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Contributions in the theory of univalent functions of one and several complex variables

Ph.D. Thesis - Summary

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Introduction

The purpose of this thesis is to present new contributions in the theory of univalent functions of one and several complex variables. The theory of univalent functions is a part of the geometric function theory that became an important point of interest for many research works done through time. Let $\mathbb{C}$ be the complex plane and let $\mathbb{C}^n = \{z = (z_1, z_2, \ldots, z_n) : z_i \in \mathbb{C}, i = 1, \ldots, n\}$, where $n \geq 2$, be the complex space equipped with the Euclidean inner product, which provides the Euclidean norm $\|\cdot\|$. Also, we denote by U the unit disc in $\mathbb{C}$, $\mathbb{B}^n$ the Euclidean unit ball in $\mathbb{C}^n$ and $\mathbb{P}^n$ the unit polydisc in $\mathbb{C}^n$. By a univalent function we mean a holomorphic and injective function. A well-known result that established new directions in the theory of univalent functions of one complex variable is the Riemann mapping theorem, which ensures the conformal equivalence of every simply connected domain Ω that is a proper subset of the complex plane $\mathbb{C}$ with the unit disc U (see [48], [66]). This result provided a direction in the study of the univalence on the unit disc U (see e.g. [48], [66]). Since any univalent function can be reduced to a normalized univalent function, i.e. univalent function f with $f(0) = f'(0) - 1 = 0$, then it suffices to study the set of normalized univalent functions on U, S (see [25], [102]). Riemann mapping theorem does not remain true in $\mathbb{C}^n$, $n \geq 2$ (see [93], [106]). The result which conducted to this statement is the fact that the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ is not biholomorphic equivalent with the unit polydisc $\mathbb{P}^n$ in $\mathbb{C}^n$, although they are homeomorphic. This observation was provided by Poincaré [100].

Along this thesis, we denote by $S(\mathbb{B}^n)$ the set of biholomorphic and normalized mappings on $\mathbb{B}^n$. Cartan H. [9] showed that $S(\mathbb{B}^n)$ is not a locally uniformly bounded family, thus $S(\mathbb{B}^n)$ is not compact in the topology of the set of holomorphic mappings on $\mathbb{B}^n$, $H(\mathbb{B}^n)$. Moreover, in view of this result, we deduce that the set $S(\mathbb{B}^n)$ does not admit growth and distortion theorems. A remarkable family introduced by Graham, Hamada and Kohr [37] is the set of mappings with parametric representation on $\mathbb{B}^n$, $S^0(\mathbb{B}^n)$. Graham et al. [37] proved that $S^0(\mathbb{B}^n)$ is a proper subset of $S(\mathbb{B}^n)$. Therefore, not any normalized biholomorphic mapping has parametric representation on $\mathbb{B}^n$. This is an essential difference between the case of several complex variables ($n \geq 2$) and the case of one complex variable ($n = 1$), where any function in the class S has parametric representation (see [102]). The set $S^0(\mathbb{B}^n)$ is not empty since any normalized starlike mapping belongs to this set.

In the complex plane, any function $f \in S$ admits parametric representation ($\mathcal{PR}$ for shortness), which means that there exists a Loewner chain ($\mathcal{LC}$ for shorteness), $f(z,t) : U \times [0, \infty) \to \mathbb{C}$, such that f is the first element of $f(z,t)$. This result is due to Pommerenke [102] and is a fundamental result in the theory of Loewner chains. Important subclasses of S with geometric properties have analytical characterization in terms of Loewner chains. Let us mention here the description through Loewner chains for spirallike functions of the type γ, $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, starlike functions, (see e.g. [102],

[48]), almost starlike functions of order α, $\alpha \in [0,1)$ (see [119]), convex functions (see [48], [102]) on U. Various problems and applications on this topic may be consulted in the work of Pommerenke [102], Duren [25], Graham and Kohr [48], Mocanu, Bulboacă and Sălăgean [90].

Another topic of great interest in the theory of biholomorphic mappings in $\mathbb{C}^n$ is represented by the study of geometric properties of univalent mappings. An analytical characterization of starlikeness on $\mathbb{B}^n$ have been obtained by Matsuno [88]. Gurganus [54] and Suffridge [114] provided a characterization of starlikeness on the unit ball of a Banach space. Suffridge [113] gave a similar characterization of starlikeness on the unit polydisc of $\mathbb{C}^n$. Growth, covering and coefficient results for the class $S^*(\mathbb{B}^n)$ of normalized starlike mappings on $\mathbb{B}^n$ have been obtained by Kubicka and Poreda [78], and Barnard, FitzGerald and Gong [5], Gong [32, 33], Graham and Kohr [48], respectively by Kohr [72], Curt [19], Graham, Hamada and Kohr [37]. The concept of order α starlikeness on the unit ball in $\mathbb{C}^n$, where $\alpha \in [0,1)$, was introduced by Kohr [70], respectively the concept of order α almost starlikeness, $\alpha \in [0,1)$, was introduced by Feng [30] on the unit ball of a Banach space. Moreover, T. Chirilă [12] defined almost starlikeness of order α and type γ on $\mathbb{B}^n$, with $0 \leq \alpha < 1$ and $0 \leq \gamma < 1$. The analytic characterization of convexity in C^n was given by Kikuchi [69], Gong, Wang and Yu [34] and Suffridge [114, 112]. Other results regarding the class of normalized convex mappings such as growth theorem, coefficient estimates, a Marx-Strohhäcker theorem have been obtained by Suffridge [115], FitzGerald and Thomas [31], Liu [79], Kohr [70, 72], Curt [18]. Gurganus K. [54] defined the notion of spirallikeness with respect to a normal linear operator, whose eigenvalues have positive real part (see also [60], [48]). Also, Suffridge [112] extended this notion to the case of a complex Banach space. Other generalizations were considered by Liu and Liu [82] and Chirilă [11].

The theory of Loewner chains in the complex plane provided numerous applications such as: analytic characterizations of the univalent functions with geometric properties, approaching different extremal problems, proving Bieberbach conjecture, etc (see [25], [48]). The first contribution in the generalization of the Loewner chains and Loewner differential equation in n-dimensions, $n \geq 2$, is due to Pfaltzgraff [96, 97], which extended the notion of Loewner chain to $\mathbb{B}^n$ in $\mathbb{C}^n$. Later, Poreda [103, 104] refined these results on the unit polydisc $\mathbb{P}^n$ in $\mathbb{C}^n$ and introduced the set of normalized univalent mappings which admit parametric representation on $\mathbb{P}^n$, $S^0(\mathbb{P}^n)$. Other important contributions have been obtained by Kubicka and Poreda [78], who analyzed the class $S^*(\mathbb{B}^n)$. Notable improvements appeared through time in the theory of Loewner chains in n-dimensions. An remarkable example in this sense is the introduction of the family $S^0(\mathbb{B}^n)$ of mappings which admit parametric representation on $\mathbb{B}^n$ in $\mathbb{C}^n$, due to Graham, Hamada and Kohr [37]. Graham et al. [37] proved the strict inclusion $S^0(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n)$, which shows that not any mapping from $S(\mathbb{B}^n)$ admits parametric representation on $\mathbb{B}^n$. An outstanding contribution to the theory of Loewner chains in $\mathbb{C}^n$ has been done by G. Kohr and her collaborators in a series of valuable publications starting with [37], [51], [23], [52], [26]. Other aspects regarding this topic can be found in [7], [2], [8], [3]. Also, univalence criteria using the method of Loewner chains can be consulted in [16, 17].

The theory of Loewner on hyperbolic complex manifolds was developed by Arosio, Bracci, Hamada and Kohr in [3]. Graham I., Hamada H., Kohr G. and Kohr M. [42] studied the notion of generalized parametric representation with respect to an operator A, which is time dependent, in the case of reflexive complex Banach spaces (see also [58]). Important results regarding Loewner chains and non-linear resolvents of $\mathcal{M}$, which is the Carathéodory family on $\mathbb{B}^n$, were considered in [40] which extends the work in [29].

Also, topological Loewner chains on Riemann surfaces were studied in [15]. The reader may also consult the survey on Loewner chains and approximation results regarding univalent mappings on the Euclidean unit ball $\mathbb{B}^n$ in [59].

An important subclass of $S^0(\mathbb{B}^n)$ is the set of mappings with g-parametric representation (g-$\mathcal{PR}$ for shortness) on $\mathbb{B}^n$, $S_g^0(\mathbb{B}^n)$, where the function g satisfies some common properties. This important class was introduced by Graham, Hamada and Kohr in [37] and is strictly connected with the notion of g-Loewner chain (g-$\mathcal{LC}$ for shortness). We say that $f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is a g-Loewner chain if it satisfies the following conditions : $f(z,t)$ is a Loewner chain, the family $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is normal on $\mathbb{B}^n$ and the mapping h which appears in the Loewner differential equation

$$\frac{\partial f}{\partial t} = Df(z,t)h(z,t), \text{ a.e. } t \geq 0, \forall z \in \mathbb{B}^n, \tag{0.0.1}$$

has the property that, for almost every $t \geq 0$, $h(\cdot,t)$ belongs to $\mathcal{M}_g$ ([37]). The class M_g is given below ([37])

$$\mathcal{M}_g = \left\{ h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n, \langle h(z), \frac{z}{\|z\|^2} \rangle \in g(U), z \in \mathbb{B}^n \right\},$$

with $\langle h(z), \frac{z}{\|z\|^2} \rangle|_{z=0} = 1$. Then f admits g-parametric representation if and only if f is the first element of a g-Loewner chain. We have important reasons to study g-parametric representation, respectively g-Loewner chains for $n \geq 2$. For example, if $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, then any g-Loewner chain becomes a Loewner chain. However, for $n \geq 2$, there exists Loewner chains that are not g-Loewner chains. A growth and coefficient estimates for the family $S_g^0(\mathbb{B}^n)$ were obtained in [37], [73]. Also, g-Loewner chains been studied in [39], [43], [62].

The ability to build biholomorphic mappings with geometric properties in n-dimensions, $n \geq 2$, seems to require sometimes a great deal of effort, such in the case of convex mappings in $\mathbb{C}^n$. A first step in this direction was done by Roper and Suffridge [108], who proposed the extension operator $\Phi_n : \mathcal{L}S \to \mathcal{L}S_n$, defined by

$$\Phi_n(f)(z) = (f(z_1), \tilde{z}\sqrt{f'(z_1)}), \; z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where $\sqrt{f'(z_1)}|_{z_1=0} = 1$, as a way of constructing convex mappings on $\mathbb{B}^n$ using convex functions on U. Therefore, the operator Φ_n preserves convexity. This property of the operator Φ_n was also obtained by Graham and Kohr [47] by using a different method. In [47], Graham I. and Kohr G. first showed that Φ_n preserves the notion of starlikeness. A few years later, Hamada, Kohr and Kohr [63] proved that Φ_n preserves starlikeness of order 1/2. Further, Liu [80] proved the conservation of starlikeness of order $\alpha \in (0,1)$ (a different proof in terms of g-Loewner chains was given by Chirilă in [12]). Moreover, Graham, Kohr and Kohr [51] proved that spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is preserved under the operator Φ_n. Using g-Loewner chains, Chirilă [12] showed that the same operator preserves also the spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and order $\alpha \in (0,1)$ (also [82]). All of these properties are consequences of the following result due to Graham, Kohr and Kohr [51]: if $f \in S$ then $\Phi_n(f) \in S^0(\mathbb{B}^n)$.

Other extension operators that map a locally univalent function on U onto a mapping with the same properties on $\mathbb{B}^n$ in $\mathbb{C}^n$ were given through time. An example is the following extension operator which extends the operator Φ_n:

$$\Phi_{n,\alpha,\beta}(f)(x) = \left(f(z_1), \tilde{z}\Big(\frac{f(z_1)}{z_1}\Big)^{\alpha} (f'(z_1))^{\beta} \right), \; \forall z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where $\alpha \geq 0$, $\beta \geq 0$. We consider the branches of the power functions such that

$$\left(\frac{f(z_1)}{z_1}\right)^\alpha\Big|_{z_1=0} = 1,\ (f'(z_1))^\beta\big|_{z_1=0} = 1.$$

This remarkable extension operator was introduced by Graham, Hamada, Kohr and Suffridge in [46]. If $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1/2$ and $\alpha+\beta \leq 1$, then $\Phi_{n,\alpha,\beta}$ preserves parametric representation, starlikeness, order γ starlikeness, $\gamma \in (0,1)$, almost starlikeness of type $\gamma \in (0,1)$ and order $\delta \in [0,1)$, spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and order $\delta \in (0,1)$ (see [46], [80], [81], [11]). An important question related to the operator $\Phi_{n,\alpha,\beta}$ is that of the preservation of convexity. Graham, Hamada, Kohr and Suffridge [46] showed that $\Phi_{n,\alpha,\beta}$ preserves convexity only if $(\alpha,\beta) = (0,1/2)$. Motivated to find a way to provide extreme points of $K(\mathbb{B}^n)$, Muir [92] gave a generalization of the extension operator due to Roper and Suffridge, which maps extreme points of the class K to extreme points of the class $K(\mathbb{B}^n)$. This extension operator is defined by

$$\Phi_{n,Q}(f)(z) = (f(z_1) + Q(\tilde{z})f'(z_1), \tilde{z}\sqrt{f'(z_1)}),\ z = (z_1,\tilde{z}) \in \mathbb{B}^n,$$

where we choose the branch of the square root such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$ and $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2. For $\|Q\| \leq 1/4$, we have that $\Phi_{n,Q}$ conserves parametric representation, starlikeness as it was proved by Kohr [75]. Also, if $\|Q\| \leq 1/2$, the Muir extension operator conserves convexity (see [92]). For $\|Q\| \leq \frac{1-|2\alpha-1|}{8\alpha}$, starlikeness of order $\alpha \in (0,1)$ is also preserved under $\Phi_{n,Q}$ (see [116], [12]).

The Pfaltzgraff-Suffridge extension operator is a generalization of the extension operator due to Roper-Suffridge, which maps a locally biholomorphic and normalized mapping on $\mathbb{B}^n$ into a mapping with the same properties on $\mathbb{B}^{n+1}$. This Pfaltzgraff-Suffridge operator is described as follows ([99]): $\Psi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ with

$$\Psi_n(f)(z) = \left(f(\tilde{z}), z_{n+1}[J_f(\tilde{z})]^{\frac{1}{n+1}}\right),\ z = (\tilde{z}, z_{n+1}) \in \mathbb{B}^{n+1}.$$

We consider the branch of the power function to be $[J_f(\tilde{z})]^{\frac{1}{n+1}}\Big|_{\tilde{z}=0} = 1$. This extension operator satisfies properties like: $\Psi_n(S^0(\mathbb{B}^n)) \subseteq S^0(\mathbb{B}^{n+1})$, $\Psi_n(S^*(\mathbb{B}^n)) \subseteq S^*(\mathbb{B}^{n+1})$ (see [53]). The preservation of convexity under the extension operator Ψ_n remained until now an open problem. However, a partial result was obtained by Graham, Kohr and Pfaltzgraff in [53], where the authors showed that for a convex mapping on $\mathbb{B}^n$ the image of that mapping through Ψ_n includes the convex envelope of the image of an egg domain in $\mathbb{B}^n$.

Roper-Suffridge type extension operators with alike properties are studied in [27], [28], [36], [38], [46], [47], [48], [49], [50], [75], etc. Pfaltzgraff-Suffridge type extension operators were considered in [13], [63] on Reinhardt domains and [41] on bounded symmetric domains in $\mathbb{C}^n$.

In [65], the authors adapted the extension operators $\Phi_{n,Q}$ and Ψ_n to non-normalized mappings and chains and showed that these extension operators conserve L^d-Loewner chains. Recent contributions regarding Roper-Suffridge and Pfaltzgraff-Suffridge type extension operators on complex Banach spaces obtained by Graham et al. are gathered in [45] (see also [44], [41]). In this work, the authors studied the preservation of g-Loewner chains. More recent results concerning extended Loewner chains and the extension operator $\Phi_{n,Q}$, as well as other preservation results have been obtain in [91].

Our main objective is represented by the study of certain preservation results concerning the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ and subclasses of $S_g^0(\mathbb{B}^n)$, where g is the function given by

$$g(\zeta) = \frac{1+A\zeta}{1+B\zeta},\ \zeta \in U,\ \text{where } -1 \leq B < A \leq 1. \tag{0.0.2}$$

Given this form of g, we shall refer in this thesis to the set of mappings with g-parametric representation on $\mathbb{B}^n$, the set of g-starlike mappings on $\mathbb{B}^n$, the set of g-almost starlike mappings of order α on $\mathbb{B}^n$ and the set of g-spirallike mappings of type γ on $\mathbb{B}^n$. For a suitable choice of parameters A, B in the expression of g, these classes can be reduced to some well-known subclasses of $S(\mathbb{B}^n)$. We especially remark the case $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$, when g-parametric representation on $\mathbb{B}^n$ reduces to parametric representation $\mathbb{B}^n$, g-starlikeness on $\mathbb{B}^n$ reduces to standard starlikeness on $\mathbb{B}^n$ and so on. We are also interested on a particular type of starlikeness on $\mathbb{B}^n$, namely Janowski (almost) starlikeness with real coefficients. These notions coincides with g-starlikeness on $\mathbb{B}^n$ for a proper selection of A and B in (0.0.2). Also, we shall refer to a generalization of these notions, namely to Janowski (almost) starlikeness with complex coefficients.

Let the function g be described by (0.0.2)*. In this thesis, we prove that g-parametric presentation is preserved under the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$. Also, we show that g-starlikeness and g-spirallikeness of type γ is conserved through $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$. Moreover, g-almost starlikeness of order α is preserved under $\Phi_{n,\alpha,\beta}$. A direct consequence of the preservation of g-starlikeness is represented by the fact that $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ preserve Janowski (almost) starlikeness with real coefficients.*

Let $|1-a| < b \leq a$ with $a, b \in \mathbb{R}$. Let $\mathcal{J}^{(a,b)}$ be the class of Janowski starlike functions on U and let $\mathcal{AJ}^{(a,b)}$ be the class of Janowski almost starlike functions on U. Also, let S^* be the set of normalized starlike functions on U and let S_g^* be the set of g-starlike functions on U. We give the $\mathcal{J}^{(a,b)}$ radius of the classes S, S^*. Then, we determine the $\mathcal{J}^{(a,b)}$ radius of the classes $\Phi_{n,\alpha,\beta}(S)$ and $\Phi_{n,\alpha,\beta}(S^*)$. We next obtain growth theorems for the class $\Phi_{n,Q}(S_g^0)$, the class $\Phi_{n,Q}(S_g^*)$, the class $\Phi_{n,Q}(\mathcal{J}^{(a,b)})$ and the class $\Phi_{n,Q}(\mathcal{AJ}^{(a,b)})$. We give estimates of $\det D\Phi_{n,Q}(f)(z)$*, where f belongs to S_g^*, $\mathcal{J}^{(a,b)}$ or $\mathcal{AJ}^{(a,b)}$. Some distortion results along a vector of norm equal to* 1 *in $\mathbb{C}^n$ for certain subclasses of $\Phi_{n,Q}(S_g^*)$ will be given. Particular cases that derive from the previous mentioned growth and distortion results will be mentioned.*

In the last part, we prove that the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ preserve Janowski (almost) starlikeness with complex coefficients. Also, we obtain the expression of the Herglotz vector field associated with a particular Loewner chain $F(z,t) : \mathbb{B}^3 \times [0,\infty) \to \mathbb{C}^3$, where its first element is $\Psi_2(f)$ with $f \in S^0(\mathbb{B}^2)$. The Loewner chain $F(z,t)$ is mentioned in [53] (see the proof of Theorem 2.1*).*

The content of this thesis is divided in four chapters. We present a brief introduction of these chapters in the following.

In **Chapter 1**, we begin with some preliminary results regarding holomorphic functions in $\mathbb{C}$, respectively holomorphic functions and holomorphic mappings in $\mathbb{C}^n$. We present some basic properties of holomorphic functions in the case of one complex variable and, then, analyze if the extension of these results to higher dimensions, $n \geq 2$, remains valid. We next describe general results for the univalent functions in $\mathbb{C}$ and biholomorphic mappings in $\mathbb{C}^n$. We present the Carathédory class $\mathcal{P}$ in $\mathbb{C}$ (see [48], [102]) and its extension to n-dimensions, $n \geq 2$, the family $\mathcal{M}$ (see [96], [48]).

In another section we present certain subclasses of S. The first part is devoted to the class of univalent and normalized functions on U, denote by S. Then, we refer to the class of starlike and normalized functions on U with respect to the origin, denoted by S^*, the class of normalized convex functions on U, denoted by K, the class of normalized starlike functions of order α on U, denoted by S^*_α, the class of normalized spirallike functions of type γ on U, denoted by $\hat{S}_\gamma$, and the class of normalized almost starlike functions of order α on U, denoted by $\mathcal{A}S^*_\alpha$. We present analytical characterizations, growth, covering and distortion theorems, coefficients bounds for the previous mentioned subclasses. We continue in the next section with the generalization to higher dimensions, $n \geq 2$, of these subclasses of normalized univalent functions and investigate similar properties as in the case of one complex variable, which may or may not remain true.

In the last section we present Loewner chains in $\mathbb{C}$ and $\mathbb{C}^n$. We include general results regarding Loewner chains on U. We shall present the Loewner differential equation on U and, further, give an analytical characterization of some subclasses of normalized univalent functions through Loewner chains. In a separate part of this chapter, we mention general results regarding Loewner chains in $\mathbb{C}^n$. We describe Loewner chains and the Loewner differential equation in $\mathbb{C}^n$, $n \geq 2$. We give the characterization of some subclasses of $S(\mathbb{B}^n)$ through Loewner chains.

This chapter contains important and useful results for the forthcoming chapters. We intend to present only the statement of these results and omit their proofs.

In **Chapter 2**, we present the parametric representation on the unit disc U and on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. First, we describe the notion of a function with parametric representation on U. Then, we shall see that any function $f \in S$ admits parametric representation on U (see [102], [48]). Further, we present the class of mappings which admit parametric representation on $\mathbb{B}^n$, $S^0(\mathbb{B}^n)$. This class was described in [37]. We give growth and coefficient estimates (see [37], [73]) and state the compactness of this class (see [51]). We next continue with a subclass of the family $\mathcal{M}$, the set $\mathcal{M}_g$, where $g : U \to \mathbb{C}$ fulfills certain assumptions. The class $\mathcal{M}_g$ was considered by Graham, Hamada and Kohr in [37]. We present the class of mappings with g-parametric representation on $\mathbb{B}^n$, $S^0_g(\mathbb{B}^n)$. We shall consider growth and coefficient estimates for this class (see [37]). We describe the concept of a g-Loewner chain due to Graham, Hamada and Kohr [37].

Let g be a function described by (0.0.2). In the next section, we are concerned about some preservation results concerning the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ and a subclass of mappings with g-parametric representation. We give a brief introduction and some well-known properties of these extension operators. We are also concerned about certain radii problems regarding subclasses of biholomorphic mappings generated by these extension operators. The main results represented by Theorem 2.3.2 and Theorem 2.3.3 are included in the last section and state that the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ preserve g-parametric representation on $\mathbb{B}^n$, where g is defined by (0.0.2).

These original results have been obtained by the author of this thesis in [85, 86].

In **Chapter 3**, we present certain subclasses of $S(\mathbb{B}^n)$ which have geometric properties and admit g-parametric representation, where the function $g : U \to \mathbb{C}$ satisfies certain assumptions. We describe g-starlikeness, g-almost starlikeness of order $0 \leq \alpha < 1$ and g-spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on $\mathbb{B}^n$. We give the characterization in terms of g-Loewner chains of the mappings described by these concepts. We prove that these notions are conserved under the extension operator $\Phi_{n,\alpha,\beta}$ when g has the particular form (0.0.2). For the same function g, we show that $\Phi_{n,Q}$ preserves g-starlikeness and g-spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on $\mathbb{B}^n$. We state these properties in Theorem 3.1.10, Theorem 3.1.12, Theorem 3.1.13, respectively Theorem 3.1.15, Theorem 3.1.17. These results are original and have been obtained in [85, 86], except Theorem 3.1.13 which was obtained after the publication of the paper [85].

Let $a, b \in \mathbb{R}$ satisfying the condition $|1-a| < b \leq a$. We next study two subclasses of functions that admit g-parametric representation on U and have interesting geometric properties, namely the set of Janowski starlike functions on U, $\mathcal{J}^{(a,b)}$, respectively the set of Janowski almost starlike functions on U, $\mathcal{AJ}^{(a,b)}$. The set $\mathcal{J}^{(a,b)}$ was defined by Silverman [109] (see also [110]), respectively the set $\mathcal{AJ}^{(a,b)}$ was defined by Curt [21]. Then, we present their natural generalization to the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ due to Curt [21]. The results contained in Theorem 3.2.5, Theorem 3.2.6, Theorem 3.2.7 and Theorem 3.2.8 are original and show that the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ preserve the Janowski (almost) starlikeness on $\mathbb{B}^n$. These results have been obtained in [85, 86].

In a separate section, we investigate certain radii problems regarding the extension operator $\Phi_{n,\alpha,\beta}$ and Janowski starlikeness on the unit disc U. We give the $\mathcal{J}^{(a,b)}$ radius of the classes S, S^*, where $|1-a| < b \leq a$, in Theorem 3.3.3 and Theorem 3.3.5. Particular cases of Theorem 3.3.3 are Corollary 3.3.4 and Remark 3.3.6. We compute the $\mathcal{J}^{(a,b)}$ radius of the classes $\Phi_{n,\alpha,\beta}(S)$, $\Phi_{n,\alpha,\beta}(S^*)$ in Theorem 3.3.8, Theorem 3.3.9. Also, we deduce the radius of almost starlikeness of order α, with $\alpha \in (0,1)$, of the classes $\Phi_{n,\alpha,\beta}(S)$, $\Phi_{n,\alpha,\beta}(S^*)$ in Theorem 3.3.9. These results are original and are included in [85].

The last section is devoted to growth and distortion results for certain families of mappings with g-parametric representation generated under the operator $\Phi_{n,Q}$, where g is described by (0.0.2). First, we mention the growth result for the class $S_g^0(\mathbb{B}^n)$ due to Graham, Hamada and Kohr [37], which generalizes Theorem 2.3 in [73]. We next give growth theorems for the class $\Phi_{n,Q}(S_g^0)$ stated in Theorem 3.4.2, the class $\Phi_{n,Q}(S_g^*)$ stated in Corollary 3.4.3 and the classes $\Phi_{n,Q}(\mathcal{J}^{(a,b)})$, respectively $\Phi_{n,Q}(\mathcal{AJ}^{(a,b)})$, stated in Corollary 3.4.4, Corollary 3.4.5.

The next part provides distortion theorems for certain subclasses of $\Phi_{n,Q}(S_g^*)$, when g is described by (0.0.2). We give estimates for $\det D\Phi_{n,Q}(f)(z)$, where f belongs to S_g^*, $\mathcal{J}^{(a,b)}$ or $\mathcal{AJ}^{(a,b)}$ in Theorem 3.4.9, Corollary 3.4.10, Corollary 3.4.11. Particular cases of these results are stated in Corollary 3.4.12, Corollary 3.4.13. Some distortion results along a vector of norm 1 in $\mathbb{C}^n$ for certain subclasses of $\Phi_{n,Q}(S_g^*)$ are obtained in Theorem 3.4.14, Corollary 3.4.15, Corollary 3.4.16, and their consequences in Corollary 3.4.17 and Corollary 3.4.18.

The above results are original and are included in [86].

In **Chapter 4**, we consider the concept of g-parametric representation, g-Loewner chains and g-starlikeness where $g : U \to \mathbb{C}$ is univalent on U, $g(0) = 1$ and $\mathrm{Re}\, g(\zeta) > 0$, $\zeta \in U$ (see [44]). We state that g-parametric representation and g-starlikeness are conserved under the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ (see [44]). We give particular selections of this general function g that depend on two complex parameters A and B, which will serve as a connection between g-starlikeness and Janowski (almost) starlikeness with complex coefficients (see [22]). This type of starlikeness generalizes the Janowski (almost) starlikeness with real coefficients defined in [21]. We give a brief presentation of Janowski (almost) starlikeness with complex coefficients. Then we shall refer to the preservation of these notions under $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ in Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.2.4. These preservation properties generalize the results obtained in [85, 86], which regard Janowski classes with real coefficients. Also, we prove a useful result that we will use to prove some of the above properties and it is described by Remark 4.2.1.

In the final part of this chapter we obtain the expression of the Herglotz vector field associated with a particular Loewner chain $F(z,t) : \mathbb{B}^3 \times [0, \infty) \to \mathbb{C}^3$ where its first element is the image of a mapping $f \in S^0(\mathbb{B}^2)$ through the Pfaltzgraff-Suffridge extension operator Ψ_2. We refer to the Loewner chain $F(z,t)$ from the proof of Theorem 2.1 in [53] when $n = 2$. This result is included in Theorem 4.3.7. First, we present the Pfaltzgraff-Suffridge extension operator Ψ_n and mention important results concerning this operator.

The new results presented in this chapter have been obtained in [87], except Theorem 4.3.7.

We include below a list with the main results presented in this thesis:

- **Chapter 2**: Theorem 2.3.2, Theorem 2.3.3.
- **Chapter 3**: Theorem 3.1.10, Theorem 3.1.12, Theorem 3.1.13, Theorem 3.1.15, Theorem 3.1.17, Theorem 3.2.5, Theorem 3.2.6, Theorem 3.2.7, Theorem 3.2.8, Theorem 3.3.3, Theorem 3.3.5, Corollary 3.3.4, Remark 3.3.6, Theorem 3.3.8, Theorem 3.3.9, Theorem 3.4.2, Corollary 3.4.3, Corollary 3.4.4, Corollary 3.4.5, Theorem 3.4.9, Corollary 3.4.10, Corollary 3.4.11, Corollary 3.4.12, Corollary 3.4.13, Theorem 3.4.14, Corollary 3.4.15, Corollary 3.4.16, Corollary 3.4.17, Corollary 3.4.18.
- **Chapter 4**: Remark 4.2.1, Theorem 4.2.2, Theorem 4.2.3, Theorem 4.2.4, Theorem 4.3.7.

The new results presented in the content of this thesis are published in the following articles:

- **Manu, A.:** *Extension Operators Preserving Janowski Classes of Univalent Functions*, Taiwanese J. Math., **24**:1 (2020), 97 – 117, Impact Factor/2020: 1.136, Accession Number: WOS:000508232900007, DOI: 10.11650/tjm/190407
- **Manu, A.:** *The Muir extension operator and Janowski univalent functions*, Complex Var. Elliptic Equ., **65**:6 (2020), 897 – 919, Impact Factor/2020: 0.846, Accession Number: WOS:000476259700001, DOI: 10.1080/17476933.2019.1636788
- **Manu, A.:** *Extension operators and Janowski starlikeness with complex coefficients*, Stud. Univ. Babeş-Bolyai Math., submitted, ISSN: 2065-961x.

The new results presented in this thesis were communicated to the following conferences:

- October 15 – 18, 2021, 16th International Symposium on Geometric Function Theory and Applications (GFTA 2021), online, Lucian Blaga University, Sibiu, Romania; communication: *Roper-Suffridge extension operators and Janowski univalent functions.*
- October 22 – 24, 2020, Conferinţa Şcolilor Doctorale din Consorţiul Universitaria (CSDCU-MIF2020), Third Edition, online, Alexandru Ioan Cuza University, Iaşi, Romania; communication: *Roper-Suffridge extension operators and Janowski univalent functions.*

- June 25 – 27, 2018, The 5th Conference of PhD Students in Mathematics (CSM), University of Szeged, Bolyai Institute, Szeged, Hungary; communication: *Extension operators preserving Janowski classes of univalent functions.*

- June 14 – 16, 2018, International Conference on Mathematics and Computer Science (MACOS 2018), Transilvania University, Braşov, Romania; communication: *Extension operators preserving Janowski classes of univalent functions.*

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Chapter 1

Univalence in one and several complex variables

In this chapter, we present general results regarding holomorphic functions in $\mathbb{C}$, respectively holomorphic functions and holomorphic mappings in $\mathbb{C}^n$. First, we shall refer to some important results in the theory of holomorphic functions on $\mathbb{C}$. Next, we consider the extensions of these results to holomorphic functions in $\mathbb{C}^n$. In the case of holomorphic mappings in C^n, these results may not be valid.

We shall refer to basic properties of the univalent functions in $\mathbb{C}$ and biholomorphic mappings in $\mathbb{C}^n$. A fundamental result in the theory of univalent functions is the Riemann mapping theorem, which states the conformal equivalence of every simply connected domain in $\mathbb{C}$ that is a proper subset of $\mathbb{C}$ with the unit disc U. Poincaré H. [100] showed that the Euclidean unit ball in $\mathbb{C}^n$ is not biholomorphic equivalent with the unit polydisc in $\mathbb{C}^n$, although they are homeomorphic, which leads to the failure of Riemann mapping theorem in higher dimensions, $n \geq 2$.

In the next part, we present the Carathédory class $\mathcal{P}$ in $\mathbb{C}$ and its extension to n-dimensions, $n \geq 2$, the family $\mathcal{M}$. A key result in the theory of Loewner chains in n-dimensions was proving the compactness of the set $\mathcal{M}$, due to Graham, Hamada and Kohr [37].

Another part is dedicated to the study of certain subclasses of univalent functions on the unit disc U in $\mathbb{C}$. First, we present the class of univalent and normalized functions on U, denoted by S. Then, we shall refer to the class of normalized starlike function on U with respect to 0, denoted by S^*, the class of normalized convex functions on U, denoted by K, the class of normalized starlike functions of order α on U, denoted by S^*_α, the class of normalized spirallike functions of type γ on U, denoted by $\hat{S}_\gamma$, and the class of normalized almost starlike functions of order α on U, denoted by $\mathcal{A}S^*_\alpha$. We aim to give analytical characterizations and other important properties of these subclasses. Next, we shall consider the generalization to higher dimensions, $n \geq 2$, of these subclasses of normalized univalent functions and investigate similar properties as in the case of one complex variable, which may or may not be valid.

Let $S(\mathbb{B}^n)$ be the set of biholomorphic and normalized mappings on $\mathbb{B}^n$ for $n \geq 2$. The compactness of the class $S(\mathbb{B}^n)$ does no longer hold for $n \geq 2$ as in the case of the class S.

We are also concerned about the study of Loewner chains in $\mathbb{C}$ and $\mathbb{C}^n$. We include important results regarding Loewner chains on U. We shall present the Loewner differential equation on U and, further, give an analytical characterization of some subclasses of normalized univalent functions through Loewner chains. We next continue with the generalization of Loewner chains and the Loewner differential equation in $\mathbb{C}^n$, $n \geq 2$. We mention important results in the theory of Loewner chains in $\mathbb{C}^n$, including the characterization of some subclasses of $S(\mathbb{B}^n)$ through Loewner chains.

The principal sources used to document and prepare this chapter are [102], [66], [48], [76], [90].

1.1 Preliminaries

First, we give certain useful notations that we will use throughout the content of this thesis.

Let $\mathbb{C}$ be the complex plane. Let $a \in \mathbb{C}$ and $r > 0$. Let

$$U(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$$

be the disc of center a and radius r. Also, let $\overline{U}(a, r)$, respectively $\partial U(a, r)$, be the closure, respectively the boundary, of the disc $U(a, r)$. In the case $a = 0$, we use the notation U_r instead of $U(0, r)$, respectively U instead of $U(0, 1)$.

Let $n \in \mathbb{N} = 1, 2, 3 \ldots$, and let $\mathbb{C}^n = \{z = (z_1, z_2, \ldots, z_n) : z_i \in \mathbb{C}, i = 1, \ldots, n\}$ be the complex space equipped with the Euclidean inner product

$$\langle z, w\rangle = \sum_{i=1}^{n} z_i \overline{w}_i,$$

which provides the Euclidean norm $\|z\|^2 = \langle z, z\rangle$. Let $a \in \mathbb{C}^n$ and let $R > 0$. Let

$$\mathbb{B}^n(a, R) = \{z \in \mathbb{C}^n : \|z - a\| < R\}$$

be the open ball of radius R and center a. Let $\overline{\mathbb{B}^n}(a, R)$, respectively $\partial\mathbb{B}^n(a, R)$, be the closure, respectively the boundary, of the open ball $\mathbb{B}^n(a, R)$. If $a = 0$ then we denote the open ball $\mathbb{B}^n(0, R)$ by $\mathbb{B}^n_R$, respectively the open unit ball $\mathbb{B}^n(0, 1)$ by $\mathbb{B}^n$.

Let us consider the multiradius $r = (r_1, \ldots, r_n) \in \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$. Let

$$\mathbb{P}^n(z^0, r) = U(z^0_1, r_1) \times \cdots \times U(z^0_n, r_n)$$

be the open polydisc of center $z^0 = (z^0_1, \ldots, z^0_n) \in \mathbb{C}^n$ and multiradius r. If $r = (1, \ldots, 1)$ then we denote the unit polydisc $\mathbb{P}^n(0, r)$ by $\mathbb{P}^n$.

1.2 Holomorphic function theory in one and several complex variables

This section is dedicated to the presentation of holomorphic functions in $\mathbb{C}$ and $\mathbb{C}^n$, respectively the holomorphic mappings in $\mathbb{C}^n$. We begin by presenting the notion of holomorphic function in $\mathbb{C}$ and then we refer to the holomorphism in higher dimensions. In the first section, we include main results that have an important role in the theory of holomorphic function in $\mathbb{C}$. In a succeeding part, we shall present certain generalizations of these results in higher dimensions. The main sources used to prepare this section are [66], [76], [74], [77], and [106].

1.2.1 Holomorphic functions in $\mathbb{C}$

In the followings we present certain elementary properties of holomorphic functions on an open set in $\mathbb{C}$. Let Ω be an open set in $\mathbb{C}$. Then $H(\Omega)$ means the set of holomorphic functions defined on Ω with values in $\mathbb{C}$.

Assume that $0 \in \Omega$. Then a holomorphic function on Ω is *normalized* if $f(0) = f'(0) - 1 = 0$.

Two main results well known in the theory of the holomorphic functions are *open mapping theorem*, respectively *maximum(minimum) modulus theorem* (see [66], [76]). These properties play a main role in the development of the results presented in the next chapters. They state the followings:

Theorem 1.2.1. (Open mapping theorem) *Let Ω be a domain in $\mathbb{C}$ and let $f \in H(\Omega)$ be such that f is a nonconstant function on Ω. Then $f(\Omega)$ is also a domain in $\mathbb{C}$.*

The above result also holds for holomorphic functions from domains in $\mathbb{C}^n$ into $\mathbb{C}$, and for locally biholomorphic mappings defined on domains in $\mathbb{C}^n$ with values in $\mathbb{C}^n$, $n \geq 2$. (see e.g. [106]).

Theorem 1.2.2. (Maximum(minimum) modulus theorem) *Assume that $\Omega \subseteq \mathbb{C}$ is a domain and $f \in H(\Omega)$. If there exists $z_0 \in \Omega$ such that*

$$|f(z_0)| = max\{|f(z)| : z \in \Omega\} \ \Big(|f(z_0)| = min\{|f(z)| : z \in \Omega\}\Big),$$

then f is constant on Ω.

The maximum modulus theorem have various applications in the theory of holomorphic functions. One of them is known as the *Schwarz's lemma* (see [66], [76]). We present its statement below:

Corollary 1.2.3. (Schwarz's lemma) *Assume $f \in H(U)$ such that $f(0) = 0$ and $|f(z)| < 1$, $z \in U$. Then $|f(z)| \leq |z|$ for $z \in U$ and $|f'(0)| \leq 1$. Moreover, if there exists some $w \in U\backslash\{0\}$ such that $|f(w)| = |w|$, or if $|f'(0)| = 1$, then there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $f(z) = \lambda z$, $z \in U$.*

We next present two notions that will be useful in the followings sections. We shall describe the locally uniformly bounded family, respectively the normal family on an open set of $\mathbb{C}$ (see e.g. [66], [76]).

Definition 1.2.4. Assume that $\Omega \subseteq \mathbb{C}$ is an open set. Let $\mathcal{F} \subseteq H(\Omega)$. We say that the family $\mathcal{F}$ is *locally uniformly bounded* if for each compact subset of Ω, K, there is $M_K > 0$ (the constant could depend on K) with the property that for all $f \in \mathcal{F}$ we have that $\|f|_K\| \leq M_K$. Here, we consider the norm $\|f|_K\|$ to be $\|f|_K\| = \max\{|f(z)| : z \in K\}$.

Definition 1.2.5. Assume that $\Omega \subseteq \mathbb{C}$ is an open set and let $\mathcal{F} \subseteq H(\Omega)$. We say that the family $\mathcal{F}$ is *normal* if each sequence $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$ has a subsequence which converges locally uniformly on Ω.

It was proved that the notions of locally uniformly bounded family, respectively of normal family are equivalent. The result was obtained by Montel (see e.g. [66], [76]). It is important to note that the result remains valid when $n \geq 2$ (see [93]).

Theorem 1.2.6. *Assume that $\Omega \subseteq \mathbb{C}$ is an open set and $\mathcal{F} \subseteq H(\Omega)$. Then the family $\mathcal{F}$ is normal if and only if $\mathcal{F}$ is locally uniformly bounded.*

Another useful result that derives from Montel's theorem is stated in following corollary (see [66], [76]). The result remains true when $n \geq 2$ (see [93]).

Corollary 1.2.7. *If $\Omega \subseteq \mathbb{C}$ is an open set and $\mathcal{F} \subseteq H(\Omega)$, then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is locally uniformly bounded and closed.*

1.2.2 Holomorphic functions in $\mathbb{C}^n$. Holomorphic mappings in $\mathbb{C}^n$

In this part, we propose to study holomorphic functions and holomorphic mappings on a open set in $\mathbb{C}^n$. We mainly focus on the generalizations to n-dimensions, $n \geq 2$, of the results presented in the previous part for holomorphic functions on an domain in $\mathbb{C}$. Further, we assume that $m > 1$.

First, we recall the notion of a holomorphic function in $\mathbb{C}^n$. (see e.g. [48]).

Definition 1.2.8. Let $\Omega \subseteq \mathbb{C}^n$ be an open set and let $f : \Omega \to \mathbb{C}$. If f is continuous on Ω and holomorphic in each variable separately, then we say that f is *holomorphic* on Ω.

In view of the Hartogs result, the condition of continuity in Definition 1.2.8 is not needed. Therefore, we deduce that any holomorphic function in each variable separately is holomorphic on the entire open set Ω (see [10], [77]). Let $H(\Omega, \mathbb{C})$ denote the set of holomorphic functions on the open set $\Omega \subseteq \mathbb{C}^n$ into $\mathbb{C}$.

In the followings, we shall highlight basic properties of the holomorphic functions on a domain in C^n. First, we give the generalization of the *open mapping theorem* for holomorphic functions a open set in $\mathbb{C}^n$ (see [93]).

Theorem 1.2.9. *If $\Omega \subseteq \mathbb{C}^n$ is a domain and $f : \Omega \to \mathbb{C}$ is a nonconstant holomorphic function then $f(\Omega)$ is a domain in $\mathbb{C}$.*

Since the locally uniformly bounded families, respectively the normal families are defined similarly in $\mathbb{C}^n$, $n \geq 2$, we skip their presentation in higher dimensions (see e.g. [93], [106]).

We will proceed further with the generalization of Montel's theorem in n-dimensions, $n \geq 2$ (see e.g. [93]).

Theorem 1.2.10. *Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $\mathcal{F} \subseteq H(\Omega, \mathbb{C})$. Then $\mathcal{F}$ is a normal family if and only if $\mathcal{F}$ is locally uniformly bounded.*

We next give a characterization of compact families of holomorphic functions on a open set in $\mathbb{C}^n$ (see e.g. [93], [106]).

Corollary 1.2.11. *Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $\mathcal{F} \subseteq H(\Omega, \mathbb{C})$. Then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is locally uniformly bounded and closed.*

Next, we want to present the notion of a holomorphic mapping from open subsets of C^n into $\mathbb{C}^m$ (see e.g. [48]).

Definition 1.2.12. Assume that $\Omega \subseteq \mathbb{C}^n$ is an open set and $f : \Omega \to \mathbb{C}^m$. We say that the mapping $f = (f_1, \ldots, f_m)$ is *holomorphic* if each of its components, f_j, $j = \overline{1, m}$, is an holomorphic function from Ω into $\mathbb{C}$.

Let $H(\Omega, \mathbb{C}^m)$ be the set of holomorphic mappings from the open set $\Omega \subseteq \mathbb{C}^n$ into $\mathbb{C}^m$. For $n = m$, we use the notation $H(\Omega)$ instead of $H(\Omega, \mathbb{C}^n)$.

Assuming that Ω is a domain in $\mathbb{C}^n$ and that $f \in H(\Omega, \mathbb{C}^m)$, then the differential $Df(z)$ at $z \in \Omega$ is a complex linear mapping from $\mathbb{C}^n$ into $\mathbb{C}^m$ and is linked to the complex Jacobian matrix

$$Df(z) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_n} \end{bmatrix}.$$

In the case $n = m$, the determinant of the matrix $Df(z)$, $z \in \Omega$, denoted by $J_f(z)$, represents the complex Jacobian determinant of f at z. We mean by a *normalized* mapping on Ω a mapping f with the properties: $f(0) = 0$ and $Df(0) = I_n$, when $0 \in \Omega$ and I_n is the identity matrix $n \times n$.

Further, we shall present certain properties that are satisfies by the holomorphic mappings in $\mathbb{C}^n$. We saw in the begining of this part that certain results obtained for holomorphic functions in $\mathbb{C}$ still remain true for holomorphic functions in $\mathbb{C}^n$. In the case of holomorphic mappings in $\mathbb{C}^n$, these results may not be valid. We illustrate in the following remark one example of this kind.

Remark 1.2.13. Let $\Omega \subseteq \mathbb{C}^n$ be a domain. The generalization of the Theorem 1.2.9 does not hold in the case of a mapping from $H(\Omega, \mathbb{C}^m)$, where $m > 1$ (see e.g. [106]). However, this result remains true for locally biholomorphic mappings from Ω into $\mathbb{C}^n$(see [48]).

Note that the maximum(minimum) modulus theorem can be extended to holomorphic mappings in the complex space $\mathbb{C}^n$ equipped with an arbitrary norm $\|\cdot\|$ (see [48], [93]).

Theorem 1.2.14. *Assume that $\Omega \subseteq \mathbb{C}^n$ is a domain and let $f \in H(\Omega, \mathbb{C}^m)$. If there exists a point $z_0 \in \Omega$ such that*

$$\|f(z_0)\| = max\{\|f(z)\| : z \in \Omega\} \ \Big(\|f(z_0)\| = min\{\|f(z)\| : z \in \Omega\}\Big),$$

then $\|f(z)\|$ is constant on Ω.

Remark 1.2.15. Assume the conditions in the Theorem 1.2.14 hold. If the norm $\|\cdot\|$ in the space $\mathbb{C}^m$ is the Euclidean norm then the mapping f is constant on Ω.

In the followings, we present the analogous of Schwarz's lemma for holomorphic mappings on $\mathbb{B}^n$. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{C}^n$ (see e.g. [48], [93]).

Corollary 1.2.16. *Let $f : \mathbb{B}^n \to \mathbb{C}^m$ be such that $f \in H(\mathbb{B}^n, \mathbb{C}^m)$, $f(0) = 0$ and $\|f(z)\| < 1$, $z \in \mathbb{B}^n$. Then $\|f(z)\| \leq \|z\|$, $z \in \mathbb{B}^n$, and $\|Df(0)\| \leq 1$. Moreover, if there exists $z_0 \in \mathbb{B}^n \setminus \{0\}$ such that $\|f(z_0)\| = \|z_0\|$, then $\|f(\lambda z_0)\| = \|\lambda z_0\|$, for all $\lambda \in \mathbb{C}$, $|\lambda| < 1/\|z_0\|$.*

1.3 Univalence in one and several complex variables

In this section we include important results in the theory of the univalent function in $\mathbb{C}$ and $\mathbb{C}^n$. First, we introduce the notion of univalent function in $\mathbb{C}$ and present certain well known results regarding univalent functions (see e.g. [89], [90], [102]). In the following part, we define the notions of univalent mapping in C^n and biholomorphic mapping in C^n and, next, show the connection between these notions. This part will be preceded by certain important results regarding biholomorphic mappings in C^n (see e.g. [48], [77], [93]).

1.3.1 Univalent functions on $\mathbb{C}$

This part is allotted to the study of univalent function of one complex variables. Well known results regarding univalence on the complex plane will be mentioned.

The following definition presents the notion of an univalent function in the complex plane $\mathbb{C}$ (see e.g. [90], [102]).

Definition 1.3.1. Assume that $\Omega \subseteq \mathbb{C}$ is a domain. We say that the function $f : \Omega \to \mathbb{C}$ is *univalent* on Ω if f is injective and holomorphic on Ω.

We next present some suggestive examples of univalent functions that play an important role in the univalent functions theory (see e.g. [90], [102]).

Example 1.3.2. (i) A well known example of a univalent function that plays an important role in many extremal problems is described by $k(z) = \frac{z}{(1-z)^2}$, $z \in U$ (*Koebe function*), which extends U onto $\mathbb{C}\backslash(-\infty, -1/4]$.

(ii) Let $\theta \in \mathbb{R}$. The rotation of the Koebe function is still a univalent function

$$k_\theta(z) = \frac{z}{(1 - e^{i\theta} z)^2},\ z \in U.$$

In the next statement, we provide a necessary condition for univalence. However, this requirement is not a sufficient condition for univalence (see [48]).

Theorem 1.3.3. *Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be a univalent function on Ω. Then $f'(z) \neq 0$, $z \in \Omega$.*

Still, one have a criterion for global univalence for holomorphic functions. The result is due to Alexander [1], Noshiro [95], Warschawski [117] and Wolff [118] and is presented in the following theorem.

Theorem 1.3.4. *Let $\Omega \subseteq \mathbb{C}$ be a convex domain and let $f \in H(\Omega)$. If $\mathrm{Re} f'(z) > 0$, $z \in \Omega$, then f is univalent on Ω.*

We next recall the notions of a conformal function, conformal equivalence of domains in $\mathbb{C}$ and then point up certain observations regarding these notions(see e.g. [25], [66], [76], [102]).

Definition 1.3.5. Assume that Ω_1 and Ω_2 are domains in $\mathbb{C}$. The domains Ω_1 and Ω_2 are *conformally equivalent* if there is a function $f : \Omega_1 \to \Omega_2$ satisfying the conditions: f is univalent on Ω_1 and $f(\Omega_1) = \Omega_2$. A function f with these properties is called a *conformal mapping*. Moreover, if $\Omega_1 = \Omega_2$, then f is a *conformal automorphism* of Ω_1 (see e.g. [66]).

We present a main result in the theory of univalent functions in $\mathbb{C}$, known as the *Riemann mapping theorem* (see e.g. [48], [66]). However, Riemann mapping theorem is not true in $\mathbb{C}^n$, for $n \geq 2$ (see [93], [106]).

Theorem 1.3.6. *Assume that $\Omega \subsetneq \mathbb{C}$ is a simply connected domain. Then Ω and the unit disc U are conformally equivalent. In addition, if $z_0 \in \Omega$, there is a unique conformal mapping $f : \Omega \to U$ satisfying the conditions: $f(z_0) = 0$, $f'(z_0) > 0$.*

We present in the next statement a consequence of the Riemann mapping theorem (see e.g. [66], [76]).

Corollary 1.3.7. *Any two simply connected domains in $\mathbb{C}$, that are different by the entire complex plane $\mathbb{C}$, are conformally equivalent.*

1.3.2 Biholomorphic mappings in $\mathbb{C}^n$

In this part, we propose to study the biholomorphic mappings in $\mathbb{C}^n$, as well as certain properties that are satisfied by these mappings.

In the followings, we introduce the notion of univalence, respectively the notion of biholomorphy in n-dimensions, $n \geq 2$ (see [48], [77], [93]).

Definition 1.3.8. Assume that $\Omega \subseteq \mathbb{C}^n$ is a domain. Let $f : \Omega \to \mathbb{C}^n$.

(i) If f is holomorphic and injective on Ω, then we say that f is *univalent* on Ω.

(ii) If $f \in H(\Omega)$ and the inverse mapping f^{-1} exists and is holomorphic on the domain $\Omega' = f(\Omega)$, then we say that f is *biholomorphic*. In such a case, we say that the domains Ω and Ω' are *biholomorphically equivalent*. Moreover, if $\Omega = \Omega'$ then the mapping f is called an *automorphism* of Ω.

In higher dimensions, $n \geq 2$, the equivalence between biholomorphy and univalence still holds (see e.g. [93], [106]), but for infinite dimensional complex Banach spaces, is no longer valid (see [112]).

Theorem 1.3.9. *Assume that $\Omega \subseteq \mathbb{C}^n$ is a domain. Let $f : \Omega \to \mathbb{C}^n$. Then f is biholomorphic from Ω onto $f(\Omega)$ if and only if f is univalent on Ω.*

The fact that the $\mathbb{B}^n$ is not biholomorphic equivalent with the unit polydisc in $\mathbb{C}^n$ leads to the failure of Riemann mapping theorem in higher dimensions, $n \geq 2$ (see [93], [106]). The following result is due to Poincaré [100] and illustrates this context.

Theorem 1.3.10. *Let $n \geq 2$. Then $\mathbb{B}^n$ is not biholomorphically equivalent to $\mathbb{P}^n$. But, these domains are homeomorphic.*

Now, we describe the notion of locally univalence on a domain from $\mathbb{C}^n$(see e.g. [48]).

Definition 1.3.11. Assume that $\Omega \subseteq \mathbb{C}^n$ is a domain. Let $f \in H(\Omega)$. The mapping f is called *locally biholomorphic* on Ω if for every $z \in \Omega$ one can find an open and connected set $V \subset \Omega$, which is a neighborhood of z, such that $f|_V : V \to f(V)$ is biholomorphic mapping.

Remark 1.3.12. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $f \in H(\Omega)$. Then $J_f(z) \neq 0$, $z \in \Omega$, if and only if f is locally biholomorphic on Ω (see e.g. [48], [106]).

1.4 The Carathéodory class in $\mathbb{C}$ and $\mathbb{C}^n$.

In this section, we shall present the Carathéodory class in $\mathbb{C}$ and its generalization to higher dimensions and give important properties of these classes. The main sources used to prepare this section are [90] and [102] in the case of one complex variable, respectively [48], [37] and [96] in the case of several complex variables.

1.4.1 Holomorphic functions with positive real part

In the followings, we shall introduce the notion of subordination in $\mathbb{C}$ (see e.g. [90]). First, we need the to describe the class of Schwarz functions on U, $\mathcal{V}$. We have that $\varphi \in \mathcal{V}$ if and only if φ is holomorphic on the unit disc U, $\varphi(0) = 0$ and $|\varphi(z)| < 1$, $z \in U$.

Definition 1.4.1. Let $f, g \in H(U)$. We say that f is *subordinate* to g, and write $f \prec g$, if there is a Schwarz function φ such that $f(z) = g(\varphi(z))$, $z \in U$.

We have the following characterization of subordination (see e.g. [90], [102]):

Theorem 1.4.2. *Assume that $f, g : U \to \mathbb{C}$ are holomorphic functions on U. Also, g is univalent on U. The subordination condition $f \prec g$ is equivalent to $f(U) \subseteq g(U)$ and $f(0) = g(0)$.*

Assume that the conditions in the previous theorem hold and $f(U) \subseteq g(U)$, $f(0) - g(0) = 0$. Then, the condition $f(U_R) \subseteq g(U_R)$ holds for any $R \in (0, 1)$. This result is known as the *subordination principle.*

Let be the following class of holomorphic functions on U (see e.g. [48], [90], [102]):

$$\mathcal{P} = \{p \in H(U) : p(0) = 1, \mathrm{Re}p(z) > 0, z \in U\}.$$

The class $\mathcal{P}$ is known as *Carathéodory class* and has an major contribution in the description of some subclasses of univalent functions on U and in the theory of Loewner chains.

Next, we present growth and distortion results for the Carathéodory class $\mathcal{P}$ (see [90]).

Theorem 1.4.3. *Let $p \in \mathcal{P}$. Then the following estimates are true:*

$$\frac{1 - |z|}{1 + |z|} \leq \mathrm{Re}p(z) \leq |p(z)| \leq \frac{1 + |z|}{1 - |z|}, \ z \in U,$$
$$|p'(z)| \leq \frac{2\mathrm{Re}p(z)}{1 - |z|^2} \leq \frac{2}{(1 - |z|)^2}, \ z \in U.$$

These estimates are sharp.

The function $p(z) = \frac{1+\lambda z}{1-\lambda z}$, $z \in U$, for some complex number λ with $|\lambda| = 1$, yields equality in the above estimates.

Any function $p \in \mathcal{P}$ can be written as: $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in U$. We next give the following estimates for the coefficients p_k, $k \in \mathbb{N}$, (see e.g. [90]):

Theorem 1.4.4. *Assume that $p \in \mathcal{P}$. Let p_k, $k \in \mathbb{N}$ denote the coefficients of the power series expansion of p. Then the estimates $|p_k| \leq 2$, $k \geq 1$ hold and are sharp.*

The function $p(z) = \frac{1+\lambda z}{1-\lambda z}$, $z \in U$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, is extremal for the above estimates (see e.g. [90]).

1.4.2 Holomorphic mappings in the class $\mathcal{M}$

The notion of subordination can be extended to holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{C}^n$ (see e.g. [48]). First, we say that a mapping φ defined on $\mathbb{B}^n$ is a *Schwarz mapping* if $\varphi \in H(\mathbb{B}^n)$ and $\|\varphi(z)\| \leq \|z\|$, $z \in \mathbb{B}^n$.

Definition 1.4.5. Assume that $f, g \in H(\mathbb{B}^n)$. Then f is *subordinate* to g ($f \prec g$), if there exists a Schwarz mapping φ such that $f(z) = g(\varphi(z))$, $z \in \mathbb{B}^n$.

The condition of subordination from the above definition can be described as follows (see [48]):

Theorem 1.4.6. *Let $f, g \in H(\mathbb{B}^n)$. If the mapping g is biholomorphic on $\mathbb{B}^n$ then $f \prec g$ if and only if $f(\mathbb{B}^n) \subseteq g(\mathbb{B}^n)$ and $f(0) = g(0)$.*

The following set represents the Carathéodory class on $\mathbb{C}^n$ (see [96], [114] and, also [48], [74])

$$\mathcal{M} = \{h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n, \mathrm{Re}\langle h(z), z\rangle > 0, z \in \mathbb{B}^n \backslash \{0\}\}. \tag{1.4.1}$$

This set has a significant role in the theory of Loewner chains in $\mathbb{C}^n$ and in the description of some classes of mappings, which are biholomorphic on $\mathbb{B}^n$ (see [48]).

It can be seen that, when $n = 1$, h is a mapping in the class $\mathcal{M}$ if and only if $p \in \mathcal{P}$, where $h(z) = zp(z)$, $z \in U$. This points out that the class $\mathcal{M}$ extends the class $\mathcal{P}$ in n-dimensions, $n \geq 2$.

A simple example of a mapping from $\mathcal{M}$ is the mapping described by $h(z) = (z_1p_1(z_1), \dots, z_np_n(z_n))$, $z = (z_1, \dots, z_n) \in \mathbb{B}^n$, where $p_i \in \mathcal{P}$, $i = 1, \dots, n$.

We next give a growth result due to Pfaltzgraff [96] (see also [48]).

Theorem 1.4.7. *Any mapping $h \in \mathcal{M}$ satisfies the estimates*

$$\|z\|^2 \frac{1 - \|z\|}{1 + \|z\|} \leq \mathrm{Re}\langle h(z), z\rangle \leq \|z\|^2 \frac{1 + \|z\|}{1 - \|z\|}, \ z \in \mathbb{B}^n. \tag{1.4.2}$$

The inequalities are sharp.

Graham et al. [37] obtained a stronger upper bound than that in (1.4.2).

Theorem 1.4.8. *If $h \in \mathcal{M}$ then*

$$\|z\| \frac{1 - \|z\|}{1 + \|z\|} \leq \|h(z)\| \leq \frac{4\|z\|}{(1 - \|z\|)^2}, \ z \in \mathbb{B}^n. \tag{1.4.3}$$

The compactness of the class $\mathcal{M}$ was proved by Graham et al. [37] (see also [61]).

Corollary 1.4.9. *The class $\mathcal{M}$ is compact in $H(\mathbb{B}^n)$.*

1.5 Subclasses of univalent functions on unit disc

In this section we shall set forth certain subclasses of univalent functions on the unit disc U. We start by presenting the class S of normalized univalent functions on U. In this section, we shall also refer to the set of normalized starlike function on U with respect to 0, S^*, the set of normalized convex functions on U, K, the set of normalized starlike functions of order α on U, S^*_α , the set of normalized spirallike functions of type γ on U, $\hat{S}_\gamma$, and the set of normalized almost starlike functions of order α on U, $\mathcal{A}S^*_\alpha$. Our main focus will be to give analytical and geometric characterizations of the previous mentioned classes. The main bibliographic sources used to prepare this section are [25], [48], [90] and [102].

1.5.1 The class S

We study univalence on the unit disc U, as it suffices to investigate only on U, according to Riemann mapping theorem.

For this purpose, we introduce the class

$$S = \{f \in H(U) : f \text{ univalent}, \ f(0) = f'(0) - 1 = 0\}.$$

Note that all the functions that are mentioned in Examples 1.3.2 are univalent and normalized.

A function $f \in S$ can be written as follows:

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \ldots, \quad z \in U. \tag{1.5.1}$$

An estimate for second coefficient a_2 in the above power series expansion was obtained by Bieberbach [6].

Theorem 1.5.1. *If $f \in S$ has the power series expansion* (1.5.1)*, then $|a_2| \leq 2$. Equality occurs if and only if f is a rotation of the Koebe function.*

On the basis that the coefficients a_k, $k = 2, 3, \ldots$ satisfy the relation $|a_k| = k$, $k = 2, 3, \ldots$, for a rotation of the Koebe function, let be the next conjecture formulated by Bieberbach [6]:

Conjecture 1.5.2. (Bieberbach's conjecture) *If $f \in S$ has the power series expansion* (1.5.1) *then*

$$|a_k| \leq k, \;\; k = 2, 3, \ldots \tag{1.5.2}$$

Equality occurs in (1.5.2) *if and only if f is a rotation of the Koebe function.*

The above conjecture was given in 1916 and was solved years later by de Branges [24] (1985).

Another important consequence of Theorem 1.5.1 is the *Koebe distortion theorem* given in (1.5.4) (see [6]). Starting from this distortion theorem, one may deduce the estimates (1.5.3), (1.5.5) (see e.g [48]).

Theorem 1.5.3. *If $f \in S$ then:*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \;\; \forall z \in U, \tag{1.5.3}$$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}, \;\; \forall z \in U, \tag{1.5.4}$$

and

$$\frac{1-|z|}{1+|z|} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|}, \;\; \forall z \in U. \tag{1.5.5}$$

These estimates are sharp. Equality in each of the above relations holds if and only if f is a rotation of the Koebe function.

The following statement can be seen as covering theorem for the class S and is an application of Theorem 1.5.1. It is well known as *Koebe* 1/4*-covering theorem* for the class S (see [48], [90]).

Theorem 1.5.4. *Let $f \in S$. Then $f(U) \supseteq U_{1/4}$.*

The following theorem states the compactness of the class S, which was proved using the upper bound from the estimates (1.5.3) (see [90], [48]).

Corollary 1.5.5. *The class S is a compact subset of $H(U)$.*

1.5.2 The class S^*

An important subclass of S is the set of starlike and normalized functions on U, S^*. Different results and properties related to starlike functions may be found in [102], [25], [35], [48], [90].

The notion of a starlike function on U was introduced by Alexander [1].

Definition 1.5.6. Assume $f \in H(U)$, $f(0) = 0$. The function f is called *starlike* on U if f is univalent on U and $f(U)$ is a starlike domain with respect to 0 (origin).

The notion of starlikeness can be described in an analytical way, as stated below (see e.g. [25], [102], [48]).

Theorem 1.5.7. *Assume that $f \in H(U)$, $f(0) = 0$. Then $f \in S^*$ if and only if $f'(0) \neq 0$ and the following condition holds:*

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \; z \in U.$$

Remark 1.5.8. The growth and distortion results from Theorem 1.5.3 remain true and sharp for the class S^*. The class S^* is compact in $H(U)$. Also, the Koebe constant for S^* is 1/4 ([83], [94], [48]). Moreover, Bieberbach conjecture remains true for the class S^* (see [83], [94]).

1.5.3 The class S^*_α

Another important subclass of S is the set of normalized starlike functions of order α on U, S^*_α.

The concept of order α starlikeness was introduced by Robertson [107].

Definition 1.5.9. Let $0 \leq \alpha < 1$ and $f \in H(U)$. The function f is called *starlike of order α* on U if $f(0) = 0$, $f'(0) = 1$ and

$$\mathrm{Re}\Big[\frac{zf'(z)}{f(z)}\Big] > \alpha, \; z \in U.$$

Note that any function from S^*_α is also starlike on U and the set S^*_0 becomes S^*.

The following statement shows a way to generate a function from S^*_α using a starlike function on U and vice versa (see [48]).

Theorem 1.5.10. *Let $0 \leq \alpha < 1$. Then $f \in S^*_\alpha$ if and only if the following function*

$$g(z) = z\Big[\frac{f(z)}{z}\Big]^{\frac{1}{1-\alpha}}, \; z \in U,$$

belongs to S^*, where $\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\alpha}}\Big|_{z=0} = 1$.

Next, we give a growth theorem for functions in S^*_α (see [48], [35], [90]).

Theorem 1.5.11. *Let $f \in S^*_\alpha$ and let $0 \leq \alpha < 1$. Then:*

$$\frac{|z|}{(1+|z|)^{2(1-\alpha)}} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^{2(1-\alpha)}}.$$

These inequalities are sharp.

1.5.4 The class $\mathcal{A}S^*_\alpha$

The notion of almost starlikeness of order α was introduced first in the case of complex Banach spaces (see [119]).

Definition 1.5.12. Assume that $0 \leq \alpha < 1$. Let $f \in H(U)$ be such that f is normalized. The function f is called *almost starlike of order* α on U if

$$\mathrm{Re}\left[\frac{f(z)}{zf'(z)}\right] > \alpha, \; z \in U. \tag{1.5.6}$$

Let $\mathcal{A}S^*_\alpha$ be the set of normalized almost starlike functions of order α on U. Note that $\mathcal{A}S^*_\alpha \subseteq S^*$.

1.5.5 The class K

We give below the definition of a convex function on U (see e.g. [48]).

Definition 1.5.13. Let $f \in H(U)$. We say that f is *convex* if f is univalent on U and $f(U)$ is a convex domain.

We denote the set of convex and normalized functions on U by K. Also, we have the following inclusions $K \subset S^* \subset S$.

Let $f \in K$. Then

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \ldots, \; z \in U. \tag{1.5.7}$$

Also, the convexity can be described in an analytical way on U (see [25], [48]):

Theorem 1.5.14. *Assume that $f \in H(U)$. Then $f \in K$ if and only if the next conditions hold:*

$$\mathrm{Re}\Big[1 + \frac{zf''(z)}{f'(z)}\Big] > 0, \; z \in U,$$

and $f'(0) \neq 0$.

Next, we shall give growth and distortion results for the set K (see e.g. [48]).

Theorem 1.5.15. *Let $f \in K$. The following relations hold:*

$$\frac{|z|}{1+|z|} \leq |f(z)| \leq \frac{|z|}{1-|z|}, \; z \in U,$$
$$\frac{1}{(1+|z|)^2} \leq |f'(z)| \leq \frac{1}{(1-|z|)^2}, \; z \in U.$$

These inequalities are sharp and equality yields at $z \neq 0$ for $f(z) = \frac{z}{1-\lambda z}$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

In view of the growth theorem and since K is closed, we deduce the compactness of the class K (see e.g. [90]).

Next result provides sharp estimates of the convex and normalized function coefficients([83]).

Theorem 1.5.16. *Let $f \in K$ and let (1.5.7) be its power series expansion. Then $|a_k| \leq 1$, $k = 2, 3, \ldots$. These inequalities are sharp and equality occurs if and only if $f(z) = \frac{z}{1-\lambda z}$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$.*

Next, we state the duality theorem obtained by Alexander [1], which establishes a bound between convex and starlike functions on U.

Theorem 1.5.17. *Let $f \in H(U)$, $f(0) = 0$. Then the relation $f \in K$ is equivalent to $g(z) = zf'(z) \in S^*$.*

Note that the duality theorem is not true for $n \geq 2$ (see [112] and also [48]).

Next, we highlight the bound between convexity and starlikeness of order 1/2 due to Marx and Strohhäcker (see e.g. [48], [90]). This result is true in n-dimensions, $n \geq 2$ (see [18], [70]).

Theorem 1.5.18. *If $f \in K$ then $f \in S^*_{1/2}$. This result is sharp.*

1.5.6 The class $\hat{S}_\gamma$

Next, we consider the class of normalized spirallike functions on U, introduced by Špaček [111].

Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $z_0 \in \mathbb{C}\setminus\{0\}$. The curve given by

$$z = z_0 e^{-e^{-i\gamma}t},\ t \in \mathbb{R},$$

is a *logarithmic* $\gamma-$*spiral* (or $\gamma-$*spiral*).

First we define the notion of a spirallike domain (see [111]; see also [48]).

Definition 1.5.19. Let $\Omega \subset \mathbb{C}$ be a domain and $0 \in \Omega$. We say that Ω is *spirallike of type* γ, $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if for every $z \in \Omega$, $z \neq 0$, the arc of γ-spiral connecting z with the origin lies entirely in Ω.

Now we can define the notion of type γ spirallike function on U (see [111]).

Definition 1.5.20. Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Assume that $f \in H(U)$, $f(0) = 0$.

1. The function f is called *spirallike of type* γ on the unit disc if f is univalent on U and $f(U)$ is a spirallike domain of type γ.

2. The function f is called *spirallike* if there is $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that f is spirallike of type γ.

Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $\hat{S}_\gamma$ be the class of normalized spirallike functions of type γ on U. In this case, $\hat{S}_\gamma \subset S^*$ and $\hat{S}_0 = S^*$.

The following theorem, due to Špaček [111], presents a necessary and sufficient condition for spirallikeness of type γ on U (see also [48]).

Theorem 1.5.21. *Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Assume that $f \in H(U)$ with $f(0) = 0$, $f'(0) \neq 0$. Then the function f is spirallike of type γ if and only if*

$$\mathrm{Re}\Big[e^{i\gamma}\ \frac{zf'(z)}{f(z)}\Big] > 0,\ z \in U.$$

The next characterization of spirallikeness can be used to establish to connection between classes S^* and $\hat{S}_\gamma$ (see e.g. [48], [90]).

Theorem 1.5.22. *Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\theta = e^{-i\gamma}\cos\gamma$. The condition $f \in \hat{S}_\gamma$ is equivalent to the fact that there is a function $g \in S^*$ with the property*

$$f(z) = z\Big[\frac{g(z)}{z}\Big]^\theta,\ z \in U,$$

where $\left[\frac{g(z)}{z}\right]^\theta\Big|_{z=0} = 1.$

1.5.7 Radius problems regarding univalent functions on U

In this part we present certain radii problems for some subclasses of S. The reader may consult [35], [48] and [90] for more details on this topic.

Definition 1.5.23. Let $\mathcal{F}$ be a family of functions from the class S. Let $\mathcal{P}$ be a certain property that we investigate on the family $\mathcal{F}$. We denote by $r(\mathcal{P},\mathcal{F})$ the *radius for the property* $\mathcal{P}$ in the set $\mathcal{F}$, which represents the largest radius $r>0$ such that each function in $\mathcal{F}$ has the property $\mathcal{P}$ on the open disc of radius r centered at origin.

Let $r(S^*,S)$ be the radius of starlikeness of S and let $r(K,S)$ be the radius of convexity of S.

The radius $r(S^*,S)$ was determined by Nenvalinna and Campbell and the radius $r(S^*,S)$ was obtained by Grunsky (see e.g. [35]).

Theorem 1.5.24. *1.* $r(S^*,S)=\tanh\frac{\pi}{4}=\frac{e^{\pi/2}-1}{e^{\pi/2}+1}$.

2. $r(K,S)=r(K,S^*)=2-\sqrt{3}$.

1.6 Subclasses of biholomorphic mappings in $\mathbb{C}^n$

In this section, we present certain families of biholomorphic mappings on the Euclidean unit ball of $\mathbb{C}^n$ that have geometric properties. We shall present the set of starlike mappings, the set of starlike mappings of order α, the set of almost starlike mappings of order α, the set of convex mappings and the set of spirallike mappings of type γ. Moreover, we shall give analytical and geometric properties of these classes.

The main bibliographic sources used for preparing this section are [48], [74], [112] and [12].

In the followings, let $S(\mathbb{B}^n)$ be the set of normalized biholomorphic mappings on $\mathbb{B}^n$ in $\mathbb{C}^n$. Also, let $\mathcal{L}S_n(\mathbb{B}^n)$ be the set of normalized locally biholomorphic mappings on $\mathbb{B}^n$. For $n=1$, we use the notation $\mathcal{L}S$ instead of $\mathcal{L}S_1(\mathbb{B}^1)$.

1.6.1 The class $S^*(\mathbb{B}^n)$

This part is dedicated to the study of the set of normalized starlike mappings on $\mathbb{B}^n$, $S^*(\mathbb{B}^n)$. We shall refer to the extensions of some properties of starlike functions on U to $\mathbb{B}^n$ în $\mathbb{C}^n$.

We give the definition of starlikeness on $\mathbb{B}^n$ (see [48], [74]).

Definition 1.6.1. Let $f\in H(\mathbb{B}^n)$. We say that f is *starlike* if f is biholomorphic on $\mathbb{B}^n$, $f(0)=0$, and $f(\mathbb{B}^n)$ is a starlike domain with respect to zero.

We denote the class of starlike and normalized mappings on $\mathbb{B}^n$ by $S^*(\mathbb{B}^n)$. For $n=1$, $S^*(\mathbb{B}^1)$ becomes S^*.

An analytical characterization of starlikeness on $\mathbb{B}^n$ have been obtained by Matsuno [88]. Other extensions of this result have been given through time. We can mention here the extension to the unit ball of a Banach space obtained by Gurganus [54] and Suffridge [114] and to unit polydisc of $\mathbb{C}^n$ due to Suffridge [113].

Theorem 1.6.2. *Assume that $f\in\mathcal{L}S(\mathbb{B}^n)$, $f(0)=0$. Then $f\in S^*(\mathbb{B}^n)$ if and only if the mapping $h(z)=[Df(z)]^{-1}f(z)$ belongs to the class $\mathcal{M}$, i.e.*

$$\mathrm{Re}\langle[Df(z)]^{-1}f(z),z\rangle>0,\ z\in\mathbb{B}^n\backslash\{0\}. \tag{1.6.1}$$

The following statement presents a growth result for the mappings in $S^*(\mathbb{B}^n)$, due to Kubicka and Poreda [78], and Barnard, FitzGerald and Gong [5]. Other generalizations of this result can be found in [32, 33], [48].

Theorem 1.6.3. *If $f \in S^*(\mathbb{B}^n)$ then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \ z \in \mathbb{B}^n.$$

These estimates are sharp. Consequently, $f(\mathbb{B}^n) \supseteq \mathbb{B}^n_{1/4}$.

1.6.2 The class $S^*_\alpha(\mathbb{B}^n)$

In this part we shall refer to the class of order α starlike mappings on $\mathbb{B}^n$ in $\mathbb{C}^n$, $S^*_\alpha(\mathbb{B}^n)$.

First, the concept of order α starlikeness on $\mathbb{B}^n$ was introduced by Kohr [70] (see also [18]).

Definition 1.6.4. Assume that $0 \leq \alpha < 1$ and $f \in \mathcal{L}S(\mathbb{B}^n)$. We say that f is *starlike of order α* if

$$\mathrm{Re}\Big[\frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z\rangle}\Big] > \alpha, \ z \in \mathbb{B}^n \setminus \{0\}. \tag{1.6.2}$$

Note that $S^*_0(\mathbb{B}^n) = S^*(\mathbb{B}^n)$ and $S^*_\alpha(\mathbb{B}^n) \subseteq S^*(\mathbb{B}^n)$.

We next present a growth result for the set $S^*_\alpha(\mathbb{B}^n)$ (see [70], [18]).

Theorem 1.6.5. *Let $f \in S^*_\alpha(\mathbb{B}^n)$, where $0 \leq \alpha < 1$. Then the following relation holds:*

$$\frac{\|z\|}{(1+\|z\|)^{2(1-\alpha)}} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^{2(1-\alpha)}}, \ z \in \mathbb{B}^n.$$

These inequalities are sharp.

1.6.3 The class $\mathcal{A}S^*_\alpha(\mathbb{B}^n)$

In the followings, we describe the notion of order α almost starlikeness on $\mathbb{B}^n$ in $\mathbb{C}^n$. This concept was first introduced on the unit ball of a Banach space by Xu and Liu [119].

Definition 1.6.6. Assume that $0 \leq \alpha < 1$. Let $f \in \mathcal{L}S(\mathbb{B}^n)$. We say that f is *almost starlike of order α* if

$$\mathrm{Re}\Big[\frac{\langle [Df(z)]^{-1} f(z), z\rangle}{\|z\|^2}\Big] > \alpha, \ z \in \mathbb{B}^n \setminus \{0\}. \tag{1.6.3}$$

We denote the set of almost starlike mappings of order α on $\mathbb{B}^n$ by $\mathcal{A}S^*_\alpha(\mathbb{B}^n)$.

In [12], T. Chirilă introduced the concept of almost starlikeness of order α and type γ, where $0 \leq \alpha < 1$ and $0 \leq \gamma < 1$.

Definition 1.6.7. Assume that $0 \leq \alpha < 1$, $0 \leq \gamma < 1$. Let $f \in \mathcal{L}S(\mathbb{B}^n)$. We say that f is *almost starlike of order α and type γ* if

$$\mathrm{Re}\Big(1\Big/\Big[\frac{1}{(1-\alpha)\|z\|^2}\langle [Df(z)]^{-1} f(z), z\rangle - \frac{\alpha}{1-\alpha}\Big]\Big) > \gamma, \ z \in \mathbb{B}^n \setminus \{0\}.$$

We denote the set of mappings which are almost starlike of order α and type γ by $\mathcal{A}S^*_{\alpha,\gamma}(\mathbb{B}^n)$. For $n = 1$, we use the notation $\mathcal{A}S^*_{\alpha,\gamma}$ instead of $\mathcal{A}S^*_{\alpha,\gamma}(\mathbb{B}^1)$.

The following equivalence holds: $f \in \mathcal{A}S^*_{\alpha,0}(\mathbb{B}^n)$ if and only if $f \in \mathcal{A}S^*_\alpha(\mathbb{B}^n)$. Also, any mapping from the set $\mathcal{A}S^*_{\alpha,\gamma}(\mathbb{B}^n)$ is also from $\mathcal{A}S^*_\alpha(\mathbb{B}^n) \subseteq S^*(\mathbb{B}^n)$.

1.6.4 The class $K(\mathbb{B}^n)$

Next, we describe a convex mapping on $\mathbb{B}^n$ (see [48], [70]).

Definition 1.6.8. We say that $f \in H(\mathbb{B}^n)$ is *convex* if f is biholomorphic on $\mathbb{B}^n$ and the domain $f(\mathbb{B}^n)$ is convex.

Let $K(\mathbb{B}^n)$ be the set of normalized convex mappings on the unit ball of $\mathbb{B}^n$.

The following result states the analytical characterization of a convex mapping on $\mathbb{B}^n$. The result is due to Kikuchi [69]. An equivalent result was obtained by Gong, Wang and Yu in [34].

Theorem 1.6.9. *Let $f \in \mathcal{L}S(\mathbb{B}^n)$. Then the condition $f \in K(\mathbb{B}^n)$ is equivalent to*

$$1 - \mathrm{Re}\langle [Df(z)]^{-1} D^2 f(z)(v, v), z\rangle > 0, \tag{1.6.4}$$

for every $z \in \mathbb{B}^n$ and $v \in \mathbb{C}^n$ with $\|v\| = 1$ and $\mathrm{Re}\langle z, v\rangle = 0$.

In the following remark, we highlight important observations regarding convexity on $\mathbb{B}^n$, and, also, give an example of a convex mapping on $\mathbb{B}^n$.

Remark 1.6.10. For $n \geq 2$, it is more difficult to build a convex mapping on $\mathbb{B}^n$ than for $n = 1$. Let $f(z) = (f_1, \dots, f_n)$, where $f_i : U \to \mathbb{C}$, $i = \overline{1, n}$ are convex functions on U. However, f is not necessary convex on $\mathbb{B}^n$, for $n \geq 2$ (see [34]). But, in particular, the mapping described by

$$f(z) = \Big(\frac{z_1}{1 - z_1}, \dots, \frac{z_n}{1 - z_1}\Big), \ z = (z_1, \dots, z_n) \in \mathbb{B}^n$$

is convex.

The following statement represents a growth result for the set $K(\mathbb{B}^n)$ and was obtained by Suffridge [115], FitzGerald and Thomas [31] and Liu [79].

Theorem 1.6.11. *Let $f \in K(\mathbb{B}^n)$. Then*

$$\frac{\|z\|}{1 + \|z\|} \leq \|f(z)\| \leq \frac{\|z\|}{1 - \|z\|}, \ z \in \mathbb{B}^n.$$

The above inequalities are sharp.

The Marx-Strohhäcker theorem for one complex variable, presented in Theorem 1.5.18, was extended to n-dimensions, $n \geq 2$, by Kohr[70] and Curt [18].

Theorem 1.6.12. $K(\mathbb{B}^n) \subseteq S^*_{1/2}(\mathbb{B}^n)$. *The result is sharp.*

1.6.5 The class $\hat{S}_\gamma(\mathbb{B}^n)$

Next, we shall describe the concept of spirallikeness on $\mathbb{B}^n$ in $\mathbb{C}^n$. Gurganus K. [54] defined this concept with respect to a normal linear operator, whose eigenvalues have positive real part. Also, Suffridge [112] extended this concept to a complex Banach space.

Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ and $t \geq 0$. Also, let

$$m(A) = \min\{\mathrm{Re}\langle A(z), z\rangle : \|z\| = 1\},$$
$$e^{-tA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k A^k.$$

We next state the notion of spirallikeness on $\mathbb{B}^n$ by following the definition given by Suffridge [112].

Definition 1.6.13. Let $f \in S(\mathbb{B}^n)$. Also, assume $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $m(A) > 0$. We say that f is *spirallike relative to* A if $e^{-tA} f(\mathbb{B}^n) \subseteq f(\mathbb{B}^n)$ for all $t \geq 0$.

Let be the liniar operator $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $m(A) > 0$. In the next statement, we present an analytical characterization of spirallikeness relative to the operator A. This characterization was given by Suffridge [112] (see also [54]).

Theorem 1.6.14. *Let $f \in \mathcal{L}S(\mathbb{B}^n)$. Then the mapping f is spirallike relative to A if and only if*

$$\operatorname{Re}\langle [Df(z)]^{-1} A f(z), z\rangle > 0, \ z \in \mathbb{B}^n \backslash \{0\}. \tag{1.6.5}$$

In particular, if A is the operator $e^{-i\gamma} I_n$, where $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then one obtains the class $\hat{S}_\gamma(\mathbb{B}^n)$ of *spirallike mappings of type* γ, which was considered by Hamada and Kohr [60]. Then the condition (1.6.5) becomes $\operatorname{Re}(e^{-i\gamma}\langle [Df(z)]^{-1} f(z), z\rangle) > 0$, $z \in \mathbb{B}^n \backslash \{0\}$.

Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\alpha \in [0, 1)$. An extension of the notion of spirallikeness of type γ is the concept of spirallikeness of type γ and order α, introduced by Liu and Liu [82] and Chirilă [11].

Definition 1.6.15. Assume that $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\alpha \in [0, 1)$. Let $f \in \mathcal{L}S(\mathbb{B}^n)$. We say that f is *spirallike of type* γ and order α if the following condition is satisfied

$$\operatorname{Re}\Big[\frac{1}{(1 - i\tan\alpha)\langle [Df(z)]^{-1} f(z), z\rangle / \|z\|^2 + i\tan\alpha}\Big] > \gamma, \ z \in \mathbb{B}^n \backslash \{0\}.$$

The set of mappings that satisfy the above definition is denoted by $\hat{S}_{\gamma,\alpha}(\mathbb{B}^n)$. For $n = 1$, we use the notation $\hat{S}_{\gamma,\alpha}$ instead of $\hat{S}_{\gamma,\alpha}(\mathbb{B}^1)$.

Any mapping from the set $\hat{S}_{\gamma,\alpha}(\mathbb{B}^n)$ is also from $\hat{S}_\gamma(\mathbb{B}^n) \subseteq S^*(\mathbb{B}^n)$. Also, $\hat{S}_{\gamma,0}(\mathbb{B}^n) = \hat{S}_\gamma(\mathbb{B}^n)$.

1.7 Loewner chains in one and several complex variables

In this section we study the Loewner chains in $\mathbb{C}$ and $\mathbb{C}^n$. We begin with general results in the theory of Loewner chains on the unit disc U. We give the Loewner differential equation on U and next continue with the analytical characterization of some subclasses of normalized univalent functions using the method of Loewner chains. More results in this direction are presented in [102], [48], [90] and also in [25].

Next, we shall consider the generalization of Loewner chains and of the Loewner differential equation in $\mathbb{C}^n$. An important application of Loewner chains in n dimensions is represented by the characterizations of some subclasses of $S(\mathbb{B}^n)$ using Loewner chains. An remarkable contribution in the theory of Loewner chains în $\mathbb{C}^n$ is the introduction of the family $S^0(\mathbb{B}^n)$ of mappings which admit parametric representation on $\mathbb{B}^n$ in $\mathbb{C}^n$ due to Graham, Hamada and Kohr [37]. Graham et al. [37] proved the strict inclusion $S^0(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n)$. An outstanding contribution to the theory of Loewner chains in $\mathbb{C}^n$ has been done by G. Kohr and her collaborators in a series of valuable publications starting with [37], [51], [23], [52], [26].

Throughout this section, we shall use the following abbreviations:

Notation 1.7.1. We use the following shorter notations: $\mathcal{LC}$ for a Loewner chain, $\mathcal{LDE}$ for the Loewner differential equation, $\mathcal{PR}$ for the parametric representation.

1.7.1 The theory of Loewner chains in one complex variable

In this part, we present a brief introduction of $\mathcal{LC}s$ on the unit disc U. We shall first recall the definition of an univalent subordination chain, followed by the definition of a $\mathcal{LC}$.

The principal sources used to prepare this part are [102], [48], [90].

1.7.1.1 General results regarding Loewner chains in $\mathbb{C}$

We begin by presenting preliminary notions regarding the theory of $\mathcal{LC}s$ on U.

We first need to give the definition of a univalent subordination chain (see e.g. [48]).

Definition 1.7.2. The function $f : U \times [0, \infty) \to \mathbb{C}$ is said to be a *univalent subordination chain* if the followings conditions hold:

(i) $f(\cdot, t)$ is univalent on U,

(ii) $f(0, t) = 0$, for $t \geq 0$,

(iii) $f(\cdot, s) \prec f(\cdot, t)$, whenever $0 \leq s \leq t < \infty$.

If, in addition, $f'(0, t) = e^t$, for all $t \geq 0$, then f is a $\mathcal{LC}$.

In the above definition one have used the notation $f'(z, t)$ instead of $\frac{\partial f}{\partial z}(z, t)$.

Assume that $f(z, t)$ is a $\mathcal{LC}$. In this case, there exists a unique $v = v(z, s, t)$ such that v is an univalent Schwarz function and satisfies the following property (see e.g. [48]):

$$f(z, s) = f(v(z, s, t), t), \tag{1.7.1}$$

where $z \in U$ and $0 \leq s \leq t < \infty$. The function v is called the *transition function* associated to f.

Next, we present an important result in the theory of $\mathcal{LC}s$ (see [48]).

Theorem 1.7.3. *Assume that $p : U \times [0, \infty) \to \mathbb{C}$ is a function which satisfies the following properties:*

(i) $p(\cdot, t) \in \mathcal{P}$, for all $t \geq 0$,

(ii) $p(z, \cdot)$ is measurable on $[0, \infty)$, for all $z \in U$.

Under this conditions and for all $z \in U$, $s \geq 0$, the Cauchy problem

$$\begin{cases} \frac{\partial v}{\partial t} & = -vp(v, t), \ \text{a.e. } t \geq s, \\ v(z, s, s) & = z \end{cases} \tag{1.7.2}$$

admits a unique solution $v(z, s, \cdot)$, which is locally absolutely continuous and $v'(0, s, t) = e^{s-t}$.

Moreover, if $s \geq 0$ and $z \in U$ then $v(z, s, \cdot)$ is Lipschitz continuous on $[s, \infty)$, locally uniformly with respect to z. Also, for all $t \geq s$, $v(\cdot, s, t)$ is a univalent Schwarz function.

Furthermore, for every $s \geq 0$, the following limit exists:

$$f(z, s) := \lim_{t \to \infty} e^t v(z, s, t) \tag{1.7.3}$$

locally uniformly on U, where $f(z, t)$ is a $\mathcal{LC}$ satisfying the following differential equation

$$\frac{\partial f}{\partial t}(z, t) = zf'(z, t)p(z, t), \ \text{a.e. } t \geq 0, \ \forall z \in U. \tag{1.7.4}$$

The differential equation (1.7.4) is known as $\mathcal{LDE}$ (or *Loewner-Kufarev differential equation*).

We next present the characterization of the $\mathcal{LC}s$ due to Pommerenke [101] (see also [48]).

Theorem 1.7.4. *Assume $f : U \times [0,\infty) \to \mathbb{C}$ such that $f(0,t) = 0$, $f'(0,t) = e^t$, $t \geq 0$. Then f is a $\mathcal{LC}$ if and only if the following requirements are satisfied:*

(i) There exist $r \in (0,1)$ and $M > 0$ such that $f(\cdot,t) \in H(U(0,r))$ (where $U(0,r) = \{z \in \mathbb{C} : |z| < r\}$) for all $t \geq 0$, $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in U(0,r)$, and $|f(z,t)| \leq Me^t$, for all $z \in U(0,r)$, $t \geq 0$.

(ii) There exists $p : U \times [0,\infty) \to \mathbb{C}$ satisfying the conditions (i) and (ii) from Theorem 1.7.3 such that

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \ \text{a.e. } t \geq 0, \ z \in U(0,r).$$

1.7.1.2 Loewner chains and subclasses of univalent functions on U

In the followings, we give the characterization of some subclasses of S using $\mathcal{LC}s$ (see [48]).

First, we present the characterization of the functions from $\hat{S}_\gamma$ through $\mathcal{LC}s$. Since the set $\hat{S}_0$ coincides with the set S^*, then we can give a characterization of starlike functions on U using $\mathcal{LC}s$ (see [102], [48]).

Theorem 1.7.5. *Assume that $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let $f \in H(U)$ be a normalized function. Then the relation $f \in \hat{S}_\gamma$ on U is equivalent to the property that the function*

$$f(z,t) = e^{(1-ia)t} f(e^{iat}z), \ z \in U, t \geq 0,$$

is a $\mathcal{LC}$, with $a = \tan\gamma$. Particularly, the relation $f \in S^*$ is equivalent to the property that the function

$$f(z,t) = e^t f(z), \ z \in U, \ t \geq 0,$$

is a $\mathcal{LC}$.

We next state the characterization of almost starlikeness of order α using $\mathcal{LC}s$ (see [119]).

Theorem 1.7.6. *Assume that $0 \leq \alpha < 1$. Let $f \in H(U)$ be a normalized function. Then the relation $f \in \mathcal{A}S^*_\alpha$ is equivalent to the property that the function*

$$f(z,t) = e^{\frac{1}{1-\alpha}t} f(e^{\frac{\alpha}{\alpha-1}t} z), \ z \in U, \ t \geq 0$$

is a $\mathcal{LC}$.

The statement below gives a characterization of convex function on U in terms of $\mathcal{LC}s$ (see [48], [102]).

Theorem 1.7.7. *Let $f \in H(U)$ be a normalized function. Then the relation $f \in K$ is equivalent to the property that the function*

$$f(z,t) = f(z) + (e^t - 1)zf'(z), \ z \in U, \ t \geq 0$$

is a $\mathcal{LC}$.

1.7.2 The theory of Loewner chains in several complex variables

In this part, we shall consider the generalization of the $\mathcal{LC}s$ and $\mathcal{LDE}$ to n dimensions. Further, we shall provide important results regarding $\mathcal{LC}s$ in higher dimensions ($n \geq 2$). Also, we are concerned about various applications of $\mathcal{LC}s$ in n dimensions. We include here characterizations of some subclasses of $S(\mathbb{B}^n)$.

The main sources used to prepare this section are [48], [20], [37], [51].

1.7.2.1 General results regarding Loewner chains on several complex variables

We first give the definition of a $\mathcal{LC}$ in higher dimensions ($n \geq 2$) (see [96], [48]).

Definition 1.7.8. We say that $f : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ is a *univalent subordination chain* if the next statements are true:

(i) $f(\cdot, t)$ is biholomorphic on $\mathbb{B}^n$,

(ii) $f(0, t) = 0$ for $t \geq 0$, and

(iii) $f(\cdot, s) \prec f(\cdot, t)$, $0 \leq s \leq t < \infty$.

Furthermore, if $Df(0, t) = e^t I_n$, with I_n the $n \times n$-identity matrix and $t \geq 0$, then the mapping $f(z, t)$ is called a $\mathcal{LC}$.

The subordination condition from the above definition is equivalent to the following statement (see [96], [48]): there exists a unique biholomorphic mapping $v = v(z, s, t)$, called the *transition mapping*, such that $\|v(z, s, t)\| \leq \|z\|$, $z \in \mathbb{B}^n$, and the following relation holds

$$f(z, s) = f(v(z, s, t), t),\ z \in \mathbb{B}^n, 0 \leq s \leq t.$$

The following result was obtained by Pfaltzgraff [96]. The result was also studied on Banach spaces by Poreda [105].

Theorem 1.7.9. [96] *Let $h : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ be such that the next requirements hold:*

(i) $h(\cdot, t) \in \mathcal{M}$, $t \geq 0$,

(ii) $h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in \mathbb{B}^n$.

Then the next Cauchy problem admits a unique locally absolutely continuous solution $v(t)$ ($= v(z, s, t) = e^{s-t}z + \dots$):

$$\begin{cases} \frac{\partial v}{\partial t} & = -h(v, t),\ a.e.\ t \geq s \\ v(s) & = z, \end{cases} \tag{1.7.5}$$

for every $s \geq 0$ and $z \in \mathbb{B}^n$. In addition, v is a univalent Schwarz mapping on $\mathbb{B}^n$ with respect to the first variable and, for fixed $s \geq 0$ and $z \in \mathbb{B}^n$, it is a Lipschitz function of $t \geq s$ locally uniformly with respect to z.

The mapping h satisfying the requirements (i), (ii) of the Theorem 1.7.9 is known as a *Herglotz vector field.* The differential equation from (1.7.5) is known as *Loewner (ordinary) differential equation* associated to h.

Next, we give the following important result obtained by Poreda[105], Hamada and Kohr [61] (see also [48]). The following remarkable result presents a way to obtain a $\mathcal{LC}$ through its transition mapping, which is the solution of the Cauchy problem (1.7.5).

Theorem 1.7.10. *Assume that h is a Herglotz vector field and v is the solution of the Cauchy problem* (1.7.5). *Then the following limit exists*

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s),$$

locally uniformly on $\mathbb{B}^n$ for every $s \geq 0$. Moreover, $f(\cdot,s)$ is univalent on $\mathbb{B}^n$ and $f(z,s) = f(v(z,s,t),t)$ for all $z \in \mathbb{B}^n$, $0 \leq s \leq t < \infty$. Thus $f(z,t)$ is a $\mathcal{LC}$ such that the family $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is normal on $\mathbb{B}^n$ and $f(z,\cdot)$ is a locally Lipschitz mapping on $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$. Also, f satisfies the following equation:

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \ \text{a.e. } t \geq 0, \forall z \in \mathbb{B}^n. \tag{1.7.6}$$

The differential equation (1.7.6) is known as the *(generalized) $\mathcal{LDE}$* associated to h.

The following statement represents a principal result in studying the theory of $\mathcal{LC}s$. The result was obtained by Pfaltzgraff [96] and by Poreda [105] on Banach spaces. Other important connected contributions have been obtained by Hamada and Kohr in [61].

Theorem 1.7.11. *Assume that h is a Herglotz vector field. Also, assume that $f = f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ satisfies the following properties: $f(\cdot,t) \in H(\mathbb{B}^n)$, $f(0,t) = 0$, $Df(0,t) = e^t I_n$, where $t \geq 0$, $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$ and the $\mathcal{LDE}$* (1.7.6) *holds.*

Let $\{t_m\}_{m\in\mathbb{N}}$ be an increasing sequence of strictly positive real numbers with $t_m \to \infty$ and $\lim_{m\to\infty} e^{-t_m} f(z,t_m) = F(z)$ locally uniformly on $\mathbb{B}^n$. Let v be the solution of the Cauchy problem (1.7.5) *for all $z \in \mathbb{B}^n$. Then $f(z,t)$ is a $\mathcal{LC}$ and*

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$$

locally uniformly on $\mathbb{B}^n$ for all $s \geq 0$.

The next statement shows that any $\mathcal{LC}$ on $\mathbb{B}^n$ satisfies the $\mathcal{LDE}$ (1.7.6) and is due to Graham et al. [37] (see also [23], [48]).

Theorem 1.7.12. *Assume that f is a $\mathcal{LC}$ on $\mathbb{B}^n$. Then there exists a unique Herglotz vector field h such that the $\mathcal{LDE}$* (1.7.6) *is satisfied by f.*

1.7.2.2 Loewner chains and subclasses of biholomorphic mappings on $\mathbb{B}^n$

The following results represents characterizations of certain subclasses of $S(\mathbb{B}^n)$ by using $\mathcal{LC}s$.

The next result gives a characterization through $\mathcal{LC}s$, obtained by Hamada and Kohr [60], for the mappings from the set $\hat{S}_\gamma(\mathbb{B}^n)$. In particular, this result also presents a characterization for starlike mappings on $\mathbb{B}^n$ obtained by Pfaltzgraff and Suffridge [98].

Theorem 1.7.13. *Let $f \in \mathcal{L}S(\mathbb{B}^n)$ and let $\gamma \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the relation $f \in \hat{S}_\gamma(\mathbb{B}^n)$ is equivalent to the property that the mapping*

$$f(z,t) = e^{(1-ia)t} f(e^{iat}z), \ z \in \mathbb{B}^n, \ t \geq 0,$$

is a $\mathcal{LC}$, with $a = \tan\gamma$.

Particularly, the relation $f \in S^*(\mathbb{B}^n)$ is equivalent to the property that the mapping $f(z,t) = e^t f(z)$ is a $\mathcal{LC}$.

Any almost starlike mapping of order α on $\mathbb{B}^n$ can be characterized using $\mathcal{LC}s$. The following characterization was obtained in the case of Banach spaces by Xu and Liu [119].

Theorem 1.7.14. *Let $\alpha \in [0,1)$. Assume that $f \in \mathcal{L}S(\mathbb{B}^n)$. Then the relation $f \in \mathcal{A}S^*_\alpha(\mathbb{B}^n)$ is equivalent to the property that the mapping $f(z,t) = e^{\frac{t}{1-\alpha}} f(e^{\frac{\alpha t}{\alpha-1}} z)$, $z \in \mathbb{B}^n$, $t \geq 0$, is a $\mathcal{LC}$.*

Chapter 2

Extension operators that preserve g-parametric representation on $\mathbb{B}^n$

In this chapter, we study the parametric representation on the unit disc U and on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. Then, we recall that any function $f \in S$ admits parametric representation on U (see [102], [48]). Further, we present the class $S^0(\mathbb{B}^n)$ of mappings that have parametric representation on the Euclidean unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, introduced by Graham, Hamada and Kohr [37]. We remark that the set $S^0(\mathbb{B}^n)$ is compact as shown in [51] and, thus, is a proper subset of $S(\mathbb{B}^n)$, which is the natural generalization of the class S to higher dimensions. In the next part, let g be a function satisfying the requirements of Assumption 2.1.6. We shall present the class of holomorphic mappings $\mathcal{M}_g$ introduced by Graham et. al in [37]. We next present the class $S_g^0(\mathbb{B}^n)$, which includes the mappings with g-parametric representation on $\mathbb{B}^n$ (see [37]). In addition, we shall give the definition of a g-Loewner chain due to Graham, Hamada and Kohr [37].

In another part of this chapter, we present some preservation results concerning the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ and the subclass of mappings with g-parametric representation, where g is described by $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, with $-1 \leq B < A \leq 1$. These preservation results represent the novelty of this chapter. We give a short introduction and some well-known properties of these extension operators. In the last section of this chapter, we shall prove that $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ preserve g-parametric representation on $\mathbb{B}^n$, where g is described as above. These results are due to Manu [85, 86].

The main sources used for preparing this chapter are [102], [108], [36], [47], [46], [51], [92], [63], [75], [11], [12].

Throughout this chapter, we use the shorter notations from Notation 1.7.1. Thus, the following abbreviations are used: $\mathcal{LC}$ for a Loewner chain, $\mathcal{LDE}$ for the Loewner differential equation and $\mathcal{PR}$ for the parametric representation. In addition, the abbreviation g-$\mathcal{LC}$, respectively g-$\mathcal{PR}$, is used for a g-Loewner chain, respectively for g-parametric representation.

2.1 Loewner chains and parametric representations in one and higher dimensions

We start with a short introduction of univalent functions that have $\mathcal{PR}$ on the unit disc. We shall also refer to mappings which admit $\mathcal{PR}$ on $\mathbb{B}^n$. Further, we will introduce the notion of g-$\mathcal{PR}$ on $\mathbb{B}^n$.

2.1.1 Normalized univalent functions with parametric representation on U

We first give the definition of a univalent function on U that has $\mathcal{PR}$ (see [102]).

Definition 2.1.1. A normalized holomorphic function f on U has $\mathcal{PR}$ if there exists a $\mathcal{LC}$, $f(z,t) : U \times [0,\infty) \to \mathbb{C}$, such that $f(z,0) = f(z)$.

The next statement shows that every function from the class S has $\mathcal{PR}$ on U (see [102]). Still, the property does not hold true for the family $S(\mathbb{B}^n)$ for $n \geq 2$, as we shall see in the next section (see [37]).

Theorem 2.1.2. *If $f \in S$ then f has $\mathcal{PR}$.*

2.1.2 Normalized univalent mappings with parametric representation on $\mathbb{B}^n$

This part addresses the class of mappings which admit $\mathcal{PR}$ on $\mathbb{B}^n$ in $\mathbb{C}^n$, which have been studied by Kohr in [73]. Generalizations of the $\mathcal{PR}$ with respect to an arbitrary norm were regarded by Graham, Hamada and Kohr in [37]. We next continue with the definition of g-$\mathcal{PR}$ introduced by Graham et al. in [37], which is a natural generalization of the notion of $\mathcal{PR}$.

We now present the definition of $\mathcal{PR}$ on $\mathbb{B}^n$ in $\mathbb{C}^n$ (see [37]).

Definition 2.1.3. A mapping $f \in S(\mathbb{B}^n)$ has $\mathcal{PR}$ if there exists a $\mathcal{LC}$, $f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$, with the property that the family $\{e^{-t}f(z,t)\}_{t\geq 0}$ is normal on $\mathbb{B}^n$ and $f(z) = f(z,0)$.

We denote the set of mappings which admit $\mathcal{PR}$ by $S^0(\mathbb{B}^n)$.

Graham et al. [37] proved the strict inclusion $S^0(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n)$. Also, the authors showed that the families $S^*(\mathbb{B}^n)$, $\hat{S}_\gamma(\mathbb{B}^n)$ are subclasses of $S(\mathbb{B}^n)$ which admit $\mathcal{PR}$ on $\mathbb{B}^n$.

We next give a growth and a covering result for the family $S^0(\mathbb{B}^n)$ due to Graham, Hamada and Kohr [37], where the considered norm is arbitrary (see also [73]). Moreover, this result is not true for the class $S(\mathbb{B}^n)$ (see [48]).

Theorem 2.1.4. *Let $f \in S^0(\mathbb{B}^n)$. Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2},\ z \in \mathbb{B}^n.$$

The above inequalities are sharp. Thus, $f(\mathbb{B}^n) \supseteq \mathbb{B}^n_{1/4}$, where $\mathbb{B}^n_{1/4} = \{z \in \mathbb{C}^n : \|z\| < 1/4\}$.

We next state the compactness of the class $S^0(\mathbb{B}^n)$ on $H(\mathbb{B}^n)$ due to Graham, Kohr and Kohr [51].

Corollary 2.1.5. *The family $S^0(\mathbb{B}^n)$ is compact on $H(\mathbb{B}^n)$.*

2.1.3 Mappings that admit g-parametric representation on $\mathbb{B}^n$

We next introduce the g-$\mathcal{PR}$ on the unit ball in $\mathbb{C}^n$, where the function g satisfies the properties mentioned in the following assumption (see [37]).

Assumption 2.1.6. Assume that $g : U \to \mathbb{C}$ is an univalent function on U, which satisfies the following properties: $g(0) = 1$, $g(\overline{\zeta}) = \overline{g(\zeta)}$, Re $g(\zeta) > 0$, $\zeta \in U$, and, for $r \in (0, 1)$, the next relations hold:

$$\min_{|\zeta|=r} \text{Re } g(\zeta) = \min\{g(r), g(-r)\},$$
$$\max_{|\zeta|=r} \text{Re } g(\zeta) = \max\{g(r), g(-r)\}.$$

In the followings, we consider that g is a function satisfying Assumption 2.1.6.

Next, let $\mathcal{M}_g$ be the following nonempty subset of the Carathéodory class $\mathcal{M}$ given by the following definition:

Definition 2.1.7.

$$\mathcal{M}_g = \left\{ h \in H(\mathbb{B}^n) : h \text{ normalized }, \langle h(z), \frac{z}{\|z\|^2} \rangle \in g(U), z \in \mathbb{B}^n \right\}.$$

We choose the branch $\langle h(z), \frac{z}{\|z\|^2} \rangle|_{z=0} = 1$. This class was defined by Graham, Hamada and Kohr [37]. It is immediate that $id_{\mathbb{B}^n} \in \mathcal{M}_g$ (therefore, $\mathcal{M}_g$ is a nonempty set) and $\mathcal{M}_g = \mathcal{M}$ when $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$.

The notion of a g-$\mathcal{LC}$ was introduced by Graham et al. in [37].

Definition 2.1.8. Assume that $f(z, t) : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$. Then f is a g-$\mathcal{LC}$ if the next requirements are fulfilled:

(i) $f(z, t)$ is a $\mathcal{LC}$,

(ii) the family $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is normal on $\mathbb{B}^n$,

(iii) the mapping h from the following $\mathcal{LDE}$

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t), \ \forall z \in \mathbb{B}^n, \text{ a.e. } t \geq 0, \tag{2.1.1}$$

has the property $h(\cdot, t) \in \mathcal{M}_g$, for almost every $t \geq 0$.

The next statement gives the definition of a mapping with g-$\mathcal{PR}$ on $\mathbb{B}^n$ due to Graham et al. [37] (see also [51] for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$).

Definition 2.1.9. Let Assumption 2.1.6 hold and let $f \in S(\mathbb{B}^n)$. A mapping f has g-$\mathcal{PR}$ if there exists a g-$\mathcal{LC}$, $f(z, t)$, such that $f = f(\cdot, 0)$.

We consider $S_g^0(\mathbb{B}^n)$ to be the class of mappings with g-$\mathcal{PR}$ on $\mathbb{B}^n$. Next, we present certain observations concerning the family $S_g^0(\mathbb{B}^n)$ due to Graham, Hamada and Kohr [37]. First, let g be a function satisfying Assumption 2.1.6.

Remark 2.1.10. (i) $S_g^0(\mathbb{B}^n) \subseteq S^0(\mathbb{B}^n) \subseteq S(\mathbb{B}^n)$.

(ii) If $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, then $S_g^0(\mathbb{B}^n) = S^0(\mathbb{B}^n)$.

The next remark due to Graham, Hamada and Kohr [37] illustrates an important reason to study g-$\mathcal{PR}$, respectively g-$\mathcal{LC}s$ for $n \geq 2$.

Remark 2.1.11. Assume that $g(\zeta) = 1 - \zeta$, $\zeta \in U$. Then any normalized convex mapping on B admits g-$\mathcal{PR}$.

A growth theorem for the family $S_g^0(\mathbb{B}^n)$ was obtained by Graham, Hamada and Kohr [37], and implies that $S_g^0(\mathbb{B}^n)$ is a locally uniformly bounded (see also [73]).

2.2 Introduction in the theory of extension operators

This section presents certain extension operators that conserve geometric and analytic properties on the unit ball in $\mathbb{C}^n$. More specific, we shall study the extension operator Φ_n introduced by Roper and Suffridge [108], and two well known generalizations of this extension operator, $\Phi_{n,\alpha,\beta}$, introduced by Graham, Hamada, Kohr and Suffridge [46], respectively $\Phi_{n,Q}$, introduced by Muir [92]. First of all, these extension operators map a locally univalent function on U onto a mapping with the same properties on $\mathbb{B}^n$.

In the followings sections, we denote by $\tilde{z} = (z_2, \dots, z_n) \in \mathbb{C}^{n-1}$.

2.2.1 The extension operator Φ_n

The operator Φ_n was defined by Roper and Suffridge in [108] and its purpose was to build convex mappings on $\mathbb{B}^n$ using convex functions on U. The idea of constructing an extension operator of this kind started from the fact that if $f_1, \dots, f_n \in K$ then the mapping $f = (f_1, \dots, f_n)$ is not necessary convex on $\mathbb{B}^n$. The following well-known mapping on $\mathbb{B}^n$ illustrates this situation:

$$F(z) = \left(\frac{z_1}{1 - z_1}, \dots, \frac{z_n}{1 - z_n} \right), \ z = (z_1, \dots, z_n) \in \mathbb{B}^n,$$

with the mention that the function $\frac{\zeta}{1-\zeta}$, $\zeta \in U$, is convex on U.

In [108], the Roper-Suffridge extension $\Phi_n : \mathcal{LS} \to \mathcal{LS}_n$ was defined as follows:

$$\Phi_n(f)(z) = (f(z_1), \tilde{z}\sqrt{f'(z_1)}), \ z = (z_1, \tilde{z}) \in \mathbb{B}^n. \tag{2.2.1}$$

We consider here the branch of the square root function to be $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

In [108], Roper K. and Suffridge T. proved that the extension operator Φ_n preserves the notion of convexity. Graham and Kohr [47] obtained the result in a different way.

Theorem 2.2.1. *Let $f \in K$. Then $\Phi_n(f) \in K(\mathbb{B}^n)$. Therefore, $\Phi_n(K) \subseteq K(\mathbb{B}^n)$.*

The operator Φ_n also preserves starlikeness of order $\alpha \in (0, 1)$. In [47], Graham I. and Kohr G. first showed that the operator preserves the concept of starlikeness. A few years later, Hamada H., Kohr G. and Kohr M.[63] proved that Φ_n preserves the starlikeness of order 1/2. Further, Liu X.[80] proved that Φ_n preserves starlikeness of order $\alpha \in (0, 1)$ (a different proof using g-$\mathcal{LC}s$ was given by Chirilă in [12]).

Theorem 2.2.2. *If $f \in S^*_\alpha$, where $\alpha \in [0, 1)$, then $\Phi_n(f) \in S^*_\alpha(\mathbb{B}^n)$. Therefore,*

$$\Phi_n(S^*_\alpha) \subseteq S^*_\alpha(\mathbb{B}^n).$$

In [12], Chirilă proved the following preservation result regarding the functions from the set $\mathcal{A}S^*_{\alpha,\gamma}$.

Theorem 2.2.3. *Assume $0 \le \alpha < 1$ and $0 < \gamma < 1$. If $f \in \mathcal{A}S^*_{\alpha,\gamma}$ then $\Phi_n(f) \in \mathcal{A}S^*_{\alpha,\gamma}(\mathbb{B}^n)$.*

Graham, Kohr and Kohr [51] proved that the operator Φ_n conserves type γ spirallikeness, $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Using g-$\mathcal{LC}s$, Chirilă [12] obtained that the operator preserves spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and order $\alpha \in (0, 1)$.

Theorem 2.2.4. *Assume that $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $0 < \alpha < 1$. If $f \in \hat{S}_{\gamma,\alpha}$, then $\Phi_n(f) \in \hat{S}_{\gamma,\alpha}(\mathbb{B}^n)$.*

The following result plays an important role as it was used to prove some of the above results. For example, the conservation of starlikeness and spirallikeness of type α was proved using the characterization of these notions in terms of $\mathcal{LC}s$ (see [47], [51]). The result was obtained by Graham, Kohr and Kohr [51] and states that the operator Φ_n maps a function that has $\mathcal{PR}$ on U onto a mapping that has $\mathcal{PR}$ on $\mathbb{B}^n$.

Theorem 2.2.5. *If $f \in S$ then $\Phi_n(f) \in S^0(\mathbb{B}^n)$. Therefore, $\Phi_n(S) \subseteq S^0(\mathbb{B}^n)$.*

2.2.2 The extension operator $\Phi_{n,\alpha,\beta}$

We consider the following extension operator:

Definition 2.2.6. Assume that $\alpha \geq 0$, $\beta \geq 0$. Let $\Phi_{n,\alpha,\beta} : \mathcal{L}S \to \mathcal{L}S_n$ be given by

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z} \left(\frac{f(z_1)}{z_1} \right)^{\alpha} (f'(z_1))^{\beta} \right), \ z = (z_1, \tilde{z}) \in \mathbb{B}^n. \tag{2.2.2}$$

We consider the branches of the power functions to be

$$\left(\frac{f(z_1)}{z_1} \right)^{\alpha} \Big|_{z_1=0} = 1, \ (f'(z_1))^{\beta} \big|_{z_1=0} = 1.$$

This extension operator was introduced by Graham, Hamada, Kohr and Suffridge in [46]. For the pair $(\alpha, \beta) = (0, 1/2)$, the operator $\Phi_{n,\alpha,\beta}$ becomes the operator Φ_n.

We consider the following:

Assumption 2.2.7. Let $0 \leq \alpha \leq 1$, $0 \leq \beta \leq \frac{1}{2}$ and $\alpha + \beta \leq 1$.

The next statement presents important preservation properties satisfied by the extension operator $\Phi_{n,\alpha,\beta}$ (see [46]):

Theorem 2.2.8. *Under Assumption 2.2.7, the following statements hold:*

(i) $\Phi_{n,\alpha,\beta}(S) \subseteq S^0(\mathbb{B}^n)$.

(ii) $\Phi_{n,\alpha,\beta}(S^*) \subseteq S^*(\mathbb{B}^n)$.

(iii) $\Phi_{n,\alpha,\beta}(S^*_\gamma) \subseteq S^*_\gamma(\mathbb{B}^n)$, *with* $\gamma \in (0, 1)$.

(iv) The operator $\Phi_{n,\alpha,\beta}$ conserves spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and order $\delta \in (0, 1)$.

(v) The operator $\Phi_{n,\alpha,\beta}$ conserves almost starlikeness of type $\gamma \in (0, 1)$ and order $\delta \in [0, 1)$.

The statement of item (iii) was obtained by Liu in [80], respectively the statement of item (iv) was proved by Liu and Liu in [81] (see also [11], where the result was proved using g-$\mathcal{LC}s$). The last item is due to Chirilă [11].

An important question that arises is if the extension operator $\Phi_{n,\alpha,\beta}$ can preserve convexity and, if yes, under which conditions. Graham, Hamada, Kohr and Suffridge [46] showed that $\Phi_{n,\alpha,\beta}$ preserves convexity only if $(\alpha, \beta) = (0, 1/2)$.

Theorem 2.2.9. *If $f \in K$, then $\Phi_{n,\alpha,\beta}(f) \in K(\mathbb{B}^n)$ only if $\alpha = 0$, $\beta = \frac{1}{2}$ (i.e. only when $\Phi_{n,\alpha,\beta}$ becomes Φ_n).*

2.2.3 The extension operator $\Phi_{n,Q}$

Motivated to find a way to provide extreme points of the set $K(\mathbb{B}^n)$, Muir [92] gave a modification of the extension operator provided by Roper and Suffridge, which maps the extreme points of K onto extreme points of $K(\mathbb{B}^n)$. Before presenting the definition of this extension operator, we shall first recall the definition of a *homogeneous polynomial of degree* k (see [48], [67]).

Definition 2.2.10. We say that a mapping $Q : \mathbb{C}^n \to \mathbb{C}$ is called a *homogeneous polynomial of degree* $k \in \mathbb{N}\backslash\{0\}$ if there is a continuous multilinear of degree k mapping $L : \prod_{i=1}^{k} \mathbb{C}^n \to \mathbb{C}^n$ such that $Q(z) = L(\underbrace{z, \dots, z}_{k\text{-times}})$, $z \in \mathbb{C}^n$.

We can easily deduce that $Q \in H(\mathbb{C}^n)$ and, for all $z \in \mathbb{C}^n$, $Q(\lambda z) = \lambda^k Q(z)$, $\lambda \in \mathbb{C}$, respectively $DQ(z)(z) = kQ(z)$. Also, we have that $Q(0) = 0$.

We consider the following:

Assumption 2.2.11. Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2.

Now, we present the extension operator $\Phi_{n,Q}$ (see [92]).

Definition 2.2.12. Let Assumption 2.2.11 hold. Let $\Phi_{n,Q} : \mathcal{L}S \to \mathcal{L}S_n$ be given by

$$\Phi_{n,Q}(f)(z) = (f(z_1) + Q(\tilde{z})f'(z_1), \tilde{z}\sqrt{f'(z_1)}),\ z = (z_1, \tilde{z}) \in \mathbb{B}^n. \tag{2.2.3}$$

We consider the branch of the square root of $f'(z_1)$ to be $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

It is immediate that, for $Q \equiv 0$, the operator $\Phi_{n,Q}$ becomes the operator Φ_n.

Next, we present important preservation properties satisfied by the extension operator $\Phi_{n,Q}$. Note that the results of items (i) and (ii) were proved by Kohr [75], the result of item (iii) was obtained by Muir [92] and the result of item (iv) was shown by Wang and Liu [116]. The last result was also obtained by Chirilă in [12] using a different method.

Theorem 2.2.13. *(i)* $\Phi_{n,Q}(S) \subseteq S^0(\mathbb{B}^n)$, *if and only if* $\|Q\| \leq 1/4$;

(ii) $\Phi_{n,Q}(S^*) \subseteq S^*(\mathbb{B}^n)$, *if and only if* $\|Q\| \leq 1/4$;

(iii) $\Phi_{n,Q}(K) \subseteq K(\mathbb{B}^n)$, *if and only if* $\|Q\| \leq 1/2$;

(iv) $\Phi_{n,Q}$ *preserves starlikeness of order* $\alpha \in (0, 1)$ *if and only if* $\|Q\| \leq \frac{1-|2\alpha-1|}{8\alpha}$.

2.2.4 Radii of some families of holomorphic mappings associated with extension operators

In this part, we present certain radii problems associated with the extension operators Φ_n, $\Phi_{n,\alpha,\beta}$. First, we mention that Definition 1.5.23 can be extended from U to $\mathbb{B}^n$ in $\mathbb{C}^n$. Assume that $\mathcal{F}$ is a nonempty subset of $S(\mathbb{B}^n)$. Let $r(\mathcal{P}, \mathcal{F})$ be the radius of the property $\mathcal{P}$ in $\mathcal{F}$.

In the followings, we include some well-known radii problems concerning the operator Φ_n due to Graham, Kohr and Kohr [51] (see also [47]). See also Theorem 1.5.24.

Theorem 2.2.14. *(i)* $r(S^*, \Phi_n(S)) = r(S^*, S)$.

(ii) $r(K, \Phi_n(S)) = r(K, \Phi_n(S^*)) = r(K, S)$.

Note that, for $n \geq 2$, we have that $r(K, S^0(\mathbb{B}^n)) \leq r(K, S^*(\mathbb{B}^n)) < 2 - \sqrt{3}$ (see [48], [51]).

The next result is due to Graham et al. [46] (see also [11] for other radii problems concerning the operator $\Phi_{n,\alpha,\beta}$).

Theorem 2.2.15. *Under Assumption 2.2.7, the following statement holds:*

$$r(S^*, \Phi_{n,\alpha,\beta}(S)) = r(S^*, S).$$

2.3 Generalized Roper-Suffridge extension operators and g-parametric representation

In the following part, we are interested to investigate under which conditions the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ preserve g-$\mathcal{PR}$ on $\mathbb{B}^n$, where the function g has a particular form defined by (2.3.1). In the next chapter, we shall consider other important consequences of these results concerning certain subclasses of mappings with g-$\mathcal{PR}$. The original results presented in this section have been obtained in [85] and [86].

Throughout this section, we consider the following:

Assumption 2.3.1. Let $A, B \in \mathbb{R}$ with $-1 \leq B < A \leq 1$. Let $g : U \to \mathbb{C}$ be a function described by:

$$g(\zeta) = \frac{1 + A\zeta}{1 + B\zeta}, \ \zeta \in U.$$

It is important to mention that the above function g meets the requirements of Assumption 2.1.6.

2.3.1 The extension operator $\Phi_{n,\alpha,\beta}$ and g-parametric representation

In this part, we prove that the extension operator $\Phi_{n,\alpha,\beta}$ given in Definition (2.2.6) preserves g-$\mathcal{PR}$ when g is given by Assumption 2.3.1.

The next statement is due to Manu [85] and shows that g-$\mathcal{PR}$ is preserved under the extension operator $\Phi_{n,\alpha,\beta}$, when g is given by Assumption 2.3.1. In the case $(A, B) = (1, -1)$ (i.e. for $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$), the result was obtained by Graham et al. in [46, Theorem 2.1] (see also [50], when $\alpha = 0$). In [11], Chirilă obtained this result for $(A, B) = (1, 2\gamma - 1)$, with $0 < \gamma < 1$ (i.e. for $g(\zeta) = \frac{1+\zeta}{1+(2\gamma-1)\zeta}$, $\zeta \in U$, $0 < \gamma < 1$; see also [12], for $\alpha = 0$).

Theorem 2.3.2. *Let Assumptions 2.2.7 and 2.3.1 be satisfied. Let $f \in S_g^0$. Then $F = \Phi_{n,\alpha,\beta}(f)$ belongs to the set $S_g^0(\mathbb{B}^n)$.*

In the next chapter, we will present in details certain consequences of this main result.

2.3.2 The extension operator $\Phi_{n,Q}$ and g-parametric representation

We want to show in the next part that g-$\mathcal{PR}$ is preserved under the extension operator $\Phi_{n,Q}$ described in Definition (2.2.12), when g satisfying Assumption 2.3.1.

The following statement due to Manu [86] shows that the notion of g-$\mathcal{PR}$ is conserved through the operator $\Phi_{n,Q}$, where g satisfying Assumption 2.3.1. Particular cases of this result were proved by Kohr [75] and Chirilă [12]. For $(A, B) = (1, -1)$, the below result reduces to [75, Theorem 2.1], due to Kohr. For $(A, B) = (1, 2\gamma - 1)$, with $\gamma \in (0, 1)$, this result becomes [12, Theorem 3.1] due to Chirilă. Recall that Q is a homogeneous polynomial as in Assumption 2.2.11.

Theorem 2.3.3. *Let Assumptions 2.2.11 and 2.3.1 hold. Assume that $f \in S_g^0$. If $\|Q\| \leq \frac{A-B}{4(1+|B|)}$ then $F = \Phi_{n,Q}(f)$ belongs to the set $S_g^0(\mathbb{B}^n)$.*

We will state important consequences of this result in the forthcoming chapter.

Chapter 3

Janowski starlikeness and Janowski almost starlikeness

In this chapter, we study certain subclasses of normalized biholomorphic mappings which have geometric characterization and, moreover, admit g-parametric representation. We start by presenting the definitions of a g-starlike mapping, g-almost starlike mapping of order α, respectively g-spirallike mapping of type γ on the Euclidean ball $\mathbb{B}^n$. Further, we study some preservation properties regarding the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ and these notions. Another important part is dedicated to the study of Janowski (almost) starlikeness and of the connection with g-starlikeness on $\mathbb{B}^n$. The novelty of this chapter is represented by the preservation of Janowski (almost) starlikeness on $\mathbb{B}^n$ under the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$. Some radii problems concerning Janowski starlikeness will be also included. In the final part of the chapter, we present growth and distortion results for the above mentioned subclasses.

The main bibliographic sources used throughout this chapter are [68], [109], [110], [46], [37], [55], [11], [21], [14], [12], see also [63], [56], [57].

Our original results presented in this chapter have been obtained in [85], [86].

Throughout this chapter, we use the shorter notations from Notation 1.7.1. Hence, the following abbreviations are used: $\mathcal{LC}$ for a Loewner chain, $\mathcal{LDE}$ for the Loewner differential equation and $\mathcal{PR}$ for the parametric representation. In addition, the abbreviation g-$\mathcal{LC}$, respectively g-$\mathcal{PR}$, is used for a g-Loewner chain, respectively for g-parametric representation.

3.1 Certain subclasses of biholomorphic mappings that have g-parametric representation

We start this section by presenting the definitions of certain classes of mappings which admit g-$\mathcal{PR}$ on $\mathbb{B}^n$. Further, we show that these mappings can be characterized through g-$\mathcal{LC}s$. Also, some preservations results concerning the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ due to Manu [85, 86] will be included.

3.1.1 Preliminaries

Throughout this section, let Assumption 2.1.6 be satisfied.

Graham, Hamada and Kohr [37], respectively Hamada and Honda [55] introduced the definition of g-starlikeness (see Definition 3.1.1). Various properties of this concept were studied in [56], [57], and also in [11], [12] and [14]. Extensions to complex Banach spaces have been recently obtained in [64].

Definition 3.1.1. Let $f \in \mathcal{L}S(\mathbb{B}^n)$. The mapping f is *g-starlike* on $\mathbb{B}^n$ if

$$\langle [Df(z)]^{-1}f(z), \frac{z}{\|z\|^2}\rangle \in g(U),\ \forall z \in \mathbb{B}^n\setminus\{0\}.$$

Let $S_g^*(\mathbb{B}^n)$ be the set of g-starlike mappings on $\mathbb{B}^n$. For $n=1$, we use the notation S_g^* instead of $S_g^*(U)$.

In the following remark, we establish a connection between g-starlikeness and standard starlikeness on $\mathbb{B}^n$ (see [37], [55]).

Remark 3.1.2. Assume that $\gamma \in [0,1)$.

(i) If $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$, then $S_g^*(\mathbb{B}^n) = S^*(\mathbb{B}^n)$.

(ii) If $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, then $S_g^*(\mathbb{B}^n) = S_\gamma^*(\mathbb{B}^n)$.

(iii) If $g(\zeta) = \frac{1+(1-2\gamma)\zeta}{1-\zeta}$, $\zeta \in U$, then $S_g^*(\mathbb{B}^n) = \mathcal{A}S_\gamma^*(\mathbb{B}^n)$.

Chirilă T. [14] proved that g-starlike mappings on $\mathbb{B}^n$ are also mappings from $S_g^0(\mathbb{B}^n)$.

Theorem 3.1.3. *Let $f \in \mathcal{L}S(\mathbb{B}^n)$. The condition $f \in S_g^*(\mathbb{B}^n)$ is equivalent to the property that $e^t f(z)$ is a g-$\mathcal{LC}$. Hence, any mapping from $S_g^*(\mathbb{B}^n)$ admits g-$\mathcal{PR}$.*

The definition of g-almost starlikeness of order α was given by Chirilă in [14].

Definition 3.1.4. Let $0 \leq \alpha < 1$. Assume that $f \in \mathcal{L}S(\mathbb{B}^n)$. The mapping f is *g-almost starlike of order α* on $\mathbb{B}^n$ if

$$\frac{1}{1-\alpha}\left(\left\langle [Df(z)]^{-1}f(z), \frac{z}{\|z\|^2}\right\rangle - \alpha\right) \in g(U),\ z \in \mathbb{B}^n\setminus\{0\}.$$

The set of g-almost starlike mappings of order α on $\mathbb{B}^n$ is denoted by $\mathcal{A}S_g^*(\mathbb{B}^n)$ and, for $n=1$, by $\mathcal{A}S_g^*$.

The next remark illustrates the relation between the classes $\mathcal{A}S_g^*(\mathbb{B}^n)$ and some well-known classes (see [14]).

Remark 3.1.5. Let $0 \leq \alpha < 1$ and $0 < \gamma < 1$.

(i) If $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$ then $\mathcal{A}S_g^*(\mathbb{B}^n) = \mathcal{A}S^*(\mathbb{B}^n)$.

(ii) If $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$ then $\mathcal{A}S_g^*(\mathbb{B}^n)$ reduces to the set $\mathcal{A}S_{\alpha,\gamma}^*(\mathbb{B}^n)$.

(iii) $\mathcal{A}S_g^*(\mathbb{B}^n) \subseteq \mathcal{A}S_\alpha^*(\mathbb{B}^n)$.

(iv) If $\alpha = 0$ then $\mathcal{A}S_g^*(\mathbb{B}^n) = S_g^*(\mathbb{B}^n)$.

The next result states the characterization of the mappings from $\mathcal{A}S_g^*(\mathbb{B}^n)$ through g-$\mathcal{LC}s$ and is due to Chirilă [14].

Theorem 3.1.6. *Let $0 \leq \alpha < 1$ and let $f \in \mathcal{L}S(\mathbb{B}^n)$. The condition $f \in \mathcal{A}S_g^*(\mathbb{B}^n)$ is equivalent to the property that $e^{\frac{1}{1-\alpha}t}f(e^{\frac{\alpha}{\alpha-1}t}z)$ is a g-$\mathcal{LC}$. Thus, any mapping from the class $\mathcal{A}S_g^*(\mathbb{B}^n)$ admits g-$\mathcal{PR}$.*

Further, we give the definition of g-spirallikeness of type γ on $\mathbb{B}^n$, $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (see [14]).

Definition 3.1.7. Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Assume that $f \in \mathcal{LS}(\mathbb{B}^n)$. The mapping f is *g-spirallike of type γ* on $\mathbb{B}^n$ if

$$i\frac{\sin\gamma}{\cos\gamma} + \frac{e^{-i\gamma}}{\cos\gamma}\left\langle [Df(z)]^{-1}f(z), \frac{z}{\|z\|^2}\right\rangle \in g(U),\ z \in \mathbb{B}^n\backslash\{0\}.$$

We denote the set of g-spirallike mappings of type γ on $\mathbb{B}^n$ by $\hat{S}_g(\mathbb{B}^n)$. If $n = 1$, we use the notation $\hat{S}_g$ instead of $\hat{S}_g(U)$.

The following remarks show the connection between the classes $\hat{S}_g(\mathbb{B}^n)$ and $\hat{S}_\gamma(\mathbb{B}^n)$. Other important observations will be included (see [14]).

Remark 3.1.8. Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $0 < \alpha < 1$.

(i) If $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$ then $\hat{S}_g(\mathbb{B}^n) = \hat{S}_\gamma(\mathbb{B}^n)$.

(ii) If $g(\zeta) = \frac{1-\zeta}{1+(1-2\alpha)\zeta}$, $\zeta \in U$ then $\hat{S}_g(\mathbb{B}^n)$ becomes the set $\hat{S}_{\gamma,\alpha}(\mathbb{B}^n)$.

(iii) $\hat{S}_g(\mathbb{B}^n) \subseteq \hat{S}_\gamma(\mathbb{B}^n)$ (see Definition 1.6.13 when the operator $A = e^{-i\gamma}I$, where $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$).

(iv) For $\gamma = 0$, $\hat{S}_g(\mathbb{B}^n)$ becomes $S_g^*(\mathbb{B}^n)$.

In the following, we describe the g-spirallikeness of type γ on $\mathbb{B}^n$, $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, in terms of g-$\mathcal{LC}s$. This characterization was obtained in [14].

Theorem 3.1.9. *Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $f \in \mathcal{LS}(\mathbb{B}^n)$. The condition $f \in \hat{S}_g(\mathbb{B}^n)$ is equivalent to the property that $e^{(1-ia)t}f(e^{iat}z)$ is a g-$\mathcal{LC}$, with $a = \tan\gamma$. Thus, any mapping from the class $\hat{S}_g(\mathbb{B}^n)$ admits g-$\mathcal{PR}$.*

3.1.2 Extension operators that preserve geometric properties of mappings with g-parametric representation

Next, we present certain preservation results concerning the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$, defined in the Chapter 2 (see Definition 2.2.6 and Definition 2.2.12), and notions like of g-starlikeness, g-almost starlikeness of order α and g-spirallikeness of type γ on the Euclidean unit ball $\mathbb{B}^n$, when g is a function given by Assumption 2.3.1.

Throughout this part, we consider that Assumption 2.3.1 holds.

Also, let the extension operator $\Phi_{n,\alpha,\beta}$ be defined by Definition 2.2.6 and let the extension operator $\Phi_{n,Q}$ be given by Definition 2.2.12.

In view of Theorem 2.3.2 and from the characterization in terms of g-$\mathcal{LC}s$ of g-starlikeness, we deduce the following result, due to Manu [85]:

Theorem 3.1.10. *Let Assumptions 2.2.7 and 2.3.1 be satisfied. If $f \in S_g^*$, then $F = \Phi_{n,\alpha,\beta}(f) \in S_g^*(\mathbf{B}^n)$.*

The next remark follows from Theorem 3.1.10 for a suitable choice of A, B in the definition of function g defined by Assumption 2.3.1.

Remark 3.1.11. Assume that $g : U \to \mathbb{C}$ is satisfying Assumption 2.3.1.

(i) For the pair $(A, B) = (1, -1)$, we have that $S_g^*(\mathbb{B}^n) = S^*(\mathbb{B}^n)$. Thus, $\Phi_{n,\alpha,\beta}(S^*) \subset S^*(\mathbb{B}^n)$. The result is due to Graham et al. [46].

(ii) Let $\gamma \in (0, 1)$. For the pair $(A, B) = (1, 2\gamma - 1)$, we have that $S_g^*(\mathbb{B}^n) = S_\gamma^*(\mathbb{B}^n)$. Thus, $\Phi_{n,\alpha,\beta}(S_\gamma^*) \subset S_\gamma^*(\mathbb{B}^n)$. This property was obtained by Hamada, Kohr and Kohr [63], when $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, and also by Liu [80], when Assumption 2.2.7 holds and $\gamma \in (0, 1)$. Using the method of g-$\mathcal{LC}s$, Chirilă T. [11] showed that the same property holds.

We have the next preservation property due to Manu [85]. We remark that this result is a consequence of Theorem 2.3.2.

Theorem 3.1.12. *Let Assumptions 2.2.7 and 2.3.1 be satisfied. Let $\gamma \in (-\pi/2, \pi/2)$. If $f \in \hat{S}_g$, then $F = \Phi_{n,\alpha,\beta}(f) \in \hat{S}_g(\mathbb{B}^n)$.*

The following statement is also due to Manu and was obtained after the publication of the paper [85].

Theorem 3.1.13. *Let Assumptions 2.2.7 and 2.3.1 be satisfied. Let $0 \le \gamma < 1$. If f is a g-almost starlike function of order γ on U, then $F = \Phi_{n,\alpha,\beta}(f)$ is a g-almost starlike mapping of order γ on $\mathbb{B}^n$.*

In the next remark, we include certain consequences of the above results.

Remark 3.1.14. Assume that g satisfies Assumption 2.3.1. Also, let $\delta \in (0, 1)$. If we take $A = 1$ and $B = 2\delta - 1$ then we get the followings:

(i) Let $\gamma \in [0, 1)$. The operator $\Phi_{n,\alpha,\beta}$ preserves almost starlikeness of order γ and type δ. (see [11]).

(ii) Let $\gamma \in (-\pi/2, \pi/2)$. The operator $\Phi_{n,\alpha,\beta}$ preserves spirallikenes of type γ and order δ. This property was obtained by Liu and Liu [82] (see also [80], [11]).

Further, we prove that g-starlikeness and g-spirallikeness of type γ on $\mathbb{B}^n$ are conserved under the extension operator $\Phi_{n,Q}$, when the function g is given by Assumption 2.3.1.

Since any g-starlike mapping has g-$\mathcal{PR}$, we deduce the following preservation property due to Manu [86]. Recall that Q is a homogeneous polynomial as in Assumption 2.2.11.

Theorem 3.1.15. *Let Assumptions 2.2.11 and 2.3.1 hold. Assume that $f \in S_g^*$. If $\|Q\| \le \frac{A-B}{4(1+|B|)}$ then $F = \Phi_{n,Q}(f) \in S_g^*(\mathbb{B}^n)$.*

Note that the above result follows from Theorem 2.3.3. Other particular cases are presented in the following remark.

Remark 3.1.16. Assume that g is satisfying Assumption 2.3.1.

(i) For the pair $(A, B) = (1, -1)$, one have $S_g^*(\mathbb{B}^n) = S^*(\mathbb{B}^n)$. Kohr G. proved the property: $\Phi_{n,Q}(S^*) \subset S^*(\mathbb{B}^n)$ (see also [71]).

(ii) Let $\gamma \in (0, 1)$. For the pair $(A, B) = (1, 2\gamma - 1)$, one have $S_g^*(\mathbb{B}^n) = S_\gamma^*(\mathbb{B}^n)$. Thus, $\Phi_{n,Q}(S_\gamma^*) \subset S_\gamma^*(\mathbb{B}^n)$. The property was prove by Wang and Liu [116] (see also [12], where the author used a different method).

Manu A. [86] proved that g-spirallikeness of type γ, where $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is conserved under the operator $\Phi_{n,Q}(f)$.

Theorem 3.1.17. *Let Assumptions 2.2.11 and 2.3.1 hold. Assume that $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $f \in \hat{S}_g$. If $\|Q\| \leq \frac{A-B}{4(1+|B|)}$ then $F = \Phi_{n,Q}(f) \in \hat{S}_g(\mathbb{B}^n)$.*

3.2 Janowski starlike and Janowski almost starlike mappings

In this section we study two subclasses of functions that admit g-$\mathcal{PR}$ on U and have interesting geometric properties, namely the class of Janowski starlike functions and the class of Janowski almost starlike functions on U. Then, we present their natural generalization to the unit ball $\mathbb{B}^n$ of $\mathbb{C}$. Further, we highlight the connection between these concepts and g-starlikeness, when g meets the conditions of Assumption 2.3.1. We show that the Janowski (almost) starlikeness is preserved under the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$. This section includes original results from [85, 86].

Various results regarding Janowski starlikeness on U can be found in [68], [109], [110]. Regarding Janowski (almost) starlikeness on $\mathbb{B}^n$ of $\mathbb{C}^n$, the reader may consult [21].

3.2.1 Preliminaries

First, we define Janowski starlikeness on U of $\mathbb{C}$. Assume that $-1 \leq B < A \leq 1$.

In this part, we consider $g : U \to \mathbb{C}$ defined by Assumption 2.3.1.

W. Janowski [68] defined the following set:

$$\mathcal{J}^{[A,B]} = \left\{ f \in H(U) : f \text{ normalized}, \frac{zf'(z)}{f(z)} \prec g \right\} \tag{3.2.1}$$

Note that $\mathcal{J}^{[1,-1]}$ becomes the set S^* and $\mathcal{J}^{[1-2\alpha,-1]}$ becomes the set S^*_α, where $0 \leq \alpha < 1$.

Assume that $a, b \in R$ with $|1-a| < b \leq a$. Let be the following classes:

$$\mathcal{J}^{(a,b)} = \left\{ f \in H(U) : f \text{ normalized}, \left| \frac{zf'(z)}{f(z)} - a \right| < b,\ z \in U \right\},$$

introduced by Silverman in [109] (see also [110]) and

$$\mathcal{AJ}^{(a,b)} = \left\{ f \in H(U) : f \text{ normalized}, \left| \frac{f(z)}{zf'(z)} - a \right| < b,\ z \in U \right\},$$

defined by Curt in [21].

Assume that g is satisfying Assumption 2.3.1. The following remark due to Manu [85] points out the connection between the set S^*_g and the sets $\mathcal{J}^{[A,B]}$, $\mathcal{J}^{(a,b)}$.

Remark 3.2.1. Assume that g is defined by Assumption 2.3.1. Then

(i) $S^*_g = \{f \in H(U) : f \text{ normalized },\, f(z)/(zf'(z)) \prec g,\ z \in U\}$.

(ii) $\mathcal{J}^{[-B,-A]} = S^*_g$.

(iii) $S^*_g = \mathcal{J}^{(a,b)}$, with $a = \frac{1-AB}{1-A^2}$, $b = \frac{A-B}{1-A^2}$ and $A \neq 1$. For $A = 1$, we have $S^*_g = S^*_{(1+B)/2}$.

Natural extensions of the classes $\mathcal{J}^{(a,b)}$ and $\mathcal{AJ}^{(a,b)}$ to the unit ball $\mathbb{B}^n$ of $\mathbb{C}$ were obtained by Curt [21] and they will be presented in the next definition.

Definition 3.2.2. Let $a, b \in \mathbb{R}$ be such that $|1-a| < b \leq a$. Let

$$\mathcal{J}^{(a,b)}(\mathbb{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z\rangle} - a \right| < b,\ z \in \mathbb{B}^n \backslash \{0\} \right\},$$

be the set of *Janowski starlike mappings* on $\mathbb{B}^n$ and let

$$\mathcal{AJ}^{(a,b)}(\mathbb{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\langle [Df(z)]^{-1} f(z), z\rangle}{\|z\|^2} - a \right| < b,\ z \in \mathbb{B}^n \backslash \{0\} \right\},$$

be the set of *Janowski almost starlike mappings* on $\mathbb{B}^n$.

Note that $\mathcal{J}^{(a,b)}(\mathbb{B}^n)$ and $\mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$ are subsets of $S^*(\mathbb{B}^n)$.

In the following remark, we highlight the connection between the sets $\mathcal{J}^{(a,b)}(\mathbb{B}^n)$, $\mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$ and the set $S_g^*(\mathbb{B}^n)$ (see [21]).

Remark 3.2.3. Let Assumption 2.3.1 be satisfied. Let $a, b \in R$ such that $|1-a| < b \leq a$. Then

(i) $S_g^*(\mathbb{B}^n) = \mathcal{J}^{(a,b)}(\mathbb{B}^n)$, for $A = \frac{a-1}{b}$ and $B = \frac{a^2-b^2-a}{b}$.

(ii) $S_g^*(\mathbb{B}^n) = \mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$, for $A = \frac{a-a^2+b^2}{b}$ and $B = \frac{1-a}{b}$.

(iii) Let $\alpha \in (0,1)$ and let $a = \frac{1}{2\alpha}$ and $b = \frac{1}{2\alpha}$. Then

$$\mathcal{AJ}^{\left(\frac{1}{2\alpha}, \frac{1}{2\alpha}\right)}(\mathbb{B}^n) = S_\alpha^*(\mathbb{B}^n) \text{ and } \mathcal{J}^{\left(\frac{1}{2\alpha}, \frac{1}{2\alpha}\right)}(\mathbb{B}^n) = \mathcal{A}S_\alpha^*(\mathbb{B}^n).$$

3.2.2 Extension operators and Janowski starlike and Janowski almost starlike mappings

In the next part we show that the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ preserve the Janowski (almost) starlikeness on the unit ball $\mathbb{B}^n$. These results are obtained by Manu in [85, 86].

Let the operator $\Phi_{n,\alpha,\beta}$ be given by Definition 2.2.6 and let the operator $\Phi_{n,Q}$ be given by Definition 2.2.12.

We consider the following:

Assumption 3.2.4. Let $a, b \in R$ be such that $|1-a| < b \leq a$.

Assume that g is a function defined by Assumption 2.3.1. For a suitable selection of A and B, we get the next particular cases of Theorem 3.1.10. These results are due to Manu [85].

Theorem 3.2.5. *Let Assumptions 2.2.7 and 3.2.4 hold. Let* $f \in \mathcal{J}^{(a,b)}$. *Then* $F = \Phi_{n,\alpha,\beta}(f)$ *belongs to the set* $\mathcal{J}^{(a,b)}(\mathbb{B}^n)$.

Theorem 3.2.6. *Let Assumptions 2.2.7 and 3.2.4 hold. Let* $f \in \mathcal{AJ}^{(a,b)}$. *Then* $F = \Phi_{n,\alpha,\beta}(f)$ *belongs to the set* $\mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$.

The next two results due to Manu [86] are direct consequences of Theorem 3.1.15. Recall that Q is a homogeneous polynomial as in Assumption 2.2.11.

Theorem 3.2.7. *Let Assumptions 2.2.11 and 3.2.4 be satisfied. Let* $f \in \mathcal{J}^{(a,b)}$. *If* $\|Q\| \leq \frac{b^2-(1-a)^2}{4(b+|a^2-b^2-a|)}$ *then* $F = \Phi_{n,Q}(f)$ *belongs to the set* $\mathcal{J}^{(a,b)}(\mathbb{B}^n)$.

Theorem 3.2.8. *Let Assumptions 2.2.11 and 3.2.4 be satisfied. Let* $f \in \mathcal{AJ}^{(a,b)}$. *If* $\|Q\| \leq \frac{b-|1-a|}{4}$ *then* $F = \Phi_{n,Q}(f)$ *belongs to the set* $\mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$.

3.3 Radii problems and Janowski starlikeness

We propose to investigate certain radii problems regarding the extension operator $\Phi_{n,\alpha,\beta}$ and Janowski starlikeness on the unit disc. We assume that the conditions from Assumption 3.2.4 hold. First, we give the $\mathcal{J}^{(a,b)}$ radius of the classes S and S^*. Then, we compute the $\mathcal{J}^{(a,b)}$ radius of the classes $\Phi_{n,\alpha,\beta}(S)$ and $\Phi_{n,\alpha,\beta}(S^*)$. These results are due to Manu [85]. We shall mention other cases which derive from this results.

Let $r \in (0,1]$. We assume that the requirements from Assumption 3.2.4 hold.

Let be the open ball $\mathbb{B}_r^n = \{z \in \mathbb{C}^n : \|z\| < r\}$. We denote by $\mathcal{L}S(\mathbb{B}_r^n)$ the set of normalized locally biholomorphic mappings on $\mathbb{B}_r^n$.

Let

$$\mathcal{J}^{(a,b)}(\mathbb{B}_r^n) = \left\{ f \in \mathcal{L}S(\mathbb{B}_r^n) : \left| \frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z \rangle} - a \right| < b, z \in \mathbb{B}_r^n \setminus \{0\} \right\}.$$

When $n = 1$, we denote $\mathcal{J}^{(a,b)}(\mathbb{B}_r^n)$ by $\mathcal{J}^{(a,b)}(U_r)$.

Assume that $f_r(\zeta) = f(r\zeta)/r$, $\zeta \in U$. By using the below useful property due to Graham et al. [46]:

$$\Phi_{n,\alpha,\beta}(f_r)(z) = \frac{1}{r} \Phi_{n,\alpha,\beta}(f)(rz), \; z \in \mathbb{B}^n, \tag{3.3.1}$$

we deduce the following statements:

Remark 3.3.1. Let Assumptions 2.2.7 and 3.2.4 be satisfied.

(i) If $\Phi_{n,\alpha,\beta}(f) \in \mathcal{J}^{(a,b)}(B_r^n)$, then f belongs to $J^{(a,b)}(U_r)$, for all $0 < r < 1$.

(ii) If $f \in \mathcal{J}^{(a,b)}(U_r)$, then $\Phi_{n,\alpha,\beta}(f)$ belongs to $\mathcal{J}^{(a,b)}(B_r^n)$, for all $0 < r < 1$.

First, we denote by $r_{a,b}$ the $\mathcal{J}^{(a,b)}$ radius of S and by $r_{a,b}^*$ the $\mathcal{J}^{(a,b)}$ radius of S^*, defined as follows (see [110]).

Definition 3.3.2. The Janowski radius $r_{a,b}$ (respectively $r_{a,b}^*$) represents the radius of the largest disc $U(0, r_{a,b})$ (respectively $U(0, r_{a,b}^*)$) such that the following relation is true

$$\left| zf'(z)/f(z) - a \right| < b, \tag{3.3.2}$$

on $U(0, r_{a,b})$ (respectively $U(0, r_{a,b}^*)$) for any function f from S (respectively S^*).

We next intend to obtain the radii $r_{a,b}$ and $r_{a,b}^*$. The next result states the radius $r_{a,b}$ and is due to Manu [85].

Theorem 3.3.3. *Under Assumption 3.2.4, the radius $r_{a,b}$ is given by*

$$r_{a,b} = \min \left\{ \tanh \frac{\pi}{4}, \frac{-1+a+b}{1+a+b}, \frac{1-a+b}{1+a-b} \right\}. \tag{3.3.3}$$

We present the next particular case of Theorem 3.3.3:

Corollary 3.3.4. [85] *Then $r_{a,a} = \frac{2a-1}{2a+1}$, for $a \in (1/2, r)$, and $r_{a,a} = \tanh \frac{\pi}{4}$, for $a \geq r$, where $r = \frac{1}{2} \cdot e^{\pi/2}$.*

Next, we obtain the radius $r_{a,b}^*$. This result was obtained by Manu in [85].

Theorem 3.3.5. *Under Assumption 3.2.4, the radius* $r^*_{a,b}$ *is given by*

$$r^*_{a,b} = \min\left\{\frac{-1+a+b}{1+a+b}, \frac{1-a+b}{1+a-b}\right\}. \tag{3.3.4}$$

Moreover, $r^*_{a,a} = \frac{2a-1}{2a+1}$.

In view of Remark 3.2.3 (iii), for $n = 1$, and Corollary 3.3.4, one have the next remarks (see [85]):

Remark 3.3.6. Assume that $\alpha \in (0,1)$. Let q_α (q^*_α) denote the radius of almost starlikeness of order α of the class S (respectively S^*).

(i) If $0 < \alpha \le e^{-\pi/2}$, then $q_\alpha = \tanh\frac{\pi}{4}$. For $e^{-\pi/2} < \alpha < 1$, we have that $q_\alpha = \frac{1-\alpha}{1+\alpha}$.

(ii) $q^*_\alpha = \frac{1-\alpha}{1+\alpha}$.

In the following definition, assume that the conditions of Assumption 3.2.4 hold. Also, we use the notation $r_{a,b}(\Phi_{n,\alpha,\beta}(S))$ (respectively $r_{a,b}(\Phi_{n,\alpha,\beta}(S^*))$) for the $\mathcal{J}^{(a,b)}$ radius of the class $\Phi_{n,\alpha,\beta}(S)$ (respectively $\Phi_{n,\alpha,\beta}(S^*)$).

Definition 3.3.7. The Janowski radius $r_{a,b}(\Phi_{n,\alpha,\beta}(S))$ (respectively $r_{a,b}(\Phi_{n,\alpha,\beta}(S^*))$) is the radius $r \in (0,1]$ of the largest ball $\mathbb{B}^n_r$ with the property that if $F \in \Phi_{n,\alpha,\beta}(S)$ (respectively $F \in \Phi_{n,\alpha,\beta}(S^*)$) then F belongs to the family $\mathcal{J}^{(a,b)}(\mathbb{B}^n_r)$.

The next result due to Manu [85] gives the radius $r_{a,b}(\Phi_{n,\alpha,\beta}(S))$.

Theorem 3.3.8. *Let Assumptions 2.2.7 and 3.2.4 hold. The radius* $r_{a,b}(\Phi_{n,\alpha,\beta}(S))$ *is equal to the quantity given in* (3.3.3).

The next statements are consequences of Theorem 3.3.8 and are due to Manu [85]. First, let $q_\lambda(\Phi_{n,\alpha,\beta}(S))$ (respectively $q_\lambda(\Phi_{n,\alpha,\beta}(S^*))$) denote the radius of almost starlikeness of order λ of $\Phi_{n,\alpha,\beta}(S)$ (respectively $\Phi_{n,\alpha,\beta}(S^*)$), where $\lambda \in (0,1)$.

Theorem 3.3.9. *Let Assumptions 2.2.7 and 3.2.4 be satisfied. Let* $\lambda \in (0,1)$.

(i) The radius $r_{a,b}(\Phi_{n,\alpha,\beta}(S^*))$ *is equal to the quantity given in* (3.3.4).

(ii) If $0 < \lambda \le e^{-\pi/2}$ *then* $q_\lambda(\Phi_{n,\alpha,\beta}(S)) = \tanh\frac{\pi}{4}$. *For* $e^{-\pi/2} < \lambda < 1$, *we have that* $q_\lambda(\Phi_{n,\alpha,\beta}(S)) = \frac{1-\lambda}{1+\lambda}$. *Moreover,* $q_\lambda(\Phi_{n,\alpha,\beta}(S^*)) = \frac{1-\lambda}{1+\lambda}$.

3.4 Growth and distortion results regarding subclasses of $S_g^0(\mathbb{B}^n)$

This section presents growth results for certain families of mappings which admit g-$\mathcal{PR}$ generated under the operator $\Phi_{n,Q}$. Here, we consider g satisfying the conditions of Assumption 2.3.1. For the same families, we shall present certain distortion results. Also, we mention other particular cases that are direct consequences of these results. The section contains original results from [86].

Throughout this section, let Assumption 2.3.1 hold.

3.4.1 Growth results

In this part, we present growth results for some subclasses of the family $\Phi_{n,Q}(S_g^0)$, where the function g is given by Assumption 2.3.1.

Further, we present a growth result for the class $S_g^0(\mathbb{B}^n)$ due to Graham, Hamada and Kohr [37]. This result is more general than Theorem 2.3 in [73].

Theorem 3.4.1. *Let Assumption 2.3.1 be satisfied. Assume that $F \in S_g^0(\mathbb{B}^n)$. Then*

$$\|z\| \exp \int_0^{\|z\|} \Big[1/\max\{g(x), g(-x)\} - 1\Big] \frac{dx}{x} \leq \|F(z)\| \tag{3.4.1}$$
$$\leq \|z\| \exp \int_0^{\|z\|} \Big[1/\min\{g(x), g(-x)\} - 1\Big] \frac{dx}{x}, \ z \in \mathbb{B}^n.$$

The following statement gives a growth result for the class $\Phi_{n,Q}(S_g^0)$ and was obtained by Manu [86].

Theorem 3.4.2. *Let Assumptions 2.2.11 and 2.3.1 be satisfied. Assume that $\|Q\| \leq \frac{A-B}{4(1+|B|)}$ and let f be a function that admits g-$\mathcal{PR}$ on U. Let $F = \Phi_{n,Q}(f)$.*

(i) In the case $A = 0$, the following estimates hold

$$\|z\| e^{B\|z\|} \leq \|F(z)\| \leq \|z\| e^{-B\|z\|}, \ \forall z \in \mathbb{B}^n. \tag{3.4.2}$$

(ii) In the case $A \neq 0$, the following estimates hold

$$\|z\| \left(1 + A\|z\|\right)^{\frac{B-A}{A}} \leq \|F(z)\| \leq \|z\| \left(1 - A\|z\|\right)^{\frac{B-A}{A}}, \ \forall z \in \mathbb{B}^n. \tag{3.4.3}$$

The inequalities (3.4.2) *and* (3.4.3) *are sharp.*

Next we state some consequences of Theorem 3.4.2. The first one is a growth theorem for the class $\Phi_{n,Q}(S_g^*)$ and was obtained by Manu [86] (see also [21]).

Corollary 3.4.3. *Let Assumptions 2.2.11 and 2.3.1 be satisfied. Assume that $\|Q\| \leq \frac{A-B}{4(1+|B|)}$ and let $f \in S_g^*$. Let $F = \Phi_{n,Q}(f)$.*

(i) In the case $A = 0$, the following estimates hold

$$\|z\| e^{B\|z\|} \leq \|F(z)\| \leq \|z\| e^{-B\|z\|}, \ \forall z \in \mathbb{B}^n.$$

(ii) In the case $A \neq 0$, the following estimates hold

$$\|z\| \Big(1 + A\|z\|\Big)^{(B-A)/A} \leq \|F(z)\| \leq \|z\| \Big(1 - A\|z\|\Big)^{(B-A)/A}, \ \forall z \in \mathbb{B}^n.$$

These inequalities are sharp.

The following two statements represent growth theorems for the classes $\Phi_{n,Q}(\mathcal{J}^{(a,b)})$, respectively $\Phi_{n,Q}(\mathcal{AJ}^{(a,b)})$. These results have been obtained by Manu [86] (see also [21]).

Corollary 3.4.4. *Let Assumptions 2.2.11 and 3.2.4 hold. Assume that $\|Q\| \leq \frac{b^2-(1-a)^2}{4(b+|a^2-b^2-a|)}$ and let $f \in \mathcal{J}^{(a,b)}$. Also, let $c = b^2 - (a-1)^2$ and $F = \Phi_{n,Q}(f)$.*

(i) In the case $a = 1$, the following estimates hold

$$\|z\|e^{-b\|z\|} \leq \|F(z)\| \leq \|z\|e^{b\|z\|}, \ \forall z \in \mathbb{B}^n.$$

(ii) In the case $a \neq 1$, the following estimates hold

$$\|z\|\Big(1 - \frac{1-a}{b}\|z\|\Big)^{c/(1-a)} \leq \|F(z)\| \leq \|z\|\Big(1 + \frac{1-a}{b}\|z\|\Big)^{c/(1-a)}, \ \forall z \in \mathbb{B}^n.$$

These inequalities are sharp.

Corollary 3.4.5. *Let Assumptions 2.2.11 and 3.2.4 hold. Assume that $\|Q\| \leq \frac{b-|1-a|}{4}$ and let $f \in \mathcal{AJ}^{(a,b)}$. Also, let $c = b^2 - (a-1)^2$ and $F = \Phi_{n,Q}(f)$.*

(i) In the case $a = \frac{1+\sqrt{4b^2+1}}{2}$, the following estimates hold

$$\|z\|e^{\frac{1-\sqrt{4b^2+1}}{2b}\|z\|} \leq \|F(z)\| \leq \|z\|e^{\frac{\sqrt{4b^2+1}-1}{2b}\|z\|}, \ \forall z \in \mathbb{B}^n. \tag{3.4.4}$$

(ii) In the case $a \neq \frac{1+\sqrt{4b^2+1}}{2}$, the following estimates hold

$$\begin{aligned} \|z\|\Big(1 + \|z\|(a - a^2 + b^2)/b\Big)^{c/(a^2-b^2-a)} &\leq \|F(z)\| \\ \leq \|z\|\Big(1 - \|z\|(a - a^2 + b^2)/b\Big)^{c/(a^2-b^2-a)}&, \ \forall z \in \mathbb{B}^n. \end{aligned} \tag{3.4.5}$$

These inequalities are sharp.

Next, we present some direct consequences of Theorem 3.4.2 and Corollary 3.4.3. Since the classes S_g^0, respectively S_g^* reduce to S, respectively S^* when $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $z \in U$, then we have the below result obtained by Kohr (see [75, Corollary 2.4]):

Remark 3.4.6. Let Assumption 2.2.11 be satisfied. Assume that $\|Q\| \leq \frac{1}{4}$ and let $f \in S$ ($f \in S^*$). Then, for $F = \Phi_{n,Q}(f)$, the following estimates hold

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|F(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \ \forall z \in \mathbb{B}^n.$$

These inequalities are sharp.

In view of Remark 3.2.3 (iii) when $n = 1$, we have the following consequences of Corollary 3.4.4 and Corollary 3.4.5:

Remark 3.4.7. Let $\alpha \in (0,1)$ and let Assumption 2.2.11 hold.

(i) Assume that $\|Q\| \leq \frac{1-|2\alpha-1|}{8\alpha}$ and let $f \in S_\alpha^*$. Let $F = \Phi_{n,Q}(f)$. Then (see [18, 70] and e.g. [48, Chapter 10]):

$$\frac{\|z\|}{(1+\|z\|)^{2(1-\gamma)}} \leq \|F(z)\| \leq \frac{\|z\|}{(1-\|z\|)^{2(1-\gamma)}}, \ \forall z \in \mathbb{B}^n.$$

These inequalities are sharp.

(ii) Assume that $\|Q\| \leq \frac{1-\alpha}{4}$ and let $f \in \mathcal{A}S_\alpha^*$. Let $F = \Phi_{n,Q}(f)$. (see [21], [71] for $\alpha = \frac{1}{2}$):

(a) In the case $\alpha = \frac{1}{2}$, the following estimate holds

$$\|z\|e^{-\|z\|} \leq \|F(z)\| \leq \|z\|e^{\|z\|}, \ \forall z \in \mathbb{B}^n.$$

(b) In the case $\alpha \neq \frac{1}{2}$, the following estimate holds

$$\frac{\|z\|}{(1-(2\alpha-1)\|z\|)^{2(\alpha-1)/(2\alpha-1)}} \leq \|F(z)\| \leq \frac{\|z\|}{(1+(2\alpha-1)\|z\|)^{2(\alpha-1)/(2\alpha-1)}},$$

for all $z \in \mathbb{B}^n$.

These inequalities are sharp.

3.4.2 Distortion results

In this part we provide distortion theorems for certain subclasses of $\Phi_{n,Q}(S_g^*)$, when the function g is given by Assumption 2.3.1.

We consider that Assumption 2.3.1 holds.

Further, we intend to give a distortion result for the set S_g^*. Let $\varphi : U \to \mathbb{C}$ be a univalent function on U, with the following properties: Re $\varphi(\zeta) > 0$, $\zeta \in U$, $\varphi(U)$ is a symmetric domain with respect to the real axis and $\varphi(U)$ is a starlike domain with respect to $\varphi(0) = 1$. Also, let $\varphi'(0) > 0$.

Let be the set $S^*(\varphi) = \{f \in S : \frac{zf'(z)}{f(z)} \prec \varphi\}$ introduced by Ma and Minda in [84]. Note that $S_g^* = S^*(g)$, since the function g satisfies the same properties as the function φ.

We next give a particular case of Theorem 2 from [84] due to Ma and Minda.

Lemma 3.4.8. *Let Assumption 2.3.1 be satisfied. Let $f \in S_g^*$.*

(i) In the case $B = 0$, the following estimates hold

$$(3.4.6) \qquad (1 - A|z|)e^{-A|z|} \leq |f'(z)| \leq (1 + A|z|)e^{A|z|}, \ \forall z \in U.$$

(ii) In the case $B \neq 0$, the following estimates hold

$$(3.4.7) \quad (1 - A|z|)(1 - B|z|)^{\frac{A}{B}-2} \leq |f'(z)| \leq (1 + A|z|)(1 + B|z|)^{\frac{A}{B}-2}, \ \forall z \in U.$$

These inequalities are sharp.

Let Assumption 2.2.11 hold and let $f \in \mathcal{L}S$. Using elementary computations, one have that

$$(3.4.8) \qquad \det D\Phi_{n,Q}(f)(z) = [f'(z_1)]^{\frac{n+1}{2}}, \ z = (z_1, \tilde{z}) \in \mathbb{B}^n.$$

In the next statement we give estimates of $\det D\Phi_{n,Q}(f)(z)$ with f from the class S_g^*. The result was obtained by Manu [86].

Theorem 3.4.9. *Let Assumptions 2.2.11 and 2.3.1 be satisfied. Assume that $f \in S_g^*$ and let $d(z) = \det D\Phi_{n,Q}(f)(z)$, $z \in \mathbb{B}^n$.*

(i) In the case $B = 0$, the following estimates hold

$$(3.4.9) \qquad \Big[(1 - A\|z\|)e^{-A\|z\|}\Big]^{\frac{n+1}{2}} \leq |d(z)| \leq \Big[(1 + A\|z\|)e^{A\|z\|}\Big]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

(ii) In the case $B \neq 0$, the following estimates hold

(3.4.10)
$$\left[(1-A\|z\|)(1-B\|z\|)^{\frac{A}{B}-2}\right]^{\frac{n+1}{2}} \leq |d(z)| \leq \left[(1+A\|z\|)(1+B\|z\|)^{\frac{A}{B}-2}\right]^{\frac{n+1}{2}},$$

for all $z \in \mathbb{B}^n$.

These inequalities are sharp.

The following two results derive from Theorem 3.4.9 and have been obtained by Manu [86].

Corollary 3.4.10. *Let Assumptions 2.2.11 and 3.2.4 be satisfied. Let $d(z) = \det D\Phi_{n,Q}(f)(z)$, where $z \in \mathbb{B}^n$ and $f \in \mathcal{J}^{(a,b)}$.*

(i) In the case $a = \frac{1+\sqrt{1+4b^2}}{2}$, the following estimates hold

$$\left[\left(1+\frac{1-\sqrt{1+4b^2}}{2b}\|z\|\right)e^{\frac{1-\sqrt{1+4b^2}}{2b}\|z\|}\right]^{\frac{n+1}{2}} \leq |d(z)|$$
$$\leq \left[\left(1+\frac{\sqrt{1+4b^2}-1}{2b}\|z\|\right)e^{\frac{\sqrt{1+4b^2}-1}{2b}\|z\|}\right]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

(ii) In the case $a \neq \frac{1+\sqrt{1+4b^2}}{2}$, the following estimates hold

$$\left[\left(1+\frac{1-a}{b}\|z\|\right)\left(1+\frac{b^2-a^2+a}{b}\|z\|\right)^{\frac{a-1}{a^2-b^2-a}-2}\right]^{\frac{n+1}{2}} \leq |d(z)|$$
$$\leq \left[\left(1+\frac{a-1}{b}\|z\|\right)\left(1+\frac{a^2-b^2-a}{b}\|z\|\right)^{\frac{a-1}{a^2-b^2-a}-2}\right]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

These inequalities are sharp.

Corollary 3.4.11. *Let Assumptions 2.2.11 and 3.2.4 be satisfied. Let $d(z) = \det D\Phi_{n,Q}(f)(z)$, where $z \in \mathbb{B}^n$ and $f \in \mathcal{AJ}^{(a,b)}$.*

(i) In the case $a = 1$, the following estimates hold

(3.4.11)
$$\left[(1-b\|z\|)e^{-b\|z\|}\right]^{\frac{n+1}{2}} \leq |d(z)| \leq \left[(1+b\|z\|)e^{b\|z\|}\right]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

(ii) In the case $a \neq 1$, the following estimates hold

(3.4.12)
$$\left[\left(1+\frac{a^2-b^2-a}{b}\|z\|\right)\left(1+\frac{a-1}{b}\|z\|\right)^{\frac{b^2-a^2+a}{1-a}-2}\right]^{\frac{n+1}{2}} \leq |d(z)|$$
$$\leq \left[\left(1+\frac{b^2-a^2+a}{b}\|z\|\right)\left(1+\frac{1-a}{b}\|z\|\right)^{\frac{b^2-a^2+a}{1-a}-2}\right]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

These inequalities are sharp.

The following statement is a particular case of Theorem 3.4.9. The result is due to Manu [86].

Corollary 3.4.12. *Assume that $f \in S^*$. Then*

$$\left[\frac{1-\|z\|}{(1+\|z\|)^3}\right]^{\frac{n+1}{2}} \leq |\det D\Phi_{n,Q}(f)(z)| \leq \left[\frac{1+\|z\|}{(1-\|z\|)^3}\right]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

The inequalities are sharp.

In view of Corollaries 3.4.10, 3.4.11, one obtain the following statements due to Manu [86]:

Corollary 3.4.13. *Let $0 < \alpha < 1$.*

(i) Let $f \in S^*_\alpha$.

(a) For $\alpha = \frac{1}{2}$, the following estimates hold

$$\left[(1-\|z\|)e^{-\|z\|}\right]^{\frac{n+1}{2}} \leq |\det D\Phi_{n,Q}(f)(z)| \leq \left[(1+\|z\|)e^{\|z\|}\right]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

(b) For $\alpha \neq \frac{1}{2}$, the following estimates hold

$$\left[(1-\|z\|)(1-(2\alpha-1)\|z\|)^{\frac{1}{2\alpha-1}-2}\right]^{\frac{n+1}{2}} \leq |\det D\Phi_{n,Q}(f)(z)|$$
$$\leq \left[(1+\|z\|)(1+(2\alpha-1)\|z\|)^{\frac{1}{2\alpha-1}-2}\right]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

(ii) Let $f \in \mathcal{A}S^*_\alpha$. Then

$$\left[(1+(2\alpha-1)\|z\|)(1+\|z\|)^{2\alpha-3}\right]^{\frac{n+1}{2}} \leq |\det D\Phi_{n,Q}(f)(z)|$$
$$\leq \left[(1-(2\alpha-1)\|z\|)(1-\|z\|)^{2\alpha-3}\right]^{\frac{n+1}{2}}, \ \forall z \in \mathbb{B}^n.$$

The above inequalities are sharp.

In the following part, we present some distortion results along a vector with norm equal to 1 in $\mathbb{C}^n$ for certain mappings which admit g-$\mathcal{PR}$ on $\mathbb{B}^n$ generated under the operator $\Phi_{n,Q}$. These results are due to Manu [86].

First, we remark that for $f \in S^*_g$, with $F = \Phi_{n,Q}(f) \in S^*_g(\mathbb{B}^n)$, we have that $[DF(z)]^{-1}F(z) \neq 0$, for $z \neq 0$. In this case, if $z \in \mathbb{B}^n \backslash \{0\}$, then we can construct an unit vector in terms of the mapping F given by

$$v(z) = \frac{[DF(z)]^{-1}F(z)}{\|[DF(z)]^{-1}F(z)\|}. \tag{3.4.13}$$

The above statement is also true if $f \in \mathcal{J}^{(a,b)}$ or $f \in \mathcal{AJ}^{(a,b)}$.

The next statement gives a distortion theorem due to Manu [86] for the class $\Phi_{n,Q}(S^*_g)$. A similar result was obtained by Curt [21, Theorem 3.8] for the entire class $S^*_g(\mathbb{B}^n)$, where the function g is given by Assumption 2.3.1.

Theorem 3.4.14. *Let Assumptions 2.2.11 and 2.3.1 hold. Assume that* $\|Q\| \leq \frac{A-B}{4(1+|B|)}$. *Let* $f \in S_g^*$ *and let* $G(z) = D\Phi_{n,Q}(f)(z)v(z)$, *where* v *is given by* (3.4.13).

For $A \neq 0$, *the following estimate holds*

$$\|G(z)\| \leq \frac{1 - B\|z\|}{(1 - A\|z\|)^{2-B/A}}, \ \forall z \in \mathbb{B}^n \backslash \{0\}. \tag{3.4.14}$$

For $A = 0$, *the following estimate holds*

$$\|G(z)\| \leq (1 - B\|z\|) \cdot e^{-B\|z\|}, \ \forall z \in \mathbb{B}^n \backslash \{0\}. \tag{3.4.15}$$

These inequalities are sharp.

The following consequences of Theorem 3.4.14 are due to Manu [86]. Curt [21] obtained similar results for the families $\mathcal{J}^{(a,b)}(\mathbb{B}^n)$, $\mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$.

Corollary 3.4.15. *Let Assumptions 2.2.11 and 3.2.4 hold. Assume that* $\|Q\| \leq \frac{b^2-(1-a)^2}{4(b+|a^2-b^2-a|)}$. *Let* $f \in \mathcal{J}^{(a,b)}$ *and let* $G(z) = D\Phi_{n,Q}(f)(z)v(z)$, *where* v *is given by* (3.4.13).

For $a \neq 1$, *the following estimate holds*

$$\|G(z)\| \leq \frac{1 + \frac{b^2-a^2+a}{b}\|z\|}{\left(1 + \frac{1-a}{b}\|z\|\right)^{2-\frac{a^2-b^2-a}{a-1}}}, \ \forall z \in \mathbb{B}^n \backslash \{0\}.$$

For $a = 1$, *the following estimate holds*

$$\|G(z)\| \leq (1 + b\|z\|)e^{b\|z\|}, \ \forall z \in \mathbb{B}^n \backslash \{0\}.$$

These inequalities are sharp.

Corollary 3.4.16. *Let Assumptions 2.2.11 and 3.2.4 hold. Assume that* $\|Q\| \leq \frac{b-|1-a|}{4}$. *Let* $f \in \mathcal{AJ}^{(a,b)}$ *and let* $G(z) = D\Phi_{n,Q}(f)(z)v(z)$, *where* v *is given by* (3.4.13).

For $a \neq \frac{1+\sqrt{4b^2+1}}{2}$, *the following estimate holds*

$$\|G(z)\| \leq \frac{1 - \frac{1-a}{b}\|z\|}{\left(1 - \frac{b^2-a^2+a}{b}\|z\|\right)^{2-\frac{1-a}{b^2-a^2+a}}}, \ \forall z \in \mathbb{B}^n \backslash \{0\}.$$

For $a = \frac{1+\sqrt{4b^2+1}}{2}$, *the following estimate holds*

$$\|G(z)\| \leq \left(1 + \frac{\sqrt{4b^2+1}-1}{2b}\|z\|\right) \cdot e^{\frac{\sqrt{4b^2+1}-1}{2b}\|z\|}, \ \forall z \in \mathbb{B}^n \backslash \{0\}.$$

These inequalities are sharp.

In the followings, we consider that the Assumption 2.2.11 holds. We give below some consequences of Theorem 3.4.14 due to Manu [86] (see also [21]).

Corollary 3.4.17. *Let* $z \in \mathbb{B}^n \backslash \{0\}$. *Assume that* $f \in S^*$ *and let* $\|Q\| \leq \frac{1}{4}$. *Let* $G(z) = D\Phi_{n,Q}(f)(z)v(z)$, *where* v *is given by* (3.4.13). *Then*

$$\|G(z)\| \leq \frac{1 + \|z\|}{(1 - \|z\|)^3}, \ \forall z \in \mathbb{B}^n \backslash \{0\}.$$

The estimate is sharp.

Assume that $\alpha \in (0,1)$. From Corollaries 3.4.15, 3.4.16 and Remark 3.2.3 (iii), when $n=1$, we deduce the following consequences due to Manu [86](see also [21]):

Corollary 3.4.18. *(i) Assume that* $\|Q\| \leq \frac{1-|2\alpha-1|}{8\alpha}$ *and let* $f \in S^*_\alpha$. *Let* $G(z) = D\Phi_{n,Q}(f)(z)v(z)$, *where* v *is given by* (3.4.13).

Then

$$\|G(z)\| \leq \frac{1-(2\alpha-1)\|z\|}{(1-\|z\|)^{3-2\alpha}}, \ \forall z \in \mathbb{B}^n \setminus \{0\}.$$

(ii) Assume that $\|Q\| \leq \frac{1-\alpha}{4}$ *and let* $f \in \mathcal{A}S^*_\alpha$. *Let* $G(z) = D\Phi_{n,Q}(f)(z)v(z)$, *where* v *is given by* (3.4.13).

For $\alpha = \frac{1}{2}$, *the following estimate holds*

$$\|G(z)\| \leq (1+\|z\|)e^{\|z\|}, \ \forall z \in \mathbb{B}^n \setminus \{0\}.$$

For $\alpha \neq \frac{1}{2}$, *the following estimate holds*

$$\|G(z)\| \leq \frac{1+\|z\|}{(1+(2\alpha-1)\|z\|)^{2-\frac{1}{2\alpha-1}}}, \ \forall z \in \mathbb{B}^n \setminus \{0\}.$$

These inequalities are sharp.

Chapter 4

Extensions of Janowski starlikeness to the case of complex coefficients

In this chapter, we consider the notion of g-parametric representation, g-Loewner chains and g-starlikeness, where $g : U \to \mathbb{C}$ is univalent on U, $g(0) = 1$ and $\mathrm{Re}g(\zeta) > 0$, $\zeta \in U$. These notions have been introduced in the case of Banach spaces by Graham, Hamada, Kohr and Kohr in [44]. Let be the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$. We next state that g-parametric representation and g-starlikeness are conserved under these operators (see [44]). Based on the preservation of the g-starlikeness, we shall prove that the concept of Janowski (almost) starlikeness with complex coefficients introduced by Curt in [22] is preserved under these operators. These properties generalize the results obtained in [85, 86], which regard Janowski classes with real coefficients.

In the final part of this chapter we are interested about the preservation of g-parametric representation through the Pfaltzgraff-Suffridge extension operator Ψ_n ($n \geq 2$), where the function g is satisfying Assumption 2.3.1. For the case $n = 2$, we consider a particular Loewner chain $F(z,t) : \mathbb{B}^3 \times [0,\infty) \to \mathbb{C}^3$, where its first element is the image of a mapping $f \in S^0(\mathbb{B}^2)$ through the operator Ψ_2. We refer to the Loewner chain $F(z,t)$ mentioned in the proof of [53, Theorem 2.1] for $n = 2$. We will obtain the expression of the Herglotz vector field $H(z,t)$ of $F(z,t)$, which is a first step in approaching this preservation property regarding operator Ψ_n, for the general case $n \geq 2$. We leave the effective study of this topic to be addressed in another paper. Also, we shall present the Pfaltzgraff-Suffridge extension operator Ψ_n ($n \geq 2$) and mention important results concerning this operator.

The original results presented in this chapter have been obtained in [87].

The main sources used to prepare the first part of this chapter are [44], [45], [22] and [53], [41], [45] for the second part.

Throughout this chapter, we use the shorter notations from Notation 1.7.1. Hence, the following abbreviations are used: $\mathcal{LC}$ for a Loewner chain, $\mathcal{LDE}$ for the Loewner differential equation and $\mathcal{PR}$ for the parametric representation. In addition, the abbreviation g-$\mathcal{LC}$, respectively g-$\mathcal{PR}$, is used for a g-Loewner chain, respectively for g-parametric representation.

4.1 Preliminaries

Let us consider the following function defined on U (see [44]):

Assumption 4.1.1. Let $g : U \to \mathbb{C}$ be a univalent function on U such that $g(0) = 1$ and $\mathrm{Re}g(\zeta) > 0$, $\zeta \in U$.

We remark that the function from Assumption 4.1.1 is more general than the one from Assumption 2.1.6. This function was regarded in the work of Graham, Hamada, Kohr and Kohr [44] in order to introduce g-$\mathcal{PR}$ on the unit ball of a complex Banach space, when g satisfies Assumption 4.1.1.

Let be the following subset of the class $\mathcal{M}$ (see [44]):

Definition 4.1.2.

$$\mathcal{M}_g = \left\{ h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n, \left\langle h(z), \frac{z}{\|z\|^2} \right\rangle \in g(U), z \in \mathbb{B}^n \right\},$$

where the function g meets the conditions of Assumption 4.1.1.

In the followings, we shall regard the notion of g-$\mathcal{PR}$ described in Definition 2.1.9 and the notion of g-$\mathcal{LC}$ defined in Definition 2.1.8, where g satisfies Assumption 4.1.1. This generalizations have been approached by Graham, Hamada, Kohr and Kohr in [44].

An example of a function that fulfills the requirements of Assumption 4.1.1 is $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$.

Further, let be the extension operator $\Phi_{n,\alpha,\beta}$ given by Definition 2.2.6 and let be the Muir extension operator $\Phi_{n,Q}$ given by Definition 2.2.12.

In a more recent work, it has been shown that g-$\mathcal{PR}$ and g-starlikeness are preserved under the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$, where g is a convex function on U satisfying Assumption 4.1.1 (see [44]).

Theorem 4.1.3. **[44]** *Let g be a convex function on U satisfying Assumption 4.1.1. Let Assumption 2.2.7 hold and let $f \in S_g^0$. Then $\Phi_{n,\alpha,\beta}(f)$ belongs to the set $S_g^0(\mathbb{B}^n)$.*

Until now, certain cases of the above result have been studied and proved to be true in: [46] for $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$, (see Theorem 2.2.8 (i)), [11] for $g(\zeta) = \frac{1+\zeta}{1+(2\gamma-1)\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$ (see also [12], in the case $\alpha = 0$) and [85], when g is described by Assumption 2.3.1.

In the next statement, let $dist(1, \partial g(U))$ denote the expression $\inf_{\zeta \in \partial g(U)} |\zeta - 1|$. Recall that Q is a homogeneous polynomial as in Assumption 2.2.11.

Theorem 4.1.4. **[44]** *Let g be a convex function on U satisfying Assumption 4.1.1. Let $f \in S_g^0$ and let Assumption 2.2.11 hold. If $\|Q\| \le dist(1, \partial g(U))/4$, then $\Phi_{n,Q}(f)$ belongs to the set $S_g^0(\mathbb{B}^n)$.*

For different selections of the function g, particular cases of this result have been proved in : [75], for $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$, in [12] for $g(\zeta) = \frac{1+\zeta}{1+(2\gamma-1)\zeta}$, $\zeta \in U$, $\gamma \in (0, 1)$ and [86], when g is described by Assumption 2.3.1.

We next continue with two properties studied in [44] regarding g-starlikeness.

Theorem 4.1.5. **[44]** *Let g be a convex function on U which meets the requirements of Assumption 4.1.1. Let Assumption 2.2.7 be satisfied and let $f \in S_g^*$. Then $\Phi_{n,\alpha,\beta}(f)$ belongs to the set $S_g^*(\mathbb{B}^n)$.*

This result is a generalization of the results presented in Remark 3.1.11 and Theorem 3.1.10.

Theorem 4.1.6. **[44]** *Let g be a convex function on U which meets the requirements of Assumption 4.1.1. Let Assumption 2.2.11 be satisfied and let $f \in S_g^*$. If $\|Q\| \leq dist(1, \partial g(U))/4$, then $\Phi_{n,Q}(f)$ belongs to the set $S_g^*(\mathbb{B}^n)$.*

The above mentioned property represents a generalization of some results mentioned in Remark 3.1.16 and Theorem 3.1.15.

Let be the following function $g : U \to \mathbb{C}$ considered by Curt in [22], which satisfies Assumption 4.1.1.

Assumption 4.1.7. Let $A, B \in \mathbb{C}$, $A \neq B$. Let $g : U \to \mathbb{C}$ be a holomorphic function with positive real part on U given by:

$$g(\zeta) = \frac{1 + A\zeta}{1 + B\zeta},\ \zeta \in U. \tag{4.1.1}$$

The condition that requires that the function g from Assumption 4.1.7 to have positive real part on U implies certain constraints on the parameters A and B. In the following remark we present these constraints obtained by Curt in [22].

Remark 4.1.8. **[22]** Under Assumption 4.1.7, one of the following two relations holds:

$$|B| < 1,\ |A| \leq 1 \text{ and } \operatorname{Re}(1 - A\overline{B}) \geq |A - B|, \tag{4.1.2}$$

or

$$|B| = 1,\ |A| \leq 1 \text{ and } -1 \leq A\overline{B} < 1. \tag{4.1.3}$$

We consider the following:

Assumption 4.1.9. Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1 - a| < b \leq \operatorname{Re} a$.

In [22], the author considered the following subclasses of $S^*(\mathbb{B}^n)$:

Definition 4.1.10. Under Assumption 4.1.9, we consider the following classes

$$\mathcal{J}^{(a,b)}(\mathbb{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z \rangle} - a \right| < b,\ z \in \mathbb{B}^n \backslash \{0\} \right\},$$

and

$$\mathcal{A}\mathcal{J}^{(a,b)}(\mathbb{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\langle [Df(z)]^{-1} f(z), z \rangle}{\|z\|^2} - a \right| < b,\ z \in \mathbb{B}^n \backslash \{0\} \right\}.$$

The above subclasses represent the set of Janowski starlike mappings on $\mathbb{B}^n$ and Janowski almost starlike mappings on $\mathbb{B}^n$. These families of starlike mappings generalize the classes given in Definition 3.2.2, which have been introduced by Curt in [21]. It is clear that if $a \in \mathbb{R}$ (or equivalently Re $a = a$), the above definition coincides with Definition 3.2.2.

For $n = 1$, let $\mathcal{J}^{(a,b)}$ denote $\mathcal{J}^{(a,b)}(U)$ and let $\mathcal{A}\mathcal{J}^{(a,b)}$ denote $\mathcal{A}\mathcal{J}^{(a,b)}(U)$.

In the below results we highlight the connection between g-starlikeness and Janowski (almost) starlikeness on $\mathbb{B}^n$, when g satisfies Assumption 4.1.7. The following observations are due to Curt [22].

Remark 4.1.11. **[22]** Let Assumption 4.1.9 be satisfied.

(i) For $g(\zeta) = \frac{1+(\overline{a}-1)/b\zeta}{1+(|a|^2-b^2-a)/b\zeta}$, $\zeta \in U$, $S_g^*(\mathbb{B}^n)$ reduces to $\mathcal{J}^{(a,b)}(\mathbb{B}^n)$.

(ii) For $g(\zeta) = \frac{1+(a-|a|^2+b^2)/b\zeta}{1+(1-\overline{a})/b\zeta}$, $\zeta \in U$, $S_g^*(\mathbb{B}^n)$ reduces to $\mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$.

(iii) For $b = a \in R$ $(b > 0)$, we get

$$\mathcal{AJ}^{(b,b)}(\mathbb{B}^n) = S^*_{\frac{1}{2b}}(\mathbb{B}^n) \text{ and } \mathcal{J}^{(b,b)}(\mathbb{B}^n) = \mathcal{A}S^*_{\frac{1}{2b}}(\mathbb{B}^n).$$

4.2 Extension operators and Janowski starlike mappings with complex coefficients

In this part, we are concerned about the preservation of Janowski (almost) starlikeness with complex coefficients under the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$. These properties generalize the results obtained in [85, 86], which regard Janowski classes with real coefficients. The original results presented in this part have been obtained in [87].

Let us first consider $n \geq 2$.

Let $\Phi_{n,\alpha,\beta} : \mathcal{LS} \to \mathcal{LS}_n$ be the extension operator described by Definition 2.2.6 and let $\Phi_{n,Q} : \mathcal{LS} \to \mathcal{LS}_n$ be the extension operator described by Definition 2.2.12.

Next, we present the below remark which will be useful in the following results. This statement is due to Manu [87].

Remark 4.2.1. Let Assumption 4.1.7 hold. Then $dist(1, \partial g(U)) = \frac{|A-B|}{1+|B|}$.

The next statement results from Theorem 4.1.5 and Remark 4.1.11, and states that the extension operator $\Phi_{n,\alpha,\beta}$ conserves Janowski (almost) starlikeness with complex coefficients from U to $\mathbb{B}^n$. This result generalizes Theorem 3.2.5, Theorem 3.2.6 (see also [85]) and was obtained by Manu [87].

Theorem 4.2.2. *Under Assumptions 2.2.7 and 4.1.9, the following statements hold:*

(i) Let $f \in \mathcal{J}^{(a,b)}$. Then $\Phi_{n,\alpha,\beta}(f)$ belongs to the set $\mathcal{J}^{(a,b)}(\mathbb{B}^n)$,

(ii) Let $f \in \mathcal{AJ}^{(a,b)}$. Then $\Phi_{n,\alpha,\beta}(f)$ belongs to the set $\mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$.

Using Theorem 4.1.5 and Remark 4.1.11, we obtained the following results. The below statement shows that the extension operator $\Phi_{n,Q}$ conserves Janowski starlikeness with complex coefficients.

Theorem 4.2.3. *Let Assumptions 2.2.11 and 4.1.9 hold. Assume that $f \in \mathcal{J}^{(a,b)}$. If*

$$\|Q\| \leq \frac{b^2 - (1-a)(1-\overline{a})}{4(b + ||a|^2 - b^2 - a|)},$$

then $\Phi_{n,Q}(f) \in \mathcal{J}^{(a,b)}(\mathbb{B}^n)$.

It is immediate that, for $a \in \mathbb{R}$, the above result reduces to Theorem 3.2.7 (see also [86]).

We present another conservation property of the Muir extension operator $\Phi_{n,Q}$, which shows that this operator conserves Janowski almost starlikeness with complex coefficients.

Theorem 4.2.4. *Let Assumptions 2.2.11 and 4.1.9 be satisfied. Assume that* $f \in \mathcal{AJ}^{(a,b)}$. *If*

$$\|Q\| \leq \frac{b^2 - (1-a)(1-\overline{a})}{4(b + |1-\overline{a}|)},$$

then $\Phi_{n,Q}(f) \in \mathcal{AJ}^{(a,b)}(\mathbb{B}^n)$.

If we consider $a, b \in \mathbb{R}$ in Assumption 4.1.9, the above property reduces to Theorem 3.2.8 (see also [86]).

4.3 A note on Loewner chains and Herglotz vector fields related to the Pfaltzgraff-Suffridge operator

In this part, we investigate the following property regarding the Pfaltzgraff-Suffridge extension operator Ψ_n: if $f \in S_g^0$ then $\Psi_n(f)$ is the first element of a g-$\mathcal{LC}$, $F(z,t)$, where the function g is satisfying Assumption 2.3.1. We make a first step in this sense and, for the case $n = 2$, obtain the expression of the Herglotz vector field $H(z,t)$ of a particular $\mathcal{LC}$, $F(z,t) : \mathbb{B}^3 \times [0,\infty) \to \mathbb{C}^3$, which has the property that $F(\cdot, 0) = \Psi_2(f)$, where $f \in S^0(\mathbb{B}^2)$. We shall use the $\mathcal{LC}$, $F(z,t)$, mentioned in the proof of [53, Theorem 2.1] for $n = 2$. The expression of $H(z,t)$ have been obtained for $n \geq 2$ by Hamada, Kohr and Muir in [65], where the authors considered the case of L^d-Loewner chains. We leave the actual study of the preservation of g-$\mathcal{PR}$ under the operator Ψ_n, $n \geq 2$, for a forthcoming paper. First, we shall present the Pfaltzgraff-Suffridge extension operator Ψ_n ($n \geq 2$) and mention important properties of this operator.

The main bibliographic sources used to prepare this section are [53], [41], [45].

We present the following extension operator:

Definition 4.3.1. Let $\Psi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ be the extension operator given by:

$$\Psi_n(f)(z) = \left(f(\tilde{z}), z_{n+1}[J_f(\tilde{z})]^{\frac{1}{n+1}}\right),\ z = (\tilde{z}, z_{n+1}) \in \mathbb{B}^{n+1}. \tag{4.3.1}$$

We consider the branch of the power function to be

$$[J_f(\tilde{z})]^{\frac{1}{n+1}}\Big|_{\tilde{z}=0} = 1.$$

This operator was introduced by Pfaltzgraff and Suffridge [99]. Also, it is immediate that $f \in S(\mathbb{B}^n) \to \Psi_n(f) \in S(\mathbb{B}^{n+1})$. For $n = 1$, the operator Ψ_1 reduces to the operator Φ_2 (see relation (2.2.1)).

The following result states that a normalized biholomorphic mapping with $\mathcal{PR}$ on $\mathbb{B}^n$ can be expanded to mapping with the same properties on $\mathbb{B}^{n+1}$. This result is due to Graham, Kohr and Pfaltzgraff [53]. This property was also obtained by Graham et al. in [41], but in a more general case, for bounded symmetric domains with $n \geq 2$ (see also [38], [13]).

Theorem 4.3.2. *If* $f \in S^0(\mathbb{B}^n)$ *then* $\Psi_n(f) \in S^0(\mathbb{B}^{n+1})$.

The next result states that the notion of starlikeness is preserved under the extension operator Ψ_n due to Graham, Kohr and Pfaltzgraff [53] (see also [41]).

Theorem 4.3.3. *Assume that* $f \in S^*(\mathbb{B}^n)$. *Then* $\Psi_n(f)$ *is a mapping from* $S^*(\mathbb{B}^{n+1})$.

We have the following conjecture proposed by Pfaltzgraff and Suffridge [99]:

Conjecture 4.3.4. *Assume that* $f \in K(\mathbb{B}^n)$*. Then* $\Psi_n(f) \in K(\mathbb{B}^{n+1})$*.*

Even if there is no complete answer to this conjecture, a partial positive answer has been given by Graham, Kohr and Pfaltzgraff in [53]. Let us state this result.

Let $0 < a \leq 1$. Also, let

$$\Omega_{a,n} = \{z = (\tilde{z}, z_{n+1}) \in \mathbb{C}^{n+1} : |z_{n+1}|^2 < a^{\frac{2n}{n+1}}(1 - \|\tilde{z}\|^2)\}.$$

We remark that $\Omega_{1,n} = \mathbb{B}^{n+1}$ and $\Omega_{a,n} \subseteq \mathbb{B}^{n+1}$.

Theorem 4.3.5. *Assume that* $f \in K(\mathbb{B}^n)$ *and* $a_1, a_2 > 0$ *such that* $a_1 + a_2 \leq 1$*. Then we have that* $\gamma\Psi_n(f)(z) + (1-\gamma)\Psi_n(f)(w)$ *is an element from* $\Psi_n(f)(\Omega_{a_1+a_2,n})$*, with* $z \in \Omega_{a_1,n}$*,* $w \in \Omega_{a_2,n}$ *and* $\gamma \in [0,1]$*.*

The following preservation results are due to Chirilă [13] (see also [41]).

Theorem 4.3.6. *(i) Assume that* $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$*. Let* $f \in \hat{S}_\gamma(\mathbb{B}^n)$*. Then* $F = \Psi_n(f) \in \hat{S}_\gamma(\mathbb{B}^{n+1})$*.*

(i) Assume that $\alpha \in [0,1)$*. Let* $f \in \mathcal{A}S^*_\alpha(\mathbb{B}^n)$*. Then* $F = \Psi_n(f) \in \mathcal{A}S^*_\alpha(\mathbb{B}^{n+1})$*.*

In the following statement, we obtain the expression of the Herglotz vector field $H(z,t)$ of the $\mathcal{LC}$, $F(z,t) : \mathbb{B}^3 \times [0,\infty) \to \mathbb{C}^3$, mentioned in the proof of [53, Theorem 2.1] for $n = 2$ (see also [65] in the case of L^d-Loewner chains and $n \geq 2$). Note that the existence and uniqueness of $H(z,t)$ is assured by Theorem 1.7.12.

Theorem 4.3.7. *Let* $n = 2$*. Assume that* $f \in S^0(\mathbb{B}^2)$*. Let* $f_t(z)$ *be a* $\mathcal{LC}$ *such that* f *represents its first element. Let* $F(z,t) : \mathbb{B}^3 \times [0,\infty) \to \mathbb{C}^3$ *be the* $\mathcal{LC}$ *described by*

$$F(z,t) = \left(f_t(\tilde{z}), z_3 e^{\frac{t}{3}}[J_{f_t}(\tilde{z})]^{1/3}\right), \; z = (\tilde{z}, z_3) \in \mathbb{B}^3, \; t \geq 0, \tag{4.3.2}$$

where $[J_{f_t}(\tilde{z})]^{1/3}\Big|_{\tilde{z}=0} = e^{\frac{2t}{3}}$*. Also,* $F(z,0) = \Psi_2(f)(z)$*,* $z \in \mathbb{B}^3$*. Then the associated Herglotz vector field* $H(z,t) : \mathbb{B}^3 \times [0,\infty) \to \mathbb{C}^3$ *of* $F(z,t)$ *is given by*

$$H(z,t) = \left(h(\tilde{z},t), \frac{z_3}{3}\left(1 + \frac{\partial h_1}{\partial z_1}(\tilde{z},t) + \frac{\partial h_2}{\partial z_2}(\tilde{z},t)\right)\right),$$

where $z = (\tilde{z}, z_3) \in \mathbb{B}^3$*,* $\tilde{z} = (z_1, z_2)$*, a.e.* $t \geq 0$*.*

Conclusions

In this thesis we give new contributions in the theory of univalent functions of one and several complex variables. We shall present them in the following paragraphs.

The first chapter presents important results which are useful for the forthcoming chapters and it does not include original results.

Further let g be a function satisfying Assumption 2.1.6.

We shall refer in the following two chapters to the g-parametric representation introduced in [37], g-starlikeness introduced in [37, 55], g-almost starlikeness of order $\alpha \in [0,1)$ and g-spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ introduced in [14], on the Euclidean unit ball $\mathbb{B}^n$, where the function g is defined as above.

Also, let $\Phi_{n,\alpha,\beta}$ be the extension operator defined by Definition 2.2.6 and let $\Phi_{n,Q}$ be the Muir extension operator defined by Definition 2.2.12.

We choose a particular selection of the function g given by Assumption 2.3.1.

In **Chapter 2**, we prove that g-parametric is conserved under the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ in Theorem 2.3.2 and Theorem 2.3.3, where the function g is given by Assumption 2.3.1.

In **Chapter 3**, we show that g-starlikeness, g-almost starlikeness of order $\alpha \in [0,1)$ and g-spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ are conserved under the extension operator $\Phi_{n,\alpha,\beta}$, where g is given by Assumption 2.3.1. Moreover, we prove that $\Phi_{n,Q}$ preserves g-starlikeness and g-spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, where g is defined by Assumption 2.3.1. These preservation properties are presented in Theorem 3.1.10, Theorem 3.1.12, Theorem 3.1.13, Theorem 3.1.15, Theorem 3.1.17.

Let $a, b \in \mathbb{R}$ such that $|1-a| < b \leq a$. We shall refer to the Janowski classes of starlike functions on U, $\mathcal{J}^{(a,b)}$ (see [109], see also [110]), and $\mathcal{AJ}^{(a,b)}$ (see [21]). Then, we refer to their natural generalization to the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ (see [21]). These Janowski classes can be reduced to g-starlikeness for a suitable choice of function g satisfying Assumption 2.3.1, for $n \geq 1$. We prove that $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ preserve Janowski (almost) starlikeness from U to $\mathbb{B}^n$ in $\mathbb{C}^n$. The results are presented in Theorem 3.2.5, Theorem 3.2.6, Theorem 3.2.7 and Theorem 3.2.8.

We compute the $\mathcal{J}^{(a,b)}$ radius of the classes S, S^* in Theorem 3.3.3 and Theorem 3.3.5. Particular cases of these results are included in Corollary 3.3.4 and Remark 3.3.6. We also deduce the $\mathcal{J}^{(a,b)}$ radius of the classes $\Phi_{n,\alpha,\beta}(S)$ and $\Phi_{n,\alpha,\beta}(S^*)$ in Theorem 3.3.8, Theorem 3.3.9. Also, we obtain the radius of almost starlikeness of order α, with $\alpha \in (0,1)$, of the classes $\Phi_{n,\alpha,\beta}(S)$ and $\Phi_{n,\alpha,\beta}(S^*)$ in Theorem 3.3.9.

We next continue with growth results for certain families of mappings which admit g-parametric representation and are obtained under the operator $\Phi_{n,Q}$, where g is given by Assumption 2.3.1. We give growth theorems for the class $\Phi_{n,Q}(S_g^0)$ stated in Theorem 3.4.2, the class $\Phi_{n,Q}(S_g^*)$ stated in Corollary 3.4.3 and the classes $\Phi_{n,Q}(\mathcal{J}^{(a,b)})$, respectively $\Phi_{n,Q}(\mathcal{AJ}^{(a,b)})$, stated in Corollary 3.4.4, Corollary 3.4.5.

Then we provide distortion results for certain subclasses of $\Phi_{n,Q}(S_g^*)$, where g is described by Assumption 2.3.1. We give estimates for the expression $\det D\Phi_{n,Q}(f)(z)$, when f belongs to S_g^*, $\mathcal{J}^{(a,b)}$ or $\mathcal{AJ}^{(a,b)}$ in Theorem 3.4.9, Corollary 3.4.10, Corollary 3.4.11. Certain consequences of these results are stated in Corollary 3.4.12, Corollary 3.4.13. Some distortion results along a vector of norm equal to 1 in $\mathbb{C}^n$ for certain subclasses of $\Phi_{n,Q}(S_g^*)$ are obtained in Theorem 3.4.14, Corollary 3.4.15, Corollary 3.4.16, and their consequences in Corollary 3.4.17 and Corollary 3.4.18.

The above original results have been obtained in [85, 86], except Theorem 3.1.13 which was proved after the publication of the article [85].

In **Chapter 4**, we shall refer to a more general function g satisfying Assumption 4.1.1. Then we refer to notions like g-parametric representation, g-Loewner chains and g-starlikeness introduced in [44] in a certain Banach space, where g is this general function. Further, we present a particular choice of the function g which meets the conditions of Assumption 4.1.7.

For a proper selection of parameters A and B, we can establish a connection between g-starlikeness and Janowski (almost) starlikeness with complex coefficients. This type of starlikeness was recently introduced in [22] and generalizes the Janowski (almost) starlikeness with real coefficients defined by the same author in [21]. Making use of this connection between g-starlikeness and Janowski (almost) starlikeness, we prove that Janowski (almost) starlikeness with complex coefficients is preserved under the operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ in Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.2.4 (see also Remark 4.2.1). The results obtained in [85, 86] and presented in Chapter 3, which refer to the Janowski classes with real coefficients, are particular cases of these preservation results.

These new results have been obtained in [87].

Let Ψ_n be the operator given by Definition 4.3.1, which was considered by Pfaltzgraff and Suffridge.

Let $n = 2$ and let $f \in S^0(\mathbb{B}^2)$ be such that there exists a Loewner chain $f_t(z)$ with $f_0(z) = f(z)$, $z \in \mathbb{B}^2$. Next, we want obtain the expression of the Herglotz vector field $H(z,t)$ of the following Loewner chain $F(z,t) : \mathbb{B}^3 \times [0,\infty) \to \mathbb{C}^3$ given by

$$F(z,t) = \left(f_t(\tilde{z}), z_3 e^{\frac{t}{3}}[J_{f_t}(\tilde{z})]^{1/3}\right),\ z = (\tilde{z}, z_3) \in \mathbb{B}^3,\ t \geq 0,$$

where we consider the branch of the power function to be: $[J_{f_t}(\tilde{z})]^{1/3}\Big|_{\tilde{z}=0} = e^{\frac{2t}{3}}$. Also, $F(z,0) = \Psi_2(f)(z)$, $z \in \mathbb{B}^3$. The Loewner chain $F(z,t)$ is chosen from the proof of [53, Theorem 2.1]. Therefore, we obtain the following simple expression of $H(z,t)$:

$$H(z,t) = \left(h(\tilde{z},t), \frac{z_3}{3}\left(1 + \frac{\partial h_1}{\partial z_1}(\tilde{z},t) + \frac{\partial h_2}{\partial z_2}(\tilde{z},t)\right)\right),$$

where $h = (h_1, h_2)$ is the associated Herglotz vector field for $f_t(\tilde{z})$ and $z = (\tilde{z}, z_3) \in \mathbb{B}^3$, $\tilde{z} = (z_1, z_2)$, for a.e. $t \geq 0$. This result is obtained in Theorem 4.3.7 (see also [65, Theorem 6.3] for $n \geq 2$).

Concluding the above description of the main results, let us mention that the purpose of this thesis was the study of some preservation properties regarding the extension operators $\Phi_{n,\alpha,\beta}$, $\Phi_{n,Q}$ and some subclasses of $S(\mathbb{B}^n)$ which admit g-parametric representation on $\mathbb{B}^n$, where the function g meets certain conditions. Moreover, based on these preservation properties, we have studied radii problems and have obtained growth and distortion results. The new results presented in this thesis have been obtained using methods from the geometric function theory of one and several complex variables, especially from the theory of Loewner chains. Also, we have used methods of functional analysis and from the theory of partial differential equations (see [4]).

Further research directions

In the followings, we present certain research directions that may be approached in order to extend the results obtained in this thesis. They are mainly based on the idea of the preservation of geometric or analytic properties of some special classes of univalent mappings through extension operators.

- ⋄ In the last part of this thesis, we have referred to the Janowski classes with complex coefficients. Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ such that $|1-a| < b \leq \operatorname{Re} a$. Let $\mathcal{J}^{(a,b)}$ ($\mathcal{AJ}^{(a,b)}$) be the class of Janowski (almost) starlike functions on U. Let $\Phi_{n,\alpha,\beta}$ be the extension operator described by Definition 2.2.6 and let $\Phi_{n,Q}$ be the extension operator described by Definition 2.2.12.

  It would be of interest to compute the $\mathcal{J}^{(a,b)}$ radius of the classes S, S^* and then to obtain the radii for the classes $\Phi_{n,\alpha,\beta}(S)$, $\Phi_{n,\alpha,\beta}(S^*)$. Note that the operator $\Phi_{n,\alpha,\beta}$ preserves the classes $\mathcal{J}^{(a,b)}$ and $\mathcal{AJ}^{(a,b)}$ as shown in Theorem 4.2.2.

  Also, we are interested to obtain growth and distortion results, similar to those presented in the last section of Chapter 3, for the following classes: $\Phi_{n,\alpha,\beta}(\mathcal{J}^{(a,b)})$, $\Phi_{n,\alpha,\beta}(\mathcal{AJ}^{(a,b)})$, $\Phi_{n,Q}(\mathcal{J}^{(a,b)})$ and $\Phi_{n,Q}(\mathcal{AJ}^{(a,b)})$.

- ⋄ An interesting generalization of the Muir extension operator was given by Graham, Hamada, Kohr and Kohr in [44]. First, let Y be a complex Banach space and let the function $P_k : Y \to \mathbb{C}$ be a homogeneous polynomial of degree k, where $k \geq 2$. Let Ω_k be the domain defined as $\{(z_1, z') \in \mathbb{C}\times Y : |z_1|^2 + \|z'\|_Y^k < 1\}$. We consider the following extension operator $\Phi_{P_k} : \mathcal{L}S \to \mathcal{L}S(\Omega_k)$ given by (see [44]):

  $$\Phi_{P_k}(f)(z) = (f(z_1) + P_k(z')f'(z_1), (f'(z_1))^{1/k}z'),\ z = (z_1, z') \in \Omega_k,$$

  where we choose the branch of the power function such that $(f'(z_1))^{1/k}\Big|_{z_1=0} = 1$.

  Also, the authors referred to the operator $\Phi_{n,\alpha,\beta}$ given by Definition 2.2.6 with the mention that $\Phi_{n,\alpha,\beta}$ maps a function from $\mathcal{L}S$ into a mapping from $\mathcal{L}S(\Omega_k)$.

  It has been proved in [44] that the operators Φ_{P_k} and $\Phi_{n,\alpha,\beta}$ preserve g-parametric presentation and Bloch functions, where $g : U \to \mathbb{C}$ is an univalent function with $g(0) = 1$, $\operatorname{Re} g(\zeta) > 0$, $\zeta \in U$, and g is convex on U. We are interested if other subclasses of biholomorphic mappings can be preserved under these extension operators.

- ⋄ In [99], Pfaltzgraff and Suffridge proposed the extension operator $\Psi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ defined by:

  $$\Psi_n(f)(z) = \left(f(\tilde{z}), z_{n+1}[J_f(\tilde{z})]^{\frac{1}{n+1}}\right),\ z = (\tilde{z}, z_{n+1}) \in \mathbb{B}^{n+1},$$

where we consider the branch of the power function to be: $[J_f(\tilde{z})]^{\frac{1}{n+1}}\big|_{\tilde{z}=0} = 1$. This extension operator preserve parametric representation from $\mathbb{B}^n$ to $\mathbb{B}^{n+1}$ as shown by Graham, Kohr and Pfaltzgraff in [53]. Then we are interested to investigate if the operator Ψ_n preserves g-parametric representation, with g satisfying Assumption 2.3.1. In addition, we are interested to study if other subclasses of $S_g^0(\mathbb{B}^n)$ are conserved through this operator, like g-starlikeness, g-almost starlikeness of order $\alpha \in [0,1)$, g-spirallikeness of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and Janowski (almost) starlikeness on $\mathbb{B}^n$ with real coefficients.

A generalization of the Ψ_n operator was considered in [41] by Graham, Hamada and Kohr. Assume that $\mathbb{B}_X$ is the open unit ball of an n-dimensional JB*-triple (which is a complex Banach space where its open unit ball is homogeneous), denoted by X, $\mathbb{B}_Y$ is the open unit ball of a complex Banach space Y and $\mathbb{D}_a \subseteq \mathbb{B}_X \times \mathbb{B}_Y$ is a domain such that $\mathbb{B}_X \times \{0\} \subset \mathbb{D}_a$, and $a > 0$.

Let $\Psi_{n,a} : \mathcal{L}S(\mathbb{B}_X) \to \mathcal{L}S(\mathbb{D}_a)$ be the following extension operator:

$$\Psi_{n,a}(f)(z) = \left(f(\tilde{z}), [J_f(\tilde{z})]^{\frac{1}{2a\, c(\mathbb{B}_X)}} w\right),\ z = (\tilde{z}, w) \in \mathbb{D}_a,$$

where $[J_f(\tilde{z})]^{\frac{1}{2a\, c(\mathbb{B}_X)}}\big|_{\tilde{z}=0} = 1$ ($J_f(z) = det Df(z)$, $z \in \mathbb{D}_a$) and $c(\mathbb{B}_X)$ is a constant determined by the Bergman metric on X. In the same paper [41], the authors proved that $\Psi_{n,a}$ preserves parametric representation from $\mathbb{B}_X$ to $\mathbb{D}_a$ if $a \geq \frac{n}{2c(\mathbb{B}_{\mathbb{X}})}$. An interesting related problem could be the analysis of the preservation of g-parametric representation from $\mathbb{B}_X$ to $\mathbb{D}_a$ through the operator $\Psi_{n,\alpha}$, where g is given by Assumption 2.3.1. Another important problem is that of preservation through the operator $\Psi_{n,\alpha}$ of other subclasses of normalized biholomorphic mappings, in particular those with g-parametric representation.

◇ We remark the recent work of Muir in [91], where the author considered Loewner chains $F(z,t)$ of order p on $\mathbb{B}^n$ and for $t \geq 0$, but not normalized, in other words, Loewner chains $F(z,t)$ which satisfy a locally uniform local L^p-continuity condition with respect to t. Also, in this paper [91] are approached two extension operators: the Muir operator $\Phi_{n,Q}$ (see Definition 2.2.12) and a modification of the Pfaltzgraff–Suffridge operator. The author uses this type of Loewner chains to define a generalized form of spirallikeness with respect to A, where A is locally integrable operator-valued function on $[0,\infty)$. An important result of this paper is the fact that generalized spiralshaped mappings with respect to A are generated using the above mentioned extension operators. We wonder if other subclasses of biholomorphic mappings can be enhanced and connected to these extended Loewner chains. It would be interesting to generate this kind of mappings using the above operators.

◇ In view of Elin's work [27], which refers to generating extension operators using the semigroup theory, it would be of interest to study extension operators and their mapping properties through semigroup theory.

◇ An new approach of the theory of the Loewner chains was developed by Arosio, Bracci, Hamada and Kohr in [3], by considering Loewner chains on complete hyperbolic complex manifolds. They obtained a new geometric construction of Loewner chains in one and several complex variables, which holds on such manifolds, and showed that there is a one-to-one correspondence between L^d-Loewner chains and L^d-evolution families. Also, they gave examples of L^d-Loewner chains generated by the Roper-Suffridge extension operator. There would be valuable to generate and study L^d-Loewner chains in one and several complex variables by using other extension operators.

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