrix} {\scriptstyle 1} & {\scriptstyle 2n+1} \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array};$$

$\underline{\dim} I(8n,7) = (4n, 4n, 4n+1, 4n+1, 8n, 8n, 8n+1)$,

$I(0,7) = \left(0,0,0,0,\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\end{bmatrix}\right)$,

$M_{\alpha}^{I(8n,7)} = M_{\alpha}^{I(8n,1)} \oplus M_{\alpha}^{I(8n+5,1)[n\mapsto n-1]}, \quad \text{for } \alpha \neq (4 \to 7),$

$$M_{4\to 7}^{I(8n,7)} = \left(M_{4\to 7}^{I(8n,1)} \oplus M_{4\to 7}^{I(8n+5,1)[n\mapsto n-1]}\right) \boxplus \begin{array}{c|cc} & 1 & 2n-1 \\ \hline 4n & 0 & 0 \\ 1 & 1 & 0 \end{array}, \quad n>0;$$

$\underline{\dim} I(8n+1,7) = (4n, 4n, 4n+1, 4n+1, 8n, 8n+1, 8n+2)$,

$I(1,7) = \left(0,0,0,\begin{bmatrix}0 & 1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$,

$M_{\alpha}^{I(8n+1,7)} = M_{\alpha}^{I(8n+1,1)} \oplus M_{\alpha}^{I(8n+6,1)[n\mapsto n-1]}, \quad \text{for } \alpha \neq (5 \to 2),$

$$M_{5\to 2}^{I(8n+1,7)} = \left(M_{5\to 2}^{I(8n+1,1)} \oplus M_{5\to 2}^{I(8n+6,1)[n\mapsto n-1]}\right) \boxplus \begin{array}{c|cc} & 4n+2 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2n-3 & 0 & 0 \end{array}, \quad n>0;$$

$\underline{\dim} I(8n+2,7) = (4n, 4n, 4n+1, 4n+1, 8n+1, 8n+2, 8n+2)$,

$I(2,7) = \left(0,0,\begin{bmatrix}0 & 1\end{bmatrix},\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$,

$M_{\alpha}^{I(8n+2,7)} = M_{\alpha}^{I(8n+2,1)} \oplus M_{\alpha}^{I(8n+7,1)[n\mapsto n-1]}, \quad \text{for } \alpha \neq (5 \to 1),$

$$M_{5\to 1}^{I(8n+2,7)} = \left(M_{5\to 1}^{I(8n+2,1)} \oplus M_{5\to 1}^{I(8n+7,1)[n\mapsto n-1]}\right) \boxplus \begin{array}{c|cc} & 4n-1 & 1 \\ \hline 2n & 0 & 0 \\ 1 & 0 & 1 \end{array}, \quad n>0;$$

$\underline{\dim} I(8n+3,7) = (4n+1, 4n+1, 4n+2, 4n+2, 8n+3, 8n+3, 8n+3)$,

$M_{\alpha}^{I(8n+3,7)} = M_{\alpha}^{I(8n+3,1)} \oplus M_{\alpha}^{I(8n,1)}, \quad \text{for } \alpha \neq (5 \to 2),$

$$M_{5\to 2}^{I(8n+3,7)} = \left(M_{5\to 2}^{I(8n+3,1)} \oplus M_{5\to 2}^{I(8n,1)}\right) \boxplus \begin{array}{c|cc} & 4n & 1 \\ \hline 2n & 0 & 0 \\ 1 & 0 & 1 \end{array};$$

$\underline{\dim} I(8n+4,7) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+4, 8n+4, 8n+5)$,

$M_{\alpha}^{I(8n+4,7)} = M_{\alpha}^{I(8n+4,1)} \oplus M_{\alpha}^{I(8n+1,1)}, \quad \text{for } \alpha \neq (3 \to 7),$

$$M_{3\to 7}^{I(8n+4,7)} = \left(M_{3\to 7}^{I(8n+4,1)} \oplus M_{3\to 7}^{I(8n+1,1)}\right) \boxplus \begin{array}{c|cc} & 2n & 1 \\ \hline 1 & 0 & 1 \\ 4n+2 & 0 & 0 \end{array};$$

$\underline{\dim} I(8n+5,7) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+4, 8n+5, 8n+6)$,

$M_{\alpha}^{I(8n+5,7)} = M_{\alpha}^{I(8n+6,4)} \oplus M_{\alpha}^{I(8n+5,4)}, \quad \text{for } \alpha \neq (3 \to 7),$

$$M_{3\to 7}^{I(8n+5,7)} = \left(M_{3\to 7}^{I(8n+6,4)} \oplus M_{3\to 7}^{I(8n+5,4)}\right) \boxplus \begin{array}{c|ccc} & 1 & 1 & 2n \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 4n+1 & 0 & 0 & 0 \end{array};$$

$\underline{\dim} I(8n+6,7) = (4n+2, 4n+2, 4n+3, 4n+3, 8n+5, 8n+6, 8n+6),$

$M_{\alpha}^{I(8n+6,7)} = M_{\alpha}^{I(8n+6,1)} \oplus M_{\alpha}^{I(8n+3,1)}, \quad \text{for } \alpha \neq (3 \to 7),$

$$M_{3\to 7}^{I(8n+6,7)} = \left(M_{3\to 7}^{I(8n+6,1)} \oplus M_{3\to 7}^{I(8n+3,1)}\right) \boxplus \begin{array}{c} \\ 1 \\ 1 \\ 4n+2 \end{array} \begin{array}{cc} 1 & 2n \\ \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{array}\right] \end{array};$$

$\underline{\dim} I(8n+7,7) = (4n+3, 4n+3, 4n+4, 4n+4, 8n+7, 8n+7, 8n+7),$

$M_{\alpha}^{I(8n+7,7)} = M_{\alpha}^{I(8n+7,1)} \oplus M_{\alpha}^{I(8n+4,1)}, \quad \text{for } \alpha \neq (3 \to 7),$

$$M_{3\to 7}^{I(8n+7,7)} = \left(M_{3\to 7}^{I(8n+7,1)} \oplus M_{3\to 7}^{I(8n+4,1)}\right) \boxplus \begin{array}{c} \\ 1 \\ 4n+3 \end{array} \begin{array}{cc} 1 & 2n+1 \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] \end{array}.$$

III.4.3 The exceptional regular modules

There are only a finite number of exceptional regular modules. These are the non-homogeneous indecomposable regulars with dimension vector falling below $\delta = (1,1,1,1,2,2,2)$, marked with green in Figure III.6. Note that $\underline{\dim} R_0^l(2) = \underline{\dim} R_1^{l'}(4) = \underline{\dim} R_\infty^l(2) = \delta$, where $l \in \{1,2\}$, $l' \in \{1,2,3,4\}$.

Representations of regular simples of $\Delta(\widetilde{\mathbb{D}}_6)$ are also given in [34], we include them here only for the sake of completeness:

$$\underline{\dim} R_\infty^1(1) = (0,1,0,1,1,1,1),$$
$$R_\infty^1(1) = \left(0, \left[1\right], \left[1\right], \left[1\right], 0, \left[1\right]\right);$$

$$\underline{\dim} R_\infty^2(1) = (1,0,1,0,1,1,1),$$
$$R_\infty^2(1) = \left(\left[1\right], 0, \left[1\right], \left[1\right], \left[1\right], 0\right);$$

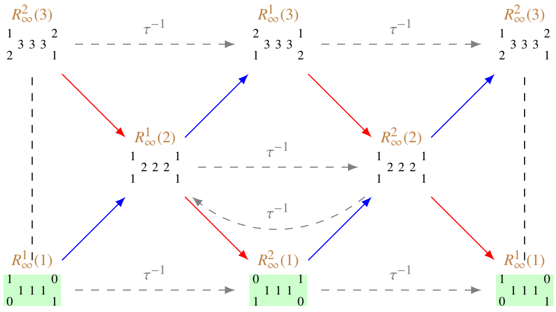
$$\underline{\dim} R_0^1(1) = (0,1,1,0,1,1,1),$$
$$R_0^1(1) = \left(0, \left[1\right], \left[1\right], \left[1\right], \left[1\right], 0\right);$$

$$\underline{\dim} R_0^2(1) = (1,0,0,1,1,1,1),$$
$$R_0^2(1) = \left(\left[1\right], 0, \left[1\right], \left[1\right], 0, \left[1\right]\right);$$

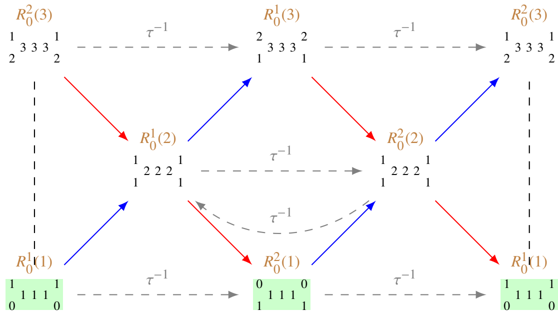
$$\underline{\dim} R_1^1(1) = (0,0,0,0,0,0,1),$$
$$R_1^1(1) = (0,0,0,0,0,0);$$

$$\underline{\dim} R_1^1(2) = (1,1,1,1,1,1,2),$$
$$R_1^1(2) = \left(\left[1\right], \left[1\right], \left[1\right], \left[0 \quad 1\right], \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right);$$

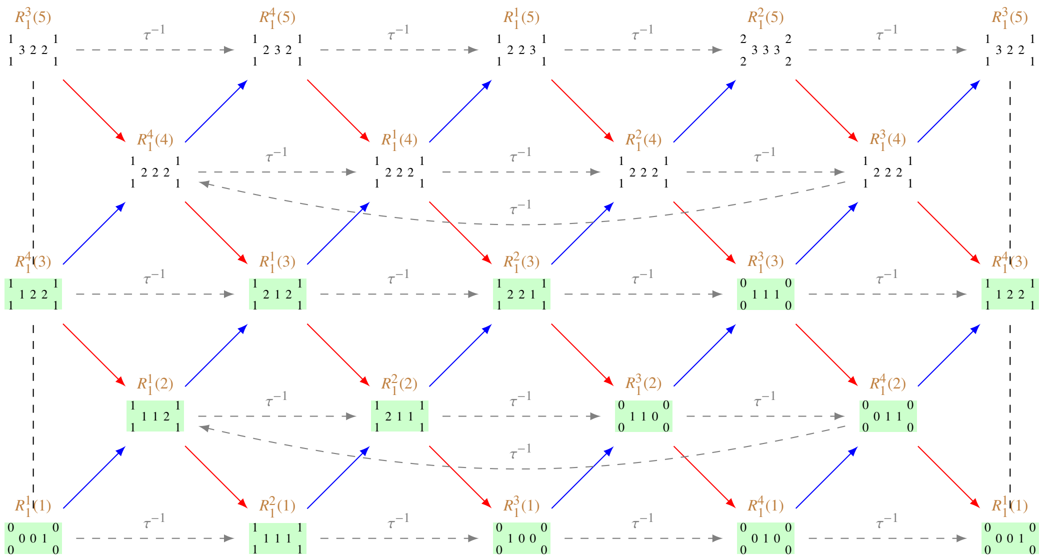
The regular non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_6)}$



The regular non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_6)}$



The regular non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_6)}$



Regular non-homogeneous tubes in the case of $\Delta(\widetilde{\mathbb{D}}_6)$

$$\underline{\dim} R^1_1(3) = (1,1,1,1,2,1,2),$$
$$R^1_1(3) = \left(\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right);$$

$$\underline{\dim} R^2_1(1) = (1,1,1,1,1,1,1),$$
$$R^2_1(1) = \left(\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix} \right);$$

$$\underline{\dim} R^2_1(2) = (1,1,1,1,2,1,1),$$
$$R^2_1(2) = \left(\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix} \right);$$

$$\underline{\dim} R_1^2(3) = (1,1,1,1,2,2,1),$$
$$R_1^2(3) = \left(\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right);$$

$$\underline{\dim} R_1^3(1) = (0,0,0,0,1,0,0),$$
$$R_1^3(1) = (0,0,0,0,0,0);$$

$$\underline{\dim} R_1^3(2) = (0,0,0,0,1,1,0),$$
$$R_1^3(2) = \left(0,0,\begin{bmatrix}1\end{bmatrix},0,0,0\right);$$

$$\underline{\dim} R_1^3(3) = (0,0,0,0,1,1,1),$$
$$R_1^3(3) = \left(0,0,\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\end{bmatrix},0,0\right);$$

$$\underline{\dim} R_1^4(1) = (0,0,0,0,0,1,0),$$
$$R_1^4(1) = (0,0,0,0,0,0);$$

$$\underline{\dim} R_1^4(2) = (0,0,0,0,0,1,1),$$
$$R_1^4(2) = \left(0,0,0,\begin{bmatrix}1\end{bmatrix},0,0\right);$$

$$\underline{\dim} R_1^4(3) = (1,1,1,1,1,2,2),$$
$$R_1^4(3) = \left(\begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0\\ 1\end{bmatrix}, \begin{bmatrix}1\\ 1\end{bmatrix}\right).$$

Chapter IV

On the combinatorial nature of tree representations of Euclidean quivers

As we have mentioned earlier, Ringel proved that every exceptional module has a tree representation, but one of the steps in his proof involves a choice of basis, which seems to depend on the underlying field. He posed the question (see Problems 1. and 2. from Section 9. of [27]) whether there exist tree representations that are independent of this choice of basis, hence being field independent. This problem remains open in general, but as we have seen in Chapters II and III in some particular cases it has been settled: the tree representations for the canonically oriented Euclidean quivers $\widetilde{\mathbb{E}}_6$ and $\widetilde{\mathbb{D}}_m$ listed in the previous chapter are indeed field independent, thus giving an affirmative answer to Ringel's question in these cases.

We recall that the representations from the previous two chapters were obtained by experimentation in $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and were not specifically constructed to be field independent. This is probably not a lucky coincidence, and begs the question whether every tree representation is field independent or not.

In this chapter, based on the article [18] we verify computationally this question in the case of Euclidean quivers of type $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{D}}_5$ and $\widetilde{\mathbb{E}}_6$ with dimension vector bounded by the minimal radical vector of the quiver. This includes a large class of exceptional representations, in particular all the regular non-homogeneous exceptionals.

IV.1 Computational findings and conjectures

In the following let k be an arbitrary field, Q a Euclidean quiver, and x an exceptional root over Q. We introduce the following notation for the set of all tree representations with dimension vector x over k:

$$T_k(x) = \{ M \in \operatorname{rep} kQ \mid \underline{\dim} M = x \text{ and } M \text{ is a tree representation } \}.$$

Proposition IV.1.1. *Let Q denote a canonically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{D}}_5$ or $\widetilde{\mathbb{E}}_6$. Let x be an exceptional root over Q, smaller than the minimal radical δ. If we regard the matrices of the representations as formal 2-dimensional arrays of the symbols 0 and 1, then the set $T_k(x)$ has the same elements over any field, that is $T_k(x) = T_{k'}(x)$ for any two fields k and k'.*

As a result of the previous proposition we formulate the following conjecture:

Conjecture IV.1.2. *Let x be an exceptional root over an arbitrary Euclidean quiver Q smaller than the minimal radical vector δ. If we regard the matrices of the representations as formal 2-dimensional arrays* *of the symbols* 0 *and* 1*, then the set* $T_k(x)$ *has the same elements over any field, that is* $T_k(x) = T_{k'}(x)$ *for any two fields* k *and* k'*.*

In the case of the (computationally verified) quivers we could omit the index k and denote the set only as $T(x)$.

Let z be an exceptional root of the quiver Q and $Z \in T(z)$ a tree representation. We define the set $S(z)$, which will contain the pairs of dimension vectors of every (non-special) Schofield pair belonging to Z. More precisely:

$$\begin{aligned} S(z) = \{\ (x,y) \mid\ & x, y \text{ are exceptional roots of } Q \text{ and } (Y,X) \text{ is a Schofield} \\ & \text{pair belonging to } Z, \text{ where } Z \in T(z),\ Y \in T(y) \text{ and} \\ & X \in T(x) \text{ with } \underline{\dim} X = x,\ \underline{\dim} Y = y,\ \underline{\dim} Z = z\ \} \end{aligned}$$

Note that while the representations $X, Y, Z \in \operatorname{mod} kQ$ exist within the context of a base field k, the conditions stated in Proposition 7 from [39] depend only on the value of the roots (dimension vectors), hence the set $S(z)$ may be used in a field independent context.

If the root z is smaller than the minimal radical vector δ, then we have only so-called non-special Schofield sequences of the form $0 \to X \to Z \to Y \to 0$ (see Propositions 7 and 9 from [39]) and the set $S(z)$ may be given in the following way:

$$S(z) = \{\ (x,y) \mid x, y \text{ are exceptional roots of } Q,\ x + y = z,\ \langle x, y\rangle = 0\ \}$$

In what follows we define a set of representations constructed using Schofield pairs. Let x and y be exceptional roots of the quiver Q and consider arbitrary tree representations $X \in T(x)$ and $Y \in T(y)$. We construct a new representation $R^{\alpha ij}_{XY}$, as follows ($\alpha \in Q_1$ and i, j being row respectively column indices in the upper right block of the matrix M_α):

$$R^{\alpha ij}_{XY} = (M_v, M_a)_{\substack{v \in Q_0 \\ a \in Q_1}} = \left((X_v \oplus Y_v)_{v \in Q_0}, \left(\begin{bmatrix} X_a & E^{ij}_a \\ 0 & Y_a \end{bmatrix} \right)_{a \in Q_1} \right)$$

where for the upper right block E^{ij}_a is true that $E^{ij}_a = 0$ for $a \neq \alpha$ and E^{ij}_α contains exactly one non-zero entry 1 in the ith row and jth column and it is zero elsewhere. Using this notaion we introduce the following set $E_k(x, y) \subseteq \operatorname{mod} kQ$:

$$\begin{aligned} E_k(x,y) = \{\ R^{\alpha ij}_{XY} \mid\ & \alpha \in Q_1,\ i, j \text{ are row resp. column indices}, \\ & X \in T_k(x),\ Y \in T_k(y),\ R^{\alpha ij}_{XY} \in T_k(x+y)\ \} \end{aligned}$$

For given tree representations X and Y, the representation $R^{\alpha ij}_{XY}$ is the construction given by Ringel in Section 6 of [27]. As mentioned there, the position of the single nonzero entry specified by α, i and j involves a choice of basis and could very well depend on the base field k. To our surprise, however, this seems not to be the case:

Proposition IV.1.3. *Let Q denote a canonically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{D}}_5$ or $\widetilde{\mathbb{E}}_6$. Let x and y be exceptional roots over Q, smaller than the minimal radical δ. If we regard the matrices of the representations as formal 2-dimensional arrays of the symbols 0 and 1, then the set $E_k(x, y)$ has the same elements over any field, that is $E_k(x, y) = E_{k'}(x, y)$ for any two fields k and k'.*

Based on our findings we conjecture that Proposition IV.1.3 holds for arbitrary tame quivers and exceptional roots. In the case of the (computationally verified) quivers we could omit the index k and denote the set only as $E(x, y)$.

Further advancing with our "computational inquiry" into the problem of field independence we may ask for a method to construct the set of tree representations, other than the "exhaustive search" we have performed. Ringel in his proof used Schofield induction to construct tree representations (see Section 6. of [27]), and we may ask the question whether there are other methods for obtaining them, or does his construction result in every possible tree representation. Permuting the basis vectors is a field independent operation, so we introduce the following:

Definition IV.1.4. Let $M = (M_i, M_\alpha)$ and $N = (N_i, N_\alpha)$ be representations of a quiver Q. Then we call them *permutation-similar*, provided there exists a family of permutation matrices $\{A_i \mid i \in Q_0\}$ such that the following diagram is commutative for every arrow $\alpha \in Q_1$:

$$\begin{array}{ccc} M_i & \xrightarrow{M_\alpha} & M_j \\ \downarrow{\scriptstyle A_i} & & \downarrow{\scriptstyle A_j} \\ N_i & \xrightarrow{N_\alpha} & N_j \end{array}$$

Let $Z \in T(z)$ be a tree representation, we denote by $\pi(Z)$ the set of all tree representations that are permutation-similar to Z.

Using the notations introduced above, we state the following proposition, giving a method to inductively construct the sets of tree representations:

Proposition IV.1.5. *Let z be an exceptional root of a caconically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{D}}_5$ or $\widetilde{\mathbb{E}}_6$, such that $z < \delta$. Then we have*

$$T(z) = \bigcup_{\substack{(x,y) \in S(z) \\ Z \in E(x,y)}} \pi(Z).$$

We conjecture that Proposition IV.1.5 also holds true for every exceptional root of any Euclidean quiver.

Bibliography

[1] *Clean 3.0*, https://wiki.clean.cs.ru.nl/Clean.

[2] *GAP*, http://www.gap-system.org/.

[3] I. Assem, A. Skowroński, and D. Simson, *Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory*, London Mathematical Society Student Texts, vol. 1, Cambridge University Press, 2006.

[4] M. Auslander and I. Reiten, *Representation theory of Artin algebras, III almost split sequences*, Communications in Algebra **3** (1975), no. 3, 239–294.

[5] W. W. Crawley-Boevey, *Matrix problems and Drozd's theorem*, Banach Center Publications **26** (1990), no. 1, 199–222.

[6] V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, vol. 173 = Vol. 6,[3], American Mathematical Society, 1976.

[7] P. Dowbor, H. Meltzer, and A. Mróz, *An algorithm for the construction of exceptional modules over tubular canonical algebras*, Journal of Algebra **323** (2010), no. 10, 2710–2734.

[8] P. Dowbor, H. Meltzer, and M. Schmidmeier, *The "0, 1-property" of exceptional objects for nilpotent operators of degree 6 with one invariant subspace*, Journal of Pure and Applied Algebra **223** (2019), no. 7, 3150–3203.

[9] E. Dynkin, *Classification of the simple Lie groups*, Rec. Math. [Mat. Sbornik] N. S. **18(60)** (1946), 347–352.

[10] P. Fahr, *Infinite Gabriel-Roiter measures for the 3-Kronecker quiver*, Ph.D. thesis, Bielefeld University, 2008.

[11] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Mathematica **6** (1972), no. 1, 71–103.

[12] M. Grzecza, S. Kasjan, and A. Mróz, *Tree Matrices and a Matrix Reduction Algorithm of Belitskii*, Fundamenta Informaticae **118** (2012), no. 3, 253–279.

[13] D. Kędzierski and H. Meltzer, *Indecomposable representations for extended Dynkin quivers of type* $\widetilde{E}_8$, Colloquium Mathematicum **124** (2011), no. 1, 95–116.

[14] D. Kussin and H. Meltzer, *Indecomposable representations for extended Dynkin quivers*, 2006.

[15] Sz. Lénárt, Á. Lőrinczi, Cs. Szántó, and I. Szöllősi, *Proof of the tree module property for exceptional representations of tame quivers*, ArXiv **abs/2001.00016v3** (2021).

[16] Sz. Lénárt, Á. Lőrinczi, Cs. Szántó, and István Szöllősi, *Tree representations of the quiver $\widetilde{D}_m$*, Colloquium Mathematicum **167** (2022), no. 2, 261–302.

[17] Sz. Lénárt, Á. Lőrinczi, and I. Szöllősi, *Tree representations of the quiver $\widetilde{E}_6$*, Colloquium Mathematicum **164** (2021), no. 2, 221–250.

[18] Á. Lőrinczi, *On the combinatorial nature of tree representations of Euclidean quivers*, accepted for publication in Mathematica.

[19] Á. Lőrinczi and Cs. Szántó, *The indecomposable preprojective and preinjective representations of the quiver $\widetilde{D}_n$*, Mathematica **57 (80)** (2015), 95–116.

[20] I. Reiten M. Auslander and S. Smalø, , Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[21] A. Mróz, *On the Multiplicity Problem and the Isomorphism Problem for the Four Subspace Algebra*, Communications in Algebra **40** (2012), no. 6, 2005–2036.

[22] A. Mróz, *The dimensions of the homomorphism spaces to indecomposable modules over the four subspace algebra*, 2012.

[23] M. Plasmeijer and M. Eekelen, *Functional Programming and Parallel Graph Rewriting*, 1993.

[24] C. M. Ringel, *Representations of k-species and bimodules*, Journal of Algebra **41** (1976), no. 2, 269–302.

[25] C. M. Ringel, *The braid group action on the set of exceptional sequences of a hereditary Artin algebra*, Abelian group theory and related topics: Conference on Abelian Groups, August 1 - 7, 1993, Oberwolfach, Germany (Rüdiger Göbel, ed.), vol. 171, American Mathematical Soc., 1994, 339–352.

[26] C. M. Ringel, *Exceptional objects in hereditary categories*, Analele Stiintifice ale Universitatii Ovidius Constanta. Proceedings: Representation Theory of Groups, Algebras, and Orders. September 25 - October 6, 1995, Constanta (Klaus W. Roggenkamp and Mirela Stefanescu, eds.), no. 2, Faculty of Mathematics and Computer Science, Ovidius University, Constanta, Romania, 1996, 150–158.

[27] C. M. Ringel, *Exceptional modules are tree modules*, Linear Algebra and its Applications **275-276** (1998), 471–493.

[28] C. M. Ringel, *Combinatorial Representation Theory. History and Future*, Representations of algebra. Vol. I, II, Beijing Norm. Univ. Press, Beijing, 2002, 122–144.

[29] C. M. Ringel, *Indecomposable representations of the kronecker quivers*, Proceedings of the American Mathematical Society **141** (2012), no. 1, 115–121.

[30] C. M. Ringel, *Introduction to the representation theory of quivers*, https://www.math.uni-bielefeld.de/~sek/kau, 2012.

[31] C. M. Ringel, *Representations of Quivers. An Introduction*, https://www.math.uni-bielefeld.de/~sek/shanghai/sjtu.html, 2015.

[32] R. Schiffler, *Quiver representations*, Springer International Publishing, 2014.

[33] A. Schofield, *Semi-Invariants of Quivers*, Journal of the London Mathematical Society **s2-43** (1991), no. 3, 385–395.

[34] D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras*, London Mathematical Society Student Texts, vol. 2, Cambridge University Press, 2007.

[35] A. Skowroński and K. Yamagata, *Frobenius Algebras I*, European Mathematical Society Publishing House, 2011.

[36] A. Skowroński and K. Yamagata, *Frobenius algebras II*, European Mathematical Society Publishing House, 2017.

[37] S. Smetsers, E. Barendsen, M. Eekelen, and R. Plasmeijer, *Guaranteeing safe destructive updates through a type system with uniqueness information for graphs*, Graph Transformations in Computer Science, Springer Berlin Heidelberg, 1994, 358–379.

[38] Cs. Szántó, *On some Ringel–Hall products in tame cases*, Journal of Pure and Applied Algebra **216** (2012), no. 10, 2069–2078.

[39] Cs. Szántó and I. Szöllősi, *Schofield sequences in the Euclidean case*, Journal of Pure and Applied Algebra **225** (2021), no. 5, 106586.

[40] T. Weist, *Tree modules of the generalised Kronecker quiver*, Journal of Algebra **323** (2010), no. 4, 1107–1138.

[41] T. Weist, *Tree modules*, Bulletin of the London Mathematical Society **44** (2012), no. 5, 882–898.

[42] P. Zhang, Y.-B. Zhang, and J.-Y. Guo, *Minimal Generators of Ringel–Hall Algebras of Affine Quivers*, Journal of Algebra **239** (2001), no. 2, 675–704.