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Contributions to the study of the hyperbolic geometry

PhD thesis Summary

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Introduction

The name of non Euclidean geometry has been used for the first time by Karl Friedrich Gauss (1777-1855) to distinguish the geometries that differ from Euclid's geometry through the axiom of parallelism . According to some dated letters, Gauss started to develop a non Euclidian geometry in the year 1792. He had to overcome the prejudices against a non Euclidian geometry and to accept a geometrical system that was against his intuition. For the first time he realized that the axiom of parallelism was independent of the other axioms of geometry, therefore, the problem of proving it did not have sense, that removing this axiom, the geometry obtained is not contradictory. Nevertheless he never published anything from this domain fearing ridicule. Karl Ferdinand Schweikart (1780-1859) had also developed a non Euclidian geometry and in 1818 he sent a memorandum to Gauss. Schweikart affirmed that beside the Euclidian geometry there was a new geometry called astral geometry, in whitch the sum of the measures of the angles in a triangle is less than the sum of two straight angles. But, like Gauss, he didn't published anything. It took some time until mathematicians accepted the existence of non Euclidian geometry, especially because of the belief in the Kantian theory. Immanuel Kant (1724-1804) believed that geometry was an absolute science. Therefore, the idea of the existence of another geometry would challenge the Kantian philosophy .

The non Euclidian geometry does not contradict the classical geometry but it includes it as a limited case. It is more advanced because it corresponds to a more abstract faze in the process of acknowledging. Such a type of geometry was independently discovered by János Bolyai (1802-1860) and Nikolai Lobacevski (1793-1856). N. Lobacevski published in 1829 his work in "Principles of Geometry" in which he affirmed that if we presume a given line l and a point P which does not belong to the line, then there are at least two lines which go through P and are parallel to l. Even more, he demonstrated that the new geometry is just as logical and not contradictory as Euclidian geometry.

J. Bolyai developes a geometry, which he names *absolute*, independent of the truth or falsity of the axiom of parallelism and creates hyperbolic geometry following the same road and obtaining the same result as Lobacevski. The work of J. Bolyai , *Appendix Scientiam Spatii Absolute Veram Exhibens* (the annex in which is exposed the absolute and true science of space) appeared at Târgu Mureş in 1831, as an annex of the *Tentamen* volume of Farkas Bolyai .

The name of hyperbolic geometry has its justification through the fact that when an object of a new theory is classified by the nature of some generator elements – in our case, the number of parallels drawn through a point to a line – problem equivalent in general with the interpretation of the roots of a second grade equation, it is called *elliptic*, the case which corresponds to the complex values (when we don't have parallels) and analogue, *parabolic* or *hyperbolic*, the cases that correspond to the equal roots (when we have a single parallel) or different real roots (when we have two parallels).

In 1868, Eugenio Beltrami (1835-1900) shows that hyperbolic geometry can be realized on the pseudo sphere, which is a real surface of rotation, obtained through the rotation of the curve called *traction*. In this case, the lines of geometry are geodetics of the *pseudosphere*, so curves of the shortest distance. Beltrami's result is particularly important, because it gives us a real model, in ordinary space, on which hyperbolic geometry is valid, so it completely solves the problem of plane non contradiction of Bolyai-Lobacevski .

This paper is structured in three chapters, one is introductive and the other two chapters in which our contributions in the study of hyperbolic geometry are presented.

In **Chapter 1** is presented Beltrami-Klein's disc model of hyperbolic geometry using the gyrovector notion introduced by American mathematician Abraham Ungar [133] in 1997. Firstly we introduced the gyrogrup notion, the gyrocomutativ gyrogrup notion and their main properties. Then we presented the gyrovector notion, the space of gyrovectors and their main properties. Further, on the space of gyrovectors $(G, \oplus, \otimes)$ we introduced the gyrometric $d_{\oplus}(\mathbf{a}, \mathbf{b})$, we defined gyrolinia, gyrosegment, gyromidpoint of gyrosegment, gyrocosinus of angle between two gyrovectors and we gave the conditions for three points to be gyrocollinear. In the next paragraph of the first chapter we introduced adding and multiplying Einstein gave the main properties and presented Beltrami-Klein model of hyperbolic geometry seen through the concept of gyrovector. In the last part of this chapter we set out some useful theorems in our scientific approach, whose proof is found in [134], [135] and [136]

In **Chapter 2,** named *The hyperbolic version of some classic geometric results* we transcend in the hyperbolic version – using Einstein relativistic velocity, on the disc model of Poincaré and the model of upper half-plan of Poincaré – different famous theorems of euclidean geometry. So, in Subsection 2.1, C. Barbu and F. Smarandache [38] presented the hyperbolic form of Menelaus' theorem and used this new form to prove a theorem of Ţiţeica in the hyperbolic geometry.

In Subsection 2.2, using the concept of hyperbolic ration – introduced by W. Stothers [130], C. Barbu [18] gave the hyperbolic version of Menelaus theorem for quadrilaterals, transversality theorem for a triangle and Menelaus' theorem for a convex polygon.

In Subsection 2.3, D. Andrica and C. Barbu [7] treated Ceva's theorem in the Poincaré disc model, we showed that in some conditions it admits converse and we presented some applications of it for a hyperbolic triangle.

In Subsection 2.4, D. Andrica and C. Barbu [9] presented the hyperbolic version of Desargues' theorem in the Poincaré disc model and showed that in some conditions it admits converse.

In Subsection 2.5, C. Barbu [16] defined the hyperbolic podar poligon and presented the hyperbolic version of the pedal polygon of Smarandache, this being a generalization of Carnot's theorem.

In Subsection 2.6, C. Barbu [15] gave a trigonometrical proof of Steiner-Lehmus in the Poincaré disc model.

In Subsection 2.7, C. Barbu [19] introduced the concept of hyperbolic isogonal, we demonstrated Steiner's theorem that refers to isogonals in the Poincaré disc model, and we exposed the consequences of this theorem and demonstrated a theorem of Andreescu refering at isogonal lines in a hyperbolic triangle.

In Subsection 2.8, C. Barbu [19] introduced the concept of hyperbolic simedian, we presented Mathieu's theorem in the Poincaré disc model and we give some consequences of it in the hyperbolic geometry.

In Subsection 2.9, C Barbu and L. Pişcoran [33] was given the demonstration of Nobbs's theorem in the Poincaré disc model of hyperbolic geometry.

In Subsection 2.10, C. Barbu [20] presented the hyperbolic version of isotomic transversal theorem in the Poincaré disc model of hyperbolic geometry.

In Subsection 2.11, C. Barbu [20] presented the hyperbolic version of Neuberg's theorem in the Poincaré disc model of hyperbolic geometry and some consequences that derive from it.

In Subsection 2.12, C. Barbu and L. Pişcoran [34] presented the hyperbolic version of Gülicher's theorem in the Poincaré disc model of hyperbolic geometry.

In Subsection 2.13, C. Barbu and L. Pişcoran [34] presented the hyperbolic version of the internal bisector's theorem, its converse, the hyperbolic version of the external bisector's theorem and its converse, and some applications of these theorems.

In Subsection 2.14, C. Barbu and L. Pişcoran [30] presented the hyperbolic version of Zajic's theorem in the Poincaré disc model of hyperbolic geometry.

In Subsection 2.15, C. Barbu and N. Sönmez [41] presented the hyperbolic version of Carnot's theorem in the model of upper half-plan of Poincaré of hyperbolic geometry, we showed that in some conditions it admits converse and presented some applications of it in a hyperbolic triangle.

In Subsection 2.16, C. Barbu and N. Sönmez [40] presented the hyperbolic version of ortopol's theorem in diferent models of hyperbolic geometry.

In Subsection 2.17, C. Barbu [22] presented the hyperbolic version of Stewart's theorem in the Einstein relativistic velocity model of hyperbolic geometry and gave some of its consequences.

In Subsection 2.18, C. Barbu [25] presented the hyperbolic version of Van Aubel's theorem in the Einstein relativistic velocity model of hyperbolic geometry and gave some consequences of it.

In Subsection 2.19, C. Barbu [23] presented the hyperbolic version of a theorem of minimum of Smarandache in the Einstein relativistic velocity model of hyperbolic geometry.

In Subsection 2.20, C. Barbu [29] presented the hyperbolic version of a theorem of Pappus in the Einstein relativistic velocity model of hyperbolic geometry, we defined the antibisector for a hyperbolic triangle and gave some consequences of Pappus's theorem.

In Subsection 2.21, C. Barbu [13] presented the hyperbolic version of the cevian triangle theorem of Smarandache in the Einstein relativistic velocity model of hyperbolic geometry.

In Subsection 2.22, C. Barbu [31] treated some inequalities for a hyperbolic triangle.

In Subsection 2.23, C. Barbu and L. Pişcoran [35] presented the hyperbolic version of the inequality of Andrica-Iwata and the consequences that derives from it.

In Subsection 2.24, C. Barbu and L. Pişcoran [37] treated some applications of Cusa-Huygens inequality in a hyperbolic triangle.

In Subsection 2.25, C. Barbu and L. Pişcoran [36] presented the hyperbolic form of a inequality of Panaitopol and made some considerations on Jordan's inequality.

In **Chapter 3**, entitled *The fundamental inequality of the triangle between Euclidean geometry and hyperbolic geometry* we treated Blundon's inequality in euclidian geometry and in hyperbolic geometry. So, in Subsection 3.1, D. Andrica and C. Barbu [8] presented a geometric proof to the so-called fundamental triangle inequality. Given a triangle ABC, denote by s the semiperimeter, R the circumradius, and r the inradius of ABC. The necessary and sufficient condition for the existence of a triangle, with elements s, R, and r, is

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \leq s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}.$$

contains a simple geometric proof of the fundamental inequality. We gave the formula for $\cos\widehat{ION}$ in terms of the symmetric invariants of triangle s, R, r, where denote by O the circumcenter, I the incenter, N the Nagel point of ABC. This formula contains in fact the fundamental triangle inequalities and it is connected to the famous Euler's determination problem. So, assume that the triangle ABC is not equilateral, the following relation holds:

$$\cos\widehat{ION} = \frac{2R^2+10Rr-r^2-s^2}{2(R-2r)\sqrt{R^2-2Rr}},$$

From previous theorem we obtain a natural and important problem to construct the triangle ABC from its incenter I, circumcenter O, and its Nagel point N. Taking into account that points I, G, N are collinear determining the Nagel line of triangle, it follows that we get the centroid G on the segment IN such that $IG = \frac{1}{3}IN$. Then, using the Euler's line of the triangle, we get the orthocenter H on the ray $(OG$ such that $OH = 3OG$. Now we have reduced the construction problem to the famous Euler's determination problem i.e. to construct a triangle from its incenter I, circumcenter O, and orthocenter H [68]. Some new approaches involving this problem are given by B.Scimemi [120], G.C.Smith [126], J.Stern [129] and P.Yiu [144].

In Subsection 3.2, D. Andrica and C. Barbu [8] proved a dual version to the Blundon's inequality, which gives a formula for $\cos\widehat{I_aON_a}$ in terms of α, R, r_a, where denote by I_a the $A-$ excenter, N_a the $A-$adjoint point to the Nagel point N, $\alpha = \frac{a^2+b^2+c^2}{4}$ and r_a is $A-$exradius of ABC. We prove that the following relations hold:

$$0 \leq \frac{a^2+b^2+c^2}{4} \leq R^2-3Rr_a-r_a^2+(R+2r_a)\sqrt{R^2+2Rr_a},$$

and

$$\cos\widehat{I_aON_a} = \frac{R^2-3Rr_a-r_a^2-\alpha}{(R+2r_a)\sqrt{R^2+2Rr_a}}.$$

In Subsection 3.3, D. Andrica and C. Barbu [8] presented some inequalities involving s, R and the exradii of the triangle.

In Subsection 3.4, D. Andrica and C. Barbu [10] we presented a geometric proof to approach to Blundon - Wu inequality. S. Wu [140] gave a new sharp version of the Blundon's inequality by introducing a parameter, as follows: for any triangle $A_1A_2A_3$, the following inequalities hold

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\cos\phi \leq s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\cos\phi.$$

where $\phi = \min_{1\le i<j\le 3} |A_i - A_j|$.

In Subsection 3.5. we presented the hyperbolic version of Blundon's inequalities. This form was given by D. Svrtan and D. Veljan [131]. So, they have showed that for a hyperbolic triangle that has a circumcircle of radius R, incircle of radius r, semiperimeter s, and excess ε, we have

$$\frac{D}{s'^2} \ge 0,$$

where

$$D = s'^2[(r'^2R'^2\varepsilon'^2 + 4r'^4R'^4\varepsilon'^2 - 4r'^3R'^3\varepsilon'^2 - 1 + 6r'R' - 12r'^2R'^2 + 8r'^3R'^3)s'^4$$
$$+r'^2R'\varepsilon'(1 - 4r'R' + 4r'^2R'^2\varepsilon' - 8r'^2R'^2\varepsilon'^2 + 9\varepsilon' + 18r'R'\varepsilon')s'^3$$
$$+r'^2(r'^2R'^2 - 10r'R' - 12r'^2R'^2\varepsilon'^2 - 2)s'^2 - 6r'^4R'\varepsilon's' - r'^4],$$

and $r' = \tanh\frac{r}{k}$, $R' = \tanh\frac{R}{k}$, $\varepsilon' = \cot\frac{\varepsilon}{2}$, $s' = \sinh\frac{s}{k}$.

In Subsection 3.6, D. Andrica, C. Barbu and N. Minculete [26], [11] presented a geometric way to generate Blundon type inequalities, using the barycentric coordinates. We have introduced the Cevians of rank $(k;l;m)$. The line AD is called *ex-Cevian of rank* $(k;l;m)$ or *exterior Cevian of rank* $(k;l;m)$, if the point D is situated on side (BC) of the non-isosceles triangle ABC and the following relation holds

$$\frac{BD}{DC} = \left(\frac{c}{b}\right)^k \cdot \left(\frac{s-c}{s-b}\right)^l \cdot \left(\frac{a+b}{a+c}\right)^m$$

$k, l, m \in \mathbb{R}$, and we presented some properties of this cevian. We have showed that if I_1, I_2, I_3 are three Cevian points of rank $(k;l;m)$ with barycentric coordinates as follows:

$$I_i[a^{k_i}(s-a)^{l_i}(b+c)^{m_i} : b^{k_i}(s-b)^{l_i}(a+c)^{m_i} : c^{k_i}(s-c)^{l_i}(a+b)^{m_i}],$$

$i = \overline{1,3}$, and $t_i^1 = a^{k_i}(s-a)^{l_i}(b+c)^{m_i}, t_i^2 = b^{k_i}(s-b)^{l_i}(a+c)^{m_i}, t_i^3 = c^{k_i}(s-c)^{l_i}(a+b)^{m_i}$,

$$\alpha_{ij} = \frac{t_j^1}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^1}{t_i^1 + t_i^2 + t_i^3},$$

$$\beta_{ij} = \frac{t_j^2}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^2}{t_i^1 + t_i^2 + t_i^3}$$

$$\gamma_{ij} = \frac{t_j^3}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^3}{t_i^1 + t_i^2 + t_i^3}$$

for all $i, j \in \{1,2,3\}$, then

$$\cos\widehat{I_1I_2I_3} =$$
$$\frac{-a^2(\beta_{12}\gamma_{12} + \beta_{23}\gamma_{23} - \beta_{31}\gamma_{31}) - b^2(\gamma_{12}\alpha_{12} + \gamma_{23}\alpha_{23} - \gamma_{31}\alpha_{31}) + c^2(\alpha_{12}\beta_{12} + \alpha_{23}\beta_{23} - \alpha_{31}\beta_{31})}{2\sqrt{-\beta_{12}\gamma_{12}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2}}$$

and from here we obtain:

$$-2\sqrt{-\beta_{12}\gamma_{12}a^2-\gamma_{12}\alpha_{12}b^2-\alpha_{12}\beta_{12}c^2}\cdot\sqrt{-\beta_{23}\gamma_{23}a^2-\gamma_{23}\alpha_{23}b^2-\alpha_{23}\beta_{23}c^2}\leq$$

$$-a^2(\beta_{12}\gamma_{12}+\beta_{23}\gamma_{23}-\beta_{31}\gamma_{31})-b^2(\gamma_{12}\alpha_{12}+\gamma_{23}\alpha_{23}-\gamma_{31}\alpha_{31})+c^2(\alpha_{12}\beta_{12}+\alpha_{23}\beta_{23}-\alpha_{31}\beta_{31})\leq$$

$$2\sqrt{-\beta_{12}\gamma_{12}a^2-\gamma_{12}\alpha_{12}b^2-\alpha_{12}\beta_{12}c^2}\cdot\sqrt{-\beta_{23}\gamma_{23}a^2-\gamma_{23}\alpha_{23}b^2-\alpha_{23}\beta_{23}c^2},$$

the Blundon's type inequalities are simple direct consequences of the inequalities $-1\leq \cos\widehat{I_1I_2I_3}\leq 1$.

I would like, on this occasion, to express my appreciation to all the persons that, over time, directly or by implication, contributed with their advice to the creation of this scientific demarche. I offer thanks to Professor Doctor **Dorin Andrica** who, beside scientifically managing my doctorate thesis, guided me carefully and rigorously to give maximum definition to the script, all the working stage long.

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Chapter 1

The Einstein relativistic velocity model of hyperbolic geometry

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common, but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models etc. Following [134] and [136] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model.

The special theory of relativity was originally formulated by Einstein in 1905, [65], to explain the massive experimental evidence against ether as the medium for propagating electromagnetic waves, and Varičak in 1908 discovered the connection between the special theory of relativity and hyperbolic geometry [138]. The Einstein relativistic velocity model is another model of hyperbolic geometry. Many of the theorems of Euclidean geometry have a relative similar form in the Einstein relativistic velocity model

We present this model of hyperbolic geometry.

A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, $\mathbf{0}$, called a left identity, satisfying

(G1) $\mathbf{0} \oplus a = a$, for all $a \in G$. There is an element $\mathbf{0} \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a,satisfying

(G2) $\ominus a \oplus a = \mathbf{0}$

Moreover, for any $a, b, c \in G$ there exists a unique element $gyr[a, b]c \in G$ so that the binary operation obeys the left gyroassociative law

(G3) $a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$.

The map $gyr[a, b] : G \to G$ given by $c \to gyr[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$,

(G4) $gyr[a, b] \in Aut(G, \oplus)$

and the automorphism $gyr[a;b]$ of G is called the *gyroautomorphism* of G generated by $a, b \in G$ The operator $gyr : G \times G \to Aut(G, \oplus)$ is called the *gyrator* of G. Finally, the gyroautomorphism $gyr[a, b]$ generated by any $a, b \in G$ possesses the left loop property

(G5) $gyr[a, b] = gyr[a \oplus b, b]$.

As in group theory, we use the notation

$$a \ominus b = a \oplus (\ominus b)$$

in gyrogroup theory as well.

A gyrogrup $(G, \oplus)$ is *gyrocommutative* if its binary operation obeys the gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a),$$

for all $a, b \in G$.

Let $(G, \oplus)$ be a gyrogroup, and let $a, b \in G$. The unique solution of the equation

$$a \oplus x = b$$

in G for the unknown x is

$$x = \ominus a \oplus b.$$

Let $(G, \oplus)$ be a gyrogroup. Then, for any $a, b, c \in G$ we have

$$a \oplus (\ominus a \oplus b) = b$$

$$gyr[a, b] = gyr[a, b \oplus a]$$

$$gyr[a, b] = gyr[a \oplus b, \ominus a]$$

$$gyr[b, a] = gyr[\ominus a, a \oplus b]$$

Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Then,

$$\ominus(a \oplus b) = \ominus a \ominus b$$

$$gyr[a, b]\{b \oplus (a \oplus c)\} = (a \oplus b) \oplus c$$

$$gyr[a, b]b = (a \oplus b) \ominus a$$

$$(a \oplus b) \ominus (a \oplus c) = gyr[a, b](b \ominus c)$$

$$gyr[a, b]gyr[b \oplus a, c] = gyr[a, b \oplus c]gyr[b, c]$$

$$gyr[a, \ominus b]gyr[b, \ominus c]gyr[c, \ominus a] = gyr[\ominus a \oplus b, \oplus a \ominus c]$$

$$gyr[a, \ominus b]gyr[b, \ominus c]gyr[c, \ominus d] = gyr[a, \ominus d]$$

$$gyr[a, \ominus b] = gyr[\ominus a \oplus b, a \oplus b]gyr[a, b]$$

for all $a, b, c, d \in G$

A rooted gyrovector PQ in a gyrocommutative gyrogroup $(G, \oplus)$ is an ordered pair of points $P, Q \in G$. The rooted gyrovector PQ is rooted at the point P. The points P and Q of the rooted gyrovector PQ are called, respectively, the tail and the head of the rooted gyrovector. The value in G of the rooted gyrovector PQ is $\ominus P \oplus Q$. Accordingly, we write

$$\mathbf{v} = PQ = \ominus P \oplus Q$$

and call $\mathbf{v} = \ominus P \oplus Q$ the rooted gyrovector, rooted at P, with tail P and head Q in G. The rooted gyrovector PQ is nonzero if $P \neq Q$.

Let $PQ = \ominus P \oplus Q$, $P'Q' = \ominus P' \oplus Q'$ be two rooted gyrovectors in a gyrocommutative gyrogroup $(G, \oplus)$ with respective tails P and P' and respective heads Q and Q'. The two rooted gyrovectors are equivalent,

$$\ominus P \oplus Q \sim \ominus P' \oplus Q'$$

if they have the same value in G, that is, if

$$\ominus P \oplus Q = \ominus P' \oplus Q'.$$

The relation "$\sim$" is an equivalence relation. The resulting equivalence classes are called *gyrovectors*.

A real inner product gyrovector space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) G is a subset of a real inner product vector space V called the carrier of G, $\subset V$, from which it inherits its inner product, $\cdot$, and norm, $\|\cdot\|$, which are invariant under gyroautomorphisms, that is,

(V0) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$, for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(V1) $1 \otimes \mathbf{a} = \mathbf{a}$

(V2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(V3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(V4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$, $\mathbf{a} \neq \mathbf{0}, r \neq 0$

(V5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(V6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real, one-dimensional vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$, with vector addition $\oplus$ and scalar multiplication $\otimes$, for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(V7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(V8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$.

One can readily verify that $(-1) \otimes \mathbf{a} = \ominus \mathbf{a}$, $\|\ominus \mathbf{a}\| = \|\mathbf{a}\|$, $\mathbf{a} \otimes r = r \otimes \mathbf{a}$. Owing to the scalar distributive law, the condition for $1 \otimes \mathbf{a}$ in *(V1)* is equivalent to the condition

$$n \otimes \mathbf{a} = \underbrace{\mathbf{a} \oplus \mathbf{a} \oplus ... \oplus \mathbf{a}}_{n \text{ times}}$$

and

$$\mathbf{a} \otimes (-t) = \ominus \mathbf{a} \otimes t.$$

A gyrovector space $G = (G, \oplus, \otimes)$ possesses the monodistributive law

$$r \otimes (r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}) = r \otimes (r_1 \otimes \mathbf{a}) \oplus r \otimes (r_2 \otimes \mathbf{a}).$$

Let $G = (G, \oplus, \otimes)$ be a gyrovector space. Its *gyrometric* is given by the gyrodistance function $d_{\oplus}(\mathbf{a}, \mathbf{b}) : G \times G \to \mathbb{R}_{+}$,

$$d_{\oplus}(\mathbf{a}, \mathbf{b}) = \|\ominus \mathbf{a} \oplus \mathbf{b}\| = \|\mathbf{b} \ominus \mathbf{a}\|$$

where $d_{\oplus}(\mathbf{a}, \mathbf{b})$ is the gyrodistance of $\mathbf{a}$ to $\mathbf{b}$.

Let $\mathbf{a}$,$\mathbf{b}$ be any two distinct points in a gyrovector space $(G, \oplus, \otimes)$. The *gyroline* in G that passes through the points $\mathbf{a}$ and $\mathbf{b}$ is the set of all points

$$L = \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \otimes t$$

in G with $t \in \mathbb{R}$. Two gyrolines that share two distinct points are coincident. A *gyrosegment* $\mathbf{ab}$ with endpoints $\mathbf{a}$ and $\mathbf{b}$ is the set of all points with $t \in [0, 1]$ which verifies the relation $L = \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \otimes t$. The gyrolength $|\mathbf{ab}|$ of the gyrosegment $\mathbf{ab}$ is the gyrodistance between $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{ab}| = d_{\oplus}(\mathbf{a}, \mathbf{b}) = \|\ominus \mathbf{a} \oplus \mathbf{b}\|$$

Two gyrosegments are congruent if they have the same gyrolength.

A point $\mathbf{b}$ lies between the points $\mathbf{a}$ and $\mathbf{c}$ in a gyrovector space $(G, \oplus, \otimes)$ (i) if the points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are gyrocollinear, means that, they are related by the equations

$$\mathbf{a}_k = \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \otimes t_k$$

$k = \overline{1, 3}$, for some $\mathbf{a}, \mathbf{b} \in G$, $\mathbf{a} \neq \mathbf{b}$, and some $t_k \in \mathbb{R}$, and (ii) if, in addition, either $t_1 < t_2 < t_3$ or $t_3 < t_2 < t_1$.

If the three points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in a gyrovector space $G = (G, \oplus, \otimes)$ are gyrocollinear then

$$gyr[\mathbf{a}_1, \ominus \mathbf{a}_2] gyr[\mathbf{a}_2, \ominus \mathbf{a}_3] = gyr[\mathbf{a}_1, \ominus \mathbf{a}_3].$$

The gyromidpoint $\mathbf{p}^m_{\mathbf{ac}}$ of any two distinct points $\mathbf{a}$ and $\mathbf{c}$ in a gyrovector space $(G, \oplus, \otimes)$ is given by the equations

$$\mathbf{p}^m_{\mathbf{ac}} = \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{c}) \otimes \frac{1}{2}$$

Two nonzero gyrovectors $\ominus \mathbf{a}_1 \oplus \mathbf{b}_1$ and $\ominus \mathbf{a}_2 \oplus \mathbf{b}_2$ in a gyrovector space $(G, \oplus, \otimes)$. The gyrocosine of the gyroangle α, $0 \leq \alpha \leq \pi$, between the two gyrovectors is given by the equation

$$\cos \alpha = \frac{\ominus \mathbf{a}_1 \oplus \mathbf{b}_1}{\|\ominus \mathbf{a}_1 \oplus \mathbf{b}_1\|} \cdot \frac{\ominus \mathbf{a}_2 \oplus \mathbf{b}_2}{\|\ominus \mathbf{a}_2 \oplus \mathbf{b}_2\|} \tag{1.1}$$

Let s be any positive constant and let $(\mathbb{R}^n, +, \cdot)$ be the Euclidean $n-$space, and $\mathbb{R}^n_s = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < s\}$ be the $s-$ ball of all relativistically admissible velocities of material particles. It is the open ball of radius s, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than s. Einstein velocity addition is a binary operation, $\oplus$, in the $s-$ball $\mathbb{R}^n_s$ given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{s^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{s^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\},$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, where $\gamma_{\mathbf{u}}$ is the *gamma factor* given by the equation

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}} = \frac{1}{\sqrt{1 - \frac{\mathbf{u}^2}{s^2}}} \tag{1.2}$$

Here $\mathbf{u} \cdot \mathbf{v}$ are the inner product, and $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = \mathbf{u}^2$. A frequently used identity that follows immediately from (1.2) is

$$\frac{\mathbf{u}^2}{s^2} = \frac{\|\mathbf{u}\|^2}{s^2} = \frac{\gamma_{\mathbf{u}}^2 - 1}{\gamma_{\mathbf{u}}^2}.$$

$(\mathbb{R}_s^n, \oplus)$ is Einstein's grupoid. We naturally use the abbreviation $\mathbf{a} \ominus \mathbf{b} = \mathbf{a} \oplus (\ominus \mathbf{b})$ for Einstein subtraction.

Einstein addition is noncommutative and Einstein addition is also nonassociative. Einstein addition satisfies the mutually equivalent gamma identities

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} \left(1 + \frac{\mathbf{u} \cdot \mathbf{v}}{s^2}\right),$$

and

$$\gamma_{\mathbf{u}\ominus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} \left(1 - \frac{\mathbf{u} \cdot \mathbf{v}}{s^2}\right) \tag{1.3}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$. The gamma identity (1.3) signaled the emergence of hyperbolic geometry in special relativity when it was first studied by Sommerfeld and Varičak in terms of *rapidities*. Then, the rapidity $\phi_{\mathbf{v}}$ of a relativistically admissible velocity $\mathbf{v}$ is defined by the equation

$$\phi_{\mathbf{v}} = \tanh^{-1} \frac{\|\mathbf{v}\|}{s} \tag{1.4}$$

so that

$$\cosh \phi_{\mathbf{v}} = \gamma_{\mathbf{v}}$$
$$\sinh \phi_{\mathbf{v}} = \gamma_{\mathbf{v}} \frac{\|\mathbf{v}\|}{s}.$$

Sommerfeld and Varičak proved that

$$\cosh \phi_{\mathbf{u}\ominus\mathbf{v}} = \cosh \phi_{\mathbf{u}} \cosh \phi_{\mathbf{v}} - \sinh \phi_{\mathbf{u}} \sinh \phi_{\mathbf{v}} \cos A$$

where angle A is a hyperbolic angle in the relativistic "triangle of velocities" in the Beltrami-Klein ball model of hyperbolic geometry. By (1.3) we obtain

$$\frac{\mathbf{u} \cdot \mathbf{v}}{s^2} = 1 - \frac{\gamma_{\mathbf{u}\ominus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}$$

Two nonzero gyrovectors $\mathbf{u}$ and $\mathbf{v}$ from $\mathbb{R}_s^n$ are *parallel* if exists a constant $\lambda \in \mathbb{R}$ so that $\mathbf{u} = \lambda \mathbf{v}$, the Einstein addition reduces to

$$\mathbf{u} \oplus \mathbf{v} = \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{1}{s^2} \|\mathbf{u}\| \|\mathbf{v}\|}, \tag{1.5}$$

and, accordingly,

$$\|\mathbf{u}\| \oplus \|\mathbf{v}\| = \frac{\|\mathbf{u}\| + \|\mathbf{v}\|}{1 + \frac{1}{s^2} \|\mathbf{u}\| \|\mathbf{v}\|}.$$

Einstein addition determined by (1.5) is commutative and associative. Accordingly, the restricted Einstein addition is a group operation $\mathbb{R}^n_s$, as Einstein noted in [Einstein, 1]. Einstein made no remark about group properties of his addition of velocities that don't need to be parallel. Indeed, the general Einstein addition is not a group operation but, rather, a gyrocommutative gyrogroup operation, a structure that was discovered only in 1988 by Ungar [Ungar]. Einstein gyrations $gyr[\mathbf{u},\mathbf{v}] : \mathbb{R}^n_s \to \mathbb{R}^n_s$ are automorphisms of the Einstein gyrogroup $(\mathbb{R}^n_s, \oplus)$, given by the equation

$$gyr[\mathbf{u},\mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\}, \tag{1.6}$$

and they preserve the inner product that the ball $\mathbb{R}^n_s$ inherits from its real inner product space V,

$$gyr[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot gyr[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b},$$

for all $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^n_s$.

From the equation (1.6) we obtain the gyration equation

$$gyr[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w}+\frac{A\mathbf{u}+B\mathbf{v}}{D},$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^n_s$, where

$$A = -\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}^2}{\gamma_{\mathbf{u}}+1}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w})+\frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})$$

$$+\frac{2}{s^4}\frac{\gamma_{\mathbf{u}}^2\gamma_{\mathbf{v}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})$$

$$B = -\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}+1)(\mathbf{u}\cdot\mathbf{w})+(\gamma_{\mathbf{u}}-1)\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\}$$

$$D = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\right)+1 = \gamma_{\mathbf{u}\oplus\mathbf{v}}+1>1$$

An Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is an Einstein gyrogroup $(\mathbb{R}^n_s, \oplus)$ with scalar multiplication $\otimes$ given by

$$r\otimes\mathbf{v} = s\frac{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r-\left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r+\left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$= s\tanh(r\tanh^{-1}\frac{\|\mathbf{v}\|}{s})\frac{\mathbf{v}}{\|\mathbf{v}\|},$$

where $r\in\mathbb{R}, \mathbf{v}\in\mathbb{R}^n_s, \mathbf{v}\neq\mathbf{0}$ and $r\otimes\mathbf{0}=\mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$.

Einstein scalar multiplication can also be written in terms of the gamma factor as

$$r\otimes\mathbf{v} = \frac{1-(\gamma_{\mathbf{v}}-\sqrt{\gamma_{\mathbf{v}}^2-1})^{2r}}{1+(\gamma_{\mathbf{v}}-\sqrt{\gamma_{\mathbf{v}}^2-1})^{2r}}\frac{\gamma_{\mathbf{v}}}{\sqrt{\gamma_{\mathbf{v}}^2-1}}\mathbf{v} \tag{1.7}$$

with $\mathbf{v}\neq\mathbf{0}$.

An Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is a gyrometric space with gyrometric given by the Einstein gyrodistance function

$$d(\mathbf{a}, \mathbf{b}) = |\mathbf{b} \ominus \mathbf{a}| . \tag{1.8}$$

Gyrolines in the Beltrami ball model of hyperbolic geometry, that is, in Einstein gyrovector spaces, are Euclidean straight lines.

An Einstein gyrotriangle $A_1A_2A_3$ in Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ has (i) three vertices $A, B, C \in \mathbb{R}_s^n$, (ii) three gyroangles α_1, α_2 and α_3, (iii) three sides, which form the three gyrovectors $\mathbf{a}_{12}, \mathbf{a}_{23}$ and a_{31}, with respective (iv) three sidegyrolengths a_{12}, a_{23}, and a_{31} as defined as follows

$$\begin{aligned} a_{12} &= \|\mathbf{a}_{12}\| = \|\ominus A_1 \oplus A_2\| , \\ a_{23} &= \|\mathbf{a}_{23}\| = \|\ominus A_2 \oplus A_3\| , \\ a_{31} &= \|\mathbf{a}_{31}\| = \|\ominus A_3 \oplus A_1\| . \end{aligned}$$

Let $\ominus A_1 \oplus A_2$ and $\ominus A_1 \oplus A_3$ be two rooted gyrovectors with a common tail A_1. They include $\alpha_1 = \angle A_2A_1A_3$, the radian measure of which is given by the equation (1.1).

The following are some theorems proved by A. Ungar, used by us to obtain other results.

The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Spaces

Let $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$. If a gyroline meets the sides of gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ at points $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23,}$, then

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{12}\|}\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{23}\|}\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{13}\|} = 1 \tag{1.8}$$

where $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{s^2}}}$. ([135], p. 463)

The Hyperbolic Theorem of Ceva in Einstein Gyrovector Spaces

Let $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$. Let $\mathbf{a}_{123}$ be a point in their gyroplane, which is off the gyroline $\mathbf{a}_1\mathbf{a}_2, \mathbf{a}_2\mathbf{a}_3, \mathbf{a}_3\mathbf{a}_1$. If $\mathbf{a}_1\mathbf{a}_{123}$ meets $\mathbf{a}_2\mathbf{a}_3$ at $\mathbf{a}_{23}$, etc., then

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{12}\|}\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{23}\|}\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{13}\|} = 1, \tag{1.9}$$

where $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{s^2}}}$. ([135], p. 461)

Law of Gyrosines in Einstein Gyrovector Spaces

Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ having the vertices A, B and C, the sides $\mathbf{a} = -B \oplus C, \mathbf{b} = -C \oplus A$ and $\mathbf{c} = -A \oplus B$. Let $a = \|\mathbf{a}\|, b = \|\mathbf{b}\|, c = \|\mathbf{c}\|$ be sidegyrolengths of ABC, $a, b, c \in (-s, s)$, and $\alpha = \angle BAC, \beta = \angle CBA$, $\gamma = \angle ACB$ are gyroangles of gyrotriangle ABC. Then,

$$\frac{\sin\alpha}{\gamma_a a} = \frac{\sin\beta}{\gamma_b b} = \frac{\sin\gamma}{\gamma_c c}, \tag{1.10}$$

where $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{s^2}}}$. ([135], p. 544)

Law of Gyrocosines in Einstein Gyrovector Spaces

Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ having the vertices A, B and C, the sides $\mathbf{a} = -B \oplus C, \mathbf{b} = -C \oplus A$ and $\mathbf{c} = -A \oplus B$. Let $a = \|\mathbf{a}\|, b = \|\mathbf{b}\|, c = \|\mathbf{c}\|$ be sidegyrolengths of ABC, $a, b, c \in (-s, s)$, and $\alpha = \angle BAC, \beta = \angle CBA$, $\gamma = \angle ACB$ are gyroangles of gyrotriangle ABC. Then,

$$\gamma_a = \gamma_b \gamma_c (1 - b_s c_s \cos\alpha), \tag{1.11}$$

where $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{s^2}}}$. ([135], p.542)

The Gyrotriangle Bisector Theorem in Einstein Gyrovector Spaces

Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ having the vertices A, B and C, the sides $\mathbf{a} = -B \oplus C, \mathbf{b} = -C \oplus A$ and $\mathbf{c} = -A \oplus B$. Let $a = \|\mathbf{a}\|, b = \|\mathbf{b}\|, c = \|\mathbf{c}\|$ be sidegyrolengths of ABC, $a, b, c \in (-s, s)$, and let D be a point lying on the side BC so that AD is the bisector of gyroangle $\angle BAC$. Then,

$$\frac{\gamma_{|BD|}\,|BD|}{\gamma_{|CD|}\,|CD|} = \frac{\gamma_{|AB|}\,|AB|}{\gamma_{|AC|}\,|AC|}, \tag{1.12}$$

where $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{s^2}}}$. ([136], p.151)

Chapter 2

The hyperbolic version of the classic geometric results

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both of these geometries have many results in common but many are different.

2.1 The hyperbolic Menelaus theorem in the Poincaré disc model

Menelaus of Alexandria was a Greek mathematician and astronomer, the first to recognize geodesics on a curved surface as natural analogs of straight lines. Here, in this study, we present a proof of Menelaus's theorem in the Poincaré disc model of hyperbolic geometry. The well-known Menelaus theorem states that if l is a line not through any vertex of a triangle ABC so that l meets BC in D, CA in E, and AB in F, then $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$ [79]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by A. Johnson [84], N. A. Court [55], C. Coşniţă [53], A. Ungar [135]. Here we present a proof of Menelaus's theorem in the Poincaré disc model of hyperbolic geometry. The author has published the results in the paper [38].

Theorem 2.1.1. (The hyperbolic Menelaus theorem) If l is a gyroline not through any vertex of an gyrotriangle ABC so that l meets BC in D, CA in E, and AB in F, then

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1.$$

Theorem 2.1.2. (Converse of Menelaus's theorem for hyperbolic triangle) If D lies on the gyroline BC, E on CA, and F on AB so that

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

then D, E, and F are collinear.

Theorem 2.1.3. Let O be a point situated in the interior of a gyrotriangle ABC. Let l be a gyroline not through any vertex of a gyrotriangle ABC so that O is on gyroline l, and l meets the gyrosides BC, CA, and AB in the points A', B', and C', respectively. If A'', B'', C'' are the symetrics of the points A', B', and C', respectively from the point O, and two of the gyrolines AA'', BB'', CC'' are concurrent, then all three are concurrent.

2.2 The Menelaus's theorem for hyperbolic quadrilaterals

The well-known Menelaus theorem for quadrilateral states that if X, Y, Z, W are collinear points on the sides AB, BC, CD, and DA respectively,of quadrilateral $ABCD$, then $\frac{AX}{BX} \cdot \frac{BY}{CY} \cdot \frac{CZ}{DZ} \cdot \frac{DW}{AW} = 1$ [17]. Here we present a proof of Menelaus's theorem for hyperbolic quadrilateral in the Poincaré disc model of hyperbolic geometry. The author has published this results in the paper [18] and [39].

Definition. If A, B and X are distinct points on an h-line, then their hyperbolic ratio is $h(A, X, B) = \frac{\sinh(d(A,X))}{\sinh(d(X,B))}$, if X is between A and B, and $h(A, X, B) = -\frac{\sinh(d(A,X))}{\sinh(d(X,B))}$, otherwise.

Basic properties of hyperbolic ratio:

1. $h(A, X, B) = \frac{1}{h(B,X,A)}$,2. if X is between A and B,then $h(A, X, B) \in (0, 1)$,

3. if X is on AB, beyond B,then $h(A, X, B) \in (-\infty, -1)$,

4. if X is on AB, beyond A,then $h(A, X, B) \in (-1, 0)$,

5. if X and Y are points on the h-line AB so that $h(A, X, B) = h(A, Y, B)$, then $X = Y$ (the hyperbolic ratio theorem).

For more details see [130].

Theorem 2.2.1. If l is a h-line not through any vertex of a h-quadrilateral $ABCD$ so that l meets AB in X, BC in Y, CD in Z, and DA in W, then

$$h(A, X, B) \cdot h(B, Y, C) \cdot h(C, Z, D) \cdot h(D, W, A) = 1$$

F. Smarandache (1983) has generalized the Theorem of Menelaus for any polygon with $n \geq 4$ sides as follows: If a line l intersects the n-gon $A_1A_2...A_n$ sides

$A_1A_2, A_2A_3, ...,$ and A_nA_1 respectively in the points $M_1, M_2, ...,$ and M_n, then $\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot ... \cdot \frac{M_nA_n}{M_nA_1} = 1.$ [122].

Theorem 2.2.2. If l is a gyroline not through any vertex of a gyroquadrilateral $ABCD$ so that l meets AB in X, BC in Y, CD in Z, and DA in W, then

$$\frac{\gamma_{|AX|}|AX|}{\gamma_{|BX|}|BX|} \cdot \frac{\gamma_{|BY|}|BY|}{\gamma_{|CY|}|CY|} \cdot \frac{\gamma_{|CZ|}|CZ|}{\gamma_{|DZ|}|DZ|} \cdot \frac{\gamma_{|DW|}|DW|}{\gamma_{|AW|}|AW|} = 1$$

Corollary 2.2.3. (Transversal theorem for triangles) Let D be on gyroside BC, and l is a gyroline not through any vertex of a gyrotriangle ABC such that l meets AB in M, AC in N, and AD in P, then

$$\frac{\gamma_{|AM|}|AM|}{\gamma_{|AB|}|AB|} \cdot \frac{\gamma_{|AC|}|AC|}{\gamma_{|AN|}|AN|} \cdot \frac{\gamma_{|PN|}|PN|}{\gamma_{|PM|}|PM|} \cdot \frac{\gamma_{|DB|}|DB|}{\gamma_{|DC|}|DC|} = 1$$

Theorem 2.2.4. (The Hyperbolic Theorem of Menelaus for $n-gons$ in Einstein Gyrovector Space) If l is a gyroline not through any vertex of a $n-gyrogon$ $A_1A_2...A_n$ so that l meets A_1A_2 in M_1, A_2A_3 in $M_2, ...,$ and A_nA_1 in M_n, then

$$\frac{\gamma_{|M_1A_1|}|M_1A_1|}{\gamma_{|M_1A_2|}|M_1A_2|} \cdot \frac{\gamma_{|M_2A_2|}|M_2A_2|}{\gamma_{|M_2A_3|}|M_2A_3|} \cdot ... \cdot \frac{\gamma_{|M_nA_n|}|M_nA_n|}{\gamma_{|M_nA_1|}|M_nA_1|} = 1$$

2.3 The Hyperbolic Ceva Theorem in The Poincaré Disc Model

Here, in this study, we present a proof of Ceva's theorem in the Poincaré disc model of hyperbolic geometry. The Euclidean version of this well-known theorem states that if three lines from the vertices of a triangle $A_1A_2A_3$ are concurrent at M, and meet the opposite sides at P, Q, R respectively, then $\frac{A_1P}{PA_2} \cdot \frac{A_2R}{RA_3} \cdot \frac{A_3Q}{QA_1} = 1$ [84]. This result has a simple statement but it is of great interest. We just mention here few different proofs given N.A.Court [54], D.Grindberg [75], R.Honsberg [79], A.Ungar [135]. The author has published this results in the paper [7].

Theorem 2.3.1. (The Ceva's theorem for hyperbolic triangle) If M is a point not on any side of a gyrotriangle $A_1A_2A_3$ so that A_3M and A_1A_2 meet in P, A_2M and A_3A_1 in Q, and A_1M and A_2A_3 meet in R, then

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1$$

Naturally, one may wonder whether the converse of the Ceva theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 2.3.2. (Converse of Ceva's theorem for hyperbolic triangle) If P lies on the gyroline A_1A_2, R on A_2A_3, and Q on A_3A_1 so that

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1,$$

and two of the gyrolines A_1R, A_2Q and A_3P meet, then all three are concurrent.

Definition 2.3.3. The symmetric of the median of a triangle with respect to the internal bisector issued from the same vertex is called symmedian.

Corollary 2.3.4. The gyromedians of a gyrotriangle $A_1A_2A_3$ are concurrent.

Definition 2.3.5. The gyrolines AM and AM' are isogonals of the gyroangle $\widehat{BAC}$ if the gyrolines are symmetrical to the gyroangle $\widehat{BAC}$.

Theorem 2.3.6. If the gyrolines A_1P and A_1Q are two isogonals of a vertex A_1 of a gyrotriangle $A_1A_2A_3$, and the gyropoints P and Q are on the gyroside A_2A_3, then

$$\frac{(CQ)_\gamma}{(A_2Q)_\gamma} \cdot \frac{(CP)_\gamma}{(A_2P)_\gamma} = \left(\frac{(CA_1)_\gamma}{(A_2A_1)_\gamma}\right)^2.$$

Corollary 2.3.7. If the gyroline A_1P is a gyrosymmedian of a gyrotriangle $A_1A_2A_3$, and the point P is on the gyroside A_2A_3, then

$$\frac{(A_3P)_\gamma}{(A_2P)_\gamma} = \left(\frac{(A_3A_1)_\gamma}{(A_2A_1)_\gamma}\right)^2.$$

Corollary 2.3.8. The gyrosymedians of a gyrotriangle are concurrent.

Corollary 2.3.9. The internal angle bisectors of a gyrotriangle $A_1A_2A_3$ are concurrent.

Theorem 2.3.10. **(**The Ţiţeica's Theorem for Hyperbolic Gyrotriangle**)**. Let $A_1B_1C_1$ be the cevian gyrotriangle of gyropoint P with respect to the gyrotriangle ABC, and let l be a gyroline not through any vertex of a gyrotriangle ABC so that l meets the gyrosides BC, CA, and AB in the points A_2, B_2, and C_2, respectively. If the gyrolines B_1C_2 and BC meet in the gyropoint A_3, the gyrolines C_1A_2 and CA meet in the gyropoint B_3, and the gyrolines A_1B_2 and AB meet in the gyropoint C_3, then the gyrolines AA_3, BB_3, and CC_3 are concurrent.

2.4 The Hyperbolic Desargues Theorem in The Poincaré Disc Model

Here, in this study, we present a proof of Desargues's theorem in the Poincaré disc model of hyperbolic geometry. The well-known Desargues theorem states that if the three straight lines joining the corresponding vertices of two triangles and also meeting in a point, then the three intersections of pairs of corresponding sides lie on a straight line [84]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by N. A. Court [54], H. Coxeter [57], C. Durell [64], H. Eves [67], C.Ogilvy [111], W. Graustein [74]. The author has published this results in the paper [9].

Theorem 2.4.1. (The Desargues's theorem for hyperbolic triangle) If ABC, $A'B'C'$ are two gyrotriangles so that the gyrolines AA', BB', CC' meet in O, and BC and $B'C'$ meet at L, CA and $C'A'$ at M, AB and $A'B'$ at N, then L, M, and N are collinear.

Naturally, one may wonder whether the converse of the Desargues theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 2.4.2. (Converse of Desargues's theorem for hyperbolic triangle) Let ABC, $A'B'C'$ be two gyrotriangles so that the gyrolines BC and $B'C'$ meet at L, CA and $C'A'$ at M, AB and $A'B'$ at N, and the gyropoints L, M, and N are collinear. If two of the gyrolines AA', BB', CC' meet, then all three are concurrent.

2.5 The Smarandache's pedal polygon theorem

In this note we choose the Poincaré disc model in order to present the hyperbolic version of the Smarandache's pedal polygon theorem. The Euclidean version of this well-known theorem states that if the points $M_i, i = \overline{1,n}$ are the projections of a point M on the sides $A_iA_{i+1}, i = \overline{1,n}$, where $A_{n+1} = A_1$, of the polygon $A_1A_2...A_n$, then

$$M_1A_1^2 + M_2A_2^2 + ... + M_nA_n^2 = M_1A_2^2 + M_2A_3^2 + ... + M_{n-1}A_n^2 + M_nA_1^2$$

[123]. This theorem is generalized theorem of Carnot. O. Demirel and E. Soytürk [60] gived the hyperbolic form of Carnot's theorem. In the same maner we present a proof for Smarandache's pedal polygon theorem in the Poincaré disc model of hyperbolic geometry. The author has published this results in the paper [16].

Observation 2.5.1. (The Möbius hyperbolic Pythagorean theorem) Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$, with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V_s}$ and side gyrolenghts $a, b, c \in (-s, s)$, $\mathbf{a} = -B \oplus C, \mathbf{b} = -C \oplus A$, $\mathbf{c} = -A \oplus B$, $a = \|\mathbf{a}\|, b = \|\mathbf{b}\|, c = \|\mathbf{c}\|$ and with gyroangles α, β, and γ at the vertices A, B, and C. If $\gamma = \pi/2$, then

$$\frac{c^2}{s} = \frac{a^2}{s} \oplus \frac{b^2}{s}$$

([135], p 290)

Theorem 2.5.2. Let $A_1A_2...A_n$ be a hyperbolic convex polygon in the Poincaré disc, whose vertices are the points $A_1, A_2, ..., A_n$ of the disc and whose sides (directed counterclockwise) are $\mathbf{a}_1 = -A_1 \oplus A_2$, $\mathbf{a}_2 = -A_2 \oplus A_3, ..., \mathbf{a}_n = -A_n \oplus A_1$. Let the points $M_i, i = \overline{1, n}$ be located on the sides $\mathbf{a}_1$, $\mathbf{a}_2, ..., \mathbf{a}_n$ of the hyperbolic convex polygon $A_1A_2...A_n$ respectively. If the perpendiculars to the sides of the hyperbolic polygon at the points $M_1, M_2, ...,$ and M_n are concurrent, then the following holds:

$$|-A_1 \oplus M_1|^2 \ominus |-M_1 \oplus A_2|^2 \oplus |-A_2 \oplus M_2|^2 \ominus |-M_2 \oplus A_3|^2 \oplus ... \oplus |-A_n \oplus M_n|^2 \ominus |-M_n \oplus A_1|^2 = 0.$$

Observation 2.5.3. For $n = 3$ we obtain the hyperbolic Carnot theorem [60].

Theorem 2.5.4. Let ABC be a gyrotriangle in the Poincaré disc, whose gyropoints are A, B and C of the disc and whose sides (directed counterclockwise) are $a = -B \oplus C, b = -C \oplus A$ and $c = -A \oplus B$. Let the gyropoints A', B' and C' be located on the gyrosides a, b and c of the gyrotriangle ABC respectively, and let A'' be the reflection of A' about the midpoint of gyrosegment BC, and construct B'' and C'' similarly. If the perpendiculars to the gyrosides of the gyrotriangle at the points A', B' and C' are concurrent, and two of the three perpendiculars to the sides of the hyperbolic triangle at the points A'', B'' and C'' are concurrent, then the three perpendiculars are concurrent.

2.6 Trigonometric proof of Steiner-Lehmus theorem in hyperbolic geometry

In this note we choose the Poincaré disc model in order to present the hyperbolic version of the Steiner-Lehmus theorem. We mention that N.Sonmez [127] has presented a trigonometric proof for the Poincaré half plane model but his approach is different than ours. The Euclidean version of this well-known theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles [57]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by O.A.AbuArqob, H.E.Rabadi, J.S.Khitan [1], G.Gilbert, D.MacDonnell [71], H.Hajja [78], M.Levin [91], J.V.Malesevic [95] and A.P.Pargeter

[113]. We prove that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is not isosceles. The author has published this result in the paper [15].

To prove the Steiner-Lehmus's theorem we use the cosinus theorem, as follow:

Let ABC be a gyro-triangle in a Möbius gyro-vector space $(V_s, \oplus, \otimes)$,with vertices A, B, C, corresponding gyro-angles $\alpha, \beta, \gamma, 0 < \alpha + \beta + \gamma < \pi$, and side gyro-lengths (or, simply, sides) a, b, c. The gyro-angles of the gyro-triangle ABC are determined by its sides :

$$\begin{aligned}
\cos\alpha &= \frac{-a_s^2 + b_s^2 + c_s^2 - a_s^2 b_s^2 c_s^2}{2 b_s c_s} \cdot \frac{1}{1 - a_s^2}, \\
\cos\beta &= \frac{a_s^2 - b_s^2 + c_s^2 - a_s^2 b_s^2 c_s^2}{2 a_s c_s} \cdot \frac{1}{1 - b_s^2}, \\
\cos\gamma &= \frac{a_s^2 + b_s^2 - c_s^2 - a_s^2 b_s^2 c_s^2}{2 b_s a_s} \cdot \frac{1}{1 - c_s^2},
\end{aligned}$$

with $a_s = \frac{a}{s}$. ([134], p. 259])

Theorem 2.6.1. If the internal angle bisectors of two angles of a triangle are equal, then the triangle is not isosceles.

2.7 The Steiner's isogonals theorem for a hyperbolic triangle

Here we give hyperbolic version of Steiner isogonals theorem in the Einstein relativistic velocity model. The Steiner theorem states that the product of ratios in which two isogonals from a given vertex of a triangle cut the opposite side is constant, and equal to the ratio of the squares of adjacent sides [17]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by R.Johnson [84], N.A.Court [54], C. Coşniţă [53]. The author has published these results in the paper [19].

We use the law of sines, as follows:

In the hyperbolic triangle let α, β, γ denote at A, B, C and a, b, c denote the hyperbolic lengths of the sides opposite A, B, C, respectively, then

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}$$

([98], p.112)

Theorem 2.7.1. (The Hyperbolic Theorem of Steiner in Poincaré disc model). If the hyperbolic lines AP and AQ are two isogonals of a vertex A of a hyperbolic triangle ABC, and the points P and Q are on the side BC, then

$$\frac{\sinh(d(C,P))}{\sinh(d(B,P))}\cdot\frac{\sinh(d(C,Q))}{\sinh(d(B,Q))}=\left(\frac{\sinh(d(A,C))}{\sinh(d(A,B))}\right)^2$$

Corollary 2.7.2. If the hyperbolic line AP is a symmedian of a hyperbolic triangle ABC, and the point P is on the side BC, then

$$\frac{\sinh(d(C,P))}{\sinh(d(B,P))}=\left(\frac{\sinh b}{\sinh c}\right)^2. \tag{8}$$

In the same way, we present the hyperbolic version of Steiner theorem in the Einstein gyrovector space. The author has published these results in the paper [31].

Theorem 2.7.3. If the gyrolines AP and AQ are two isogonals of a vertex A of a gyrotriangle ABC, and the points P and Q are on the gyroside BC, then

$$\frac{\gamma_{|CQ|}\,|CQ|}{\gamma_{|BQ|}\,|BQ|}\cdot\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|BP|}\,|BP|}=\left(\frac{\gamma_{|CA|}\,|CA|}{\gamma_{|BA|}\,|BA|}\right)^2$$

Corollary 2.7.4. If the gyroline AP is a symmedian of a gyrotriangle ABC, and the point P is on the gyroside BC, then

$$\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|BP|}\,|BP|}=\left(\frac{\gamma_{|CA|}\,|CA|}{\gamma_{|BA|}\,|BA|}\right)^2.$$

Theorem 2.7.5. (The Hyperbolic Theorem of Steiner in Einstein Gyrovector Space). If the gyrolines AP and AQ are two isogonals of a vertex A of a gyrotriangle ABC, and the points P and Q are on the gyroside BC, then

$$\frac{\gamma_{|CQ|}\,|CQ|}{\gamma_{|BQ|}\,|BQ|}\cdot\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|BP|}\,|BP|}=\left(\frac{\gamma_{|CA|}\,|CA|}{\gamma_{|BA|}\,|BA|}\right)^2$$

Corollary 2.7.6. If the gyrolines AP and AQ are two isogonals of a vertex A of a gyrotriangle ABC, and the points P and Q are on the gyroside BC, then

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|AQ|}\,|AQ|}=\frac{\gamma_{|BA|}\,|BA|}{\gamma_{|BQ|}\,|BQ|}\cdot\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|CA|}\,|CA|}.$$

Corollary 2.7.7. If the gyrolines AP and AQ are two isogonals of a vertex A of a gyrotriangle ABC, and the points P and Q are on the gyroside BC, then

$$\frac{\gamma_{|CQ|}\,|CQ|}{\gamma_{|BQ|}\,|BQ|}+\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|BP|}\,|BP|}\geq 2\frac{\gamma_{|CA|}\,|CA|}{\gamma_{|BA|}\,|BA|}$$

2.8 Hyperbolic version of Mathieu's theorem

Here we give hyperbolic version of Mathieu theorem in the Poincaré Model of Hyperbolic Geometry. The Mathieu theorem states that if three lines from the vertices of a triangle are concurrent, their isogonals are also concurrent [85]. We just mention here few different proofs given by T. Lalescu [88], C. Barbu [17], C. Coşniţă [53]. The author has published these results in the paper [19].

Theorem 2.8.1. If three gyrolines from a gyrotriangle ABC, and concurrent at P, meet the opposite gyrosides at P_1, P_2, P_3 respectively, and the gyrolines $AQ_1, BQ_2,$ and CQ_3 are their isogonal gyrolines, and two of the gyrolines $AQ_1, BQ_2,$ and CQ_3 meet, then all three are also concurrent.

Observation 2.8.2. The point of concurrence (Q) of gyrolines AQ_1, BQ_2 and CQ_3 is called the isogonal conjugate of P.

Corollary 2.8.3. The incenter I of a gyrotriangle is its own isogonal conjugate of I.

Observation 2.8.4. From Mathieu's theorem result that the symmedians of the triangle ABC are concurrent. The point of concurrence of symmedians is Lemoine's point of triangle ABC.

Corollary 2.8.5. The isogonal conjugate of centroid of triangle ABC is Lemoine's point of triangle ABC.

2.9 The hyperbolic Nobbs theorem in the Poincaré disc model

Here we give hyperbolic version of Nobbs theorem. The well-known Nobbs theorem states that if A', B', C' are the points of contact of the incircle of triangle ABC: A' on side BC, B' on side CA, C' on side AB , and denote the intersection of AB and $A'B'$ as C'', and of AC and $A'C'$ as B'', and let A'' be the intersection of BC and $B'C'$, then the points A'', B'', and C'' are collinear. The points A'', B'', and C'' are known as *Nobbs' points*. The theorem tell us that the Nobbs' points of a triangle are collinear. The line on which the points lie on was named the *Gergonne line.* We just mention here few different proofs given by C. Barbu [17], C. Coşniţă [53].The author has published this result in the paper [33].

Lemma 2.9.1. Let A', B', C' be the points of contact of the h-incircle of the triangle ABC: A' on h-side BC, B' on h-side CA, C' on side h-AB, and I center of the h-incircle.Then $d(A, B') = d(A, C'), d(B, C') = d(B, A')$, and $d(C, A') = d(C, B')$.

Lemma 2.9.2. The h-lines joining the vertices of a h-triangle ABC to the tangent points A', B', and C' of the inscribed circle are h-concurrent at point Γ called the Gergonne Point of the h-triangle ABC.

Theorem 2.9.3. Let A', B', C' be the points of contact of the h-incircle of the h-triangle ABC: A' on h-side BC, B' on h-side CA, C' on h-side AB. If $B'C'$ and BC meet in A'', $C'A'$ and CA in B'', $A'B'$ and AB in C'', then A'', B'', and C'' are h-collinear.

2.10 The isotomic transversal theorem in hyperbolic geometry

Here we give hyperbolic versions of the isotomic transversal theorem. The well-known isotomic transversal theorem states that if l is a line not through any vertex of a triangle ABC so that l meets sidelines BC, CA, and AB in points A', B', and C', respectively, and let A'' be the reflection of A' about the midpoint of segment BC, and construct B'' and C'' similarly, then A'', B'', and C'' are collinear in a line known as the isotomic transversal of l [86]. We just mention here few different proofs given by K. Kimberling [86], C. Barbu [17], C. Coşniţă [53]. The author has published this result in the paper [20].

Theorem 2.10.1. (The isotomic transversal theorem). Let l be a gyroline not through any gyrovertex of a gyrotriangle ABC such that l meets gyroside BC, CA, and AB in gyropoints A', B', and C', respectively, and let A'' be the reflection of A' about the midpoint of gyrosegment BC, and construct B'' and C'' similarly, then the gyropoints A'', B'', and C'' are collinear.

2.11 The theorem of Neuberg in hyperbolic version

In this note we present the hyperbolic version of Neuberg theorem. The Neuberg theorem states that if three lines from a triangle ABC, and concurrent at P, meet the opposite sides at P_1, P_2, P_3 respectively; and if we cut off BQ_1, CQ_2, AQ_3 equal respectively P_1C, P_2A, P_3B, then AQ_1, BQ_2, and CQ_3 are concurrent [84]. We just mention here few different proofs given by C. Barbu [17], C. Coşniţă [53]. The author has published these results in the paper [20].

Theorem 2.11.1. If three gyrolines from a gyrotriangle ABC, and concurrent at P, meet the opposite gyrosides at P_1, P_2, P_3 respectively; and if we cut off $BQ_1, CQ_{2,}AQ_3$ equal respectively P_1C, P_2A, P_3B, and two of the gyrolines $AQ_1, BQ_{2,}$ and CQ_3 meet, then all three are concurrent.

Definition 2.11.2. The gyrolines $AQ_1, BQ_{2,}$ and CQ_3 are concurrent in a gyropoint Q, called the isotomic conjugate of P.

Corollary 2.11.3. The centroid G of a gyrotriangle is its own isotomic conjugate of G.

Corollary 2.11.4. If the Nagel point of the gyrotriangle ABC exists, then the Gergonne gyropoint is your isotomic conjugate.

2.12 The theorem of Gülicher in the Poincaré disc model

We present a proof of Gülicher's theorem in the Poincaré disc model of hyperbolic geometry. Gülicher's theorem states that if $Q_1Q_2Q_3$ is the cevian triangle of point Q with respect to the triangle $P_1P_2P_3$, and $R_1R_2R_3$ is the cevian triangle of point R with respect to the triangle $Q_1Q_2Q_3$, then the lines P_1R_1, P_2R_2, and P_3R_3 are concurrent [77]. The author has published this result in the paper [34].

Theorem 2.12.1 Let $Q_1Q_2Q_3$ be the cevian gyrotriangle of gyropoint Q with respect to the gyrotriangle $P_1P_2P_3$, and Q is located inside the gyrotriangle $P_1P_2P_3$. Let $R_1R_2R_3$ be the cevian gyrotriangle of gyropoint R with respect to the gyrotriangle $Q_1Q_2Q_3$, and R is located inside the gyrotriangle $Q_1Q_2Q_3$. Then the gyrolines P_1R_1, P_2R_2, and P_3R_3 are concurrent.

2.13 The bisector theorem in hyperbolic geometry

In this note we present the hyperbolic version of bisector theorem in the Poincaré disc model of hyperbolic geometry. The bisector theorem states that if AD is a bisector of interior angle $\widehat{BAC}$ of the triangle ABC, then $\frac{DB}{DC} = \frac{AB}{AC}$. We mention some different demonstrations of this theorem, given by C. Coşniţă [53], A. Johnson [84]. The author has published these results in the paper [34].

Theorem 2.13.1. Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and let D be a point lying on side BC of the gyrotriangle so that AD is a interior bisector of gyroangle $\angle BAC$. Then

$$\frac{(DB)_\gamma}{(DC)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma},$$

where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$.

Theorem 2.13.2. (**Converse of the interior bisector theorem**). Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and let D be a point lying on side BC of the gyrotriangle so that $\frac{(DB)_\gamma}{(DC)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma}$, then AD is a bisector of gyroangle $\angle BAC$, where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$.

Theorem 2.13.3. Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and let E be a point lying on gyroline BC so that AE is the exterior bisector of gyroangle $\angle BAC$. Then

$$\frac{(EB)_\gamma}{(EC)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma},$$

where $v_\gamma = \frac{v}{1-v^2}$.

Theorem 2.13.4. Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and let E be a point lying on the gyroline BC of the gyrotriangle so that $\frac{(EB)_\gamma}{(EC)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma}$, and the exterior bisector of angle $\angle BAC$ intersect the gyroline BC, then AE is the exterior bisector of gyroangle $\angle BAC$.

Corollary 2.13.5. Let ABC be a gyrotriangle in a Möbius gyrovector space. If D is a point on BC such that AD is a exterior bisector of angle $\angle BAC$, and BE, CF are the interior bisectors of the angles $\angle ABC$ and $\angle BCA$ respectively, then the points D, E and F are colliniar.

Corollary 2.13.6. (Pătraşcu's Theorem for Hyperbolic Gyrotriangle). Let D be a point on the gyroside BC of a gyrotriangle ABC, and let E and F be the points lying on sides CA and AB of the gyrotriangle ABC so that DE is a bisector of gyroangle $\angle ADC$, and DF is a bisector of gyroangle $\angle ADB$, then the gyrolines AD, BE, and CF are concurrent.

Theorem 2.13.7. Let $Q_1Q_2Q_3$ be the cevian gyrotriangle of gyropoint Q with respect to the gyrotriangle $P_1P_2P_3$, and Q is located inside the gyrotriangle $P_1P_2P_3$. If the bisectors of gyroangles of gyrotriangle $P_1P_2P_3$ meet the gyrosides Q_2Q_3, Q_3Q_1, and Q_1Q_3 at the gyropoints R_1, R_2, and R_3, respectively, then the gyrolines Q_1R_1, Q_2R_2, and Q_3R_3 are concurrent.

Observation 2.13.8. Many of the theorems of Euclidean geometry have a relatively similar form in the Poincaré model of hyperbolic geometry, Gülicher's theorem is an example in this respect. In the Euclidean limit of large s, $s \to \infty$, v_γ reduces to v, so Gülicher's theorem for hyperbolic triangle reduces to the Gülicher's theorem of euclidian geometry.

2.14 Zajic's theorem in the Poincaré disc model

Here we give hyperbolic version of Zajic's theorem. The Zajic theorem states that if A' is the point of contact of the incircle of triangle ABC on the side BC, X is a point on the side BC, and T_1 and T_2 are the points of contact of the incircles of the triangles ABX and ACX respectively, on the side BC, then the segments $A'X$ and T_1T_2 are equal [79]. The author has published this result in the paper [30].

Theorem 2.14.1. If A' is the point of contact of the incircle of $h-$triangle ABC on the $h-$side BC, X is a point on the $h-$side BC, and T_1 and T_2 are the points of contact of the incircles of the $h-$triangles ABX and ACX respectively, on the $h-$side BC, then the hyperbolic distances $d(A', X)$ and $d(T_1, T_2)$ are equal.

Corollary 2.14.2. Let X be a point on the $h-$side BC of $h-$triangle ABC, and A', A_1, A_2 are the points of contact of the incircles of the $h-$triangles ABC, ABX, and ACX, respectively, on the $h-$side BC. Then, $d(A', A_1) = d(X, A_2)$ and $d(A', A_2) = d(X, A_1)$.

Corollary 2.14.3. If A' is the point of contact of the incircle of $h-$triangle ABC on the $h-$side BC, then both incircles of $h-$triangles ABA' and ACA' touch $h-$line AA' at the same point.

2.15 The Carnot's theorem in the Poincaré upper half-plane model

Here we give a hyperbolic version of the Carnot theorem in the Poincaré upper half-plane model. The well-known Carnot theorem states that if the points A', B', and C' are located on the sides BC, AC, and respectively AB of the triangle ABC, then the perpendiculars to the sides of the triangle at points A', B', and C' are concurrent if and only if

$$AC'^2 - BC'^2 + BA'^2 - CA'^2 + CB'^2 - AB'^2 = 0.$$

The standard simple proof is based on the theorem of Pythagoras. For more details we refer to the monograph of C. Barbu [17]. We mention that O. Demirel and E. Soytürk [60] gave the hyperbolic form of Carnot's theorem in the Poincaré disc model of hyperbolic geometry. The author has published these results in the paper [41].

We use the hyperbolic Pitagora's theorem:

Let ABC be a hyperbolic triangle with a right angle at C. If a, b, c, are the hyperbolic lengths of the sides opposite A, B, C, respectively, then

$$\cosh c = \cosh a \cdot \cosh b. \tag{2}$$

For the proof of the theorem see [98].

Theorem 2.15.1. Let $\triangle ABC$ be a hyperbolic triangle. Let the points A', B', and C' be located on the sides BC, CA and AB of the hyperbolic triangle ABC respectively. If the perpendiculars to the sides of the hyperbolic triangle at the points A', B', and C' are concurrent in the point M, then the following relations hold:

$$i)\ \cosh MA'(\cosh A'B - \cosh A'C) + \cosh MB'(\cosh B'C - \cosh B'A) +$$

$$\cosh MC'(\cosh C'A - \cosh C'B) = 0,$$

$$ii)\ \frac{\cosh A'B}{\cosh A'C} \cdot \frac{\cosh B'C}{\cosh B'A} \cdot \frac{\cosh C'A}{\cosh C'B} = 1.$$

Naturally, one may wonder whether the converse of the Carnot theorem exists. Indeed, a partially converse theorem does exist as we will show in the following theorem.

Theorem 2.15.2. Let $\triangle ABC$ be a hyperbolic triangle. Let the points A', B', and C' be located on the sides BC, CA and AB of the hyperbolic triangle ABC respectively. If the perpendiculars to the sides of the hyperbolic triangle at the points B' and C' are concurrent in the point M and the following relation holds

$$\frac{\cosh A'B}{\cosh A'C} \cdot \frac{\cosh B'C}{\cosh B'A} \cdot \frac{\cosh C'A}{\cosh C'B} = 1,$$

then the point M is on the perpendicular to BC at the point A'.

Observation 2.15.3. We gave the generalisation of the Carnot theorem:

Let $A_1A_2...A_n$ be a hyperbolic convex polygon in the Poincaré upper half-plane, whose vertices are the points $A_1, A_2, ..., A_n$ and whose sides (directed counterclockwise) are $a_1 = A_1A_2$, $a_2 = A_2A_3, ..., a_n = A_nA_1$. Let the points M_i, $i = \overline{1,n}$ be located on the sides $a_1, a_2, ..., a_n$ of the hyperbolic convex polygon $A_1A_2...A_n$ respectively. If the perpendiculars to the sides of the hyperbolic polygon at the points $M_1, M_2, ..., M_n$ are concurrent in a point M, then the following equalities hold

$$\sum_{i=1}^{n} \cosh MA_i(\cosh M_iA_i - \cosh M_iA_{i+1}) = 0$$

where $A_{n+1} = A_1$,

$$\frac{\cosh M_1A_1}{\cosh M_1A_2} \cdot \frac{\cosh M_2A_2}{\cosh M_2A_3} \cdot ... \cdot \frac{\cosh M_nA_n}{\cosh M_nA_1} = 1.$$

Let $A_1A_2...A_n$ be a hyperbolic convex polygon in the Poincaré upper half-plane, whose vertices are the points $A_1, A_2, ..., A_n$ and whose sides (directed counterclockwise) are $a_1 = A_1A_2$, $a_2 = A_2A_3, ..., a_n = A_nA_1$. Let the points M_i, $i = \overline{1,n}$ be located on the sides $a_1, a_2, ..., a_n$ of the hyperbolic convex polygon $A_1A_2...A_n$ respectively. If the perpendiculars to the sides of the hyperbolic polygon at the points $M_1, M_2, ..., M_{n-1}$ are concurrent in a point M and the following relation holds

$$\frac{\cosh M_1A_1}{\cosh M_1A_2} \cdot \frac{\cosh M_2A_2}{\cosh M_2A_3} \cdot ... \cdot \frac{\cosh M_nA_n}{\cosh M_nA_1} = 1,$$

then the point M is on the perpendicular to A_nA_1 at the point M_n.

2.16 The orthopole theorem in the Poincaré upper half-plane model

Here, in this study, we give hyperbolic version of the orthopole theorem in the Poincaré upper half-plane of hyperbolic geometry. The well-known orthopole theorem states that if A', B', C' are the projections of the vertices A, B, C of the triangle ABC on a straight line d, the perpendiculars from A' on BC, from B' on CA, and from C' on AB are concurrent at a point called the orthopole of d for the triangle ABC [73]. This result has a simple statement but it is of great interes. We just mention here few different proofs given by J. Neuberg [105], W. Gallaty [70]. The author has published this result in the paper [40].

Theorem 2.16.1. Let A', B', C' be the projections of the vertices A, B, C of the hyperbolic triangle ABC on a geodesic segment d. If two of the three perpendiculars from A' on BC, from B' on CA, and from C' on AB are concurrent, then the three perpendiculars are concurrent.

Observation 2.16.2. The point of concurrence of the geodesic segments $A'A''$, $B'B''$, and $C'C''$ is called the orthopol of the geodesic line d in relation to the triangle ABC.

2.17 The theorem of Stewart in hyperbolic version

Here, in this study, we present a proof of the hyperbolic Stewart theorem in the Einstein relativistic velocity model of hyperbolic geometry. The well-known Stewart theorem states that if a point D lies between the vertices A and C of the triangle ABC, then

$$AB^2 \cdot DC + BC^2 \cdot AD - BD^2 \cdot AC = AC \cdot DC \cdot AD$$

[58]. We just mention here few different proofs given by O. Demirel [61], W. Stothers [130] The author has published these results in the paper [22].

Theorem 2.17.1. (The theorem of Stewart in hyperbolic version). If a point D lies between the vertices A and C of the gyrotriangle ABC, then

$$\gamma_{|AB|} \cdot \gamma_{|DC|} \cdot |DC| + \gamma_{|AC|} \cdot \gamma_{|BD|} \cdot |BD| - \gamma_{|AD|} \cdot \gamma_{|DC|} \cdot \gamma_{|BD|} \cdot [|BD| + |DC|] = 0,$$

where $|DC|, |BD|$, and $|BC|$ noted the gyrolengths of gyrosegments DC, BD, and BC, respectively.

Corollary 2.17.2. (Median theorem in hyperbolic geometry). Let ABC be a gyrotriangle, and D is a gyromidpoint of the gyrosegment BC. Then,

$$\gamma_{|AD|} = \frac{\gamma_{|AB|} + \gamma_{|AC|}}{2 \cdot \gamma_{|DC|}}.$$

Corollary 2.17.3. (The gamma factor of an angle bisector). Let ABC be a gyrotriangle, and let D be a point lying on side BC of the gyrotriangle so that AD is a bisector of gyroangle $\angle BAC$. Then

$$\gamma_{|AD|} = \frac{\gamma_{|AB|} \cdot |DC|}{\gamma_{|BD|} \cdot [|BD| + |DC|]} \cdot \left(1 + \frac{|AB|}{|AC|}\right).$$

2.18 The theorem of Van Aubel in hyperbolic version

Here, in this study, we give hyperbolic version of Van Aubel theorem. The well-known Van Aubel theorem states that if ABC is a triangle and AD, BE, CF are concurrent cevians at P, then $\frac{AP}{PD} = \frac{AE}{EC} + \frac{AF}{FB}$ [17]. We just mention here few different proofs given by C. Barbu [17], N. Minculete [99]. The author has published these results in the paper [25].

Theorem 2.18.1. If the point P does lie on any side of the hyperbolic triangle ABC, and BC meets AP in D, CA meets BP in E, and AB meets CP in F, then

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PD|}\,|PD|} = \frac{\gamma_{|BC|}\,|BC|}{2}\left[\frac{\gamma_{|AE|}\,|AE|}{\gamma_{|EC|}\,|EC|}\cdot\frac{1}{\gamma_{|BD|}\,|BD|} + \frac{\gamma_{|FA|}\,|FA|}{\gamma_{|FB|}\,|FB|}\cdot\frac{1}{\gamma_{|CD|}\,|CD|}\right].$$

Corollary 2.18.2. Let G be the centroid of the hyperbolic triangle ABC, and D, E, F are the midpoints of hyperbolic sides BC, CA, and AC respectively. Then,

$$\frac{\gamma_{|AG|}\,|AG|}{\gamma_{|GD|}\,|GD|} = \frac{\gamma_{|BC|}\,|BC|}{2}\left[\frac{1}{\gamma_{|BD|}\,|BD|} + \frac{1}{\gamma_{|CD|}\,|CD|}\right].$$

Corollary 2.18.3. Let I be the incenter of the hyperbolic triangle ABC, and let the angle bisectors be AD, BE, and CF. Then,

$$\frac{\gamma_{|AI|}\,|AI|}{\gamma_{|ID|}\,|ID|} = \frac{1}{2}\left[\frac{\gamma_{|AB|}\,|AB|}{\gamma_{|BD|}\,|BD|} + \frac{\gamma_{|AC|}\,|AC|}{\gamma_{|CD|}\,|CD|}\right].$$

2.19 Smarandache's minimum theorem in hyperbolic geometry

In this note we give hyperbolic version of Smarandache minimum theorem in the Einstein relativistic velocity model of hyperbolic geometry. The well-known Smarandache minimum theorem states that if ABC is a triangle and AA', BB', CC' are concurrent cevians at P, then $\frac{PA}{PA'}\cdot\frac{PB}{PB'}\cdot\frac{PC}{PC'} \geq 8$ and $\frac{PA}{PA'} + \frac{PB}{PB'} + \frac{PC}{PC'} \geq 6$ [125]. The author has published this result in the paper [23].

Theorem 2.19.1. If ABC is a gyrotriangle and AA', BB', CC' are concurrent cevians at P, then

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PA'|}\,|PA'|}\cdot\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PB'|}\,|PB'|}\cdot\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|PC'|}\,|PC'|} \geq 1,$$

and

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PA'|}\,|PA'|}+\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PB'|}\,|PB'|}+\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|PC'|}\,|PC'|}\geq 3.$$

Many of the theorems of Euclidean geometry have a relatively similar form in the Einstein relativistic velocity model, Smarandache minimum theorem is an example in this respect. In the Euclidean limit of large s, $s\rightarrow\infty$, gamma factor γ_v reduces to 1, so that the gyroinequalities (11) and (12) reduces to the

$$\frac{PA}{PA'}\cdot\frac{PB}{PB'}\cdot\frac{PC}{PC'}\geq 1,$$

and

$$\frac{PA}{PA'}+\frac{PB}{PB'}+\frac{PC}{PC'}\geq 3,$$

in Euclidean geometry. We observe that the previous inequalities are "weaker" than the inequalities of Smarandache's theorem of minimum.

2.20 Pappus's harmonic theorem in the Einstein relativistic velocity model

Here, in this study, we present a proof of Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry. Pappus's harmonic theorem states that if $A'B'C'$ is the cevian triangle of point M with respect to the triangle ABC so that the lines $B'C'$ and BC meet at A'', then $\frac{A''B}{A''C}=\frac{A'B}{A'C}$ [57]. The author has published these results in the paper [29].

Theorem 2.20.1. (Pappus's harmonic theorem). If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC so that the gyrolines $B'C'$ and BC meet at A'', then

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|}=\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}.$$

Corollary 2.20.2. If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC so that the gyrolines $B'C'$ and BC meet at A'', and AA' is a bisector of gyroangle $\measuredangle BAC$, then

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}=\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}.$$

Definition 2.20.3. The symmetric of the median with respect to the internal bisector issued from the same vertex is called symmedian.

Corollary 2.20.4. If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC so that the gyrolines $B'C'$ and BC meet at A'', and AA' is a bisector of gyroangle $\measuredangle BAC$, and AA_1 is a antibisector of gyroangle $\measuredangle BAC$, then

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|A_1B|}|A_1B|}{\gamma_{|A_1C|}|A_1C|}\right)^{-1}.$$

Corollary 2.20.5. If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC so that the gyrolines $B'C'$ and BC meet at A'', and AA' is a symmedian of gyroangle $\measuredangle BAC$, and the point A' is on the gyroside BC, then

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}\right)^{2}.$$

Theorem 2.20.6. If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC so that the gyrolines $B'C'$ and BC meet at A'', and AA' is a bisector of gyroangle $\measuredangle BAC$, the gyrolines $A'C'$ and BB' meet at D, $A'B'$ and CC' meet at E,,AD and BC meet at D', and AE and BC meet in E', then

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|E'A'|}|E'A'|}{\gamma_{|E'C|}|E'C|}.$$

2.21 Smarandache's cevian triangle theorem in hyperbolic geometry

In this study we present a proof of Smarandache's cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry. Smarandache's cevian triangle theorem states that if $A_1B_1C_1$ is the cevian triangle of point P with respect to the triangle ABC, then

$$\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B \cdot B_1C \cdot C_1A}$$

[124]. The author has published this result in the paper [13].

Teorema 2.21.1. If $A_1B_1C_1$ is the cevian gyrotriangle of gyropoint P with respect to the gyrotriangle ABC, then

$$\frac{\gamma_{|PA|}|PA|}{\gamma_{|PA_1|}|PA_1|} \cdot \frac{\gamma_{|PB|}|PB|}{\gamma_{|PB_1|}|PB_1|} \cdot \frac{\gamma_{|PC|}|PC|}{\gamma_{|PC_1|}|PC_1|} = \frac{\gamma_{|AB|}|AB| \cdot \gamma_{|BC|}|BC| \cdot \gamma_{|CA|}|CA|}{\gamma_{|AB_1|}|AB_1| \cdot \gamma_{|BC_1|}|BC_1| \cdot \gamma_{|CA_1|}|CA_1|}.$$

2.22 Inegalities on hyperbolic triangle

In this section we present some inequalities in a hyperbolic triangle. The author has published these results in the paper [31].

Theorem 2.22.1. Let I be the incenter of the hyperbolic triangle ABC. If $\widehat{A} < \widehat{B} < \widehat{C}$, then $d(A, I) > d(B, I) > d(C, I)$.

Theorem 2.22.2. If the perpendicular bisector of the side CA of hyperbolic triangle ABC intersect the side BC in D, and M is a point of perpendicular bisector of CA, then

$$d(M, A) + d(M, B) > d(D, A) + d(D, B).$$

Theorem 2.22.3. Let AA', BB', CC' be the medians of hyperbolic triangle ABC. Then,

$$d(A, A') + d(B, B') + d(C, C') < d(A, B) + d(B, C) + d(C, A)$$

Theorem 2.22.4. Let AA', BB', CC' be the medians of hyperbolic triangle ABC. Then,

$$2 \cdot [d(A', B') + d(B', C') + d(C', A')] < d(A, B) + d(B, C) + d(C, A)$$

Theorem 2.22.5. In any hyperbolic triangle the following inequalities hold:

$$\cosh a + \cosh b + \cosh c \leq \alpha + 2\left(\sinh^2 \frac{a}{2} + \sinh^2 \frac{b}{2} + \sinh^2 \frac{c}{2}\right),$$

$$\cosh a + \cosh b \leq 2\left(\cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2}\right),$$

$$\sinh a + \sinh b + \sinh c <$$

$$4\left(\sinh \frac{a}{2} + \sinh \frac{b}{2} + \sinh \frac{c}{2} + \sinh \frac{a}{2} \sinh^2 \frac{a}{4} + \sinh \frac{b}{2} \sinh^2 \frac{b}{4} + \sinh \frac{c}{2} \sinh^2 \frac{c}{4}\right)$$

where $\alpha \geq 3$, and denote by a, b, c the hyperbolic lenghtsides of the triangle.

Theorem 2.22.6. In any hyperbolic triangle the following inequality hold:

$$3 \cosh \frac{a + b + c}{3} \leq \cosh a + \cosh b + \cosh c$$

Observation 2.22.7. From Theorems 2.22.5 and 2.22.6 we obtain:

$$3 \cosh \frac{a + b + c}{3} \leq \cosh a + \cosh b + \cosh c \leq \alpha + 2\left(\sinh^2 \frac{a}{2} + \sinh^2 \frac{b}{2} + \sinh^2 \frac{c}{2}\right),$$

where $\alpha \geq 3$.

Corollary 2.22.8. In any hyperbolic triangle the following inequality hold:

$$\sinh^2 \frac{a}{2} + \sinh^2 \frac{b}{2} + \sinh^2 \frac{c}{2} \geq \frac{3}{2}\left(1 - \cosh \frac{a+b+c}{3}\right).$$

Theorem 2.22.9. In any hyperbolic triangle the following inequality hold:

$$\cosh \frac{c}{2} < \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2}.$$

Theorem 2.22.10. In any hyperbolic triangle the following inequality hold:

$$\frac{\sinh a}{a} + \frac{\sinh b}{b} + \frac{\sinh c}{c} < 2 + \frac{2}{3}\left(\cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} + \cosh^2 \frac{c}{2}\right).$$

Corollary 2.22.11. In any hyperbolic triangle the following inequality hold:

$$\frac{\sinh a}{a} + \frac{\sinh b}{b} + \frac{\sinh c}{c} < \frac{2}{3}\left(5 + \sinh^2 \frac{a}{2} + \sinh^2 \frac{b}{2} + \sinh^2 \frac{c}{2}\right).$$

Theorem 2.22.12. In any hyperbolic triangle the following inequality hold:

$$6 \sinh \frac{a+b+c}{6} \leq \sinh a + \sinh b + \sinh c.$$

Observation 2.22.13. In any hyperbolic triangle the following inequality hold:

$$\sinh \frac{a+b+c}{6} < \frac{2}{3}\left(\sinh \frac{a}{2} + \sinh \frac{b}{2} + \sinh \frac{c}{2} + \sinh \frac{a}{2} \sinh^2 \frac{a}{4} + \sinh \frac{b}{2} \sinh^2 \frac{b}{4} + \sinh \frac{c}{2} \sinh^2 \frac{c}{4}\right)$$

Theorem 2.22.14. In any hyperbolic triangle the following inequality hold:

$$\sinh \frac{a}{3} < \frac{2}{3}\left(\sinh \frac{a}{2} + \sinh \frac{b}{2} + \sinh \frac{c}{2} + \sinh \frac{a}{2} \sinh^2 \frac{a}{4} + \sinh \frac{b}{2} \sinh^2 \frac{b}{4} + \sinh \frac{c}{2} \sinh^2 \frac{c}{4}\right)$$

2.23 Andrica-Iwata's inequality in Hyperbolic Triangle

In the studies by Andrica [5] and Iwata [81], a basic theorem is established to be a source of inequalities from a euclidian triangle. Andrica-Iwata's theorem states that if ABC is a triangle, and the segments BC, CA, AB have lengths a, b, c, respectively, then

$$\frac{a}{b+c} \geq \sin \frac{A}{2}. \tag{2.1}$$

This result has a simple statement but it is of great interest. We just mention here few different proofs given by D. Mitrinović, J. Pečarić, V. Volenec [102], C. Ţiu [Aplications in trigonometry, 1992]. In what follows we are going to present the counterpart of these results for the hyperbolic triangle. The author has published these results in the paper [35].

We use in our demonstrations this theorems:

Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\sinh(b) \cdot \sinh(c) \cdot \cos(A) = \cosh(b) \cdot \cosh(c) - \cosh(a)$$

([57], p. 238)

Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\frac{\sinh(a)}{\sin A} = \frac{\sinh(b)}{\sin B} = \frac{\sinh(c)}{\sin C}$$

([57], p. 238).

If AD is a median of the hyperbolic triangle ABC and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, $d(A,D) = d$, then

$$\cosh(d) = \frac{\cosh(b) + \cosh(c)}{2\cosh\left(\frac{a}{2}\right)}$$

[130]

Theorem 2.23.1. Let ABC be a hyperbolic acute triangle or a right hyperbolic triangle in A, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then the following inequality holds

$$\frac{\sinh(a)}{\sinh(b) + \sinh(c)} < \frac{1}{\sqrt{2}\cos\frac{\varepsilon + A}{2}},$$

where $\varepsilon = \pi - (A + B + C)$ is the defect of the triangle ABC.

Corollary 2.23.2. Let ABC be a hyperbolic acute triangle or a right hyperbolic triangle in A, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then the following inequality holds

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)} < \frac{1}{\sqrt{2}\cos\frac{A}{2}}.$$

Corollary 2.23.3. Let ABC be a hyperbolic acute triangle or a right hyperbolic triangle in A, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then the following inequality holds

$$\sinh(a) < \sinh(b) + \sinh(c).$$

Corollary 2.23.4. Let ABC be a hyperbolic acute triangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then the following inequality holds

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)} < \frac{1}{\cos\frac{\varepsilon}{2}}.$$

Corollary 2.23.5. Let ABC be a hyperbolic acute triangle, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)} + \frac{\sinh(b)}{\sinh(a)+\sinh(c)} + \frac{\sinh(c)}{\sinh(b)+\sinh(a)} < \frac{3}{\cos\frac{\varepsilon}{2}}.$$

Theorem 2.23.6. Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)} + \frac{\sinh(b)}{\sinh(a)+\sinh(c)} + \frac{\sinh(c)}{\sinh(b)+\sinh(a)} \geq \frac{3}{2}.$$

Observation 2.23.7. The equality

$$\frac{\sinh(a)}{\sinh(b)+\sinh(c)} + \frac{\sinh(b)}{\sinh(a)+\sinh(c)} + \frac{\sinh(c)}{\sinh(b)+\sinh(a)} = \frac{3}{2}$$

holds if and only if ABC is a equilateral triangle.

Theorem 2.23.8. Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\cosh(b) + \cosh(c) > 2\cosh\left(\frac{a}{2}\right).$$

Theorem 2.23.9. If AD is a median of the hyperbolic triangle ABC and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, $d(A,D) = d$, then

$$\sinh(d) > \frac{\cosh(b)-\cosh(c)}{2\sinh\left(\frac{a}{2}\right)}.$$

Corollary 2.23.10. If AD is a median of the hyperbolic triangle ABC and the segments have hyperbolic lengths $d(B,C)=a$, $d(C,A)=b$, $d(A,B)=c$, $d(A,D)=d$, then

$$2\sqrt{\sinh\frac{b+c}{2}\left|\sinh\frac{b-c}{2}\right|}-\sinh\left(\frac{a}{2}\right)<\sinh(d)<\frac{1}{\sqrt{2}\cos C}\left[\sinh\left(\frac{a}{2}\right)+\sinh(b)\right]$$

Theorem 2.23.11. Let ABC be a hyperbolic triangle, and the segments have hyperbolic lengths $d(B,C)=a$, $d(C,A)=b$, $d(A,B)=c$, then

$$\sinh(a)\geq\sqrt{\cosh(a)-\cosh(b-c)}$$

Observation 2.23.12. Using the formula

$$\cosh(b)-\cosh(c)=2\sinh\left(\frac{b+c}{2}\right)\sinh\left(\frac{b-c}{2}\right),$$

in the previous result we can write

$$\sinh(a)\geq\sqrt{2\sinh\left(\frac{a+b-c}{2}\right)\sinh\left(\frac{a+c-b}{2}\right)}.$$

and the similar relation for $\sinh(b)$ and $\sinh(c)$.

Corollary 2.23.13. Let ABC be a hyperbolic triangle and the segments have hyperbolic lengths $d(B,C)=a$, $d(C,A)=b$, $d(A,B)=c$, then the following inequalities hold

$$2\sqrt{2}\prod_{ciclic}\sinh(s-a)\leq\prod_{ciclic}\sinh(a)<\frac{\prod\limits_{ciclic}[\sinh(a)+\sinh(b)]}{2\sqrt{2}\prod\limits_{ciclic}\cos\frac{A}{2}}$$

where s is the semiperimeter of the triangle ABC.

2.24 Cusa's inequality in hyperbolic triangle

In this paper we gave some inequalities in a hyperbolic triangle using Cusa's inequality. The Cusa-Huygens inequality is presented as follows

$$(\cos x)^{1/3}<\frac{\sin x}{x}<\frac{2+\cos x}{3}\quad\left(0<x<\frac{\pi}{2}\right).$$

The left-hand side inequality first appeared in [101], while the right-hand side inequality was first mentioned by Nicolaus de Cusa (1401-1464). We just mention here few different proofs given by Baricz [42], Huygens [80], Klén, Visuri, Vuorinen [87], Mortici [104], Neuman, Sándor [108]. Newman has given in [106] a hyperbolic form for Cusa's inequality as follows:

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2+\cosh x}{3} \quad (x \neq 0).$$

We use the following inequalities [87]:

$$\cosh\sqrt{xy} \leq \frac{\cosh x + \cosh y}{2},$$

$$\frac{\sinh x}{x} < \frac{1}{2} + \frac{1}{2}\cosh x,$$

$$\frac{\sinh\sqrt{xy}}{\sqrt{xy}} < \frac{1}{2}\left(\frac{\sinh x}{x} + \frac{\sinh y}{y}\right),$$

where $x, y \in (0, \infty)$. The author has published these results in the paper [37].

Theorem 2.24.1. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$3\cos^{\frac{1}{3}}(\varepsilon + A) < 1 + \left(\frac{\sinh b + \sinh c}{\sinh a}\right)^2,$$

where $\varepsilon = \pi - (A+B+C)$ is the defect of the triangle ABC.

Theorem 2.24.2. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\cos A + \cos B + \cos C < \frac{3(1+R^2)^2}{4R^4} \cdot \frac{\sin\left(\frac{\pi+\varepsilon/2}{3}\right)}{\sin(\varepsilon/2)},$$

where $R = \sqrt{\frac{\sin(A+\varepsilon/2)\sin(B+\varepsilon/2)\sin(C+\varepsilon/2)}{\sin(\varepsilon/2)}}$.

Corollary 2.24.3. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\cos A + \cos B + \cos C < \frac{3(1+R^2)^2}{4R^4\sin(\varepsilon/2)}.$$

Theorem 2.24.4. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\cos A\cos B\cos C < \frac{(1+R^2)^6}{64R^{10}} \cdot \frac{1}{\sin^2(\varepsilon/2)}.$$

Theorem 2.24.5. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\cos\frac{A}{2} \leq \frac{1+R^2}{2R^2} \cdot \sqrt{\frac{\sin\left(A+\varepsilon/2\right)}{\sin(\varepsilon/2)}},$$

where $R = \sqrt{\frac{\sin(A+\varepsilon/2)\sin(B+\varepsilon/2)\sin(C+\varepsilon/2)}{\sin(\varepsilon/2)}}$.

Corollary 2.24.6. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\cos\frac{A}{2} \leq \frac{1+R^2}{2R^2} \cdot \frac{1}{\sqrt{\sin(\varepsilon/2)}}$$

Corollary 2.24.7. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq \frac{\left(1+R^2\right)^3}{R^5} \cdot \frac{1}{\sqrt{\sin(\varepsilon/2)}}.$$

Corollary 2.24.8. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3(1+R^2)}{2R^2} \cdot \sqrt{\frac{\sin\left(\frac{\pi+\varepsilon/2}{3}\right)}{\sin(\varepsilon/2)}}$$

Theorem 2.24.9. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\sin A + \sin B + \sin C \leq \frac{3(1+R^2)}{R^2} \cdot \sqrt{\frac{\sin\left(\frac{\pi+\varepsilon/2}{3}\right)}{\sin(\varepsilon/2)}}.$$

Theorem 2.24.10. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C) = a$, $d(C,A) = b$, $d(A,B) = c$, then

$$\sin A \sin B \sin C \leq \frac{\left(1+R^2\right)^3}{R^5} \cdot \frac{1}{\sqrt{\sin(\varepsilon/2)}}.$$

Theorem 2.24.11. Let $x > 0$. Then the inequality

$$\frac{\sinh\sqrt{xy}}{\sqrt{xy}} < \frac{1}{4}\left(2+\cosh x+\cosh y\right)$$

holds.

Corollary 2.24.12. Let ABC be a hyperbolic acutetriangle and the segments have hyperbolic lengths $d(B,C)=a$, $d(C,A)=b$, $d(A,B)=c$, then the inequality

$$\frac{\sinh\sqrt{ab}}{\sqrt{ab}}+\frac{\sinh\sqrt{bc}}{\sqrt{bc}}+\frac{\sinh\sqrt{ca}}{\sqrt{ca}}<\frac{3}{2}+\frac{1}{2}(\cosh a+\cosh b+\cosh c)$$

holds.

Corollary 2.24.13. Let ABC be a hyperbolic triangle and the segments have hyperbolic lengths $d(B,C)=a$, $d(C,A)=b$, $d(A,B)=c$, then the inequality

$$\cosh\sqrt{ab}+\cosh\sqrt{bc}+\cosh\sqrt{ca}\leq\cosh a+\cosh b+\cosh c$$

holds.

Theorem 2.24.14. Let $x>0$. Then the inequality

$$\sqrt{\frac{\sinh x}{x}}<\frac{1}{2\sqrt{2}}\left(1+2\cosh\frac{x}{2}\right)$$

holds.

Corollary 2.24.15. Let ABC be a hyperbolic triangle and the segments have hyperbolic lengths $d(B,C)=a$, $d(C,A)=b$, $d(A,B)=c$, then the inequality

$$\sqrt{\frac{\sinh a}{a}}+\sqrt{\frac{\sinh b}{b}}+\sqrt{\frac{\sinh c}{c}}<\frac{3}{2\sqrt{2}}+\frac{1}{\sqrt{2}}(\cosh a+\cosh b+\cosh c)$$

holds.

2.25 On Panaitopol and Jordan type inequalities

In this paper we will consider inequalities of Panaitopol and Jordan type by using hyperbolic trigonometric functions. We will find upper and lower bounds for the product

$$\left(1+\frac{\alpha}{\sinh^m x}\right)\left(1+\frac{\beta}{\cosh^n x}\right),$$

where m,n,α, and β are positive numbers. We present some Jordan's type inequalities with hyperbolic trigonometric functions. Panaitopol [Problems by geometry (romanian)] proposed two inequalities as the following statements.

Problem 1. If $0<x<\pi/2$ and $a,b\in(0,\infty)$, then

$$\left(1+\frac{a}{\sin x}\right)\left(1+\frac{b}{\cos x}\right)\geq\left(1+\sqrt{2ab}\right)^2.$$

Problem 2. If $0 < x < \pi/2$ and n is a natural number, then

$$\left(1+\frac{1}{\sin^n x}\right)\left(1+\frac{1}{\cos^n x}\right) \geq \left(1+2^{\frac{n}{2}}\right)^2.$$

In this paper we will consider inequalities of previous type by using hyperbolic trigonometric functions.

Lazarević [89] (or see Mitrinović [101]) gives us the following inequality:

$$\left(\frac{\sinh x}{x}\right)^q < \cosh x, \quad (x \neq 0,\ q \geq 3).$$

The following inequalities

$$\sinh x < x + \frac{x^3}{5}, \quad (0 < x < 1),$$

$$\frac{\sinh kx}{kx} \leq \frac{\sinh x}{x}, \quad (x > 0),$$

$$\frac{1}{\cosh x} < 1 - \frac{x^2}{3}, \quad (0 < x < 1),$$

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}, \quad (0 < x < \frac{\pi}{2}),$$

have established by R. Klén, M. Visuri şi M. Vuorinen [87].

The inequality

$$\left(\frac{x}{\sinh x}\right)^\alpha < (1-\eta) + \eta\left(\frac{1}{\cosh x}\right)^\alpha,$$

where $x > 0, \alpha > 0$ and $\eta \leq 1/3$ was studied recently by Zhu in [139].

The following inequalities are due to Jordan [Jordan]. He had attracted attention of several researchers

$$\frac{2}{\pi}x \leq \sin x \leq x,$$

The author has published this results in the paper [36].

Theorem 2.25.1. Let x, α and β be the positive numbers. Then the inequality

$$\left(1+\sqrt{\frac{2\alpha\beta}{\sinh 2x}}\right)^2 \leq \left(1+\frac{\alpha}{\sinh x}\right)\left(1+\frac{\beta}{\cosh x}\right)$$

holds.

Using Theorem 2.25.1, we formulate the following more general result:

Theorem 2.25.2. Let $x > 0$ and $k \in \mathbb{N}$. If $\alpha > 0$ and $\beta > 0$, then the following inequality

$$\left(1+\sqrt{\frac{2\alpha\beta}{\sinh 2^{k+1}x}}\right)^2 \leq \left(1+\frac{\alpha}{\sinh 2^k x}\right)\left(1+\frac{\beta}{\cosh 2^k x}\right)$$

holds.

Theorem 2.25.3. Let $x > 0$ and $k \in \mathbb{N}$. If m, n, α and β are positive numbers, then the following inequalities hold

$$\gamma \leq \left(1 + \frac{\alpha}{\sinh^m x}\right)\left(1 + \frac{\beta}{\cosh^n x}\right) \leq \delta,$$

where

$$\gamma = \left(1 + \sqrt{\frac{\alpha\beta}{\sinh^m x \cosh^n x}}\right)^2$$

and

$$\delta = 1 + \sqrt{(\alpha^2 + \beta^2)\left(\frac{1}{\sinh^{2m} x} + \frac{1}{\cosh^{2n} x}\right)} + \frac{\alpha\beta}{\sinh^m x \cosh^n x}.$$

Theorem 2.25.4. Let $x \in \left(0, \frac{\pi}{2}\right)$. If $\alpha > 0$ and $\beta > 0$, then the following inequality

$$\left(1 + \sqrt{\frac{2\alpha\beta}{\sinh 2x}}\right)^2 < 1 + \frac{\alpha}{\sinh x} + \frac{\beta x}{\sinh x} + \frac{\alpha\beta x}{\sinh^2 x}$$

holds.

Corollary 2.25.5. One has that

$$\frac{2x}{\sinh 2x} < \left(\frac{x}{\sinh x}\right)^2$$

holds for all $x \in \left(0, \frac{\pi}{2}\right)$.

Corollary 2.25.6. One has that

$$\tanh x < x$$

holds for all $x \in \left(0, \frac{\pi}{2}\right)$.

Observation 2.25.7. The previous inequality is a weaker form of a result of Mitrinović [101] since this inequality holds for $x \in \left(0, \frac{\pi}{2}\right)$ while it holds for $x > 0$.

Corollary 2.25.8. One has that

$$\left(1 + \sqrt{\frac{2}{\sinh 2x}}\right)^2 < \left(1 + \frac{1}{\sinh x}\right)\left(1 + \frac{x}{\sinh x}\right)$$

holds for all $x \in \left(0, \frac{\pi}{2}\right)$.

Theorem 2.25.9. Let $x \in (0, 1)$ and $q \geq 3$. Then the inequalities

$$\sqrt[q]{\frac{3}{3 - k^2 x^2}} < \frac{\sinh x}{x} < 1 + \frac{x^2}{5}$$

holds.

Theorem 2.25.10. Let $x \in (0,1), \alpha > 0$ and $\eta \leq 1/3$, then the following inequality

$$\left(\frac{x}{\sinh x}\right)^{\alpha} < (1-\eta) + \eta\left(1 - \frac{x^2}{3}\right)^{\alpha}$$

holds.

Corollary 2.25.11. Let $x \in (0,1)$, and $\eta \leq 1/3$, then the following inequality

$$\frac{x}{\sinh x} < 1 - \eta\frac{x^2}{3}$$

holds.

Theorem 2.25.12. Let $x \in (0,1)$. Then the following inequalities

$$1 - \frac{x^2}{2} \leq \frac{1}{\cosh x} \leq 1 - \frac{x^2}{3}$$

hold.

Theorem 2.25.13. Let $x > 0$. Then the function

$$f(t) = \frac{1}{\cosh^t \frac{x}{t}}$$

is decreasing on $(0, \infty)$.

Corollary 2.25.14. Let $x \in (0, \infty)$. Then the following inequality

$$\cosh\frac{x}{3} < \frac{2}{3} + \frac{1}{3}\cosh x$$

holds.

Theorem 2.25.15. Let $x \in (0, \infty)$. Then the following inequality

$$\frac{\sinh x}{x} > \frac{1}{\cosh\frac{x}{3}}$$

holds.

Corollary 2.25.16. Let $x \in (0, \infty)$. Then the following inequality

$$\frac{x}{\sinh x} < \frac{2}{3} + \frac{1}{3}\cosh x$$

holds.

Theorem 2.25.17. Let $x \in (0, \infty)$. Then the following inequality

$$\frac{\sin x}{x} < \cosh x.$$

holds.

Observation 2.25.18. Klén, Visuri, and Vuorinen [87] have shown that the inequalities

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}$$

are true for $x \in (0, \pi/2)$.

Theorem 2.25.19. For $x \in (0, \pi/2)$

$$\frac{\sin x}{x} < \sqrt{\cosh x}.$$

Theorem 2.25.20. For $x, k \in (0, \infty)$

$$\frac{\sin x}{x} < \frac{\sinh kx}{kx}.$$

Theorem 2.25.21. For $x \in (0, \infty)$ and $k \in [1, \infty)$ the following inequality

$$\frac{\sin x}{x} < \frac{\sinh kx}{x}$$

holds.

Theorem 2.25.22. Let $x, y, z \in \left(0, \frac{1}{2}\right)$. Then the following inequality

$$\frac{1}{\cosh \frac{x+y}{2}} + \frac{1}{\cosh \frac{y+z}{2}} + \frac{1}{\cosh \frac{z+x}{2}} \geq \frac{1}{\cosh x} + \frac{1}{\cosh y} + \frac{1}{\cosh z}$$

holds.

Theorem 2.25.23. Let $x, y, z \in (0, 1)$. Then the following inequality

$$\frac{1}{\sinh \frac{x+y}{2}} + \frac{1}{\sinh \frac{y+z}{2}} + \frac{1}{\sinh \frac{z+x}{2}} \leq \frac{1}{\sinh x} + \frac{1}{\sinh y} + \frac{1}{\sinh z}$$

holds.

Corollary 2.25.24. Let $x, y, z \in (0, 1)$. Then the following inequality

$$\frac{6}{\sinh x + \sinh y + \sinh z + \sinh \frac{x+y+z}{3}} \leq \frac{1}{\sinh x} + \frac{1}{\sinh y} + \frac{1}{\sinh z}$$

holds.

Chapter 3

The fundamental triangle inequality between Euclidean geometry and hyperbolic geometry

3.1 Euclidean version of Blundon's inequalities

Given a triangle ABC, denote by O the circumcenter, I the incenter , G the centroid, N the Nagel point, s the semiperimeter, R the circumradius, and r the inradius of ABC. In this note, we present a geometric proof to the so-called fundamental triangle inequality. This relation contains in fact two inequalities and it was first time proved by E. Rouché in 1851 (see [117]), answering a question of Ramus concerning the necessary and sufficient conditions for three positive real numbers s, R, r to be the semiperimeter, circumradius, and inradius of a triangle. The standard simple proof was given for the first time by W.J.Blundon [45] and it is based on the following algebraic property of the roots of a cubic equation: The roots x_1, x_2, x_3 to the equation

$$x^3 + a_1x^2 + a_2x + a_3 = 0$$

are the side lengths of a (nondegenerate) triangle if, and only if, the following three conditions are verified: i) $18a_1a_2a_3 + a_1^2a_2^2 - 27a_3^3 - 4a_2^3 - 4a_1^3a_3 > 0$; ii) $-a_1 > 0, a_2 > 0, -a_3 > 0$; iii) $a_1^3 - 4a_1a_2 + 8a_3 > 0$. For more details we refer to the monograph of D. Mitrinović, J. Pečarić, V. Volenec [102], and to the papers of C.Niculescu [109], [110], R.A.Satnoianu [119], S.Wu [142] We mention that G. Dospinescu, M. Lascu, C. Pohoaţă, M Tetiva have proposed a new algebraic proof to the weaker Blundon's inequality $s \leq 2R + (3\sqrt{3} - 4)r$. This inequality is a direct consequence of the fundamental triangle inequality.

In order to state our main results we need to recall some important distances in triangle ABC. The famous formula for the distance OI is called Euler's relation and it is given by

$$OI^2 = R^2 - 2Rr$$

For the standard geometric proof to this relation we refer to the books of H.S.M.Coxeter and S.L.Greitzer [56] or T.Lalescu [88]. For a proof by using complex numbers we mention the book of T.Andreescu and D.Andrica [4]. The next important distance is ON and it is given by

$$ON = R - 2r$$

The previous relation gives in geometric way the difference between the quantities involved in the Euler's inequality $R \geq 2r$ and it will play an important role in the proof of our main results. A proof by using complex numbers is given in the book of T.Andreescu and D.Andrica [4]. Another useful distance is OG and the following relation holds

$$OG^2 = R^2 - \frac{a^2 + b^2 + c^2}{9},$$

where a, b, and c are the side lengths of triangle ABC. The standard proof uses the Leibniz's relation combined with the median formula (see [4])

The sum $a^2 + b^2 + c^2$ can be expressed in terms of the symmetric invariants s, R, r of triangle ABC as follows:

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$$

This formula can be found in different references, for instance in D.S.Mitrinović, J.E. Pečarić, V.Volenec [102]. The author has published this results in the paper [8].

The following result contains a simple geometric proof to the fundamental inequality of a triangle.

Theorem 3.1.1. Assume that the triangle ABC is not equilateral. The following relation holds :

$$\cos \widehat{ION} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R - 2r)\sqrt{R^2 - 2Rr}}. \tag{3.1.}$$

Theorem 3.1.2. (Rouché) The necessary and sufficient condition for the existence of a triangle, with elements s, R, and r is

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2 - 2Rr} \leq s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2 - 2Rr}.$$

Equilateral triangles give the trivial situation where we have equality in the previous inequalities. Suppose that we are not working with equilateral triangles, i.e we have $R-2r \neq 0$. Denote by $\mathcal{T}(R, r)$ the family of all triangles having the circumradius R and the inradius r. The previous inequalities give in terms of R and r the exact interval for the semiperimeter s of triangles in family $\mathcal{T}(R, r)$. We have

$$s^2_{min} = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}$$

and

$$s^2_{max} = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}.$$

If we fix the circumcenter O and the incenter I so that $OI = \sqrt{R^2 - 2Rr}$, then the triangle in the family $\mathcal{T}(R, r)$ with minimal semiperimeter corresponds to the equality case $\cos\widehat{ION} = 1$, i.e. points I, O, N are collinear and I and N belong to the same ray with the origin O. Taking into account the well-known property that points O, G, H are collinear on the Euler's line of triangle, this implies that O, I, G must be collinear, hence in this case triangle ABC is isosceles. In Figure 3.1 this triangle is denoted by $A_{min}B_{min}C_{min}$. Also, the triangle in the family $\mathcal{T}(R, r)$ with maximal semiperimeter corresponds to the equality case $\cos\widehat{ION} = -1$, i.e. points I, O, N are collinear and O is situated between I and N. Using again the Euler's line of the triangle, it follows that triangle ABC is isosceles. In Figure 1 this triangle is denoted by $A_{max}B_{max}C_{max}$. Note that we have $B_{min}C_{min} > B_{max}C_{max}$. The triangles in the family $\mathcal{T}(R, r)$ are "between" these two extremal triangles (see Figure 3.1). According to the Poncelet's closure Theorem, they are inscribed in the circle $\mathcal{C}(O; R)$ and their sides are tangent externally to the circle $\mathcal{C}(I; r)$.

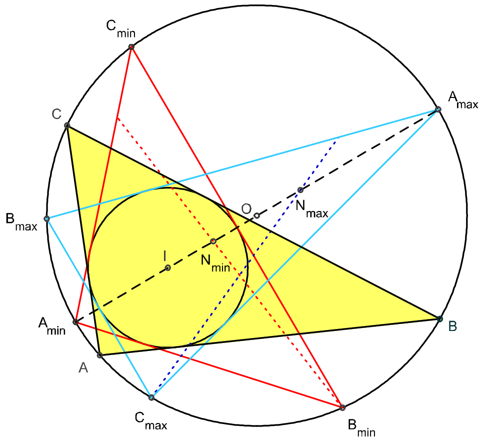

Figure 3.1

From Theorem 3.1.2. it follows that it is a natural and important problem to construct the triangle ABC from its incenter I, circumcenter O, and its Nagel point N. Taking into account that points I, G, N are collinear determining the Nagel line of triangle, it follows that we get the centroid G on the segment IN so that $IG = \frac{1}{3}IN$. Then, using the Euler's line of the triangle, we get the orthocenter H on the ray $(OG$ such that $OH = 3OG$. Now we have reduced the construction problem to the famous Euler's determination problem i.e. to construct a triangle from its incenter I, circumcenter O, and orthocenter H [68]. Some new approaches involving this problem are given by B.Scimemi [120], G.C.Smith [126], J.Stern [129] and P.Yiu [144].

3.2 A dual version to Blundon's inequalities

In this section we consider a triangle ABC with the circumcenter O, the incenter I, the excenters I_a, I_b, I_c, and N_a, N_b, N_c the adjoint points to the Nagel point N. For the definition and some properties of the adjoint points N_a, N_b, N_c we refer to the paper of D.Andrica and K.L.Nguyen [12].

Let s, R, r, r_a, r_b, r_c be the semiperimeter, the circumradius, the inradius, and the exradii of triangle ABC, respectively. We know that points N_a, G, I_a are collinear and we have $N_aI_a = 3GI_a$. The similar properties hold for the triples of points N_b, G, I_b and N_c, G, I_c. The following relations

$$OI_a^2 = R^2 + 2Rr_a, OI_b^2 = R^2 + 2Rr_b, OI_c^2 = R^2 + 2Rr_c$$

and

$$ON_a = R + 2r_a, ON_b = R + 2r_b, ON_c = R + 2r_c$$

hold. For a proof by using complex numbers we refer to the paper [12].

Theorem 3.2.1. The following relation holds

$$\cos \widehat{I_aON_a} = \frac{R^2 - 3Rr_a - r_a^2 - \alpha}{(R + 2r_a)\sqrt{R^2 + 2Rr_a}}, \tag{3.2.}$$

where $\alpha = \frac{a^2+b^2+c^2}{4}$.

Theorem 3.2.2. (Dual form to Blundon's inequalities) The following inequalities hold

$$0 \leq \frac{a^2 + b^2 + c^2}{4} \leq R^2 - 3Rr_a - r_a^2 + (R + 2r_a)\sqrt{R^2 + 2Rr_a}.$$

Denote by $\mathcal{T}(R, r_a)$ the family of all triangles having the circumradius R and the exradius r_a. The inequalities (15) give in terms of R and r_a the exact interval for α, for triangles in family $\mathcal{T}(R, r_a)$. We have $\alpha_{min} = 0$ and $\alpha_{max} = R^2 - 3Rr_a - r_a^2 + (R + 2r_a)\sqrt{R^2 + 2Rr_a}$. If we fix the circumcenter O and the excenter I_a so that we have $OI_a = \sqrt{R^2 + 2Rr_a}$, then the triangles in the family $\mathcal{T}(R, r_a)$ with minimal α are degenerated to a point and they correspond to the intersection points of the circles. In Figure 3.2 these points are denoted by A'_{min} and $A"_{min}$. Also, the triangle in the family $\mathcal{T}(R, r_a)$ with maximal α corresponds to the equality case $\cos \widehat{I_aON_a} = -1$, i.e. points I_a, O, N_a are collinear and O is between I_a and N_a. Using again the Euler's line of the triangle, it follows that triangle ABC is isosceles. Figure 3.2 presents the standard geometric configuration illustrating the extremal triangles.

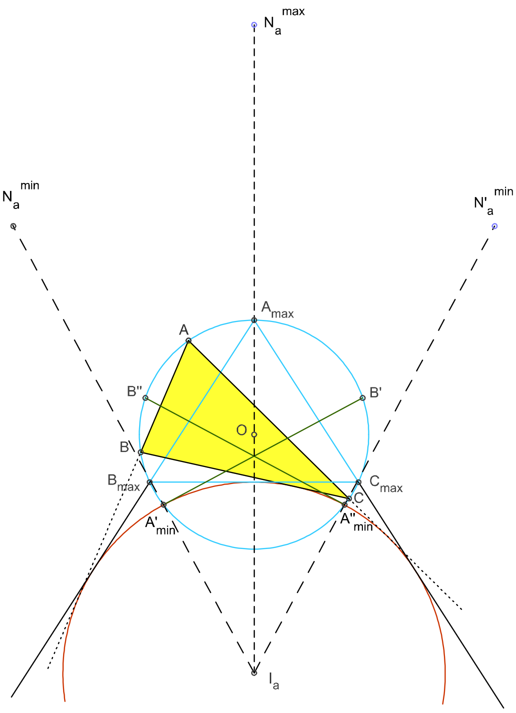

Figure 3.2

In this case we have also an exterior Poncelet's closure Theorem, that is the triangles "between" these two extremal triangles belong to the family $\mathcal{T}(R, r_a)$. We refer to the paper of L.Emelyanov and T.Emelyanov [7] for a complicate proof of this property.

From Theorem 3.2.1 it follows that it is a natural problem to construct the triangle ABC from its excenter I_a, circumcenter O, and its adjoint Nagel point N_a. Taking into account that points I_a, G, N_a are collinear, it follows that we get the centroid G on the segment I_aN_a so that $I_aG = \frac{1}{3}I_aN_a$. Then, using the Euler's line of the triangle, we get the orthocenter H on the ray $(OG$ so that $OH = 3OG$. Now we have reduced again our construction problem to the famous Euler's determination problem: to construct a triangle from its incenter I, circumcenter O, and orthocenter H (we refer to the original reference [68]).

Observation 3.2.3.

1) From Theorem 3.2.1 it follows that it is a natural question to express α in terms of s, R, r_a, in order to obtain a similar formula (3.1). In order to answer to this question we shall prove that

$$ab + bc + ca = \frac{s^6 + r_a(4R + 3r_a)s^4 + r_a^2(3r_a^2 - 16R^2)s^2 + r_a^5(r_a - 4R)}{(s^2 + r_a^2)^2}.$$

2) The express for α is a complicate function of s, R, r_a, that way in order to have a duality between formulas (3.1) and (3.2) we have to express $\cos\widehat{ION}$ in terms of α, R, r. In this respect, from formula $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$ we get $s^2 = 2\alpha + 4Rr + r^2$.

Replacing in formula (3.1) we obtain

$$\cos \widehat{ION} = \frac{R^2 + 3Rr - r^2 - \alpha}{(R - 2r)\sqrt{R^2 - 2Rr}}.$$

The formal transformation $r \mapsto -r_a$ gives the duality between formulas (3.1) and (3.2). We have similar duality relations for r_a, r_b.

3.3 Applications to some inequalities in s, R, and the exradii

In this section we will obtain as consequences to Theorem 3.2.1 some inequalities involving s, R and the exradii of the triangle.

Corollary 3.3.1. In any triangle with semiperimeter s the following inequalities hold

$$2R^2 + r^2 + 4Rr - 6Rr_a - 2r_a^2 - 2(R + 2r_a)\sqrt{R^2 + 2Rr_a} \le s^2 \le$$

$$2R^2 + r^2 + 4Rr - 6Rr_a - 2r_a^2 - 2(R + 2r_a)\sqrt{R^2 + 2Rr_a}.$$

Using the fact that $r < r_a$ from the right inequality in the previous result we obtain

$$s^2 < 2R^2 - 2Rr_a - r_a^2 + 2(R + 2r_a)\sqrt{R^2 + 2Rr_a}.$$

Corollary 3.3.2. In any triangle with semiperimeter s the following inequality holds

$$s^2 < min\{4R^2 + 4Rr_a + 3r_a^2, 4R^2 + 4Rr_b + 3r_b^2, 4R^2 + 4Rr_c + 3r_c^2\}.$$

Corollary 3.3.3. The following inequalities hold

$$a^2 + b^2 + c^2 < min\{8R^2 + 4r_a^2, 8R^2 + 4r_b^2, 8R^2 + 4r_c^2\}.$$

Theorem 3.3.4. The following inequalities hold:

$$s^2 \le 2\sqrt{2}(2R + r_a)r_a$$

Observation 3.3.5. The following inequalities hold

$$i)\ s^2 \le 2\sqrt{2}(2R + r_b)r_b;$$

$$ii)\ s^2 \le 2\sqrt{2}(2R + r_c)r_c.$$

Theorem 3.3.6. In any triangle with semiperimeter s the following inequality holds

$$s^2 \le 2\sqrt{3}R(R + 2r_a).$$

Observation 3.3.7. In any triangle the following inequalities hold

$$i)\ s^2 \le 2\sqrt{3}R(R + 2r_b);$$

$$ii)\ s^2 \le 2\sqrt{3}R(R + 2r_c).$$

Observation 3.3.8. Because $OI_a^2 = R^2 + 2Rr_a$, the inequality from Theorem 3.3.6 can be rewritten as following $s \le \sqrt[4]{12}OI_a$.

Theorem 3.3.9. In any triangle the following inequality holds: $s \le 2 \cdot OI_a$.

Theorem 3.3.10. In any triangle the following inequality holds:

$$i) s^2 \le \min\{12Rr_a, 12Rr_b, 12Rr_c\}.$$

3.4 The natural approach of Blundon - Wu inequalities

S. Wu [141] gave a new sharp version of the Blundon's inequality by introducing a parameter, as follows: for any triangle $A_1A_2A_3$, the following inequalities hold

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\cos\phi \le s^2 \le 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\cos\phi. \tag{3.3}$$

where $\phi = \min\limits_{1\le i<j\le 3} |A_i - A_j|$. Next, we called (3.3) the Blundon - Wu inequality. In this paper we present a geometric proof to approach to Blundon - Wu inequality. It is well-known that distance ON is given by $ON = R - 2r$. Denote by $\mathcal{T}(R, r)$ the family of all triangles having the circumradius R and the inradius r. The Blundon's inequalities give in terms of R and r the exact interval for the semiperimeter s for triangles in family $\mathcal{T}(R, r)$. We have

$$s_{min}^2 = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}$$

and

$$s_{max}^2 = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}.$$

The triangles in the family $\mathcal{T}(R, r)$ are "between" two extremal triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$ determined by $s_{\min}$ and $s_{\max}$. Denote by $N_{\min}$ and $N_{\max}$ the Nagel's points of the triangles $A_{\min}B_{\min}C_{\min}$ and $A_{\max}B_{\max}C_{\max}$, respectively.

Obviously, because the distance ON is constant, the Nagel's point N moves on the circle of diameter $N_{\min}N_{\max}$, and $\widehat{ION}$ varies from 0 to 180°. The author has published these results in the paper [10].

Theorem 3.4.1. For any triangle ABC, the following inequalities hold

$$-\cos\phi \leq \cos\widehat{ION} \leq \cos\phi, \tag{3.4}$$

where $\phi = \min\{|A-B|, |B-C|, |C-A|\}$. Both equalities hold in (3.4) if and only if the triangle is equilateral.

Observation 3.4.2. In fact ϕ divides the triangles of family $\mathcal{T}(R,r)$ in function of the position of the point A on the circle $O(R)$.

3.5 Hyperbolic version of Blundon's inequalities

In this section we present the hyperbolic version of Blundon's inequalities. This form was given by D. Svrtan and D. Veljan [131]. So, they have showed that for a hyperbolic triangle that has a circumcircle of radius R, incircle of radius r, semiperimeter s, and excess ε, we have

$$\frac{D}{s'^2} \geq 0,$$

where

$$\begin{aligned} D = s'^2[&(r'^2R'^2\varepsilon'^2 + 4r'^4R'^4\varepsilon'^2 - 4r'^3R'^3\varepsilon'^2 - 1 + 6r'R' - 12r'^2R'^2 + 8r'^3R'^3)s'^4 \\ &+ r'^2R'\varepsilon'(1 - 4r'R' + 4r'^2R'^2\varepsilon' - 8r'^2R'^2\varepsilon'^2 + 9\varepsilon' + 18r'R'\varepsilon')s'^3 \\ &+ r'^2(r'^2R'^2 - 10r'R' - 12r'^2R'^2\varepsilon'^2 - 2)s'^2 - 6r'^4R'\varepsilon's' - r'^4], \end{aligned}$$

and $r' = \tanh\frac{r}{k}, R' = \tanh\frac{R}{k}, \varepsilon' = \cot\frac{\varepsilon}{2}, s' = \sinh\frac{s}{k}$.

3.6 A Geometric way to generate Blundon type inequalities

Let P be a point situated in the plane of the triangle ABC. The Cevian triangle DEF is defined by the intersection of the Cevian lines through the point P and the sides BC, CA, AB of triangle. If the point P has barycentric coordinates $t_1 : t_2 : t_3$, then the vertices of the Cevian triangle DEF have barycentric coordinates given by: $D(0 : t_2 : t_3), E(t_1 : 0 : t_3)$ and $F(t_1 : t_2 : 0)$. The barycentric coordinates were introduced in 1827 by Möbius. The using of barycentric coordinates defines a distinct part of Geometry called Barycentric Geometry. More details can be found in the monographs of C. Bradley [48], C. Coandă [51], C. Coşniţă [53], C. Kimberling [86], O. Bottema [47], J. Scott [121] and P. Yiu [144]. The author has published these results in the paper [11].

In the paper [26] we introduced the Cevians of rank $(k;l;m)$. The line AD is called *ex-Cevian of rank* $(k;l;m)$ or *exterior Cevian of rank* $(k;l;m)$, if the point D is situated on side (BC) of the non-isosceles triangle ABC and the following relation holds:

$$\frac{BD}{DC}=\left(\frac{c}{b}\right)^k\cdot\left(\frac{s-c}{s-b}\right)^l\cdot\left(\frac{a+b}{a+c}\right)^m.$$

In the paper [26] it is proved that the Cevians of rank $(k;l;m)$ are concurrent in the point $I(k,l,m)$ called *the Cevian point of rank* $(k;l;m)$ and the barycentric coordinates of $I(k,l,m)$ are:

$$a^k(s-a)^l(b+c)^m : b^k(s-b)^l(a+c)^m : c^k(s-c)^l(a+b)^m.$$

For every point M in the plane of triangle ABC, the following relation holds:

$$(t_1+t_2+t_3)\overrightarrow{MP}=t_1\overrightarrow{MA}+t_2\overrightarrow{MB}+t_3\overrightarrow{MC}.$$

In the particular case when $M\equiv P$, we obtain

$$t_1\overrightarrow{PA}+t_2\overrightarrow{PB}+t_3\overrightarrow{PC}=\overrightarrow{0}.$$

Theorem 3.6.1. If M is a point situated in the plane of triangle ABC, then

$$(t_1+t_2+t_3)^2MP^2=(t_1MA^2+t_2MB^2+t_3MC^2)(t_1+t_2+t_3)-(t_2t_3a^2+t_3t_1b^2+t_1t_2c^2).$$

Observation 3.6.2. If we consider that t_1,t_2,t_3, and $t_1+t_2+t_3$ are nonzero real numbers, then the previous relation becomes the Lagrange's relation

$$MP^2=\frac{t_1MA^2+t_2MB^2+t_3MC^2}{t_1+t_2+t_3}-\frac{t_1t_2t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right).$$

Corollary 3.6.3. If we consider $M\equiv O$, the circumcenter of the triangle, then it follows

$$R^2-OP^2=\frac{t_1t_2t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right)$$

with $t_1,t_2,t_3>0$.

Corollary 3.6.4. In any triangle with semiperimeter s the following inequality holds:

$$R^2 - OP^2 \geq \frac{4s^2 t_1 t_2 t_3}{(t_1+t_2+t_3)^3}$$

with $t_1, t_2, t_3 > 0$.

C. Coşniţă give in [53] this theorem: If the points P and Q have barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$, respectively, with respect to the triangle ABC, and $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$, then

$$PQ^2 = -\alpha\beta\gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right)$$

where the numbers α, β, γ are defined by

$$\alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}.$$

Theorem 3.6.5. Let P and Q be two points different from the circumcircle O, having the barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$ with respect to the triangle ABC and let $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$. If $u_1, u_2, u_3, t_1, t_2, t_3 \neq 0$, then the following relation holds

$$\cos\widehat{POQ} = \frac{2R^2 - \frac{t_1t_2t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right) - \frac{u_1u_2u_3}{(u_1+u_2+u_3)^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right) + \alpha\beta\gamma\left(\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right)}{2\sqrt{\left[R^2 - \frac{t_1t_2t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right)\right] \cdot \left[R^2 - \frac{u_1u_2u_3}{(u_1+u_2+u_3)^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right)\right]}}$$

where a, b, c are the length sides of the triangle and

$$\alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}.$$

Theorem 3.6.6. Let P and Q be two points different from the circumcircle O, having the barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$ with respect to the triangle ABC and let $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$. If $u_1, u_2, u_3, t_1, t_2, t_3 \neq 0$, then the following relation holds

$$-2\sqrt{\left[R^2 - \frac{t_1t_2t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right)\right] \cdot \left[R^2 - \frac{u_1u_2u_3}{(u_1+u_2+u_3)^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right)\right]} \leq$$

$$\alpha\beta\gamma\left(\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right) + 2R^2 - \left[\frac{t_1t_2t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right) + \frac{u_1u_2u_3}{(u_1+u_2+u_3)^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right)\right] \leq$$

$$2\sqrt{\left[R^2 - \frac{t_1t_2t_3}{(t_1+t_2+t_3)^2}\left(\frac{a^2}{t_1}+\frac{b^2}{t_2}+\frac{c^2}{t_3}\right)\right] \cdot \left[R^2 - \frac{u_1u_2u_3}{(u_1+u_2+u_3)^2}\left(\frac{a^2}{u_1}+\frac{b^2}{u_2}+\frac{c^2}{u_3}\right)\right]}$$

where a, b, c are the length sides of the triangle and

$$\alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}.$$

Corollary 3.6.7. The necessary and sufficient condition for the existence of a triangle, with elements s, R, and r is

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \leq s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}.$$

Observation 3.6.8. Let I_1, I_2, I_3 be three Cevian points of rank $(k;l;m)$ with barycentric coordinates as follows:

$$I_i[a^{k_i}(s-a)^{l_i}(b+c)^{m_i} : b^{k_i}(s-b)^{l_i}(a+c)^{m_i} : c^{k_i}(s-c)^{l_i}(a+b)^{m_i}],\ i=\overline{1,3}.$$

Then,

$$\cos\widehat{I_1I_2I_3} = \frac{-a^2(\beta_{12}\gamma_{12}+\beta_{23}\gamma_{23}-\beta_{31}\gamma_{31})-b^2(\gamma_{12}\alpha_{12}+\gamma_{23}\alpha_{23}-\gamma_{31}\alpha_{31})+c^2(\alpha_{12}\beta_{12}+\alpha_{23}\beta_{23}-\alpha_{31}\beta_{31})}{2\sqrt{-\beta_{12}\gamma_{12}a^2-\gamma_{12}\alpha_{12}b^2-\alpha_{12}\beta_{12}c^2}\cdot\sqrt{-\beta_{23}\gamma_{23}a^2-\gamma_{23}\alpha_{23}b^2-\alpha_{23}\beta_{23}c^2}}$$

Theorem 3.6.9. The following inequalities hold

$$-2\sqrt{-\beta_{12}\gamma_{12}a^2-\gamma_{12}\alpha_{12}b^2-\alpha_{12}\beta_{12}c^2}\cdot\sqrt{-\beta_{23}\gamma_{23}a^2-\gamma_{23}\alpha_{23}b^2-\alpha_{23}\beta_{23}c^2} \leq$$

$$-a^2(\beta_{12}\gamma_{12}+\beta_{23}\gamma_{23}-\beta_{31}\gamma_{31})-b^2(\gamma_{12}\alpha_{12}+\gamma_{23}\alpha_{23}-\gamma_{31}\alpha_{31})+c^2(\alpha_{12}\beta_{12}+\alpha_{23}\beta_{23}-\alpha_{31}\beta_{31}) \leq$$

$$2\sqrt{-\beta_{12}\gamma_{12}a^2-\gamma_{12}\alpha_{12}b^2-\alpha_{12}\beta_{12}c^2}\cdot\sqrt{-\beta_{23}\gamma_{23}a^2-\gamma_{23}\alpha_{23}b^2-\alpha_{23}\beta_{23}c^2}$$

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