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Doctoral Thesis Summary

Variational and topological methods in the study of elliptic inclusions and equations

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1 Introduction

One of the basic problems in calculus of variations is to study the existence of extrema or critical/equilibrium points of functionals. Many non-linear problems in partial differential equations can be reduced to finding the extrema of the associated energy functional, which is more easier than solving the original problems. This method is applicable since if an extremum can be detected then it will be a weak solution to the original problem. This approach is called the direct method of calculus of variations and can be derived from Physics, where in general the study of the equilibrium state of a process can be reduced to the minimization of the total energy that characterizes the underlying physical process. In proving the existence of extrema of (energy) functionals the Palais–Smale, or shortly, the (PS) condition [Palais and Smale, 1964] is the most frequently used tool, since it provides a sufficient condition for the existence of a minimum. Thus, if we show that the (PS) condition is satisfied, then we have at least a critical point.

The variational principle of Ekeland introduced in [Ekeland, 1972, 1974] for lower semi-continuous functionals on metric spaces is the non-linear version of the Bishop–Phelps theorem [Bishop and Phelps, 1963] and it is a useful tool in the construction of the minimizer of lower semi-continuous functionals on complete metric spaces. Actually, the variational principle of Ekeland characterizes the completeness of a metric space and it can be used successfully to establish the existence of an approximate minimizer and, used in conjunction with the (PS) condition, the functional must achieve its minimum. Depending on the variational framework, several different versions and extensions of this principle are known, of which we will focus on its symmetric version introduced and proved by [Squassina, 2012].

It is easier to find the extrema of functionals, than to localize those critical points which are not minima or maxima. At first, [Ambrosetti and Rabinowitz, 1973] presented a method for finding non-extremal critical points of continuously differentiable functionals that are unbounded from either above, or below. Due to their geometrical interpretation, we refer to these critical points as mountain pass points. To detect such critical points, they proposed a minimax method that is based on a deformation lemma and consists of two basic steps, namely the proof of the existence of a (PS) sequence and the verification of the (PS) condition. Their mountain pass theorem of positive altitude has been intensively studied and applied in order to establish the existence of critical points of such types of functionals. [Pucci and Serrin, 1984, 1985] developed a weaker version of the theorem, which is called mountain pass theorem of zero altitude that can be used to prove the existence of critical points in the case when the separating mountain range has zero altitude. Another important extension of the theorem is due to [Ghoussoub and Preiss, 1989], which in addition to proving the existence of critical points, also derives information about their location. [Willem, 1996] proved a quantitative deformation lemma that can be used in the construction of (PS) sequences independently of their compactness conditions. This approach is more general than the original one and it can be used to many problems where the (PS) condition fails. Based on the general minimax principle of [Willem, 1996], [Schaftingen, 2005] proved a symmetric minimax principle for continuously differentiable functionals in order to obtain symmetric critical points. Moreover, he used this new method to study the symmetry properties of solutions to elliptic partial differential equations. In addition to these results, the mountain pass theorem has a wide range of applications to non-linear partial differential equations – without providing an exhaustive survey, we mention here, e.g., the works of [Ambrosetti and Malciodi, 2007; Aubin and Ekeland, 1984; Brezis and Nirenberg, 1991; Jabri, 2003; Kristály et al., 2010; Mahwin and Willem, 1989; Rabinowitz, 1986; Schechter, 1999; Struwe, 2008] and many references therein. In the last decades, different areas of critical point theory have been developed very intensively and it became a natural framework in several modern mathematical fields with important applications in Mechanics, Engineering, Biology and Economics.

Another way to determine the extrema of functionals is possible by using fixed point theory, since many applications can be reformulated as a fixed point problem. In this regard, one has to define sufficient conditions guaranteeing the existence of fixed points of functionals and also to study the nature of the obtained fixed points, i.e., whether they are extrema or not. Due to its usefulness, the application of fixed point theorems has become a very popular tool in studying the existence (and in some cases the multiplicity) of solutions to partial differential inclusions/equations. Among of the well-known fixed point theorems, we will use both the Leray–Schauder alternative [Granas and Dugundji, 2003] and a variant of the Krasnoselkii's fixed point theorem proved in [Precup, 2012] for establishing the existence of solutions to certain non-linear partial differential equations, which are described as Dirichlet problems that involve Finsler–Laplacian operators.

The thesis is devoted to the study of non-linear problem

$$\begin{cases} \mathrm{L}\, u \in \partial F\,(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a non-empty open subset of the Euclidean space $\mathbb{R}^n$ $(n \in \mathbb{N}_{\geq 1})$, $u : \Omega \to \mathbb{R}$ is a sufficiently smooth function, L may correspond to the classical Laplacian, p-Laplacian or Finsler–Laplacian differential operator and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is either a continuously differentiable or only a locally Lipschitz function.

Depending on the type of the differential operator L and the properties of the function F, we use either smooth or non-smooth analysis methods presented above to prove the existence of the solution(s). Moreover, in some cases, we can also either localize them or to show that they have special properties, e.g., they are invariant by spherical cap symmetrization.

At first, we will provide a brief summary of the preliminary notions and existing theoretical results, then – by combining variational and topological methods with the elements of either critical or fixed point theory – in each subsequent chapter of the thesis we will study the existence, multiplicity (in some cases the symmetry property) and localization of the solutions to differently parametrized problems of type (1.1).

Apart from the current introductory chapter, the present thesis consists of five main chapters, their brief summary and structure are presented below.

- In Chapter 2, we summarize the basic preliminary notions and results related to the theoretical background of the methods which will be presented in further chapters of the thesis.
- In Chapter 3, based on article [Mezei, Molnár and **Vas**, 2014], at first – considering a semi-linear

elliptic differential inclusion problem – we study the case when $\Omega = B(0,1)$ is the unit ball in the Euclidean space $\mathbb{R}^N$ ($N \in \mathbb{N}_{\geq 2}$), $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function that is super-linear at the origin and also fulfils a sub-linear growth condition at infinity and $\mathrm{L} = \Delta_p u = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian operator defined on the Sobolev space $W_0^{1,p}(\Omega)$ with $p \in (1, N)$ fixed. Then, in the case when $\Omega = \mathbb{R}^N$ ($N \in \mathbb{N}_{\geq 2}$), $(X, \|\cdot\|_X)$ is a real separable reflexive Banach space with its topological dual space $(X^*, \|\cdot\|_{X^*})$, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and $A : X \to X^*$ is a potential operator we deal with a hemivariational inequality. In addition to proving the existence of solutions to the above problems, by using both the symmetric version of the variational principle of Ekeland proved by [Squassina, 2012] and the non-smooth version of the symmetric minimax principle developed by [Schaftingen, 2005], we also study the multiplicity property of the solutions. Moreover, we prove a very important qualitative property of them, namely that they are invariant by spherical cap symmetrization.

Our contributions in this chapter are: Theorems 3.1, 3.2 and Lemmas 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8.

- Chapter 4 is dedicated to the study of the critical point theory developed by Schechter [1992, 1999]. Based on article [**Vas**, 2015], in Section 4.1 at first we prove a Schechter-type critical point result for locally Lipschitz functions defined on a ball of a Hilbert space and we also provide a concrete application of it. Using our Schechter-type critical point theorem, we prove the existence of the solutions to problem (1.1), when Ω is a bounded open set in $\mathbb{R}^N$ that has C^1 regular boundary $\partial\Omega$, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and $\mathrm{L} = \Delta u$ is the classical Laplacian operator. The second part of the chapter relies on [Lisei and **Vas**, 2016]. In Section 4.2, we apply the smooth version of the previously discussed Schechter-type critical point theorem developed for C^1 functionals in Banach spaces to localize the solutions to problems which contain the p-Laplacian operator on either bounded or unbounded domains.

Our contributions in this chapter are: Theorems 4.1, 4.2, 4.3, 4.4 and Propositions 4.1, 4.2, 4.3.

- Chapter 5 is based on article [Lisei, Varga and **Vas**, 2018]. After a brief summary of preliminary notions and properties, Section 5.2 presents some auxiliary results that are required to prove the main statements of Section 5.3 for the case of a wedge intersected with a ball in a reflexive locally uniformly convex smooth Banach space. More precisely, we mention the new variants of: the deformation lemma and the bounded version of the general minimax theorem of [Willem, 1996]; and the mountain pass theorem of [Ambrosetti and Rabinowitz, 1973] as well. In Section 5.4, we apply our previously discussed results to localize two non-trivial solutions to Dirichlet problems involving non-homogeneous operators in the context of Orlicz–Sobolev spaces. The chapter concludes with three concrete examples of the considered Dirichlet problem.

Our contributions in this chapter are: Propositions 5.1, 5.2, 5.3, 5.4, Lemmas 5.1, 5.2 and Theorems 5.1, 5.2, 5.3, 5.4, 5.5, 5.6.

- Chapter 6 relies on [Mezei and **Vas**, 2019] and studies the existence and localization results in the case of two Dirichlet problems that involve the Finsler–Laplacian operator, i.e., $\mathrm{L} = \Delta_F u$. At first, in the case of our first problem, based on the results of [Dinca et al., 2001], we show the existence of the solutions in two different ways: by applying the direct method of the calculus of variations, then by using the Leray–Schauder alternative. Then, in the case of our second problem, we prove an existence and localization result, by the combined use of the Harnack inequality and a Krasnosel'skii-type fixed point theorem of [Precup, 2012].

Our contributions in this chapter are: Lemma 6.1 and Theorems 6.1, 6.3, 6.7.

Our results

The thesis is based on the articles:

- I. I. Mezei, A. É. Molnár and O. **Vas**, 2014. *Multiple symmetric solutions for some hemivariational inequalities*, Studia Universitatis Babeş–Bolyai Mathematica, **59**, No. 3, 369–384.
  URL: http://193.0.225.37/download/pdf/877.pdf
- O. **Vas**, 2015. *A Schechter-type critical point result for locally Lipschitz functions*, Mathematica, Tome 57 (**80**), No. 1–2, 117–125.
  URL: http://math.ubbcluj.ro/~mathjour/articles/2015/vas.pdf
- H. Lisei and O. **Vas**, 2016. *Critical point result of Schechter type in a Banach space*, Electronic Journal of Qualitative Theory of Differential Equations, No. 14, 1–16, $\mathbf{IF}_{2016} = 0.926$, $\mathbf{JCR}_{2016}$ Category: Mathematics. Rank in Category: 73/311. Quartile in Category: Q1.
  DOI: https://doi.org/10.14232/ejqtde.2016.1.14
- H. Lisei, Cs. Varga and O. **Vas**, 2018. *Localization method for the solutions of nonhomogeneous operator equations*, Applied Mathematics and Computation, 329, 64–83, $\mathbf{IF}_{2018} = 3.092$, $\mathbf{JCR}_{2018}$ Category: Mathematics, Applied. Rank in Category: 14/254. Quartile in Category: Q1.
  DOI: https://doi.org/10.1016/j.amc.2018.01.031
- I. I. Mezei and O. **Vas**, 2019. *Existence results for some Dirichlet problems involving Finsler–Laplacian operator*, Acta Mathematica Hungarica, **157** (**1**), 39–53, $\mathbf{IF}_{2019} = 0.588$, $\mathbf{JCR}_{2019}$ Category: Mathematics, Applied. Rank in Category: 233/325. Quartile in Category: Q3.
  DOI: https://doi.org/10.1007/s10474-018-0894-8

Some of our results were presented at the following workshops and scientific conferences:

- 11^{th} Joint Conference on Mathematics and Computer Science, May 20–22, 2016, Eger, Hungary;
- Geometry and PDEs, June 13–14, 2017, West University of Timişoara, Romania.

Keywords

locally Lipschitz functions; variational principle of Ekeland; Palais–Smale condition; Schechter-type critical points; spherical cap symmetrization; Willem-type deformation lemma; mountain pass theorem; classical-, p- and Finsler–Laplacian operator; elliptic inclusion and equation problems; hemivariational inequalities; fixed point theorems.

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2
Preliminary notions and results

This chapter is devoted to collect the basic preliminary notions and results related to the theoretical background of the methods which will be used in further chapters.

2.1 Lebesgue spaces

2.2 Sobolev spaces

2.3 Orlicz–Sobolev spaces

2.4 Elements of calculus of variations

2.5 Locally Lipschitz functions

2.6 Symmetrization and polarization

3

Multiple symmetric solutions to some hemivariational inequalities

Based on article [Mezei, Molnár and **Vas**, 2014], this chapter presents some multiplicity results for hemivariational inequalities defined either on the unit ball or on the whole space $\mathbb{R}^N$. Using the symmetric version of the variational principle of Ekeland introduced by [Squassina, 2012] and a non-smooth version of the symmetric minimax principle due to [Schaftingen, 2005], we prove that the solutions to these inequalities are invariant by spherical cap symmetrization. In Section 3.1 we study the existence of multiple symmetric solutions to a semi-linear elliptic differential inclusion problem defined on the unit ball of the space $\mathbb{R}^N$, while in Section 3.2 we deal with the case of hemivariational inequality defined on the whole space $\mathbb{R}^N$.

3.1 The first problem

Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^N$ and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Fix the parameters $p \in (1, N)$ and $\lambda > 0$, moreover, by using the p-Laplacian operator $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, consider the semi-linear elliptic differential inclusion problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \lambda \partial_y F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.P_λ}$$

where $\partial_y F(x,s)$ is the generalized gradient of F at the point $s \in \mathbb{R}$ with respect to the second variable.

In the study of the existence of the solutions to problem (3.1.P_λ), we combine methods of calculus of variations and symmetrization techniques. [Schaftingen, 2005] developed an abstract framework for symmetrizations and, relying on his work, [Squassina, 2012] formulated the symmetric versions of classic variational principles. Many papers related to symmetrizations have been published, where the solutions are eighter radially [Squassina, 2011] or axially symmetric [Kristály and Mezei, 2012] functions, or which have some symmetry properties with respect to certain group actions [Farkas and Mezei, 2013]. Moreover, [Filippucci, Pucci and Varga, 2015] proved the existence of multiple symmetric solutions to some eigenvalue problems, and [Farkas and Varga, 2014] obtained multiplicity results for a model quasi-linear elliptic system in the case of C^1 functionals.

Our aim was to extend the above-mentioned results for the case of locally Lipschitz functions, by proving the existence of multiple spherical cap symmetric solutions to problem (3.1.P_λ) on the Sobolev space $W_0^{1,p}(\Omega)$, endowed with its standard norm.

Our result is valid provided that the function F satisfies the conditions:

$(\mathbf{C}_F^1)$ $\lim\limits_{|s|\to 0} \dfrac{\max\{|\xi| : \xi \in \partial_y F(x,s)\}}{|s|^{p-1}} = 0$;

$(\mathbf{C}_F^2)$ $\lim\limits_{|s|\to +\infty} \dfrac{\max\{|\xi| : \xi \in \partial_y F(x,s)\}}{|s|^{p-1}} = 0$;

$(\mathbf{C}_F^3)$ there exists an $u_0 \in W_0^{1,p}(\Omega)$, $u_0 \neq 0$ for which $\int\limits_\Omega F(x, u_0(x))\mathrm{d}x > 0$;

$(\mathbf{C}_F^4)$ $F(x,s) = F(y,s)$ for a.e. $x, y \in \Omega$ with $|x| = |y|$ and all $s \in \mathbb{R}$;

$(\mathbf{C}_F^5)$ $F(x,s) \leq F(x,-s)$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}_-$.

We can state an existence and multiplicity result related to problem (3.1.P_λ) as follows.

Theorem 3.1 ([Mezei, Molnár and **Vas**, 2014]). *Assume that $p \in (1, N)$ is fixed. Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^N$ and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function with $F(x,0) = 0$ which satisfies conditions $(\mathbf{C}_F^1)$–$(\mathbf{C}_F^5)$. Then,*

(*a*) *there exists a scalar λ_F such that, for every $0 < \lambda \leq \lambda_F$ the problem (3.1.P_λ) has only the trivial solution;*

(*b*) *there exists a real number λ_1 such that, for every $\lambda > \lambda_1$ the problem (3.1.P_λ) has at least two weak solutions in $W_0^{1,p}(\Omega)$, which are invariant by spherical cap symmetrization.*

Remark 3.1. *For $p = 3$, the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ with the formula*

$$F(x,s) = \begin{cases} |x|\left(s^4 - s^2\right), & |s| \leq 1, \\ |x| \ln\left(s^2\right), & |s| > 1 \end{cases} \tag{3.2}$$

fulfils conditions $(\mathbf{C}_F^1)$–$(\mathbf{C}_F^5)$.

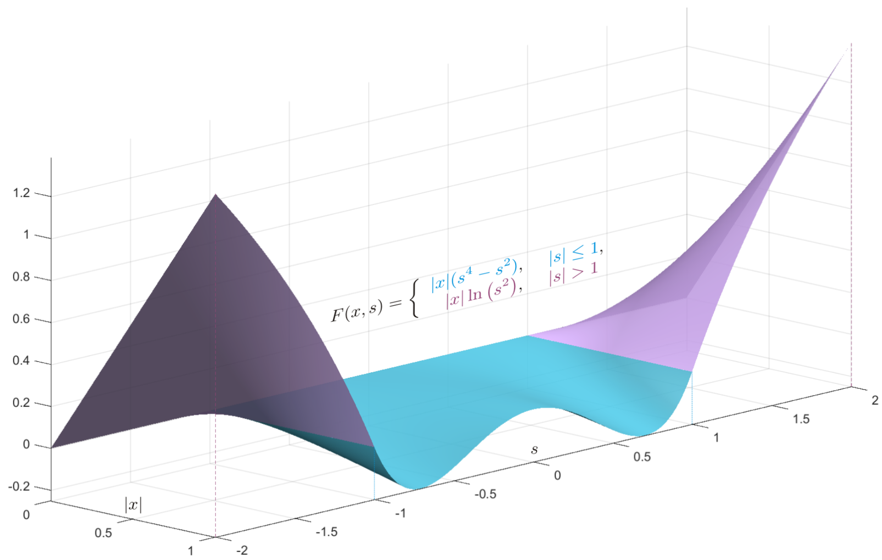

Fig. 3.1: Graph of the function (3.2).

Definition 3.1 (Weak solution to problem (3.1.P_λ)). *A function $u \in W_0^{1,p}(\Omega)$ is a weak solution to problem (3.1.P_λ) if there exists $\xi_F \in \partial_y F(x, u(x))$ for a.e. $x \in \Omega$ such that*

$$\int_\Omega \left(|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) + |u(x)|^{p-2}u(x)v(x)\right) \mathrm{d}x = \lambda \int_\Omega \xi_F(x)v(x)\mathrm{d}x \tag{3.3}$$

for all $v \in W_0^{1,p}(\Omega)$.

We consider the functionals $I, \mathcal{F} : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u(x)|^p + |u(x)|^p)\, \mathrm{d}x \text{ and } \mathcal{F}(u) = \int_\Omega F(x, u(x))\mathrm{d}x,$$

by means of which one can associate the energy functional $\mathcal{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$\mathcal{E}_\lambda(u) = I(u) - \lambda\mathcal{F}(u) \tag{3.4}$$

with the problem (3.1.P_λ).

Remark 3.2. *If Ω is bounded, due to* [Motreanu and Panagiotopoulos, 1999, Theorem 1.3]*, we have that*

$$\partial\mathcal{F}(u) \subset \int_\Omega \partial_y F(x, u(x))\mathrm{d}x.$$

Thus, the critical points of the energy functional $\mathcal{E}_\lambda$ are the weak solutions to problem (3.1.P_λ). Consequently, instead of looking for the solutions to problem (3.1.P_λ), it is enough to find the critical points of $\mathcal{E}_\lambda$.

Using the properties of the energy functional $\mathcal{E}_\lambda$ stated in the following lemmas and applying the non-smooth version of the symmetric minimax principle of [Schaftingen, 2005, Theorem 3.5], one can prove the existence and multiplicity results of Theorem 3.1 related to the solutions to problem (3.1.P_λ).

Lemma 3.1 ([Mezei, Molnár and **Vas**, 2014]). *The energy functional $\mathcal{E}_\lambda$ is coercive for every $\lambda \geq 0$, i.e., $\mathscr{E}_\lambda(u) \to \infty$ as $\|u\|_{W_0^{1,p}(\Omega)} \to \infty$, for all $u \in W_0^{1,p}(\Omega)$.*

Lemma 3.2 ([Mezei, Molnár and **Vas**, 2014]). *For every $\lambda > 0$, the energy functional $\mathcal{E}_\lambda$ satisfies the non-smooth (PS) condition.*

Starting from the symmetric version of the variational principle of Ekeland, proved by [Squassina, 2012, Theorem 2.8], we can formulate the following lemma for the case of locally Lipschitz functions.

Lemma 3.3 ([Mezei, Molnár and **Vas**, 2014]). *Using the notations $V := L^p(\Omega)$ and $X := W_0^{1,p}(\Omega)$ assume that $(X, V, *, \mathcal{H}_*, S)$ satisfies the conditions of* [Squassina, 2012, Definition 2.1] *with the further property that if $(u_n)_n \subset W_0^{1,p}(\Omega)$ for which $u_n \to u$ in $L^p(\Omega)$, then $u_n^* \to u^*$ in $L^p(\Omega)$. We suppose that $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a bounded from below locally Lipschitz functional such that*

$$\Phi(u^H) \leq \Phi(u), \ \forall u \in S, \ \forall H \in \mathcal{H}_*, \tag{3.5}$$

and for all $u \in W_0^{1,p}(\Omega)$ there exists $\xi \in S$ for which $\Phi(\xi) \leq \Phi(u)$.

If the $(PS)_{\inf\Phi}$ condition holds for the functional Φ, then there exists a function $v \in W_0^{1,p}(\Omega)$ for which $\Phi(v) = \inf\Phi$ and $v = v^*$ in $L^p(\Omega)$.

Lemma 3.4 ([Mezei, Molnár and **Vas**, 2014]). *The energy functional $\mathcal{E}_\lambda$ satisfies the inequality*

$$\mathcal{E}_\lambda(u^H) \leq \mathcal{E}_\lambda(u).$$

3.2 The second problem

Let $\Omega = \mathbb{R}^N$ and consider a real separable reflexive Banach space $(X, \|\cdot\|_X)$ together with its topological dual space $(X^*, \|\cdot\|_{X^*})$. Additionally, let $p \in [2, N)$ be fixed and let us denote by $p^* := \frac{Np}{N-p}$ the critical

Sobolev exponent. We impose on X the following conditions:

($\mathbf{C}_X^1$) suppose that the embedding $X \hookrightarrow L^r\left(\mathbb{R}^N\right)$ is continuous with the embedding constant C_r for $r \in [p, p^*]$;

($\mathbf{C}_X^2$) assume that the embedding $X \hookrightarrow L^r\left(\mathbb{R}^N\right)$ is compact for $r \in (p, p^*)$.

Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that conditions ($\mathbf{C}_F^1$), ($\mathbf{C}_F^4$) and ($\mathbf{C}_F^5$) hold. In what follows, we do not require ($\mathbf{C}_F^3$), but we suppose that:

($\widetilde{\mathbf{C}}_F^1$) there exist a positive constant c and $r \in (p, p^*)$ for which $|\xi| \leq c(|s|^{p-1} + |s|^{r-1})$, $\forall s \in \mathbb{R}$, $\xi \in \partial_y F(x, s)$ and a.e. $x \in \mathbb{R}^N$; and

($\widetilde{\mathbf{C}}_F^2$) instead of condition ($\mathbf{C}_F^2$), there exist $q \in (0, p)$, $\nu \in (p, p^*)$, $\alpha \in L^{\frac{\nu}{\nu - q}}(\mathbb{R}^N)$ and $\beta \in L^1(\mathbb{R}^N)$ such that

$$F(z, s) \leq \alpha(z)|s|^q + \beta(z), \ \forall s \in \mathbb{R} \text{ and a.e. } z \in \mathbb{R}^N.$$

Remark 3.3. *When $\Omega = B(0,1)$ (as it was the case in our first problem (3.1.P_λ)), we can deduce assumption ($\widetilde{\mathbf{C}}_F^1$) by means of conditions ($\mathbf{C}_F^1$) and ($\mathbf{C}_F^2$), but if $\Omega = \mathbb{R}^N$ (as it is in the current section), condition ($\widetilde{\mathbf{C}}_F^1$) is necessary.*

Consider also the potential operator $A : X \to X^*$ with the potential $a : X \to \mathbb{R}$, i.e., a is Gâteaux-differentiable and for every $u, v \in X$ we have that

$$\lim_{t \to 0} \frac{a(u + tv) - a(u)}{t} = \langle A(u), v \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and X. For a potential we always suppose that $a(0) = 0$. Additionally, we also assume that A fulfils the next conditions:

($\mathbf{C}_A^1$) A is hemi-continuous, i.e., A is continuous on line segments in X and X^*, endowed with the weak topology;

($\mathbf{C}_A^2$) A is homogeneous of degree $p - 1$, i.e., $A(tu) = t^{p-1}A(u)$, $\forall u \in X$, $\forall t > 0$;

($\mathbf{C}_A^3$) A is a strongly monotone operator, i.e., there exists a continuous function $\tau : [0, \infty) \to [0, \infty)$ that is strictly positive on $(0, \infty)$, furthermore $\tau(0) = 0$, $\lim_{t \to \infty} \tau(t) = \infty$ and

$$\langle A(u) - A(v), u - v \rangle \geq \tau(\|u - v\|_X)\|u - v\|_X, \ \forall u, v \in X;$$

($\mathbf{C}_A^4$) $a(u) \geq c\|u\|_X^p$ for all $u \in X$, where $c > 0$ is a constant;

($\mathbf{C}_A^5$) $a(u^H) \leq a(u)$ for all $u \in X$, where u^H is the polarization of the function u.

Remark 3.4. *Conditions ($\mathbf{C}_A^1$) and ($\mathbf{C}_A^2$) imply that $a(u) = \frac{1}{p}\langle A(u), u \rangle$.*

Our second problem is formulated as follows: find $u \in X$ for which

$$\langle Au, v \rangle + \int_{\mathbb{R}^N} F_y^\circ(x, u(x); -v(x))\mathrm{d}x \geq 0, \ \forall v \in X, \tag{3.6.P_λ}$$

where F_y° denotes the generalized directional derivative of F in the second variable.

Under these conditions, we can state our second main existence and multiplicity result related to the solutions to problem (3.6.P_λ).

Theorem 3.2 ([Mezei, Molnár and **Vas**, 2014])**.** *Assume that $p \in [2, N)$ is fixed. Let $\Omega = \mathbb{R}^N$, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function and $A : X \to X^*$ be a potential operator such that the conditions ($\mathbf{C}_A^1$)–($\mathbf{C}_A^5$), ($\mathbf{C}_X^1$)–($\mathbf{C}_X^2$), ($\mathbf{C}_F^1$), ($\widetilde{\mathbf{C}}_F^1$)–($\widetilde{\mathbf{C}}_F^2$) and ($\mathbf{C}_F^4$)–($\mathbf{C}_F^5$) are satisfied. Then, there exists $\lambda_2 > 0$ such that for every $\lambda > \lambda_2$ the problem (3.6.P_λ) has two non-trivial solutions, which are invariant by spherical cap symmetrization.*

Consider the functional $\widetilde{\mathcal{F}} : X \to \mathbb{R}$,

$$\widetilde{\mathcal{F}}(u) = \int_{\mathbb{R}^N} F(x, u(x))\mathrm{d}x \tag{3.7}$$

and the energy functional $\mathcal{A}_\lambda : X \to \mathbb{R}$,

$$\mathcal{A}_\lambda(u) = a(u) - \lambda\widetilde{\mathcal{F}}(u) \tag{3.8}$$

related to the problem (3.6.P_λ).

Remark 3.5. *Using* [Kristály and Varga, 2004, Proposition 5.1.2], *due to condition* $(\widetilde{\mathbf{C}}_F^1)$, *we have the inequality*

$$\widetilde{\mathcal{F}}^\circ(u; v) \leq \int_{\mathbb{R}^N} F_y^\circ(x, u(x); v(x))\mathrm{d}x. \tag{3.9}$$

Thus, the critical points of the energy functional $\mathcal{A}_\lambda$ *are the weak solutions to problem* (3.6.P_λ).

Similarly to the case of the previous problem, the existence and multiplicity result of Theorem 3.2 related to the solutions to problem (3.6.P_λ) can be proved by using the properties of the energy functional $\mathcal{A}_\lambda$ stated in the following lemmas.

Lemma 3.5 ([Mezei, Molnár and **Vas**, 2014])**.** *If conditions* $(\widetilde{\mathbf{C}}_F^2)$ *and* $(\mathbf{C}_A^4)$ *hold, then the energy functional* $\mathcal{A}_\lambda$ *is coercive for each* $\lambda > 0$, *i.e.,* $\mathcal{A}_\lambda(u) \to \infty$ *as* $\|u\|_X \to \infty$, *for all* $u \in X$.

Lemma 3.6 ([Mezei, Molnár and **Vas**, 2014])**.** *For every* $\lambda > 0$ *the energy functional* $\mathcal{A}_\lambda$ *satisfies the* (PS) *condition.*

Lemma 3.7 ([Mezei, Molnár and **Vas**, 2014])**.** *Assume that conditions* $(\mathbf{C}_F^4)$–$(\mathbf{C}_F^5)$ *and* $(\mathbf{C}_A^5)$ *hold. Then, for all* $H \in H_*$, *we have that*

$$\mathcal{A}_\lambda(u^H) \leq \mathcal{A}_\lambda(u),\ \forall u \in X.$$

3.2.1 A particular case

This subsection presents a particular case of the previously studied problem (3.6.P_λ).

Let $V : \mathbb{R}^N \to \mathbb{R}$ be a function that fulfils the following assumptions:

$(\mathbf{C}_V^1)$ $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$;

$(\mathbf{C}_V^2)$ $\mathrm{meas}\left(\{x \in \mathbb{R}^N : V(x) \leq M\}\right) < \infty,\ \forall M > 0$;

$(\mathbf{C}_V^3)$ $V(x) \leq V(y),\ \forall x, y \in \mathbb{R}^N$ with $|x| \leq |y|$.

The space

$$H := \left\{u \in W^{1,2}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + V(x)u^2(x)\right)\mathrm{d}x < \infty\right\}$$

equipped with the inner product

$$\langle u, v\rangle := \int_{\mathbb{R}^N} \left(\nabla u(x)\nabla v(x) + V(x)u(x)v(x)\right)\mathrm{d}x$$

is a Hilbert space and, by means of [Bartsch and Wang, 1995], we know that H is compactly embedded into $L^s\left(\mathbb{R}^N\right)$ for $s \in [2, 2^*)$.

We can also state a particular case of problem (3.6.P_λ): find a positive $u \in H$ for which

$$\int_{\mathbb{R}^N} \left(\nabla u(x)\nabla v(x) + V(x)u(x)v(x)\right)\mathrm{d}x + \int_{\mathbb{R}^N} F_y^\circ(x, u(x); -v(x))\mathrm{d}x \geq 0,\ \forall v \in H. \tag{3.10.P'_λ}$$

Lemma 3.8 ([Mezei, Molnár and **Vas**, 2014])**.** *If* $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *satisfies conditions* $(\mathbf{C}_F^1)$, $(\widetilde{\mathbf{C}}_F^2)$, $(\mathbf{C}_F^4)$–$(\mathbf{C}_F^5)$ *and* $(\mathbf{C}_V^1)$–$(\mathbf{C}_V^3)$, *then there exist two non-trivial solutions to problem* (3.10.P'_λ), *which are invariant by the spherical cap symmetrization.*

4

Schechter-type critical point results

In this chapter, we focus to the critical point theory developed by [Schechter, 1992, 1999]. Of the methods he presents in [Schechter, 1999], we will deal with those that are related to the existence of a minimizer for C^1 functionals defined on a closed ball of an appropriate Hilbert or Banach space.

[Precup, 2009, 2013] studies the critical point theorems of Schechter-type for C^1 functionals defined on a closed ball, on annular domains and also on a closed conical shell of a Hilbert space. Using the variational principle of Bishop–Phelps, he gives in [Precup, 2013] a new proof to Schechter's theorem for extrema.

Based on the variational principle of Ekeland, in [Lisei and **Vas**, 2016], we improved the aforementioned Schechter-type results on a closed ball of Precup for sublevel sets in locally uniformly convex Banach spaces and we applied our result for localizing the solutions to problems which contain the p-Laplacian operator on both bounded and unbounded domains. Then, in article [**Vas**, 2015], we extended the Schechter-type results of Precup for locally Lipschitz functions defined on a closed ball of a Hilbert space and, to illustrate the applicability of our result, we have presented an inclusion problem.

From our results mentioned above, in Section 4.1, we present the Schechter-type critical point theorem for the case of locally Lipschitz functions, as well a concrete application of it. Thereafter, in Section 4.2, we deal with the critical point theorem of Schechter-type for C^1 functionals in Banach spaces, emphasizing the applicability of the theorem by presenting two applications.

4.1 A Schechter-type critical point result for locally Lipschitz functions

This section discusses a critical point result of Schechter-type for locally Lipschitz functions defined on a ball of a Hilbert space.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|_X = \sqrt{\langle\cdot,\cdot\rangle}$. Consider the origin-centered closed ball $\overline{X}_R := \{x \in X : \|x\|_X \leq R\}$ of the space X with radius $R > 0$ and the corresponding sphere $\partial X_R := \{x \in X : \|x\|_X = R\}$.

Using the above notations, the main result of the section can be formulated as follows.

Theorem 4.1 ([Vas, 2015]). *Let $F : \overline{X}_R \to \mathbb{R}$ be a locally Lipschitz function, which is bounded from below. There exists a sequence $(x_n)_n \subset \overline{X}_R$, such that $F(x_n) \to \inf F(\overline{X}_R) := \inf_{x\in\overline{X}_R} F(x)$ and one of the following two situations holds:*

(a) $\lambda_F(x_n) \to 0$;

(b) $\|x_n\|_X = R$, $\langle w_n^*, x_n\rangle \le 0$ *for all* n *and* $w_n^* \in \partial F(x_n)$, *and* $\lambda_{F,\partial X_R}(x_n) \to 0$, *where* $\partial F(x_n)$ *is the generalized gradient of the locally Lipschitz function* F *and*

$$\lambda_{F,\partial X_R}(x_n) := \inf\left\{ w^* - \frac{1}{R^2}\left\langle w^*_, x_n\right\rangle \Lambda x_n : w^* \in \partial F(x_n)\right\}.$$

If in addition $\langle x^*, x\rangle \ge -a > -\infty$ *for all* $x \in \partial X_R$ *and* $x^* \in \partial F(x)$, *furthermore* F *satisfies a* (PS)*-type compactness condition and the boundary condition*

$$x^* + \mu\Lambda x \neq 0 \tag{4.1}$$

holds for all $x \in \partial X_R$ *and* $\mu > 0$, *then there exists* $x \in \overline{X}_R$ *such that*

$$F(x) = \inf F\left(\overline{X}_R\right).$$

4.1.1 An application

This subsection gives a concrete application of our Schechter-type critical point Theorem 4.1.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \in \mathbb{N}_{\ge 2})$ that has C^1 regular boundary $\partial\Omega$. Consider the Sobolev space $W_0^{1,2}(\Omega)$ endowed with the norm $\|u\|_{W_0^{1,2}(\Omega)} := \left(\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$ and let $W^{-1,2}(\Omega)$ denote the topological dual space $\left(W_0^{1,2}(\Omega)\right)^*$.

Due to the Rellich–Kondrachov Theorem, the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in \left(1, 2^* = \frac{2N}{N-2}\right)$ and there exists a constant $C_q > 0$ for which

$$\|u\|_{L^q(\Omega)} \le C_q \|u\|_{W_0^{1,2}(\Omega)}, \ \forall u \in W_0^{1,2}(\Omega). \tag{4.2}$$

Consider the Carathéodory function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ that fulfils the conditions:

(a) $F(\cdot, u)$ is measurable for each $u \in \mathbb{R}$;

(b) $F(x, \cdot)$ is locally Lipschitz for each $x \in \Omega$;

(c) $F(\cdot, 0) \in L^1(\Omega)$;

and, also assume that the growth condition

$$|z| \le a(x) + b(x)\,|y|^{q-1}, \ \forall z \in \partial_y F(x, y), \ (x, y) \in \Omega \times \mathbb{R}, \tag{4.3}$$

holds, where $a \in L^{\frac{q}{q-1}}(\Omega)$, $b \in L^\infty(\Omega)$ are positive functions and $q \in \left(1, 2^* = \frac{2N}{N-2}\right)$.

Under the above conditions, we consider the non-smooth Dirichlet inclusion problem

$$\begin{cases} -\Delta u \in \partial_y F(x, u) & \text{a.e. } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4.P}$$

We recall the notations

$$\overline{X}_R := \left\{u \in W_0^{1,2}(\Omega) : \|u\|_{W_0^{1,2}(\Omega)} \le R\right\}$$

and

$$\partial X_R := \left\{u \in W_0^{1,2}(\Omega) : \|u\|_{W_0^{1,2}(\Omega)} = R\right\}.$$

Definition 4.1 (Weak solution to problem (4.4.P)). *A function* $u \in W_0^{1,2}(\Omega)$ *is a weak solution to problem*

(4.4.P) *if there exists* $w_F(x) \in \partial_y F(x, u(x))$ *for a.e.* $x \in \Omega$ *such that*

$$\int_\Omega \nabla u(x)\nabla v(x)\mathrm{d}x = \int_\Omega w_F(x) \cdot v(x)\mathrm{d}x,\ \forall v \in W_0^{1,2}(\Omega).$$

Let $\mathcal{E} : W_0^{1,2}(\Omega) \to \mathbb{R}$,

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x - \int_\Omega F(x, u(x))\mathrm{d}x \tag{4.5}$$

be the energy functional related to problem (4.4.P), whose critical points are the weak solutions to (4.4.P).

Proposition 4.1 ([**Vas**, 2015])**.** *If* $R > 0$ *is the solution to the inequality*

$$R - C_q^q \|b\|_{L^\infty(\Omega)} R^{q-1} > C_q \|a\|_{L^{\frac{q}{q-1}}(\Omega)} \tag{4.6}$$

over $\mathbb{R}$*, then*

$$\langle u^*, u\rangle + \mu \cdot \langle \Lambda u, u\rangle \neq 0,\ \forall u^* \in \partial\mathcal{E}(u)$$

for any $\mu > 0$*, where* $u \in \partial X_R$*.*

Using the conditions of Proposition 4.1 and our related Schechter-type critical point Theorem 4.1 we can formulate the next result related to the solutions to problem (4.4.P).

Theorem 4.2 ([**Vas**, 2015])**.** *If we choose* $R > 0$ *to be the solution to the inequality*

$$R - C_q^q \|b\|_{L^\infty(\Omega)} R^{q-1} > C_q \|a\|_{L^{\frac{q}{q-1}}(\Omega)}$$

over $\mathbb{R}$*, then problem* (4.4.P) *admits a weak solution* $u \in \overline{X}_R$*, which minimizes* $\mathcal{E}$ *on* $\overline{X}_R$*.*

4.2 A critical point result of Schechter-type in a Banach space

Based on article [Lisei and **Vas**, 2016], this section presents two applications of our Schechter-type critical point theorem [Lisei and **Vas**, 2016, Theorem 3.1] for C^1 functionals in Banach spaces. In both examples, the theorem will be used to localize the solutions to partial differential equations which contain the p-Laplacian operator, but in the first one we will work on bounded, while in the second one on unbounded domain.

4.2.1 The first example

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}_{\geq 2}$) that has Lipschitz continuous boundary and let $p \in (1, \infty)$ be fixed. Consider the uniformly convex smooth Banach space $\left(W_0^{1,p}(\Omega), \|\cdot\|_{W_0^{1,p}(\Omega)}\right)$ with its uniformly convex topological dual space $W^{-1,p'}(\Omega)$.

By means of the Rellich–Kondrachov Theorem, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $q \in (1, p^*)$ (where $p^* = \frac{Np}{N-p}$, if $p < N$ and $p^* = \infty$, provided that $p \geq N$), thus there exists $C_q > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C_q \|u\|_{W_0^{1,p}(\Omega)},\ \forall u \in W_0^{1,p}(\Omega). \tag{4.7}$$

Let $J_\varphi : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the duality mapping corresponding to the normalization function $\varphi(t) = t^{p-1}$ for $t \in \mathbb{R}_+$ and consider the mapping $\bar{J} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$, $\bar{J} = J_\varphi^{-1}$. Due to [Dinca et al., 2001, Theorem 5] $\bar{J}$ is bounded, continuous and monotone, moreover, for each $w \in W^{-1,p'}(\Omega)$ one has that

$$\langle w, \bar{J}w\rangle = \varphi^{-1}\left(\|w\|_{W_0^{1,p}(\Omega)}\right)\|w\|_{W_0^{1,p}(\Omega)} \text{ and } \|\bar{J}w\|_{W_0^{1,p}(\Omega)} = \varphi^{-1}\left(\|w\|_{W_0^{1,p}(\Omega)}\right). \tag{4.8}$$

We also consider the p-Laplacian operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$,

$$\langle -\Delta_p(u), v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\mathrm{d}x,\ \forall u, v \in W_0^{1,p}(\Omega).$$

Let the functional $H : W_0^{1,p}(\Omega) \to \mathbb{R}$ be given by $H(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}$, which is continuously Fréchet-differentiable on $W_0^{1,p}(\Omega)$ and $H' = -\Delta_p$. Since the operator $-\Delta_p$ is the above duality mapping J_φ, we have that $H' = J_\varphi$, see [Dinca et al., 2001, Theorems 7 and 9]. In this example we will also use the notations

$$\overline{X}_R := \left\{u \in W_0^{1,p}(\Omega) : \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} \leq R\right\}$$

and

$$\partial X_R := \left\{u \in W_0^{1,p}(\Omega) : \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} = R\right\}.$$

Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) \neq 0$ for a.e. $x \in \Omega$ and it also fulfils the growth condition

$$|f(x, s)| \leq a(x)|s|^{q-1} + b(x), \ \forall x \in \Omega, \ s \in \mathbb{R}, \tag{4.9}$$

where $a \in L^\infty(\Omega)$, $b \in L^{\frac{q}{q-1}}(\Omega)$ are positive functions and $q \in (1, p^*)$. The Nemytskii operator N_f : $W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ introduced by the function f is

$$N_f\,(u)\,(x) = f(x, u(x)).$$

Under the above conditions, we have that $N_f(W_0^{1,p}(\Omega)) \hookrightarrow N_f(L^q(\Omega)) \subset L^{\frac{q}{q-1}}(\Omega) = (L^q(\Omega))^* \hookrightarrow W^{-1,p'}(\Omega)$ and N_f is a continuous function which maps bounded sets into bounded sets (see [Goldberger et al., 1992]).

Using the p-Laplacian operator, consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x, u) & \text{a.e. } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.10.P}$$

Definition 4.2 (Weak solution to problem (4.10.P))**.** *A function $u \in W_0^{1,p}(\Omega)$ is a weak solution to problem (4.10.P) if*

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\mathrm{d}x = \int_\Omega f(x, u(x))v(x)\mathrm{d}x, \ \forall v \in W_0^{1,p}(\Omega). \tag{4.11}$$

Let $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$\mathcal{E}\,(u) = \frac{1}{p}\,\|u\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega h\,(x, u(x))\,\mathrm{d}x \tag{4.12}$$

be the energy functional associated with problem (4.10.P), where $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is $h\,(x, t) = \int_0^t f\,(x, s)\,\mathrm{d}s$. Then, due to [Goldberger et al., 1992, Theorem 7], we have that

$$\mathcal{E}'\,(u) = H'\,(u) - N_f\,(u)\,.$$

Moreover, the critical points of the energy functional $\mathcal{E}$ are the solutions to (4.11), consequently the weak solutions to problem (4.10.P).

Now, we formulate some assumptions for R: denote by C an upper bound for the constant C_q and suppose that one of the following three assumptions is satisfied:

$(\mathbf{C}_R^1)$ if $p > q$, let $R > 0$ be a solution to the inequality

$$R^{\frac{p-1}{p}} > C^q p^{\frac{q-p}{p}}\,\|a\|_{L^\infty(\Omega)} R^{\frac{q-1}{p}} + Cp^{\frac{1-p}{p}}\,\|b\|_{L^{\frac{q}{q-1}}(\Omega)}$$

over $\mathbb{R}$;

$(\mathbf{C}_R^2)$ if $p = q$, assume that $1 > C^p\|a\|_{L^\infty(\Omega)}$ and let R be such that

$$R > \left(\frac{Cp^{\frac{1-p}{p}}\,\|b\|_{L^{\frac{p}{p-1}}(\Omega)}}{1 - C^p\|a\|_{L^\infty(\Omega)}}\right)^{\frac{p}{p-1}};$$

$(\mathbf{C}_R^3)$ if $q > p$, assume that $1 > C^q p^{\frac{q-p}{p}} \|a\|_{L^\infty(\Omega)} + C p^{\frac{1-p}{p}} \|b\|_{L^{\frac{q}{q-1}}(\Omega)}$ and let $R > 0$ be a solution to the inequality

$$R^{\frac{p-1}{p}} - C^q p^{\frac{q-p}{p}} \|a\|_{L^\infty(\Omega)} R^{\frac{q-1}{p}} > C p^{\frac{1-p}{p}} \|b\|_{L^{\frac{q}{q-1}}(\Omega)}$$

over $\mathbb{R}$.

Proposition 4.2 ([Lisei and **Vas**, 2016]). *If R satisfies one of the above conditions $(\mathbf{C}_R^1)$–$(\mathbf{C}_R^3)$, the relation*

$$\mathcal{E}'(u) + \mu H'(u) \neq 0, \ \forall \mu > 0, \ u \in \partial X_R$$

holds.

Proposition 4.3 ([Lisei and **Vas**, 2016]). *Assume that R satisfies one of the conditions $(\mathbf{C}_R^1)$–$(\mathbf{C}_R^3)$. Then, $\mathcal{E}$ satisfies the following (PS)-type compactness condition: if $(u_n)_n$ is a sequence from $\overline{X}_R$ such that one of the following statements hold*

(a) $\mathcal{E}'(u_n) \to 0$ *as* $n \to \infty$;

(b) *for each* $n \in \mathbb{N}$ *we have that* $H(u_n) = R$, $\langle H'(u_n), \bar{J}\mathcal{E}'(u_n)\rangle \leq 0$ *and*

$$\mathcal{E}'(u_n) - \frac{\langle \mathcal{E}'(u_n), u_n\rangle}{\langle H'(u_n), u_n\rangle} H'(u_n) \to 0 \ \text{ as } n \to \infty,$$

then $(u_n)_n$ admits a convergent subsequence.

In order to localize the solutions to problem (4.10.P), we apply our smooth version of the Schechter-type critical point theorem, see [Lisei and **Vas**, 2016, Theorem 3.1].

Theorem 4.3 ([Lisei and **Vas**, 2016]). *Assume that R satisfies one of the conditions $(\mathbf{C}_R^1)$–$(\mathbf{C}_R^3)$. Then, problem (4.10.P) admits a weak solution $u \in \overline{X}_R$, which minimizes $\mathcal{E}$ on $\overline{X}_R$.*

We study situations when the best Sobolev constant C_q admits an upper estimate which can be computed as follows.

Denoting by $\lambda_p(\Omega) := \min\limits_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} \dfrac{\int_\Omega |\nabla v(x)|^p \mathrm{d}x}{\int_\Omega |v(x)|^p \mathrm{d}x}$ the first eigenvalue of the p-Laplacian operator, one has the inequality

$$\|u\|^p_{L^p(\Omega)} \leq \frac{1}{\lambda_p(\Omega)} \|u\|^p_{W_0^{1,p}(\Omega)}, \ \forall u \in W_0^{1,p}(\Omega).$$

Therefore, the best embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is $C_p = \left(\frac{1}{\lambda_p(\Omega)}\right)^{\frac{1}{p}}$, while for $q < p$ the best embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ due to the Hölder's inequality verifies $C_q \leq |\Omega|^{\frac{p-q}{pq}} \left(\frac{1}{\lambda_p(\Omega)}\right)^{\frac{1}{p}}$, where $|\Omega|$ denotes the Lebesgue measure, i.e., the N-dimensional volume of the set Ω. In order to obtain upper bounds for the constant C_q $(q \leq p)$, we need to determine lower bounds for the first eigenvalue $\lambda_p(\Omega)$.

Due to the Faber-Kahn inequality [Bhattacharya, 1999, Theorem 1], we have that $\lambda_p(\Omega) \geq \lambda_p(\Omega^*)$, where Ω^* denotes the N-dimensional origin-centered ball, the volume of which coincides with that of Ω, consequently, it has the radius $r = \frac{1}{\sqrt{\pi}} \left(|\Omega| \Gamma\left(\frac{N}{2} + 1\right)\right)^{\frac{1}{N}}$.

By [Lefton and Wei, 1997], for the ball $\Omega^* \subset \mathbb{R}^N$ with radius r we have the inequality $\lambda_p(\Omega^*) \geq \left(\frac{N}{rp}\right)^p$, accordingly the best Sobolev constant C_p has the upper estimate

$$C_p \leq \frac{p}{N\sqrt{\pi}} \left(|\Omega| \Gamma\left(\frac{N}{2} + 1\right)\right)^{\frac{1}{N}},$$

and for $q \in (1,p)$ one has that

$$C_q \leq \frac{p}{N\sqrt{\pi}} \left(|\Omega|^{\frac{(p-q)}{pq}+\frac{1}{N}} \Gamma\left(\frac{N}{2}+1\right)\right)^{\frac{1}{N}}.$$

In one-dimension, for $\Omega = (0,T) \subset \mathbb{R}$ the first eigenvalue is (see [Drábek and Manásevich, 1999]) $\lambda_p(\Omega) = (p-1)\left(\frac{2\pi}{Tp\sin\left(\frac{\pi}{p}\right)}\right)^p$, consequently

$$C_p = \frac{Tp\sin\left(\frac{\pi}{p}\right)}{2\pi(p-1)^{\frac{1}{p}}}.$$

Also, for $\Omega = (0,T) \subset \mathbb{R}$ – by the sharp Poincaré inequality (see [Talenti, 1976, p. 357]) – for each $p \in (1,\infty)$, $q \in [1,\infty)$ and $u \in W_0^{1,p}(\Omega)$ the inequality

$$\|u\|_{L^q(\Omega)} \leq C_q \|u\|_{W_0^{1,p}(\Omega)}$$

is satisfied with the embedding constant

$$C_q = \frac{T^{\frac{1}{q}+\frac{1}{p'}}}{2B\left(\frac{1}{q},\frac{1}{p'}\right)} (p')^{\frac{1}{q}}\, q^{\frac{1}{p'}}\, (p'+q)^{\frac{1}{p}-\frac{1}{q}},$$

where $p' = \frac{p}{p-1}$ and B is the Beta function.

4.2.2 The second example

For fixed $p \in (1,\infty)$, we define the closed subspace

$$W_r^{1,p}\left(\mathbb{R}^N\right) = \left\{u \in W^{1,p}\left(\mathbb{R}^N\right) : u(x) = u(x'),\ \forall\, x, x' \in \mathbb{R}^N : |x| = |x'|\right\}$$

of radially symmetric functions of the separable reflexive uniformly convex smooth Banach space $W^{1,p}\left(\mathbb{R}^N\right)$ equipped with the norm

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \left(|\nabla u(x)|^p + |u(x)|^p\right) \mathrm{d}x\right)^{\frac{1}{p}} \tag{4.13}$$

induced from $W^{1,p}\left(\mathbb{R}^N\right)$. Then, $W_r^{1,p}\left(\mathbb{R}^N\right)$ is also uniformly smooth and its topological dual space $\left(W_r^{1,p}\left(\mathbb{R}^N\right)\right)^*$ is uniformly convex.

Let $J_\varphi : W_r^{1,p}\left(\mathbb{R}^N\right) \to \left(W_r^{1,p}\left(\mathbb{R}^N\right)\right)^*$ be the duality mapping that corresponds to the normalization function $\varphi(t) = t^{p-1}$, $t \in \mathbb{R}_+$ (see [Chabrowski, 1997, Proposition 2.2.4]). Under the above conditions, for the duality mapping J_φ the properties

$$\|J_\varphi u\|_{W^{1,p}(\mathbb{R}^N)} = \varphi\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}\right) \quad \text{and} \quad \langle J_\varphi u, u\rangle = \|J_\varphi u\|_{W^{1,p}(\mathbb{R}^N)} \|u\|_{W^{1,p}(\mathbb{R}^N)}$$

hold for all $u \in W_r^{1,p}\left(\mathbb{R}^N\right)$. Moreover, the functional $H : W_r^{1,p}\left(\mathbb{R}^N\right) \to \mathbb{R}$, $H(u) = \frac{1}{p}\|u\|^p_{W^{1,p}(\mathbb{R}^N)}$ is convex and Fréchet-differentiable with $H' = J_\varphi$. We consider also $\bar{J} : \left(W_r^{1,p}\left(\mathbb{R}^N\right)\right)^* \to W_r^{1,p}\left(\mathbb{R}^N\right)$, $\bar{J} = J_\varphi^{-1}$.

Due to [Lions, 1982, Théorème II.1], the embedding $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ is compact for $q \in (p, p^*)$ (where $N \in \mathbb{N}_{\geq 2}$, $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = \infty$, if $p \geq N$) and there exists the best embedding constant $C_q > 0$ such that

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C_q \|u\|_{W^{1,p}(\mathbb{R}^N)},\ \forall u \in W_r^{1,p}\left(\mathbb{R}^N\right). \tag{4.14}$$

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,0) \neq 0$ for a.e. $x \in \mathbb{R}^N$ and which fulfils the growth condition

$$|f(x,s)| \leq a(x)|s|^{q-1} + b(x),\ \forall (x,s) \in \mathbb{R}^N \times \mathbb{R},$$

where $a \in L^{\infty}\left(\mathbb{R}^N\right)$, $b \in L^{\frac{q}{q-1}}\left(\mathbb{R}^N\right)$ are positive functions, $q \in (p, p^*)$ and $f(x, \cdot) = f(x', \cdot)$ for all $x, x' \in \mathbb{R}^N$ for which $|x| = |x'|$ (i.e., f is radially symmetric in the first variable).

Using the p-Laplacian operator, consider the problem

$$-\Delta_p u + |u|^{p-2}u = f(x, u) \quad \text{a.e. } x \in \mathbb{R}^N. \tag{4.15.P}$$

Definition 4.3 (Weak solution to problem (4.15.P)). *A function $u \in W^{1,p}\left(\mathbb{R}^N\right)$ is a weak solution to problem (4.15.P) if*

$$\int_{\Omega} |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) + |u(x)|^{p-2}u(x)v(x)\mathrm{d}x = \int_{\Omega} f(x, u(x))v(x)\mathrm{d}x \tag{4.16}$$

holds for all $v \in W^{1,p}\left(\mathbb{R}^N\right)$.

Define the Nemytskii operator $N_f : W_r^{1,p}\left(\mathbb{R}^N\right) \to \left(W_r^{1,p}\left(\mathbb{R}^N\right)\right)^*$ by $N_f\left(u\right)\left(x\right) = f(x, u(x))$. Let the energy functional $\mathcal{E} : W_r^{1,p}\left(\mathbb{R}^N\right) \to \mathbb{R}$ related to the problem (4.15.P) be defined by

$$\mathcal{E}\left(u\right) = \frac{1}{p}\left\|u\right\|^p_{W^{1,p}(\Omega)} - \int_{\mathbb{R}^N} h\left(x, u(x)\right)\mathrm{d}x, \tag{4.17}$$

where $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is $h\left(x, t\right) = \displaystyle\int_0^t f\left(x, s\right)\mathrm{d}s$. Hence, one has that

$$\mathcal{E}'\left(u\right) = H'\left(u\right) - N_f\left(u\right).$$

Let $\mathbb{G} = O\left(\mathbb{R}^N\right)$ be the set of all rotations on $\mathbb{R}^N$, whose elements leave $\mathbb{R}^N$ invariant, i.e., $g(\mathbb{R}^N) = \mathbb{R}^N$ for all $g \in \mathbb{G}$. Observe that $\mathbb{G}$ induces an isometric linear action on the space $W^{1,p}\left(\mathbb{R}^N\right)$ by the formula

$$(gu)(x) = u(g^{-1}x),\ g \in \mathbb{G},\ u \in W^{1,p}\left(\mathbb{R}^N\right) \quad \text{a.e. } x \in \mathbb{R}^N.$$

A function $\phi \in W^{1,p}\left(\mathbb{R}^N\right)$ is $\mathbb{G}$-invariant if

$$\phi(gu) = \phi(u),\ \forall g \in \mathbb{G},\ u \in W^{1,p}\left(\mathbb{R}^N\right).$$

Actually, $W_r^{1,p}\left(\mathbb{R}^N\right)$ is the fixed point set of $W^{1,p}\left(\mathbb{R}^N\right)$ under $\mathbb{G}$ and the norm (4.13) is $\mathbb{G}$-invariant on $W^{1,p}\left(\mathbb{R}^N\right)$.

Based on the assumptions set for the function f and the above remark, the energy functional $\mathcal{E}$ is $\mathbb{G}-$invariant, consequently, by means of the principle of symmetric criticality [Palais, 1979], every critical point of $\mathcal{E}$ is also a solution to (4.16).

Considering the set

$$\overline{X}_R = \left\{u \in W_r^{1,p}\left(\mathbb{R}^N\right) : \frac{1}{p}\|u\|^p_{W^{1,p}(\mathbb{R}^N)} \leq R\right\},$$

the next result can be proved similarly to the case of our previous example in Section 4.2.1.

Theorem 4.4 ([Lisei and **Vas**, 2016]). *Suppose that R satisfies one of the conditions* $(\mathbf{C}_R^1)$–$(\mathbf{C}_R^3)$. *Then, $\mathcal{E}$ admits a critical point $u \in \overline{X}_R$, which minimizes $\mathcal{E}$ on $\overline{X}_R$. Moreover, this critical point is also a weak solution to the problem (4.15.P).*

We present situations when the Sobolev constant C_q admits an upper estimate, which can be computed as follows.

Due to [Talenti, 1976], we have the following result: for $p \in (1, N)$ and $p^* = \frac{Np}{N-p}$ the inequality

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C_{\mathbb{R}}\left(\int_{\mathbb{R}^N} |\nabla u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}},\ \forall u \in W^{1,p}\left(\mathbb{R}^N\right)$$

holds, where

$$C_{\mathbb{R}} = \frac{1}{\sqrt{\pi} N^{\frac{1}{p}}} \left(\frac{p-1}{N-p} \right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(1+N-\frac{N}{p}\right)} \right)^{\frac{1}{N}}.$$

Consequently,

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C_{\mathbb{R}} \|u\|_{W^{1,p}(\mathbb{R}^N)}, \ \forall u \in W^{1,p}\left(\mathbb{R}^N\right).$$

For any $q \in (p, p^*)$ there exists $\theta \in (0,1)$ such that $q = \theta p + (1-\theta)p^*$, then, by using Hölder's inequality, for each $u \in W^{1,p}\left(\mathbb{R}^N\right)$ we obtain that

$$\|u\|^q_{L^q(\mathbb{R}^N)} \leq \|u\|^{\theta p}_{L^p(\mathbb{R}^N)} \|u\|^{(1-\theta)p^*}_{L^{p^*}(\mathbb{R}^N)} \leq C_{\mathbb{R}}^{Nq\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|^q_{W^{1,p}(\mathbb{R}^N)}.$$

Thus, the Sobolev constant C_q has the upper estimate

$$C_q \leq C_{\mathbb{R}}^{N\left(\frac{1}{p}-\frac{1}{q}\right)}, \ q \in \left(p, \frac{Np}{N-p}\right), \ p \in (1,\infty).$$

5

A localization method for the solutions to non-homogeneous operator equations

This chapter is based on article [Lisei, Varga and **Vas**, 2018]. After a brief summary of preliminary notions and properties, Section 5.2 presents some auxiliary results that are required to prove the main statements of Section 5.3, in which we discuss the new variants of: the deformation lemma and the bounded version of the general minimax theorem of [Willem, 1996]; the mountain pass theorem of [Ambrosetti and Rabinowitz, 1973]; and the variational principle of Ekeland [Ekeland, 1974] as well for the case of a wedge intersected with a ball in a reflexive locally uniformly convex smooth Banach space. Finally, in Section 5.4, we apply our results to localize two non-trivial solutions to Dirichlet problems that involve non-homogeneous operators in the context of Orlicz–Sobolev spaces.

5.1 Preliminaries

Consider the real Banach space $(X, \|\cdot\|_X)$, its topological dual space X^* and denote by $\langle\cdot,\cdot\rangle$ the duality between X^* and X.

Consider also the set-valued operator $J_\varphi : X \to \mathcal{P}(X^*)$,

$$J_\varphi x = \{x^* \in X^* : \langle x^*, x\rangle = \varphi(\|x\|_X)\|x\|_X,\ \|x^*\|_X = \varphi(\|x\|_X)\},\ x \in X,$$

which is the duality mapping that corresponds to the normalization function φ.

Hereafter, we suppose that:

$(\mathbf{C}^1_X)$ X is a locally uniformly convex reflexive smooth Banach space.

Since X is smooth, due to [Ciorănescu, 1990, Corollary 4.5, p. 27], $\operatorname{card}(J_\varphi x) = 1$. Hence, $J_\varphi : X \to X^*$ and we have that $\langle J_\varphi x, x\rangle = \varphi(\|x\|_X)\|x\|_X$ and $\|J_\varphi x\|_X = \varphi(\|x\|_X)$. Under these conditions, by means of [Dinca and Matei, 2007, Theorem 5], the duality mapping J_φ is bijective and its inverse J_φ^{-1} is bounded, continuous and monotone. Moreover, by using the canonical isomorphism $\chi : X \to X^{**}$ and the duality mapping $J^*_{\varphi^{-1}} : X^* \to X^{**}$ that corresponds to the normalization function φ^{-1}, we have that $J_\varphi^{-1} = \chi^{-1}J^*_{\varphi^{-1}}$.

We consider the mapping $\bar{J} : X^* \to X$ defined by $\bar{J} = J_\varphi^{-1}$, which due to the above result is bounded, continuous and monotone. Additionally, we have that

$$\langle w, \bar{J}w \rangle = \varphi^{-1}(\|w\|_X)\|w\|_X \text{ and } \|\bar{J}w\|_X = \varphi^{-1}(\|w\|_X), \ \forall w \in X^*. \tag{5.1}$$

In the case of the normalization function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ we will use the notation $\Psi(t) := \int_0^t \varphi(s)\mathrm{d}s$, which is a convex function by means of [Cioränescu, 1990, Lemma 4.3, p. 25]. Since X satisfies condition $(\mathbf{C}_X^1)$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(\|x+ty\|_X)\Big|_{t=0} = \langle J_\varphi x, y \rangle, \ \forall x, y \in X,$$

due to [Cioränescu, 1990, Corollary 4.5, p. 27]. Consequently, the Gâteaux derivative of $x \in X \mapsto \Psi(\|x\|_X) \in \mathbb{R}$ in the direction $y \in X$ is

$$\langle \Psi'(\|x\|_X), y \rangle = \langle J_\varphi x, y \rangle, \ \forall x, y \in X. \tag{5.2}$$

We also assume that:

$(\mathbf{C}_X^2)$ the duality mapping $J_\varphi : X \to X^*$ is continuous.

Remark 5.1. *Due to* [Cioränescu, 1990, Corollary 5.3, p. 77], *if X^* is locally uniformly convex and X is reflexive, then J_φ is continuous. In the case of the applications we studied, it is easier to show the continuity property of the duality mapping J_φ than to prove the locally uniformly convex property of the space X^*, thus we require condition $(\mathbf{C}_X^2)$. This way we can also avoid theorems which involve equivalent renormings, because we need the concrete expression and certain properties of the duality mapping in our applications.*

Let $K \subset X$, $K \neq \{0\}$ be a wedge of X, i.e., it is a convex closed non-empty set such that $\lambda u \in K$ for every $u \in K$ and $\lambda \geq 0$. For $R > 0$ we introduce the notations

$$X_R := \{x \in X : \|u\|_X < R\}, \ \overline{X}_R := \{x \in X : \|u\|_X \leq R\}$$

and

$$K_R := \{x \in K : \|u\|_X \leq R\}, \ \partial K_R := \{x \in K : \|u\|_X = R\}.$$

Consider the subset $S \subset K$ and for a fixed $\rho > 0$ let

$$S_\rho := \{x \in K : \mathrm{dist}(x, S) = \inf\{\|x - y\|_X : y \in S\} \leq \rho\}.$$

be the ρ-neighbourhood of S. Consider the C^1 functional $E : X \to \mathbb{R}$ and for any $a, b \in \mathbb{R}$ for which $a \leq b$ we also define the sets

$$E_a := \{x \in K_R : E(x) \geq a\}, \ E^b := \{x \in K_R : E(x) \leq b\}$$

and

$$E_a^b := \{x \in K_R : a \leq E(x) \leq b\}.$$

5.2 Auxiliary results

This section presents some auxiliary results, which play an important role in proving our main results for the case of a wedge in a Banach space.

Proposition 5.1 ([Lisei, Varga and **Vas**, 2018]). *Assume that the condition $(\mathbf{C}_X^1)$ holds. Let α and θ be real numbers such that $0 < \alpha < \frac{1}{2}(1-\theta)$. Then, for each $x^*, y^* \in X^* \setminus \{0\}$ such that*

$$-\langle x^*, \bar{J}y^* \rangle \leq \theta \|x^*\|_X \|\bar{J}y^*\|_X, \tag{5.3}$$

there exists $h \in X$ for which

$$\langle x^*, h\rangle \leq -\alpha\|x^*\|_X\|h\|_X \quad \text{and} \quad \langle y^*, h\rangle < 0. \tag{5.4}$$

Additionally, if $K \subset X$ is a wedge and $\bar{J}y^*, \bar{J}y^* - \bar{J}x^* \in K$, then $\bar{J}y^* + \mu h \in K$ for all $\mu \in \left[0, \frac{\theta+\alpha}{\theta+1}\right]$.

Next, by using Proposition 5.1, we state a pseudo-gradient lemma which generalizes the results of [Schechter, 1999, Lemma 5.9.2] for the case of a wedge in a Banach space.

Lemma 5.1 ([Lisei, Varga and **Vas**, 2018])**.** *Assume that conditions $(\mathbf{C}^1_X)$ and $(\mathbf{C}^2_X)$ hold. Let $E : X \to \mathbb{R}$ be a C^1 functional and for $a > 0$ consider both the set $U := \{u \in K_R : \|E'(u)\|_X > a\} \neq \emptyset$ and the closed subset $U_0 \subseteq U \cap \partial K_R$. Suppose that there exists $\theta \in (0, 1)$ such that*

$$-\langle E'(u), u\rangle \leq \theta\|E'(u)\|_X\|u\|_X, \ \forall u \in U_0$$

and

$$u - \bar{J}E'(u) \in K, \ \forall u \in K_R.$$

Then, there exist $\alpha \in (0, 1)$ and a locally Lipschitz continuous map $H : U \to X$ such that

$$u + H(u) \in K, \ \|H(u)\|_X \leq 1, \ \langle E'(u), H(u)\rangle \leq -\alpha\|E'(u)\|_X, \ \forall u \in U \tag{5.5}$$

and

$$\langle J_\varphi u, H(u)\rangle < 0, \ \forall u \in U_0. \tag{5.6}$$

The next corollary discusses the case when in Lemma 5.1 we assume that $U \cap \partial K_R = \emptyset$ and property (5.6) is dropped.

Corollary 5.1. *Suppose that the condition $(\mathbf{C}^1_X)$ holds. Let $E : X \to \mathbb{R}$ be a C^1 functional and for $a > 0$ consider the set $U := \{u \in K_R : \|E'(u)\|_X > a\} \neq \emptyset$ with $U \cap \partial K_R = \emptyset$. Suppose that $u - \bar{J}E'(u) \in K$ for each $u \in K_R$. Then, there exist $\alpha > 0$ and a locally Lipschitz continuous map $H : U \to X$ for which*

$$u + H(u) \in K, \ \|H(u)\|_X \leq 1 \quad \text{and} \quad \langle E'(u), H(u)\rangle \leq -\alpha\|E'(u)\|_X, \ \forall u \in U. \tag{5.7}$$

5.3 A deformation lemma and minimax theorems

Based on the auxiliary results presented in Section 5.2, in this section we provide a deformation lemma [Willem, 1996], a bounded version of the general minimax theorem of [Willem, 1996] and of the mountain pass theorem of [Ambrosetti and Rabinowitz, 1973] and a particular case of the variational principle of Ekeland [Ekeland, 1974] for the case of a wedge in Banach spaces.

First, we present a Willem-type deformation lemma on a wedge, which is a generalization of [Willem, 1996, Lemma 2.3].

Lemma 5.2 ([Lisei, Varga and **Vas**, 2018])**.** *Assume that conditions $(\mathbf{C}^1_X)$ and $(\mathbf{C}^2_X)$ hold. Consider the C^1 functional $E : X \to \mathbb{R}$, suppose that $u - \bar{J}E'(u) \in K$ for all $u \in K_R$ and there exists $\theta \in (0, 1)$ for which*

$$\langle E'(u), u\rangle + \theta R\|E'(u)\|_X \geq 0, \ \forall u \in \partial K_R. \tag{5.8}$$

Consider the closed subset $S \subset K_R$ and assume that for some $c \in \mathbb{R}$ and $\varepsilon, \rho > 0$ the functional E satisfies the condition

$$\|E'(u)\|_X \geq \frac{2\varepsilon}{\rho}, \ \forall u \in E^{c+2\varepsilon}_{c-2\varepsilon} \cap S_{2\rho} \tag{5.9}$$

and $E^{c+\varepsilon}_{c-\varepsilon} \cap S_\rho \neq \emptyset$.

Then, there exists a continuous deformation $\sigma : [0, 1] \times K_R \to K_R$ such that the following properties hold:

($\mathbf{C}_\sigma^1$) $\sigma(0,\cdot) = \mathrm{id}_{K_R}$;

($\mathbf{C}_\sigma^2$) $\sigma(t,\cdot) : K_R \to K_R$ *is an homeomorphism for all* $t \in [0,1]$;

($\mathbf{C}_\sigma^3$) $\sigma(t,\cdot) = \mathrm{id}$ *on* $K_R \setminus (E_{c-2\varepsilon}^{c+2\varepsilon} \cap S_{2\rho})$ *for all* $t \in [0,1]$;

($\mathbf{C}_\sigma^4$) *for every* $u \in K_R$ *the mapping* $t \in [0,1] \mapsto E(\sigma(t,u))$ *is non-increasing;*

($\mathbf{C}_\sigma^5$) *there exists* $\alpha \in (0,1)$ *such that* $\sigma(\alpha, E^{c+\alpha\varepsilon} \cap S) \subset E^{c-\alpha\varepsilon} \cap S_\rho$.

Remark 5.2. *If* $E : X \to \mathbb{R}$ *is a* C^1 *functional for which there exists* $\theta \in (0,1)$ *such that the inequality*

$$\langle E'(u), u\rangle + \theta R\|E'(u)\|_X \geq 0, \ \forall u \in \partial K_R \tag{5.10}$$

holds, then

$$E'(u) + \lambda J_\varphi u \neq 0, \ \forall \lambda > 0 \text{ and } \ \forall u \in \partial K_R.$$

To prove this statement let us suppose that there exist a function $v \in \partial K_R$ *and* $\lambda > 0$ *such that* $E'(v) = -\lambda J_\varphi v$. *Then, inequality* (5.10) *implies that*

$$-\lambda\langle Jv, v\rangle + \theta R\lambda\|Jv\|_X \geq 0,$$

from where follows that $\theta \geq 1$, *which contradicts that* $\theta \in (0,1)$.

Next, we state a bounded version on a wedge of the general minimax theorem of [Willem, 1996, Theorem 2.8] which can be proved by using our Willem-type deformation Lemma 5.2.

Theorem 5.1 ([Lisei, Varga and **Vas**, 2018])**.** *Let* $(X, \|\cdot\|_X)$ *be a Banach space which satisfies conditions* ($\mathbf{C}_X^1$) *and* ($\mathbf{C}_X^2$). *Consider the* C^1 *functional* $E : X \to \mathbb{R}$. *We suppose that* $u - \bar{J}E'(u) \in K$ *for all* $u \in K_R$ *and that there exists* $\theta \in (0,1)$ *such that*

$$\langle E'(u), u\rangle + \theta R\|E'(u)\|_X \geq 0, \ \forall u \in \partial K_R.$$

Consider the closed subspace M_0 *of the metric space* M *and* $\Gamma_0 \subset \mathcal{C}(M_0, K_R)$ *and define the set*

$$\Gamma := \{\gamma \in \mathcal{C}(M, K_R) : \ \gamma|_{M_0} \in \Gamma_0\}.$$

If E *satisfies*

$$\infty > c := \inf_{\gamma\in\Gamma} \sup_{u\in M} E(\gamma(u)) > a := \sup_{\gamma_0\in\Gamma_0} \sup_{u\in M_0} E(\gamma(u)), \tag{5.11}$$

then for every $(\gamma_n)_n \subset \Gamma$ *such that*

$$\lim_{n\to\infty} \sup_{u\in M} E(\gamma_n(u)) = c, \tag{5.12}$$

and for $n \in \mathbb{N}_{>\frac{2}{c-a}}$ *there exists* $u_n \in K_R$ *such that* $u_n \in E_{c-\frac{2}{n}}^{c+\frac{2}{n}}$, $\mathrm{dist}(u_n, \gamma_n(M)) \leq \frac{2}{\sqrt{n}}$ *and* $\|E'(u_n)\|_X < \dfrac{1}{\sqrt{n}}$.

From Theorem 5.1 we derive the following bounded version of the mountain pass theorem of [Ambrosetti and Rabinowitz, 1973] for the case of a wedge.

Theorem 5.2 ([Lisei, Varga and **Vas**, 2018])**.** *Let* $(X, \|\cdot\|_X)$ *be a Banach space which satisfies conditions* ($\mathbf{C}_X^1$) *and* ($\mathbf{C}_X^2$). *Consider the* C^1 *functional* $E : X \to \mathbb{R}$. *We suppose that* $u - \bar{J}E'(u) \in K$ *for all* $u \in K_R$ *and there exists* $\theta \in (0,1)$ *such that*

$$\langle E'(u), u\rangle + \theta R\|E'(u)\|_X \geq 0, \ \forall u \in \partial K_R.$$

Let $e \in K_R$ *and* $r > 0$ *be fixed, with* $\|e\|_X > r$ *for which*

$$\inf\{E(u) : \ u \in K_R, \ \|u\|_X = r\} > \max\{E(0), E(e)\}.$$

We will use the notation

$$\Gamma := \{\gamma \in C([0,1], K_R) : \gamma(0) = 0,\ \gamma(1) = e\}$$

and

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)).$$

Then, there exists a sequence $(u_n)_n \subset K_R$ *for which*

$$E(u_n) \to c \text{ and } E'(u_n) \to 0.$$

A particular case of the variational principle of Ekeland [Ekeland, 1974] can be proved by following the steps of the proof given in [Willem, 1996, Theorem 2.4].

Theorem 5.3 ([Lisei, Varga and **Vas**, 2018]). *Let* $(X, \|\cdot\|_X)$ *be a Banach space which satisfies conditions* $(\mathbf{C}^1_X)$ *and* $(\mathbf{C}^2_X)$. *Let* $E : X \to \mathbb{R}$ *be a* C^1 *functional which is bounded from below on* K_R. *We suppose that* $u - \bar{J}E'(u) \in K$ *for all* $u \in K_R$ *and there exists* $\theta \in (0,1)$ *for which*

$$\langle E'(u), u\rangle + \theta R\|E'(u)\|_X \geq 0,\ \forall u \in \partial K_R.$$

Let $(v_n)_n \subset K_R$ *be such that*

$$\lim_{n\to\infty} E(v_n) = \inf E(K_R). \tag{5.13}$$

Then, there exists $(w_n)_n \subset K_R$ *such that*

$$E(w_n) \leq \inf E(K_R) + \frac{2}{n},\ \|E'(w_n)\|_X < \frac{1}{\sqrt{n}} \text{ and } \operatorname{dist}(w_n, S) \leq \frac{2}{\sqrt{n}},$$

where $S = \operatorname{cl}(\{v_n : n \in \mathbb{N}\})$.

Theorem 5.4 ([Lisei, Varga and **Vas**, 2018]). *Let* $(X, \|\cdot\|_X)$ *be a Banach space which satisfies conditions* $(\mathbf{C}^1_X)$ *and* $(\mathbf{C}^2_X)$. *Consider the* C^1 *functional* $E : X \to \mathbb{R}$ *such that* $E(0) = 0$. *We suppose that* $u - \bar{J}E'(u) \in K$ *for all* $u \in K_R$ *and there exists* $\theta \in (0,1)$ *for which*

$$\langle E'(u), u\rangle + \theta R\|E'(u)\|_X \geq 0,\ \forall u \in \partial K_R. \tag{5.14}$$

Let $e \in K_R$ *and* $r > 0$, *with* $\|e\|_X > r$ *such that* $E(e) < 0$ *and*

$$\inf\{E(u) :\ u \in K_R,\ \|u\|_X = r\} > 0. \tag{5.15}$$

Assume that E *is bounded from below on* K_R *and it also fulfils the* (PS) *condition on* K_R. *Then, the functional* E *has two non-trivial critical points located in* K_R *and one of them is the global minimum of* E *on* K_R.

5.4 Application to non-homogeneous operator equations

In this section, we apply the results presented in the previous section to the case of non-homogeneous operator equations.

Based on the results of [Adams and Fournier, 2003; Dinca and Matei, 2007; Pick et al., 2013], we enumerate some basic assumptions necessary to work in Orlicz–Sobolev spaces.

Let $a : \mathbb{R} \to \mathbb{R}$ be an admissible function. Consider the N-function $A : \mathbb{R} \to \mathbb{R}_+,\ A(t) = \displaystyle\int_0^t a(s)\mathrm{d}s$ and its complementary N-function $\bar{A} : \mathbb{R} \to \mathbb{R}_+,\ \bar{A}(t) = \displaystyle\int_0^t a^{-1}(s)\mathrm{d}s$ determined by the admissible functions a and a^{-1}, respectively. Assume that the following properties hold:

$(\mathbf{C}_A^1)$ $p_0 = \inf_{t>0} \frac{ta(t)}{A(t)} > 1$ and $p^* = \sup_{t>0} \frac{ta(t)}{A(t)} < \infty$;

$(\mathbf{C}_A^2)$ the function $t \in (0,\infty) \mapsto \frac{a(t)}{t}$ is non-decreasing;

$(\mathbf{C}_A^3)$ there exists a constant $C > 0$ for which $A(t) \geq C \cdot t^{p_0}$ for all $t \in (0,1)$;

$(\mathbf{C}_A^4)$ $\int_0^1 \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,\mathrm{d}\tau < \infty$ and $\int_1^\infty \frac{A^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,\mathrm{d}\tau = \infty$;

$(\mathbf{C}_A^5)$ for the constant p_0 defined in $(\mathbf{C}_A^1)$ the inequality $p_0 < p_* := \liminf_{t\to\infty} \frac{tA_*'(t)}{A_*(t)}$ holds, where A_* denotes the Sobolev conjugate of A.

Due to [Pick et al., 2013, Theorem 4.4.4], [Clément et al., 2004, Lemma C.8] and $(\mathbf{C}_A^1)$, the Δ_2-condition holds for both N-functions A and $\bar{A}$.

Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}_{\geq 2}$) be a bounded open set and consider the Orlicz space $L_A(\Omega)$ associated with the N-function A, which is a reflexive separable Banach space with respect to the Luxemburg norm

$$\|u\|_{L_A(\Omega)} := \inf\left\{k > 0 : \int_\Omega A\left(\frac{u(x)}{k}\right)\,\mathrm{d}x \leq 1\right\}. \tag{5.16}$$

Due to condition $(\mathbf{C}_A^1)$, at first we have that $1 < p_0 = \inf_{t>0} \frac{ta(t)}{A(t)}$, thus by means of [Clément et al., 2004, Lemma C.9] one obtains that

$$\int_\Omega A(u(x))\mathrm{d}x \leq \|u\|_{L_A(\Omega)}^{p_0},\ \forall u \in L_A(\Omega) \text{ with } \|u\|_{L_A(\Omega)} \leq 1; \tag{5.17}$$

secondly, we have that $1 < p^* = \sup_{t>0} \frac{ta(t)}{A(t)} < \infty$, which by [Dinca and Matei, 2007, Remark 7.2] implies that

$$A(t) \leq t^{p^*} A(1),\ \forall t \geq 1, \tag{5.18}$$

and by [Dinca and Matei, 2007, Lemma 6.5] it follows that

$$\int_\Omega A(u(x))\mathrm{d}x \leq \|u\|_{L_A(\Omega)}^{p^*},\ \forall u \in L_A(\Omega) \text{ with } \|u\|_{L_A(\Omega)} > 1. \tag{5.19}$$

Consequently, by (5.17) and (5.19), for all $u \in L_A(\Omega)$ we have the inequality

$$\int_\Omega A(u(x))\,\mathrm{d}x \leq \|u\|_{L_A(\Omega)}^{p_0} + \|u\|_{L_A(\Omega)}^{p^*}. \tag{5.20}$$

Remark 5.3. *If besides conditions $(\mathbf{C}_A^1)$–$(\mathbf{C}_A^4)$ for the N-function A we also have that*

$$\lim_{t\to\infty} \frac{ta(t)}{A(t)} = \ell,$$

then, by means of a property given in [Clément et al., 2000, p. 55], we obtain that

$$\lim_{t\to\infty} \frac{tA_*'(t)}{A_*(t)} = \frac{N\ell}{N-\ell}. \tag{5.21}$$

For fixed $m \in \mathbb{N}^*$ consider the Orlicz–Sobolev space $W_0^m L_A(\Omega)$ and let $T[\cdot,\cdot]$ be a non-negative symmetric bilinear form on the space $W_0^m L_A(\Omega)$, involving only generalized derivatives of order m, satisfying the condition

$(\mathbf{C}_T^1)$
$$c_1 \sum_{|\alpha|=m} (D^\alpha u)^2 \leq T[u,u] \leq c_2 \sum_{|\alpha|=m} (D^\alpha u)^2$$

a.e. on Ω for all $u \in W_0^m L_A(\Omega)$, where $c_1, c_2 > 0$ are constants.

We denote by $\|\cdot\|_{W_0^m L_A(\Omega)} := \left\|\sqrt{T[\cdot,\cdot]}\right\|_{L_A(\Omega)}$ the norm on the Orlicz–Sobolev space $W_0^m L_A(\Omega)$. We also denote by $C_A > 0$ the constant such that

$$\left(\sum_{|\alpha|<m} \|D^\alpha u\|^2_{L_A(\Omega)}\right)^{\frac{1}{2}} \leq C_A \|u\|_{W_0^m L_A(\Omega)}. \tag{5.22}$$

Since the embedding $W_0^m L_A(\Omega) \hookrightarrow L_{A_*}(\Omega)$ is continuous, i.e., there exists a positive constant C_* such that

$$\|D^\alpha u\|_{L_{A_*}(\Omega)} \leq C_* \|u\|_{W_0^m L_A(\Omega)} \tag{5.23}$$

for any α for which $|\alpha| < m$.

Let $J_a : W_0^m L_A(\Omega) \to \left(W_0^m L_A(\Omega)\right)^*$ be the duality mapping corresponding to the normalization function a such that

$$J_a(0) = 0 \ \text{ and } \ J_a u = a(\|\cdot\|_{W_0^m L_A(\Omega)})\|\cdot\|'_{W_0^m L_A(\Omega)}(u), \ \forall u \in W_0^m L_A(\Omega) \setminus \{0\}. \tag{5.24}$$

Consider the set $\{f_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}_+ : |\alpha| < m\}$ of Carathéodory functions which have primitives

$$F_\alpha(x,s) = \int_0^s f_\alpha(x,\tau)\,\mathrm{d}\tau$$

for all α with $|\alpha| < m$ and suppose that:

$\left(\mathbf{C}^1_{f_\alpha}\right)$ for each α with $|\alpha| < m$, there exist N-functions M_α, that increase essentially more slowly than A_* near infinity,

$$1 < q_\alpha = \inf_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} \leq q^*_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)}$$

and

$$f_\alpha(x,s) \leq c_\alpha + d_\alpha \overline{M}_\alpha^{-1}\left(M_\alpha(s)\right), \ \forall s \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where $\overline{M}_\alpha$ are the complementary N-functions to M_α and $c_\alpha, d_\alpha > 0$ are constants;

$\left(\mathbf{C}^2_{f_\alpha}\right)$ for each $|\alpha| < m$ it holds

$$\limsup_{s\to 0} \frac{f_\alpha(x,s)}{a(s)} < \frac{C}{2N_0 C_A^{p_0}} \text{ uniformly for a.e. } x \in \Omega,$$

where $N_0 \coloneqq \sum_{|\alpha|<m} 1$, C and p_0 are the constants from $\left(\mathbf{C}^1_A\right)$ and $\left(\mathbf{C}^3_A\right)$, while C_A is the constant from inequality (5.22);

$\left(\mathbf{C}^3_{f_\alpha}\right)$ for each multi-index α, there exist $s_\alpha > 0$ and $\theta_\alpha > p^*$ (p^* defined in $\left(\mathbf{C}^1_A\right)$) such that

$$0 < \theta_\alpha F_\alpha(x,s) \leq s f_\alpha(x,s), \ \forall |s| \geq s_\alpha \text{ and a.e. } x \in \Omega.$$

Under these conditions, our aim is to localize the solutions to the boundary value problem

$$\begin{cases} J_a u = \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha f_\alpha(x, D^\alpha u) & \text{in } \Omega, \\ D^\alpha u = 0 & \text{on } \partial\Omega, \ |\alpha| \leq m-1, \end{cases} \tag{5.25.P}$$

Moreover, when $m = 1$ and $a(t) = |t|^{p-2}t$, $t \in \mathbb{R}$ with $p \in (1, N)$ fixed, we are able to localize the positive solutions to the above problem, see Example 5.3.

Due to $\left(\mathbf{C}^1_A\right)$ and $\left(\mathbf{C}^2_A\right)$, by means of [Dinca and Matei, 2007, Theorems 3.6, 3.14 and 4.5], the Banach space $W_0^m L_A(\Omega)$ is smooth and uniformly convex, moreover the duality mapping on $\left(W_0^m L_A(\Omega), \|\cdot\|_{W_0^m L_A(\Omega)}\right)$

subordinated to the normalization function a is

$$\langle J_a u, h\rangle = \frac{a\left(\|u\|_{W_0^m L_A(\Omega)}\right)\cdot \int_\Omega a\left(\frac{\sqrt{T[u,u](x)}}{\|u\|_{W_0^m L_A(\Omega)}}\right)\frac{T[u,h](x)}{\sqrt{T[u,u](x)}}\mathrm{d}x}{\int_\Omega a\left(\frac{\sqrt{T[u,u](x)}}{\|u\|_{W_0^m L_A(\Omega)}}\right)\frac{\sqrt{T[u,u](x)}}{\|u\|_{W_0^m L_A(\Omega)}}\mathrm{d}x} \tag{5.26}$$

for $u,h\in W_0^m L_A(\Omega),\ u\neq 0$. Moreover, J_a is bijective and its inverse $\bar{J}=J_a^{-1}$ is continuous.

Also consider the functional $\mathcal{E}: W_0^m L_A(\Omega)\to\mathbb{R}$,

$$\mathcal{E}(u) = A\left(\|u\|_{W_0^m L_A(\Omega)}\right) - \sum_{|\alpha|<m}\int_\Omega F_\alpha(x, D^\alpha u(x))\mathrm{d}x, \tag{5.27}$$

whose critical points are the weak solutions to problem (5.25.P). By means of [Dinca and Matei, 2007, Proposition 7.5], one has that

$$\langle \mathcal{E}'(u), v\rangle = \langle J_a(u), v\rangle - \sum_{|\alpha|<m}\int_\Omega f_\alpha(x, D^\alpha u(x))D^\alpha v(x)\mathrm{d}x,\ \forall u,v\in W_0^m L_A(\Omega).$$

Let $K\subseteq W_0^m L_A(\Omega)$ be a wedge and for $R>0$ we recall the notations

$$K_R := \{u\in W_0^m L_A(\Omega) : u\in K \text{ and } \|u\|_{W_0^m L_A(\Omega)}\leq R\}$$

and

$$\partial K_R := \{u\in W_0^m L_A(\Omega) : u\in K \text{ and } \|u\|_{W_0^m L_A(\Omega)} = R\}.$$

Proposition 5.2 ([Lisei, Varga and **Vas**, 2018])**.** *Assume that conditions* $(\mathbf{C}_A^1)$–$(\mathbf{C}_A^2)$, $(\mathbf{C}_A^4)$, $(\mathbf{C}_T^1)$ *and* $\left(\mathbf{C}_{f_\alpha}^1\right)$ *are satisfied. Then, the functional* $\mathcal{E}$ *fulfils the* (PS) *condition on* K_R.

Proposition 5.3 ([Lisei, Varga and **Vas**, 2018])**.** *Suppose that conditions* $(\mathbf{C}_A^1)$–$(\mathbf{C}_A^2)$, $(\mathbf{C}_A^4)$, $(\mathbf{C}_T^1)$ *and* $\left(\mathbf{C}_{f_\alpha}^1\right)$ *hold. Then, for all* $s\in\mathbb{R}$ *and a.e.* $x\in\Omega$ *the inequality*

$$|F_\alpha(x,s)|\leq c_\alpha|s| + 2d_\alpha M_\alpha(|s|) \tag{5.28}$$

holds and $\mathcal{E}$ *maps bounded sets into bounded sets.*

Proposition 5.4 ([Lisei, Varga and **Vas**, 2018])**.** *Assume that conditions* $(\mathbf{C}_A^1)$–$(\mathbf{C}_A^2)$, $(\mathbf{C}_A^4)$, $(\mathbf{C}_T^1)$ *and* $\left(\mathbf{C}_{f_\alpha}^1\right)$–$\left(\mathbf{C}_{f_\alpha}^2\right)$ *hold. For any* α *with* $|\alpha|<m$ *there exist* $\mu_\alpha\in\left(0,\frac{C}{2N_0 C_A^{p_0}}\right)$ *and* $t_\alpha>0$ *such that*

$$F_\alpha(x,s)\leq \mu_\alpha A(s),\ \forall |s|<t_\alpha \text{ and a.e. } x\in\Omega, \tag{5.29}$$

and

$$|F_\alpha(x,s)|\leq\left(\frac{c_\alpha t_\alpha}{M_\alpha(t_\alpha)}+2d_\alpha\right)M_\alpha(|s|),\ \forall |s|\geq t_\alpha \text{ and a.e. } x\in\Omega. \tag{5.30}$$

Since, by our assumptions, the N-function M_α increases essentially more slowly than A_* near infinity, in particular one obtains that

$$\lim_{s\to\infty}\frac{M_\alpha(s)}{A_*(s)} = 0.$$

Accordingly, there exists $t'_\alpha\geq t_\alpha$ such that

$$M_\alpha(s)\leq A_*(s),\ \forall s\geq t'_\alpha. \tag{5.31}$$

Due to the definition of p_* from $(\mathbf{C}_A^1)$ for any $\mu\in(0, p_*-p_0)$ there exists $t''_\alpha\geq t'_\alpha$ for which

$$\frac{A'_*(s)}{A_*(s)}\geq\frac{p_*-\mu}{s},\ \forall s\geq t''_\alpha. \tag{5.32}$$

For each $|\alpha| < m$ we introduce the notation

$$k_\alpha := \frac{t''_\alpha}{t_\alpha} > 1. \tag{5.33}$$

We formulate the following assumptions for R:

$(\mathbf{C}_R^1)$
$$2C_W \sum_{|\alpha|<m} \left(a_\alpha + d_\alpha((C_W R)^{q_\alpha - 1} + (C_W R)^{q_\alpha^* - 1})\right) \le a(R),$$
where $a_\alpha := \frac{c_\alpha}{M_\alpha^{-1}\left(\frac{1}{\mathrm{vol}(\Omega)}\right)}$ and C_W is the constant from the compact embedding $W_0^m L_A(\Omega) \hookrightarrow W := \bigcap_{|\alpha|<m} W^{m-1} L_{M_\alpha}(\Omega)$;

$(\mathbf{C}_R^2)$
$$R > \rho_0 := \min\left\{1, \frac{1}{C_A}, \frac{1}{\left(\max\limits_{|\alpha|<m} k_\alpha\right) C_*}, \left(\frac{C}{3D}\right)^{\frac{1}{p_* - \mu - p_0}}\right\},$$
where C_* is the constant from (5.23), C_A the constant from (5.22),
$$D := C_*^{p_* - \mu} \sum_{|\alpha|<m} \left(\frac{c_\alpha t_\alpha}{M_\alpha(t_\alpha)} + 2d_\alpha\right) k_\alpha^{p_* - \mu},$$
with t_α obtained in Proposition 5.4, k_α defined in (5.33) and $\mu \in (0, p_* - p_0)$ is fixed arbitrarily;

$(\mathbf{C}_R^3)$ let the function $v \in W_0^m L_A(\Omega)$ be such that for (at least one) multi-index α with $|\alpha| < m$ the set
$$\widehat{\Omega}_\alpha^1 := \{x \in \Omega : |D^\alpha v(x)| \ge s_\alpha\}$$
has $\mathrm{vol}\left(\widehat{\Omega}_\alpha^1\right) > 0$, and let λ be the smallest real number with $\lambda > \max\left\{1, \frac{\rho_0}{\|v\|_{W_0^m L_A(\Omega)}}\right\}$ for which
$$A(1)\lambda^{p^*} \|v\|^{p^*}_{W_0^m L_A(\Omega)} - \sum_{|\alpha|<m} \lambda^{\theta_\alpha} \gamma_\alpha + \sum_{|\alpha|<m} b_\alpha < 0,$$
where
$$\gamma_\alpha := \int_{\widehat{\Omega}_\alpha^1} \min\{F_\alpha(x, s_\alpha), F_\alpha(x, -s_\alpha)\} \mathrm{d}x$$
and
$$b_\alpha := (c_\alpha s_\alpha + 2d_\alpha M_\alpha(s_\alpha)) \,\mathrm{vol}(\Omega)$$
with s_α given in condition $\left(\mathbf{C}_{f_\alpha}^3\right)$; moreover, also assume that $R \ge \lambda \|v\|_{W_0^m L_A(\Omega)}$.

Following the ideas of articles [Clément et al., 2000; Dinca and Matei, 2007], the next existence and localization result related to the solutions to problem (5.25.P) can be proved by applying Theorem 5.4.

Theorem 5.5 ([Lisei, Varga and **Vas**, 2018])**.** *We assume that $u - \bar{J}\mathcal{E}'(u) \in K$ for all $u \in K_R$ and the assumptions $(\mathbf{C}_A^1)$–$(\mathbf{C}_A^5)$, $(\mathbf{C}_T^1)$ and $\left(\mathbf{C}_{f_\alpha}^1\right)$–$\left(\mathbf{C}_{f_\alpha}^3\right)$ hold. Let R be the smallest positive number such that conditions $(\mathbf{C}_R^1)$–$(\mathbf{C}_R^3)$ are satisfied. Then, problem (5.25.P) admits two weak non-trivial solutions in K_R and one of them is the global minimum of the functional $\mathcal{E}$ on K_R.*

5.5 Examples

This section presents three concrete examples in order to show the applicability of the results discussed in Section 5.4.

In the examples below, the non-negative symmetric bilinear form T is

$$T[u, u] = \sum_{|\alpha|=m} (D^\alpha u)^2, \ \forall u \in W_0^m L_A(\Omega)$$

and the wedge $K = W_0^m L_A(\Omega)$ denotes the whole Orlicz–Sobolev space. In this case, $K_R = \overline{X}_R$ and evidently $u - \bar{J}\mathcal{E}'(u) \in K$ for all $u \in K_R$.

Example 5.1. *Let $1 < p_1 < p_2 < \cdots < p_n < N$ and consider the admissible function*

$$a : \mathbb{R} \to \mathbb{R},\ a(t) = \sum_{i=1}^{n} |t|^{p_i - 2} t \tag{5.34}$$

with the corresponding N-function

$(\mathbf{E}_{5.1}|\mathbf{C}_A)$ $$A : \mathbb{R} \to \mathbb{R},\ A(t) = \sum_{i=1}^{n} \frac{1}{p_i} |t|^{p_i},$$

where $p_0 = p_1 > 1$ and $p^* = p_n < N$.

Consider also the set $\{f_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}_+ : |\alpha| < m\}$ of Carathéodory functions which have primitives

$$F_\alpha(x, s) = \int_0^s f_\alpha(x, \tau)\,\mathrm{d}\tau$$

for all α with $|\alpha| < m$ and satisfy the following conditions:

$\left(\mathbf{E}_{5.1}|\mathbf{C}^1_{f_\alpha}\right)$ *for each multi-index α with $|\alpha| < m$ there exist $q_\alpha \in \left(1, \frac{Np_n}{N-p_n}\right)$ such that*

$$|f_\alpha(x, s)| \le c_\alpha + d_\alpha |s|^{q_\alpha - 1},\ \forall s \in \mathbb{R} \text{ and a.e. } x \in \Omega; \tag{5.35}$$

$\left(\mathbf{E}_{5.1}|\mathbf{C}^2_{f_\alpha}\right)$ *using the notation $N_0 := \sum_{|\alpha|<m} 1$, assume that*

$$\limsup_{s \to 0} \frac{f_\alpha(x, s)}{a(s)} < \frac{1}{2p_1 N_0 C_A^{p_1}} \text{ uniformly for a.e. } x \in \Omega;$$

$\left(\mathbf{E}_{5.1}|\mathbf{C}^3_{f_\alpha}\right)$ *for each multi-index α with $|\alpha| < m$ there exist $s_\alpha > 0$ and $\theta_\alpha > p_n$ such that*

$$0 < \theta_\alpha F_\alpha(x, s) \le s f_\alpha(x, s),\ \forall s \in \mathbb{R} \text{ with } |s| \ge s_\alpha \text{ and for a.e. } x \in \Omega. \tag{5.36}$$

Under the above conditions, problem (5.25.P) has two non-trivial weak solutions in $\overline{X}_R$, where R is the smallest positive number such that conditions $(\mathbf{C}^1_R)$–$(\mathbf{C}^3_R)$ hold.

Remark 5.4. *In the case when the admissible function is $a(t) = |t|^{p-2} \cdot t$, $p \in (1, N)$, $m = 1$ and $T[u, v] = \nabla u \cdot \nabla v$, the space X becomes the ordinary Sobolev space $W_0^{1,p}(\Omega)$ and the duality mapping is $J_a = -\Delta_p$. The existence result obtained in this Example 5.1 completes the localization results of Section 4.2 of the thesis, by ensuring that the Dirichlet problem*

$$\begin{cases} -\Delta_p u = f_0(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has two non-trivial weak solutions located in $\overline{X}_R \subset W_0^{1,p}(\Omega)$, where R is the smallest positive number such that assumptions $(\mathbf{C}^1_R)$–$(\mathbf{C}^3_R)$ hold and the function f_0 satisfies the conditions $\left(\mathbf{E}_{5.1}|\mathbf{C}^1_{f_\alpha}\right)$–$\left(\mathbf{E}_{5.1}|\mathbf{C}^3_{f_\alpha}\right)$. For the localization of positive solutions see Example 5.3.

Example 5.2. *Let $p \in (1, N-1)$ be fixed and consider the admissible function*

$$a : \mathbb{R} \to \mathbb{R},\ a(t) = |t|^{p-2} t \sqrt{t^2 + 1}. \tag{5.37}$$

Consider also the set $\{f_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}_+ : |\alpha| < m\}$ of Carathéodory functions which have primitives

$$F_\alpha(x, s) = \int_0^s f_\alpha(x, \tau)\,\mathrm{d}\tau$$

for all α with $|\alpha| < m$ and satisfy the following conditions:

$\left(\mathbf{E}_{5.2}|\mathbf{C}^1_{f_\alpha}\right)$ *for each multi-index α with $|\alpha| < m$ there exist $q_\alpha \in \left(1, \frac{Np}{N-p}\right)$ such that (5.35) holds;*

$\left(\mathbf{E}_{5.2}|\mathbf{C}^2_{f_\alpha}\right)$ *using the notation $N_0 := \sum_{|\alpha|<m} 1$, assume that*

$$\limsup_{s\to 0} \frac{f_\alpha(x,s)}{a(s)} < \frac{1}{2pN_0C_A^p}, \text{ uniformly for a.e. } x \in \Omega;$$

$\left(\mathbf{E}_{5.2}|\mathbf{C}^3_{f_\alpha}\right)$ *for each multi-index α with $|\alpha| < m$ there exist $s_\alpha > 0$ and $\theta_\alpha > p+1$ such that (5.36) holds.*

Under the above conditions, problem (5.25.P) has two non-trivial weak solutions in $\overline{X}_R$, where R is the smallest positive number such that conditions $(\mathbf{C}^1_R)$–$(\mathbf{C}^3_R)$ hold.

Remark 5.5. *In Examples 5.1 and 5.2, a more explicit expression of the duality mapping J_a can be computed if the admissible function is written in the form $a(t) = b(t)t$, $\forall t \in \mathbb{R}$, and A is the corresponding N-function. Then, the duality mapping has the formula*

$$\langle J_a u, h\rangle = \frac{b\left(\|u\|_{W_0^1L_A(\Omega)}\right)}{\displaystyle\int_\Omega b\left(\frac{|\nabla u(x)|}{\|u\|_{W_0^1L_A(\Omega)}}\right)|\nabla u(x)|^2\,\mathrm{d}x}\int_\Omega b\left(\frac{|\nabla u(x)|}{\|u\|_{W_0^1L_A(\Omega)}}\right)\nabla u(x)\nabla h(x)\,\mathrm{d}x, \tag{5.38}$$

where $u, h \in W_0^1L_A(\Omega), u \neq 0$.

Example 5.3. *For fixed $p \in (1, N)$ consider the admissible function $a(t) = |t|^{p-2}\cdot t$ and we choose $m = 1$, $T[u,v] = \nabla u \cdot \nabla v$ and $X = W_0^{1,p}(\Omega)$. Then, the duality mapping is $J_a = -\Delta_p$. Choosing also $M_0(t) = \frac{|t|^q}{q}$, we have that*

$(\mathbf{E}_{5.3}|\mathbf{C}_A)$ $\quad A(t) = \frac{|t|^p}{p}$, $\|u\|_{L_A(\Omega)} = p^{-\frac{1}{p}}\|u\|_{L^p(\Omega)}$, $\|u\|_{W_0^1L_A(\Omega)} = p^{-\frac{1}{p}}\|\nabla u\|_{L^p(\Omega)}$,

$$A_*(t) = \left(\frac{N-p}{Np}\right)^{\frac{Np}{N-p}}\cdot p^{-\frac{N}{N-p}}\cdot t^{\frac{N-p}{Np}}, \quad \|u\|_{L_{A_*}(\Omega)} = \frac{N-p}{Np}\cdot p^{-\frac{1}{p}}\|u\|_{L^{\frac{N-p}{Np}}(\Omega)}$$

and

$$p_0 = p^* = p, \; p_* = \frac{Np}{N-p}.$$

In this case, the constants C_A and C_W from (5.22) and the compact embedding $W_0^mL_A(\Omega)\hookrightarrow W := \bigcap_{|\alpha|<m} W^{m-1}L_{M_\alpha}(\Omega)$, respectively, are as follows:

- *$C_A = C_p$ is the best embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, i.e., $C_A = C_p = \left(\frac{1}{\lambda_p(\Omega)}\right)^{\frac{1}{p}}$, where $\lambda_p(\Omega)$ is the first eigenvalue of the p-Laplacian operator defined on Ω;*
- *$C_W = C_q$ is the best embedding constant of the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, since $q \in \left(1, \frac{Np}{N-p}\right)$ and $p < N$.*

Based on article [Lisei and **Vas**, 2016, Section 4], *in Section 4.2 of the thesis we presented the detailed construction of upper estimates for the constants C_p and C_q.*

Let

$$K := \left\{u \in W_0^{1,p}(\Omega) : u(x) \geq 0 \text{ for a.e. } x \in \Omega\right\}$$

be the wedge and assume that the Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}_+$ fulfils the conditions:

$\left(\mathbf{E}_{5.3}|\mathbf{C}^1_{f_0}\right)$ *there exists $q \in \left(1, \frac{Np}{N-p}\right)$ such that for constants $c_0, d_0 > 0$ one has that*

$$|f_0(x,s)| \leq c_0 + d_0|s|^{q-1}, \; \forall s \in \mathbb{R} \text{ and a.e. } x \in \Omega;$$

$\left(\mathbf{E}_{5.3}|\mathbf{C}^2_{f_0}\right)$ $\displaystyle\limsup_{s\to 0}\frac{f_0(x,s)}{s^{p-1}} < \frac{\lambda_p(\Omega)}{2p}$ *uniformly for a.e. $x \in \Omega$;*

$\left(\mathbf{E}_{5.3}|\mathbf{C}^3_{f_0}\right)$ *there exist* $s_0 > 0$ *and* $\theta_0 > p$ *for which*

$$0 < \theta_0 F_0(x,s) \le s f_0(x,s), \ \forall |s| \ge s_0 \text{ and a.e. } x \in \Omega.$$

Under the above-mentioned conditions, the Dirichlet problem

$$\begin{cases} -\Delta_p u = f_0(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has two non-trivial weak solutions in K_R*, where* R *is the smallest positive number such that the following assumptions hold:*

$\left(\mathbf{E}_{5.3}|\mathbf{C}^1_R\right)$ $$c_0 C_q (\mathrm{vol}(\Omega))^{\frac{q-1}{q}} + d_0 C_q^q R^{q-1} \le R^{p-1};$$

$\left(\mathbf{E}_{5.3}|\mathbf{C}^2_R\right)$ $$R > \rho_0 := \min\left\{1, \frac{1}{C_p}, \frac{1}{k_0 C_*}, \left(\frac{C}{3D}\right)^{\frac{1}{p_* - \mu - p_0}}\right\},$$

where k_0 *is given in* (5.33)*,* $C_* := \frac{N-p}{Np} C_{\frac{Np}{N-p}}$*,* $C := \frac{1}{p}$*,* $D := c_0 t_0^{1-q} + \frac{d_0}{q}$ *with* t_0 *obtained in Proposition* 5.4*, while* $\mu \in \left(0, \frac{p^2}{N-p}\right)$ *is arbitrarily fixed;*

$\left(\mathbf{E}_{5.3}|\mathbf{C}^3_R\right)$ *let* $v \in W_0^{1,p}(\Omega)$ *be such that the set*

$$\widehat{\Omega}^1_\alpha := \{x \in \Omega : |v(x)| \ge s_0\}$$

has $\mathrm{vol}\left(\widehat{\Omega}^1_0\right) > 0$ *and let* λ *be the smallest real number with* $\lambda > \max\left\{1, \frac{\rho_0}{\|v\|_{W_0^1 L_A(\Omega)}}\right\}$ *such that*

$$\frac{1}{p}\lambda^p \|v\|^p_{W_0^1 L_A(\Omega)} - \lambda^{\theta_0}\gamma_0 + b_0 < 0,$$

where

$$\gamma_0 := \int_{\widehat{\Omega}^1_0} \min\{F_0(x,s_0), F_0(x,-s_0)\}\mathrm{d}x$$

and

$$b_0 := \left(c_0 s_0 + \frac{d_0 s_0^q}{q}\right)\mathrm{vol}(\Omega);$$

moreover, suppose that $\lambda\|v\|_{W_0^1 L_A(\Omega)} \le R$*.*

Remark 5.6. *For this special case, conditions* $\left(\mathbf{E}_{5.3}|\mathbf{C}^1_R\right)$–$\left(\mathbf{E}_{5.3}|\mathbf{C}^3_R\right)$ *are adapted versions of* $\left(\mathbf{C}^1_R\right)$–$\left(\mathbf{C}^3_R\right)$*, consequently these estimates are better, because they were calculated for this concrete case.*

For $u \in K$ we denote $v := u - \bar{J}\mathcal{E}'(u)$. Then, $J_a(u-v) = \mathcal{E}'(u) \le J_a u$ (weakly), since by our assumptions f_0 is a positive function. By the weak comparison principle for J_a (see [Shapiro, 1980, Theorem, p. 259]), one has that $u - v \le u$. Thus, $v = u - \bar{J}\mathcal{E}'(u) \in K$.

Theorem 5.6 ([Lisei, Varga and **Vas**, 2018])**.** *Assume that* R *satisfies conditions* $\left(\mathbf{E}_{5.3}|\mathbf{C}^1_R\right)$–$\left(\mathbf{E}_{5.3}|\mathbf{C}^3_R\right)$ *and the above assumptions* $\left(\mathbf{E}_{5.3}|\mathbf{C}^1_{f_0}\right)$–$\left(\mathbf{E}_{5.3}|\mathbf{C}^3_{f_0}\right)$ *hold as well. Then, equation* (5.25.P) *admits two non-trivial weak positive solutions in* K_R *and one of them is the global minimum of the functional* $\mathcal{E}$ *on* K_R*.*

6

Existence results for some Dirichlet problems involving Finsler–Laplacian operator

Based on article [Mezei and **Vas**, 2019], in this chapter we present some existence and localization results for two Dirichlet problems which involves the Finsler–Laplacian operator.

At first, let $\Omega \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}_{\geq 2}$) be a smooth bounded domain and consider the problem

$$\begin{cases} -\Delta_F u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1.P}$$

where $-\Delta_F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ denotes the Finsler–Laplacian operator and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that fulfils some special growth condition. In the case of our first problem (6.1.P), based on the results of [Dinca et al., 2001], we show the existence of the solutions in two different ways: by applying the direct method of the calculus of variations, then by using the Leray–Schauder alternative.

Secondly, consider the problem

$$\begin{cases} -\Delta_F u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2.P}$$

where the function $g : \mathbb{R}_+ \to \mathbb{R}_+$. In the case of our second problem (6.2.P), we prove an existence and localization result, by the combined use of the Harnack inequality and a Krasnosel'skii-type fixed point theorem of [Precup, 2012].

Hereafter, we suppose that F is a norm in $\mathbb{R}^n$ for which F^2 is strongly convex in $\mathbb{R}^n \setminus \{0\}$. Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}_{\geq 2}$) be a smooth bounded domain and consider the Sobolev space $W_0^{1,2}(\Omega)$ equipped with the inner product $\langle u, v\rangle = \int_\Omega (\nabla u(x) \cdot \nabla v(x))\, dx$ that induces the norm $\|u\|_{W_0^{1,2}(\Omega)} := \left(\int_\Omega |\nabla u(x)|^2 \, dx\right)^{\frac{1}{2}}$. Then, $\left(W_0^{1,2}(\Omega), \|\cdot\|_{W_0^{1,2}(\Omega)}\right)$ is a Hilbert space and $W^{-1,2}(\Omega)$ is its topological dual space. Based on [Xia, 2012], we define the Finsler–Laplacian operator $-\Delta_F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$,

$$\Delta_F u \coloneqq \operatorname{div}\left(F(\nabla u) F_\xi(\nabla u)\right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(F(\nabla u) F_{\xi_i}(\nabla u)\right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial \xi_i}\left(\frac{1}{2}F^2(\nabla u)\right)\right).$$

6.1 Existence results for problem (6.1.P) via critical point theory

This section studies our first Dirichlet problem (6.1.P), where $-\Delta_F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ denotes the Finsler–Laplacian operator and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

In order to prove the existence of the solutions to problem (6.1.P) by applying the tools of critical point theory, we need to define the Nemytskii operator associated with the Carathéodory function g.

Let us consider the set $\mathcal{M} \coloneqq \{u : \Omega \to \mathbb{R} : u \text{ is measurable}\}$.

Proposition 6.1. *If $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, then for each measurable function $u \in \mathcal{M}$ the function $N_g(u) : \Omega \to \mathbb{R}$,*

$$N_g(u)(x) = g(x, u(x)), \ \forall x \in \Omega$$

is measurable in Ω.

Remark 6.1. *The function $N_g : \mathcal{M} \to \mathcal{M}$ is the Nemytskii operator associated with the function g.*

Proposition 6.2 ([Dinca et al., 2001, Proposition 6.]). *Assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that fulfils the growth condition*

$$|g(x,s)| \le C|s|^{q-1} + b(x), \ \forall x \in \Omega, \ \forall s \in \mathbb{R}, \tag{6.3}$$

where $C \ge 0$ is constant, $q > 1$, $b \in L^{q'}(\Omega)$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Moreover, consider also the function $G : \Omega \times \mathbb{R} \to \mathbb{R}$, $G(x,s) = \int_0^s g(x,\tau)\,\mathrm{d}\tau$. Then:

(a) *the function G is Carathéodory and there exist a constant $C_1 \ge 0$ and a function $c \in L^1(\Omega)$ for which*

$$|G(x,s)| \le C_1|s|^q + c(x), \ \forall x \in \Omega, \ \forall s \in \mathbb{R};$$

(b) *the functional $\Phi : L^q(\Omega) \to \mathbb{R}$, $\Phi(u) \coloneqq \int_\Omega N_G(u)(x)\mathrm{d}x = \int_\Omega G(x, u(x))\,\mathrm{d}x$ is continuously Fréchet-differentiable and $\Phi'(u) = N_g(u)$ for all $u \in L^q(\Omega)$.*

Remark 6.2. *Under the above-mentioned conditions, we have that $N_g(L^q(\Omega)) \subset L^{q'}(\Omega)$ and $N_G(L^q(\Omega)) \subset L^1(\Omega)$, while the Nemytskii operators N_g and N_G are continuous and bounded. Moreover, for each fixed $u \in L^q(\Omega)$ the equality $N_g(u) = \Phi'(u) \in L^{q'}(\Omega)$ holds.*

Assume that the Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfils the growth condition (6.3) with $q \in (1, 2^*)$.

Due to the restriction $q \in (1, 2^*)$, the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ is compact with the constant C_q and $N_g : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is a compact operator, i.e., $W_0^{1,2}(\Omega) \overset{I_d}{\hookrightarrow} L^q(\Omega) \xrightarrow{N_g} L^{q'}(\Omega) \overset{I_d^*}{\hookrightarrow} W^{-1,2}(\Omega)$.

6.1.1 An existence result via the direct method of calculus of variations

Define $J_G : W_0^{1,2}(\Omega) \to \mathbb{R}$ by $J_G(u) = \int_\Omega G(x, u(x))\mathrm{d}x$ and let $\mathcal{E} : W_0^{1,2}(\Omega) \to \mathbb{R}$ be the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega F^2(\nabla u(x))\mathrm{d}x - J_G(u) \tag{6.4}$$

associated with problem (6.1.P), where $\mathcal{E}$ is well-defined, moreover, the critical points of the energy functional $\mathcal{E}$ are the weak solutions to problem (6.1.P).

By means of the direct method of the calculus of variations, the existence of critical points of $\mathcal{E}$ can be proved, by using the coercivity and the sequentially weakly lower semi-continuous property of $\mathcal{E}$.

Lemma 6.1 ([Mezei and **Vas**, 2019]). *If the function g satisfies the growth condition (6.3) with $q \in (1, 2)$, then the energy functional $\mathcal{E}$ is coercive and sequentially weakly lower semi-continuous.*

Theorem 6.1 ([Mezei and **Vas**, 2019]). *If the function g satisfies the growth condition (6.3) with $q \in (1, 2)$, then the energy functional $\mathcal{E}$ is coercive and sequentially weakly lower semi-continuous, consequently $\mathcal{E}$ has at least one critical point in $W_0^{1,2}(\Omega)$, which is a solution to problem (6.1.P).*

6.1.2 An existence result via the Leray–Schauder alternative

To show the existence of the solutions to problem (6.1.P) by using the Leray–Schauder technique, we reduce our problem to a fixed point problem with a compact operator.

Definition 6.1. *If*

$$-\Delta_F u = N_g(u) \tag{6.5}$$

holds for some $u \in W_0^{1,2}(\Omega)$, i.e.,

$$\langle -\Delta_F u, v\rangle = \langle N_g(u), v\rangle = \int_\Omega g(x, u(x)) \cdot v(x)\, \mathrm{d}x,\ \forall v \in W_0^{1,2}(\Omega),$$

then u is a solution to problem (6.1.P) in the sense of $W^{-1,2}(\Omega)$.

Lemma 6.2. *The function $-\Delta_F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is a bijection and $(-\Delta_F)^{-1} : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is Lipschitzian.*

Since the operator $(-\Delta_F)^{-1}$ is bounded and continuous, we can rewrite the equation (6.5) into the form

$$u = (-\Delta_F)^{-1} \circ N_g(u),$$

where $(-\Delta_F)^{-1} \circ N_g : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is a compact operator.

Theorem 6.2 (Leray–Schauder alternative, [Granas and Dugundji, 2003]). *Let $T : X \to X$ be a compact operator and consider the set $\mathcal{S} = \{x \in X : x = \alpha T(x), \alpha \in [0, 1)\}$. Then either the set $\mathcal{S}$ is unbounded, or T has at least one fixed point.*

Based on the technique presented in [Dinca et al., 2001, Theorem 11], we can state the next existence result for the solutions to problem (6.1.P).

Theorem 6.3 ([Mezei and **Vas**, 2019]). *If the Carathéodory function g satisfies the growth condition (6.3) with $q \in (1, 2)$, then the operator $(-\Delta_F)^{-1} \circ N_g$ has at least one fixed point in $W_0^{1,2}(\Omega)$, which is a solution to problem (6.1.P).*

6.2 An existence result for problem (6.2.P) via Harnack inequality and Krasnosel'skii theorem

This section is devoted to the study of our second problem (6.2.P), in which case the existence of the solutions will be proved by the combined use of the Harnack inequality and a Krasnosel'skii-type fixed point theorem.

Taking $A(x, u, \nabla u) = \nabla_\xi \left(\frac{1}{2} F^2(\nabla u)\right)$ and $B(x, u, \nabla u) = 0$ in the weak Harnack inequality of [Pucci and Serrin, 2007, Theorem 7.1.2], all the assumptions are satisfied, thus one has the inequalities

$$\left\langle \nabla_\xi \left(\frac{1}{2} F^2(\xi)\right), \xi \right\rangle = F^2(\xi) \geq a\,|\xi|^2 \quad \text{and} \quad \left|\nabla_\xi \left(\frac{1}{2} F^2(\xi)\right)\right| \leq a_1\,|\xi|,$$

using which we can formulate the relevant form of the weak Harnack inequality.

Theorem 6.4 (Weak Harnack inequality)**.** *Fix the parameter $p \in (1,n)$ and let the function $u \in W^{1,p}_{loc}(\Omega)$ be a non-negative solution to the inequality*

$$-\Delta_F u \geq 0. \tag{6.6}$$

Then, for any ball B_{4R} in Ω and any $s \in \left(0, \frac{(p-1)n}{n-p}\right)$, we have that

$$R^{-\frac{n}{s}} \|u\|_{L^s(B_{2R})} \leq C \cdot \inf_{B_{2R}} u, \tag{6.7}$$

or equivalently,

$$M_0 \|u\|_{L^s(B_{2R})} \leq \inf_{B_{2R}} u, \tag{6.8}$$

where the constant C depends only on the parameters n, s and $M_0 := \dfrac{R^{-\frac{n}{s}}}{C}$.

[Precup, 2012, Theorem 1.3] gives a very similar estimation to (6.7) for any non-negative superharmonic function, which remains true on any bounded subdomain Ω_0 with $\overline{\Omega}_0 \subset \Omega$ (i.e., $\Omega_0 \Subset \Omega$), not only on balls, as in the case of [Pucci and Serrin, 2007, Theorem 7.1.2]. Replacing the superharmonic function with a Δ_F-superharmonic function in [Precup, 2012, Theorem 1.3], we can state the next Moser–Harnack-type inequality.

Theorem 6.5. *Let either $n \geq 3$ and $s \in \left(1, \frac{n}{n-2}\right)$, or $n = 2$ and $s \in (1, \infty)$ be arbitrarily fixed numbers and consider the domain $\Omega_0 \Subset \Omega$. Then, there exists a constant $M = M(n, s, \Omega, \Omega_0) > 0$ for which the inequality*

$$M \|u\|_{L^s(\Omega_0)} \leq \inf_{\Omega_0} u \tag{6.9}$$

holds for every non-negative Δ_F-superharmonic function $u \in \Omega$.

Hereafter, assume that $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N}_{\geq 2})$ is a bounded regular domain and consider the continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$. Relying on the results of [Precup, 2012], our goal is to prove the existence of positive, Δ_F-superharmonic solutions to problem (6.2.P), i.e., $u \in C^1\left(\overline{\Omega}\right)$, $u(x) > 0$ and $-\Delta_F u \geq 0$ for any $x \in \Omega$ and u which fulfils (6.2.P).

In order to formulate the main existence result of this section, we introduce some notations and revive some relevant preliminary results. Let the space $X = C_0\left(\overline{\Omega}\right) = \{u \in C\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\}$ be equipped with the norm $|u| = |u|_\infty = \max_{\overline{\Omega}} |u(x)|$. We fix any bounded subdomain $\Omega_0 \Subset \Omega$ and consider the space $Y = L^p(\Omega_0)$ endowed with the norm $\|v\|_{L^p(\Omega_0)} = \left(\int_{\Omega_0} |v(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$, where $p \in \left[1, \frac{n}{n-2}\right)$ if $n > 2$ and $p \in [1, \infty)$ if $n = 2$.

Define the linear map $\mathcal{I} : C_0\left(\overline{\Omega}\right) \to L^p(\Omega_0)$, $\mathcal{I}u = u|_{\Omega_0}$. For any function $u \in C_0\left(\overline{\Omega}\right)$, the inequality $\|u\|_{L^p(\Omega_0)} \leq |u| \left(\text{meas}(\Omega_0)\right)^{\frac{1}{p}}$ holds, which implies that $|\mathcal{I}| \leq \left(\text{meas}(\Omega_0)\right)^{\frac{1}{p}}$.

Due to [Azizieh and Clèment, 2002, Lemma 1.1] and [Lieberman, 1988, Theorem 1], if Ω is a bounded regular domain of class $C^{1,\beta}$ for some $\beta \in (0,1)$ and $g \in L^\infty(\Omega)$, then the weak solution in $W^{1,2}_0(\Omega)$ to the Dirichlet problem

$$\begin{cases} -\Delta_F u = g\,(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.10}$$

belongs to $C^1\left(\overline{\Omega}\right)$ and $\left(-\Delta_F\right)^{-1} : L^\infty(\Omega) \to C^1\left(\overline{\Omega}\right)$ is continuous, compact and order-preserving.

Using the function $G : C\left(\overline{\Omega}; \mathbb{R}_+\right) \to C\left(\overline{\Omega}\right)$, $G(u)(x) = g(u(x))$ we define the function $\mathcal{N} : C\left(\overline{\Omega}; \mathbb{R}_+\right) \to C_0\left(\overline{\Omega}\right)$,

$$\mathcal{N} = \left(-\Delta_F\right)^{-1} \circ G.$$

Since the function g is non-negative and $\left(-\Delta_F\right)^{-1}$ is positive, the function $\mathcal{N}$ maps $C\left(\overline{\Omega}; \mathbb{R}_+\right)$ onto $C\left(\overline{\Omega}; \mathbb{R}_+\right)$.

Let us introduce the set

$$K \coloneqq \{u \in C_0\left(\overline{\Omega};\mathbb{R}_+\right) : u(x) \geq M \|u\|_{L^p(\Omega_0)},\ \forall x \in \Omega_0\},$$

where the constant $M > 0$ comes from the Harnack inequality (6.9).

By the definition of the function $\mathcal{N}$, we have that $\mathcal{N}(u)$ is Δ_F-superharmonic, thus by applying Theorem 6.5 the function $\mathcal{N}$ maps K into itself. Consequently, we can use the following variant of the Krasnosel'skii-type fixed point theorem.

Theorem 6.6 (Krasnosel'skii-type theorem, [Precup, 2012, Theoem 2.1])**.** *Let $\mathcal{N} : K \to K$ be completely continuous, let $\phi \in K$ with $|\phi| = 1$ be any fixed element, let R_0 and R_1 be any positive numbers with $R_0 < \|\phi\|_{L^p(\Omega_0)} R_1$ and let $h \in K$ be such that $\|h\|_{L^p(\Omega_0)} > R_0$. Assume also that conditions*

$$\mathcal{N}u \neq \lambda u,\ \forall\, |u| = R_1,\ \forall \lambda \geq 1 \tag{6.11}$$

and

$$(1-\mu)\mathcal{N}\left(\min\left\{\frac{R_1}{|u|},1\right\}u\right) + \mu h \neq u,\ \forall \mu \in (0,1)\,,\ \|u\|_{L^p(\Omega_0)} = R_0,\ \forall\, |u| \leq R_2 \tag{6.12}$$

are satisfied, where $R_2 = \max\left\{R_1, |h|\,, \max_{|u|\leq R_1} |\mathcal{N}u|\right\}$.

Then, function $\mathcal{N}$ has a fixed point u in $K_{R_0R_1} \coloneqq \left\{u \in K : R_0 < \|u\|_{L^p(\Omega_0)}\,, |u| < R_1\right\}$.

Due to [Franzina, 2012, Theorem 3.2.1], we can take ϕ as the positive eigenfunction that corresponds to the first eigenvalue λ_1, i.e.,

$$\begin{cases} \Delta_F\phi + \lambda_1\phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

with $|\phi| = 1$.

Let $\chi_{\Omega_0} : \overline{\Omega} \to \mathbb{R}_+$ be the characteristic function of Ω_0, i.e,

$$\chi_{\Omega_0}(x) = \begin{cases} 1, & x \in \Omega_0, \\ 0, & x \notin \Omega_0, \end{cases}$$

and also consider the set $C = |1|_{L^p(\Omega_0)} = (\text{meas}(\Omega_0))^{\frac{1}{p}}$. The inequality

$$\left(-\Delta_F\right)^{-1}\chi_{\Omega_0} \geq M\left\|\left(-\Delta_F\right)^{-1}\chi_{\Omega_0}\right\|_{L^p(\Omega_0)} \quad \text{in } \Omega_0$$

implies that

$$\left\|\left(-\Delta_F\right)^{-1}\chi_{\Omega_0}\right\|_{L^p(\Omega_0)} \geq MC\left\|\left(-\Delta_F\right)^{-1}\chi_{\Omega_0}\right\|_{L^p(\Omega_0)},$$

which yields that $MC \leq 1$.

Introducing the notations $A \coloneqq \dfrac{1}{MC\left\|\left(-\Delta_F\right)^{-1}\chi_{\Omega_0}\right\|_{L^p(\Omega_0)}}$ and $B \coloneqq \dfrac{1}{\left|\left(-\Delta_F\right)^{-1}1\right|}$, we can formulate the main existence result of the current section related to the solutions to problem (6.2.P).

Theorem 6.7 ([Mezei and **Vas**, 2019])**.** *Suppose that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and there exist R_0, R_1 with $R_0 \in \left(0, MC\,\|\phi\|_{L^p(\Omega_0)} R_1\right)$ such that*

$$\min_{\tau\in[MR_0,R_1]} g(\tau) > A\cdot R_0 \tag{6.13}$$

and

$$\max_{\tau\in[0,R_1]} g(\tau) < B\cdot R_1. \tag{6.14}$$

Then, problem (6.2.P) has at least one solution with $R_0 < \|u\|_{L^p(\Omega_0)}$ and $|u| < R_1$.

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