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EQUIVALENCES BETWEEN MODULE TRIPLES IN FINITE GROUP REPRESENTATION THEORY

PhD Thesis Summary

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Contents

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Introduction

The representation theory of a finite group is a very important tool needed to examine the structure of said group, thereby one of the main goals of the representation theory of finite groups is the classification of all finite groups. In this sense, a main result obtained this century was the successful classification of all simple groups. For the classification of all finite groups, some of the oldest conjectures propose to relate the representation theory of a finite group to that of its local subgroups.

One of the more modern strategies used in order to tackle these so-called local-global conjectures is to reduce them to some more complicated statements on finite simple subgroups through the so-called inductive conditions. Such reduction theorems are obtained, by techniques of Clifford theory, and in particular, by using the language of character triples and of the various relations between them.

In the recent results of Britta Späth, surveyed in [37] and [38], on inductive conditions for the McKay conjecture and the Alperin weight conjecture, the developed reduction theorems involve certain order relations between two character triples, which forces the two triples to have "the same Clifford theory". More precisely, Britta Späth considers three relations between character triples: the first-order relation ([38, Definition 2.1]), the central-order relation ([38, Definition 2.7]) and the blockwise-order relation ([38, Definition 4.2]).

In this thesis we create a categorical version of character triples, named module triples and we give three relations between module triples analogous in a sense to those between character triples previously recalled. Moreover, we will prove that these relations are consequences of some group graded Morita, Rickard or derived equivalences, respectively, with some additional properties. This is motivated by the convictions, expressed primarily in the work of Michel Broué (see, for instance, [5] and [6]), that character correspondences with good properties are consequences of categorical equivalences, like Morita equivalences or Rickard equivalences between blocks of group algebras. To explain the link between these two points of view, let us introduce our context.

We consider a finite group G, a p-modular system $(\mathcal{K}, \mathcal{O}, \Bbbk)$, where $\mathcal{O}$ is a complete discrete valuation ring, $\mathcal{K}$ is the field of fractions of $\mathcal{O}$ and $\Bbbk = \mathcal{O}/\mathrm{J}(\mathcal{O})$ is its residue field, together with the assumptions that $\Bbbk$ is algebraically closed, and that $\mathcal{K}$ contains all the unity roots of order $|\mathsf{G}|$.

Let N be a normal subgroup of G, and denote $\bar{\mathsf{G}} := \mathsf{G}/\mathsf{N}$. Let also G' be a subgroup of G such that $\mathsf{G} = \mathsf{N}\mathsf{G}'$, and let $\mathsf{N}' = \mathsf{G}' \cap \mathsf{N}$. Let b and b' be $\bar{\mathsf{G}}$-invariant blocks of $\mathcal{O}\mathsf{N}$ and $\mathcal{O}\mathsf{N}'$, respectively, and consider the strongly $\bar{\mathsf{G}}$-graded $\mathcal{O}$-algebras $\mathsf{A} = \mathsf{b}\mathcal{O}\mathsf{G}$ and $\mathsf{A}' = \mathsf{b}'\mathcal{O}\mathsf{G}'$, with identity components $\mathsf{B} = \mathsf{b}\mathcal{O}\mathsf{N}$ and $\mathsf{B}' = \mathsf{b}'\mathcal{O}\mathsf{N}'$.

In Chapter 1, we start with the "Butterfly theorem", which as stated by B. Späth in [38, Theorem 2.16], gives the possibility to construct certain relations between character triples. Motivated by this, as we have published in article [28], we consider group graded Morita equivalences between block extensions and we obtain Theorem 1.5.2, which shows how to construct a group graded Morita equivalence from a given one, under very similar assumptions to those in [38]. Nevertheless, our assumption that all elements of M commute with the elements in the center of N, where M is a $(\mathsf{B}, \mathsf{B}')$-bimodule assumed to induce a Morita equivalence between B and B', provides the motivation for some further developments of this categorical version of the "Butterfly theorem", which will be presented in Chapter 2.

Moreover, one of the conditions in Späth's definition [37, Definition 3.1], [38, Definition 2.7] of the central order $\leq_c$ between two character triples is that $C_G(N) \leq G'$ and that the projective characters associated to the character triples take the same scalar values on elements of $C_G(N)$. Observe that in this situation, the group algebra $\mathcal{C} = \mathcal{O}C_G(N)$ is a $\bar{G}$-graded subalgebra of both A and A', and has an obvious $\bar{G}$-action compatible with the $\bar{G}$-grading.

Henceforth, given that we already know that equivalences induced by $\bar{G}$-graded bimodules preserve many Clifford theoretical invariants (see [25, Chapter 5] and [26]), both our initial categorical version of the "Butterfly theorem" and the relation $\leq_c$ between character triples leads us to the consideration of $\bar{G}$-graded (A, A')-bimodules $\tilde{M}$ satisfying $m_{\bar{g}}c = {}^{\bar{g}}c m_{\bar{g}}$ for all $\bar{g} \in \bar{G}$, $c \in \mathcal{C}$ and $m_{\bar{g}} \in \tilde{M}_{\bar{g}}$. It turns out that equivalences induced by such bimodules imply the relation $\leq_c$ between corresponding character triples. This type of bimodules, namely G-graded (A, A')-bimodules over a $\bar{G}$-graded $\bar{G}$-acted algebra $\mathcal{C}$, will be introduced in Chapter 2, where we will also establish their main properties. This new type of structures was first published by us in article [29]. The main result of this chapter is Theorem 2.1.13, where we show that their corresponding category, A-Gr/$\mathcal{C}$-A', is equivalent to the category of $\Delta^{\mathcal{C}}$-modules, where $\Delta^{\mathcal{C}} = \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}} \otimes_{\mathcal{C}} A'^{\mathrm{op}}_{\bar{g}}$. Actually, there are three naturally isomorphic functors giving this equivalence (see Proposition 2.1.14), and we also prove in Proposition 2.1.15 that these functors are also compatible with tensor products and with taking homomorphisms.

In the following sections of the chapter, we develop a group graded Morita Theory over $\mathcal{C}$, as we have done in article [32], and we show that G-graded Morita equivalences over $\mathcal{C}$ can be induced from certain equivariant Morita equivalences between B and B', after which we prove in Theorem 2.4.3 our final version of the "Butterfly theorem", generalizing the initial result of Chapter 1. As an application, in Section 2.5 we show how to obtain $\bar{G}$-graded Morita equivalences over $\mathcal{C}$ from the Morita equivalences induced by the Scott module $\mathrm{Sc}(N \times N', \Delta Q)$ of Koshitani and Lassueur [19], [20].

In order to take advantage of this categorical setting, in Chapter 3, as we have published in article [29], we introduce module triples (A, B, V), where V is a $\bar{G}$-invariant simple $\mathcal{K} \otimes_{\mathcal{O}} B$-module, and the relation $\geq_c$ between two module triples (A, B, V) and (A', B', V'). Our final main result is Theorem 3.2.4, where we prove that if the module triples (A, B, V) and (A', B', V') correspond under a $\bar{G}$-graded Rickard equivalence over $\mathcal{C} = \mathcal{O}C_G(N)$, then $(A, B, V) \geq_c (A', B', V')$; this, in turn, implies the relation $(G, N, \theta) \geq_c (G', N', \theta')$ between the associated character triples. Furthermore, in Definition 3.3.1 we also provide a module triple version of the blockwise-relation $\geq_b$ (see [38, Definition 4.2]), and we show in Proposition 3.3.6 that this is a consequence of a special type of group graded derived equivalences which is compatible in a certain sense with the Brauer map. These new blockwise additions to our theory of module triples were developed by us in article [30].

Finally, in Chapter 4, given that Britta Späth's research shows that a new character triple can be constructed via a wreath product construction of character triples ([38, Theorem 2.21]), we will prove that our equivalences are compatible with tensor and wreath product constructions, as we have done in articles [33, 30].

Cumulatively, this thesis is based upon our results published in [28, 29, 32, 33, 30]. Additionally certain parts of this thesis have been presented in various national and international conferences.

For any unexplained concepts and results, our general references are [23] and [25].

Chapter 1

$\bar{G}$-graded endomorphism algebras and Morita equivalences

In this chapter, we will present our basic notations and in the following sections we will prove a group graded Morita equivalences version of the "butterfly theorem" on character triples. This gives a method to construct an equivalence between block extensions from another related equivalence. Results of this chapter were published in [28].

1.1 Notations and preliminaries

In general, our notations and assumptions are standard and follow [25].

All rings in this thesis are associative with identity $1 \neq 0$ and all modules are unital and finitely generated.

Let A be a ring. We denote by A-Mod the category of all left A-modules. We shall usually write actions on the left, so in particular, by module we will usually mean a left module, unless otherwise stated. The notation ${}_AM$ (respectively, ${}_AM_{A'}$) will be used to emphasize that M is a left A-module (respectively, an (A, A')-bimodule).

To introduce our context, let G be a finite group and let $(\mathcal{K}, \mathcal{O}, \mathcal{k})$ be a p-modular system, where $\mathcal{O}$ is a complete discrete valuation ring, $\mathcal{K}$ is the field of fractions of $\mathcal{O}$ (of characteristic 0) and $\mathcal{k} = \mathcal{O}/J(\mathcal{O})$ is its residue field (of characteristic p). We assume that $\mathcal{k}$ is algebraically closed, and that $\mathcal{K}$ contains all the roots of unity of order $|G|$.

We consider N to be a normal subgroup of G, and we will denote by $\bar{G}$ the factor group G/N.

Let $A = \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}}$ be a $\bar{G}$-graded $\mathcal{O}$-algebra. For a subgroup $\bar{H}$ of $\bar{G}$, we denote by $A_{\bar{H}} := \bigoplus_{\bar{g}\in\bar{H}} A_{\bar{g}}$ the truncation of A from $\bar{G}$ to $\bar{H}$. We denote by A-Gr the category of all $\bar{G}$-graded left A-modules. For $M = \bigoplus_{\bar{g}\in\bar{G}} M_{\bar{g}} \in A$-Gr and $\bar{g} \in \bar{G}$, the $\bar{g}$-suspension of M is defined to be the $\bar{G}$-graded A-module $M(\bar{g}) = \oplus_{\bar{h}\in\bar{G}} M(\bar{g})_{\bar{h}}$, where $M(g)_{\bar{h}} = M_{\bar{g}\bar{h}}$. For any $\bar{g} \in \bar{G}$, $T^A_{\bar{g}} : A\text{-Gr} \to A\text{-Gr}$ will denote (as in [11]) the $\bar{g}$-suspension functor, i.e. $T^A_{\bar{g}}(M) = M(\bar{g})$ for all $\bar{g} \in \bar{G}$. The stabilizer of M in G is, by definition [25, §2.2.1], the subgroup

$$\bar{G}_M = \left\{\bar{g} \in \bar{G} \mid M \simeq M(\bar{g}) \text{ as } \bar{G}\text{-graded left } A\text{-modules}\right\}.$$

If $\bar{G}_M = \bar{G}$ or equivalently M is isomorphic in A-Gr to each of of its $\bar{g}$-suspensions, we say that M is $\bar{G}$-invariant.

Let $M, N \in A$-Gr. We denote by $\mathrm{Hom}_A(M, N)$, the additive group of all A-linear homomorphisms from M to N. Because $\bar{G}$ is finite, E. C. Dade showed in [9, Corollary 3.10] that $\mathrm{Hom}_A(M, N)$ is $\bar{G}$-graded. More precisely, if $\bar{g} \in \bar{G}$, the component of degree $\bar{g}$ (furthermore called the $\bar{g}$-component) is defined as in [25, §1.2]:

$$\mathrm{Hom}_A(M, N)_{\bar{g}} := \left\{f \in \mathrm{Hom}_A(M, N) \mid f(M_{\bar{x}}) \subseteq N_{\bar{x}\bar{g}}, \text{ for all } \bar{x} \in \bar{G}\right\}.$$

We denote by id_X the identity map defined on a set X.

For the sake of simplicity, in this chapter, we will mostly consider only crossed products, also because the generalization of the statements to the case of strongly graded algebras is a mere technicality. Recall that, if A is a crossed product, we can chose an invertible homogeneous element $u_{\bar{g}}$ in the component $A_{\bar{g}}$, for all $\bar{g} \in \bar{G}$.

Our main example for a $\bar{G}$-graded crossed product is obtained as follows: Regard $\mathcal{O}G$ as a $\bar{G}$-graded algebra with the 1-component $\mathcal{O}N$. Let $b \in Z(\mathcal{O}N)$ be a $\bar{G}$-invariant block idempotent. We denote $A := b\mathcal{O}G$ and $B := b\mathcal{O}N$. Then the block extension A is a $\bar{G}$-graded crossed product, with 1-component B. This will also constitute the basis of our main framework (Section 1.4) and also the reason for which we will mainly utilize "$\bar{G}$-gradings" in this thesis, although this is not always essential: in most cases, one may consider instead the gradings to be given directly by G.

1.2 Group graded Morita equivalences

In this section, we recall from [25] the main facts on group graded Morita equivalences and we state a graded variant of the second Morita Theorem [12, Theorem 12.12].

Let $A = \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}}$ and $A' = \bigoplus_{\bar{g}\in\bar{G}} A'_{\bar{g}}$ be strongly $\bar{G}$-graded algebras, with the 1-components B and B' respectively.

It is clear that $A \otimes_{\mathcal{O}} A'^{\mathrm{op}}$ is a $\bar{G} \times \bar{G}$-graded algebra. Let

$$\delta(\bar{G}) := \{(\bar{g}, \bar{g}) \mid \bar{g} \in \bar{G}\}$$

be the diagonal subgroup of $\bar{G} \times \bar{G}$, and let $\Delta^{\mathcal{O}}$ be the diagonal subalgebra of $A \otimes_{\mathcal{O}} A'^{\mathrm{op}}$

$$\Delta^{\mathcal{O}} := \Delta(A \otimes_{\mathcal{O}} A'^{\mathrm{op}}) := (A \otimes_{\mathcal{O}} A'^{\mathrm{op}})_{\delta(\bar{G})} = \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}} \otimes A'_{\bar{g}^{-1}}.$$

Then $\Delta^{\mathcal{O}}$ is a $\bar{G}$-graded algebra, with 1-component $\Delta_1^{\mathcal{O}} = B \otimes_{\mathcal{O}} B'^{\mathrm{op}}$.

Let M be a (B, B')-bimodule, or, equivalently, M is a $B \otimes_{\mathcal{O}} B'^{\mathrm{op}}$-module, thus a Δ_1-module. Let $M^* := \mathrm{Hom}_B(M, B)$ be its B-dual. Note that if B is a symmetric algebra, then we have the isomorphism

$$M^* := \mathrm{Hom}_B(M, B) \simeq \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O}),$$

where $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$, is the $\mathcal{O}$-dual of M.

Definition 1.2.1. We say that the $\bar{G}$-graded (A, A')-bimodule $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence between A and A', if $\tilde{M} \otimes_{A'} \tilde{M}^* \simeq A$ as $\bar{G}$-graded (A, A)-bimodules and $\tilde{M}^* \otimes_A \tilde{M} \simeq A'$ as $\bar{G}$-graded (A', A')-bimodules, where the A-dual $\tilde{M}^* = \mathrm{Hom}_A(\tilde{M}, A)$ of $\tilde{M}$ is a $\bar{G}$-graded (A', A)-bimodule.

By [25, Theorem 5.1.2], the following statements are equivalent:

(1) between B and B' we have a Morita equivalence given by the $\Delta_1^{\mathcal{O}}$-module M:

$$B \underset{{}_{B'}M^*_B \otimes_B -}{\overset{{}_B M_{B'} \otimes_{B'} -}{\leftrightarrows}} B';$$

and M extends to a $\Delta^{\mathcal{O}}$-module;

(2) $\tilde{M} := A \otimes_B M$ is a $\bar{G}$-graded (A, A')-bimodule and $\tilde{M}^* := A' \otimes_{B'} M^*$ is a $\bar{G}$-graded (A', A)-bimodule, which induce a $\bar{G}$-graded Morita equivalence between A and A', given by the functors:

$$A \underset{{}_{A'}\tilde{M}^*_A \otimes_A -}{\overset{{}_A \tilde{M}_{A'} \otimes_{A'} -}{\leftrightarrows}} A'.$$

In this case, by [25, Lemma 1.6.3] tells us the following are naturally isomorphic equivalences of categories:

$$A \otimes_B - \simeq - \otimes_{B'} A' \simeq ((A \otimes_{\mathcal{O}} A'^{\mathrm{op}}) \otimes_{\Delta^{\mathcal{O}}} -),$$

thus we have the natural isomorphisms of $\bar{G}$-graded bimodules

$$\tilde{M} := A \otimes_B M \simeq M \otimes_{B'} A' \simeq ((A \otimes_{\mathcal{O}} A'^{\mathrm{op}}) \otimes_{\Delta^{\mathcal{O}}} M),$$

Assume that B and B' are Morita equivalent. Then, by the second Morita Theorem [12, Theorem 12.12], we can choose the bimodule isomorphisms

$$\varphi : M^* \otimes_B M \to B', \qquad \psi : M \otimes_{B'} M^* \to B,$$

such that

$$\psi(m \otimes m^*)n = m\varphi(m^* \otimes n), \quad \forall m, n \in M,\ m^* \in M^*$$

and that

$$\varphi(m^* \otimes m)n^* = m^*\psi(m \otimes n^*), \quad \forall m^*, n^* \in M^*,\ m \in M.$$

By the surjectivity of this functions, we may choose finite sets I and J and the elements m_j^*, $n_i^* \in M^*$ and m_j, $n_i \in M$, for all $i \in I$, $j \in J$ such that:

$$\varphi(\sum_{j \in J} m_j^* \otimes_B m_j) = 1_{B'}, \qquad \psi(\sum_{i \in I} n_i \otimes_B n_i^*) = 1_B.$$

Assume that $\tilde{M}$ and $\tilde{M}^*$ give a $\bar{G}$-graded Morita equivalence between A and A'. As above, by [12, Theorem 12.12], we can choose the isomorphisms

$$\tilde{\varphi} : \tilde{M}^* \otimes_A \tilde{M} \to A', \qquad \tilde{\psi} : \tilde{M} \otimes_{A'} \tilde{M}^* \to A$$

of $\bar{G}$-graded bimodules such that

$$\tilde{\psi}(\tilde{m} \otimes \tilde{m}^*)\tilde{n} = \tilde{m}\tilde{\varphi}(\tilde{m}^* \otimes \tilde{n}), \quad \forall \tilde{m}, \tilde{n} \in \tilde{M},\ \tilde{m}^* \in \tilde{M}^*$$

and that

$$\tilde{\varphi}(\tilde{m}^* \otimes \tilde{m})\tilde{n}^* = \tilde{m}^*\tilde{\psi}(\tilde{m} \otimes \tilde{n}^*), \quad \forall \tilde{m}^*, \tilde{n}^* \in \tilde{M}^*,\ \tilde{m} \in \tilde{M}.$$

Actually, $\tilde{\varphi}_1$ and $\tilde{\psi}_1$ are the same with φ and ψ from before and are $\Delta^{\mathcal{O}}$-linear isomorphisms. Moreover, we have that $1_A = 1_B \in B$ and $1_{A'} = 1_{B'} \in B'$. Henceforth, we may choose the same finite sets I and J and the same elements m_j^*, $n_i^* \in M^*$ and m_j, $n_i \in M$, $\forall i \in I, j \in J$ such that:

$$\tilde{\varphi}(\sum_{j \in J} m_j^* \otimes_B m_j) = 1_{B'}, \qquad \tilde{\psi}(\sum_{i \in I} n_i \otimes_B n_i^*) = 1_B.$$

1.3 Centralizers and graded endomorphism algebras

This section is based on [28, §3]. In it, we show that there is a natural map, compatible with Morita equivalences, from the centralizer $C_A(B)$ of B in A to the endomorphism algebra of a $\bar{G}$-graded A-module induced from a B-module.

We will assume that A and A' are $\bar{G}$-graded crossed products, although the results of this section can be generalized to strongly graded algebras. Let $U \in B$-mod and $U' \in B'$-mod such that $U' = M^* \otimes_B U$. We denote

$$E(U) := \mathrm{End}(A \otimes_B U)^{\mathrm{op}}, \qquad E(U') := \mathrm{End}(A' \otimes_{B'} U')^{\mathrm{op}},$$

the $\bar{G}$-graded endomorphism algebras of the modules induced from U and U'.

We will prove that there exists a natural $\bar{G}$-graded algebra homomorphism between the centralizer of B in A and $E(U)$, compatible with $\bar{G}$-graded Morita equivalences.

Lemma 1.3.1. *The map*

$$\theta : C_A(B) \to E(U), \qquad \theta(c)(a \otimes u) = ac \otimes u,$$

where $c \in C_A(B)$, $a \in A$ *and* $u \in U$ *is a homomorphism of* $\bar{G}$*-graded algebras.*

By [25, Lemma 1.6.3], we have

$$A \otimes_B M \simeq M \otimes_{B'} A',$$

and we will need an explicit isomorphism between the two. We will choose invertible elements $u_{\bar{g}} \in U(A) \cap A_{\bar{g}}$ and $u'_{\bar{g}} \in U(A) \cap A'_{\bar{g}}$ of degree $\bar{g} \in \bar{G}$. We have that an arbitrary element $a'_{\bar{g}} \in A'_{\bar{g}}$ can be written uniquely in the form $a'_{\bar{g}} = u'_{\bar{g}} b'$, where $b' \in B'$. The desired $\bar{G}$-graded bimodule isomorphism is:

$$\varepsilon : M \otimes_{B'} A' \to A \otimes_B M \qquad m \otimes_{B'} a'_{\bar{g}} \mapsto u_{\bar{g}} \otimes_B u_{\bar{g}}^{-1} m a'_{\bar{g}}$$

for $m \in M$. We will also need the explicit isomorphism of $\bar{G}$-graded bimodules

$$\beta : A' \otimes_{B'} M^* \to M^* \otimes_B A \qquad a'_{\bar{g}} \otimes_{B'} m^* \mapsto a'_{\bar{g}} m^* u_{\bar{g}}^{-1} \otimes_B u_{\bar{g}}$$

for $m^* \in M^*$. Henceforth we consider the isomorphism of $\bar{G}$-graded A'-modules

$$\beta \otimes_B \mathrm{id}_U : A' \otimes_{B'} M^* \otimes_B U \to M^* \otimes_B A \otimes_B U.$$

Proposition 1.3.2. *Assume that* $\tilde{M}$ *and* $\tilde{M}^*$ *give a* $\bar{G}$*-graded Morita equivalence between* A *and* A'*. Then the diagram*

$$\begin{tikzcd} E(U) \arrow[r, "\varphi_1", "\sim"'] & E(U') \\ C_A(B) \arrow[u, "\theta"] \arrow[r, "\varphi_2", "\sim"'] & C_{A'}(B') \arrow[u, "\theta'"'] \end{tikzcd}$$

is commutative, where the maps are defined as follows:

$$\begin{aligned} \theta(c)(a \otimes u) &= ac \otimes u, \\ \theta'(c')(a' \otimes u') &= a'c' \otimes u' \\ \varphi_1(f) &= (\beta \otimes_B \mathrm{id}_U)^{-1} \circ (\mathrm{id}_{\tilde{M}^*} \otimes f) \circ (\beta \otimes_B \mathrm{id}_U), \\ \varphi_2(c) &= \tilde{\varphi}(\sum_{j \in J} m_j^* c \otimes_B m_j). \end{aligned}$$

for all $a \in A$, $a' \in A'$, $c \in C_A(B)$, $c' \in C_{A'}(B')$, $u \in U$, $u' \in U'$ *and* $f \in E(U)$.

1.4 The beginning of the main framework

Recall that G is a finite group and that N is a normal subgroup of G. Additionally, let G' be a subgroup of G and N' a normal subgroup of G'. We assume that $N' = G' \cap N$ and $G = G'N$, hence by the second isomorphism theorem

$$\bar{G} := G/N \simeq G'/(G' \cap N) = G'/N'.$$

We can represent this in the following diagram:

$$\begin{tikzcd}[column sep=small] G \arrow[d, dash] \arrow[dr, dash] & \\ N \arrow[dr, dash] & G' \arrow[d, dash] \\ & N'. \end{tikzcd}$$

We regard $\mathcal{O}G$ and $\mathcal{O}G'$ as $\bar{G}$-graded algebras with the 1-components $\mathcal{O}N$ and $\mathcal{O}N'$ respectively. It is known that $\bar{G}$ acts on $Z(\mathcal{O}N)$ and $Z(\mathcal{O}N')$.

Let $b \in Z(\mathcal{O}N)$ and $b' \in Z(\mathcal{O}N')$ be two block idempotents. We denote

$$A := b\mathcal{O}G, \qquad A' := b'\mathcal{O}G', \qquad B := b\mathcal{O}N, \qquad B' := b'\mathcal{O}N',$$

hence A and A' are $\bar{G}$-graded crossed products, with 1-components B and B' respectively.

We assume that b and b' are $\bar{G}$-invariant. This hypothesis does not represent a huge loss in generality. Indeed, for example, if we consider the block idempotent $e \in \mathrm{Blk}\,(\mathcal{O}N)$ such that b covers e, then according to the Fong-Reynolds reduction theorem, we have that $A := b\mathcal{O}G = \mathcal{O}Ge\mathcal{O}G \simeq e\mathcal{O}Ge \simeq e\mathcal{O}G_{\bar{G}_e} = e\mathcal{O}\bar{G}_e$ and e is $\bar{G}_e$-invariant.

1.5 The butterfly theorem for group graded Morita equivalences

In this last section of the chapter, published in [28, §4], we prove that a Morita equivalence between the 1-components of two block extensions always lifts to a graded equivalence between certain centralizer algebras. This is the main ingredient in the proof of the main result from this chapter, Theorem 1.5.2.

Additionally to the framework presented in Section 1.4, we assume that $C_G(N) \subseteq G'$, and we denote $\bar{C}_G(N) := NC_G(N)/N$. We consider the algebras:

$$\begin{tikzcd} A := b\mathcal{O}G \arrow[r, dash] \arrow[d, dash] & A' := b'\mathcal{O}G' \arrow[d, dash] \\ C := b\mathcal{O}NC_G(N) \arrow[r, dash] \arrow[d, dash] & C' := b'\mathcal{O}N'C_G(N) \arrow[d, dash] \\ B := b\mathcal{O}N \arrow[r, dash] & B' := b'\mathcal{O}N'. \end{tikzcd}$$

If M induces a Morita equivalence between B and B', the question that arises is what can we deduce without the additional hypothesis that M extends to a $\Delta^{\mathcal{O}}$-module. One answer is given by the following proposition.

Proposition 1.5.1. *Assume that:*

(1) $C_G(N) \subseteq G'$.

(2) M *induces a Morita equivalence between* B *and* B'.

(3) $zm = mz$ *for all* $m \in M$ *and* $z \in Z(N)$.

Then there is a $\bar{C}_G(N)$*-graded Morita equivalence between* C *and* C'

$$\begin{tikzcd}
A := b\mathcal{O}G \arrow[d, dash] & & A' := b'\mathcal{O}G' \arrow[d, dash] \\
C := b\mathcal{O}NC_G(N) \arrow[rr, dash, "{}_C\hat{M}_{C'}", "\sim"'] \arrow[d, dash] & & C' := b'\mathcal{O}N'C_G(N) \arrow[d, dash] \\
B := b\mathcal{O}N \arrow[rr, dash, "{}_BM_{B'}", "\sim"'] & & B' := b'\mathcal{O}N',
\end{tikzcd}$$

induced by the $\bar{C}_G(N)$*-graded* (C, C')*-bimodule*

$$\hat{M} := C \otimes_B M \simeq M \otimes_{B'} C' \simeq (C \otimes C'^{op}) \otimes_{\Delta(C\otimes C'^{op})} M.$$

Our main result, published in [28, Theorem 4.2], is a version for Morita equivalences of the so-called "butterfly theorem" [38, Theorem 2.16].

Theorem 1.5.2 (The butterfly theorem for group graded Morita equivalences). *Let* $\hat{G}$ *be another group with normal subgroup* N *such that the block* b *is also* $\hat{G}$*-invariant. Assume that:*

(1) $C_G(N) \subseteq G'$;

(2) $\tilde{M}$ *induces a* $\bar{G}$*-graded Morita equivalence between* A *and* A';

(3) $zm = mz$ *for all* $m \in M$ *and* $z \in Z(N)$;

(4) *the conjugation maps* $\varepsilon : G \to Aut(N)$ *and* $\hat{\varepsilon} : \hat{G} \to \mathrm{Aut}(N)$ *satisfy* $\varepsilon(G) = \hat{\varepsilon}(\hat{G})$.

Denote $\hat{G}' = \hat{\varepsilon}^{-1}(\varepsilon(G'))$. *Then there is a* $\hat{G}/N$*-graded Morita equivalence between* $\hat{A} := b\mathcal{O}\hat{G}$ *and* $\hat{A}' := b'\mathcal{O}\hat{G}'$.

Chapter 2

Equivalences over a group graded group acted algebra

This chapter is based on our results published in [29] and [32]. In it, we introduce Morita and Rickard equivalences over a group graded G-algebra between block extensions, which are needed given that a consequence of such equivalences is that Späth's central order relation holds between two corresponding character triples (as we shall see in Chapter 3).

2.1 $\bar{\mathsf{G}}$-graded bimodules over a $\bar{\mathsf{G}}$-graded $\bar{\mathsf{G}}$-acted algebra

The notions introduced in this section, together with the results have first been published in [29, §2].

2.1.1. Let $A = \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}}$ be a $\bar{G}$-graded $\mathcal{O}$-algebra with the identity component $B := A_1$. For the sake of simplicity, we assume that A is a crossed product (the generalization is not difficult, see for instance [25, §1.4.B.]). This means that we can choose invertible homogeneous elements $u_{\bar{g}}$ in the component $A_{\bar{g}}$, for any $\bar{g} \in \bar{G}$. Let also A' be another crossed product with identity component B', and choose invertible homogeneous elements $u'_{\bar{g}} \in A'_{\bar{g}}$, for all $\bar{g} \in \bar{G}$.

Definition 2.1.2. An $\mathcal{O}$-algebra $\mathcal{C}$ is a $\bar{G}$-*graded* $\bar{G}$-*acted* $\mathcal{O}$-*algebra* if

(1) $\mathcal{C}$ is $\bar{G}$-graded, and we write $\mathcal{C} = \bigoplus_{\bar{g}\in\bar{G}} \mathcal{C}_{\bar{g}}$;

(2) $\bar{G}$ acts on $\mathcal{C}$ (always on the left in this thesis);

(3) for all $\bar{g}, \bar{h} \in \bar{G}$ and for all $c \in \mathcal{C}_{\bar{h}}$ we have ${}^{\bar{g}}c \in \mathcal{C}_{\bar{g}\bar{h}}$.

We denote the identity component of $\mathcal{C}$ by $\mathcal{Z} := \mathcal{C}_1$, which is a $\bar{G}$-acted algebra.

Example 2.1.3. By Miyashita's theorem [25, p.22], we know that the centralizer $C_A(B)$ of B in A is a $\bar{G}$-graded $\bar{G}$-acted algebra, where for all $\bar{h} \in \bar{G}$,

$$C_A(B)_{\bar{h}} = \{a \in A_{\bar{h}} \mid ab = ba, \forall b \in B\},$$

and the action is given by ${}^{\bar{g}}c = u_{\bar{g}} c u_{\bar{g}^{-1}}$, for all $c \in C_A(B)_{\bar{h}}$ and $\bar{g}, \bar{h} \in \bar{G}$. Note that this definition does not depend on the choice of the elements $u_{\bar{g}}$ and that $C_A(B)_1 = Z(B)$ (the center of B) . This example goes back to Dade's work [8] on the Clifford theory of blocks.

Example 2.1.4. If we denote $\bar{C}_G(N) := NC_G(N)/N$, then $\mathcal{O}C_G(N)$ is a strongly $\bar{C}_G(N)$-graded $\bar{G}$-acted algebra and therefore by consequence a $\bar{G}$-graded $\bar{G}$-acted algebra.

Definition 2.1.5. Let $\mathcal{C}$ be a $\bar{G}$-graded $\bar{G}$-acted $\mathcal{O}$-algebra. We say the A is a $\bar{G}$-*graded* $\mathcal{O}$-*algebra over* $\mathcal{C}$ if there is a $\bar{G}$-graded $\bar{G}$-acted algebra homomorphism

$$\zeta : \mathcal{C} \to C_A(B),$$

i.e. for any $\bar{h} \in \bar{G}$ and $c \in \mathcal{C}_{\bar{h}}$, we have $\zeta(c) \in C_A(B)_{\bar{h}}$, and for every $\bar{g} \in \bar{G}$, $\zeta({}^{\bar{g}}c) = {}^{\bar{g}}\zeta(c)$.

Example 2.1.6. In our main framework given in Section 1.4, given also Example 2.1.4, we have that $A := b\mathcal{O}G$ is a $\bar{G}$-graded algebra over the $\bar{G}$-graded $\bar{G}$-acted algebra $\mathcal{C} := \mathcal{O}C_G(N)$, with structural map $\zeta : \mathcal{C} \to C_A(B)$ given by inclusion.

Moreover, if we assume, in addition, that $C_G(N) \subseteq G'$ (as needed in Section 1.5), then $A' := b'\mathcal{O}G'$ is also a $\bar{G}$-graded algebra over $\mathcal{C} := \mathcal{O}C_G(N)$, with structural map $\zeta' : \mathcal{C} \to C_{A'}(B')$ given again by inclusion.

Another important example of a $\bar{G}$-graded $\mathcal{O}$-algebra over a $\bar{G}$-graded $\bar{G}$-acted algebra, is given by the following lemma (published in [32, Lemma 1]):

Lemma 2.1.7. *Let P be a $\bar{G}$-graded A-module. Assume that P is $\bar{G}$-invariant. Let $A' = \mathrm{End}_A(P)^{op}$ be the set of all A-linear endomorphisms of P. Then A' is a $\bar{G}$-graded $\mathcal{O}$-algebra over $C_A(B)$.*

Definition 2.1.8. Let A and A' be two $\bar{G}$-graded crossed products over $\mathcal{C}$, with structure maps ζ and ζ', respectively.

a) We say that $\tilde{M}$ is a $\bar{G}$-*graded* (A, A')-*bimodule over* $\mathcal{C}$ if:

(1) $\tilde{M}$ is an (A, A')-bimodule;

(2) $\tilde{M}$ has a decomposition $\tilde{M} = \bigoplus_{\bar{g}\in\bar{G}} \tilde{M}_{\bar{g}}$ such that $A_{\bar{g}}\tilde{M}_{\bar{x}}A'_{\bar{h}} \subseteq \tilde{M}_{\bar{g}\bar{x}\bar{h}}$, for all $\bar{g}, \bar{x}, \bar{h} \in \bar{G}$;

(3) $\tilde{m}_{\bar{g}} \cdot c = {}^{\bar{g}}c \cdot \tilde{m}_{\bar{g}}$, for all $c \in \mathcal{C}, \tilde{m}_{\bar{g}} \in \tilde{M}_{\bar{g}}, \bar{g} \in \bar{G}$, where $c \cdot \tilde{m} = \zeta(c) \cdot \tilde{m}$ and $\tilde{m} \cdot c = \tilde{m} \cdot \zeta'(c)$, for all $c \in \mathcal{C}, \tilde{m} \in \tilde{M}$.

b) $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$ form a category, which we will denote by $A\text{-}\mathrm{Gr}/\mathcal{C}\text{-}A'$, where the morphisms between $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$ are just homomorphism between $\bar{G}$-graded (A, A')-bimodules.

Remark 2.1.9. Condition (3) of Definition 2.1.8 can be replaced by

(3') $m \cdot c = c \cdot m$, for all $c \in \mathcal{C}$, $m \in \tilde{M}_1$.

An example of a $\bar{G}$-graded bimodule over a $\bar{G}$-graded $\bar{G}$-acted algebra is given by the following proposition (published in [32, Proposition 1]):

Proposition 2.1.10. *Let $\mathcal{C}$ be a $\bar{G}$-graded $\bar{G}$-acted algebra and A a strongly $\bar{G}$-graded $\mathcal{O}$-algebra over $\mathcal{C}$. Let P be a $\bar{G}$-invariant $\bar{G}$-graded A-module. Let $A' = \mathrm{End}_A(P)^{op}$. Then the following statements hold:*

(1) *A' is a $\bar{G}$-graded $\mathcal{O}$-algebra over $\mathcal{C}$;*

(2) *P is a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$.*

2.1.11. We regard A'^{op} as a $\bar{G}$-graded algebra with components $(A'^{\mathrm{op}})_{\bar{g}} = A'^{\mathrm{op}}_{\bar{g}} = A'_{\bar{g}^{-1}}$, $\forall \bar{g} \in \bar{G}$. We denote by "$*$" the multiplication in A'^{op}. We consider the diagonal part of $A \otimes_{\mathcal{C}} A'^{\mathrm{op}}$, which is well-defined:

$$\Delta^{\mathcal{C}} := \Delta(A \otimes_{\mathcal{C}} A'^{\mathrm{op}}) := \bigoplus_{\bar{g} \in \bar{G}} A_{\bar{g}} \otimes_{\mathcal{C}} A'_{\bar{g}^{-1}}.$$

Lemma 2.1.12. (1) *$\Delta^{\mathcal{C}}$ is an $\mathcal{O}$-algebra.*

(2) *$A \otimes_{\mathcal{C}} A'^{\mathrm{op}}$ is a right $\Delta^{\mathcal{C}}$-module and a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$.*

Theorem 2.1.13. [29, Theorem 2.9] *The category of $\Delta^{\mathcal{C}}$-modules and the category of $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$ are equivalent:*

$$\Delta^{\mathcal{C}}\text{-Mod} \underset{(-)_1}{\overset{(A \otimes_{\mathcal{C}} A'^{\mathrm{op}}) \otimes_{\Delta^{\mathcal{C}}} -}{\rightleftarrows}} A\text{-Gr}/\mathcal{C}\text{-}A'$$

Proposition 2.1.14. *The functors*

$$\left(A \otimes_{\mathcal{C}} A'^{\mathrm{op}}\right) \otimes_{\Delta^{\mathcal{C}}} -,\ A \otimes_B -,\ - \otimes_{B'} A' : \Delta^{\mathcal{C}}\text{-Mod} \to A\text{-Gr}/\mathcal{C}\text{-}A'$$

are naturally isomorphic equivalences of categories, and their inverse is taking the 1-component $(-)_1$.

Proposition 2.1.15. *For a $\Delta^{\mathcal{C}}$-module M, we denote $\tilde{M} = (A \otimes_{\mathcal{C}} A'^{\mathrm{op}}) \otimes_{\Delta^{\mathcal{C}}} M \simeq A \otimes_B M \simeq M \otimes_{B'} A'$. Let A'' be a third $\bar{G}$-graded crossed product over $\mathcal{C}$.*

(1) *Let M be a $\Delta(A \otimes_{\mathcal{C}} A'^{\mathrm{op}})$-module and let M' be a $\Delta(A' \otimes_{\mathcal{C}} A''^{\mathrm{op}})$-module. Then $M \otimes_{B'} M'$ is a $\Delta(A \otimes_{\mathcal{C}} A''^{\mathrm{op}})$-module with the multiplication operation defined by*

$$(a_{\bar{g}} \otimes_{\mathcal{C}} a''^{\mathrm{op}}_{\bar{g}})(m \otimes_{B'} m') := (a_{\bar{g}} \otimes_{\mathcal{C}} (u'^{-1}_{\bar{g}})^{\mathrm{op}})m \otimes_{B'} (u'_{\bar{g}} \otimes_{\mathcal{C}} a''^{\mathrm{op}}_{\bar{g}})m',$$

for all $\bar{g} \in \bar{G}$ and for all $a_{\bar{g}} \in A_{\bar{g}}$, $a''^{\mathrm{op}}_{\bar{g}} \in A''^{\mathrm{op}}_{\bar{g}}$, $m \in M$, $m' \in M'$. Moreover, we have the isomorphism

$$\widetilde{M \otimes_{B'} M'} \simeq \tilde{M} \otimes_{A'} \tilde{M}'$$

of $\bar{G}$-graded (A, A'')-bimodules over $\mathcal{C}$.

(2) *Let M be a $\Delta(A' \otimes_{\mathcal{C}} A^{\mathrm{op}})$-module and let M' be a $\Delta(A' \otimes_{\mathcal{C}} A''^{\mathrm{op}})$-module. Then $\mathrm{Hom}_{B'}(M, M')$ is a $\Delta(A \otimes_{\mathcal{C}} A''^{\mathrm{op}})$-module with the following operation:*

$$((a_{\bar{g}} \otimes_{\mathcal{C}} a''^{\mathrm{op}}_{\bar{g}})f)(m) := (u'_{\bar{g}} \otimes_{\mathcal{C}} a''^{\mathrm{op}}_{\bar{g}})f((u'^{-1}_{\bar{g}} \otimes_{\mathcal{C}} a^{\mathrm{op}}_{\bar{g}^{-1}})m),$$

for all $\bar{g} \in \bar{G}$ and for all $a_{\bar{g}} \in A_{\bar{g}}$, $a''^{\mathrm{op}}_{\bar{g}} \in A''^{\mathrm{op}}_{\bar{g}}$, $m \in M$, $f \in \mathrm{Hom}_{B'}(M, M')$. Moreover, we have the isomorphism

$$\mathrm{Hom}_{B'}\widetilde{(M, M')} \simeq \mathrm{Hom}_{A'}(\tilde{M}, \tilde{M}')$$

of $\bar{G}$-graded (A, A'')-bimodules over $\mathcal{C}$.

2.2 Group graded Morita theory over $\mathcal{C}$

In this section, as published in [32], we develop a group graded Morita theory over a $\bar{G}$-graded $\bar{G}$-acted algebra $\mathcal{C}$. We will follow, in its development, the treatment of Morita theory given by C. Faith in 1973 in [12]. Significant here is the already developed graded Morita theory, which started in 1980 when R. Gordon and E. L. Green have characterized graded equivalences in the case of $G = \mathbb{Z}$, in [14]. Furthermore, in 1988 it was observed to work for arbitrary groups G by C. Menini and C. Năstăsescu, in [31]. We will make use of their results under the form given by A. del Río in 1991 in [11] and we will also use the graded Morita theory developed by P. Boisen in 1994 in [4].

In what follows, we construct the notion of a $\bar{G}$-graded Morita context over $\mathcal{C}$ and we will give an appropriate example. In Section 2.2.2 we introduce the notions of graded functors over $\mathcal{C}$ and of graded Morita equivalences over $\mathcal{C}$ and finally we state and prove two Morita-type theorems using the said notions in which we prove that by taking a G-graded bimodule over a G-graded G-acted algebra we obtain a G-graded Morita equivalence over the said G-graded G-acted algebra and that by being given a G-graded Morita equivalence over a G-graded G-acted algebra, we obtain a G-graded bimodule over the said G-graded G-acted algebra, which induces the given G-graded Morita equivalence.

2.2.1 Group graded Morita contexts over $\mathcal{C}$

We start by introducing the notion of a $\bar{G}$-graded Morita context over $\mathcal{C}$, following the treatment given in [12, §12]. Note that some authors [12] use the terminology of a set of pre-equivalence data instead of Morita context.

Definition 2.2.1. Consider the following Morita context:

$$(A, A', \tilde{M}, \tilde{M}', f, g).$$

We call it a $\bar{G}$-graded Morita context over $\mathcal{C}$ if:

(1) A and A' are strongly $\bar{G}$-graded $\mathcal{O}$-algebras over $\mathcal{C}$;

(2) ${}_A\tilde{M}_{A'}$ and ${}_{A'}\tilde{M}'_A$ are $\bar{G}$-graded bimodules over $\mathcal{C}$;

(3) $f : \tilde{M} \otimes_{A'} \tilde{M}' \to A$ and $g : \tilde{M}' \otimes_A \tilde{M} \to A'$ are $\bar{G}$-graded bimodule homomorphisms such that by setting $f(\tilde{m} \otimes \tilde{m}') = \tilde{m}\tilde{m}'$ and $g(\tilde{m}' \otimes \tilde{m}) = \tilde{m}'\tilde{m}$, we have the associative laws:

$$(\tilde{m}\tilde{m}')\tilde{n} = \tilde{m}(\tilde{m}'\tilde{n}) \quad \text{and} \quad (\tilde{m}'\tilde{m})\tilde{n}' = \tilde{m}'(\tilde{m}\tilde{n}'),$$

for all $\tilde{m}, \tilde{n} \in \tilde{M}$, $\tilde{m}', \tilde{n}' \in \tilde{M}'$.

If f and g are isomorphisms, then $(A, A', \tilde{M}, \tilde{M}', f, g)$ is called a surjective $\bar{G}$-graded Morita context over $\mathcal{C}$.

Note that, if f and g are isomorphisms, then $(A, A', \tilde{M}, \tilde{M}', f, g)$ is called in [12] a set of $\bar{G}$-graded equivalence data over $\mathcal{C}$.

As an example of a $\bar{G}$-graded Morita context over $\mathcal{C}$, we have the following proposition which arises from [12, Proposition 12.6].

Proposition 2.2.2. *Let A be a strongly $\bar{G}$-graded $\mathcal{O}$-algebra over $\mathcal{C}$, let P be a $\bar{G}$-invariant $\bar{G}$-graded A-module, let $A' = \mathrm{End}_A(P)^{op}$ and let $P^* := \mathrm{Hom}_A(P, A)$ be the A-dual of P. Then*

$$(A, A', P, P^*, (\cdot,\cdot), [\cdot,\cdot])$$

is a $\bar{G}$-graded Morita context over $\mathcal{C}$, where $(\cdot,\cdot)$ is a $\bar{G}$-graded (A, A)-homomorphism, called the evaluation map, defined by:

$$\begin{aligned}&(\cdot,\cdot) : P \otimes_{A'} P^* \to A,\\ &x \otimes \varphi \mapsto \varphi(x), \ \textit{for all } \varphi \in P^*, x \in P,\end{aligned}$$

and where $[\cdot,\cdot]$ is a $\bar{G}$-graded (A', A')-homomorphism defined by:

$$\begin{aligned}&[\cdot,\cdot] : P^* \otimes_A P \to A',\\ &\varphi \otimes x \mapsto [\varphi, x], \ \textit{for all } \varphi \in P^*, x \in P,\end{aligned}$$

where for every $\varphi \in P^*$ and $x \in P$, $[\varphi, x]$ is an element of A' such that

$$y[\varphi, x] = \varphi(y) \cdot x, \ \textit{for all } y \in P.$$

If $(A, A', \tilde{M}, \tilde{M}', f, g)$ is a surjective $\bar{G}$-graded Morita context over $\mathcal{C}$, then by Proposition 12.7 of [12], we have that A' is isomorphic to $\mathrm{End}_A(\tilde{M})^{\mathrm{op}}$ and we have a bimodule isomorphism between $\tilde{M}'$ and $\tilde{M}^* = \mathrm{Hom}_A(\tilde{M}, A)$. Henceforth, in this situation, the example given by Proposition 2.2.2 is essentially unique up to an isomorphism.

Given Corollary 12.8 of [12], the example given by Proposition 2.2.2 is a surjective $\bar{G}$-graded Morita context over $\mathcal{C}$ if and only if ${}_AP$ is a progenerator.

2.2.2 Group graded Morita theorems over $\mathcal{C}$

We denote by A and A' two strongly $\bar{G}$-graded $\mathcal{O}$-algebras over $\mathcal{C}$ (with identity components $B := A_1$ and $B' := A'_1$), each endowed with a $\bar{G}$-graded $\bar{G}$-acted algebra homomorphism $\zeta : \mathcal{C} \to C_A(B)$ and $\zeta' : \mathcal{C} \to C_{A'}(B')$ respectively. According to [11] we have the following definitions:

Definition 2.2.3. (1) We say that the functor $\tilde{\mathcal{F}} : A\text{-Gr} \to A'\text{-Gr}$ is G-graded if for every $g \in \bar{G}$, $\tilde{\mathcal{F}}$ commutes with the g-suspension functor, i.e. $\tilde{\mathcal{F}} \circ T_g^A$ is naturally isomorphic to $T_g^{A'} \circ \tilde{\mathcal{F}}$;

(2) We say that A and A' are $\bar{G}$-graded Morita equivalent if there is a $\bar{G}$-graded equivalence: $\tilde{\mathcal{F}} : A\text{-Gr} \to A'\text{-Gr}$.

Assume that A and A' are $\bar{G}$-graded Morita equivalent. Therefore, we can consider the G-graded functors:

$$A\text{-Gr} \underset{\tilde{\mathcal{G}}}{\overset{\tilde{\mathcal{F}}}{\rightleftarrows}} A'\text{-Gr}$$

which give a $\bar{G}$-graded Morita equivalence between A and A'. By Gordon and Green's result [11, Corollary 10], this is equivalent to the existence of a Morita equivalence between A and A' given by the following functors:

$$A\text{-Mod} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} A'\text{-Mod};$$

such that the following diagram is commutative:

$$\begin{tikzcd} A\text{-Gr} \arrow[r, shift left, "\tilde{\mathcal{F}}"] \arrow[d, "u"'] & A'\text{-Gr} \arrow[l, shift left, "\tilde{\mathcal{G}}"] \arrow[d, "u'"] \\ A\text{-Mod} \arrow[r, shift left, "\mathcal{F}"] & A'\text{-Mod} \arrow[l, shift left, "\mathcal{G}"] \end{tikzcd}$$

in the sense that:

$$U' \circ \tilde{F} = F \circ U, \qquad U \circ \tilde{G} = G \circ U',$$

where U is the forgetful functor from A-Gr to A-Mod and U' is the forgetful functor from A'-Gr to A'-Mod.

Lemma 2.2.4. *If $\tilde{P}$ is a $\bar{G}$-graded A-module, then $\tilde{P}$ and $\tilde{\mathcal{F}}(\tilde{P})$ have the same stabilizer in G.*

Consider $\tilde{P}$ and $\tilde{Q}$ two $\bar{G}$-graded A-modules. We have the following morphism:

$$\mathrm{Hom}_A(\tilde{P}, \tilde{Q}) \xrightarrow{\quad\tilde{\mathcal{F}}\quad} \mathrm{Hom}_{A'}(\tilde{\mathcal{F}}(\tilde{P}), \tilde{\mathcal{F}}(\tilde{Q})). \tag{$*$}$$

By following the proofs of Lemma 2.1.7 and Proposition 2.1.10, we have a G-graded homomorphism from $\mathcal{C}$ to $\mathrm{End}_A(\tilde{P})^{\mathrm{op}}$ (the composition between the structure homomorphism $\zeta : \mathcal{C} \to C_A(B)$ and the morphism $\theta : C_A(B) \to \mathrm{End}_A(\tilde{P})^{\mathrm{op}}$ from Lemma 1.3.1) and that $\tilde{P}$ is a $\bar{G}$-graded $(A, \mathrm{End}_A(\tilde{P})^{\mathrm{op}})$-bimodule. Then, by the restriction of scalars we obtain that $\tilde{P}$ is a right $\mathcal{C}$-module. Analogously $\tilde{Q}$, $\tilde{\mathcal{F}}(\tilde{P})$ and $\tilde{\mathcal{F}}(\tilde{Q})$ are also right $\mathcal{C}$-modules, thus $\mathrm{Hom}_A(\tilde{P}, \tilde{Q})$ and $\mathrm{Hom}_A(\tilde{\mathcal{F}}(\tilde{P}), \tilde{\mathcal{F}}(\tilde{Q}))$ are $\bar{G}$-graded $(\mathcal{C}, \mathcal{C})$-bimodules. This allows us to state the following definition:

Definition 2.2.5. (1) We say that the functor $\tilde{\mathcal{F}}$ is over $\mathcal{C}$ if the morphism $\tilde{\mathcal{F}}$ (see ($*$)) is a morphism of $\bar{G}$-graded $(\mathcal{C}, \mathcal{C})$-bimodules;

(2) We say that A and A' are $\bar{G}$-graded Morita equivalent over $\mathcal{C}$ if there is a $\bar{G}$-graded equivalence over $\mathcal{C}$: $\tilde{\mathcal{F}} : A\text{-Gr} \to A'\text{-Gr}$.

Theorem 2.2.6 (Graded Morita I over $\mathcal{C}$ - [32, Theorem 1])**.**
Let $(A, A', \tilde{M}, \tilde{M}', f, g)$ be a surjective $\bar{G}$-graded Morita context over $\mathcal{C}$. Then the functors

$$\begin{aligned} \tilde{M}' \otimes_A - &: A\text{-Gr} \to A'\text{-Gr} \\ \tilde{M} \otimes_{A'} - &: A'\text{-Gr} \to A\text{-Gr} \end{aligned}$$

are inverse $\bar{G}$-graded equivalences over $\mathcal{C}$.

By Proposition 2.2.2 and the observations made in Section 2.2.1, the following corollary is straightforward.

Corollary 2.2.7. *Let P be a $\bar{G}$-invariant $\bar{G}$-graded A-module and $A' = \mathrm{End}_A(P)^{\mathrm{op}}$. If ${}_AP$ is a progenerator, then $P \otimes_{A'} -$ is a $\bar{G}$-graded Morita equivalence over $\mathcal{C}$ between A'-Gr and A-Gr, with $P^* \otimes_A -$ as its inverse.*

Theorem 2.2.8 (Graded Morita II over $\mathcal{C}$ - [32, Theorem 2]). *Assume that A and A' are $\bar{\mathsf{G}}$-graded Morita equivalent over $\mathcal{C}$ and let*

$$\mathsf{A}\text{-Gr} \underset{\tilde{\mathcal{G}}}{\overset{\tilde{\mathcal{F}}}{\rightleftarrows}} \mathsf{A}'\text{-Gr}$$

be inverse $\bar{\mathsf{G}}$-graded equivalences over $\mathcal{C}$. Then this equivalence is given by the following $\bar{\mathsf{G}}$-graded bimodules over $\mathcal{C}$: $\mathsf{P} = \tilde{\mathcal{F}}(\mathsf{A})$ and $\mathsf{Q} = \tilde{\mathcal{G}}(\mathsf{A}')$. More exactly, P is a $\bar{\mathsf{G}}$-graded $(\mathsf{A}', \mathsf{A})$-bimodule over $\mathcal{C}$, Q is a $\bar{\mathsf{G}}$-graded $(\mathsf{A}, \mathsf{A}')$-bimodule over $\mathcal{C}$ and the following natural equivalences of functors hold:

$$\tilde{\mathcal{F}} \simeq \mathsf{P} \otimes_{\mathsf{A}} - \quad \textit{and} \quad \tilde{\mathcal{G}} \simeq \mathsf{Q} \otimes_{\mathsf{A}'} -.$$

Now, given both Morita theorems (Theorems 2.2.6 and 2.2.8) and Proposition 2.1.15 we can state the following equivalent definition of a group graded Morita equivalence over $\mathcal{C}$.

Definition 2.2.9. Let $\tilde{\mathsf{M}}$ be a $\bar{\mathsf{G}}$-graded $(\mathsf{A}, \mathsf{A}')$-bimodule over $\mathcal{C}$. Clearly, the A-dual $\tilde{\mathsf{M}}^* = \mathrm{Hom}_{\mathsf{A}}(\tilde{\mathsf{M}}, \mathsf{A})$ of $\tilde{\mathsf{M}}$ is a $\bar{\mathsf{G}}$-graded $(\mathsf{A}', \mathsf{A})$-bimodule over $\mathcal{C}$. We say that $\tilde{\mathsf{M}}$ induces a $\bar{\mathsf{G}}$-*graded Morita equivalence over* $\mathcal{C}$ between A and A', if $\tilde{\mathsf{M}} \otimes_{\mathsf{A}'} \tilde{\mathsf{M}}^* \cong \mathsf{A}$ as $\bar{\mathsf{G}}$-graded (A, A)-bimodules over $\mathcal{C}$ and $\tilde{\mathsf{M}}^* \otimes_{\mathsf{A}} \tilde{\mathsf{M}} \cong \mathsf{A}'$ as $\bar{\mathsf{G}}$-graded $(\mathsf{A}', \mathsf{A}')$-bimodules over $\mathcal{C}$.

2.2.3 Group graded Morita equivalences over $\mathcal{C}$

We continue with the notations of the preceding section. In this section, as published in [29, §3], we extend [25, Theorem 5.1.2] to the case of $\bar{\mathsf{G}}$-graded Morita equivalences over $\mathcal{C}$, in order to provide a link between them and Morita equivalences (Theorem 2.2.10). Moreover, continuing on the work done in Section 1.3, Proposition 2.2.13 provides the reasoning behind our choice of the definition of the central-order relation between module triples (see Definition 3.2.3).

Theorem 2.2.10. [29, Theorem 3.3] *Assume that the $(\mathsf{B}, \mathsf{B}')$-bimodule M and its B-dual $\mathsf{M}^* = \mathrm{Hom}_{\mathsf{B}}(\mathsf{M}, \mathsf{B})$ induce a Morita equivalence between B and B':*

$$\mathsf{B}\text{-Mod} \underset{\mathsf{M}\otimes_{\mathsf{B}'}-}{\overset{\mathsf{M}^*\otimes_{\mathsf{B}}-}{\rightleftarrows}} \mathsf{B}'\text{-Mod}$$

If M extends to a $\Delta^{\mathcal{C}}$-module, then we have the following:

(1) M^* *becomes a $\Delta(\mathsf{A}' \otimes_{\mathcal{C}} \mathsf{A}^{\mathrm{op}})$-module;*

(2) $\tilde{\mathsf{M}} := (\mathsf{A} \otimes_{\mathcal{C}} \mathsf{A}'^{\mathrm{op}}) \otimes_{\Delta^{\mathcal{C}}} \mathsf{M}$ *is a $\bar{\mathsf{G}}$-graded $(\mathsf{A}, \mathsf{A}')$-bimodule over $\mathcal{C}$, $\tilde{\mathsf{M}}^* \simeq (\mathsf{A}' \otimes_{\mathcal{C}} \mathsf{A}^{\mathrm{op}}) \otimes_{\Delta(\mathsf{A}' \otimes_{\mathcal{C}} \mathsf{A}^{\mathrm{op}})} \mathsf{M}^*$ as $\bar{\mathsf{G}}$-graded $(\mathsf{A}', \mathsf{A})$-bimodules over $\mathcal{C}$, and they induce a $\bar{\mathsf{G}}$-graded Morita equivalence over $\mathcal{C}$ between A and A'.*

Remark 2.2.11. $\bar{\mathsf{G}}$-graded Morita equivalence over $\mathcal{C}$ can be truncated [25, Corollary 5.1.4.]. In our case, consider $\tilde{\mathsf{M}}$ a $\bar{\mathsf{G}}$-graded $(\mathsf{A}, \mathsf{A}')$-bimodule over $\mathcal{C}$ and $\tilde{\mathsf{M}}^*$ its A-dual. Assume that $\tilde{\mathsf{M}}$ and $\tilde{\mathsf{M}}^*$ induce a $\bar{\mathsf{G}}$-graded Morita equivalence over $\mathcal{C}$ between A and A'. Let $\bar{\mathsf{H}}$ be a subgroup of $\bar{\mathsf{G}}$. Then $\tilde{\mathsf{M}}_{\bar{\mathsf{H}}} := \bigoplus_{\bar{h} \in \bar{\mathsf{H}}} \tilde{\mathsf{M}}_{\bar{h}}$ and $\tilde{\mathsf{M}}^*_{\bar{\mathsf{H}}}$ induce a $\bar{\mathsf{H}}$-graded Morita equivalence over $\mathcal{C}_{\bar{\mathsf{H}}}$ between $\mathsf{A}_{\bar{\mathsf{H}}}$ and $\mathsf{A}'_{\bar{\mathsf{H}}}$.

2.2.12. If U is a B-module, we denote by

$$E(U) := \mathrm{End}_A(A \otimes_B U)^{\mathrm{op}}$$

the $\bar{G}$-graded endomorphism algebra of the A-module induced from U.

Proposition 2.2.13. *Assume that $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence over $\mathcal{C}$ between A and A'. Let U be a B-module and let $U' = M^* \otimes_B U$ be the B'-module corresponding to U under the given equivalence. Then there is a commutative diagram:*

$$\begin{tikzcd}
E(U) \arrow[r, "\sim"] & E(U') \\
C_A(B) \arrow[u] \arrow[r, "\sim"] & C_{A'}(B') \arrow[u] \\
\mathcal{C} \arrow[u] \arrow[r, equal, "\mathrm{id}_{\mathcal{C}}"] & \mathcal{C}. \arrow[u]
\end{tikzcd}$$

2.3 Updated main framework

2.3.1. We recall our main framework given in Section 1.4: Let G' be a subgroup of G and N' a normal subgroup of G'. We assume that $N' = G' \cap N$ and $G = G'N$, hence

$$\bar{G} := G/N \simeq G'/(G' \cap N) = G'/N'.$$

Let $b \in Z(\mathcal{O}N)$ and $b' \in Z(\mathcal{O}N')$ be two $\bar{G}$-invariant block idempotents. We denote

$$A := b\mathcal{O}G, \qquad A' := b'\mathcal{O}G', \qquad B := b\mathcal{O}N, \qquad B' := b'\mathcal{O}N',$$

hence A and A' are $\bar{G}$-graded crossed products, with 1-components B and B' respectively.

2.3.2. Additionally, we assume that A and A' are $\bar{G}$-graded algebras over the $\bar{G}$-graded $\bar{G}$-acted $\mathcal{O}$-algebra $\mathcal{C}$, with structural maps $\zeta : \mathcal{C} \to C_A(B)$ and $\zeta' : \mathcal{C} \to C_{A'}(B')$, as in Definition 2.1.5.

For instance, Example 2.1.6 tells us that, if we assume, in addition, that $C_G(N) \subseteq G'$, then we may take $\mathcal{C} := \mathcal{O}C_G(N)$, which is a $\bar{G}$-graded $\bar{G}$-acted algebra, and we have that A and A' are $\bar{G}$-graded algebras over $\mathcal{C} := \mathcal{O}C_G(N)$ with structural maps $\zeta : \mathcal{C} \to C_A(B)$ and $\zeta' : \mathcal{C} \to C_{A'}(B')$ given by inclusion.

2.4 The butterfly theorem for $\bar{G}$-graded Morita equivalences over $\mathcal{C}$

2.4.1. As published in [29, §3], we will give a version for Morita equivalences over $\mathcal{C}$ of the so-called "butterfly theorem" [38, Theorem 2.16], based on Theorem 1.5.2. We start by adapting Proposition 1.5.1, henceforth, we will work in the context of our updated framework (Section 2.3). Additionally, we assume that $C_G(N) \subseteq G'$, and we denote

$\bar{C}_G(N) := NC_G(N)/N$ and $\mathcal{C} := \mathcal{O}C_G(N)$. We consider the algebras:

$$\begin{tikzcd} A := b\mathcal{O}G \arrow[r, no head] \arrow[d, no head] & A' := b'\mathcal{O}G' \arrow[d, no head] \\ C := b\mathcal{O}NC_G(N) \arrow[r, no head] \arrow[d, no head] & C' := b'\mathcal{O}N'C_G(N) \arrow[d, no head] \\ B := b\mathcal{O}N \arrow[r, no head] & B' := b'\mathcal{O}N'. \end{tikzcd}$$

Proposition 2.4.2. *Let $\mathcal{C} = \mathcal{O}C_G(N)$, and assume that:*

(1) $C_G(N) \subseteq G'$;

(2) M *induces a Morita equivalence between* B *and* B';

(3) $zm = mz$, *for all* $m \in M$ *and* $z \in Z(N)$.

Then there is a $\bar{C}_G(N)$*-graded Morita equivalence between* C *and* C' *over* $\mathcal{C}$

$$\begin{tikzcd} A := b\mathcal{O}G \arrow[d, no head] & A' := b'\mathcal{O}G' \arrow[d, no head] \\ C := b\mathcal{O}NC_G(N) \arrow[r, no head, "{}_C\hat{M}_{C'}", "\sim"'] \arrow[d, no head] & C' := b'\mathcal{O}N'C_G(N) \arrow[d, no head] \\ B := b\mathcal{O}N \arrow[r, no head, "{}_BM_{B'}", "\sim"'] & B' := b'\mathcal{O}N', \end{tikzcd}$$

induced by the $\bar{C}_G(N)$*-graded* (C, C')*-bimodule over* $\mathcal{C}$:

$$\hat{M} := C \otimes_B M \simeq M \otimes_{B'} C' \simeq (C \otimes_{\mathcal{C}} C'^{op}) \otimes_{\Delta(C \otimes_{\mathcal{C}} C'^{op})} M.$$

Theorem 2.4.3 (Butterfly theorem for group graded Morita equivalences over $\mathcal{C}$ - [29])**.** *Let $\hat{G}$ be another group with normal subgroup N, such that the block b is also $\hat{G}$-invariant. Let $\mathcal{C} = \mathcal{O}C_G(N)$. Assume that:*

(1) $C_G(N) \subseteq G'$,

(2) $\tilde{M}$ *induces a* $\bar{G}$*-graded Morita equivalence over* $\mathcal{C}$ *between* A *and* A';

(3) *the conjugation maps* $\varepsilon : G \to \mathrm{Aut}(N)$ *and* $\hat{\varepsilon} : \hat{G} \to \mathrm{Aut}(N)$ *satisfy* $\varepsilon(G) = \hat{\varepsilon}(\hat{G})$.

Denote $\hat{G}' = \hat{\varepsilon}^{-1}(\varepsilon(G'))$. *Then there is a* $\hat{G}/N$*-graded Morita equivalence over* $\hat{\mathcal{C}} := \mathcal{O}C_{\hat{G}}(N)$ *between* $\hat{A} := b\mathcal{O}\hat{G}$ *and* $\hat{A}' := b'\mathcal{O}\hat{G}'$.

2.4.4. Condition (1) of Proposition 2.4.2 may be replaced with the assumption that the (B, B)-bimodule M is $\bar{G}$-invariant, that is, $A_{\bar{g}} \otimes_B M \otimes_{B'} A'_{\bar{g}^{-1}} \simeq M$ as (B, B')-bimodules, for all $\bar{g} \in \bar{G}$, to get a slightly stronger statement.

Let $\bar{G}[b] = G[b]/N$ be the stabilizer of B as a (B, B)-bimodule, that is, $\bar{G}[b]$ is the largest subgroup $\bar{H}$ of $\bar{G}$ such that $C_A(B)_{\bar{H}}$ is a crossed product (see [8, 2.9]). Then $\bar{G}[b]$ is a normal subgroup of $\bar{G}$, and in particular, we have that $C_G(N) \subseteq G[b]$.

Corollary 2.4.5. *Assume that the* (B, B)*-bimodule* M *is* $\bar{G}$*-invariant. Assume also that* $zm = mz$, *for all* $m \in M$ *and* $z \in Z(N)$. *Then* $\bar{G}[b] = \bar{G}[b']$ (*hence* $C_G(N) \subseteq G'$), *and there is a* $\bar{G}[b]$*-graded Morita equivalence over* $\mathcal{C} := \mathcal{O}C_G(N)$ *between* $A_{\bar{G}[b]}$ *and* $A'_{\bar{G}[b]}$.

Remark 2.4.6. On the other hand, still without condition (1) of Proposition 2.4.2, assume that $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence between A and A' (so in particular, M is $\bar{G}$-invariant). Then, by [25, Theorem 5.1.8], $C_A(B) \simeq C_{A'}(B')$ and $\bar{G}[b] = \bar{G}[b']$. If, in addition, $zm = mz$, for all $m \in M$ and $z \in Z(N)$, then, by Corollary 2.4.5, $\tilde{M}$ is a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C} := \mathcal{O}C_G(N)$ (in fact, even over $C_A(B)_{\bar{G}[b]}$).

2.5 Scott modules

Koshitani and Lassueur constructed in [19] and [20] Morita equivalences induced by certain Scott modules. We show here that their constructions can be extended to obtain group graded Morita equivalences over $\mathcal{C} = \mathcal{O}C_G(N)$. This section was published in [29, §4].

2.5.1. Let Q be a Sylow p-subgroup of N, let $\delta(Q) = \{(q, q) \in Q \times Q \mid q \in Q\}$ be the diagonal subgroup of $Q \times Q$, let $G' = N_G(Q)$ and $N' = G' \cap N$. In this situation, we have that $C_G(N) \subseteq G'$, and let $\mathcal{C} = \mathcal{O}C_G(N)$ and $Z = Z(N)$. Denote also

$$K = \{(g, g') \mid \bar{g} = \bar{g}' \text{ in } \bar{G} = G/N \simeq G'/N'\}.$$

Let $b \in Z(\mathcal{O}N)$ and $b' \in Z(\mathcal{O}N')$ be the principal block idempotents, and denote

$$A := b\mathcal{O}G, \qquad A' := b'\mathcal{O}G', \qquad B := b\mathcal{O}N, \qquad B' := b'\mathcal{O}N'.$$

Recall that for a subgroup $H \leq G$, the (Alperin)-Scott $\mathcal{O}G$-module with respect to H (denoted by $\mathrm{Sc}(G, H)$) is the unique indecomposable direct summand of the induced module $\mathrm{Ind}_H^G \mathcal{O}_H = \mathcal{O}G \otimes_{\mathcal{O}H} \mathcal{O}_H$ which contains $\mathcal{O}G$ in its socle. Here $\mathcal{O}_H$ represents the trivial $\mathcal{O}H$-module.

As in [19] and [20], we consider the Scott module $\mathrm{Sc}(N \times N', \delta(Q))$, and we refer to [34, §4.8] for the properties of Scott modules.

Proposition 2.5.2. *With the above notations, assume that p does not divide the order of $\bar{G}$ and denote $M = \mathrm{Sc}(N \times N', \delta(Q))$. Then*

$$M \simeq \mathrm{Res}^K_{N \times N'} \mathrm{Sc}(K, \delta(Q)).$$

In particular, M may be regarded as a $\Delta(A \otimes A'^{\mathrm{op}})$-module, and moreover, M may be chosen such that $\tilde{M} := A \otimes_B M$ is a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C} = \mathcal{O}C_G(N)$ between A and A'.

2.6 Rickard equivalences over $\mathcal{C}$

By using the remarks made in [25, §5.2.1], we may extend the results of Section 2.2.3 to the case of Rickard equivalences. We keep the notations and assumptions of our framework from Section 2.3.1, and we use $\mathcal{H}^b$ to denote a bounded homotopy category. Note that by a Rickard equivalence, we mean an equivalence between the bounded chain homotopy categories $\mathcal{H}^b(A)$ and $\mathcal{H}^b(A')$ induced by a split endomorphism tilting complex, as presented by Rickard in [21, §9.2.2]. In this case it is essential that A and A' are symmetric algebras. This section was published in [29, §5].

2.6.1. Let $\tilde{M}$ be a bounded complex of $\bar{G}$-graded (A, A')-bimodules, with 1-component M, which is a bounded complex of $\Delta(A \otimes_{\mathcal{O}} A'^{\mathrm{op}})$-modules. Recall that $\tilde{M}$ and its dual $\tilde{M}^*$ induce a $\bar{G}$-graded Rickard equivalence between A and A', if there are isomorphisms

$$\tilde{\varphi} : \tilde{M}^* \otimes_A \tilde{M} \to A' \qquad \text{and} \qquad \tilde{\psi} : \tilde{M} \otimes_{A'} \tilde{M}^* \to A,$$

in the appropriate bounded homotopy categories of $\bar{G}$-graded bimodules.

We say that this equivalence is over $\mathcal{C}$ if $\tilde{M}$ is a complex of $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$.

2.6.2. It is easy to see that Remark 2.2.11, Proposition 2.2.13 and Proposition 2.4.2 still hold by replacing "Morita equivalence" with "Rickard equivalence" (and "modules" with "bounded complexes of modules"). On the other hand, according to [25, Theorem 5.2.5], in Theorem 2.2.10 we have to assume in addition that p does not divide the order of $\bar{G}$. Corollary 2.4.5 and Remark 2.4.6 can also be adapted to this situation.

Consequently, we have the following Rickard equivalence variant of the Butterfly Theorem (Theorem 2.4.3).

Theorem 2.6.3. *Let $\hat{G}$ be another group with normal subgroup N, such that the block b is also $\hat{G}$-invariant. Let $\mathcal{C} = \mathcal{O}C_G(N)$. Assume that:*

1. $C_G(N) \subseteq G'$,
2. *$\tilde{M}$ induces a $\bar{G}$-graded Rickard equivalence over $\mathcal{C}$ between A and A';*
3. *the conjugation maps $\varepsilon : G \to \mathrm{Aut}(N)$ and $\hat{\varepsilon} : \hat{G} \to \mathrm{Aut}(N)$ satisfy $\varepsilon(G) = \hat{\varepsilon}(\hat{G})$;*
4. *p does not divide the order of $\bar{G}$.*

Denote $\hat{G}' = \hat{\varepsilon}^{-1}(\varepsilon(G'))$. Then there is a $\hat{G}/N$-graded Rickard equivalence over $\hat{\mathcal{C}} := \mathcal{O}C_{\hat{G}}(N)$ between $\hat{A} := b\mathcal{O}\hat{G}$ and $\hat{A}' := b'\mathcal{O}\hat{G}'$.

Chapter 3

Character triples and module triples

In this chapter, we introduce the concept of a module triple and we show that the relations $\geq$ and $\geq_c$ given in [38, Definition 2.1.] and [38, Definition 2.7.] are consequences of Rickard equivalences over $\mathcal{C}$ between such module triples. Additionally, in Definition 3.3.1 we also provide a module triple version of the relation $\geq_b$ (see [38, Definition 4.2]), and we prove in Proposition 3.3.6 that this too is a consequence of a special type of group graded derived equivalences which is compatible in a certain sense with the Brauer map. Finally, note that our approach to character triples is different from that of Turull [39]. Results of this chapter are published in [29] and [30].

3.1 Module triples

3.1.1. Let $\mathcal{K}$ be a field, let G be a finite group, and let N be a normal subgroup of G.
We denote by $\mathrm{Irr}_{\mathcal{K}}(G)$ the set of all irreducible $\mathcal{K}$-valued characters of G.
It is clear that the group G acts on $\mathrm{Irr}_{\mathcal{K}}N$: for $\theta \in \mathrm{Irr}_{\mathcal{K}}N$ and $g \in G$ we have:

$$\begin{aligned} &{}^{g}\theta : N \to \mathcal{K} \\ &{}^{g}\theta(n) = \theta\left(gng^{-1}\right), \forall n \in N. \end{aligned}$$

We say that θ is G-*invariant* if ${}^{g}\theta = \theta$ for all $g \in G$.
We recall the definition of a character triple ([18, p.186], [38, Definition 1.6]):

Definition 3.1.2. Let G be a finite group, let N be a normal subgroup of G and let $\theta \in \mathrm{Irr}_{\mathcal{K}}N$. We say that (G, N, θ) is *a character triple* if θ is G-invariant.

We know that $\theta \in \mathrm{Irr}_{\mathcal{K}}N$ is a character associated to a simple $\mathcal{K}N$-module V and that θ determines the isomorphism class of V.

3.1.3. Let $\mathcal{K}$ be a field and V a $\mathcal{K}N$-module. Similarly, for a $g \in G$ we can consider the g-*conjugate* of V:

$${}^{g}V = \mathcal{K}Ng \otimes_{\mathcal{K}N} V.$$

Definition 3.1.4. We say that a $\mathcal{K}N$-module V is G-*invariant* (or G-*stable*) if V is isomorphic to ${}^{g}V$ as $\mathcal{K}N$-modules, for all $g \in G$.

Remark 3.1.5. If $g \in N$, then V is isomorphic to ${}^{g}V$ as $\mathcal{K}N$-modules.

Thus, N acts trivially on the isomorphism classes of $\mathcal{K}N$-modules. Moreover, we have that for all $g \in G$ and $n \in N$:

$${}^{gn}V \simeq {}^{g}({}^{n}V) \simeq {}^{g}V,$$

so in fact $G/N := \bar{G}$ acts on the isomorphism classes of $\mathcal{K}N$-modules.

3.1.6. We continue in the previously introduced main framework (Section 2.3). In addition, let V be a G-invariant simple $\mathcal{K}B$-module, and let V' be a G-invariant simple $\mathcal{K}B'$-module, where

$$\mathcal{K}B = \mathcal{K} \otimes_{\mathcal{O}} B = (1 \otimes b)\mathcal{K}N \qquad \text{and} \qquad \mathcal{K}B' = \mathcal{K} \otimes_{\mathcal{O}} B' = (1 \otimes b')\mathcal{K}N'.$$

Let $\theta \in \mathrm{Irr}_{\mathcal{K}}(\mathsf{B})$ be a G-invariant irreducible character associated to V, and let $\theta \in \mathrm{Irr}_{\mathcal{K}}(\mathsf{B}')$ be a G'-invariant irreducible character associated to V'. Thus, $(\mathsf{G},\mathsf{N},\theta)$ and $(\mathsf{G}',\mathsf{N}',\theta')$ are character triples.

We can now define the notion of a module triple, a correspondent to the notion of a character triple, but in terms of modules and not of characters.

Definition 3.1.7. We say that $(\mathsf{A},\mathsf{B},\mathsf{V})$ is a *module triple*, and we will consider its endomorphism algebra

$$\mathsf{E}(\mathsf{V}) := \mathrm{End}_{\mathcal{K}\mathsf{A}}\left(\mathcal{K}\mathsf{A} \otimes_{\mathcal{K}\mathsf{B}} \mathsf{V}\right)^{\mathrm{op}}.$$

Because we assumed that $\mathcal{K}$ contains all the unity roots of order $|\mathsf{G}|$ (see Section 1.1), $\mathsf{E}(\mathsf{V})$ is a twisted group algebra of the form $\mathcal{K}_\alpha\bar{\mathsf{G}}$, with $\alpha \in \mathsf{Z}^2(\bar{\mathsf{G}},\mathcal{K}^\times)$. We know that the class $[\alpha] \in \mathsf{H}^2(\bar{\mathsf{G}},\mathcal{K}^\times)$ depends only on the isomorphism class of V, thus, in fact, $[\alpha]$ is determined by θ.

3.2 First-order and central-order relations

We continue with the notations and assumptions given in the previous section. The following notions and results have been published in [29, §6].

Proposition 3.2.1. *Let $\Delta(\mathsf{V}) := \Delta(\mathcal{K}\mathsf{G} \otimes \mathsf{E}(\mathsf{V})^{\mathrm{op}})$. Then $\Delta(\mathsf{V})$ is isomorphic to $\mathcal{K}_{\mathrm{Inf}_{\bar{\mathsf{G}}}^{\mathsf{G}}\alpha}\mathsf{G}$ as $\bar{\mathsf{G}}$-graded algebras. In particular, the $\mathcal{K}\mathsf{N}$-module structure of V extends to a $\mathcal{K}_{\mathrm{Inf}_{\bar{\mathsf{G}}}^{\mathsf{G}}\alpha}\mathsf{G}$-module structure.*

Remark 3.2.2. The $\Delta(\mathsf{V})$-module structure of V gives rise to the projective $\mathcal{K}$-representation of G associated to θ. Two projective representations P and P' are similar if and only if $\mathsf{E}(\mathsf{V}) = \mathcal{K}_\alpha\bar{\mathsf{G}}$ and $\mathsf{E}(\mathsf{V}') = \mathcal{K}_{\alpha'}\bar{\mathsf{G}}$ are isomorphic as $\bar{\mathsf{G}}$-graded algebras, or if and only if $[\alpha] = [\alpha']$ in $\mathsf{H}^2(\bar{\mathsf{G}},\mathcal{K}^\times)$. This holds if and only if the $\Delta(\mathsf{V})$-module V is isomorphic to the $\Delta(\mathsf{V}')$-module V', via the isomorphism as $\Delta(\mathsf{V}) \simeq \Delta(\mathsf{V}')$ of $\bar{\mathsf{G}}$-graded algebras.

Now, we can formulate a version in terms of modules of the first-order and central-order relations between character triples.

Definition 3.2.3. Let $(\mathsf{A},\mathsf{B},\mathsf{V})$ and $(\mathsf{A}',\mathsf{B}',\mathsf{V}')$ be two module triples.

a) We write $(\mathsf{A},\mathsf{B},\mathsf{V}) \geq (\mathsf{A}',\mathsf{B}',\mathsf{V}')$ if

(1) $\mathsf{G} = \mathsf{N}\mathsf{G}'$ and $\mathsf{N}' = \mathsf{N} \cap \mathsf{G}'$;

(2) there exists a $\bar{\mathsf{G}}$-graded algebra isomorphism

$$\mathsf{E}(\mathsf{V}) = \mathrm{End}_{\mathcal{K}\mathsf{A}}(\mathcal{K}\mathsf{A} \otimes_{\mathcal{K}\mathsf{B}} \mathsf{V})^{\mathrm{op}} \to \mathsf{E}(\mathsf{V}') = \mathrm{End}_{\mathcal{K}\mathsf{A}'}(\mathcal{K}\mathsf{A}' \otimes_{\mathcal{K}\mathsf{B}'} \mathsf{V}')^{\mathrm{op}}.$$

b) We write $(\mathsf{A},\mathsf{B},\mathsf{V}) \geq_c (\mathsf{A}',\mathsf{B}',\mathsf{V}')$ if

(1) $\mathsf{G} = \mathsf{G}'\mathsf{N}$, $\mathsf{N}' = \mathsf{N} \cap \mathsf{G}'$

(2) $\mathsf{C}_{\mathsf{G}}(\mathsf{N}) \subseteq \mathsf{G}'$

(3) there exists a $\bar{G}$-graded algebra isomorphism

$$\mathsf{E}(V) = \mathrm{End}_{\mathcal{K}A}(\mathcal{K}A \otimes_{\mathcal{K}B} V)^{\mathrm{op}} \to \mathsf{E}(V') = \mathrm{End}_{\mathcal{K}A'}(\mathcal{K}A' \otimes_{\mathcal{K}B'} V')^{\mathrm{op}}$$

such that the diagram

$$\begin{tikzcd} \mathsf{E}(V) \arrow[r, "\sim"] & \mathsf{E}(V') \\ \mathcal{K}\mathcal{C} \arrow[u] \arrow[r, equal, "\mathrm{id}_{\mathcal{K}\mathcal{C}}"] & \mathcal{K}\mathcal{C}, \arrow[u] \end{tikzcd}$$

of $\bar{G}$-graded $\mathcal{K}$-algebras is commutative, where $\mathcal{K}\mathcal{C} = \mathcal{K}C_G(N)$ is regarded as a $\bar{G}$-graded $\bar{G}$-acted $\mathcal{K}$-algebra, with 1-component $\mathcal{K}Z(N)$.

We now give a link between the relation $\geq_c$ for character triples, the relation $\geq_c$ for module triples and Rickard equivalences over $\mathcal{C}$. Recall that since $\mathcal{K}B$ is a semisimple algebra, the indecomposable objects in $\mathcal{D}^b(\mathcal{K}B)$ are the simple $\mathcal{K}B$-modules regarded as complexes concentrated in some degree $n \in \mathbb{Z}$.

Theorem 3.2.4. [29, Theorem 6.7] *Assume that $C_G(N) \subseteq G'$, and that the complex $\tilde{M}$ induces a $\bar{G}$-graded Rickard equivalence over $\mathcal{C} := \mathcal{O}C_G(N)$ between A and A'.*

Let V be a G-invariant simple $\mathcal{K}B$-module with character θ, and let V' be a G'-invariant simple $\mathcal{K}B'$-module corresponding to V via the given correspondence, with character θ'. Then we have that $(A, B, V) \geq_c (A', B', V')$ and $(G, N, \theta) \geq_c (G', N', \theta')$.

3.3 Blockwise relations

Späth also considered in [36], [37] and [38] the blockwise relation ($\geq_b$) between character triples. This relation is a refinement of the central-order relation ($\geq_c$), and it involves block induction, see [38, Definition 4.2].

On a similar idea, as published in [30, §5], we introduce the blockwise relation, $\geq_b$, between module triples as a refinement of the central-order relation, $\geq_c$, by using the Harris-Knörr correspondence (see Definition 3.3.1). Note that our definition does not fully cover [38, Definition 4.2], because there block induction in a more general situation is considered. We also introduce in Definition 3.3.5 a notion of a derived equivalence compatible with the Brauer map. This a a weaker condition that that of a splendid or basic equivalence, and is inspired by the results of [27], which connect basic Morita equivalences with the main result of Dade [8]. We prove in Proposition 3.3.6 that the relation $\geq_b$ between module triples is a consequence of a certain group graded derived equivalence compatible with the Brauer map.

Note that by a derived equivalence we mean an equivalence between the bounded derived categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(A')$ induced by a two-sided tilting complex as in [21, Section 6.2].

We continue with notations and assumptions of the preceding sections of this chapter, and in addition we require that $C_G(N) \subseteq G'$. Recall from Section 2.3.2, that this gives us that A and A' are $\bar{G}$-graded algebras over the $\bar{G}$-graded $\bar{G}$-acted $\mathcal{O}$-algebra $\mathcal{C} := \mathcal{O}C_G(N)$, with structural maps $\zeta : \mathcal{C} \to C_A(B)$ and $\zeta' : \mathcal{C} \to C_{A'}(B')$, as in Definition 2.1.5, given by inclusion.

We are going to use the Brauer map and basic equivalences between blocks, introduced by L. Puig in [35]. Then [38, Remark 4.3 (c)] leads us to the following setting.

Definition 3.3.1. We assume that the block b has defect group Q, $G' = N_G(Q)$, $N' = N_N(Q)$, and b' is the Brauer correspondent of b. Let (A, B, V) and (A', B', V') be two module triples. We write

$$(A, B, V) \geq_b (A', B', V')$$

if the following conditions are satisfied:

(1) $(A, B, V) \geq_c (A', B', V')$;

(2) For any subgroup $N \leq J \leq G$, if the simple $\mathcal{O}J$-module W covering V corresponds (via condition (1)) to the simple $\mathcal{O}J'$-module W' covering V' (where $J' = G' \cap J$), then the block β of $\mathcal{O}J$ to which W belongs is the Harris-Knörr correspondent of the block β' of $\mathcal{O}J'$ to which W' belongs.

3.3.2. Recall that the Harris-Knörr correspondence [17] is a bijection between the blocks of A with defect group D (where $Q \leq D$) and the blocks of A' with defect group D. This bijection in induced by the Brauer map

$$\mathrm{Br}_Q : A^Q \to A(Q).$$

3.3.3. Denote

$$\bar{C} = \bar{C}_A(B) = C_A(B)/J_{\mathrm{gr}}(C_A(B)).$$

we know from [8, 2.9] that $\bar{C}$ is a $\bar{G}[b]$-graded crossed product, where

$$\bar{G}[b] = \{\bar{g} \in \bar{G} \mid A_{\bar{g}} \simeq B \text{ as } (B, B)\text{-bimodules}\} = \{\bar{g} \in \bar{G} \mid A_{\bar{g}} A_{\bar{g}^{-1}} = B\}.$$

Denote also $\bar{C}' = \bar{C}_{A'}(B') = C_{A'}(B')/J_{\mathrm{gr}}(C_{A'}(B'))$.

The main result of Dade [8] says that the Brauer map Br_Q induces an isomorphism $\bar{C} \simeq \bar{C}'$ of $\bar{G}[b]$-graded $\bar{G}$-acted algebras. Moreover, by [27, Theorem 3.7], this isomorphism induces the same Harris-Knörr correspondence between the blocks of A and the blocks of A'.

3.3.4. Recall also from [25, Corollary 5.2.6] that a $\bar{G}$-graded derived equivalence between A and A' induces yet another isomorphism $\bar{C} \simeq \bar{C}'$ of $\bar{G}[b]$-graded $\bar{G}$-acted algebras.

Definition 3.3.5. We say that a $\bar{G}$-graded derived equivalence between A and A' is *compatible with the Brauer map* if the induced isomorphism $\bar{C} \simeq \bar{C}'$ of $\bar{G}[b]$-graded $\bar{G}$-algebras from 3.3.4 coincides with the isomorphism induced by the Brauer map Br_Q from 3.3.3.

Proposition 3.3.6. *Assume that the complex $\tilde{X}$ induces a $\bar{G}$-graded derived equivalence between A and A' compatible with the Brauer map Br_Q, such that the simple $\mathcal{K}B$-module V corresponds to the simple $\mathcal{K}B'$-module V'. Then $(A, B, V) \geq_b (A', B', V')$.*

Remark 3.3.7. By [27, Corollary 4.4], a $\bar{G}$-graded basic Morita equivalence between A and A' is compatible with the Brauer map Br_Q in the sense of the above definition.

Chapter 4

Tensor products and wreath products

This chapter is based on our results published in [30] and [33]. In it, we obtain equivalences for tensor products (Proposition 4.1.3) and wreath products (Theorem 4.3.3 and Theorem 4.4.4).

Such constructions are again motivated by the reduction methods, which require the compatibility of the relations between character triples and the wreath product constructions: in order to prove most reduction theorems, recent results of Britta Späth, surveyed in [36], [37] and [38], show that a new character triple can be constructed via a wreath product construction of character triples ([36, Theorem 5.2] and [38, Theorem 2.21]). In Theorem 3.2.4, we have proved that there is a link between character triples and group graded equivalences over a group graded group acted algebra, therefore we want to prove that a similar wreath product construction can also be made for the corresponding group graded equivalences.

Another motivation comes from the fact that it is already known by [25, Theorem 5.1.21] that Morita equivalences can be extended to wreath products.

In Section 4.1, we prove that the algebraic constructions introduced in Section 2.1 are compatible with tensor products and the main proposition in this section, Proposition 4.1.3, proves that the tensor products of some group graded Morita equivalent algebras over some group graded group acted algebras remain group graded Morita equivalent over a group graded group acted algebra.

In Section 4.2, we prove that the previously enumerated algebra types are also compatible with wreath products.

In Section 4.3, one of our main result, Theorem 4.3.3, proves that the wreath product between a G-graded bimodule over $\mathcal{C}$ and S_n (the symmetric group of order n) is also a group graded bimodule over $\mathcal{C}^{\otimes n}$, and moreover, if this bimodule induces a G-graded Morita equivalence over $\mathcal{C}$, then its wreath product with S_n will induce a group graded Morita equivalence over $\mathcal{C}^{\otimes n}$.

In Section 4.4, we build group graded derived and Rickard equivalences for wreath products. In order to do this, we extend the construction of wreath product for group graded algebras and bimodules over $\mathcal{C}$ to chain complexes of G-graded bimodules over $\mathcal{C}$. Our main result of this section, Theorem 4.4.4, says that the wreath product between a chain complex of G-graded bimodules over $\mathcal{C}$ and the symmetric group of order n, S_n, is a complex of $G\wr S_n$-graded bimodules over $\mathcal{C}^{\otimes n}$, and moreover, if the given complex induces a G-graded derived (respectively Rickard) equivalence over $\mathcal{C}$, then its wreath product with S_n (respectively a p'-subgroup of S_n) will induce a group graded derived (respectively Rickard) equivalence over $\mathcal{C}^{\otimes n}$. Our group graded algebras here are block extensions, but it is clear that most of the statements are true for more general group graded algebras. Theorem 4.4.4 improves [25, Theorem 5.2.12] in several ways, by taking into account all the additional structure that we deal with. As already noted by Zimmermann [40], a certain "p'-condition" on the order of the grading groups, which appears in [25, Theorem 5.2.12], is actually not needed in the case of derived equivalences, but is needed in the case of Rickard equivalences.

Finally, in Section 4.5, in Proposition 4.5.1 we prove that the relation $\geq_c$ is compatible with wreath products of G-graded derived equivalences over $\mathcal{C}$. Moreover, Theorem 4.5.2 and Corollary 4.5.3 are the main results of this chapter, and establish the compatibility of the relation $\geq_b$ between module triples with wreath products of derived equivalences.

4.1 Tensor products

4.1.1. Throughout this chapter n will represent an arbitrary nonzero natural number. Consider G_i to be a finite group, N_i to be a normal subgroup of G_i and denote by $\bar{G}_i = G_i/N_i$, for all $i \in \{1, \dots, n\}$. We denote by

$$\bar{\mathbf{G}} := \prod_{i=1}^{n} \bar{G}_i.$$

Lemma 4.1.2. *Let A_i be $\bar{G}_i$-graded algebras and $\mathcal{C}_i$ be $\bar{G}_i$-graded $\bar{G}_i$-acted algebras, for all $i \in \{1, \dots, n\}$. The following affirmations hold:*

(1) The tensor product $\mathbf{A} := A_1 \otimes \dots \otimes A_n$ is a $\bar{\mathbf{G}}$-graded algebra;

(2) If A_i are $\bar{G}_i$-graded crossed products, for all $i \in \{1, \dots, n\}$, then $\mathbf{A}$ is a $\bar{\mathbf{G}}$-graded crossed product;

(3) The tensor product $\boldsymbol{\mathcal{C}} := \mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_n$ is a $\bar{\mathbf{G}}$-graded $\bar{\mathbf{G}}$-acted algebra;

(4) If A_i are $\bar{G}_i$-graded algebras over $\mathcal{C}_i$, for all $i \in \{1, \dots, n\}$, then $\mathbf{A}$ is a $\bar{\mathbf{G}}$-graded algebra over $\boldsymbol{\mathcal{C}}$.

Proposition 4.1.3. *Assume that $\mathcal{C}_i$ are $\bar{G}_i$-graded $\bar{G}_i$-acted algebras and that A_i and A'_i are $\bar{G}_i$-graded crossed products over $\mathcal{C}_i$, for all $i \in \{1, \dots, n\}$. If A_i and A'_i are $\bar{G}_i$-graded Morita equivalent over $\mathcal{C}_i$, and if $\tilde{M}_i$ is a $\bar{G}_i$-graded (A_i, A'_i)-bimodule over $\mathcal{C}_i$, that induces the said equivalence, for all i, then:*

(1) $\tilde{\mathbf{M}} := \tilde{M}_1 \otimes \dots \otimes \tilde{M}_n$ is a $\bar{\mathbf{G}}$-graded $(\mathbf{A}, \mathbf{A}')$-bimodule over $\boldsymbol{\mathcal{C}}$, where $\mathbf{A} := A_1 \otimes \dots \otimes A_n$, $\mathbf{A}' := A'_1 \otimes \dots \otimes A'_n$ and $\boldsymbol{\mathcal{C}} := \mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_n$;

(2) $\tilde{\mathbf{M}}$ induces a $\bar{\mathbf{G}}$-graded Morita equivalence over $\boldsymbol{\mathcal{C}}$ between $\mathbf{A}$ and $\mathbf{A}'$.

4.2 Wreath products for algebras

4.2.1. We denote $\bar{G}^n := \bar{G} \times \dots \times \bar{G}$ (n times).

We recall the definition of a wreath product as in [38, Definition 2.19] and [25, Section 5.1.C]:

Definition 4.2.2. The wreath product $\bar{G} \wr S_n$ is the semidirect product $\bar{G}^n \rtimes S_n$, where the symmetric group of order n, S_n, acts on $\bar{G}^n$ (on the left) by permuting the components:

$${}^{\sigma}(g_1, \dots, g_n) := (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(n)}),$$

for all $g_1, \dots, g_n \in \bar{G}$ and $\sigma \in S_n$. More exactly, the elements of $\bar{G} \wr S_n$ are of the form $((g_1, \dots, g_n), \sigma)$, and the multiplication is:

$$((g_1, \dots, g_n), \sigma)((h_1, \dots, h_n), \tau) := ((g_1, \dots, g_n) \cdot {}^{\sigma}(h_1, \dots, h_n), \sigma\tau),$$

for all $g_1, \dots, g_n, h_1, \dots, h_n \in \bar{G}$ and $\sigma, \tau \in S_n$.

Definition 4.2.3. Let A be an $\mathcal{O}$-algebra. We denote by $A^{\otimes n} := A \otimes \ldots \otimes A$ (n times). The wreath product $A \wr S_n$ is the skew group algebra

$$A \wr S_n := A^{\otimes n} \otimes \mathcal{O}S_n$$

between $A^{\otimes n}$ and S_n, with multiplication

$$\begin{aligned}&((a_1 \otimes \ldots \otimes a_n) \otimes \sigma)((b_1 \otimes \ldots \otimes b_n) \otimes \tau)\\ &\quad := ((a_1 \otimes \ldots \otimes a_n) \cdot {}^{\sigma}(b_1 \otimes \ldots \otimes b_n)) \otimes (\sigma\tau),\end{aligned}$$

where

$${}^{\sigma}(b_1 \otimes \ldots \otimes b_n) := b_{\sigma^{-1}(1)} \otimes \ldots \otimes b_{\sigma^{-1}(n)},$$

for all $(a_1 \otimes \ldots \otimes a_n) \otimes \sigma$, $(b_1 \otimes \ldots \otimes b_n) \otimes \tau \in A \wr S_n$.

Note also that if A is a symmetric algebra, $A \wr S_n$ is also symmetric (see [25, Lemma 5.1.8]).

Lemma 4.2.4. *Let A be a $\bar{G}$-graded algebra and $\mathcal{C}$ be a $\bar{G}$-graded $\bar{G}$-acted algebra. The following affirmations hold:*

(1) $A \wr S_n$ is a $\bar{G} \wr S_n$-graded algebra, with $((g_1, \ldots, g_n), \sigma)$-component

$$(A \wr S_n)_{((g_1,\ldots,g_n),\sigma)} := ((A_{g_1} \otimes \ldots \otimes A_{g_n}) \otimes \mathcal{O}\sigma),$$

for each $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$;

(2) If A is a $\bar{G}$-graded crossed product, then $A \wr S_n$ is a $\bar{G} \wr S_n$-graded crossed product;

(3) $\mathcal{C}^{\otimes n}$ is a $\bar{G} \wr S_n$-acted $\bar{G}^n$-graded algebra, where

$${}^{((g_1,\ldots,g_n),\sigma)}(c_1 \otimes \ldots \otimes c_n) := {}^{g_1}c_{\sigma^{-1}(1)} \otimes \ldots \otimes {}^{g_n}c_{\sigma^{-1}(n)}.$$

(4) If A is a $\bar{G}$-graded algebra over $\mathcal{C}$, then $A \wr S_n$ is a $\bar{G} \wr S_n$-graded algebra over $\mathcal{C}^{\otimes n}$, with structural $\bar{G} \wr S_n$-graded $\bar{G} \wr S_n$-acted algebra homomorphism

$$\zeta_{wr} : \mathcal{C}^{\otimes n} \to C_{A \wr S_n}(B^{\otimes n})$$

given by the composition

$$\zeta^{\otimes n} : \mathcal{C}^{\otimes n} \to C_A(B)^{\otimes n} \subseteq C_{A \wr S_n}(B^{\otimes n}).$$

4.3 Morita equivalences for wreath products

We recall the definition of a wreath product between a module and S_n.

Definition 4.3.1. Let A and A' be two algebras. Assume that $\tilde{M}$ is an (A, A')-bimodule. The action of S_n on $\tilde{M}^{\otimes n}$ is defined by

$${}^{\sigma}(\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n) := \tilde{m}_{\sigma^{-1}(1)} \otimes \ldots \otimes \tilde{m}_{\sigma^{-1}(n)},$$

for all $\tilde{m}_1, \ldots, \tilde{m}_n \in \tilde{M}$ and $\sigma \in S_n$. As an $\mathcal{O}$-module, the wreath product $\tilde{M} \wr S_n$ is

$$\tilde{M} \wr S_n := \tilde{M}^{\otimes n} \otimes \mathcal{O}S_n,$$

with scalar multiplication

$$\begin{aligned}((a_1 \otimes \ldots \otimes a_n) \otimes \sigma)\,((\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n) \otimes \tau)\,((a'_1 \otimes \ldots \otimes a'_n) \otimes \pi) \\ = (a_1 \otimes \ldots \otimes a_n) \cdot {}^{\sigma}(\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n) \cdot {}^{\sigma\tau}(a'_1 \otimes \ldots \otimes a'_n) \otimes \sigma\tau\pi,\end{aligned}$$

for all $(a_1\otimes\ldots\otimes a_n)\otimes\sigma \in A\wr S_n$, $(\tilde{m}_1\otimes\ldots\otimes\tilde{m}_n)\otimes\tau \in \tilde{M}\wr S_n$ and $(a'_1\otimes\ldots\otimes a'_n)\otimes\pi \in A'\wr S_n$.

4.3.2. Let $\mathcal{C}$ be a $\bar{G}$-graded $\bar{G}$-acted algebra and A and A' be two $\bar{G}$-graded crossed products over $\mathcal{C}$, with identity components B and B' respectively.

If $\tilde{M}$ is an (A, A')-bimodule which induces a Morita equivalence between A and A', by the results of [25, Section 5.1.C], we already know that $\tilde{M}\wr S_n$ induces a Morita equivalence between $A\wr S_n$ and $A'\wr S_n$. The question that arises is whether this result can be extended to give a graded Morita equivalence over a group graded group acted algebra. An answer to this question is presented in Theorem 4.3.3, which extends [25, Theorem 5.1.21] to the case of group graded Morita equivalences over a group graded group acted algebra, which we published in [33, Theorem 5.3].

Theorem 4.3.3. *Let $\tilde{M}$ be a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$. Then, the following affirmations hold:*

(1) $\tilde{M}\wr S_n$ is a $\bar{G}\wr S_n$-graded $(A\wr S_n, A'\wr S_n)$-bimodule over $\mathcal{C}^{\otimes n}$, with $((g_1, \ldots, g_n), \sigma)$-component

$$(\tilde{M}\wr S_n)_{((g_1,\ldots,g_n),\sigma)} = (\tilde{M}_{g_1} \otimes \ldots \otimes \tilde{M}_{g_n}) \otimes \mathcal{O}\sigma;$$

(2) $(A\wr S_n)\otimes_{B^{\otimes n}} M^{\otimes n} \simeq M^{\otimes n}\otimes_{B'^{\otimes n}} (A'\wr S_n) \simeq \tilde{M}\wr S_n$ as $\bar{G}\wr S_n$-graded $(A\wr S_n, A'\wr S_n)$-bimodules over $\mathcal{C}^{\otimes n}$, where M is the identity component of $\tilde{M}$;

(3) If $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence over $\mathcal{C}$ between A and A', then $\tilde{M}\wr S_n$ induces a $\bar{G}\wr S_n$-graded Morita equivalence over $\mathcal{C}^{\otimes n}$ between $A\wr S_n$ and $A'\wr S_n$.

4.4 Derived equivalences for wreath products

4.4.1. Let A and A' be two $\bar{G}$-graded crossed products, hence $A\otimes A'$ is a $\bar{G}\times\bar{G}$-graded crossed product with 1-component $B\otimes B'$. We assume from now on, that A and A' are free and finitely generated as $\mathcal{O}$-modules.

4.4.2. Now, if $\tilde{X}$ is a chain complex of $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$ which induces a $\bar{G}$-graded derived or Rickard equivalence between A and A', we want to extend the results of [25, Section 5.1.C], to obtain a $\bar{G}\wr S_n$-graded derived or Rickard equivalence over $\mathcal{C}^{\otimes n}$ between $A\wr S_n$ and $A'\wr S_n$. In the case of Rickard equivalences, some additional condition will be needed.

4.4.3. Recall (see, for instance [3, Section 4.1]) that S_n acts on $\tilde{X}^{\otimes n} := \tilde{X}\otimes\ldots\otimes\tilde{X}$ (n times). By [25, Lemma 5.2.11], this action can be defined as follows: Denote $C_2 = \{\pm 1\}$, and observe that S_n acts on the abelian group $\mathrm{Fun}(C_2^n, C_2)$ of functions from C_2^n to C_2; for $i \in \mathbb{Z}$ denote also $\hat{i} = (-1)^i$. Then there is a 1-cocycle $\epsilon \in Z^1(S_n, \mathrm{Fun}(C_2^n, C_2))$ such that

$${}^{\sigma}(x_{i_1}\otimes\cdots\otimes x_{i_n}) = \epsilon_\sigma(\hat{i}_1, \ldots, \hat{i}_n)x_{i_{\sigma^{-1}(1)}}\otimes\cdots\otimes x_{i_{\sigma^{-1}(n)}},$$

where x_{i_j} belongs to the j-th factor of $\tilde{X}^{\otimes n}$, and has degree $i_j \in \mathbb{Z}$. In our situation, $\tilde{X}^{\otimes n}$ is a complex of $\bar{G}^n$-graded $(A^{\otimes n}, A'^{\otimes n})$-bimodules over $\mathcal{C}^{\otimes n}$, and even more, a complex of $\bar{G}^n$-graded $(A \otimes A'^{op}) \wr S_n$-modules.

We may therefore consider the wreath product

$$\tilde{X} \wr S_n = \tilde{X}^{\otimes n} \otimes \mathcal{O}S_n.$$

Theorem 4.4.4. [30, Theorem 3.7] *Let $\tilde{X}$ be a complex of $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$, with identity component X. Then, the following statements hold:*

(1) *$\tilde{X} \wr S_n$ is a complex of $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodules over $\mathcal{C}^{\otimes n}$, isomorphic to $(A \wr S_n) \otimes_{B^{\otimes n}} X^{\otimes n}$ and to $X^{\otimes n} \otimes_{B'^{\otimes n}} (A' \wr S_n)$.*

(2) *If $\tilde{X}$ induces a $\bar{G}$-graded derived equivalence between A and A', then $\tilde{X} \wr S_n$ induces a $\bar{G} \wr S_n$-graded derived equivalence over $\mathcal{C}^{\otimes n}$ between $A \wr S_n$ and $A' \wr S_n$.*

(3) *If $\tilde{X}$ induces a $\bar{G}$-graded Rickard equivalence between A and A', and if Σ is a p'-subgroup of S_n, then $\tilde{X} \wr \Sigma$ induces a $\bar{G} \wr \Sigma$-graded Rickard equivalence over $\mathcal{C}^{\otimes n}$ between $A \wr \Sigma$ and $A' \wr \Sigma$.*

4.5 Relations between module triples induced by wreath products

We again consider the assumptions and notations of our main example 2.3.1.

The next result is motivated by [38, Theorem 2.21].

Proposition 4.5.1. *Consider the module triples (A, B, V) and (A', B', V'). If A and A' are $\bar{G}$-graded derived equivalent over $\mathcal{C}$ such that V corresponds to V', then*

$$(A \wr S_n, B^{\otimes n}, V^{\otimes n}) \geq_c (A' \wr S_n, B'^{\otimes n}, V'^{\otimes n}).$$

Now we will do a similar wreath product construction for the blockwise relation $\geq_b$. In order to do this, we rely on the results of Harris [15, 16], which extend the results of Külshammer [22], and of Alghamdi and Khammash [1].

Theorem 4.5.2. [30, Theorem 5.8] *Assume that the complex $\tilde{X}$ induces a $\bar{G}$-graded derived equivalence between A and A' compatible with the Brauer map Br_Q. Then the $\bar{G} \wr S_n$-graded derived equivalence between $A \wr S_n$ and $A' \wr S_n$ induced by $\tilde{X} \wr S_n$ is compatible with the Brauer map Br_{Q^n}.*

From Proposition 4.5.1 and Theorem 4.5.2 we immediately deduce:

Corollary 4.5.3. *Assume that the complex $\tilde{X}$ induces a $\bar{G}$-graded derived equivalence over $\mathcal{C}$ between A and A', and that this equivalence is compatible with the Brauer map Br_Q. Assume also that the simple $\mathcal{K}B$-module V corresponds to the simple $\mathcal{K}B'$-module V'. Then*

$$(A \wr S_n, B^{\otimes n}, V^{\otimes n}) \geq_b (A' \wr S_n, B'^{\otimes n}, V'^{\otimes n}).$$

Remark 4.5.4. We are interested in the relation $\geq_b$ when induced by derived equivalences. However, it is not difficult to show directly, with the methods already used here, that similarly to [36, Theorem 5.2], if $(A, B, V) \geq_b (A', B', V')$, then $(A \wr S_n, B^{\otimes n}, V^{\otimes n}) \geq_b (A' \wr S_n, B'^{\otimes n}, V'^{\otimes n})$

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