"BABEȘ-BOLYAI" UNIVERSITY CLUJ NAPOCA
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

INTEGRAL TRANSFORMATIONS OF SOME SPECIAL CLASSES OF COMPLEX FUNCTIONS

PhD Thesis Summary

Scientific coordinator:
Prof. Univ. Dr. **Daniel Breaz**

PhD:
Maria Camelia Pode

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Contents

Introduction

The complex analysis dates from the eighteenth century, being a field of interest that has evolved rapidly due to multiple applications in various branches of science and technology.

The geometric theory of functions of a complex variable is a special branch of complex analysis that correlates rigorous reasoning with intuitive geometric models.The first significant works were published in the early twentieth century by P. Koebe in 1907, T.H. Gromwell in 1914, I.W. Alexander in 1915, L. Bieberbach in 1916. Bieberbach's conjecture, proved by Louis de Branges in 1984 led to the emergence of new directions of study in the geometric theory of analytic functions, one of which is the definition of new classes of univalent functions for which conjecture could be verified.

The complex analysis also occupies a place of honor in the Romanian school of mathematics. Romanian researchers have contributed decisively to the progress of this scientific branch through D. Pompeiu, Gh. Călugăreanu and P.T. Mocanu. Gh. Călugăreanu, the father of the Romanian school of The Theory of Univalent Functions, obtained for the first time necessary and sufficient conditions of univalence, and P.T. Mocanu introduced the $alpha$ -convex function class, addressed the issue of non-analytic injectivity, and co-created with S.S. Miller's well-known method of studying univalent classes of functions called the "admissible function method," the differential subordination method, and more recently the theory of differential subordination. Acad. P.T. Mocanu carried on the School of Theory of Univalent Functions at Univ. Babeș-Bolyai, founded by Gh. Călugăreanu.

The univalence property of functions of a complex variable is the object of research of many mathematicians, by determining necessary and sufficient conditions of univalence. The geometric interpretation of univalent functions is done by conformal transformations. Thus, it might be analysed the parameters of a certain conformal transformation, which verifies the additional conditions of univalence.

The necessary and sufficient conditions of univalence are usually presented in the form of differential inequalities, but the property of univalence can also be demonstrated by other methods such as: Loewner's parametric method, the method of integral representations, the method of differential subordinations (also known as the method of functions admissible) and the method of differential superorders.

A central role in the geometric theory of functions of a complex variable is played by the study of integral operators, defined on certain spaces of analytic functions. The first integral operator by univalent function classes was introduced in 1915 by J.W. Alexander, and since then several researchers have shown interest in this study. Among them we mention R. Libera, S. Bernardi, S.S. Miller, P.T. Mocanu, M.O. Reade, R. Singh, N.N. Pascu and many others.

The novelty of the results presented in this paper include: the introduction of integral operators as an extension of already known operators, represented as particular cases, as well as the investigation of geometric properties of univalence, convexity, starlikeness, but also of other classes of special functions, for integral operators. we. The integral operators introduced here are generalizations of some integral operators consecrated in the specialized literature, represented by constructions of several functions. Their details are presented in section 2.1. For these integral operators we obtained conditions of: univalence, starlikeness, convexity, but also of partnership to some special classes of analytic functions. This thesis is structured on 5 chapters, an introduction and a bibliography and is based on 35 papers published or in the process of being published (written in collaboration with D. Breaz) as well as some ideas not yet published.

The first chapter entitled ”Preliminary results” is divided into four paragraphs. Fundamental definitions and results are presented here the necessary basis for the following chapters. More precisely, definitions and properties are presented regarding analytic functions, univalent functions, functions with the real positive part, starlike functions, convex functions, but also other special classes of functions. The last section of this chapter contains the univalence criteria used to demonstrate the main results in the following chapters, given by Pascu [89], [90], Pescar [93], [94], Becker [42], Ozaki and Nunokawa [88].

The following four chapters present original results that represent the author's contribution to the field of geometric theory of analytic functions.

Chapter 2 it contains 8 section and begin with the presentation of four new general integral operators, which are extensions of known integral operators and which consist of several functions. The connection between these operators and other important results in the field is also marked. We specify that all the results obtained in this paper refer to the four integral operators introduced here.

The second paragraph treats the obtaining sufficient univalence criteria for full operators, when the functions are analytic. The justification of these results is made based on the univalence criteria given by Pascu and Pescar and are contained in the works [5], [6], [7], [8].

Section 2.3, contained in the works [9], [10], presents new conditions of univalence for integral operators general, $\mathcal{M}_{\delta,n}$ and $\mathcal{T}_{\delta}, n$, using an extension of Becker's univalence criterion, given by Pascu in [90], but also the result given by Mocanu and Șerb in [76].

Paragraph 2.4 studied some properties of preserving the class of univalent functions by integral operators. The tools that led to these results, included in the works [23], [24], [25], [26], are Becker's criteria of univalence and Pascu, but also the well-known inequalities of Nehari.

Paragraph 2.5 verifies the univalence of integral operators when the functions involved belong to the class $\mathcal{G}_b$, defined by Silverman in [109], for integral operators $\mathcal{M}_{\delta,n}$ and $\mathcal{T}_{\delta,n}$, using the univalence criteria of Pascu [90] and Pescar [93]. These contents can be found in the works [11], [12].

Section 2.6 considers sufficient univalence conditions for integral operators defined here, having functions in the class $\mathcal{S}(p)$ studied by Ozaki, Nunokawa, Yang, Liu, Singh and others. Evidence of these results uses Pescar's [93] univalence criterion and is printed in [13], [14].

Next, paragraph 2.7, analyzes the univalence of integral operators for functions that are part of the $\mathcal{B}(\mu)$ classes defined by Ash and Darus in [69] and $\mathcal{S}_\mu$ studied by Ponnusamy and Sing in [105]. The demonstration of these results, from the works [15], [16], [17], [18], comes with the help of the inequalities given by Deniz [60] and Frasin [63], as well as due to the univalence criteria Pascu [90] and Pescar [93].

The last section of this chapter addresses the univalence of integral operators for functions that are members of the class $\mathcal{B}(\mu, \alpha)$ defined by Frasin and Jahangiri in [70] and represents the content of the works [19], [20], [21], [22]. The univalence criteria Pascu [90] and Pescar [93], but also the Mocanu-Șerb Theorem [76] leads to these results.

Chapter 3 addresses the issue of convexity of integral operators, being divided into 6 paragraphs. In each paragraph the origin of the functions differs from one class to another.

Section 3.1 has as a starting point functions belonging to the classes $\mathcal{G}_b$, $\mathcal{B}(\mu, \alpha)$ and $\mathcal{S}^*_\beta$. These contents are in the works [27], [30], [33], [36].

The following is the row of star function classes in section 3.2, the results being included in the works [27], [30], [33], [36], [29], [32], [35], [39]. In this case, the convexity order was found for each fully investigated operator.

The study of convexity continues in paragraph 3.3 with functions belonging to the class $\mathcal{SP}(\alpha, \beta)$; information are contained in the works [28], [31], [34], [37], [29], [32], [35], [39].

Paragraph 3.4 presents the approach of the convexity of the integral operators from the perspective of $\mathcal{S}_b^*$ class functions. News that can be found in the works [28], [31], [34], [37].

Then we present in section 3.5 the convexity orders of integral operators for the case when the functions are members of the class $\mathcal{SH}(\beta)$, studies included in the works [29], [32] , [35], [39].

The end of this chapter comes with the approach of the convexity of integral operators for $alpha$ - convex functions. In the works [29], [32], [35], [39] we present this information.

În **Chapter 4**, structured in two paragraphs, we illustrate some conditions of membership of integral operators to the function class $\mathcal{N}(\beta)$. The first section here use analytic functions, and the other functions in class $\mathcal{SP}(\alpha, \beta)$. The results obtained in this chapter can be found in the works [28], [31], [34], [37].

The last chapter it encompasses six sections and refers to the study of p-valence functions. These studies are grouped into four separate articles, dedicated to each operator in full and are to be sent for publication.

Section 5.1 identifies conditions for integral operators to belong to the class of p-valent convex functions. For this purpose, the functions involved are included in the class of p-valent star functions.

The content of section 5.2 consists of conditions of belonging to class $\mathcal{N}_p(\beta)$, while functions are members of classes $\mathcal{N}_p(\beta)$ and $\mathcal{M}_p(\beta)$.

Follow the framing of the integral operators, through specific conditions to the class $\mathcal{K}_p(a, \alpha)$ presented in paragraph 5.3. The role of the functions here is played in the class $\mathcal{S}_p^*(a, \alpha)$.

Paragraph 5.4 takes the p-valent analytic functions of integral operators into the class of p-valent star functions by means of specific conditions.

Furthermore, section 5.5 transposes under certain conditions the p-valence analytic functions into the class of p-valence quasi-convex functions.

The final results set the functions in the class of p-valence analytic functions and present conditions for which integral operators belong to the class of uniformly p-valence functions almost convex.

Chapter 1

Preliminary results

1.1 Definitions, notations and elementary results from the theory of univalent functions

The notions and results described in this paragraph are part of the basic elements of the literature: the notion of holomorphic function, univalent function, analytic function, Mocanu-Serb Theorem, Nehari's Theorem, and the General Schwarz Lemma often encountered in demonstrating main results.

Definition 1.1.1. *A function f is called **holomorph at the point** z_0, if there is a neighborhood $V \in \mathcal{V}(z_0)$, so that f is derivable in this neighborhood.*

Definition 1.1.2. *A holomorphic and injective function on a domain D, from $\mathbb{C}$ is called **univalent on** D. Note with $\mathcal{H}_{\mathbb{U}}(D)$ the set of univalent functions on D.*

Definition 1.1.3. *Let $f : D \to \mathbb{C}$, $z \in D$. We say that the function f is **analytic at point** z_0 or **expandable in Taylor series in** z_0, if there is a disk $\mathbb{U}(z_0, r) = \{z \in \mathbb{C} :\mid z - z_0 \mid < r\} \subset D$, so that f is the sum a Taylor series, meaning:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in \mathbb{U}(z_0, r).$$

*We say that the function f is **analytic in the field** D, if it is analytic in every point of D.*

We further consider the following notations: $\mathbb{U} = \{z \in \mathbb{C} :\mid z \mid < 1\}$ - *the unit disk in the complex plane*.

We will also consider for $a \in \mathbb{C}$ şi $n \in \mathbb{N}^*$, the set

$$\mathcal{H}[a, n] = \left\{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} ...\right\},$$

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_{n+1} z^{n+1} ...\right\}$$

and

$$\mathcal{A} = \mathcal{A}_1.$$

We will note with

$$\mathcal{S} = \{f \in \mathcal{A} : f \in \mathcal{H}(\mathbb{U})\},$$

class of univalent functions in the unit disk and normed with conditions

$$f(0) = f^{'}(0) - 1 = 0,$$

hence the holomorphic and univalent functions in $\mathbb{U}$, which have developed in series of form powers

$$f(z) = z + a_2 z^2 + ..., \quad |z| < 1.$$

We note with

$$\mathcal{P} = \{p \in \mathcal{H}(\mathbb{U}) : p(0) = 1, \mathrm{Re}p(z) > 0, z \in \mathbb{U}\},$$

class of Caratheodory functions.

The following theorem was proved by Mocanu and Șerb.

Theorem 1.1.1. *(Mocanu - Șerb [76]) Let $M_0 = 1,5936...$ the positive solution of the equation*

$$(2 - M)\, e^M = 2. \tag{1.1.1}$$

If $f \in \mathcal{A}$ and

$$\left|\frac{f''(z))}{f'(z)}\right| \leq M_0,$$

for all $z \in \mathbb{U}$, then

$$\left|\frac{zf'(z))}{f(z)} - 1\right| \leq 1, \qquad (z \in \mathbb{U}).$$

The edge M_0 is sharp.

Theorem 1.1.2. *(General Schwarz Lemma)[73] Let f be the function regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$,one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} z^m.$$

The equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

On the other hand, Nehari proved other important results.

Lemma 1.1.3. [77] If the function g is regular in unit disk $\mathbb{U}$ and $|g(z)| < 1$ in $\mathbb{U}$, then for all $\xi \in \mathbb{U}$ the following inequalities hold

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \bar{z}\xi}\right|$$

and

$$\left|g^{'}(z)\right| \leq \frac{1 - |g(z)|^2}{1 - |z|^2},$$

the equalities hold in case $g(z) = \varepsilon \frac{z+u}{1+\bar{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 1.1.4. [77] For $z = 0$,from inequality (1) we obtain for every $\xi \in \mathbb{U}$

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)|\,|g(\xi)|}.$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|},$$

for all $z \in \mathbb{U}$.

In the next two sections we will mention important results regarding the following function classes: star function class, order star function class α, class of starlike functions of complex order b and type λ, class of convex functions, class of convex functions of order α, class of convex functions of complex order b and type λ, $alpha$ -convex function class, but also various special classes of analytic functions such as: function class $\mathcal{B}_(\mu)$, class of functions $\mathcal{B}(\mu, \alpha)$, class of functions $\mathcal{S}_\mu$, class of functions $\mathcal{SP}$, class of functions $\mathcal{SP}(\alpha, \beta)$, class of functions $\mathcal{SH}(\beta)$, class of functions $\mathcal{S}(p)$, class of functions $\mathcal{G}_b$, class of functions $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$, classes $\mathcal{S}^*_\beta$ and $\mathcal{S}_\beta$, class of functions $\mathcal{A}_p$, classes of functions $\mathcal{S}^*_p(\beta)$ and $\mathcal{S}^*_p(a, \alpha)$, classes of functions $\mathcal{K}_p(\beta)$ and $\mathcal{K}_p(a, \alpha)$, classes of functions $\mathcal{M}_p(\beta)$ and $\mathcal{N}_p(\beta)$, class of functions $\mathbb{U}_p(\beta, k)$.

1.2 Class of starlike functions and class of convex functions

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is a starlike function of order β, $0 \leq \beta < 1$ and we denote this class by $\mathcal{S}^*(\beta)$ if it satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathbb{U}.$$

We denote by $\mathcal{K}(\beta)$ the class of convex functions of order β, $0 \leq \beta < 1$ that satisfies the inequality

$$\text{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \beta,\ z \in \mathbb{U}.$$

A function $f \in \mathcal{A}$ is a convex function of the complex order b, $(b \in \mathbb{C} - \{0\})$ and type λ , $(0 \leq \lambda < 1)$, if and only if

$$\text{Re}\left\{1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > \lambda,\ z \in \mathbb{U}.$$

We denote by $\mathcal{C}^*_\lambda(b)$ the class of these functions.

A function $f \in A$ is a starlike function of the complex order b, $(b \in \mathbb{C} - \{0\})$ and type λ, $(0 \leq \lambda < 1)$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \lambda,\ z \in \mathbb{U}.$$

We denote by $\mathcal{S}_{\lambda}^{*}(b)$ the class of these functions.

A function $f \in \mathcal{K}(\beta)$ if and only if $zf' \in \mathcal{S}^{*}(\beta)$.

1.3 Special classes of analytic functions

Frasin and Jahangiri [70] studied the class $\mathcal{B}(\mu, \lambda)$, $\mu \geq 0$, $0 \leq \lambda < 1$, which consists of functions $f \in \mathcal{A}$ that satisfy the following conditions:

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1 - \lambda, \quad z \in \mathbb{U}.$$

This class $\mathcal{B}(\mu, \lambda)$ is a comprehensive class of normalized analytic functions in $\mathbb{U}$. For instance, we have $\mathcal{B}(1, \lambda) = \mathcal{S}^{*}(\lambda)$, $\mathcal{B}(0, \lambda) = \mathcal{R}(\lambda)$ and $\mathcal{B}(2, \lambda) = \mathcal{B}(\lambda)$. In particular, the analytic and univalent function class $\mathcal{B}(\lambda)$ was studied by Frasin and Darus [69].

A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu)$ if it satisfies

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 - \mu$$

for some μ $(0 \leq \mu < 1)$ and for all $z \in \mathbb{U}$.

Lemma 1.3.1. [60] Let $f(z) \in \mathcal{B}(\mu)$, then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{(1-\mu)(1+|z|)}{1-|z|}, \quad 0 \leq \mu < 1, \quad z \in \mathbb{U}.$$

Lemma 1.3.2. [64] Let $f(z) \in \mathcal{B}(\mu)$, then

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{(1-\mu)(2+|z|)}{1-|z|}, \quad 0 \leq \mu < 1, \quad z \in \mathbb{U}.$$

The subclass $\mathcal{S}_{\mu}$ of analytic functions was studied by Ponnusamy and Sing [105] is defined as follows

$$\mathcal{S}_{\mu} = \left\{f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < \mu\,|z|\,,\ 0 \leq \mu < 1,\ z \in \mathbb{U}\right\}$$

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}(\lambda)$, $0 \leq \lambda < 1$, if

$$\operatorname{Re}\left[f'(z)\right] > \lambda, \quad z \in \mathbb{U}.$$

F. Ronning introduce in [108] the class of univalent functions $\mathcal{SP}(\alpha,\beta)$, $\alpha>0$, $\beta\in[0,1)$.

The function $f\in\mathcal{S}$ is in the class $\mathcal{SP}(\alpha,\beta)$, $\alpha>0$ if only if

$$\left|\frac{zf^{'}(z)}{f(z)}-(\alpha+\beta)\right|\leq \text{Re}\frac{zf^{'}(z)}{f(z)}+\alpha-\beta,\ z\in\mathbb{U}.$$

In the paper [108], F. Ronning introduce the class of univalent functions $\mathcal{SP}(\alpha,\beta)$, $\alpha>0$, $\beta\in[0,1)$. The function $f\in S$ is in the class $\mathcal{SP}(\alpha,\beta)$, $\alpha>0$ if only if

$$\left|\frac{zf^{'}(z)}{f(z)}-(\alpha+\beta)\right|\leq \text{Re}\frac{zf^{'}(z)}{f(z)}+\alpha-\beta,\ z\in\mathbb{U}.$$

J. Stankiewicz and A. Wisniowska in [114] introduce the class of univalent functions $\mathcal{SH}(\beta)$, for some $\beta>0$. If $f\in\mathcal{SH}(\beta)$, then f verifies the next inequality:

$$\text{Re}\left(\sqrt{2}\frac{zf^{'}(z)}{f(z)}\right)+2\beta\left(\sqrt{2}-1\right)>\left|\frac{zf^{'}(z)}{f(z)}-2\beta\left(\sqrt{2}-1\right)\right|,f\in S,\ z\in\mathbb{U}.$$

In [120], it is defined the class $\mathcal{S}(p)$, which for $0<p\leq 2$, includes the functions $f\in\mathcal{A}$ which satisfy the conditions:

$$f(z)\neq 0 \quad for \quad 0<|z|<1$$

and

$$\left|\left(\frac{z}{f(z)}\right)^{''}\right|\leq p$$

for all $z\in\mathbb{U}$.

Lemma 1.3.3. [110] If $f\in\mathcal{S}(p)$, then the following inequality is true

$$\left|\frac{z^2f^{''}(z)}{[f(z)]^2}-1\right|\leq p\,|z|^2\,,z\in\mathbb{U}$$

This inequality was demonstrated by Sigh in the paper [110].

In [109] Silverman define the class G_b. Precisely, for $0<b\leq 1$ he considered the class

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<b\left|\frac{zf'(z)}{f(z)}\right|,\quad z\in\mathbb{U}.$$

Uralegaddi in [119], Owa and Srivastava in [87] define the class $\mathcal{N}(\beta)$.

A function $f\in\mathcal{A}$ is in the class $\mathcal{N}(\beta)$ if it verifies the inequality

$$\text{Re}\left(\frac{zf^{''}(z)}{f^{'}(z)}+1\right)<\beta,\ z\in\mathbb{U},\ \beta>1.$$

Let $\mathcal{A}_p$ the class of all p-valent analytic functions

$$f(z) = z^p + a_{p+1}z^{p+1} + ..., p \in \mathbb{N}$$

If we consider $p = 1$ we obtain that $\mathcal{A}_1 = \mathcal{A}$.

We consider the classes introduced and studied by R. Ali and V.Ravichandranin [2]. A function $f \in \mathcal{A}_p$ is said to be p-valenlty starlike of order β $(0 \leq \beta < p)$ if and only if

$$\frac{1}{p}\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathbb{U}.$$

We denote by $\mathcal{S}_p^*(\beta)$ the class of all such functions.

A function $f \in \mathcal{A}_p$ is said to be p-valently convex of order β $(0 \leq \beta < p)$ if and only if

$$\frac{1}{p}\text{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in \mathbb{U}.$$

Let $\mathcal{K}_p(\beta)$ denote the class of all those functions which are p-valently convex of order β in $\mathbb{U}$. We note that $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$ and $\mathcal{K}_p(0) = \mathcal{K}_p$ are respectively, the classes of p-valently starlike and p-valently convex functions in $\mathbb{U}$. Also, we note that $\mathcal{S}_1^* = \mathcal{S}^*$ and $\mathcal{K}_1 = \mathcal{K}$ are, respectively the usual classes of starlike and convex functions in $\mathbb{U}$.

Starting from the classes of starlike and convex functions of complex order a and type α, R. Ali and V.Ravichandranin [2] defined the classes $\mathcal{S}_p^*(a, \alpha)$ and $\mathcal{K}_p(a, \alpha)$ as follows:

$$\mathcal{S}_p^*(a, \alpha) = \left\{ f \in \mathcal{A}_p, \alpha < 1 : \text{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)\right) > \alpha \right\}.$$

and

$$\mathcal{K}_p(a, \alpha) = \left\{ f \in \mathcal{A}_p, \alpha < 1 : \text{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) + 1\right)\right) > \alpha \right\}.$$

In the case of $p = 1$ the classes were studied by Breaz [50], Frasin [64], etc.

Next we will consider the classes $\mathcal{M}_p(\beta)$ and $\mathcal{N}_p(\beta)$.

A function $f \in \mathcal{A}_p$ is in the classes $\mathcal{M}_p(\beta)$ if

$$\frac{1}{p}\text{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta,\ z \in \mathbb{U}$$

for $\beta > 1$.

The class $\mathcal{N}_p(\beta)$ contains all the functions that satisfy the condition

$$\frac{1}{p}\text{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) < \beta,\ z \in \mathbb{U}$$

for $f \in \mathcal{A}_p$ and $\beta > 1$.

If we consider $p = 1$, we obtain the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ that were studied by many others, for example Breaz [46], Ularu, Breaz and Frasin in [116] and Uralegaddi, Ganigi and Sarangi in [119].

Also they have defined in a analogue mode the classes $\mathcal{M}_p(a, \alpha)$ and $\mathcal{N}_p(a, \alpha)$.

A function $f \in \mathcal{A}_p$ is in the class $\mathcal{M}_p(a, \alpha)$ if

$$\text{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf^{'}(z)}{f(z)} - 1\right)\right) < \alpha$$

for $\alpha > 1$.

The class $\mathcal{N}_p(a, \alpha)$ contains all the functions $f \in \mathcal{A}_p$ that satisfy

$$\text{Re}\left(1 + \frac{1}{b}\left(\frac{1}{p}\left(1 + \frac{zf^{''}(z)}{f'(z)}\right) - 1\right)\right) < \alpha$$

for $\alpha > 1$.

A function $f \in \mathcal{A}_p$ is in the class $\mathcal{U}_p(\beta, k)$ of k-uniformly p-valent starlike of order β, with $-1 \leq \beta < p$ in the open disk $\mathcal{U}$, if the condition

$$\text{Re}\left(\frac{zf^{'}(z)}{f(z)} - \beta\right) \geq k\left|\frac{zf^{'}(z)}{f(z)} - p\right|, \quad k \geq 0, z \in \mathbb{U},$$

is satisfied. This class was introduced by Goodman in [71].

The class of uniformly p-valent close-to convex functions of order β with $-1 \leq \beta < p$ in the open disk $\mathbb{U}$ contains all the functions that satisfy

$$\text{Re}\left(\frac{zf^{'}(z)}{g(z)} - \beta\right) \geq \left|\frac{zf^{'}(z)}{g(z)} - p\right|, \quad k \geq 0, z \in U,$$

for $z \in \mathbb{U}$ and the function g from the class of p-valent starlike functions of order β.

To prove that our functions are p-valently starlike and p-valently close-to-convex in the open unit disk we will use the following lemmas:

Lemma 1.3.4. [80] If $f \in \mathcal{A}_p$ satisfies

$$\text{Re}\left(1 + \frac{zf^{''}(z)}{f'(z)}\right) < p + \frac{1}{4} \quad for \quad z \in \mathbb{U},$$

then f is p-valently starlike in $\mathbb{U}$.

Lemma 1.3.5. [62] If $f \in \mathcal{A}_p$ satisfies

$$\left|\frac{zf^{''}(z)}{f'(z)} + 1 - p\right| < p + 1 \quad for \quad z \in \mathbb{U},$$

then f is p-valently starlike in $\mathbb{U}$.

Lemma 1.3.6. [106] If $f \in \mathcal{A}_p$ satisfies

$$\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < p + \frac{a+b}{(1+a)(1-b)},$$

for $z \in \mathbb{U}$, where $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then f is p-valently close-to-convex in $\mathbb{U}$.

Lemma 1.3.7. [3] If $f \in \mathcal{A}_p$ satisfies

$$\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < p + \frac{1}{3} \quad for \quad z \in \mathbb{U},$$

then f is uniformly p-valent close-to-convex in $\mathbb{U}$.

1.4 Univalence criteria

An essential role in the study of integral operators is played by the univalence criteria, obtaining with their help remarkable results in the geometric theory of univalent functions.

In the following we will present some univalence criteria necessary for the demonstrations in the following chapters.

Lemma 1.4.1. [90] Let $f \in \mathcal{A}$ and $\gamma \in \mathbb{C}$. If Re$\gamma > 0$ and

$$\frac{1-|z|^{2\text{Re}\gamma}}{\text{Re}\gamma}\left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma\int_0^z t^{\gamma-1}f'(t)\text{dt}\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Lemma 1.4.2. [90] Let $\delta \in \mathbb{C}$ with Re$\delta > 0$. If $f \in \mathcal{A}$ satisfies

$$\frac{1-|z|^{2\text{Re}\delta}}{\text{Re}\delta}\left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then, for any complex γ with Re$\gamma \geq$ Reδ, the integral operator

$$F_\gamma(z) = \left(\gamma\int_0^z t^{\gamma-1}f'(t)\text{dt}\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Pescar gave the following univalence criteria for an integral operator:

Lemma 1.4.3. [93] Let γ be complex number, Re$\gamma > 0$ and c a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + ...$ If

$$\left| c\,|z|^{2\gamma} + \left(1 - |z|^{2\gamma}\right) \frac{zf''(z))}{\gamma f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)\mathrm{dt}\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Becker's gave the following univalence criterion:

Lemma 1.4.4. [42] If the function f is regular in unit disk $\mathbb{U}$ and $f(z) = z + a_2z^2 + ...$ and

$$\left(1 - |z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad for\ all\ z \in \mathbb{U},$$

then the function f is univalent in $\mathbb{U}$.

Ozaki and Nunokawa demonstrated the following condition of univalence:

Lemma 1.4.5. [88] Let $f \in \mathcal{A}$, satisfy the condition

$$\left| \frac{z^2 f'(z))}{[f(z)]^2} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, then f is regular and univalent in $\mathbb{U}$.

Pescar demonstrated the following two conditions of univalence in [95]:

Lemma 1.4.6. [95] Let $g \in \mathcal{A}$, α a real number, and c a complex number, $|c| \leq \frac{1}{\alpha}$, $c \neq -1$. If

$$\left| \frac{g''(z))}{g'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then the function

$$G_\alpha(z) = \left(\alpha \int_0^z \left[t^{\alpha-1} g'(t)\right]^{\alpha-1} \mathrm{dt}\right)^{\frac{1}{\alpha}},$$

is in the class in $\mathcal{S}$.

Lemma 1.4.7. [95] Let the function g, satisfy (2), M a positive real number fixed, and c a complex number. If $\alpha \in \left[\frac{2M+1}{2M+2}, \frac{2M+1}{2M}\right]$

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| (2M + 1), \qquad c \neq -1,$$

$$|g(z)| \leq M,$$

for all $z \in \mathbb{U}$, then the function

$$G_\alpha(z) = \left(\alpha \int_0^z [g(t)]^{\alpha-1} \mathrm{dt}\right)^{\frac{1}{\alpha}},$$

is in the class in $\mathcal{S}$.

Chapter 2

Sufficient conditions of univalence for new integral operators

2.1 New integral operators

We will briefly present below four integral general operators introduced by Bărbatu and Breaz in the works [5] - [8], which are integral operators such as those defined by Pfaltzgraff, Kim-Merkes and Oversea, as well as the links between them and other integral operators known in the literature.

We consider the first integral general operator $\mathcal{M}_{\delta,n}$, defined by [5]:

$$\mathcal{M}_{\delta,n}(z)=\left\{\delta\int_0^z t^{\delta-1}\prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\left(g_i{}'(t)\right)^{\beta_i}\left(\frac{g_i(t))}{t}\right)^{\gamma_i}\right]\mathrm{dt}\right\}^{\frac{1}{\delta}}, \tag{2.1.1}$$

where f_i, g_i are analytic in $\mathbb{U}$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for all $i=\overline{1,n}$, $n\in\mathbb{N}\setminus\{0\}$, $\delta\in\mathbb{C}$, with $\mathrm{Re}\delta>0$.

Remark 2.1.1. The integral operator $\mathcal{M}_{\delta,n}$ defined by (2.1.1) represents an extension of other operators as follows:

i) For $n=1$, $\delta=1$, $\alpha_1-1=\alpha_1$ and $\beta_1=\gamma_1=0$ we obtain the integral operator which was studied by Kim-Merkes [72]

$$\mathcal{F}_\alpha(z)=\int_0^z\left(\frac{f(t)}{t}\right)^\alpha \mathrm{dt}.$$

ii) For $n=1$, $\delta=1$ and $\alpha_1-1=\gamma_1=0$ we obtain the integral operator which was studied by Pfaltzgraff [104]

$$\mathcal{G}_\alpha(z)=\int_0^z\left(f'(t)\right)^\alpha \mathrm{dt}.$$

iii) For $\alpha_i-1=\alpha_i$ and $\beta_i=\gamma_i=0$, we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [47]

$$\mathcal{D}_n(z)=\left[\delta\int_0^z t^{\delta-1}\prod_{i=1}^n\left(\frac{f_i(t)}{t}\right)^{\alpha_i}\mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pascu and Pescar [91].

iv) For $\alpha_i-1=\gamma_i=0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [51]

$$\mathcal{I}_n(z)=\left[\delta\int_0^z t^{\delta-1}\prod_{i=1}^n[f_i'(t)]^{\alpha_i}\,\mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pescar and Owa in [102].

v) For $\alpha_i-1=0$ we obtain the integral operator which was defined and studied by Pescar in [97]

$$\mathcal{F}_n(z)=\left[\delta\int_0^z t^{\delta-1}\prod_{i=1}^n\left(\frac{f_i(t)}{t}\right)^{\alpha_i}\left(f_i{}'(t)\right)^{\beta_i}\mathrm{dt}\right]^{\frac{1}{\delta}}.$$

this integral operator is a generalization of the integral operator introduced by Frasin in [66] and by Oversea in [86].

vi) For $\alpha_i-1=\alpha_i$ and $\gamma_i=0$ we obtain the integral operator which was studied by Ularu in [115]

$$\mathcal{I}_n(z)=\left[\delta\int_0^z t^{\delta-1}\prod_{i=1}^n\left(\frac{f_i(t)}{t}\right)^{\alpha_i}\left(g_i{}'(t)\right)^{\beta_i}\mathrm{dt}\right]^{\frac{1}{\delta}}.$$

Let it now be the second integral general operator $\mathcal{C}_{\delta,n}$, defined by [6]:

$$\mathcal{C}_{\delta,n}=\left\{\delta\int_0^z t^{\delta-1}\prod_{i=1}^n\left[\left(\frac{f_i(t)}{t}e^{g_i(t)}\right)^{\alpha_i-1}\left(h_i{}'(t)\right)^{\beta_i}\left(\frac{h_i(t))}{t}\right)^{\gamma_i}\right]\mathrm{dt}\right\}^{\frac{1}{\delta}},\tag{2.1.2}$$

where f_i, g_i, h_i are analytic in $\mathbb{U}$ and $\delta, \alpha_i, \beta_i, \gamma_i$ are complex numbers, for all $i=\overline{1,n}$, $n\in\mathbb{N}\setminus\{0\}$, $\delta\in\mathbb{C}$, with Re$\delta>0$.

And this integral operator given by the relation (2.1.2) constitutes the extension of integral operators, thus:

Remark 2.1.2. i) For $n=1$, $\delta=1$ and $\alpha_1-1=\beta_1=0$ we obtain the integral operator which was studied by Kim-Merkes [72]

$$\mathcal{F}_\alpha(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}\mathrm{dt}.$$

ii) For $n=1$, $\delta=1$ and $\alpha_1-1=\gamma_1=0$ we obtain the integral operator which was studied by Pfaltzgraff [104]

$$\mathcal{G}_\alpha(z)=\int_0^z\left(f'(t)\right)^{\alpha}\mathrm{dt}.$$

iii) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [47]

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pascu and Pescar [91].

iv) For $\alpha_i - 1 = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [51]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n [f_i'(t)]^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pescar and Owa in [102].

v) For $\alpha_i - 1 = 0$ we obtain the integral operator which was defined and studied by Pescar [97]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(f_i{}'(t)\right)^{\beta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

this integral operator is a generalization of the integral operator introduced by Frasin in [66] and by Oversea in [86].

vi) For $n = 1$, $\delta = \beta$, $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by Stanciu in [111]

$$\mathcal{H}_1(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} e^{g(t)}\right)^{\alpha} \mathrm{dt}\right]^{\frac{1}{\beta}}.$$

We still take the third integral general operator $\mathcal{G}_{\delta,n}$, defined by [7]:

$$\mathcal{G}_{\delta,n} = \left\{\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[\left(f_i{}'(t) e^{g_i(t)}\right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)}\right)^{\beta_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\gamma_i}\right] dt\right\}^{\frac{1}{\delta}}, \tag{2.1.3}$$

where f_i, g_i, h_i, k_i are analytic in $\mathbb{U}$ and $\delta, \alpha_i, \beta_i, \gamma_i$ are complex numbers, for all $i = \overline{1,n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with Re$\delta > 0$.

The connection between the integral operator in the relation (2.1.3) and otherintegral operators look as follows:

Remark 2.1.3. i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \gamma_1 = 0$ and $k_1(z) = z$ we obtain the integral operator which was studied by Kim-Merkes [72]

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} \mathrm{dt}.$$

ii) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$, $\beta_1 = \gamma_1 = 0$ and $g_1(z) = 0$, we obtain the integral operator which was studied by Pfaltzgraff [104]

$$\mathcal{G}_\alpha(z) = \int_0^z \left(f'(t)\right)^{\alpha} \mathrm{dt}.$$

iii) For $\alpha_i - 1 = \gamma_i = 0$ and $k_i(z) = z$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [47]

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pascu and Pescar [91].

iv) For $\alpha_i - 1 = \alpha_i$, $\beta_i = \gamma_i = 0$ and $g_i(z) = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [51]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n [f_i'(t)]^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pescar and Owa in [102].

v) For $\alpha_i - 1 = 0$, $k_i(z) = z$ and $k_i'(z) = 1$ we obtain the integral operator which was defined and studied by Pescar [97]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} (f_i{}'(t))^{\beta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

this integral operator is a generalization of the integral operator introduced by Frasin in [66] and by Oversea in [86].

vi) For $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = 0$, operatorul $\mathcal{G}_{\delta,n}$ definit prin (2.1.3), se reduce la operatorul integral care a fost definit and studiat de A. Oprea and D. Breaz în [83]

$$\mathcal{G}_n(z) = \int_0^z \prod_{i=1}^n \left[\left(f_i{}'(t) e^{g_i(t)}\right)^{\alpha_i}\right] \mathrm{dt}.$$

This integral operator is a generalization of the integral operator introduced by Ularu and Breaz in [117].

vii) For $\alpha_i - 1 = 0$ we obtain the integral operator which was defined and studied by Pescar in [97]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i} \left(\frac{f_i'(t)}{g_i'(t)}\right)^{\delta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

The last integral general operator was defined in the paper [8]:

$$\mathcal{T}_{\delta,n} = \left\{\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} (g_i'(t))^{\beta_i} \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] \mathrm{dt}\right\}^{\frac{1}{\delta}}, \tag{2.1.4}$$

where f_i, g_i, h_i, k_i are analytic in $\mathbb{U}$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$ for all $i = \overline{1,n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\mathrm{Re}\delta > 0$.

The way in which this operator operates from (2.1.4) can be reduced to another known integral operator is shown below:

Remark 2.1.4. i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [72]

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha \mathrm{dt}.$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [104]

$$\mathcal{G}_\alpha(z) = \int_0^z (f'(t))^\alpha \mathrm{dt}.$$

iii) For $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [47]

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pascu and Pescar [91].

iv) For $\alpha_i - 1 = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [51]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n [f_i'(t)]^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

This integral operator is a generalization of the integral operator introduced by Pescar and Owa in [102].

v) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = \delta_i = 0$ we obtain the integral operator which was studied by Ularu in [115]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} (g_i{}'(t))^{\beta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

vi) For $\alpha_i - 1 = \beta_i = 0$, $k_i(z) = z$ and $k_i'(z) = 1$ we obtain the integral operator which was defined and studied by Pescar [97]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} (f_i{}'(t))^{\beta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

this integral operator is a generalization of the integral operator introduced by Frasin in [66] and by Oversea in [86].

vii) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by Pescar în [97]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i} \left(\frac{f_i'(t)}{g_i'(t)}\right)^{\delta_i} \mathrm{dt}\right]^{\frac{1}{\delta}}.$$

viii) For $\delta = 1$, $\alpha_i - 1 = \gamma_i = 0$, $\beta_i = \delta_i$ and $h_i(z) = \frac{z^2}{2}$ we obtain the integral operator which was defined and studied by Bucur and Breaz in [53]

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^n \left[\frac{tg_i'(t)}{k_i'(t)}\right]^{\beta_i} \text{dt}.$$

this integral operator is a generalization of the integral operator introduced by Bucur, Andrei and Breaz in [57] and [58].

xi) For $\delta = 1$, $\alpha_i - 1 = \delta_i = 0$, $\beta_i = \gamma_i$ and $h_i(z) = f_i(z)$ and $h_i(z) = f_i(z)$ we obtain the integral operator which was defined and studied by Nguyen, Oprea and Breaz in [78]

$$\mathcal{H}_{n,\alpha}(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{h_i(t)} g_i'(t)\right)^{\alpha_i} \text{dt}.$$

2.2 Sufficient conditions of univalence for analytic functions

In this paragraph we will present sufficient conditions to ensure the univalence of the integral operators described in the previous section when the functions involved are analytic.

Theorem 2.2.1. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$ and M_i, N_i, P_i real positive numbers, $f_i, g_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \qquad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \le N_i, \qquad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^n \left[|\alpha_i - 1| M_i + |\beta_i| P_i + |\gamma_i| N_i\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then for all δ complex numbers $Re\delta \ge Re\gamma$, the integral operator $\mathcal{M}_{\delta,n}$ given by (2.1.1) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.2.1, obtain the next corollary:

Corollary 2.2.1.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \le 1$, $c = Re\gamma$ and M_i, N_i, P_i real positive numbers, $f_i, g_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \qquad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \le N_i, \qquad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^n \left[|\alpha_i - 1| M_i + |\beta_i| P_i + |\gamma_i| N_i\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{M}_n$, defined in

$$\mathcal{M}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left({g_i}'(t) \right)^{\beta_i} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{2.2.1}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.2.1, obtain the next corollary:

Corollary 2.2.1.2. *Let $\gamma, \alpha_i, \beta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$ and M_i, P_i real positive numbers, $f_i, g_i \in \mathcal{A}$. If*

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \qquad \left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \, M_i + |\beta_i| \, P_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{F}_n$, defined in

$$\mathcal{F}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left({g_i}'(t) \right)^{\beta_i} \right] dt \tag{2.2.2}$$

is in the class $\mathcal{S}$.

Remark 2.2.2. The integral operator in the relation (2.2.2) is a known result, proven in [115].

If we consider $\delta = 1$ and $\beta_i = 0$ in Theorem 2.2.1, obtain the next corollary:

Corollary 2.2.2.1. *Let $\gamma, \alpha_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$ and M_i, N_i real positive numbers, $f_i, g_i \in \mathcal{A}$. If*

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \qquad \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \leq N_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \, M_i + |\gamma_i| \, N_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{G}_n$, defined in

$$\mathcal{G}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{2.2.3}$$

is in the class $\mathcal{S}$.

Remark 2.2.3. The integral operator in the relation (2.2.3) is another known result, introduced in [83].

If we consider $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.2.1, obtain the next corollary:

Corollary 2.2.3.1. *Let $\gamma, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \le 1$, $c = Re\gamma$, N_i, P_i real positive numbers, $g_i \in \mathcal{A}$. If*

$$\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \le N_i, \qquad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\beta_i| P_i + |\gamma_i| N_i\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{I}_n$, defined in

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[(g_i{}'(t))^{\beta_i} \cdot \left(\frac{g_i(t))}{t}\right)^{\gamma_i}\right] dt \tag{2.2.4}$$

is in the class $\mathcal{S}$.

Remark 2.2.4. The integral operator in the relation (2.2.4) was studied in [97].

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.2.1, obtain the next corollary:

Corollary 2.2.4.1. *Let α be complex number, $Re\alpha > 0$ and M, N, P real positive numbers, $f, g \in \mathcal{A}$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le M, \qquad \left|\frac{zg'(z)}{g(z)} - 1\right| \le N, \qquad \left|\frac{zg''(z)}{g'(z)}\right| \le P,$$

for all $z \in \mathbb{U}$, and

$$|\alpha - 1| (M + N + P) \le \frac{(2Re\alpha + 1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then the integral operator $\mathcal{M}$, defined in

$$\mathcal{M}(z) = \left\{\alpha \int_0^z \left[f(t)g'(t)\frac{g(t))}{t}\right]^{\alpha-1} dt\right\}^{\frac{1}{\alpha}}, \tag{2.2.5}$$

is in the class $\mathcal{S}$.

Theorem 2.2.5. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$ and $f_i, g_i \in \mathcal{S}$, $g_i' \in \mathcal{P}$. If*

$$2\sum_{i=1}^{n} |\alpha_i - 1| + \sum_{i=1}^{n} |\beta_i| + 2\sum_{i=1}^{n} |\gamma_i| \le \frac{c}{2}, \quad for \quad 0 < c < 1$$

or

$$2\sum_{i=1}^{n} |\alpha_i - 1| + \sum_{i=1}^{n} |\beta_i| + 2\sum_{i=1}^{n} |\gamma_i| \le \frac{1}{2}, \quad for \quad c \ge 1$$

then for any complex numbers δ, $Re\delta \ge c$, the integral operator $\mathcal{M}_{\delta,n}$ defined by (2.1.1) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.2.5, we obtain the next corollary:

Corollary 2.2.5.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \le 1$ and $f_i, g_i \in \mathcal{S}$, $g_i' \in \mathcal{P}$. If*

$$2\sum_{i=1}^{n} |\alpha_i - 1| + \sum_{i=1}^{n} |\beta_i| + 2\sum_{i=1}^{n} |\gamma_i| \le \frac{Re\gamma}{2},$$

then the integral operator $\mathcal{M}_n$ given by (2.2.1) is in the class $\mathcal{S}$.

Theorem 2.2.6. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$ and M_i, N_i, P_i, Q_i real positive numbers, $f_i, g_i, h_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \quad |g_i(z)| \le N_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le P_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le Q_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i\right) + |\beta_i| P_i + |\gamma_i| Q_i\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then for any complex numbers δ with $Re\delta \ge Re\gamma$, the integral operator $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.2.6, we obtain the next corollary:

Corollary 2.2.6.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \le 1$, $c = Re\gamma$ and M_i, N_i, P_i, Q_i real positive numbers, $f_i, g_i, h_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \quad |g_i(z)| \le N_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le P_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le Q_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i\right) + |\beta_i| P_i + |\gamma_i| Q_i\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{C}_n$, defined in

$$\mathcal{C}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)}\right)^{\alpha_i - 1} \left({h_i}'(t)\right)^{\beta_i} \left(\frac{h_i(t))}{t}\right)^{\gamma_i}\right] dt, \qquad (2.2.6)$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.2.6, we obtain the next corollary:

Corollary 2.2.6.2. *Let $\gamma, \alpha_i, \beta_i$ be complex numbers, $0 < Re\gamma \le 1$, $c = Re\gamma$ and M_i, N_i, P_i real positive numbers, $f_i, g_i, h_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \quad |g_i(z)| \le N_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le P_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i\right) + |\beta_i| \, P_i \right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{T}_n$*, defined in*

$$\mathcal{T}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left({h_i}'(t) \right)^{\beta_i} \right] dt, \tag{2.2.7}$$

is in the class $\mathcal{S}$.

Remark 2.2.7. The integral operator defined in (2.2.7) if we put them $\beta_i = 0$, we get a known proven result in [111].

If we consider $\delta = 1$ and $\beta_i = 0$ in Theorem 2.2.6, we obtain the next corollary:

Corollary 2.2.7.1. *Let* $\gamma, \alpha_i, \gamma_i$ *be complex numbers,* $0 < Re\gamma \le 1$, $c = Re\gamma$ *and* M_i, N_i, Q_i *real positive numbers,* $f_i, g_i, h_i \in \mathcal{A}$*. If*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| \le M_i, \quad |g_i(z)| \le N_i, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| \le Q_i, \quad \left| \frac{z g_i'(z)}{g_i(z)} \right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i\right) + |\gamma_i| \, Q_i \right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{R}_n$*, defined in*

$$\mathcal{R}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left(\frac{h_i(t))}{t} \right)^{\gamma_i} \right] dt, \tag{2.2.8}$$

is in the class $\mathcal{S}$.

Remark 2.2.8. If we consider $\gamma_i = 0$ in (2.2.8) we get the same integral operator introduced in [111].

If we consider $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.2.6, we obtain the next corollary:

Corollary 2.2.8.1. *Let* $\gamma, \beta_i, \gamma_i$ *be complex numbers,* $0 < Re\gamma \le 1$, $c = Re\gamma$ *and* P_i, Q_i *real positive numbers,* $h_i \in \mathcal{A}$*. If*

$$\left| \frac{z h_i''(z)}{h_i'(z)} \right| \le P_i, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| \le Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\beta_i| \, P_i + |\gamma_i| \, Q_i \right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{I}_n$*, defined in*

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left({h_i}'(t) \right)^{\beta_i} \cdot \left(\frac{h_i(t))}{t} \right)^{\gamma_i} \right] dt, \tag{2.2.9}$$

is in the class $\mathcal{S}$.

Remark 2.2.9. The integral operator given by the relation (2.2.9) is a known result demonstrated in [97].

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.2.6, we obtain the next corollary:

Corollary 2.2.9.1. *Let α be complex number, Re$\alpha > 0$, M, N, P, Q real positive numbers, $f, g, h \in \mathcal{A}$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le M, \quad |g(z)| \le N, \quad \left|\frac{zh''(z)}{h'(z)}\right| \le P, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| \le Q, \quad \left|\frac{zg'(z)}{g(z)}\right| \le 1,$$

for $z \in \mathbb{U}$, and

$$|\alpha - 1|\,(M + N + P + Q) \le \frac{(2Re\alpha + 1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then the integral operator $\mathcal{C}$ defined in

$$\mathcal{C}(z) = \left\{\alpha \int_0^z \left[f(t)e^{g(t)}h'(t)\frac{h(t)}{t}\right]^{\alpha-1} dt\right\}^{\frac{1}{\alpha}}, \tag{2.2.10}$$

is in the class $\mathcal{S}$.

Theorem 2.2.10. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c =$ Re$\gamma > 0$ and $f_i, h_i \in \mathcal{S}$, $h_i{}' \in \mathcal{P}$, $g_i \in \mathcal{R}$. If*

$$4\sum_{i=1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i| + 2\sum_{i=1}^{n}|\gamma_i| \le \frac{c}{2}, \quad for \quad 0 < c < 1$$

or

$$4\sum_{i=1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i| + 2\sum_{i=1}^{n}|\gamma_i| \le \frac{1}{2}, \quad for \quad c \ge 1$$

then for any complex numbers δ, Re$\delta \ge c$, the integral operator $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.2.10, we obtain the next corollary:

Corollary 2.2.10.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 <$ Re$\gamma \le 1$ and $f_i, h_i \in \mathcal{S}$, $g_i, h_i{}' \in \mathcal{P}$, $g_i \in \mathcal{R}$. If*

$$4\sum_{i=1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i| + 2\sum_{i=1}^{n}|\gamma_i| \le \frac{Re\gamma}{2},$$

then the integral operator $\mathcal{C}_n$ defined by (2.2.6) is in the class $\mathcal{S}$.

Theorem 2.2.11. *Let α be complex number, Re$\alpha > 0$ and M_i, N_i, P_i real positive numbers, $f_i, g_i, h_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \le M_i, \quad |g_i(z)| \le N_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le 1 \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le P_i, \quad \left|\frac{z^2g_i'(z)}{[g_i(z)]^2} - 1\right| < 1,,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| \left(M_i + 2N_i^2 + P_i + 3\right), \qquad c \in \mathbb{C}, \qquad c \neq -1,$$

then the integral operator $\mathcal{C}_{\alpha,n}$, defined in

$$\mathcal{C}_{\alpha,n}(z) = \left[\alpha \int_0^z t^{\alpha-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} e^{g_i(t)} {h_i}'(t) \frac{h_i(t))}{t}\right)^{\alpha-1} dt\right]^{\frac{1}{\alpha}} \tag{2.2.11}$$

is in the class $\mathcal{S}$.

Theorem 2.2.12. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$ and $M_i, N_i, P_i, Q_i, R_i, S_i$ real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i\right) + |\beta_i| \left(P_i + Q_i\right) + |\gamma_i| \left(R_i + S_i\right)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then for any complex numbers δ, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{G}_{\delta,n}$, given by (2.1.3) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.2.12, we obtain the next corollary:

Corollary 2.2.12.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$ and $M_i, N_i, P_i, Q_i, R_i, S_i$ real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i\right) + |\beta_i| \left(P_i + Q_i\right) + |\gamma_i| \left(R_i + S_i\right)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{G}_n$, defined in

$$\mathcal{G}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left({f_i}'(t) e^{g_i(t)}\right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)}\right)^{\beta_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)}\right)^{\gamma_i}\right] dt, \tag{2.2.12}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.2.12, we obtain the next corollary:

Corollary 2.2.12.2. *Let $\gamma, \alpha_i, \beta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$ and M_i, N_i, P_i, Q_i real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i\right) + |\beta_i| \left(P_i + Q_i\right)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{V}_n$, defined in

$$\mathcal{V}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(f_i{}'(t) e^{g_i(t)}\right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)}\right)^{\beta_i}\right] dt, \tag{2.2.13}$$

is in the class $\mathcal{S}$.

Remark 2.2.13. The integral operator defined in (2.2.12), if we put them $\beta_i = 0$, we get the known proven result in [83].

If we consider $\delta = 1$ and $\beta_i = 0$ in Theorem 2.2.12, we obtain the next corollary:

Corollary 2.2.13.1. *Let $\gamma, \alpha_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$ and M_i, N_i, R_i, S_i real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i\right) + |\gamma_i| \left(R_i + S_i\right)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{W}_n$, defined in

$$\mathcal{W}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(f_i{}'(t) e^{g_i(t)}\right)^{\alpha_i - 1} \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\gamma_i}\right] dt, \tag{2.2.14}$$

is in the class $\mathcal{S}$.

Remark 2.2.14. Putting $\gamma_i = 0$ in (2.2.13) we get the same integral operator that was introduced in [83].

If we consider $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.2.12, we obtain the next corollary:

Corollary 2.2.14.1. *Let $\gamma, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$ and P_i, Q_i, R_i, S_i real positive numbers, $h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\beta_i| (P_i + Q_i) + |\gamma_i| (R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{I}_n$, defined in

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{h_i(t)}{k_i(t)}\right)^{\beta_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)}\right)^{\gamma_i}\right] dt\,, \tag{2.2.15}$$

is in the class $\mathcal{S}$.

Remark 2.2.15. The integral operator given by the relation (2.2.14) is a known proven result in [97].

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.2.12, we obtain the next corollary:

Corollary 2.2.15.1. *Let α be complex number, $Re\alpha > 0$ and M, N, P, Q, R, S real positive numbers, $f, g, h, k \in \mathcal{A}$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq M, \quad |g(z)| \leq N, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| \leq P, \quad \left|\frac{zk'(z)}{k(z)} - 1\right| \leq Q,$$

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq R, \quad \left|\frac{zk''(z)}{k'(z)}\right| \leq S, \quad \left|\frac{zg'(z)}{g(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$ and

$$|\alpha - 1| (M + N + P + Q + R + S) \leq \frac{(2Re\alpha + 1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then the integral operator $\mathcal{G}$, defined in

$$\mathcal{G}(z) = \left[\alpha \int_0^z t^{\alpha-1} \left(f'(t)e^{g(t)} \frac{h(t)}{k(t)} \frac{h'(t))}{k'(t)}\right)^{\alpha-1} dt\right]^{\frac{1}{\alpha}}, \tag{2.2.16}$$

is in the class $\mathcal{S}$.

Theorem 2.2.16. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$ and $h_i, k_i \in \mathcal{S}$, ${f_i}', {h_i}', {k_i}' \in \mathcal{P}$, $g_i \in \mathcal{R}$. If*

$$3\sum_{i=1}^{n} |\alpha_i - 1| + 4\sum_{i=1}^{n} |\beta_i| + 2\sum_{i=1}^{n} |\gamma_i| \leq \frac{c}{2}, \quad for \quad 0 < c < 1$$

or

$$3\sum_{i=1}^{n} |\alpha_i - 1| + 4\sum_{i=1}^{n} |\beta_i| + 2\sum_{i=1}^{n} |\gamma_i| \leq \frac{1}{2}, \quad for \quad c \geq 1$$

then for any complex numbers δ, $Re\delta \geq c$, the integral operator $\mathcal{G}_{\delta,n}$ given by (2.1.3) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.2.16, we obtain the next corollary:

Corollary 2.2.16.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $h_i, k_i \in \mathcal{S}$, ${f_i}', {h_i}', {k_i}' \in \mathcal{P}$, $g_i \in \mathcal{R}$. If*

$$3\sum_{i=1}^{n} |\alpha_i - 1| + 4\sum_{i=1}^{n} |\beta_i| + 2\sum_{i=1}^{n} |\gamma_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{G}_n$ defined by (2.2.11) is in the class $\mathcal{S}$.

Theorem 2.2.17. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $Re\gamma > 0$ and M_i, N_i, P_i, real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \leq 1, \quad |g_i(z)| \leq M_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq N_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq P_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq 1, \quad \left|\frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$|c| \leq 1 - \frac{1}{|\delta|}\left[\left(1 + 2M_i^2\right)\sum_{i-1}^{n} |\alpha_i - 1| + (N_i + P_i + 4)\sum_{i=}^{n} |\beta_i| + 2\sum_{i=1}^{n} |\gamma_i|\right], \ c \in \mathbb{C}, \ c \neq -1,$$

then the integral operator $\mathcal{G}_{\delta,n}$, given by(2.1.3) is in the class $\mathcal{S}$.

If we consider $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.2.17, we obtain the next corollary:

Corollary 2.2.17.1. *Let α be complex number, $Re\alpha > 0$ and M, N, P real positive numbers, $f, g, h, k \in \mathcal{A}$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad |g(z)| \leq M, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| \leq N, \quad \left|\frac{zk'(z)}{k(z)} - 1\right| \leq P,$$

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad \left|\frac{zk''(z)}{k'(z)}\right| \leq 1, \quad \left|\frac{z^2 g'(z)}{[g(z)]^2} - 1\right| < 1,$$

for all $z \in \mathbb{U}$ and

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right|\left(2M^2 + N + P + 7\right), \qquad c \in \mathbb{C}, \qquad c \neq -1,$$

then the integral operator $\mathcal{G}_{\alpha,n}$, defined in

$$\mathcal{G}_{\alpha,n}(z) = \left[\alpha \int_0^z t^{\alpha-1} \prod_{i=1}^{n} \left(f_i'(t) e^{g_i(t)} \frac{h_i(t)}{k_i(t)} \frac{h_i'(t))}{k_i'(t)}\right)^{\alpha-1} dt\right]^{\frac{1}{\alpha}} \tag{2.2.17}$$

is in the class $\mathcal{S}$.

Theorem 2.2.18. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = Re\gamma > 0$ and $M_i, N_i, P_i, Q_i, R_i, S_i$ real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le P_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \le Q_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \le S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then for any complex numbers δ, $Re\delta \ge Re\gamma$, the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.2.18, we obtain the next corollary:

Corollary 2.2.18.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \le 1$, $c = Re\gamma$ and $M_i, N_i, P_i, Q_i, R_i, S_i$ real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le P_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \le Q_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \le S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{T}_n$ defined in

$$\mathcal{T}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \left(g_i(t)'\right)^{\beta_i} \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{2.2.18}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\delta_i = 0$ in Theorem 2.2.18, we obtain the next corollary:

Corollary 2.2.18.2. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \le 1$, $c = Re\gamma$ and M_i, N_i, P_i, Q_i real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \le Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{S}_n$*, defined in*

$$\mathcal{S}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left(g_i(t)' \right)^{\beta_i} \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \right] dt, \tag{2.2.19}$$

is in the class $\mathcal{S}$*.*

Remark 2.2.19. The integral operator defined in (2.2.18) if we put them $\gamma_i = 0$, we obtain a known result demonstrated in [115].

If we consider $\delta = 1$ and $\beta_i = 0$ in Theorem 2.2.18, we obtain the next corollary:

Corollary 2.2.19.1. *Let* $\gamma, \alpha_i, \gamma_i, \delta_i$ *be complex numbers,* $0 < Re\gamma \leq 1$*,* $c = Re\gamma$ *and* M_i, P_i, Q_i, R_i, S_i *real positive numbers,* $f_i, h_i, k_i \in \mathcal{A}$*. If*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| \leq P_i, \quad \left| \frac{z k_i'(z)}{k_i(z)} - 1 \right| \leq Q_i, \quad \left| \frac{z h_i''(z)}{h_i'(z)} \right| \leq R_i, \quad \left| \frac{z k_i''(z)}{k_i'(z)} \right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i) \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{X}_n$*, defined in*

$$\mathcal{X}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] dt, \tag{2.2.20}$$

is in the class $\mathcal{S}$*.*

Remark 2.2.20. In the integral operator defined by (2.2.19), if we take $\alpha_i - 1 = 0$, we get another known result introduced in [97].

If we consider $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.2.18, we obtain the next corollary:

Corollary 2.2.20.1. *Let* $\gamma, \beta_i, \gamma_i, \delta_i$ *be complex numbers,* $0 < Re\gamma \leq 1$*,* $c = Re\gamma$ *and* N_i, P_i, Q_i, R_i, S_i *real positive numbers,* $g_i, h_i, k_i \in \mathcal{A}$*.*

If

$$\left| \frac{z g_i''(z)}{g_i'(z)} \right| \leq N_i, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| \leq P_i, \quad \left| \frac{z k_i'(z)}{k_i(z)} - 1 \right| \leq Q_i, \quad \left| \frac{z h_i''(z)}{h_i'(z)} \right| \leq R_i, \quad \left| \frac{z k_i''(z)}{k_i'(z)} \right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\beta_i| \, N_i + |\gamma_i| \, (P_i + Q_i) + |\delta_i| \, (R_i + S_i) \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{D}_n$*, defined in*

$$\mathcal{D}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(g_i(t)' \right)^{\beta_i} \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] dt, \tag{2.2.21}$$

is in the class $\mathcal{S}$.

Remark 2.2.21. If in (2.2.20) take $\beta_i = 0$, we get the result that was introduced in [97].

If we consider $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.2.18, we obtain the next corollary:

Corollary 2.2.21.1. *Let* $\gamma, \alpha_i, \beta_i, \delta_i$ *be complex numbers,* $0 < Re\gamma \leq 1$*,* $c = Re\gamma$ *and* M_i, N_i, R_i, S_i *real positive numbers,* $f_i, g_i, h_i, k_i \in \mathcal{A}$*. If*

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad \left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq N_i, \quad \left| \frac{zh_i''(z)}{h_i'(z)} \right| \leq R_i, \quad \left| \frac{zk_i''(z)}{k_i'(z)} \right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \, M_i + |\beta_i| \, N_i + |\delta_i| \, (R_i + S_i) \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{Y}_n$*, defined in*

$$\mathcal{Y}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left(g_i(t)' \right)^{\beta_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] dt, \tag{2.2.22}$$

is in the class $\mathcal{S}$.

Remark 2.2.22. Putting in (2.2.21) $\delta_i = 0$, we get the same proven result in [115].

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.2.18, we obtain the next corollary:

Corollary 2.2.22.1. *Let* α *be complex number,* $Re\alpha > 0$ *and* M,N, P, Q, R, S *real positive numbers,* $f, g, h, k \in \mathcal{A}$*. If*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq M, \quad \left| \frac{zg''(z)}{g(z)'} \right| \leq N, \quad \left| \frac{zh'(z)}{h(z)} - 1 \right| \leq P,$$
$$\left| \frac{zk'(z)}{k(z)} - 1 \right| \leq Q, \quad \left| \frac{zh''(z)}{h'(z)} \right| \leq R, \quad \left| \frac{zk''(z)}{k'(z)} \right| \leq S,$$

for all $z \in \mathbb{U}$*, and*

$$|\alpha - 1| \, (M + N + P + Q + R + S) \leq \frac{(2Re\alpha + 1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then the integral operator $\mathcal{T}$, defined in

$$\mathcal{T}(z) = \left[\alpha \int_0^z t^{\alpha-1} \left(f(t)g'(t)\frac{h(t)}{k(t)}\frac{h'(t))}{k'(t)}\right)^{\alpha-1} \ dt\right]^{\frac{1}{\alpha}}, \tag{2.2.23}$$

is in the class $\mathcal{S}$.

Theorem 2.2.23. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = Re\gamma > 0$ and $f_i, h_i, k_i \in \mathcal{S}$, ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$. If*

$$2\sum_{i=1}^n |\alpha_i - 1| + \sum_{i=1}^n |\beta_i| + 4\sum_{i=1}^n |\gamma_i| + 2\sum_{i=1}^n |\delta_i| \le \frac{c}{2}, \quad for \quad 0 < c < 1$$

or

$$2\sum_{i=1}^n |\alpha_i - 1| + \sum_{i=1}^n |\beta_i| + 4\sum_{i=1}^n |\gamma_i| + 2\sum_{i=1}^n |\delta_i| \le \frac{1}{2}, \quad for \quad c \ge 1$$

then for any complex numbers δ, $Re\delta \ge c$, the integral operator $\mathcal{T}_{\delta,n}$ defined by (2.1.4) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.2.23, we obtain the next corollary:

Corollary 2.2.23.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \le 1$ and $f_i, h_i, k_i \in \mathcal{S}$, ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$. If*

$$2\sum_{i=1}^n |\alpha_i - 1| + \sum_{i=1}^n |\beta_i| + 4\sum_{i=1}^n |\gamma_i| + 2\sum_{i=1}^n |\delta_i| \le \frac{Re\gamma}{2}, \quad for \ \ 0 < c < 1$$

then the integral operator $\mathcal{T}_n$ given by (2.2.18) is in the class $\mathcal{S}$.

Theorem 2.2.24. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $Re\gamma > 0$ and M_i, N_i, P_i, real positive numbers, $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le 1, \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le N_i, \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \le P_i, \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le 1, \left|\frac{zk_i''(z)}{k_i'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$|c| \le 1 - \frac{1}{|\delta|}\left[(2 + M_i)\sum_{i-1}^n |\alpha_i - 1| + \sum_{i=1}^n |\beta_i| + (N_i + P_i + 4)\sum_{i=}^n |\gamma_i| + 2\sum_{i=1}^n |\delta_i|\right],$$

$c \in \mathbb{C}$, $c \ne -1$, then the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

If we consider $\delta = \gamma = \alpha$, $\alpha_i - 1 = \beta_i = \gamma_i$ and $n = 1$ in Theorem 2.2.24, we obtain the next corollary:

Corollary 2.2.24.1. *Let α be complex number, $Re\alpha > 0$ and M, N, P real positive numbers, $f, g, h, k \in \mathcal{A}$. Dacă*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le M, \left|\frac{zg''(z)}{g'(z)}\right| \le 1, \left|\frac{zh'(z)}{h(z)} - 1\right| \le N, \left|\frac{zk'(z)}{k(z)} - 1\right| \le P, \left|\frac{zh''(z)}{h'(z)}\right| \le 1, \left|\frac{zk''(z)}{k'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$ and

$$|c| \le 1 - \left|\frac{\alpha - 1}{\alpha}\right|(M_i + N_i + P_i + 8), \qquad c \in \mathbb{C}, \qquad c \ne -1,$$

then the integral operator $\mathcal{T}$, given by (2.2.23) is in the class $\mathcal{S}$.

2.3 New univalence conditions for analytic functions

This paragraph extends the sufficient conditions of univalence for operators $\mathcal{M}_{\delta,n}$ and $\mathcal{T}_{\delta,n}$ when the functions involved are analytic, using the Mocanu-Șerb Theorem.

Theorem 2.3.1. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$, M_0 the positive solution of the equation (1.1.1), $M_0 = 1,5936...$ and $f_i, g_i \in \mathcal{A}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M_0, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq M_0,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{1}{c}\sum_{i=1}^{n}|\alpha_i - 1| + \frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}|\beta_i| + \frac{1}{c}\sum_{i=1}^{n}|\gamma_i| \leq 1,$$

then for any complex numbers δ, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{M}_{\delta,n}$, defined by (2.2.1) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.3.1, we obtain the next corollary:

Corollary 2.3.1.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_0 the positive solution of the equation (1.1.1), $M_0 = 1,5936...$ and $f_i, g_i \in \mathcal{A}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M_0, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq M_0,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{1}{c}\sum_{i=1}^{n}|\alpha_i - 1| + \frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}|\beta_i| + \frac{1}{c}\sum_{i=1}^{n}|\gamma_i| \leq 1,$$

then the integral operator $\mathcal{M}_n$, given by (2.2.1) is in the class $\mathcal{S}$.

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Theorem 2.3.1, we obtain the next corollary:

Corollary 2.3.1.2. *Let α be complex number, $a = Re\alpha > 0$, M_0 the positive solution of the equation (1.1.1), $M_0 = 1,5936...$ and $f, g \in \mathcal{A}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M_0, \quad \left|\frac{g''(z)}{g'(z)}\right| \leq M_0,$$

for all $z \in \mathbb{U}$ and

$$2(\alpha - 1)\left(\frac{1}{a} + \frac{M_0}{(2a+1)^{\frac{2a+1}{2a}}}\right) \leq 1,$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{S}$.

Theorem 2.3.2. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = Re\gamma > 0$, M_0 the positive solution of the equation (1.1.1), $M_0 = 1,5936...$ and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M_0, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \le M_0, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \le M_0, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \le M_0,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{1}{c}\sum_{i=1}^{n}|\alpha_i - 1| + \frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}|\beta_i| + \frac{2}{c}\sum_{i=1}^{n}|\gamma_i| + \frac{4M_0}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}|\delta_i| \le 1,$$

then for any complex numbers δ, $Re\delta \ge Re\gamma$, the integral operator $\mathcal{T}_{\delta,n}$, given by (2.1.4) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 2.3.2, we obtain the next corollary:

Corollary 2.3.2.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \le 1$, $c = Re\gamma$, iar M_0 the positive solution of the equation (1.1.1), $M_0 = 1,5936...$ and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M_0, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \le M_0, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \le M_0, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \le M_0,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{1}{c}\sum_{i=1}^{n}|\alpha_i - 1| + \frac{2M_0}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}|\beta_i| + \frac{2}{c}\sum_{i=1}^{n}|\gamma_i| + \frac{4M_0}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}|\delta_i| \le 1,$$

then the integral operator $\mathcal{T}_n$, defined by (2.2.18) is in the class $\mathcal{S}$.

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1 = \delta_1$ in Theorem 2.3.2, we obtain the next corollary:

Corollary 2.3.2.2. *Let α be complex number, $a = Re\alpha > 0$, M_0 the positive solution of the equation (1.1.1), $M_0 = 1,5936...$ and $f, g, h, k \in \mathcal{A}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0, \quad \left|\frac{g''(z)}{g'(z)}\right| \le M_0, \quad \left|\frac{h''(z)}{h'(z)}\right| \le M_0, \quad \left|\frac{k''(z)}{k'(z)}\right| \le M_0,$$

for all $z \in \mathbb{U}$ and

$$3\,(\alpha - 1)\left(\frac{1}{a} + \frac{2M_0}{(2a+1)^{\frac{2a+1}{2a}}}\right) \le 1,$$

then the integral operator $\mathcal{T}$, defined by (2.2.20) is in the class $\mathcal{S}$.

2.4 Univalence conditions for univalent functions

The present paragraph contains sufficient conditions of univalence for the integral operators presented in this paper when the functions involved are univalent, using Nehari's Theorem.

Theorem 2.4.1. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i \in \mathcal{S}$. If*

$$\left|\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right| \leq 1, \quad \left|\frac{zg_i'(z) - g_i(z)}{zg_i(z)}\right| \leq 1, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{\sum_{i=1}^{n} (|\alpha_i - 1| + |\beta_i| + |\gamma_i|)}{\prod_{i=1}^{n} (|\alpha_i - 1| \, |\beta_i| \, |\gamma_i|)} < 1,$$

$$\prod_{i=1}^{n} (|\alpha_i - 1| \, |\beta_i| \, |\gamma_i|) \leq \frac{1}{\underset{|z|\leq 1}{max}\left[\left(1 - |z|^2\right)|z| \frac{|z|+|k|}{1+|k||z|}\right]},$$

where

$$|k| = \frac{\left|\sum_{i=1}^{n} \left[(\alpha_i - 1) a_{2i} + 2\beta_i b_{2i} + \gamma_i b_{2i}\right]\right|}{\prod_{i=1}^{n} (|\alpha_i - 1| \, |\beta_i| \, |\gamma_i|)},$$

then for any complex numbers δ, $Re\delta \geq Re\gamma$, the function $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.4.1, we obtain the next corollary:

Corollary 2.4.1.1. *Let $\alpha \in \mathbb{C}$, $Re\alpha > 0$ and $f, g \in \mathcal{S}$. If*

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| < 1, \quad \left|\frac{zg'(z) - g(z)}{zg(z)}\right| < 1, \quad \left|\frac{g''(z)}{g'(z)}\right| < 1,$$

for all $z \in \mathbb{U}$, and the constant $|\alpha|$ satisfies the condition

$$|\alpha| \leq \frac{1}{\underset{|z|\leq 1}{max}\left[\left(1 - |z|^2\right)|z| \frac{|z|+|a_2+3b_2|}{1+|a_2+3b_2||z|}\right]},$$

then the function $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{S}$.

Theorem 2.4.2. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i \in \mathcal{S}$, $i = \overline{1,n}$ and M_i, N_i positive real numbers. If*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| < M_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| < N_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| (M_i + 1) + |\beta_i| + |\gamma_i| (N_i + 1)\right] \leq \frac{1}{\underset{|z|\leq 1}{max}\left[\frac{1-|z|^{2c}}{c} \frac{|z|+|k|}{1+|k||z|}\right]},$$

where

$$|k| = \frac{\left|\sum_{i=1}^{n} \left[(\alpha_i - 1)(a_{2i} + 1) + 2\beta_i b_{2i} + \gamma_i (b_{2i} + 1)\right]\right|}{\sum_{i=1}^{n} \left[|\alpha_i - 1| (M_i + 1) + |\beta_i| + |\gamma_i| (N_i + 1)\right]},$$

then for any complex numbers δ, $Re\delta \geq Re\gamma$, the function $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

The following corollary is a consequence of the Theorem 2.4.2:

Corollary 2.4.2.1. *Let $\delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\delta > 0$ and $f_i, g_i \in \mathcal{S}$, $i = \overline{1,n}$ and M_i, N_i positive real numbers. If*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq M_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \leq N_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq 1,$$

and

$$\left|\sum_{i=1}^{n}\left[(\alpha_i - 1)(a_{2i}+1) + 2\beta_i b_{2i} + \gamma_i(b_{2i}+1)\right]\right| \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

$$\left|\sum_{i=1}^{n}\left[(\alpha_i - 1)(a_{2i}+1) + 2\beta_i b_{2i} + \gamma_i(b_{2i}+1)\right]\right| = \sum_{i=1}^{n}\left[|\alpha_i - 1|(M_i+1) + |\beta_i| + |\gamma_i|(N_i+1)\right],$$

then the function $\mathcal{M}_{\delta,n}$, given by (2.1.1) is in the class $\mathcal{S}$.

Theorem 2.4.3. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i, h_i \in \mathcal{S}$. If*

$$\left|\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right| \leq 1, \quad \left|g_i'(z)\right| \leq 1, \quad \left|\frac{zh_i'(z) - h_i(z)}{zh_i(z)}\right| \leq 1, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{\sum_{i=1}^{n}(|\alpha_i - 1| + |\beta_i| + |\gamma_i|)}{\prod_{i=1}^{n}(|\alpha_i - 1|\,|\beta_i|\,|\gamma_i|)} < 1,$$

$$\prod_{i=1}^{n}(|\alpha_i - 1|\,|\beta_i|\,|\gamma_i|) \leq \frac{1}{\max\limits_{|z|\leq 1}\left[2\left(1-|z|^2\right)|z|\,\frac{|z|+|k|}{1+|k||z|}\right]},$$

where

$$|k| = \frac{\left|\sum_{i=1}^{n}\left[(\alpha_i - 1)(a_{2i}+1) + 2\beta_i c_{2i} + \gamma_i c_{2i}\right]\right|}{2\prod_{i=1}^{n}(|\alpha_i - 1|\,|\beta_i|\,|\gamma_i|)},$$

then for any complex numbers δ, $Re\delta \geq Re\gamma$, the function $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.4.3, we obtain the next corollary:

Corollary 2.4.3.1. *Let $\alpha \in \mathbb{C}$, $Re\alpha > 0$ and $f, g, h \in \mathcal{S}$. If*

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| < 1, \quad \left|g'(z)\right| < 1, \quad \left|\frac{zh'(z) - h(z)}{zh(z)}\right| < 1, \quad \left|\frac{h''(z)}{h'(z)}\right| < 1,$$

for all $z \in \mathbb{U}$ and the constant $|\alpha|$ satisfies the condition

$$|\alpha| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[2\left(1-|z|^2\right)|z|\,\frac{2|z|+|a_2+3c_2|}{2+|a_2+3c_2||z|}\right]},$$

then the function $\mathcal{C}$, defined by (2.2.10) is in the class $\mathcal{S}$.

Theorem 2.4.4. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i, h_i \in \mathcal{S}$ and M_i, N_i, P_i are positive real numbers. If*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| < M_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| < N_i, \quad |g_i(z)| \leq 1, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| < P_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$,

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i + 1 \right) + |\beta_i| + |\gamma_i| \left(P_i + 1 \right) \right] \leq \frac{1}{\underset{|z|\leq 1}{\mathit{max}} \left[\frac{1-|z|^{2c}}{c} \frac{|z|+|k|}{1+|k||z|} \right]},$$

where

$$|k| = \frac{\left| \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(a_{2i} + b_{2i} + 1 \right) + 2\beta_i c_{2i} + \gamma_i \left(c_{2i} + 1 \right) \right] \right|}{\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i + 1 \right) + |\beta_i| + |\gamma_i| \left(P_i + 1 \right) \right]},$$

then for any complex numbers δ, *Re*$\delta \geq$ *Re*γ, *the function* $\mathcal{C}_{\delta,n}$, *defined by (2.1.2) is in the class* $\mathcal{S}$.

The following corollary is a consequence of the Theorem 2.4.4:

Corollary 2.4.4.1. *Let* $\delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\delta > 0$ *and* $f_i, g_i, h_i \in \mathcal{S}$, $i = \overline{1,n}$ *and* M_i, N_i, P_i *positive real numbers. If*

$$\left| \frac{z f_i'(z)}{f_i(z)} \right| \leq M_i, \quad \left| \frac{z g_i'(z)}{g_i(z)} \right| \leq N_i, \quad |g_i(z)| \leq 1, \quad \left| \frac{z h_i'(z)}{h_i(z)} \right| \leq P_i, \quad \left| \frac{z h_i''(z)}{h_i'(z)} \right| \leq 1,$$

and

$$\left| \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(a_{2i} + b_{2i} + 1 \right) + 2\beta_i c_{2i} + \gamma_i \left(c_{2i} + 1 \right) \right] \right| \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

$$\left| \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(a_{2i} + b_{2i} + 1 \right) + 2\beta_i c_{2i} + \gamma_i \left(c_{2i} + 1 \right) \right] \right| =$$

$$= \sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i + 1 \right) + |\beta_i| + |\gamma_i| \left(P_i + 1 \right) \right],$$

then the function $\mathcal{C}_{\delta,n}$, *defined by (2.1.2) is in the class* $\mathcal{S}$.

Theorem 2.4.5. *Let* $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ *and* $f_i, g_i, h_i, k_i \in \mathcal{S}$. *If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq 1, \; \left| g_i'(z) \right| \leq 1, \; \left| \frac{z h_i'(z) - h_i(z)}{z h_i(z)} \right| \leq 1, \; \left| \frac{z k_i'(z) - k_i(z)}{z h_i(z)} \right| \leq 1, \; \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1, \; \left| \frac{k_i''(z)}{k_i'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\frac{\sum_{i=1}^{n} \left(|\alpha_i - 1| + |\beta_i| + |\gamma_i| \right)}{\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\beta_i| \, |\gamma_i| \right)} < 1,$$

$$\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\beta_i| \, |\gamma_i| \right) \leq \frac{1}{\underset{|z|\leq 1}{\mathit{max}} \left[2 \left(1 - |z|^2 \right) |z| \, \frac{|z|+|k|}{1+|k||z|} \right]},$$

where

$$|k| = \frac{\left| \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(2a_{2i} + 1 \right) + \beta_i \left(c_{2i} + d_{2i} \right) + 2\gamma_i \left(c_{2i} + d_{2i} \right) \right] \right|}{2 \prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\beta_i| \, |\gamma_i| \right)},$$

then for any complex numbers δ, *Re*$\delta \geq$ *Re*γ, *the function* $\mathcal{G}_{\delta,n}$, *defined by (2.1.3) is in the class* $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.4.5, we obtain the next corollary:

Corollary 2.4.5.1. *Let $\alpha \in \mathbb{C}$, $Re\alpha > 0$ and $f, g, h, k \in \mathcal{S}$. If*

$$\left|\frac{f^{''}(z)}{f^{'}(z)}\right| < 1,\ \left|g^{'}(z)\right| < 1,\ \left|\frac{zh^{'}(z)-h(z)}{zh(z)}\right| < 1,\ \left|\frac{zk^{'}(z)-k(z)}{zh(z)}\right| < 1,\ \left|\frac{h^{''}(z)}{h^{'}(z)}\right| < 1,\ \left|\frac{k^{''}(z)}{k^{'}(z)}\right| < 1,$$

for all $z \in \mathbb{U}$ and the constant $|\alpha|$ satisfies the condition

$$|\alpha| \leq \frac{1}{\underset{|z|\leq 1}{max}\left[2\left(1-|z|^2\right)|z|\,\frac{2|z|+|2a_2+3c_2+3d_2+1|}{2+|2a_2+3c_2+3d_2+1||z|}\right]},$$

then the function $\mathcal{G}$, defined by (2.2.16) is in the class $\mathcal{S}$.

Theorem 2.4.6. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i.g_i, h_i, k_i \in \mathcal{S}$ and M_i, N_i, P_i positive real numbers. If*

$$\left|\frac{zf_i^{''}(z)}{f_i^{'}(z)}\right| < 1, \quad \left|\frac{zg_i^{'}(z)}{g_i(z)}\right| < M_i, \quad |g_i(z)| \leq 1,$$

$$\left|\frac{zh_i^{'}(z)}{h_i(z)}\right| < N_i, \quad \left|\frac{zk_i^{'}(z)}{k_i(z)}\right| < P_i, \quad \left|\frac{zh_i^{''}(z)}{h_i^{'}(z)}\right| \leq 1, \quad \left|\frac{zk_i^{''}(z)}{k_i^{'}(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$,

$$\sum_{i=1}^{n}\left[|\alpha_i - 1|(M_i+1) + |\beta_i|(N_i+P_i+2) + 2|\gamma_i|\right] \leq \frac{1}{\underset{|z|\leq 1}{max}\left[\frac{1-|z|^{2c}}{c}\frac{|z|+|k|}{1+|k||z|}\right]},$$

where

$$|k| = \frac{\left|\sum_{i=1}^{n}\left[(\alpha_i-1)(2a_{2i}+b_{2i}) + \beta_i(c_{2i}+d_{2i}+2) + 2\gamma_i(c_{2i}+d_{2i})\right]\right|}{\sum_{i=1}^{n}\left[|\alpha_i-1|(M_i+1) + |\beta_i|(N_i+P_i+2) + 2|\gamma_i|\right]},$$

then for any complex numbers δ, $Re\delta \geq Re\gamma$, the function $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is in the class $\mathcal{S}$.

The following corollary is a consequence of the Theorem 2.4.6:

Corollary 2.4.6.1. *Fie $\delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\delta > 0$ and $f_i, g_i, h_i, k_i \in \mathcal{S}$ and M_i, N_i, P_i positive real numbers. If*

$$\left|\frac{zf_i^{''}(z)}{f_i^{'}(z)}\right| \leq 1, \quad \left|\frac{zg_i^{'}(z)}{g_i(z)}\right| \leq M_i, \quad |g_i(z)| \leq 1,$$

$$\left|\frac{zh_i^{'}(z)}{h_i(z)}\right| \leq N_i, \quad \left|\frac{zk_i^{'}(z)}{k_i(z)}\right| \leq P_i, \quad \left|\frac{zh_i^{''}(z)}{h_i^{'}(z)}\right| \leq 1, \quad \left|\frac{zk_i^{''}(z)}{k_i^{'}(z)}\right| \leq 1,$$

and

$$\left|\sum_{i=1}^{n}\left[(\alpha_i-1)(2a_{2i}+b_{2i}) + \beta_i(c_{2i}+d_{2i}+2) + 2\gamma_i(c_{2i}+d_{2i})\right]\right| \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

$$\left|\sum_{i=1}^{n}\left[(\alpha_i-1)(2a_{2i}+b_{2i}) + \beta_i(c_{2i}+d_{2i}+2) + 2\gamma_i(c_{2i}+d_{2i})\right]\right| =$$

$$= \sum_{i=1}^{n}\left[|\alpha_i-1|(M_i+1) + |\beta_i|(N_i+P_i+2) + 2|\gamma_i|\right],$$

then the function $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is in the class $\mathcal{S}$.

Theorem 2.4.7. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i, h_i, k_i \in \mathcal{S}$. If*

$$\left|\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right| \le 1, \ \left|\frac{g_i''(z)}{g_i'(z)}\right| \le 1, \ \left|\frac{zh_i'(z) - h_i(z)}{zh_i(z)}\right| \le 1,$$

$$\left|\frac{zk_i'(z) - k_i(z)}{zh_i(z)}\right| \le 1, \ \left|\frac{h_i''(z)}{h_i'(z)}\right| \le 1, \ \left|\frac{k_i''(z)}{k_i'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{\sum_{i=1}^n (|\alpha_i - 1| + |\beta_i| + |\gamma_i| + |\delta_i|)}{\prod_{i=1}^n (|\alpha_i - 1| |\beta_i| |\gamma_i| |\delta_i|)} < 1,$$

$$\prod_{i=1}^n (|\alpha_i - 1| |\beta_i| |\gamma_i| |\delta_i|) \le \frac{1}{\max\limits_{|z|\le 1}\left[2\left(1-|z|^2\right)|z| \frac{|z|+|k|}{1+|k||z|}\right]},$$

where

$$|k| = \frac{\left|\sum_{i=1}^n \left[(\alpha_i - 1) a_{2i} + 2\beta_i b_{2i} + \gamma_i (c_{2i} + d_{2i}) + 2\delta_i (c_{2i} + d_{2i})\right]\right|}{2\prod_{i=1}^n (|\alpha_i - 1| |\beta_i| |\gamma_i| |\delta_i|)},$$

then for any complex numbers δ, $Re\delta \ge Re\gamma$, the function $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.4.7, we obtain the next corollary:

Corollary 2.4.7.1. *Let $\alpha \in \mathbb{C}$, $Re\alpha > 0$ and $f, g, h, k \in \mathcal{S}$. If*

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| < 1, \quad \left|\frac{g''(z)}{g'(z)}\right| < 1, \quad \left|\frac{zh'(z) - h(z)}{zh(z)}\right| < 1,$$

$$\left|\frac{zk'(z) - k(z)}{zh(z)}\right| < 1, \quad \left|\frac{h''(z)}{h'(z)}\right| < 1, \quad \left|\frac{k''(z)}{k'(z)}\right| < 1,$$

for all $z \in \mathbb{U}$ and the constant $|\alpha|$ satisfies the condition

$$|\alpha| \le \frac{1}{\max\limits_{|z|\le 1}\left[2\left(1-|z|^2\right)|z| \frac{2|z|+|a_2+2b_2+3c_2+3d_2|}{2+|a_2+2b_2+3c_2+3d_2||z|}\right]},$$

then the function $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{S}$.

Theorem 2.4.8. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i, h_i, k_i \in \mathcal{S}$ and M_i, N_i, P_i positive real numbers. If*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| < M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le 1, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| < N_i, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| < P_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$,

$$\sum_{i=1}^n \left[|\alpha_i - 1| (M_i + 1) + |\beta_i| + |\gamma_i| (N_i + P_i + 2) + 2|\delta_i|\right] \le \frac{1}{\max\limits_{|z|\le 1}\left[\frac{1-|z|^{2c}}{c} \frac{|z|+|k|}{1+|k||z|}\right]},$$

where

$$|k| = \frac{\left|\sum_{i=1}^n \left[(\alpha_i - 1)(a_{2i} + 1) + 2\beta_i b_{2i} + \gamma_i (c_{2i} + d_{2i} + 2) + 2\delta_i (c_{2i} + d_{2i})\right]\right|}{\sum_{i=1}^n \left[|\alpha_i - 1| (M_i + 1) + |\beta_i| + |\gamma_i| (N_i + P_i + 2) + 2|\delta_i|\right]},$$

then for any complex numbers δ, $Re\delta \ge Re\gamma$, the function $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

The following corollary is a consequence of the Theorem 2.4.8:

Corollary 2.4.8.1. *Let $\delta, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\delta > 0$ and $f_i, g_i, h_i, k_i \in \mathcal{S}$ and M_i, N_i, P_i positive real numbers. If*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)}\right| \leq M_i, \quad \left|\frac{zg_i^{''}(z)}{g_i^{'}(z)}\right| \leq 1, \quad \left|\frac{zh_i^{'}(z)}{h_i(z)}\right| \leq N_i, \quad \left|\frac{zk_i^{'}(z)}{k_i(z)}\right| \leq P_i, \quad \left|\frac{zh_i^{''}(z)}{h_i^{'}(z)}\right| \leq 1, \quad \left|\frac{zk_i^{''}(z)}{k_i^{'}(z)}\right| \leq 1,$$

and

$$\left|\sum_{i=1}^{n}\left[(\alpha_i-1)(a_{2i}+1)+2\beta_i b_{2i}+\gamma_i(c_{2i}+d_{2i}+2)+2\delta_i(c_{2i}+d_{2i})\right]\right| \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

$$\left|\sum_{i=1}^{n}\left[(\alpha_i-1)(a_{2i}+1)+2\beta_i b_{2i}+\gamma_i(c_{2i}+d_{2i}+2)+2\delta_i(c_{2i}+d_{2i})\right]\right| =$$

$$= \sum_{i=1}^{n}\left[|\alpha_i-1|(M_i+1)+|\beta_i|+|\gamma_i|(N_i+P_i+2)+2|\delta_i|\right],$$

then the function $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

2.5 Univalence conditions for the class $\mathcal{G}_b$

In this section we present sufficient conditions of univalence of integral operators $\mathcal{M}_{\delta,n}$, $\mathcal{T}_{\delta,n}$ for the situation when the functions involved belong to the class of functions $\mathcal{G}_b$, $0 < b \leq 1$.

Theorem 2.5.1. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$, with*

$$c \geq \sum_{i=1}^{n}\left[|\alpha_i-1|+(2b_i+1)|\beta_i|+|\gamma_i|\right].$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right| < 1, \quad \left|\frac{zg_i'(z)}{g_i(z)}-1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

Putting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Theorem 2.5.1, we obtain the next corollary:

Corollary 2.5.1.1. *Let α be complex number, $Re\alpha > 0$, with*

$$Re\alpha \geq |\alpha-1|(2b+3).$$

If $g \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1, \quad \left|\frac{zg'(z)}{g(z)}-1\right| < 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{S}$.

Theorem 2.5.2. *Let $\alpha_i, \beta_i, \gamma_i$ be complex numbers, $M_i \geq 1$, $N_i \geq 1$ positive real numbers, for all $i = \overline{1,n}$ and $\gamma \in \mathbb{C}$ with $c = Re\gamma$ and*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| \left(2M_i + 1\right) + \left(b_i \left|\beta_i\right| + \left|\beta_i\right| + \left|\gamma_i\right|\right) \left(2N_i + 1\right) + b_i \left|\beta_i\right| \right].$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{z^2 f_i'(z)}{\left[f_i(z)\right]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g_i'(z)}{\left[g_i(z)\right]^2} - 1 \right| < 1, \quad |f_i(z)| \leq M_i, \quad |g_i(z)| \leq N_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then for any complex numbers δ, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{M}_{\delta,n}$, given by (2.1.1) is in the class $\mathcal{S}$.

If we consider $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1$ and $b_1 = b$ in Theorem 2.5.2, we obtain the next corollary:

Corollary 2.5.2.1. *Let α be complex number, $M \geq 1$, $N \geq 1$ positive real numbers, $Re\alpha > 0$ and*

$$Re\alpha \geq |\alpha - 1| \left(2M + 2bN + 4N + 2b + 3\right).$$

If $g \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left| \frac{z^2 f'(z)}{\left[f(z)\right]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g'(z)}{\left[g(z)\right]^2} - 1 \right| < 1 \quad |f(z)| \leq M, \quad |g(z)| \leq N,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{S}$.

Theorem 2.5.3. *Let $\alpha_i, \beta_i, \gamma_i$ be complex numbers and $\delta \in \mathbb{C}$ with*

$$Re\delta \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + \left(2b_i + 1\right) |\beta_i| + |\gamma_i| \right],$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1| + \left(2b_i + 1\right) |\beta_i| + |\gamma_i| \right].$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

Letting $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1$ and $b_1 = b$ in Theorem 2.5.3, we obtain the next corollary:

Corollary 2.5.3.1. *Let $\alpha \in \mathbb{C}^*$ with*

$$Re\alpha \geq |\alpha - 1| \left(2b + 3\right),$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\alpha} |\alpha - 1| (2b + 3) .$$

If $g \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad \left| \frac{zg'(z)}{g(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{S}$.

Theorem 2.5.4. *Let $\alpha_i, \beta_i, \gamma_i$ be complex numbers, $M_i \geq 1$, $N_i \geq 1$ positive real numbers, $\delta \in \mathbb{C}$ with*

$$Re\delta \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| (2M_i + 1) + (b_i |\beta_i| + |\beta_i| + |\gamma_i|) (2N_i + 1) + b_i |\beta_i| \right] ,$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1| (2M_i + 1) + (b_i |\beta_i| + |\beta_i| + |\gamma_i|) (2N_i + 1) + b_i |\beta_i| \right] .$$

If $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, then the integral operator $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

Letting $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1$ and $b_1 = b$ in Theorem 2.5.4, we obtain the next corollary:

Corollary 2.5.4.1. *Let $\alpha \in \mathbb{C}^*$, $M \geq 1$, $N \geq 1$ positive real numbers, with*

$$Re\alpha \geq |\alpha - 1| (2M + 2bN + 4N + 2b + 3) ,$$

and let $c \in \mathbb{C}$, be such that

$$|c| \leq 1 - \frac{1}{Re\alpha} |\alpha - 1| (2M + 2bN + 4N + 2b + 3) .$$

If $g \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| < 1$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{S}$.

Theorem 2.5.5. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = Re\gamma > 0$, with*

$$c \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1) |\beta_i| + 2 |\gamma_i| + (4b_i + 2) |\delta_i| \right] .$$

If $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| < 1, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| < 1, \quad \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, then the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1 = \delta_1$ in Theorem 2.5.5, we obtain the next corollary:

Corollary 2.5.5.1. *Let α be complex number, $Re\alpha > 0$, with*

$$Re\alpha \geq 6\left|\alpha - 1\right|(b+1).$$

If $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$ $f \in \mathcal{A}$ and

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1, \ \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1, \ \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| < 1, \ \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| < 1,$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{S}$.

Theorem 2.5.6. *Let $\alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $M_i \geq 1$, $N_i \geq 1$, $P_i \geq 1$, $Q_i \geq 1$ positive real numbers, $\gamma \in \mathbb{C}$, $c = Re\gamma$ and*

$$c \geq \sum_{i=1}^{n}\left[\left|\alpha_i - 1\right|(2M_i + 1) + \left(b_i\left|\beta_i\right| + \left|\beta_i\right|\right)(2N_i + 1)\right] +$$

$$+\sum_{i=1}^{n}\left[\left(\left|\gamma_i\right| + \left|\delta_i\right| b_i + \left|\delta_i\right|\right)(2P_i + 2Q_i + 2) + b_i\left|\beta_i\right| + 2b_i\left|\delta_i\right|\right].$$

If $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$ $f_i \in \mathcal{A}$ and

$$\left|\frac{z^2 f_i'(z)}{\left[f_i(z)\right]^2} - 1\right| < 1, \ \left|\frac{z^2 g_i'(z)}{\left[g_i(z)\right]^2} - 1\right| < 1, \ \left|\frac{z^2 h_i'(z)}{\left[h_i(z)\right]^2} - 1\right| < 1, \ \left|\frac{z^2 k_i'(z)}{\left[k_i(z)\right]^2} - 1\right| < 1,$$

$$\left|f_i(z)\right| \leq M_i, \ \left|g_i(z)\right| \leq N_i, \ \left|h_i(z)\right| \leq P_i, \ \left|k_i(z)\right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, then for any complex numbers δ, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Letting $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1$ and $b_1 = b$ in Theorem 2.5.6, we obtain the next corollary:

Corollary 2.5.6.1. *Let α be complex number, $Re\alpha > 0$, $M \geq 1$, $N \geq 1$, $P \geq 1$, $Q \geq 1$ positive real numbers, with*

$$Re\alpha \geq \left[2\left|\alpha - 1\right|(M + N + 2P + 2Q + 3) + 2\left|\alpha - 1\right| b\,(N + P + Q + 3)\right].$$

If $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left|\frac{z^2 f'(z)}{\left[f(z)\right]^2} - 1\right| < 1, \left|\frac{z^2 g'(z)}{\left[g(z)\right]^2} - 1\right| < 1, \left|\frac{z^2 h'(z)}{\left[h(z)\right]^2} - 1\right| < 1, \quad \left|\frac{z^2 k'(z)}{\left[k(z)\right]^2} - 1\right| < 1$$

$$\left|f(z)\right| \leq M, \quad \left|g(z)\right| \leq N, \quad \left|h(z)\right| \leq P, \quad \left|k(z)\right| \leq Q,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{S}$.

Theorem 2.5.7. *Let $\alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $\delta \in \mathbb{C}$ with*

$$Re\delta \geq \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1)\,|\beta_i| + 2\,|\gamma_i| + (4b_i + 2)\,|\delta_i| \right]$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1| + (2b_i + 1)\,|\beta_i| + 2\,|\gamma_i| + (4b_i + 2)\,|\delta_i| \right].$$

If $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| < 1, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| < 1, \quad \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Putting $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1$ and $b_1 = b$ in Theorem 2.5.7, we obtain the next corollary:

Corollary 2.5.7.1. *Let $\alpha \in \mathbb{C}^*$ with*

$$Re\alpha \geq 6\,|\alpha - 1|\,(b + 1)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{6}{Re\alpha}\,|\alpha - 1|\,(b + 1).$$

If $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| < 1, \left| \frac{zh'(z)}{h(z)} - 1 \right| < 1, \left| \frac{zk'(z)}{k(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{S}$.

Theorem 2.5.8. *Let $\alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $M_i \geq 1$, $N_i \geq 1$, $P_i \geq 1$, $Q_i \geq 1$ positive real numbers, $\delta \in \mathbb{C}$ with*

$$Re\delta \geq \sum_{i=1}^{n} \left[|\alpha_i - 1|\,(2M_i + 1) + (b_i\,|\beta_i| + |\beta_i|)\,(2N_i + 1) \right] +$$

$$+ \sum_{i=1}^{n} \left[(|\gamma_i| + |\delta_i|\,b_i + |\delta_i|)\,(2P_i + 2Q_i + 2) + b_i\,|\beta_i| + 2b_i\,|\delta_i| \right]$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1|\,(2M_i + 1) + (b_i\,|\beta_i| + |\beta_i|)\,(2N_i + 1) \right] -$$

$$- \frac{1}{Re\delta} \sum_{i=1}^{n} \left[(|\gamma_i| + |\delta_i|\,b_i + |\delta_i|)\,(2P_i + 2Q_i + 2) + b_i\,|\beta_i| + 2b_i\,|\delta_i| \right].$$

If $g_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$, $f_i \in \mathcal{A}$ and

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| < 1, \left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| < 1, \left| \frac{z^2 h_i'(z)}{[h_i(z)]^2} - 1 \right| < 1, \left| \frac{z^2 k_i'(z)}{[k_i(z)]^2} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{T}_{\delta,n}$ defined by (2.1.4) is in the class $\mathcal{S}$.

Letting $n = 1$, $\alpha_1 - 1 = \beta_1 = \gamma_1$ and $b_1 = b$ in Theorem 2.5.8, we obtain the next corollary:

Corollary 2.5.8.1. *Let $\alpha \in \mathbb{C}^*$, $M \geq 1$, $N \geq 1$, $P \geq 1$, $Q \geq 1$ positive real numbers, with*

$$Re\alpha \geq [2\,|\alpha - 1|\,(M + N + 2P + 2Q + 3) + 2\,|\alpha - 1|\,b\,(N + P + Q + 3)]$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re\alpha}\,[2\,|\alpha - 1|\,(M + N + 2P + 2Q + 3) + 2\,|\alpha - 1|\,b\,(N + P + Q + 3)]\,.$$

If $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$, $f \in \mathcal{A}$ and

$$\left|\frac{z^2 f'(z)}{[f(z)]^2} - 1\right| < 1,\ \left|\frac{z^2 g'(z)}{[g(z)]^2} - 1\right| < 1,\ \left|\frac{z^2 h'(z)}{[h(z)]^2} - 1\right| < 1,\ \left|\frac{z^2 k'(z)}{[k(z)]^2} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{S}$.

2.6 Univalence conditions for the class$\mathcal{S}(p)$

This section contains sufficient conditions of univalence of the four integral operators if the functions involved belong to the class of functions $\mathcal{S}_p$, $0 < p \leq 2$.

Theorem 2.6.1. *Let $f_i, g_i \in \mathcal{A}$, where f_i, h_i be in the class $\mathcal{S}\,(p_i)$, $0 < p_i \leq 2$, iar M_i, N_i are real positive numbers and $\delta, \alpha_i, \beta_i, \gamma_i, c$ be complex numbers for all $i = \overline{1, n}$, with*

$$Re\delta > \sum_{i=1}^{n} \{|\alpha_i - 1|\,[(1 + p_i)\,M_i + 1] + |\beta_i| + |\gamma_i|\,[(1 + p_i)\,N_i + 1]\}\,, \quad |c| \leq 1,\ c \neq -1.$$

If

$$|f_i\,(z)| < M_i, \quad |g_i\,(z)| < N_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1,$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \{|\alpha_i - 1|\,[(1 + p_i)\,M_i + 1] + |\beta_i| + |\gamma_i|\,[(1 + p_i)\,N_i + 1]\}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, then the integral operator $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

Letting $M_i = N_i = 1$ in Theorem 2.6.1, we obtain the next corollary:

Corollary 2.6.1.1. *Let $f_i, g_i \in \mathcal{S}\,(p_i)$, $0 < p_i \leq 2$ and $\delta, \alpha_i, \beta_i, \gamma_i, c$ be complex numbers for all $i = \overline{1, n}$, with*

$$Re\delta > \sum_{i=1}^{n} [(p_i + 2)\,(|\alpha_i - 1| + |\gamma_i|) + |\beta_i|]\,, \quad |c| \leq 1.$$

If

$$|f_i\,(z)| < 1, \quad |g_i\,(z)| < 1, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1,$$

and
$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[(p_i + 2) \left(|\alpha_i - 1| + |\gamma_i| \right) + |\beta_i| \right],$$
oricare ar fi $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

If we consider $n = 1$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Theorem 2.6.1, we obtain the next corollary:

Corollary 2.6.1.2. *Let $f, g \in \mathcal{S}(p)$, $0 < p \leq 2$, M, N positive real numbers and α, c complex numbers, with*
$$Re\alpha > |\alpha - 1| \left[(1+p) M + (1+p) N + 3 \right], \quad |c| \leq 1.$$
If
$$|f(z)| < M, \quad |g(z)| < N, \quad \left| \frac{g''(z)}{g'(z)} \right| \leq 1,$$
and
$$|c| \leq 1 - \frac{1}{Re\alpha} |\alpha - 1| \left[(1+p) M + (1+p) N + 3 \right],$$
for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{S}$.

Theorem 2.6.2. *Let $f_i, g_i, h_i \in \mathcal{A}$, where f_i, g_i, h_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \leq 2$, M_i, N_i, P_i are real positive numbers and $\delta, \alpha_i, \beta_i, \gamma_i, c$ be complex numbers for all $i = \overline{1,n}$, with*
$$Re\delta > \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[\left(M_i + N_i^2 \right) (1 + p_i) + 1 \right] + |\beta_i| + |\gamma_i| \left[(1 + p_i) P_i + 1 \right] \right\}, \quad |c| \leq 1, \ c \neq -1.$$
If
$$|f_i(z)| < M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1, \quad |h_i(z)| < P_i$$
and
$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[\left(M_i + N_i^2 \right) (1 + p_i) + 1 \right] + |\beta_i| + |\gamma_i| \left[(1 + p_i) P_i + 1 \right] \right\},$$
for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

Letting $M_i = N_i = P_i = 1$ in Theorem 2.6.2, we obtain the next corollary:

Corollary 2.6.2.1. *Let $f_i, g_i, h_i \in \mathcal{S}(p_i)$, $0 < p_i \leq 2$ and $\delta, \alpha_i, \beta_i, \gamma_i, c$ be complex numbers for all $i = \overline{1,n}$, with*
$$Re\delta > \sum_{i=1}^{n} \left[(2p_i + 3) |\alpha_i - 1| + |\beta_i| + (p_i + 2) |\gamma_i| \right], \quad |c| \leq 1.$$
If
$$|f_i(z)| \leq 1, \quad |g_i(z)| \leq 1, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1, \quad |h_i(z)| \leq 1$$
and
$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[(2p_i + 3) |\alpha_i - 1| + |\beta_i| + (p_i + 2) |\gamma_i| \right],$$
for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Theorem 2.6.2, we obtain the next corollary:

Corollary 2.6.2.2. *Let $f, g, h \in \mathcal{S}(p)$, $0 < p \leq 2$, M, N, P be real positive numbers and α, c complex numbers, with*

$$Re\alpha > |\alpha - 1| \left[\left(M + N^2 + P \right) (1 + p) + 3 \right], \quad |c| \leq 1.$$

If

$$|f(z)| < M, \quad |g(z)| < N, \quad \left| \frac{h''(z)}{h'(z)} \right| \leq 1, \quad |h(z)| < P$$

and

$$|c| \leq 1 - \frac{1}{Re\alpha} |\alpha - 1| \left[\left(M + N^2 + P \right) (1 + p) + 3 \right],$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{C}$, defined by (2.2.10) is in the class $\mathcal{S}$.

Theorem 2.6.3. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}$, where g_i, h_i, k_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \leq 2$, M_i, N_i, P_i are real positive numbers and $\delta, \alpha_i, \beta_i, \gamma_i, c$ be complex numbers for all $i = \overline{1, n}$, with*

$$Re\delta > \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[(1 + p_i) M_i^2 + 1 \right] + |\beta_i| \left[(N_i + P_i)(1 + p_i) + 2 \right] + 2 |\gamma_i| \right\}, \quad |c| \leq 1, \ c \neq -1.$$

If

$$|g_i(z)| < M_i, \quad |h_i(z)| < N_i, \quad |k_i(z)| < P_i, \quad \left| \frac{f_i''(z)}{f_i'(z)} \right| \leq 1, \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1, \left| \frac{k_i''(z)}{k_i'(z)} \right| \leq 1$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[(1 + p_i) M_i^2 + 1 \right] + |\beta_i| \left[(N_i + P_i)(1 + p_i) + 2 \right] + 2 |\gamma_i| \right\},$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, then the integral operator $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is in the class $\mathcal{S}$.

Putting $M_i = N_i = P_i = 1$ in Theorem 2.6.3, we obtain the next corollary:

Corollary 2.6.3.1. *Let $f_i, g_i, h_i, k_i \in \mathcal{S}(p_i)$, $0 < p_i \leq 2$ and $\delta, \alpha_i, \beta_i, \gamma_i, c$ be complex numbers for all $i = \overline{1, n}$, with*

$$Re\delta > \sum_{i=1}^{n} \left[|\alpha_i - 1| (2 + p_i) + 2 |\beta_i| (2 + p_i) + 2 |\gamma_i| \right], \quad |c| \leq 1.$$

If

$$|g_i(z)| < 1, \quad |h_i(z)| < 1, \quad |k_i(z)| < 1, \quad \left| \frac{f_i''(z)}{f_i'(z)} \right| \leq 1, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1, \quad \left| \frac{k_i''(z)}{k_i'(z)} \right| \leq 1$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[|\alpha_i - 1| (2 + p_i) + 2 |\beta_i| (2 + p_i) + 2 |\gamma_i| \right],$$

for all $z \in \mathbb{U}$ and $i = \overline{1, n}$, then the integral operator $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is in the class $\mathcal{S}$.

If we consider $n=1$, $\delta-1=\gamma=\alpha$ and $\alpha_1-1=\beta_1=\gamma_1$ in Theorem 2.6.3, we obtain the next corollary:

Corollary 2.6.3.2. *Let $f,g,h,k\in\mathcal{S}(p)$, $0<p\leq 2$, M,N,P are real positive numbers and α,c be complex numbers for all*

$$Re\alpha>|\alpha-1|\left[(1+p)\left(M^2+N+P\right)+5\right],\quad |c|\leq 1.$$

If

$$|g(z)|<M,\quad |h(z)|<N,\quad |k(z)|<P,\quad \left|\frac{f''(z)}{f'(z)}\right|\leq 1,\ \left|\frac{h''(z)}{h'(z)}\right|\leq 1,\ \left|\frac{k''(z)}{k'(z)}\right|\leq 1$$

and

$$|c|\leq 1-\frac{1}{Re\alpha}|\alpha-1|\left[(1+p)\left(M^2+N+P\right)+5\right],$$

for all $z\in\mathbb{U}$, then the integral operator $\mathcal{G}$, defined by (2.2.16) is in the class $\mathcal{S}$.

Theorem 2.6.4. *Let $f_i,g_i,h_i,k_i\in\mathcal{A}$, with f_i,h_i,k_i, be in the class $\mathcal{S}(p_i)$, $0<p_i\leq 2$, M_i,N_i,P_i are real positive numbers and $\delta,\alpha_i,\beta_i,\gamma_i,\delta_i,c$ be complex numbers for all $i=\overline{1,n}$, with*

$$Re\delta>\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[M_i(1+p_i)+1\right]+|\beta_i|+|\gamma_i|\left[(N_i+P_i)(1+p_i)+2\right]+2|\delta_i|\right\},$$

for $|c|\leq 1$, $c\neq -1$. If

$$|f_i(z)|<M_i,\quad |h_i(z)|<N_i,\quad |k_i(z)|<P_i,\quad \left|\frac{g_i''(z)}{g_i'(z)}\right|\leq 1,\quad \left|\frac{h_i''(z)}{h_i'(z)}\right|\leq 1,\quad \left|\frac{k_i''(z)}{k_i'(z)}\right|\leq 1,$$

and

$$|c|\leq 1-\frac{1}{Re\delta}\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[M_i(1+p_i)+1\right]+|\beta_i|+|\gamma_i|\left[(N_i+P_i)(1+p_i)+2\right]+2|\delta_i|\right\},$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$, then the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Letting $M_i=N_i=P_i=1$ in Theorem 2.6.4, we obtain the next corollary:

Corollary 2.6.4.1. *Let $f_i,g_i,h_i,k_i\in\mathcal{S}(p_i)$, $0<p_i\leq 2$ and $\delta,\alpha_i,\beta_i,\gamma_i,\delta_i,c$ be complex numbers for all $i=\overline{1,n}$, with*

$$Re\delta>\sum_{i=1}^{n}\left[|\alpha_i-1|(p_i+2)+|\beta_i|+2|\gamma_i|(p_i+2)+2|\delta_i|\right],\quad |c|\leq 1.$$

If

$$|f_i(z)|<1,\quad |h_i(z)|<1,\quad |k_i(z)|<1,\quad \left|\frac{g_i''(z)}{g_i'(z)}\right|\leq 1,\quad \left|\frac{h_i''(z)}{h_i'(z)}\right|\leq 1,\quad \left|\frac{k_i''(z)}{k_i'(z)}\right|\leq 1,$$

and

$$|c|\leq 1-\frac{1}{Re\delta}\sum_{i=1}^{n}\left[|\alpha_i-1|(p_i+2)+|\beta_i|+2|\gamma_i|(p_i+2)+2|\delta_i|\right],$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$, then the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

If we consider $n = 1$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Theorem 2.6.4, we obtain the next corollary:

Corollary 2.6.4.2. *Let $f, g, h, k \in \mathcal{S}(p)$, $0 < p \leq 2$, M, N, P are real positive numbers and α, c be complex numbers, for all*

$$Re\alpha > \{|\alpha - 1| [(M + N + P)(1 + p) + 6]\}, \quad |c| \leq 1.$$

If

$$|f(z)| < M, \quad |h(z)| < N, \quad |k(z)| < P, \quad \left|\frac{g''(z)}{g'(z)}\right| \leq 1, \quad \left|\frac{h''(z)}{h'(z)}\right| \leq 1, \quad \left|\frac{k''(z)}{k'(z)}\right| \leq 1,$$

and

$$|c| \leq 1 - \frac{1}{Re\alpha}\{|\alpha - 1| [(M + N + P)(1 + p) + 6]\},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{S}$.

2.7 Univalence conditions for classes $\mathcal{B}(\mu)$ and $\mathcal{S}_\mu$

This paragraph describes sufficient conditions of univalence for the four integral operators whose functions belong to the classes of functions. $\mathcal{B}(\mu)$, $0 \leq \mu < 1$ and $\mathcal{S}_\mu$, $0 < \mu \leq 1$.

Theorem 2.7.1. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = \mathrm{Re}\gamma > 0$, $i = \overline{1, n}$ and f_i, g_i be in the class $\mathcal{B}(\mu_i)$, $0 \leq \mu_i < 1$, $i = \overline{1, n}$ satisfy the inequality*

$$\sum_{i=1}^{n} (1 - \mu_i)(2|\alpha_i - 1| + 3|\beta_i| + 2|\gamma_i|) \leq \begin{cases} c, & if \quad 0 < c < \frac{1}{2} \\ \frac{1}{2} & if \quad \frac{1}{2} < c < \infty, \end{cases}$$

then for all δ complex numbers, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{M}_{\delta,n}$, given by (2.1.1) is analytic and univalent in $\mathcal{S}$.

Theorem 2.7.2. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers and the analytic functions f_i, g_i be in the class $\mathcal{S}_{\mu_i}$, $0 < \mu_i \leq 1$, $i = \overline{1, n}$ satisfy the inequality*

$$\left|\frac{z g_i''(z)}{g_i'(z)}\right| < |z|$$

If $\gamma \in \mathbb{C}$, with $\mathrm{Re}\gamma = c > 0$ and

$$\sum_{i=1}^{n} (\mu_i |\alpha_i - 1| + |\beta_i| + \mu_i |\gamma_i|) \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then for any complex number δ, $\mathrm{Re}\delta \leq c$, the general integral operator $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is analytic and univalent in $\mathbb{U}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.7.2, we obtain the next corollary:

Corollary 2.7.2.1. *Let be the analytic functions f, g be in the class $\mathcal{S}_\mu$, $0<\mu\le 1$ satisfy the inequality*

$$\left|\frac{zg''(z)}{g'(z)}\right| < |z|.$$

If $\alpha\in\mathbb{C}$, where $\mathrm{Re}\alpha=c>0$ *and*

$$|\alpha-1|(2\mu+1)\le\frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is analytic and univalent in $\mathbb{U}$.

Theorem 2.7.3. *Let be the analytic functions f_i, g_i be in the class $\mathcal{B}(\mu_i)$, $0\le\mu_i<1$, $i=\overline{1,n}$ and $\delta,\gamma,\alpha_i,\beta_i,\gamma_i$ be complex numbers, with $\delta\ne 0$. Suppose that $M_i\ge 1$, $N_i\ge 1$ are positive real numbers, with* $\mathrm{Re}\gamma=c>0$ *and*

$$Re\delta\ge\sum_{i=1}^{n}\{|\alpha_i-1|\,[(2-\mu_i)\,M_i+1]+|\beta_i|+|\gamma_i|\,[(2-\mu_i)\,N_i+1]\}.$$

If

$$|f_i(z)|\le M_i,\quad |g_i(z)|\le N_i,\ \left|\frac{zg_i''(z)}{g_i'(z)}\right|\le 1,\quad z\in\mathbb{U},\quad i=\overline{1,n}$$

and

$$|c|\le 1-\frac{1}{Re\delta}\sum_{i=1}^{n}\{|\alpha_i-1|\,[(2-\mu_i)\,M_i+1]+|\beta_i|+|\gamma_i|\,[(2-\mu_i)\,N_i+1]\},\quad c\in\mathbb{C},\ c\ne 0,$$

then the integral operator $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is analytic and univalent in $\mathcal{S}$.

Putting $n=1$ and $\delta=\gamma=\lambda$, in Theorem 2.7.3, we obtain the next corollary:

Corollary 2.7.3.1. *Let be the analytic functions f, g be in the class $\mathcal{B}(\mu)$, $0\le\mu<1$ and $\lambda,\alpha,\beta,\gamma$ be complex numbers, with $\lambda\ne 0$. Suppose that $M\ge 1$, $N\ge 1$ are positive real numbers, with*

$$Re\lambda\ge\{|\alpha-1|\,[(2-\mu)\,M+1]+|\beta|+|\gamma|\,[(2-\mu)\,N+1]\}.$$

If

$$|f(z)|\le M,\quad |g(z)|\le N,\ \left|\frac{zg''(z)}{g'(z)}\right|\le 1,\quad z\in\mathbb{U}$$

and

$$|c|\le 1-\frac{1}{Re\lambda}\{|\alpha-1|\,[(2-\mu)\,M+1]+|\beta|+|\gamma|\,[(2-\mu)\,N+1]\},\quad c\in\mathbb{C},\ c\ne 0$$

then the integral operator $\mathcal{M}$, defined in

$$\mathcal{M}^*(z)=\int_0^z\left[\left(\frac{f(t)}{t}\right)^{\alpha-1}(g'(t))^{\beta}\left(\frac{g(t))}{t}\right)^{\gamma}\right]dt \tag{2.7.1}$$

is in the class $\mathcal{S}$.

Theorem 2.7.4. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$, $c = \mathrm{R}e\gamma > 0$ be complex numbers and the analytic functions f_i, g_i, h_i be in the class $\mathcal{B}(\mu_i)$, $0 \le \mu_i < 1$ and $g_i(z) \in \mathcal{R}(\mu_i)$, $i = \overline{1,n}$ satisfy the inequality*

$$\sum_{i=1}^{n} (1-\mu_i)\left(2\left|\alpha_i - 1\right| + 3\left|\beta_i\right| + 2\left|\gamma_i\right|\right) + 2\left|\alpha_i - 1\right| \le \begin{cases} c, & if \quad 0 < c < \frac{1}{2}, \\ \frac{1}{2} & if \quad \frac{1}{2} < c < \infty, \end{cases}$$

then for any complex number δ, $Re\delta \ge Re\gamma$, the general integral operator $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

Theorem 2.7.5. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers and the analytic functions f_i, g_i, h_i be in the class $\mathcal{S}_{\mu_i}$, $0 < \mu_i \le 1$, $i = \overline{1,n}$ satisfy the inequality*

$$|g_i(z)| \le 1, \qquad \left|\frac{zh_i''(z)}{h_i'(z)}\right| < |z|\,.$$

If $\gamma \in \mathbb{C}$ with $\mathrm{Re}\gamma = c > 0$ and

$$\sum_{i=1}^{n} \left[(2\mu_i + 1)\left|\alpha_i - 1\right| + \left|\beta_i\right| + \mu_i\left|\gamma_i\right|\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then for any complex number δ, $\mathrm{Re}\delta \le c$, the general integral operator $\mathcal{C}_{\delta,n}$, given by (2.1.2) is analytic and univalent in $\mathbb{U}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.7.5, we obtain the next corollary:

Corollary 2.7.5.1. *Let be the analytic functions f, g and h be in the class $\mathcal{S}_\mu$, $0 < \mu \le 1$ satisfy the inequality*

$$|g(z)| \le 1, \qquad \left|\frac{zh''(z)}{h'(z)}\right| < |z|\,.$$

If $\alpha \in \mathbb{C}$ with $\mathrm{Re}\alpha = c > 0$ and

$$|\alpha - 1|\,(3\mu + 2) \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

the integral operator $\mathcal{C}$, defined by (2.2.10) is analytic and univalent in $\mathbb{U}$.

Theorem 2.7.6. *Let be the analytic functions f_i, g_i and h_i be in the class $\mathcal{B}(\mu_i)$, $0 \le \mu_i < 1$, $i = \overline{1,n}$ and $\delta, \gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, with $\delta \ne 0$. Suppose that $M_i \ge 1$, $N_i \ge 1$, $P_i \ge 1$ are positive real numbers, with $i = \overline{1,n}$, with $\mathrm{Re}\gamma = c > 0$ and*

$$Re\delta \ge \sum_{i=1}^{n} \left\{|\alpha_i - 1|\left[(2-\mu_i)\left(M_i + N_i^2\right) + 1\right] + |\beta_i| + |\gamma_i|\left[(2-\mu_i)P_i + 1\right]\right\}.$$

If

$$|f_i(z)| \le M_i, \quad |g_i(z)| \le N_i, \quad |h_i(z)| \le P_i, \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$|c| \le 1 - \frac{1}{Re\delta}\sum_{i=1}^{n} \left\{|\alpha_i - 1|\left[(2-\mu_i)\left(M_i + N_i^2\right) + 1\right] + |\beta_i| + |\gamma_i|\left[(2-\mu_i)P_i + 1\right]\right\},$$

$c \in \mathbb{C}$, $c \ne 0$, then the integral operator $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

If we consider $n = 1$ and $\delta = \gamma = \alpha$ in Theorem 2.7.6, we obtain the next corollary:

Corollary 2.7.6.1. *Let be the analytic functions f, g and h be in the class $\mathcal{B}(\mu)$, $0 \le \mu < 1$ and $\lambda, \alpha, \beta, \gamma$ be complex numbers, with $\lambda \neq 0$. Suppose that $M \ge 1$, $N \ge 1$, $P \ge 1$ are positive real numbers and*

$$Re\lambda \ge \left\{|\alpha - 1| \left[(2-\mu)\left(M + N^2\right) + 1\right] + |\beta| + |\gamma| \left[(2-\mu) P + 1\right]\right\}$$

If

$$|f(z)| \le M, \quad |g(z)| \le N, \quad |g(z)| \le P, \quad \left|\frac{zh''(z)}{h'(z)}\right| \le 1, \quad z \in \mathbb{U}$$

and

$$|c| \le 1 - \frac{1}{Re\lambda}\left\{|\alpha - 1| \left[(2-\mu)\left(M + N^2\right) + 1\right] + |\beta| + |\gamma| \left[(2-\mu) P + 1\right]\right\}, \quad c \in \mathbb{C},\ c \neq 0,$$

then the integral operator $\mathcal{C}^*$, defined in

$$\mathcal{C}^*(z) = \int_0^z \left[\left(\frac{f(t)}{t} e^{g(t)}\right)^{\alpha - 1} (h'(t))^{\beta} \left(\frac{h(t))}{t}\right)^{\gamma}\right] dt, \tag{2.7.2}$$

is in the class $\mathcal{S}$.

Theorem 2.7.7. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$, $c = \mathrm{Re}\gamma > 0$ be complex numbers and the analytic functions f_i, h_i, k_i be in the class $\mathcal{B}(\mu_i)$, $0 \le \mu_i < 1$, $g_i(z) \in \mathcal{R}(\mu_i)$, $i = \overline{1,n}$ satisfy the inequality*

$$\sum_{i=1}^{n} (1-\mu_i)\left(3|\alpha_i - 1| + 4|\beta_i| + 6|\gamma_i|\right) + 2|\alpha_i - 1| \le \begin{cases} c, & if \quad 0 < c < \frac{1}{2} \\ \frac{1}{2} & if \quad \frac{1}{2} < c < \infty, \end{cases}$$

then for any complex number δ, $Re\delta \ge Re\gamma$, the integral operator $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is in the class $\mathcal{S}$.

Theorem 2.7.8. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers and the analytic functions f_i, g_i, h_i, k_i be in the class $\mathcal{S}_{\mu_i}$, $0 < \mu_i \le 1$, $i = \overline{1,n}$ satisfy the inequality*

$$|g_i(z)| \le 1, \quad \left|\frac{zf_i''(z)}{f_i'(z)}\right| < |z|, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| < |z|, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| < |z|.$$

If $\gamma \in \mathbb{C}$, with $\mathrm{Re}\gamma = c > 0$ and

$$\sum_{i=1}^{n} \left[(2+\mu_i)|\alpha_i - 1| + 2\mu_i|\beta_i| + 2|\gamma_i|\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then for any complex number δ, $\mathrm{Re}\delta \le c$, the integral operator $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is analytic and univalent in $\mathbb{U}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.7.8, we obtain the next corollary:

Corollary 2.7.8.1. *Let be the analytic functions f, g, h, k be in the class $\mathcal{S}_\mu$, $0 < \mu \leq 1$ satisfy the inequality*

$$|g(z)| \leq 1, \quad \left|\frac{zf''(z)}{f'(z)}\right| < |z|, \quad \left|\frac{zh''(z)}{h'(z)}\right| < |z|, \quad \left|\frac{zk''(z)}{k'(z)}\right| < |z|.$$

Dacă $\alpha \in \mathbb{C}$ with $\mathrm{Re}\alpha = c > 0$ and

$$|\alpha - 1|\,(3\mu + 4) \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is analytic and univalent in $\mathbb{U}$.

Theorem 2.7.9. *Let be the analytic functions f_i, g_i, h_i, k_i be in the class $\mathcal{B}(\mu_i)$, $0 \leq \mu_i < 1$ and $\delta, \gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, with $\delta \neq 0$. Suppose that $M_i \geq 1$, $N_i \geq 1$, $P_i \geq 1$ are positive real numbers and $i = \overline{1,n}$, with $\mathrm{Re}\gamma = c > 0$ and*

$$Re\delta \geq \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[1 + (2 - \mu_i)\, M_i^2\right] + |\beta_i| \left[(2 - \mu_i)(N_i + P_i) + 2\right] + 2\,|\gamma_i| \right\}.$$

If

$$|g_i(z)| \leq M_i, \quad |h_i(z)| \leq N_i, \quad |k_i(z)| \leq P_i,$$

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \leq 1, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq 1, \quad z \in \mathbb{U}, \quad i = \overline{1,n}$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[1 + (2 - \mu_i)\, M_i^2\right] + |\beta_i| \left[(2 - \mu_i)(N_i + P_i) + 2\right] + 2\,|\gamma_i| \right\}, \; c \in \mathbb{C}, \; c \neq 0,$$

then the integral operator $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is in the class $\mathcal{S}$.

Letting $n = 1$ and $\delta = \gamma = \lambda$ in Theorem 2.7.9, we obtain the next corollary:

Corollary 2.7.9.1. *Let be the analytic functions f, g, h, k be in the class $\mathcal{B}(\mu)$, $0 \leq \mu < 1$, and $\lambda, \alpha, \beta, \gamma$ be complex numbers, with $\lambda \neq 0$. Suppose that $M \geq 1$, $N \geq 1$, $P \geq 1$ are positive real numbers and*

$$Re\lambda \geq \left\{ |\alpha - 1| \left[1 + (2 - \mu)\, M^2\right] + |\beta| \left[(2 - \mu)(N + P) + 2\right] + 2\,|\gamma| \right\}.$$

If

$$|g(z)| \leq M, \quad |h(z)| \leq N, \quad |k(z)| \leq P,$$

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad \left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad \left|\frac{zk''(z)}{k'(z)}\right| \leq 1, \quad z \in \mathbb{U}$$

and

$$|c| \leq 1 - \frac{1}{Re\lambda} \left\{ |\alpha - 1| \left[1 + (2 - \mu)\, M^2\right] + |\beta| \left[(2 - \mu)(N + P) + 2\right] + 2\,|\gamma| \right\}, \; c \in \mathbb{C}, \; c \neq 0$$

then the integral operator $\mathcal{G}^*$, defined by

$$\mathcal{G}^*(z) = \int_0^z \left[\left(f'(t) e^{g(t)}\right)^{\alpha - 1} \left(\frac{h(t)}{k(t)}\right)^{\beta} \left(\frac{h'(t))}{k'(t)}\right)^{\gamma} \right] dt, \tag{2.7.3}$$

is in the class $\mathcal{S}$.

Theorem 2.7.10. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = \mathrm{Re}\gamma > 0$, $i = \overline{1,n}$ and f_i, g_i, h_i, k_i be in the class $\mathcal{B}(\mu_i)$, $0 \le \mu_i < 1$, $i = \overline{1,n}$ satisfy the inequality*

$$\sum_{i=1}^{n} (1-\mu_i)\left(2\left|\alpha_i - 1\right| + 3\left|\beta_i\right| + 4\left|\gamma_i\right| + 6\left|\delta_i\right|\right) \le \begin{cases} c, & if \quad 0 < c < \frac{1}{2} \\ \frac{1}{2} & if \quad \frac{1}{2} < c < \infty \end{cases},$$

then for any complex number δ, Re$\delta \ge$ Reγ, the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Theorem 2.7.11. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers and f_i, g_i, h_i, k_i be in the class $\mathcal{S}_{\mu_i}$, $0 < \mu_i \le 1$, $i = \overline{1,n}$ satisfy the inequality*

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| < |z|, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| < |z|, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| < |z|.$$

If $\gamma \in \mathbb{C}$, with $\mathrm{Re}\gamma = c > 0$ and

$$\sum_{i=1}^{n} \left(\mu_i\left|\alpha_i - 1\right| + \left|\beta_i\right| + 2\mu_i\left|\gamma_i\right| + 2\left|\delta_i\right|\right) \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then for any complex number δ, $\mathrm{Re}\delta \le c$, the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is analytic and univalent in $\mathbb{U}$.

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.7.11, we obtain the next corollary:

Corollary 2.7.11.1. *Let be the analytic functions f, g, h, k be in the class $\mathcal{S}_\mu$, $0 < \mu \le 1$ satisfy the inequality*

$$\left|\frac{zg''(z)}{g'(z)}\right| < |z|, \quad \left|\frac{zh''(z)}{h'(z)}\right| < |z|, \quad \left|\frac{zk''(z)}{k'(z)}\right| < |z|.$$

If $\alpha \in \mathbb{C}$, with $\mathrm{Re}\alpha = c > 0$ and

$$3\left|\alpha - 1\right|(\mu + 1) \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is analytic and univalent in $\mathbb{U}$.

Theorem 2.7.12. *Let be the analytic functions f_i, g_i, h_i, k_i be in the class $\mathcal{B}(\mu_i)$, $0 \le \mu_i < 1$, $i = \overline{1,n}$ and $\delta, \gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, with $\delta \ne 0$. Suppose that $M_i \ge 1$, $N_i \ge 1$, $P_i \ge 1$are positive real numbers and $i = \overline{1,n}$, with $\mathrm{Re}\gamma = c > 0$ and*

$$Re\delta \ge \sum_{i=1}^{n} \left\{\left|\alpha_i - 1\right|\left[(2-\mu_i) M_i + 1\right] + \left|\beta_i\right| + \left|\gamma_i\right|\left[(2-\mu_i) N_i + (2-\mu_i) P_i + 2\right] + 2\left|\delta_i\right|\right\}.$$

If

$$\left|f_i(z)\right| \le M_i, \quad \left|h_i(z)\right| \le N_i, \quad \left|k_i(z)\right| \le P_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \le 1, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \{|\alpha_i - 1| [(2-\mu_i) M_i + 1] + |\beta_i|\} -$$

$$-\frac{1}{Re\delta} \sum_{i=1}^{n} \{|\gamma_i| [(2-\mu_i) N_i + (2-\mu_i) P_i + 2] + 2|\delta_i|\}, \quad c \in \mathbb{C}, \quad c \neq 0,$$

then the integral operator $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Letting $n = 1$ and $\delta = \gamma = \lambda$, in Theorem 2.7.12, we obtain the next corollary:

Corollary 2.7.12.1. *Let be the analytic functions f, g, h, k be in the class $\mathcal{B}(\mu)$, $0 \leq \mu < 1$, be complex numbers, with $\lambda, \alpha, \beta, \gamma, , \delta$, with $\lambda \neq 0$. Suppose that $M \geq 1$, $N \geq 1$, $P \geq 1$ are positive real numbers and*

$$Re\lambda \geq \{|\alpha - 1| [(2-\mu) M + 1] + |\beta| + |\gamma| [(2-\mu) N + (2-\mu) P + 2] + |\delta|\}.$$

If

$$|f(z)| \leq M, \quad |h(z)| \leq N, \quad |k(z)| \leq P,$$

$$\left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \quad \left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad \left|\frac{zk''(z)}{k'(z)}\right| \leq 1, z \in \mathbb{U}$$

and

$$|c| \leq 1 - \frac{1}{Re\lambda} \{|\alpha - 1| [(2-\mu) M + 1] + |\beta| + |\gamma| [(2-\mu) N + (2-\mu) P + 2] + |\delta|\}, \; c \in \mathbb{C}, \; c \neq 0,$$

then the integral operator $\mathcal{T}$, defined by

$$\mathcal{T}^*(z) = \int_0^z \left[\left(\frac{f(t)}{t}\right)^{\alpha-1} (g(t)')^{\beta} \left(\frac{h(t)}{k(t)}\right)^{\gamma} \left(\frac{h'(t))}{k'(t)}\right)^{\delta}\right] dt, \tag{2.7.4}$$

is in the class $\mathcal{S}$.

2.8 Univalence conditions for the class $\mathcal{B}(\mu, \alpha)$

This paragraph presents sufficient conditions of univalence for the integral operators of this paper in the situation when their functions belong to the class of functions. $\mathcal{B}(\mu, \alpha)$, $0 \leq \alpha < 1$, $\mu \geq 0$.

Theorem 2.8.1. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $M_i, N_i, P_i \geq 1$, $i = \overline{1,n}$ are positive real numbers, such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} |\alpha_i - 1| \left[1 + (2-\lambda_i) M_i^{\mu_i - 1}\right] + 2c \sum_{i=1}^{n} |\beta_i| N_i +$$

$$+ (2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} \left\{|\gamma_i| \left[1 + (2-\eta_i) P_i^{\nu_i - 1}\right]\right\} \leq c (2c+1)^{\frac{2c+1}{2c}}.$$

If $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{B}(\nu_i, \eta_i)$, $0 \le \lambda_i, \eta_i < 1$, $\mu_i, \nu_i \ge 0$ *satisfies*

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \le N_i, \quad |g_i(z)| < P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then for any complex number* δ, *Re*$\delta \ge$ *Re*γ, *the function* $\mathcal{M}_{\delta,n}$, *defined by (2.1.1) is in the class* $\mathcal{S}$.

Putting $\mu_i = \nu_i = M_i = N_i = P_i = 1$ and $\eta_i = \lambda_i$ for all $i = \overline{1,n}$ in Theorem 2.8.1, we obtain the next corollary:

Corollary 2.8.1.1. *Let* $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c =$ *Re*$\gamma > 0$ *and* $0 \le \lambda_i < 1$, $i = \overline{1,n}$, *such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} (3 - \lambda_i)(|\alpha_i - 1| + |\gamma_i|) + 2c \sum_{i=1}^{n} |\beta_i| \le c(2c+1)^{\frac{2c+1}{2c}}.$$

If $f_i, g_i \in \mathcal{S}^*(\lambda_i)$ *and*

$$|f_i(z)| < 1, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \le 1, \quad |g_i(z)| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then for any complex number* δ, *Re*$\delta \ge$ *Re*γ, *the function* $\mathcal{M}_{\delta,n}$, *defined by (2.1.1) is in the class* $\mathcal{S}$.

Theorem 2.8.2. *Let* $c, \delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, *Re*$\delta > 0$ *and* $M_i, N_i, P_i \ge 1$, $i = \overline{1,n}$ *are positive real numbers. Suppose that* $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{B}(\nu_i, \eta_i)$, $0 \le \lambda_i, \eta_i < 1$, $\mu_i, \nu_i \ge 0$ *satisfy*

$$|f_i(z)| < M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| < N_i, \quad |g_i(z)| < P_i.$$

If

$$Re\delta \ge \sum_{i=1}^{n} \left\{|\alpha_i - 1|\left[1 + (2-\lambda_i) M_i^{\mu_i - 1}\right] + |\beta_i| N_i + |\gamma_i|\left[1 + (2-\eta_i) P_i^{\nu_i - 1}\right]\right\}$$

and

$$|c| \le 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{|\alpha_i - 1|\left[1 + (2-\lambda_i) M_i^{\mu_i - 1}\right] + |\beta_i| N_i + |\gamma_i|\left[1 + (2-\eta_i) P_i^{\nu_i - 1}\right]\right\},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the function* $\mathcal{M}_{\delta,n}$, *defined by (2.1.1) is in the class* $\mathcal{S}$.

Letting $\mu_i = \nu_i = M_i = N_i = P_i = 1$ and $\eta_i = \lambda_i$ for all $i = \overline{1,n}$ in Theorem 2.8.3, we obtain the next corollary:

Corollary 2.8.2.1. *Let* $c, \delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ *with Re*$\delta > 0$. *Suppose that* $f_i, g_i \in \mathcal{S}^*(\lambda_i)$, $0 \le \lambda_i < 1$ *satisfy*

$$|f_i(z)| < 1, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| < 1, \quad |g_i(z)| < 1.$$

If

$$Re\delta \geq \sum_{i=1}^{n} \left[(3-\lambda_i) \left(|\alpha_i - 1| + |\gamma_i| \right) + |\beta_i| \right]$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[(3-\lambda_i) \left(|\alpha_i - 1| + |\gamma_i| \right) + |\beta_i| \right],$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the function $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

If we consider $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta$ in Theorem 2.8.3, we obtain the next corollary:

Corollary 2.8.2.2. *Let $c, \delta \in \mathbb{C}$ with $Re\delta > 0$ and $M, N, P \geq 1$ are positive real numbers. Suppose that $f \in \mathcal{B}(\mu, \lambda)$, $g \in \mathcal{B}(\nu, \eta)$, $0 \leq \lambda, \eta < 1$, $\mu, \nu \geq 0$ satisfy*

$$|f(z)| < M, \quad \left| \frac{zg''(z)}{g'(z)} \right| < N, \quad |g(z)| < P.$$

If

$$Re\delta \geq |\delta| \left[(2-\lambda) M^{\mu-1} + (2-\eta) P^{\nu-1} + N + 2 \right]$$

and

$$|c| \leq 1 - \frac{|\delta|}{Re\delta} \left[(2-\lambda) M^{\mu-1} + (2-\eta) P^{\nu-1} + N + 2 \right],$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{S}$.

Theorem 2.8.3. *Let $\delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\delta > 0$, M_0 the positive solution of the equation (1.1.1.), $M_0 = 1,5936...$ and $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{B}(\nu_i, \eta_i)$, $0 \leq \lambda_i, \eta_i < 1$, $\mu_i, \nu_i \geq 0$ for all $z \in \mathbb{U}$, $i = \overline{1,n}$. Suppose also that*

$$|f_i(z)| < M_i, \quad \left| \frac{g_i''(z)}{g_i'(z)} \right| < M_0,$$

where M_i are positive real numbers. If

$$\frac{1}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| (2-\lambda_i) M_i^{\mu_i - 1} + |\beta_i| \right] + \frac{2}{(2c+1)^{\frac{2c+1}{2c}}} \sum_{i=1}^{n} |\gamma_i| M_0 \leq 1,$$

then the function $\mathcal{M}_{\delta,n}$, defined by (2.1.1) is in the class $\mathcal{S}$.

Theorem 2.8.4. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $M_i, N_i, P_i, Q_i \geq 1$, $i = \overline{1,n}$ positive real numbers, such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[1 + (2-\lambda_i) M_i^{\mu_i - 1} \right] + |\gamma_i| \left[1 + (2-\rho_i) Q_i^{\theta_i - 1} \right] \right\} +$$

$$+ 2c \sum_{i=1}^{n} \left[|\alpha_i - 1| (2-\eta_i) N_i^{\nu_i} + |\beta_i| P_i \right] \leq c (2c+1)^{\frac{2c+1}{2c}}.$$

If $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{B}(\nu_i, \eta_i)$, $h_i \in \mathcal{B}(\theta_i, \rho_i)$, $0 \leq \lambda_i, \eta_i, \rho_i < 1$, $\mu_i, \nu_i, \theta_i \geq 0$ satisfies

$$|f_i(z)| < M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq P_i, \quad |h_i(z)| < Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then for any complex number δ, $Re\delta \geq Re\gamma$, the function $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

Letting $\mu_i = \nu_i = \theta_i = M_i = N_i = P_i = Q_i = 1$ and $\rho_i = \eta_i = \lambda_i$ for all $i = \overline{1,n}$ in Theorem 2.8.5, we obtain the next corollary:

Corollary 2.8.4.1. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $0 \leq \lambda_i < 1$, $i = \overline{1,n}$, such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} (3-\lambda_i)\left(|\alpha_i - 1| + |\gamma_i|\right) + 2c \sum_{i=1}^{n} \left[|\alpha_i - 1|(2-\lambda_i) + |\beta_i|\right] \leq c\,(2c+1)^{\frac{2c+1}{2c}}.$$

If $f_i, g_i, h_i \in \mathcal{S}^*(\lambda_i)$ and

$$|f_i(z)| < 1, \quad |g_i(z)| < 1, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \leq 1, \quad |h_i(z)| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then for any complex number δ, $Re\delta \geq Re\gamma$, the function $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

Theorem 2.8.5. *Let $c, \delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $Re\delta > 0$ and $M_i, N_i, P_i, Q_i \geq 1$, $i = \overline{1,n}$ are positive real numbers. Suppose that $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{B}(\nu_i, \eta_i)$, $h_i \in \mathcal{B}(\theta_i, \nu_i)$, $0 \leq \lambda_i, \eta_i, \rho_i < 1$, $\mu_i, \nu_i, \theta_i \geq 0$ satisfies*

$$|f_i(z)| < M_i, \quad \left|zg_i'(z)\right| < N_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| < P_i, \quad |h_i(z)| < Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$. If

$$Re\delta \geq \sum_{i=1}^{n} \left\{|\alpha_i - 1|\left[1 + (2-\lambda_i) M_i^{\mu_i - 1} + N_i\right] + |\beta_i| P_i + |\gamma_i|\left[1 + (2-\rho_i) Q_i^{\theta_i - 1}\right]\right\}$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{|\alpha_i - 1|\left[1 + (2-\lambda_i) M_i^{\mu_i - 1} + N_i\right] + |\beta_i| P_i + |\gamma_i|\left[1 + (2-\rho_i) Q_i^{\theta_i - 1}\right]\right\},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the function $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

If we consider $\mu_i = \nu_i = \theta_i = M_i = N_i = P_i = Q_i = 1$ and $\rho_i = \eta_i = \lambda_i$ for all $i = \overline{1,n}$ in Theorem 2.8.6, we obtain the next corollary:

Corollary 2.8.5.1. *Let $c, \delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ with $Re\delta > 0$. Suppose that $f_i, g_i, h_i \in \mathcal{S}^*(\lambda_i)$, $0 \leq \lambda_i < 1$ satisfies*

$$|f_i(z)| < 1, \quad \left|zg_i'(z)\right| < 1, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| < 1, \quad |h_i(z)| < 1,.$$

If

$$Re\delta \geq \sum_{i=1}^{n} \left[(4-\lambda_i)|\alpha_i - 1| + |\beta_i| + |\gamma_i|(3-\lambda_i)\right]$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left[(4-\lambda_i)|\alpha_i - 1| + |\beta_i| + |\gamma_i|(3-\lambda_i)\right],$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the function $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

Putting $n=1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta$ in Theorem 2.8.6, we obtain the next corollary:

Corollary 2.8.5.2. *Let $c, \delta \in \mathbb{C}$ with $Re\delta > 0$ and $M, N, P, Q \geq 1$ are positive real numbers. Suppose that $f \in \mathcal{B}(\mu, \lambda)$, $g \in \mathcal{B}(\nu, \eta)$, $h \in \mathcal{B}(\theta, \rho)$, $0 \leq \lambda, \eta, \rho < 1$, $\mu, \nu, \theta \geq 0$ satisfy*

$$|f(z)| < M, \quad \left|zg'(z)\right| < N, \quad \left|\frac{zh''(z)}{h'(z)}\right| < P, \quad |h(z)| < Q.$$

for all $z \in \mathbb{U}$. If

$$Re\delta \geq |\delta| \left[(2-\lambda) M^{\mu-1} + (2-\rho) Q^{\theta-1} + N + P + 2\right]$$

and

$$|c| \leq 1 - \frac{|\delta|}{Re\delta} \left[(2-\lambda) M^{\mu-1} + (2-\rho) Q^{\theta-1} + N + P + 2\right],$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is in the class $\mathcal{S}$.

Theorem 2.8.6. *Let $\delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\delta > 0$, M_0 the positive solution of the equation (1.1.1), $M_0 = 1,5936...$ and $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{B}(\nu_i, \eta_i)$, $h_i \in \mathcal{A}$, $0 \leq \lambda_i, \eta_i < 1$, $\mu_i, \nu_i \geq 0$ for all $z \in \mathbb{U}$, $i = \overline{1,n}$. Suppose also that*

$$|f_i(z)| < M_i, \quad |g_i(z)| < N_i, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| < M_0,$$

where M_i, N_i are positive real numbers. If

$$\frac{1}{c}\sum_{i=1}^{n}\left\{|\alpha_i - 1|\left[1 + (2-\lambda_i) M_i^{\mu_i - 1}\right] + |\gamma_i|\right\} +$$

$$+\frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\beta_i| M_0 + |\gamma_i| |\alpha_i - 1| (2-\eta_i) N_i^{\nu_i}\right] \leq 1,$$

then the function $\mathcal{C}_{\delta,n}$, defined by (2.1.2) is in the class $\mathcal{S}$.

Theorem 2.8.7. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and M_i, N_i, P_i, Q_i, R_i, $S_i \geq 1$, $i = \overline{1,n}$ positive real numbers, such that*

$$(2c+1)^{\frac{2c+1}{2c}}\sum_{i=1}^{n}|\beta_i|\left[(2-\eta_i) P_i^{\nu_i - 1} + (2-\rho_i) Q_i^{\theta_i - 1} + 2\right] +$$

$$+2c\sum_{i=1}^{n}\left\{|\alpha_i - 1|\left[M_i + (2-\lambda_i) N_i^{\mu_i}\right] + |\gamma_i| (R_i + S_i)\right\} \leq c(2c+1)^{\frac{2c+1}{2c}}.$$

If $f_i \in \mathcal{A}$, $\quad g_i \in \mathcal{B}(\mu_i, \lambda_i)$, $\quad h_i \in \mathcal{B}(\nu_i, \eta_i)$, $\quad k_i \in \mathcal{B}(\theta_i, \rho_i)$, $0 \leq \lambda_i, \eta_i, \rho_i < 1$, $\mu_i, \nu_i, \theta_i \geq 0$ satisfies

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M_i, \quad |g_i(z)| < N_i, \quad |h_i(z)| < P_i, \quad |k_i(z)| < Q_i, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then for any complex number δ, $Re\delta \geq Re\gamma$, the function $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is in the class $\mathcal{S}$.

Letting $\mu_i = \nu_i = \theta_i = M_i = N_i = P_i = Q_i = R_i = S_i = 1$ and $\rho_i = \eta_i = \lambda_i$ for all $i = \overline{1,n}$ in Theorem 2.8.8, we obtain the next corollary:

Corollary 2.8.7.1. *Let* $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ *and* $0 \le \lambda_i < 1$, $i = \overline{1,n}$, *such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} 2\left|\beta_i\right|(3-\lambda_i) + 2c\sum_{i=1}^{n}\left[\left|\alpha_i - 1\right|(3-\lambda_i) + 2\left|\gamma_i\right|\right] \le c\,(2c+1)^{\frac{2c+1}{2c}}.$$

If $f_i \in \mathcal{A}$, $g_i, h_i, k_i \in \mathcal{S}^*(\lambda_i)$ *and*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le 1, \quad |g_i(z)| < 1, \quad |h_i(z)| < 1, \quad |k_i(z)| < 1, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \le 1, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then for any complex number* δ, $Re\delta \ge Re\gamma$, *the function* $\mathcal{G}_{\delta,n}$, *defined by (2.1.3) is in the class* $\mathcal{S}$.

Theorem 2.8.8. *Let* $c, \delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $Re\delta > 0$ *and* $M_i, N_i, P_i, Q_i, R_i, S_i \ge 1$, $i = \overline{1,n}$ *are positive real numbers. Suppose that* $f_i \in \mathcal{A}$, $h_i \in \mathcal{B}(\mu_i, \lambda_i)$, $k_i \in \mathcal{B}(\nu_i, \eta_i)$, $g_i \in \mathcal{B}(\theta_i, \rho_i)$, $0 \le \lambda_i, \eta_i, \rho_i < 1$, $\mu_i, \nu_i, \theta_i \ge 0$ *satisfies*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| < M_i, \quad \left|zg_i'(z)\right| < N_i, \quad |h_i(z)| < P_i, \quad |k_i(z)| < Q_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| < R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| < S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$. *If*

$$Re\delta \ge \sum_{i=1}^{n}\left\{|\alpha_i - 1|(M_i + N_i) + |\beta_i|\left[(2-\lambda_i)P_i^{\mu_i - 1} + (2-\eta_i)Q_i^{\nu_i - 1} + 2\right] + |\gamma_i|(R_i + S_i)\right\}$$

and

$$|c| \le 1 - \frac{1}{Re\delta}\sum_{i=1}^{n}\left\{|\alpha_i - 1|(M_i + N_i) + |\beta_i|\left[(2-\lambda_i)P_i^{\mu_i - 1}\right]\right\} +$$
$$-\frac{1}{Re\delta}\sum_{i=1}^{n}\left\{|\beta_i|\left[(2-\eta_i)Q_i^{\nu_i - 1} + 2\right] + |\gamma_i|(R_i + S_i)\right\},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the function* $\mathcal{G}_{\delta,n}$, *defined by (2.1.3) is in the class* $\mathcal{S}$.

Putting $\mu_i = \nu_i = \theta_i = M_i = N_i = P_i = Q_i = R_i = S_i = 1$ and $\rho_i = \eta_i = \lambda_i$ for all $i = \overline{1,n}$ in Theorem 2.8.9, we obtain the next corollary:

Corollary 2.8.8.1. *Let* $c, \delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ *with* $Re\delta > 0$. *Suppose that* $g_i, h_i, k_i \in \mathcal{S}^*(\lambda_i)$, $0 \le \lambda_i < 1$ *and* $f_i \in \mathcal{A}$ *satisfies*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| < 1, \quad \left|zg_i'(z)\right| < 1, \quad |h_i(z)| < 1, \quad |k_i(z)| < 1, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| < 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| < 1.$$

If

$$Re\delta \ge 2\sum_{i=1}^{n}\left[|\alpha_i - 1| + |\beta_i|(3-\lambda_i) + |\gamma_i|\right]$$

and

$$|c| \le 1 - \frac{2}{Re\delta}\sum_{i=1}^{n}\left[|\alpha_i - 1| + |\beta_i|(3-\lambda_i) + |\gamma_i|\right],$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the function* $\mathcal{G}_{\delta,n}$, *defined by (2.1.3) is in the class* $\mathcal{S}$.

If we consider $n=1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta$ in Theorem 2.8.9, we obtain the next corollary:

Corollary 2.8.8.2. *Let $c, \delta \in \mathbb{C}$ with $Re\delta > 0$ and $M, N, P, Q, R, S \geq 1$are positive real numbers. Suppose that $f \in \mathcal{A}$, $h \in \mathcal{B}(\mu, \lambda)$, $k \in \mathcal{B}(\nu, \eta)$, $g \in \mathcal{B}(\theta, \rho)$, $0 \leq \lambda, \eta, \rho < 1$, $\mu, \nu, \theta \geq 0$ such that*

$$\left|\frac{zf''(z)}{f'(z)}\right| < M, \quad \left|zg'(z)\right| < N, \quad |h(z)| < P, \quad |k(z)| < Q, \quad \left|\frac{zh''(z)}{h'(z)}\right| < R, \quad \left|\frac{zk''(z)}{k'(z)}\right| < S,$$

for all $z \in \mathbb{U}$. If

$$Re\delta \geq |\delta| \left[M + N + (2-\lambda) P^{\mu-1} + (2-\eta) Q^{\nu-1} + R + S + 2\right]$$

and

$$|c| \leq 1 - \frac{|\delta|}{Re\delta} \left[M + N + (2-\lambda) P^{\mu-1} + (2-\eta) Q^{\nu-1} + R + S + 2\right],$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is in the class $\mathcal{S}$.

Theorem 2.8.9. *Let $\delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\delta > 0$, M_0 the positive solution of the equation (1.1.1), $M_0 = 1,5936...$ and $f_i \in \mathcal{A}$, $g_i \in \mathcal{B}(\mu_i, \lambda_i)$, $h_i \in \mathcal{B}(\nu_i, \eta_i)$, $k_i \in \mathcal{B}(\theta_i, \rho_i)$, $0 \leq \lambda_i, \eta_i, \rho_i < 1$, $\mu_i, \nu_i, \theta_i \geq 0$, for all $z \in \mathbb{U}$, $i = \overline{1,n}$. Suppose also that*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| < M_0, \quad |g_i(z)| < M_i, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| < M_0, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| < M_0,$$

where M_i are positive real numbers. If

$$\frac{1}{c}\sum_{i=1}^{n} |\beta_i| + \frac{2}{(2c+1)^{\frac{2c+1}{2c}}} \sum_{i=1}^{n} \left\{|\alpha_i - 1| \left[M_0 + (2-\lambda_i) M_i^{\mu_i}\right] + 2M_0 |\gamma_i|\right\} \leq 1,$$

then the function $\mathcal{G}_{\delta,n}$, defined by (2.1.3) is in the class $\mathcal{S}$.

Theorem 2.8.10. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\gamma > 0$ and M_i, N_i, P_i, Q_i, R_i, $S_i \geq 1$, $i = \overline{1,n}$, positive real numbers, such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} \left\{|\alpha_i - 1| \left[1 + (2-\lambda_i) M_i^{\mu_i - 1}\right] + |\gamma_i| \left[2 + (2-\eta_i) P_i^{\nu_i - 1}\right]\right\} +$$

$$+ (2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} |\gamma_i| (2-\rho_i) Q_i^{\theta_i - 1} + 2c \sum_{i=1}^{n} \left[|\beta_i| N_i + |\delta_i| (R_i + S_i)\right] \leq c(2c+1)^{\frac{2c+1}{2c}}.$$

If $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{A}$, $h_i \in \mathcal{B}(\nu_i, \eta_i)$, $k_i \in \mathcal{B}(\theta_i, \rho_i)$, $0 \leq \lambda_i, \eta_i, \rho_i < 1$, $\mu_i, \nu_i, \theta_i \geq 0$ satisfies

$$|f_i(z)| < M_i, \ \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq N_i, \ |h_i(z)| < P_i, \ |k_i(z)| < Q_i, \ \left|\frac{h_i''(z)}{h_i'(z)}\right| \leq R_i, \ \left|\frac{k_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then for any complex number δ, $Re\delta \geq Re\gamma$, the function $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Letting $\mu_i = \nu_i = \theta_i = M_i = N_i = P_i = Q_i = R_i = S_i = 1$ and $\rho_i = \eta_i = \lambda_i$ for all $i = \overline{1,n}$ in Theorem 2.8.11, we obtain the next corollary:

Corollary 2.8.10.1. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $0 \le \lambda_i < 1$, $i = \overline{1,n}$, such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} (3-\lambda_i)\left(|\alpha_i - 1| + 2|\gamma_i|\right) + 2c\sum_{i=1}^{n}\left(|\beta_i| + 2|\delta_i|\right) \le c(2c+1)^{\frac{2c+1}{2c}}.$$

If $g_i \in \mathcal{A}$, f_i,h_i, $k_i \in \mathcal{S}^*(\lambda_i)$, $0 \le \lambda_i < 1$ and

$$|f_i(z)| < 1, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \le 1, \quad |h_i(z)| < 1, \quad |k_i(z)| < 1, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \le 1, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then for any complex number δ, $Re\delta \ge Re\gamma$, the function $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Theorem 2.8.11. *Let $c, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $Re\delta > 0$ and $M_i, N_i, P_i, Q_i, R_i, S_i \ge 1$, $i = \overline{1,n}$ are positive real numbers. Suppose that $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{A}$, $h_i \in \mathcal{B}(\nu_i, \eta_i)$, $k_i \in \mathcal{B}(\theta_i, \rho_i)$, $0 \le \lambda_i, \eta_i, \rho_i < 1$, $\mu_i, \nu_i, \theta_i \ge 0$ satisfies*

$$|f_i(z)| < M_i, \ \left|\frac{zg_i''(z)}{g_i'(z)}\right| < N_i, \ |h_i(z)| < P_i, \ |k_i(z)| < Q_i, \ \left|\frac{zh_i''(z)}{h_i'(z)}\right| < R_i, \ \left|\frac{zk_i''(z)}{k_i'(z)}\right| < S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$. If

$$Re\delta \ge \sum_{i=1}^{n}\left\{|\alpha_i - 1|\left[(2-\lambda_i)M_i^{\mu_i - 1} + 1\right] + |\beta_i| N_i\right\} +$$

$$+\sum_{i=1}^{n}\left\{|\gamma_i|\left[(2-\eta_i)P_i^{\nu_i - 1} + (2-\rho_i)Q_i^{\theta_i - 1} + 2\right] + |\delta_i|(R_i + S_i)\right\}$$

and

$$|c| \le 1 - \frac{1}{Re\delta}\sum_{i=1}^{n}\left\{|\alpha_i - 1|\left[(2-\lambda_i)M_i^{\mu_i - 1} + 1\right] + |\beta_i| N_i\right\} -$$

$$-\frac{1}{Re\delta}\sum_{i=1}^{n}\left\{|\gamma_i|\left[(2-\eta_i)P_i^{\nu_i - 1} + (2-\rho_i)Q_i^{\theta_i - 1} + 2\right] + |\delta_i|(R_i + S_i)\right\},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, the function $\mathcal{T}_{\delta,n}$, defined by (2.1.4) is in the class $\mathcal{S}$.

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta$ in Theorem 2.8.12, we obtain the next corollary:

Corollary 2.8.11.1. *Let $c, \delta \in \mathbb{C}$ with $Re\delta > 0$ and $M, N, P, Q, R, S \ge 1$ are positive real numbers. Suppose that $f \in \mathcal{B}(\mu, \lambda)$, $g \in \mathcal{A}$, $h \in \mathcal{B}(\nu, \eta)$, $k \in \mathcal{B}(\theta, \rho)$, $0 \le \lambda, \eta, \rho < 1$, $\mu, \nu, \theta \ge 0$ such that*

$$|f(z)| < M, \ \left|\frac{zg''(z)}{g'(z)}\right| < N, \ |h(z)| < P, \ |k(z)| < Q, \ \left|\frac{zh''(z)}{h'(z)}\right| < R, \ \left|\frac{zk''(z)}{k'(z)}\right| < S,$$

for all $z \in \mathbb{U}$. If

$$Re\delta \ge |\delta|\left[(2-\lambda)M^{\mu-1} + (2-\eta)P^{\nu-1} + (2-\rho)Q^{\theta-1} + N + R + S + 3\right]$$

and

$$|c| \le 1 - \frac{|\delta|}{Re\delta}\left[(2-\lambda)M^{\mu-1} + (2-\eta)P^{\nu-1} + (2-\rho)Q^{\theta-1} + N + R + S + 3\right],$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathbb{U}$.

Theorem 2.8.12. *Let* $\delta, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\delta > 0$, M_0 *the positive solution of the equation (1.1.1),* $M_0 = 1,5936...$ *and* $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i, h_i, k_i \in \mathcal{A}$, $0 \le \lambda_i < 1$, $\mu_i \ge 0$, *for all* $z \in \mathbb{U}$, $i = \overline{1,n}$. *Suppose also that*

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| < M_0, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| < M_0, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| < M_0,$$

where M_i *are positive real numbers. If*

$$\frac{1}{c}\sum_{i=1}^{n}\left[|\alpha_i - 1|(2-\lambda_i) M_i^{\mu_i - 1} + 2|\gamma_i|\right] + \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\beta_i| M_0 + 2|\delta_i| M_0\right] \le 1,$$

then the function $\mathcal{T}_{\delta,n}$, *defined by (2.1.4) is in the class* $\mathcal{S}$.

Chapter 3

Sufficient convexity conditions for new integral operators

In this chapter we fix $\delta = 1$ for integral operators defined in the relation (2.1.1)-(2.1.4) and we obtain the following integral operators:

$$\mathcal{M}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} ({g_i}'(t))^{\beta_i} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] \text{dt}. \tag{3.0.1}$$

$$\mathcal{C}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} ({h_i}'(t))^{\beta_i} \left(\frac{h_i(t))}{t} \right)^{\gamma_i} \right] \text{dt}. \tag{3.0.2}$$

$$\mathcal{G}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left({f_i}'(t) e^{g_i(t)} \right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)} \right)^{\beta_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\gamma_i} \right] \text{dt}, \tag{3.0.3}$$

$$\mathcal{T}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} (g_i'(t))^{\beta_i} \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] \text{dt}. \tag{3.0.4}$$

3.1 Convexity conditions for the class $\mathcal{G}_b$

This paragraph includes the study of the convexity of the above integral operators, as long as their functions belong to the classes $\mathcal{G}_b$, $0 < b \leq 1$ and $\mathcal{B}(\mu, \alpha)$, $\mu \geq 1$, $0 \leq \alpha < 1$.

Theorem 3.1.1. *Let be the analytic functions f_i, g_i and $g_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$. For $M_i, N_i \geq 1$, which verify*

$$\left| \frac{z f_i'(z)}{f_i(z)} \right| \leq M_i, \quad \left| \frac{z g_i'(z)}{g_i(z)} \right| \leq N_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *there are* $\alpha_i, \beta_i, \gamma_i$ *real positive numbers and* $\alpha_i > 1$ *so that*

$$\lambda = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(M_i + 1) + \beta_i (b_i N_i + 2b_i + N_i + 1) + \gamma_i (N_i + 1) \right] > 0.$$

In these conditions, the integral operator $\mathcal{M}_n$*, defined by (3.0.1) is in the class* $\mathcal{K}(\lambda)$*.*

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.1.1, we obtain the next corollary:

Corollary 3.1.1.1. *Let* $f, g \in \mathcal{A}$ *and* $g \in \mathcal{G}_b$*,* $0 < b \leq 1$*. For* $M, N \geq 1$*, which verify*

$$\left| \frac{zf^{'}(z)}{f(z)} \right| \leq M, \quad \left| \frac{zg^{'}(z)}{g(z)} \right| \leq N,$$

for all $z \in \mathbb{U}$*, there is* α *a real positive number so that*

$$\lambda = 1 - \alpha (M + 2N + bN + 2b + 3) > 0.$$

In these conditions, the integral operator $\mathcal{M}$*, defined by (2.2.5) is in the class* $\mathcal{K}(\lambda)$*.*

Theorem 3.1.2. *Let* $f_i, g_i \in \mathcal{A}$ *and* $g_i \in \mathcal{G}_{b_i}$*,* $0 < b_i \leq 1$*. If*

$$\left| \frac{zf_i^{'}(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{zg_i^{'}(z)}{g_i(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *there are* $\alpha_i, \beta_i, \gamma_i$ *real positive numbers and* $\alpha_i \geq 1$ *so that*

$$\lambda = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1) + \beta_i (2b_i + 1) + \gamma_i \right] > 0,$$

then the integral operator $\mathcal{M}_n$*, defined by (3.0.1) is in the class* $\mathcal{K}(\lambda)$*.*

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.1.2, we obtain the next corollary:

Corollary 3.1.2.1. *Let* $f, g \in \mathcal{A}$ *and* $g \in \mathcal{G}_b$*,* $0 < b \leq 1$*. If*

$$\left| \frac{zf^{'}(z)}{f(z)} - 1 \right| < 1, \quad \left| \frac{zg^{'}(z)}{g(z)} - 1 \right| < 1,$$

for all $z \in \mathbb{U}$ *there is* α *a real positive number so that*

$$\lambda = 1 - \alpha (2b + 3) > 0.$$

then the integral operator $\mathcal{M}$*, defined by (2.2.5) is in the class* $\mathcal{K}(\lambda)$*.*

Theorem 3.1.3. *Let be the analytic functions* f_i, g_i, h_i *and* $g_i \in \mathcal{B}(\mu_i, \lambda_i)$*,* $\mu_i \geq 1$*,* $0 \leq \lambda_i < 1$ *and* $h_i \in \mathcal{G}_{b_i}$*,* $0 < b_i \leq 1$*. For* $M_i, N_i, P_i \geq 1$*, which verify*

$$\left| \frac{zf_i^{'}(z)}{f_i(z)} \right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{zh_i^{'}(z)}{h_i(z)} \right| \leq P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *there are* $\alpha_i, \beta_i, \gamma_i$ *real positive numbers and* $\alpha_i > 1$ *so that*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)\left(M_i + (2 - \lambda_i) N_i^{\mu_i} + 1\right) + \beta_i \left(b_i (P_i + 2) + P_i + 1\right) + \gamma_i (P_i + 1) \right] > 0.$$

In these conditions, the integral operator $\mathcal{C}_n$*, defined by (3.0.2) is in the class* $\mathcal{K}(\rho)$*.*

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.1.3, we obtain the next corollary:

Corollary 3.1.3.1. *Let $f, g, h \in \mathcal{A}$ and $g \in \mathcal{B}(\mu, \lambda)$, $\mu \geq 1$, $0 \leq \lambda < 1$ and $h \in \mathcal{G}_b$, $0 < b \leq 1$.For $M, N, P \geq 1$, which verify*

$$\left|\frac{zf^{'}(z)}{f(z)}\right| \leq M, \quad |f(z)| < N, \quad \left|\frac{zh^{'}(z)}{h(z)}\right| \leq P,$$

for all $z \in \mathbb{U}$, there is α a real positive number so that

$$\rho = 1 - \alpha\left(M + (2 - \lambda)N^{\mu} + b(P + 2) + 2P + 3\right) > 0.$$

In these conditions, the integral operator $\mathcal{C}$, defined by (2.2.10) is in the class $\mathcal{K}(\rho)$.

For $\mu_i = 0$ in Theorem 3.1.3, we obtain the next corollary:

Corollary 3.1.3.2. *Let $f_i, g_i, h_i \in \mathcal{A}$ and $g_i \in \mathcal{R}_{\lambda_i}$, $0 \leq \lambda_i < 1$ and $h_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$. For $M_i, N_i, P_i \geq 1$, which verify*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)}\right| \leq M_i, \quad |f_i(z)| < N_i, \quad \left|\frac{zh_i^{'}(z)}{h_i(z)}\right| \leq P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 - \sum_{i=1}^{n}\left[(\alpha_i - 1)(3 + M_i - \lambda_i) + \beta_i\left(b_i\left(P_i + 2\right) + P_i + 1\right) + \gamma_i\left(P_i + 1\right)\right] > 0.$$

In these conditions, the integral operator $\mathcal{C}_n$, defined by (3.0.2) is in the class $\mathcal{K}(\rho)$.

For $\mu_i = 1$ in Theorem 3.1.3, we obtain the next corollary:

Corollary 3.1.3.3. *Let $f_i, g_i, h_i \in \mathcal{A}$ and $g_i \in \mathcal{S}^*_{\lambda_i}$, $0 \leq \lambda_i < 1$ and $h_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$. For $M_i, N_i, P_i \geq 1$, which verify*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)}\right| \leq M_i, \quad |f_i(z)| < N_i, \quad \left|\frac{zh_i^{'}(z)}{h_i(z)}\right| \leq P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 - \sum_{i=1}^{n}\left[(\alpha_i - 1)\left(M_i + (2 - \lambda_i)N_i + 1\right) + \beta_i\left(b_i\left(N_i + 2\right) + P_i + 1\right) + \gamma_i\left(P_i + 1\right)\right] > 0.$$

In these conditions, the integral operator $\mathcal{C}_n$, defined by (3.0.2) is in the class $\mathcal{K}(\rho)$.

Theorem 3.1.4. *Let $f_i, g_i, h_i \in \mathcal{A}$ and $h_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$. If*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)} - 1\right| < 1, \quad \left|\frac{zg_i^{'}(z)}{g_i(z)} - 1\right| < 1, \quad |g_i(z)| \leq N_i, \quad \left|\frac{zh_i^{'}(z)}{h_i(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 - \sum_{i=1}^{n}\left[(\alpha_i - 1)(1 + 2N_i) + \beta_i(2b_i + 1) + \gamma_i\right] > 0,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is in the class $\mathcal{K}(\rho)$.

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.1.4, we obtain the next corollary:

Corollary 3.1.4.1. *Let $f, g, h \in \mathcal{A}$ and $h \in \mathcal{G}_b$, $0 < b \leq 1$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad \left|\frac{zg'(z)}{g(z)} - 1\right| < 1, \quad |g(z)| \leq N, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$ there is α a real positive number so that

$$\rho = 1 - \alpha\,(2N + 2b + 3) > 0.$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is in the class $\mathcal{K}(\rho)$.

Theorem 3.1.5. *Let be the analytic functions f_i, g_i, h_i, k_i and $g_i \in \mathcal{B}(\mu_i, \lambda_i)$, $\mu_i \geq 1$, $0 \leq \lambda_i < 1$ and $f_i \in \mathcal{G}_{b_{1i}}$, $h_i \in \mathcal{G}_{b_{2i}}$, $k_i \in \mathcal{G}_{b_{3i}}$, $0 < b_{1i}, b_{2i}, b_{3i} \leq 1$. For $M_i, N_i, P_i, Q_i \geq 1$, which verify*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 - \sum_{i=1}^{n} (\alpha_i - 1)\left(b_{1i}M_i + 2b_{1i} + 2M_i + (2 - \lambda_i)\,N_i^{\mu_i} + 1\right) -$$

$$- \sum_{i=1}^{n} \left[\beta_i\left(P_i + Q_i + 2\right) + \gamma_i\left(b_{2i}P_i + 2b_{2i} + P_i + b_{3i}Q_i + 2b_{3i} + Q_i + 2\right)\right] > 0.$$

In these conditions, the integral operator $\mathcal{G}_n$, defined by (3.0.3) is in the class $\mathcal{K}(\rho)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.1.5, we obtain the next corollary:

Corollary 3.1.5.1. *Let $f, g, h, k \in \mathcal{A}$ and $g \in \mathcal{B}(\mu, \lambda)$, $\mu \geq 1$, $0 \leq \lambda < 1$ and $f \in \mathcal{G}_{b_1}$, $h \in \mathcal{G}_{b_2}$, $k \in \mathcal{G}_{b_3}$ $0 < b_1, b_2, b_3 \leq 1$. For $M, N, P, Q \geq 1$, which verify*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq M, \quad |f(z)| < N, \quad \left|\frac{zh'(z)}{h(z)}\right| \leq P, \quad \left|\frac{zk'(z)}{k(z)}\right| \leq Q,$$

for all $z \in \mathbb{U}$, there is α a real positive number so that

$$\rho = 1 - \alpha\left(b_1M + b_2P + b_1Q + 2b_1 + 2b_2 + 2b_3 + 2M + 2P + 2Q + 6 + 2N^{\mu} - \lambda N^{\mu}\right) > 0.$$

In these conditions, the integral operator $\mathcal{G}$, defined by (2.2.16) is in the class $\mathcal{K}(\rho)$.

For $\mu_i = 0$ in Theorem 3.1.5, we obtain the next corollary:

Corollary 3.1.5.2. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}$ and $g_i \in \mathcal{R}_{\lambda_i}$, $0 \leq \lambda_i < 1$ and $f_i \in \mathcal{G}_{b_{1i}}$, $h_i \in \mathcal{G}_{b_{2i}}$, $k_i \in \mathcal{G}_{b_{3i}}$, $0 < b_{1i}, b_{2i}, b_{3i} \leq 1$. For $M_i, N_i, P_i, Q_i \geq 1$, which verify*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 - \sum_{i=1}^{n} (\alpha_i - 1)\left(b_{1i} M_i + 2b_{1i} + 2M_i + 4 - \lambda_i\right) -$$

$$- \sum_{i=1}^{n} \left[\beta_i \left(P_i + Q_i + 2\right) + \gamma_i \left(b_{2i} P_i + 2b_{2i} + P_i + b_{3i} Q_i + 2b_{3i} + Q_i + 2\right)\right] > 0.$$

In these conditions, the integral operator $\mathcal{G}$, defined by (3.0.3) is in the class $\mathcal{K}(\rho)$.

For $\mu_i = 1$ in Theorem 3.1.5, we obtain the next corollary:

Corollary 3.1.5.3. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}$ and $g_i \in \mathcal{S}^*_{\lambda_i}$, $0 \leq \lambda_i < 1$ and $f_i \in \mathcal{G}_{b_{1i}}$, $h_i \in \mathcal{G}_{b_{2i}}$, $k_i \in \mathcal{G}_{b_{3i}}$, $0 < b_{1i}, b_{2i}, b_{3i} \leq 1$. For $M_i, N_i, P_i, Q_i \geq 1$, which verify*

$$\left|\frac{z f_i'(z)}{f_i(z)}\right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left|\frac{z h_i'(z)}{h_i(z)}\right| \leq P_i, \quad \left|\frac{z k_i'(z)}{k_i(z)}\right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 - \sum_{i=1}^{n} (\alpha_i - 1)\left(b_{1i} M_i + 2b_{1i} + 2M_i + \left(2 - \lambda_i\right) N_i + 2\right) -$$

$$- \sum_{i=1}^{n} \left[\beta_i \left(P_i + Q_i + 2\right) + \gamma_i \left(b_{2i} P_i + 2b_{2i} + P_i + b_{3i} Q_i + 2b_{3i} + Q_i + 2\right)\right] > 0.$$

In these conditions, the integral operator $\mathcal{G}$, defined by (3.0.3) is in the class $\mathcal{K}(\rho)$.

Theorem 3.1.6. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}$ and $f_i, h_i, k_i \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$. If*

$$\left|\frac{z f_i'(z)}{f_i(z)} - 1\right| < 1, \quad \left|\frac{z g_i'(z)}{g_i(z)} - 1\right| < 1, \ |g_i(z)| \leq N_i, \ \left|\frac{z h_i'(z)}{h_i(z)} - 1\right| < 1, \ \left|\frac{z k_i'(z)}{k_i(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)\left(2b_i + 2N_i + 1\right) + 2\beta_i + 2\gamma_i \left(2b_i + 1\right)\right] > 0,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is in the class $\mathcal{K}(\rho)$.

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.1.6, we obtain the next corollary:

Corollary 3.1.6.1. *Let $f, g, h, k \in \mathcal{A}$ and $f, h_i, k_i \in \mathcal{G}_b$, $0 < b \leq 1$. If*

$$\left|\frac{z f'(z)}{f(z)} - 1\right| < 1, \quad \left|\frac{z g'(z)}{g(z)} - 1\right| < 1, \quad |g(z)| \leq N,$$

$$\left|\frac{z f'(z)}{f(z)} - 1\right| < 1, \ \left|\frac{z g'(z)}{g(z)} - 1\right| < 1, \quad |g(z)| \leq N, \ \left|\frac{z h'(z)}{h(z)} - 1\right| < 1, \ \left|\frac{z k'(z)}{k(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, there is α a real positive number so that

$$\rho = 1 - \alpha \left(2N + 6b + 5\right) > 0,$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is in the class $\mathcal{K}(\rho)$.

Theorem 3.1.7. *Let be the analytic functions $f_i, g_i, h_i, k_i \in \mathcal{A}$ and $g_i \in \mathcal{G}_{b_{1i}}$, $h_i \in \mathcal{G}_{b_{2i}}$, $k_i \in \mathcal{G}_{b_{3i}}$, $0 < b_{1i}, b_{2i}, b_{3i} \leq 1$. For $M_i, N_i, P_i, Q_i \geq 1$, which verify*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq M_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i, \delta_i$ real positive numbers and $\alpha_i > 1$ so that

$$\lambda = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(M_i + 1) + \beta_i (b_{1i}N_i + 2b_{1i} + N_i + 1)\right] -$$

$$- \sum_{i=1}^{n} \left[\gamma_i (P_i + Q_i + 2) + \delta_i (b_{2i}P_i + 2b_{2i} + b_{3i}Q_i + 2b_{3i} + P_i + Q_i + 2)\right] > 0.$$

In these conditions, the integral operator $\mathcal{T}_n$, defined by (3.0.4) is in the class $\mathcal{K}(\lambda)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.1.7, we obtain the next corollary:

Corollary 3.1.7.1. *Let $f, g, h, k \in \mathcal{A}$ and $g \in \mathcal{G}_{b_1}$, $h \in \mathcal{G}_{b_2}$, $k \in \mathcal{G}_{b_3}$, $0 < b_1, b_2, b_3 \leq 1$. For $M, N, P, Q \geq 1$, which verify*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq M, \quad \left|\frac{zg'(z)}{g(z)}\right| \leq N, \quad \left|\frac{zh'(z)}{h(z)}\right| \leq P, \quad \left|\frac{zk'(z)}{k(z)}\right| \leq Q,$$

for all $z \in \mathbb{U}$, there is α a real positive number so that

$$\lambda = 1 - \alpha (M + b_1N + 2b_1 + N + 2P + 2Q + b_2P + 2b_2 + b_3Q + 2b_3 + 6) > 0.$$

In these conditions, the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{K}(\lambda)$.

Theorem 3.1.8. *Let $f_i, g_i, h_i, k_i, \in \mathcal{A}$ and $g_i, h_i, k_i, \in \mathcal{G}_{b_i}$, $0 < b_i \leq 1$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1, \quad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| < 1, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ there are $\alpha_i, \beta_i, \gamma_i, \delta_i$ real positive numbers and $\alpha_i > 1$ so that

$$\lambda = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1) + \beta_i (2b_i + 1) + 2\gamma_i + 2\delta_i (2b_i + 1)\right] > 0,$$

then the integral operator $\mathcal{T}_n$, defined by (3.0.4) is in the class $\mathcal{K}(\lambda)$.

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.1.8, we obtain the next corollary:

Corollary 3.1.8.1. *Let $f, g, h, k \in \mathcal{A}$ and $g, h, k \in \mathcal{G}_b$, $0 < b \leq 1$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad \left|\frac{zg'(z)}{g(z)} - 1\right| < 1, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| < 1, \quad \left|\frac{zk'(z)}{k(z)} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, there is α a real positive number so that

$$\lambda = 1 - 6\alpha (b_i + 1) > 0,$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{K}(\lambda)$.

3.2 Convexity conditions for starlike functions

In this section we present sufficient conditions to ensure the convexity of the four integral operators but also their order of convexity, considering the functions in the class of starlike functions $\mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$.

Theorem 3.2.1. *Let $f_i \in \mathcal{S}^*(\mu_i)$, $g_i \in \mathcal{K}(\lambda_i)$ and $g_i \in \mathcal{S}^*(\nu_i)$, $0 \leq \mu_i, \lambda_i, \nu_i < 1$. If $\alpha_i, \beta_i, \gamma_i$ are strictly positive real numbers and $\alpha_i > 1$ so that*

$$\sum_{i=1}^{n} \left[(\alpha_i - 1)\, \mu_i + \beta_i\, (1 - \lambda_i) + \gamma_i \nu_i \right] < 1,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is convex by the order

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)\, \mu_i + \beta_i\, (1 - \lambda_i) + \gamma_i \nu_i \right],$$

for all $i = \overline{1, n}$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.2.1, we obtain the next corollary:

Corollary 3.2.1.1. *Let $f \in \mathcal{S}^*(\mu)$, $g \in \mathcal{K}(\lambda)$ and $g \in \mathcal{S}^*(\nu)$, $0 \leq \mu, \lambda, \nu < 1$. If α is a real positive number so that*

$$\alpha\, (\mu + \nu - \lambda + 1) < 1,$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is convex by the order

$$1 + \alpha\, (\lambda - \mu - \nu - 1)\,.$$

Theorem 3.2.2. *Let $f_i \in \mathcal{S}^*(\mu_i)$, $g_i \in \mathcal{K}(\lambda_i)$ and $g_i \in \mathcal{S}^*(\lambda_i)$, $0 \leq \mu_i, \lambda_i < 1$. If $\alpha_i, \beta_i, \gamma_i$ are strictly positive real numbers and $\alpha_i > 1$ so that*

$$\sum_{i=1}^{n} \left[(\alpha_i - 1)\, (\mu_i - 1) + \beta_i\, (\lambda_i - 1) + \gamma_i\, (\lambda_i - 1) \right] < 1,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is convex by the order

$$1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)\, (\mu_i - 1) + \beta_i\, (\lambda_i - 1) + \gamma_i\, (\lambda_i - 1) \right],$$

for all $i = \overline{1, n}$.

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.2.2, we obtain the next corollary:

Corollary 3.2.2.1. *Let $f \in \mathcal{S}^*(\mu)$, $g \in \mathcal{K}(\lambda)$ and $g \in \mathcal{S}^*(\lambda)$, $0 \leq \mu, \lambda < 1$. If α is a real positive number so that*

$$\alpha\, (\mu + 2\lambda - 3) < 1,$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is convex by the order

$$1 - \alpha\, (\mu + 2\lambda - 3)\,.$$

Theorem 3.2.3. *Let $f_i \in \mathcal{S}^*(\alpha_i - 1)$, $zg_i' \in \mathcal{S}^*(\beta_i)$ and $g_i \in \mathcal{S}^*(\gamma_i)$, $i = \overline{1,n}$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i \geq 1$, $\beta_i, \gamma_i \geq 0$ for all $i = \overline{1,n}$ so that*

$$0 < \sum_{i=1}^{n} [(\alpha_i - 1) + \beta_i + \gamma_i] < 1,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is convex by the order

$$\sum_{i=1}^{n} \left[(\alpha_i - 1)^2 + \beta_i^2 + \gamma_i^2 - \alpha_i - \beta_i - \gamma_i + 1\right] + 1.$$

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.2.3, we obtain the next corollary:

Corollary 3.2.3.1. *Let $f, zg', g \in \mathcal{S}^*(\alpha)$. If α is a real positive number so that*

$$0 < 3\alpha < 1,$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is convex by the order

$$3\alpha^2 - 3\alpha + 1.$$

Theorem 3.2.4. *Let $f_i \in \mathcal{S}^*(\mu_i)$, $g_i \in \mathcal{S}^*(\nu_i)$, $h_i \in \mathcal{K}(\lambda_i)$ and $h_i \in \mathcal{S}^*(\eta_i)$, $0 \leq \mu_i, \lambda_i, \nu_i, \eta_i < 1$. If $\alpha_i, \beta_i, \gamma_i$ are strictly positive real numbers and $\alpha_i > 1$ and $|g_i(z)| \leq 1$ so that*

$$\sum_{i=1}^{n} [(\alpha_i - 1)(\mu_i + \nu_i) + \beta_i (1 - \lambda_i) + \gamma_i \eta_i] < 1,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is convex by the order

$$\rho = 1 - \sum_{i=1}^{n} [(\alpha_i - 1)(\mu_i + \nu_i) + \beta_i (1 - \lambda_i) + \gamma_i \eta_i],$$

for all $i = \overline{1,n}$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.2.4, we obtain the next corollary:

Corollary 3.2.4.1. *Let $f \in \mathcal{S}^*(\mu)$, $g \in \mathcal{S}^*(\nu)$, $h \in \mathcal{K}(\lambda)$ and $h \in \mathcal{S}^*(\eta)$, $0 \leq \mu, \lambda, \nu, \eta < 1$. If α is a real positive number so that $|g(z)| \leq 1$ so that*

$$\alpha(\mu + \nu + \eta - \lambda + 1) < 1,$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is convex by the order

$$1 + \alpha(\lambda - \mu - \nu - \eta - 1).$$

Theorem 3.2.5. *Let $f_i \in \mathcal{S}^*(\mu_i)$, $g_i \in \mathcal{S}^*(\nu_i)$, $h_i \in \mathcal{K}(\lambda_i)$ and $h_i \in \mathcal{S}^*(\lambda_i)$, $0 \leq \mu_i, \lambda_i, \nu_i < 1$. If $\alpha_i, \beta_i, \gamma_i$ are strictly positive real numbers and $\alpha_i > 1$ and $|g_i(z)| \leq 1$ so that*

$$\sum_{i=1}^{n} [(\alpha_i - 1)(\mu_i + \nu_i - 1) + \beta_i (\lambda_i - 1) + \gamma_i (\lambda_i - 1)] < 1,$$

then the integral operator $\mathcal{C}_n$,defined by (3.0.2) is convex by the order

$$1-\sum_{i=1}^{n}\left[\left(\alpha_i-1\right)\left(\mu_i+\nu_i-1\right)+\beta_i\left(\lambda_i-1\right)+\gamma_i\left(\lambda_i-1\right)\right],$$

for all $i=\overline{1,n}$.

Putting $n=1$ and $\alpha_i-1=\beta_i=\gamma_i=\alpha$ in Theorem 3.2.5, we obtain the next corollary:

Corollary 3.2.5.1. *Let $f\in\mathcal{S}^*(\mu)$, $g\in\mathcal{S}^*(\nu)$, $h\in\mathcal{K}(\lambda)$ and $h\in\mathcal{S}^*(\lambda)$, $0\leq\mu,\lambda,\nu<1$. If α is a real positive number so that $|g(z)|\leq 1$ so that*

$$\alpha\left(\mu+\nu+2\lambda-3\right)<1,$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is convex by the order

$$1-\alpha\left(\mu+\nu+2\lambda-3\right).$$

Theorem 3.2.6. *Let $f_i,g_i\in\mathcal{S}^*\left(\alpha_i-1\right)$, $zh_i'\in\mathcal{S}^*\left(\beta_i\right)$ and $h_i\in\mathcal{S}^*\left(\gamma_i\right)$, $i=\overline{1,n}$. If $\alpha_i,\beta_i,\gamma_i$ are positive real numbers and $\alpha_i\geq 1$, $\beta_i,\gamma_i\geq 0$ for all $i=\overline{1,n}$ and Re $\left(g_i(z)\right)\geq 1$ so that*

$$0<\sum_{i=1}^{n}\left[\left(\alpha_i-1\right)+\beta_i+\gamma_i\right]<1,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is convex by the order

$$\sum_{i=1}^{n}\left[2\left(\alpha_i-1\right)^2+\beta_i^2+\gamma_i^2-\alpha_i-\beta_i-\gamma_i+1\right]+1.$$

Letting $n=1$ and $\alpha_i-1=\beta_i=\gamma_i=\alpha$ in Theorem 3.2.6, we obtain the next corollary:

Corollary 3.2.6.1. *Let $f,g,h\in\mathcal{S}^*\left(\alpha\right)$ and $zh_i'\in\mathcal{S}^*\left(\alpha\right)$. If α is a real positive number so that Re $\left(g(z)\right)\geq 1$ so that*

$$0<3\alpha<1,$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is convex by the order

$$4\alpha^2-3\alpha+1.$$

Theorem 3.2.7. *Let $f_i\in\mathcal{K}(\mu_i)$, $g_i\in\mathcal{S}^*(\nu_i)$, $h_i\in\mathcal{K}(\lambda_i)$, $h_i\in\mathcal{S}^*(\eta_i)$, $k_i\in\mathcal{K}(\omega_i)$ and $k_i\in\mathcal{S}^*(\xi_i)$, $0\leq\mu_i,\nu_i,\eta_i,\lambda_i,\omega_i,\xi_i<1$. If $\alpha_i,\beta_i,\gamma_i$ are positive real numbers and $\alpha_i>1$ and $|g_i(z)|\leq 1$ so that*

$$\sum_{i=1}^{n}\left[\left(\alpha_i-1\right)\left(\nu_i-\mu_i+1\right)+\beta_i\left(\eta_i+\xi_i\right)+\gamma_i\left(2-\lambda_i-\omega_i\right)\right]<1,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is convex by the order

$$\rho=1-\sum_{i=1}^{n}\left[\left(\alpha_i-1\right)\left(\nu_i-\mu_i+1\right)+\beta_i\left(\eta_i+\xi_i\right)+\gamma_i\left(2-\lambda_i-\omega_i\right)\right],$$

for all $i=\overline{1,n}$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.2.7, we obtain the next corollary:

Corollary 3.2.7.1. *Let $f \in \mathcal{K}(\mu)$, $g \in \mathcal{S}^*(\nu)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{S}^*(\eta)$, $k \in \mathcal{K}(\omega)$, $k \in \mathcal{S}^*(\xi)$, $0 \leq \mu, \nu, \eta, \lambda, \omega, \xi < 1$. If α is a real positive number and $|g(z)| \leq 1$ so that*

$$\alpha\left(\nu + \eta + \xi - \mu - \lambda - \omega + 3\right) < 1,$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is convex by the order

$$1 - \alpha\left(\nu + \eta + \xi - \mu - \lambda - \omega + 3\right).$$

Theorem 3.2.8. *Let $f_i \in \mathcal{K}(\mu_i)$, $g_i \in \mathcal{S}^*(\nu_i)$, $h_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{S}^*(\lambda_i)$, $k_i \in \mathcal{K}(\omega_i)$ and $k_i \in \mathcal{S}^*(\omega_i)$, $0 \leq \mu_i, \nu_i, \lambda_i, \omega_i < 1$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i > 1$ and $|g_i(z)| \leq 1$ so that*

$$\sum_{i=1}^{n}\left[\left(\alpha_i - 1\right)\left(\mu_i + \nu_i - 1\right) + \left(\beta_i + \gamma_i\right)\left(\lambda_i - \omega_i\right)\right] < 1,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is convex by the order

$$1 - \sum_{i=1}^{n}\left[\left(\alpha_i - 1\right)\left(\mu_i + \nu_i - 1\right) + \left(\beta_i + \gamma_i\right)\left(\lambda_i - \omega_i\right)\right],$$

for all $i = \overline{1, n}$.

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.2.8, we obtain the next corollary:

Corollary 3.2.8.1. *Let $f_i \in \mathcal{K}(\mu)$, $g \in \mathcal{S}^*(\nu)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{S}^*(\lambda)$, $k \in \mathcal{K}(\omega)$ and $k \in \mathcal{S}^*(\omega)$, $0 \leq \mu, \nu, \lambda, \omega < 1$. If α is a real positive number and $|g(z)| \leq 1$ so that*

$$\alpha\left(\mu + \nu + 2\lambda - 2\omega - 1\right) < 1,$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is convex by the order

$$1 - \alpha\left(\mu + \nu + 2\lambda - 2\omega - 1\right).$$

Theorem 3.2.9. *Let $zf_i, g_i \in \mathcal{S}^*\left(\alpha_i - 1\right)$, $zh_i' \in \mathcal{S}^*\left(\gamma_i\right)$, $h_i \in \mathcal{S}^*\left(\beta_i\right)$, $zk_i' \in \mathcal{S}^*\left(\gamma_i\right)$ and $k_i \in \mathcal{S}^*\left(\beta_i\right)$, $i = \overline{1, n}$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i \geq 1$, $\beta_i, \gamma_i \geq 0$ for all $i = \overline{1, n}$ and Re $\left(g_i(z)\right) \geq 1$ so that*

$$0 < \sum_{i=1}^{n}\left[\left(\alpha_i - 1\right) + \beta_i + \gamma_i\right] < 1,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is convex by the order

$$\sum_{i=1}^{n}\left[2\left(\alpha_i - 1\right)^2 - \left(\alpha_i - 1\right)\right] + 1.$$

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.2.9, we obtain the next corollary:

Corollary 3.2.9.1. *Let* $g, h, k \in \mathcal{S}^*(\alpha)$ *and* $zf_i', zh_i', zk_i' \in \mathcal{S}^*(\alpha)$*. If* α *is a real positive number and* *Re* $(g(z)) \geq 1$ *so that*

$$0 < 3\alpha < 1,$$

then the integral operator $\mathcal{G}$*, defined by (2.2.16) is convex by the order*

$$2\alpha^2 - \alpha + 1.$$

Theorem 3.2.10. *Let* $f_i \in \mathcal{S}^*(\mu_i)$*,* $g_i \in \mathcal{K}(\lambda_i)$*,* $h_i \in \mathcal{S}^*(\nu_i)$*,* $k_i \in \mathcal{S}^*(\theta_i)$*,* $h_i \in \mathcal{K}(\eta_i)$ *and* $k_i \in \mathcal{K}(\sigma_i)$*,* $0 \leq \mu_i, \nu_i, \theta_i < 1$*,* $0 \leq \lambda_i, \eta_i, \sigma_i < 1$*. If* $\alpha_i, \beta_i, \gamma_i, \delta_i$ *are positive real numbers and* $\alpha_i > 1$ *so that*

$$\sum_{i=1}^{n} \left[(\alpha_i - 1)\mu_i + \beta_i (1 - \lambda_i) + \gamma_i (\nu_i + \theta_i) + \delta_i (2 - \eta_i - \sigma_i) \right] < 1,$$

then the integral operator $\mathcal{T}_n$*, defined by (3.0.4) is convex by the order*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)\mu_i + \beta_i (1 - \lambda_i) + \gamma_i (\nu_i + \theta_i) + \delta_i (2 - \eta_i - \sigma_i) \right],$$

for all $i = \overline{1, n}$*.*

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.2.10, we obtain the next corollary:

Corollary 3.2.10.1. *Let* $f \in \mathcal{S}^*(\mu)$*,* $g \in \mathcal{K}(\lambda)$*,* $h \in \mathcal{S}^*(\nu)$*,* $k \in \mathcal{S}^*(\theta)$*,* $h \in \mathcal{K}(\eta)$ *and* $k \in \mathcal{K}(\sigma)$*,* $0 \leq \mu, \nu, \theta < 1$*,* $0 \leq \lambda, \eta, \sigma < 1$*. If* α *is a real positive number so that*

$$\alpha (\mu + \nu + \theta - \lambda - \eta - \sigma + 3) < 1,$$

then the integral operator $\mathcal{T}$*, defined by (2.2.23) is convex by the order*

$$1 + \alpha (\lambda + \eta + \sigma - \mu - \nu - \theta - 3).$$

Theorem 3.2.11. *Let* $f_i \in \mathcal{S}^*(\mu_i)$*,* $g_i \in \mathcal{K}(\lambda_i)$*,* $h_i \in \mathcal{S}^*(\nu_i)$*,* $k_i \in \mathcal{S}^*(\theta_i)$*,* $h_i \in \mathcal{K}(\nu_i)$ *and* $k_i \in \mathcal{K}(\theta_i)$*,* $0 \leq \mu_i, \lambda_i, \nu_i, \theta_i < 1$*. If* $\alpha_i, \beta_i, \gamma_i, \delta_i$ *are positive real numbers and* $\alpha_i > 1$*, so that*

$$\sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i - 1) + \beta_i (\lambda_i - 1) + (\gamma_i + \delta_i)(\nu_i - \theta_i) \right] < 1,$$

then the integral operator $\mathcal{T}_n$*, defined by (3.0.4) is convex by the order*

$$1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i - 1) + \beta_i (\lambda_i - 1) + (\gamma_i + \delta_i)(\nu_i - \theta_i) \right],$$

for all $i = \overline{1, n}$*.*

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.2.11, we obtain the next corollary:

Corollary 3.2.11.1. *Let $f \in \mathcal{S}^*(\mu)$, $g \in \mathcal{K}(\lambda)$, $h \in \mathcal{S}^*(\nu)$, $k \in \mathcal{S}^*(\theta)$, $h \in \mathcal{K}(\nu)$ and $k \in \mathcal{K}(\theta)$, $0 \leq \mu, \lambda, \nu, \theta < 1$. If α is a real positive number so that*

$$\alpha\left(\mu + \lambda + 2\nu - 2\theta - 2\right) < 1,$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is convex by the order

$$1 - \alpha\left(\mu + \lambda + 2\nu - 2\theta - 2\right).$$

Theorem 3.2.12. *Let $f_i \in \mathcal{S}^*\left(\alpha_i - 1\right)$, $zg_i' \in \mathcal{S}^*\left(\beta_i\right)$, $h_i \in \mathcal{S}^*\left(\gamma_i\right)$, $zh_i' \in \mathcal{S}^*\left(\delta_i\right)$, $k_i \in \mathcal{S}^*\left(\gamma_i\right)$, $zk_i' \in \mathcal{S}^*\left(\delta_i\right)$ $i = \overline{1,n}$. If $\alpha_i, \beta_i, \gamma_i, \delta_i$ are positive real numbers and $\alpha_i \geq 1$, $\beta_i, \gamma_i, \delta_i \geq 0$ for all $i = \overline{1,n}$ so that*

$$0 < \sum_{i=1}^{n}\left[\left(\alpha_i - 1\right) + \beta_i + \gamma_i + \delta_i\right] < 1,$$

then the integral operator $\mathcal{T}_n$, defined by (3.0.4) is convex by the order

$$\sum_{i=1}^{n}\left[\left(\alpha_i - 1\right)^2 + \beta_i^2 - \alpha_i - \beta_i + 1\right] + 1.$$

If we consider $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.2.12, we obtain the next corollary:

Corollary 3.2.12.1. *Let $f, h, k \in \mathcal{S}^*\left(\alpha\right)$ and $zg', zh', zk_i' \in \mathcal{S}^*\left(\alpha\right)$. If α is a real positive number so that*

$$0 < 4\alpha < 1,$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is convex by the order

$$2\alpha^2 - 2\alpha + 1.$$

3.3 Convexity conditions for the class $\mathcal{SP}(\alpha, \beta)$

This section contains sufficient convexity conditions for new integral operators with class functions $\mathcal{SP}(\alpha, \beta)$, $\alpha > 0$, $0 \leq \beta < 1$.

Theorem 3.3.1. *Let $f_i \in \mathcal{SP}(\alpha, \beta)$, $g_i \in \mathcal{SP}(\delta, \eta)$ and $g_i \in \mathcal{K}\left(\lambda_i\right)$, $\alpha, \delta > 0$, $0 \leq \beta, \eta, \lambda_i < 1$ for all $i = \overline{1,n}$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i > 1$ so that*

$$\rho = 1 + \sum_{i=1}^{n}\left[\left(\alpha_i - 1\right)\left(\beta - \alpha - 1\right) + \beta_i\left(\lambda_i - 1\right) + \gamma_i\left(\eta - \delta - 1\right)\right] > 0,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is in the class $\mathcal{K}(\rho)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.3.1, we obtain the next corollary:

Corollary 3.3.1.1. *Let $f \in \mathcal{SP}(\alpha, \beta)$, $g \in \mathcal{SP}(\delta, \eta)$, and $g \in \mathcal{K}(\lambda)$, $\alpha, \delta > 0$, $0 \leq \beta, \eta, \lambda < 1$. If there is a positive real number α so that*

$$\rho = 1 + \alpha (\beta + \eta - \alpha - \delta + \lambda - 3) > 0.$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{K}(\rho)$.

Theorem 3.3.2. *Let $f_i \in \mathcal{SP}(\alpha_i - 1)$, $zg_i' \in \mathcal{SP}(\beta_i)$ and $g_i \in \mathcal{SP}(\gamma_i)$, $i = \overline{1, n}$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i > 1$, $\beta_i, \gamma_i > 0$ for all $i = \overline{1, n}$ so that*

$$1 < \sum_{i=1}^{n} [\alpha_i + \beta_i + \gamma_i] < 2,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is convex by the order

$$2 - \sum_{1=1}^{n} (\alpha_i + \beta_i + \gamma_i).$$

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.3.2, we obtain the next corollary:

Corollary 3.3.2.1. *Let $f, g \in \mathcal{SP}(\alpha)$ and $zg' \in \mathcal{SP}(\alpha)$. If α is a real positive number so that*

$$0 < 3\alpha < 1$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is convex by the order

$$1 - 3\alpha.$$

Theorem 3.3.3. *Let $f_i \in \mathcal{SP}(\alpha, \beta)$, $g_i \in \mathcal{SP}(\gamma, \delta)$, $h_i \in \mathcal{K}(\lambda_i)$ and $h_i \in \mathcal{SP}(\nu, \eta)$, $\alpha, \gamma, \nu > 0$, $0 \leq \beta, \delta, \eta, \lambda_i < 1$ for all $i = \overline{1, n}$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i > 1$ and Re $(g_i(z)) \geq 1$ so that*

$$\rho = 1 + \sum_{i=1}^{n} [(\alpha_i - 1)(\beta + \delta - \alpha - \gamma - 1) + \beta_i (\lambda_i - 1) + \gamma_i (\eta - \nu - 1)] > 0,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is in the class $\mathcal{K}(\rho)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.3.3, we obtain the next corollary:

Corollary 3.3.3.1. *Let $f \in \mathcal{SP}(\alpha, \beta)$, $g \in \mathcal{SP}(\gamma, \delta)$, $h \in \mathcal{K}(\lambda)$ and $h \in \mathcal{SP}(\nu, \eta)$, $\alpha, \gamma, \nu > 0$, $0 \leq \beta, \delta, \eta, \lambda < 1$. If there is a positive real number α and Re $(g(z)) \geq 1$ so that*

$$\rho = 1 + \alpha (\beta + \delta + \eta - \alpha - \gamma - \nu + \lambda - 3) > 0.$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is in the class $\mathcal{K}(\rho)$.

Theorem 3.3.4. *Let $f_i, g_i \in \mathcal{SP}(\alpha_i - 1)$, $zh_i' \in \mathcal{SP}(\beta_i)$ and $h_i \in \mathcal{SP}(\gamma_i)$, $i = \overline{1, n}$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i > 1$, $\beta_i, \gamma_i > 0$ for all $i = \overline{1, n}$ and Re $(g_i(z)) \geq 1$ so that*

$$0 < \sum_{i=1}^{n} [(\alpha_i - 1) + \beta_i + \gamma_i] < 1,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is convex by the order

$$1 - \sum_{1=1}^{n} (\beta_i + \gamma_i).$$

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.3.4, we obtain the next corollary:

Corollary 3.3.4.1. *Let $f, g, h \in \mathcal{SP}(\alpha)$ and $zh' \in \mathcal{SP}(\alpha)$. If α is a real positive number so that Re $(g(z)) \geq 1$ so that*

$$0 < 3\alpha < 1$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is convex by the order

$$1 - 2\alpha.$$

Theorem 3.3.5. *Let $f_i \in \mathcal{N}(\mu_i)$, $g_i \in \mathcal{SP}(\gamma, \delta)$, $h_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{SP}(\nu, \eta)$, $k_i \in \mathcal{K}(\sigma_i)$, $k_i \in \mathcal{SP}(\theta, \xi)$, $\gamma, \nu, \theta > 0$, $0 \leq \mu_i, \delta, \eta, \xi, \lambda_i, \sigma_i < 1$ for all $i = \overline{1, n}$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i > 1$ and Re $(g_i(z)) \geq 1$ so that*

$$\rho = 1 + \sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i + \delta - \gamma - 1) + \beta_i(\eta + \theta - \nu - \xi) + \gamma_i(\lambda_i - \sigma_i)\right] > 0,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is in the class $\mathcal{K}(\rho)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.3.5, we obtain the next corollary:

Corollary 3.3.5.1. *Let $f \in \mathcal{K}(\mu)$, $g \in \mathcal{SP}(\gamma, \delta)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{SP}(\nu, \eta)$, $k \in \mathcal{K}(\sigma)$, $k \in \mathcal{SP}(\theta, \xi)$, $\gamma, \nu, \theta > 0$, $0 \leq \mu, \delta, \eta, \xi, \lambda, \sigma < 1$.If there is a positive real number α and Re $(g(z)) \geq 1$ so that*

$$\rho = 1 + \alpha(\mu + \delta + \eta + \theta + \lambda - \gamma - \nu - \xi - \sigma - 1) > 0.$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is in the class $\mathcal{K}(\rho)$.

Theorem 3.3.6. *Let $zf_i, g_i \in \mathcal{SP}(\alpha_i - 1)$, $h_i, k_i \in \mathcal{SP}(\beta_i)$ and $zh_i', zk_i' \in \mathcal{SP}(\gamma_i)$, $i = \overline{1, n}$. If $\alpha_i, \beta_i, \gamma_i$ are positive real numbers and $\alpha_i > 1$, $\beta_i, \gamma_i > 0$ for all $i = \overline{1, n}$ and Re $(g_i(z)) \geq 1$ so that*

$$0 < \sum_{i=1}^{n} [\alpha_i + \beta_i + \gamma_i - 1] < 1,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is convex by the order 1.

If we consider $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.3.6, we obtain the next corollary:

Corollary 3.3.6.1. *Let $g, h, k \in \mathcal{SP}(\alpha)$ and $zf', zh', zk' \in \mathcal{SP}(\alpha)$. If α is a real positive number so that Re $(g(z)) \geq 1$ so that*

$$0 < 3\alpha < 1$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is convex by the order 1.

Theorem 3.3.7. *Let $f_i \in \mathcal{SP}(\alpha, \beta)$, $g_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{SP}(\sigma, \eta)$, $k_i \in \mathcal{SP}(\theta, \xi)$, , $h_i \in \mathcal{K}(\mu_i)$, , $k_i \in \mathcal{K}(\nu_i)$, $\alpha, \sigma, \theta > 0$, $0 \leq \beta, \eta, \xi, \lambda_i, \mu_i, \nu_i < 1$ for all $i = \overline{1, n}$. If $\alpha_i, \beta_i, \gamma_i, \delta_i$ are positive real numbers and $\alpha_i > 1$ so that*

$$\rho = 1 + \sum_{i=1}^{n} \left[(\alpha_i - 1)(\beta - \alpha - 1) + \beta_i(\lambda_i - 1) + \gamma_i(\theta - \xi - \sigma + \eta) + \delta_i(\mu_i - \nu_i)\right] > 0$$

then the integral operator $\mathcal{T}_n$, defined by (3.0.4) is in the class $\mathcal{K}(\rho)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.3.7, we obtain the next corollary:

Corollary 3.3.7.1. *Let $f \in \mathcal{SP}(\alpha, \beta)$, $g \in \mathcal{K}(\lambda)$, $h \in \mathcal{SP}(\sigma, \eta)$, $k \in \mathcal{SP}(\theta, \xi)$, , $h \in \mathcal{K}(\mu)$, $k \in \mathcal{K}(\nu)$, $\alpha, \sigma, \theta > 0$, $0 \leq \beta, \eta, \xi, \lambda, \mu, \nu < 1$. If there is a positive real number α so that*

$$\rho = 1 + \alpha\left(\beta + \lambda + \theta + \eta + \mu - \alpha - \xi - \sigma - \nu - 2\right) > 0$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{K}(\rho)$.

Theorem 3.3.8. *Let $f_i \in \mathcal{SP}(\alpha_i - 1)$, $zg_i' \in \mathcal{SP}(\beta_i)$, $h_i, k_i \in \mathcal{SP}(\gamma_i)$ and $zh_i', zk_i' \in \mathcal{SP}(\delta_i)$, $i = \overline{1, n}$. If $\alpha_i, \beta_i, \gamma_i, \delta_i$ are positive real numbers and $\alpha_i > 1$, $\beta_i, \gamma_i, \delta_i > 0$ for all $i = \overline{1, n}$ so that*

$$1 < \sum_{i=1}^{n}\left[\alpha_i + \beta_i + \gamma_i + \delta_i\right] < 2,$$

then the integral operator $\mathcal{T}_n$, defined by (3.0.4) is convex by the order

$$2 - \sum_{1=1}^{n}\left(\alpha_i + \beta_i\right).$$

Putting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.3.8, we obtain the next corollary:

Corollary 3.3.8.1. *Let $f, h, k \in \mathcal{SP}(\alpha)$ and $zg', zh', zk' \in \mathcal{SP}(\alpha)$. If α is a real positive number so that*

$$0 < 4\alpha < 1,$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is convex by the order

$$1 - 2\alpha.$$

3.4 Convexity conditions for the class $\mathcal{S}_\beta^*$

This paragraph describes sufficient conditions for belonging to the class of convex functions $\mathcal{C}_\mu(b)$, for new integral operators with functions in the class $\mathcal{S}_\beta^*(b)$, $0 \leq \beta < 1$, where $b \in \mathbb{C} - \{0\}$.

Theorem 3.4.1. *Let $f_i \in \mathcal{S}_{\delta_i}^*(b)$, $g_i \in \mathcal{C}_{\lambda_i}(b)$ and $g_i \in \underline{\mathcal{S}_{\lambda_i}^*}(b)$, $0 \leq \lambda_i, \delta_i < 1$, where $b \in \mathbb{C} - \{0\}$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i > 1$, for all $i = \overline{1, n}$. If*

$$0 \leq 1 + \sum_{i=1}^{n}\left[\left(\alpha_i - 1\right)\left(1 - \delta_i\right) + \left(\beta_i + \gamma_i\right)\left(1 - \lambda_i\right)\right] < 1,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is convex by the order $\mathcal{C}_\mu(b)$, with

$$\mu = 1 + \sum_{i=1}^{n}\left[\left(\alpha_i - 1\right)\left(1 - \delta_i\right) + \left(\beta_i + \gamma_i\right)\left(1 - \lambda_i\right)\right],$$

for all $i = \overline{1, n}$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.4.1, we obtain the next corollary:

Corollary 3.4.1.1. *Let $f \in \mathcal{S}^*_\delta(b)$, $g \in \mathcal{C}_\lambda(b)$ and $g \in \mathcal{S}^*_\lambda(b)$, $0 \le \lambda, \delta < 1$, where $b \in \mathbb{C} - \{0\}$. Also, let α a real positive number. If*

$$0 \le 1 + \alpha\,(3 - \delta - 2\lambda) < 1$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is convex by the order $\mathcal{C}_\mu(b)$, with

$$\mu = 1 + \alpha\,(3 - \delta - 2\lambda)\,.$$

Theorem 3.4.2. *Let $f_i \in \mathcal{S}^*_{\delta_i}(b)$, $g_i \in \mathcal{S}^*_{\eta_i}(b)$, $h_i \in \mathcal{C}_{\lambda_i}(b)$ and $h_i \in \mathcal{S}^*_{\lambda_i}(b)$, $0 \le \lambda_i, \delta_i, \eta_i < 1$, where $b \in \mathbb{C} - \{0\}$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i > 1$ for all $i = \overline{1,n}$. If $\left|\frac{g_i(z)}{b}\right| \le 1$ and*

$$0 \le 1 + \sum_{i=1}^{n} \left[(\alpha_i - 1)\,(3 - \delta_i - \eta_i) + (\beta_i + \gamma_i)\,(1 - \lambda_i)\right] < 1,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is convex by the order $\mathcal{C}_\mu(b)$, with

$$\mu = 1 + \sum_{i=1}^{n} \left[(\alpha_i - 1)\,(3 - \delta_i - \eta_i) + (\beta_i + \gamma_i)\,(1 - \lambda_i)\right],$$

for all $i = \overline{1,n}$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.4.2, we obtain the next corollary:

Corollary 3.4.2.1. *Let $f \in \mathcal{S}^*_\delta(b)$, $g \in \mathcal{S}^*_\eta(b)$, $h \in \mathcal{C}_\lambda(b)$ and $h \in \mathcal{S}^*_\lambda(b)$, $0 \le \lambda, \delta, \eta < 1$, where $b \in \mathbb{C} - \{0\}$. Also, let α a real positive number. If $\left|\frac{g(z)}{b}\right| \le 1$ and*

$$0 \le 1 + \alpha\,(5 - \delta - \eta - 2\lambda) < 1,$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is convex by the order $\mathcal{C}_\mu(b)$, with

$$\mu = 1 + \alpha\,(5 - \delta - \eta - 2\lambda)\,.$$

Theorem 3.4.3. *Let $f_i \in \mathcal{C}_{\mu_i}(b)$, $g_i \in \mathcal{S}^*_{\eta_i}(b)$, $h_i \in \mathcal{C}_{\lambda_i}(b)$, $h_i \in \mathcal{S}^*_{\lambda_i}(b)$, $k_i \in \mathcal{C}_{\sigma_i}(b)$ and $k_i \in \mathcal{S}^*_{\sigma_i}(b)$, $0 \le \mu_i, \eta_i, \lambda_i, \sigma_i < 1$, where $b \in \mathbb{C} - \{0\}$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i > 1$ for all $i = \overline{1,n}$. If $\left|\frac{g_i(z)}{b}\right| \le 1$ and*

$$0 \le 1 + \sum_{i=1}^{n} \left[(\alpha_i - 1)\,(3 - \mu_i - \eta_i) + (\beta_i + \gamma_i)\,(2 - \lambda_i - \sigma_i)\right] < 1,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is convex by the order $\mathcal{C}_\rho(b)$, with

$$\rho = 1 + \sum_{i=1}^{n} \left[(\alpha_i - 1)\,(3 - \mu_i - \eta_i) + (\beta_i + \gamma_i)\,(2 - \lambda_i - \sigma_i)\right],$$

for all $i = \overline{1,n}$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.4.3, we obtain the next corollary:

Corollary 3.4.3.1. *Let $f \in \mathcal{C}_\mu(b)$, $g \in \mathcal{S}^*_\eta(b)$, $h \in \mathcal{C}_\lambda(b)$, $h \in \mathcal{S}^*_\lambda(b)$, $k \in \mathcal{C}_\sigma(b)$ and $k \in \mathcal{S}^*_\sigma(b)$, $0 \le \mu, \eta, \lambda, \sigma < 1$, where $b \in \mathbb{C} - \{0\}$. Also, let α a real positive number. If $\left|\frac{g(z)}{b}\right| \le 1$ and*

$$0 \le 1 + \alpha\left(5 - \mu - \eta - 2\lambda - 2\sigma\right) < 1$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is convex by the order $\mathcal{C}_\rho(b)$, with

$$\rho = 1 + \alpha\left(5 - \mu - \eta - 2\lambda - 2\sigma\right).$$

Theorem 3.4.4. *Let $f_i \in \mathcal{S}^*_{\eta_i}(b)$, $g_i \in \mathcal{C}_{\lambda_i}(b)$, $h_i \in \mathcal{S}^*_{\rho_i}(b)$, $h_i \in \mathcal{C}_{\rho_i}(b)$, $k_i \in \mathcal{S}^*_{\nu_i}(b)$, $k_i \in \mathcal{C}_{\nu_i}(b)$, $0 \le \eta_i, \lambda_i, \rho_i, \nu_i < 1$ and $b \in \mathbb{C} - \{0\}$. Also, let $\alpha_i, \beta_i, \gamma_i, \delta_i$ are real numbers with $\alpha_i > 1$ for all $i = \overline{1, n}$. If*

$$0 \le 1 + \sum_{i=1}^{n}\left[\left(\alpha_i - 1\right)\left(1 - \eta_i\right) + \beta_i\left(1 - \lambda_i\right) + \left(\gamma_i + \delta_i\right)\left(2 - \rho_i - \nu_i\right)\right] < 1,$$

then the integral operator $\mathcal{T}_n$, defined by (3.0.4) is convex by the order $\mathcal{C}_\mu(b)$, with

$$\mu = 1 + \sum_{i=1}^{n}\left[\left(\alpha_i - 1\right)\left(1 - \eta_i\right) + \beta_i\left(1 - \lambda_i\right) + \left(\gamma_i + \delta_i\right)\left(2 - \rho_i - \nu_i\right)\right],$$

for all $i = \overline{1, n}$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.4.4, we obtain the next corollary:

Corollary 3.4.4.1. *Let $f \in \mathcal{S}^*_\eta(b)$, $g \in \mathcal{C}_\lambda(b)$, $h \in \mathcal{S}^*_\rho(b)$, $h \in \mathcal{C}_\rho(b)$, $k \in \mathcal{S}^*_\nu(b)$, $k \in \mathcal{C}_\nu(b)$, $0 \le \eta, \lambda, \rho, \nu < 1$ and $b \in \mathbb{C} - \{0\}$. Also, let α a real positive number. If*

$$0 \le 1 + \alpha\left(6 - \eta - \lambda - 2\rho - 2\nu\right) < 1$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is convex by the order $\mathcal{C}_\mu(b)$, with

$$\mu = 1 + \alpha\left(6 - \eta - \lambda - 2\rho - 2\nu\right).$$

3.5 Convexity conditions for the class $\mathcal{SH}(\beta)$

This section contains sufficient convexity conditions for new integral operators with class functions $\mathcal{SH}(\beta)$, $\beta > 0$.

Theorem 3.5.1. *Let $f_i \in \mathcal{SH}(\delta_i)$, $g_i \in \mathcal{K}(\lambda_i)$, $g_i \in \mathcal{SH}(\eta_i)$, $0 < \lambda_i < 1$ and $\delta_i, \eta_i > 0$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i \ge 1$, $\beta_i, \gamma_i \ge 0$, for all $i = \overline{1, n}$. If*

$$0 < \sum_{i=1}^{n}\left[-\left(\alpha_i - 1\right)\left(2\delta_i + 1\right) + \beta_i\left(\lambda_i - 1\right) - \gamma_i\left(2\eta_i + 1\right)\right] + \sum_{i=1}^{n}\sqrt{2}\left(\delta_i\alpha_i + \gamma_i\eta_i - \delta_i\right) + 1 < 1,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is convex by the order

$$\sum_{i=1}^{n}\left[-\left(\alpha_i - 1\right)\left(2\delta_i + 1\right) + \beta_i\left(\lambda_i - 1\right) - \gamma_i\left(2\eta_i + 1\right)\right] + \sum_{i=1}^{n}\sqrt{2}\left(\delta_i\alpha_i + \gamma_i\eta_i - \delta_i\right) + 1.$$

If we have $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.5.1, we obtain the next corollary:

Corollary 3.5.1.1. *Let $f \in \mathcal{SH}(\delta)$, $g \in \mathcal{K}(\lambda)$, $g \in \mathcal{SH}(\eta)$, $0 < \lambda < 1$ and $\delta, \eta > 0$. Also, let α a real positive number. If*

$$0 \leq \alpha \left[\lambda - 2\delta - 2\eta - 3 + \sqrt{2}\,(\delta + \eta)\right] + 1 < 1$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is convex by the order

$$\alpha \left[\lambda - 2\delta - 2\eta - 3 + \sqrt{2}\,(\delta + \eta)\right] + 1.$$

Theorem 3.5.2. *Let $f_i \in \mathcal{SH}(\delta_i, \nu_i)$, $g_i \in \mathcal{K}(\lambda_i)$ and $g_i \in \mathcal{SH}(\eta_i, \theta_i)$, $0 < \lambda_i \leq 1$, $0 < \nu_i, \theta_i < 1$ and $\delta_i, \eta_i > 0$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i \geq 1$, $\beta_i, \gamma_i \geq 0$, for all $i = \overline{1, n}$. If*

$$0 < \sum_{i=1}^{n} \left[-\left(\alpha_i - 1\right)\left(\delta_i - \nu_i + 1\right) + \beta_i\left(\lambda_i - 1\right) - \gamma_i\left(\eta_i - \theta_i + 1\right)\right] + 1 < 1,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is convex by the order

$$\sum_{i=1}^{n} \left[-\left(\alpha_i - 1\right)\left(\delta_i - \nu_i + 1\right) + \beta_i\left(\lambda_i - 1\right) - \gamma_i\left(\eta_i - \theta_i + 1\right)\right] + 1.$$

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.5.2, we obtain the next corollary:

Corollary 3.5.2.1. *Let $f \in \mathcal{SH}(\delta, \nu)$, $g \in \mathcal{K}(\lambda)$, $g \in \mathcal{SH}(\eta, \theta)$, $0 < \lambda \leq 1$, $0 < \nu, \theta < 1$ and $\delta, \eta > 0$. Also, let α a real positive number. If*

$$0 < \alpha\left(\nu + \lambda + \theta - \delta - \eta - 3\right) + 1 < 1,$$

then the integral operator $\mathcal{M}$, defined by (2.2.5) is convex by the order

$$\alpha\left(\nu + \lambda + \theta - \delta - \eta - 3\right) + 1.$$

Theorem 3.5.3. *Let $f_i \in \mathcal{SH}(\delta_i)$, $g_i \in \mathcal{SH}(\nu_i)$, $h_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{SH}(\eta_i)$, $0 < \lambda_i < 1$ and $\delta_i, \nu_i, \eta_i > 0$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i \geq 1$, $\beta_i, \gamma_i \geq 0$, for all $i = \overline{1, n}$. If Re $\left(g_i(z)\right) \geq 1$ and*

$$0 < \sum_{i=1}^{n} \left[-\left(\alpha_i - 1\right)\left(2\delta_i + 2\nu_i + 1\right) + \beta_i\left(\lambda_i - 1\right) - \gamma_i\left(2\eta_i + 1\right)\right] +$$

$$+ \sum_{i=1}^{n} \sqrt{2}\left(\left(\alpha_i - 1\right)\left(\delta_i + \nu_i\right) + \gamma_i \eta_i\right) + 1 < 1,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is convex by the order

$$\sum_{i=1}^{n} \left[-\left(\alpha_i - 1\right)\left(2\delta_i + 2\nu_i + 1\right) + \beta_i\left(\lambda_i - 1\right) - \gamma_i\left(2\eta_i + 1\right)\right] + \sum_{i=1}^{n} \sqrt{2}\left(\left(\alpha_i - 1\right)\left(\delta_i + \nu_i\right) + \gamma_i \eta_i\right) + 1.$$

If we have $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.5.3, we obtain the next corollary:

Corollary 3.5.3.1. *Let $f \in \mathcal{SH}(\delta)$, $g \in \mathcal{SH}(\nu)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{SH}(\eta)$, $0 < \lambda < 1$ and $\delta, \nu, \eta > 0$. Also, let α a real positive number. If Re $(g(z)) \geq 1$ and*

$$0 \leq \alpha \left[\lambda - 2\delta - 2\nu - 2\eta - 3 + \sqrt{2}\,(\delta + \nu + \eta)\right] + 1 < 1,$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is convex by the order

$$\alpha \left[\lambda - 2\delta - 2\nu - 2\eta - 3 + \sqrt{2}\,(\delta + \nu + \eta)\right] + 1.$$

Theorem 3.5.4. *Let $f_i \in \mathcal{SH}(\delta_i, \sigma_i)$, $g_i \in \mathcal{SH}(\nu_i, \mu_i)$, $h_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{SH}(\eta_i, \theta_i)$, $0 < \lambda_i \leq 1$, $0 < \sigma_i, \mu_i, \theta_i < 1$ and $\delta_i, \nu_i, \eta_i > 0$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i \geq 1$, $\beta_i, \gamma_i \geq 0$, for all $i = \overline{1, n}$. If Re $(g_i(z)) \geq 1$ and*

$$0 < \sum_{i=1}^{n} \left[-\left(\alpha_i - 1\right)\left(\delta_i + \nu_i - \sigma_i - \mu_i + 1\right) + \beta_i\left(\lambda_i - 1\right) - \gamma_i\left(\eta_i - \theta_i + 1\right)\right] + 1 < 1,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is convex by the order

$$\sum_{i=1}^{n} \left[-\left(\alpha_i - 1\right)\left(\delta_i + \nu_i - \sigma_i - \mu_i + 1\right) + \beta_i\left(\lambda_i - 1\right) - \gamma_i\left(\eta_i - \theta_i + 1\right)\right] + 1.$$

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.5.4, we obtain the next corollary:

Corollary 3.5.4.1. *Let $f \in \mathcal{SH}(\delta, \sigma)$, $g \in \mathcal{SH}(\nu, \mu)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{SH}(\eta, \theta)$, $0 < \lambda \leq 1$, $0 < \sigma, \mu, \theta < 1$ and $\delta, \nu, \eta > 0$. Also, let α a real positive number. If Re $(g(z)) \geq 1$ and*

$$0 < \alpha\left(\sigma + \mu + \lambda + \theta - \delta - \nu - \eta - 3\right) + 1 < 1$$

then the integral operator $\mathcal{C}$, defined by (2.2.10) is convex by the order

$$\alpha\left(\sigma + \mu + \lambda + \theta - \delta - \nu - \eta - 3\right) + 1.$$

Theorem 3.5.5. *Let $f_i \in \mathcal{K}(\mu_i)$, $g_i \in \mathcal{SH}(\nu_i)$, $h_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{SH}(\eta_i)$, $k_i \in \mathcal{K}(\sigma_i)$, $k_i \in \mathcal{SH}(\theta_i)$, $0 < \mu_i, \lambda_i, \sigma_i < 1$ and $\nu_i, \eta_i, \theta_i > 0$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i \geq 1$, $\beta_i, \gamma_i \geq 0$, for all $i = \overline{1, n}$. If Re $(g_i(z)) \geq 1$ and*

$$0 < \sum_{i=1}^{n} \left[\left(\alpha_i - 1\right)\left(\mu_i - 2\nu_i - 1\right) + \beta_i\left(-2\eta_i + 2\theta_i\right) + \gamma_i\left(\lambda_i - \sigma_i\right)\right] +$$

$$+ \sum_{i=1}^{n} \sqrt{2}\left[\left(\alpha_i - 1\right)\nu_i + \beta_i\left(\eta_i - \theta_i\right)\right] + 1 < 1,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is convex by the order

$$\sum_{i=1}^{n} \left[\left(\alpha_i - 1\right)\left(\mu_i - 2\nu_i - 1\right) + \beta_i\left(-2\eta_i + 2\theta_i\right) + \gamma_i\left(\lambda_i - \sigma_i\right)\right] + \sum_{i=1}^{n} \sqrt{2}\left[\left(\alpha_i - 1\right)\nu_i + \beta_i\left(\eta_i - \theta_i\right)\right] + 1.$$

If we have $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.5.5, we obtain the next corollary:

Corollary 3.5.5.1. *Let $f \in \mathcal{K}(\mu)$, $g \in \mathcal{SH}(\nu)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{SH}(\eta)$, $k \in \mathcal{K}(\sigma)$, $k \in \mathcal{SH}(\theta)$, $0 < \mu, \lambda, \sigma < 1$ and $\nu, \eta, \theta > 0$. Also, let α a real positive number. If Re $(g(z)) \geq 1$ and*

$$0 \leq \alpha \left[2\theta + \lambda - 2\nu - 2\eta - \sigma - 1 + \sqrt{2}\,(\nu + \eta - \theta)\right] + 1 < 1$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is convex by the order

$$\alpha \left[2\theta + \lambda - 2\nu - 2\eta - \sigma - 1 + \sqrt{2}\,(\nu + \eta - \theta)\right] + 1.$$

Theorem 3.5.6. *Let $f_i \in \mathcal{K}(\mu_i)$, $g_i \in \mathcal{SH}(\nu_i, \delta_i)$, $h_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{SH}(\nu_i, \omega_i)$, $k_i \in \mathcal{K}(\sigma_i)$, $k_i \in \mathcal{SH}(\theta_i, \xi_i)$, $0 < \mu_i, \lambda_i, \sigma_i \leq 1$, $0 < \delta_i, \omega_i, \xi_i < 1$ and $\nu_i, \eta_i, \theta_i > 0$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i \geq 1$, $\beta_i, \gamma_i \geq 0$, for all $i = \overline{1, n}$. If Re $(g_i(z)) \geq 1$ and*

$$0 < \sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i + \delta_i - \nu_i - 1) + \beta_i(\omega_i + \theta_i - \eta_i - \xi_i) + \gamma_i(\lambda_i - \sigma_i)\right] + 1 < 1,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is convex by the order

$$\sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i + \delta_i - \nu_i - 1) + \beta_i(\omega_i + \theta_i - \eta_i - \xi_i) + \gamma_i(\lambda_i - \sigma_i)\right] + 1.$$

If we have $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 3.5.6, we obtain the next corollary:

Corollary 3.5.6.1. *Let $f \in \mathcal{K}(\mu)$, $g \in \mathcal{SH}(\nu, \delta)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{SH}(\nu, \omega)$, $k \in \mathcal{K}(\sigma)$, $k \in \mathcal{SH}(\theta, \xi)$, $0 < \mu, \lambda, \sigma \leq 1$, $0 < \delta, \omega, \xi < 1$ and $\nu, \eta, \theta > 0$. Also, let α a real positive number. If Re $(g(z)) \geq 1$ and*

$$0 < \alpha(\delta + \mu + \omega + \theta + \lambda - \nu - \eta - \sigma - 1) + 1 < 1,$$

then the integral operator $\mathcal{G}$, defined by (2.2.16) is convex by the order

$$\alpha(\delta + \mu + \omega + \theta + \lambda - \nu - \eta - \sigma - 1) + 1.$$

Theorem 3.5.7. *Let $f_i \in \mathcal{SH}(\mu_i)$, $g_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{SH}(\nu_i)$, $k_i \in \mathcal{SH}(\theta_i)$, $h_i \in \mathcal{K}(\eta_i)$, $k_i \in \mathcal{K}(\rho_i)$ and $0 < \lambda_i, \eta_i, \rho_i < 1$, $\mu_i, \nu_i, \theta_i > 0$. Also, let $\alpha_i, \beta_i, \gamma_i, \delta_i$ are real numbers with $\alpha_i \geq 1$, $\beta_i, \gamma_i, \delta_i \geq 0$, for all $i = \overline{1, n}$. If*

$$0 < \sum_{i=1}^{n} \left[-(\alpha_i - 1)(2\mu_i + 1) + \beta_i(\lambda_i - 1) - 2\gamma_i(\nu_i - \theta_i) + \delta_i(\eta_i - \rho_i)\right] +$$

$$+ \sum_{i=1}^{n} \sqrt{2}\left((\alpha_i - 1)\mu_i + \gamma_i(\nu_i - \theta_i)\right) + 1 < 1,$$

then the integral operator $\mathcal{T}_n$, defined by (3.0.4) is convex by the order

$$\sum_{i=1}^{n} \left[-(\alpha_i - 1)(2\mu_i + 1) + \beta_i(\lambda_i - 1) - 2\gamma_i(\nu_i - \theta_i) + \delta_i(\eta_i - \rho_i)\right] +$$

$$+ \sum_{i=1}^{n} \sqrt{2}\left[(\alpha_i - 1)\mu_i + \gamma_i(\nu_i - \theta_i)\right] + 1.$$

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.5.7, we obtain the next corollary:

Corollary 3.5.7.1. *Let $f \in \mathcal{SH}(\mu)$, $g \in \mathcal{K}(\lambda)$, $h \in \mathcal{SH}(\nu)$, $k \in \mathcal{SH}(\theta)$, $h \in \mathcal{K}(\eta)$, $k \in \mathcal{K}(\rho)$. Also, let α a real positive number. If*

$$0 < \alpha\left[\lambda + 2\theta + \eta - 2\mu - 2\nu - \rho - 2 + \sqrt{2}\,(\mu + \nu - \theta)\right] + 1 < 1,$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is convex by the order

$$\alpha\left[\lambda + 2\theta + \eta - 2\mu - 2\nu - \rho - 2 + \sqrt{2}\,(\mu + \nu - \theta)\right] + 1.$$

Theorem 3.5.8. *Let $f_i \in \mathcal{SH}(\mu_i, \varepsilon_i)$, $g_i \in \mathcal{K}(\lambda_i)$, $h_i \in \mathcal{SH}(\nu_i, \xi_i)$, $k_i \in \mathcal{SH}(\theta_i, \sigma_i)$, $h_i \in \mathcal{K}(\eta_i)$, $k_i \in \mathcal{K}(\rho_i)$, $0 \le \lambda_i, \varepsilon_i, \xi_i, \sigma_i < 1$ and $\mu_i, \nu_i, \theta_i > 0$. Also, let $\alpha_i, \beta_i, \gamma_i$ are real numbers with $\alpha_i \ge 1$, $\beta_i, \gamma_i, \delta_i \ge 0$, for all $i = \overline{1, n}$. If*

$$0 < \sum_{i=1}^{n} \left[-(\alpha_i - 1)(\mu_i - \varepsilon_i + 1) + \beta_i(\lambda_i - 1) + \gamma_i(\xi_i + \theta_i - \sigma_i - \nu_i) + \delta_i(\eta_i - \rho_i)\right] + 1 < 1,$$

then the integral operator $\mathcal{T}_n$, defined by (3.0.4) is convex by the order

$$\sum_{i=1}^{n} \left[-(\alpha_i - 1)(\mu_i - \varepsilon_i + 1) + \beta_i(\lambda_i - 1) + \gamma_i(\xi_i + \theta_i - \sigma_i - \nu_i) + \delta_i(\eta_i - \rho_i)\right].$$

If we have $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 3.5.8, we obtain the next corollary:

Corollary 3.5.8.1. *Let $f \in \mathcal{SH}(\mu, \varepsilon)$, $g \in \mathcal{K}(\lambda)$, $h \in \mathcal{SH}(\nu, \xi)$, $k \in \mathcal{SH}(\theta, \sigma)$, $h \in \mathcal{K}(\eta)$, $k \in \mathcal{K}(\rho)$, $0 \le \lambda, \varepsilon, \xi, \sigma < 1$ and $\mu, \nu, \theta > 0$. Also, let α a real positive number. If*

$$0 < \alpha(\varepsilon + \xi + \lambda + \theta + \eta - \mu - \sigma - \nu - \rho - 2) + 1 < 1$$

then the integral operator $\mathcal{T}$, defined by (2.2.23) is convex by the order

$$\alpha(\varepsilon + \xi + \lambda + \theta + \eta - \mu - \sigma - \nu - \rho - 2) + 1.$$

3.6 Convexity conditions for $alpha$ -convex functions

This paragraph describes sufficient convexity conditions for the studied integral operators, when the functions involved belong to the $alpha$ -convex function class $\mathcal{M}_\alpha$, $\alpha \in \mathbb{R}$.

Theorem 3.6.1. *Let $f_i \in \mathcal{S}^*(\alpha_i - 1)$ and $g_i \in \mathcal{M}_{\gamma_i}$, with $1 \le \alpha_i < 2$ and $\beta_i = 1 - \gamma_i$, for all $i = \overline{1, n}$. If*

$$0 < \sum_{i=1}^{n} \left[(\alpha_i - 1)^2 - \alpha_i\right] + 1 < 1,$$

then the integral operator $\mathcal{M}_n$, defined by (3.0.1) is convex.

Theorem 3.6.2. *Let $f_i, g_i \in \mathcal{S}^*(\alpha_i - 1)$ and $h_i \in \mathcal{M}_{\gamma_i}$, with $1 \leq \alpha_i < 2$ and $\beta_i = 1 - \gamma_i$, for all $i = \overline{1, n}$. If Re $(g_i(z)) \geq 1$ and*

$$0 < \sum_{i=1}^{n} \left[2(\alpha_i - 1)^2 - \alpha_i\right] + 1 < 1,$$

then the integral operator $\mathcal{C}_n$, defined by (3.0.2) is convexx.

Theorem 3.6.3. *Let $zf_i', g_i \in \mathcal{S}^*(\alpha_i - 1)$ and $h_i, k_i \in \mathcal{M}_{\beta_i}$, with $1 \leq \alpha_i < 2$ and $\beta_i = 1 - \gamma_i$, for all $i = \overline{1, n}$. If Re $(g_i(z)) \geq 1$ and*

$$0 < \sum_{i=1}^{n} \left[2(\alpha_i - 1)^2 - (\alpha_i - 1)\right] + 1 < 1,$$

then the integral operator $\mathcal{G}_n$, defined by (3.0.3) is convex.

Theorem 3.6.4. *Let $f_i \in \mathcal{S}^*(\alpha_i - 1)$, $zg_i' \in \mathcal{S}^*(\beta_i)$ and $h_i, k_i \in \mathcal{M}_{\gamma_i}$, with $1 \leq \alpha_i < 2$, $\beta_i, \gamma_i, \delta_i \geq 0$ and $\gamma_i = 1 - \delta_i$, for all $i = \overline{1, n}$. If*

$$0 < \sum_{i=1}^{n} \left[(\alpha_i - 1)^2 + \beta_i^2 - \alpha_i - \beta_i + 1\right] + 1 < 1,$$

then the integral operator $\mathcal{T}_n$, defined by (3.0.4) is convex.

Chapter 4

Conditions to belong to the class $\mathcal{N}(\beta)$

4.1 Conditions for analytic functions

This paragraph describes sufficient conditions to belong to the class $\mathcal{N}(\beta)$, $\beta > 1$, for new integral operators with analytic functions.

Theorem 4.1.1. *Let $f_i, g_i \in \mathcal{A}$, $g_i \in \mathcal{N}(\lambda_i)$. For any $\lambda_i > 1$ and f_i, g_i verifying conditions*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq 1, \quad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\mu = \sum_{i=1}^{n} \left[(\alpha_i - 1) + \beta_i (\lambda_i - 1) + \gamma_i\right] + 1.$$

In these conditions the integral operator $\mathcal{M}_n$, defined by (3.0.1) is in the class $\mathcal{N}(\mu)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 4.1.1, we obtain the next corollary:

Corollary 4.1.1.1. *Let $f, g \in \mathcal{A}$ and $g \in \mathcal{N}(\lambda)$. For any $\lambda > 1$ and f, g verifying conditions*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1, \quad \left|\frac{zg'(z)}{g(z)} - 1\right| \leq 1,$$

for all $z \in \mathbb{U}$, there is a number real positive α so that

$$\mu = \alpha (\lambda + 1) + 1.$$

In these conditions the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{N}(\mu)$.

Theorem 4.1.2. *Let $f_i, g_i, h_i \in \mathcal{A}$, $h_i \in \mathcal{N}(\lambda_i)$. For any $\lambda_i > 1$ and f_i, g_i, h_i verifying conditions*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq 1, \quad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \leq 1, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq 1, \quad |g_i(z)| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i$ and $\alpha_i > 1$ so that

$$\rho = \sum_{i=1}^{n} \left[3\left(\alpha_i - 1\right) + \beta_i \left(\lambda_i - 1\right) + \gamma_i\right] + 1.$$

In these conditions the integral operator $\mathcal{C}_n$, defined by (3.0.2) is in the class $\mathcal{N}(\rho)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 4.1.2, we obtain the next corollary:

Corollary 4.1.2.1. *Let $f, g, h \in \mathcal{A}$ and $h \in \mathcal{N}(\lambda)$. For any $\lambda > 1$ and f, g, h verifying conditions*

$$\left|\frac{zf^{'}(z)}{f(z)} - 1\right| \leq 1, \quad \left|\frac{zg^{'}(z)}{g(z)} - 1\right| \leq 1, \quad \left|\frac{zh^{'}(z)}{h(z)} - 1\right| \leq 1, \quad |g(z)| \leq 1,$$

for all $z \in \mathbb{U}$, there is a number real positive α so that

$$\rho = \alpha\left(\lambda + 3\right) + 1.$$

In these conditions the integral operator $\mathcal{C}$, defined by (2.2.10) is in the class $\mathcal{N}(\rho)$.

Theorem 4.1.3. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i \in \mathcal{N}(\mu_i)$, $h_i \in \mathcal{N}(\lambda_i)$ and $k_i \in \mathcal{N}(\sigma_i)$. For any $\mu_i, \lambda_i, \sigma_i > 1$ and g_i, h_i, k_i verifying conditions*

$$\left|\frac{zg_i^{'}(z)}{g_i(z)} - 1\right| \leq 1, \quad \left|\frac{zh_i^{'}(z)}{h_i(z)} - 1\right| \leq 1, \quad \left|\frac{zk_i^{'}(z)}{k_i(z)} - 1\right| \leq 1, \quad |g_i(z)| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = \sum_{i=1}^{n} \left[\left(\alpha_i - 1\right)\left(\mu_i + 1\right) + 2\beta_i + \gamma_i\left(\lambda_i - \sigma_i\right)\right] + 1.$$

In these conditions the integral operator $\mathcal{G}_n$, defined by (3.0.3) is in the class $\mathcal{N}(\rho)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 4.1.3, we obtain the next corollary:

Corollary 4.1.3.1. *Let $f, g, h, k \in \mathcal{A}$ and $f \in \mathcal{N}(\mu)$, $h \in \mathcal{N}(\lambda)$, $k \in \mathcal{N}(\sigma)$. For any $\mu, \lambda, \sigma > 1$ and g, h, k verifying conditions*

$$\left|\frac{zg^{'}(z)}{g(z)} - 1\right| \leq 1, \quad \left|\frac{zh^{'}(z)}{h(z)} - 1\right| \leq 1, \quad \left|\frac{zk^{'}(z)}{k(z)} - 1\right| \leq 1, \quad |g(z)| \leq 1,$$

for all $z \in \mathbb{U}$, there is a number real positive α so that

$$\rho = \alpha\left(\mu + \lambda - \sigma + 3\right) + 1.$$

In these conditions the integral operator $\mathcal{G}$, defined by (2.2.16) is in the class $\mathcal{N}(\rho)$.

Theorem 4.1.4. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}$, $g_i \in \mathcal{N}(\lambda_i)$, $h_i \in \mathcal{N}(\rho_i)$, $k_i \in \mathcal{N}(\nu_i)$, $i = \overline{1,n}$. For any $\lambda_i, \rho_i, \nu_i > 1$ and f_i, h_i, k_i verifying conditions*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)} - 1\right| \leq 1, \quad \left|\frac{zh_i^{'}(z)}{h_i(z)} - 1\right| \leq 1, \quad \left|\frac{zk_i^{'}(z)}{k_i(z)} - 1\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i, \delta_i$ real positive numbers and $\alpha_i > 1$ so that

$$\mu = \sum_{i=1}^{n} \left[(\alpha_i - 1) + \beta_i(\lambda_i - 1) + 2\gamma_i + \delta_i(\rho_i - \nu_i)\right] + 1.$$

In these conditions the integral operator $\mathcal{T}_n$, defined by (3.0.4) is in the class $\mathcal{N}(\mu)$.

Letting $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 4.1.4, we obtain the next corollary:

Corollary 4.1.4.1. *Let $f, g, h, k \in \mathcal{A}$ and $g \in \mathcal{N}(\lambda)$, $h \in \mathcal{N}(\rho)$, $k \in \mathcal{N}(\nu)$. For any $\lambda, \rho, \nu > 1$ and f, h, k verifying conditions*

$$\left|\frac{zf^{'}(z)}{f(z)} - 1\right| \leq 1, \quad \left|\frac{zh^{'}(z)}{h(z)} - 1\right| \leq 1, \quad \left|\frac{zk^{'}(z)}{k(z)} - 1\right| \leq 1,$$

for all $z \in \mathbb{U}$, there is a number real positive α so that

$$\mu = \alpha(\lambda + \rho - \nu + 2) + 1.$$

In these conditions the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{N}(\mu)$.

4.2 Conditions for belonging to the class functions $\mathcal{SP}(\alpha, \beta)$

This section contains sufficient conditions for belonging to the class $\mathcal{N}(\beta)$, $\beta > 1$,for new integral operators with functions in the function class $\mathcal{SP}(\alpha, \beta)$, $\alpha > 0$, $0 \leq \beta < 1$.

Theorem 4.2.1. *Let $f_i \in \mathcal{SP}(\alpha, \beta)$, $g_i \in \mathcal{SP}(\delta, \eta)$ and $g_i \in \mathcal{N}(\lambda_i)$, $\alpha, \delta, \lambda_i > 0$, $0 \leq \beta, \eta < 1$. For any real numbers $M_i, N_i \geq 1$, which verify*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)}\right| \leq M_i, \quad \left|\frac{zg_i^{'}(z)}{g_i(z)}\right| \leq N_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i >$ so that

$$\rho = 1 + \sum_{i=1}^{n} \left[(\alpha_i - 1)(M_i + 2\alpha - 1) + \beta_i(\lambda_i - 1) + \gamma_i(N_i + 2\delta - 1)\right] > 1.$$

In these conditions the integral operator $\mathcal{M}_n$, defined by (3.0.1) is in the class $\mathcal{N}(\rho)$.

If we consider $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 4.2.1, we obtain the next corollary:

Corollary 4.2.1.1. *Let $f \in \mathcal{SP}(\alpha, \beta)$, $g \in \mathcal{SP}(\delta, \eta)$ and $g \in \mathcal{N}(\lambda)$, $\alpha, \delta, \lambda > 0$, $0 \leq \beta, \eta < 1$. For any real numbers $M, N \geq 1$, which verify*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq M, \quad \left|\frac{zg'(z)}{g(z)}\right| \leq N,$$

for all $z \in \mathbb{U}$, there is a number real positive α so that

$$\rho = 1 + \alpha\left(M + N + 2\alpha + 2\delta + \lambda - 3\right) > 1.$$

In these conditions the integral operator $\mathcal{M}$, defined by (2.2.5) is in the class $\mathcal{N}(\rho)$.

Theorem 4.2.2. *Let $f_i \in \mathcal{SP}(\alpha, \beta)$, $g_i \in \mathcal{SP}(\gamma, \delta)$, $h_i \in \mathcal{N}(\lambda_i)$ and $h_i \in \mathcal{SP}(\nu, \eta)$, $\alpha, \gamma, \nu, \lambda_i > 0$, $0 \leq \beta, \delta, \eta < 1$. For any real numbers $M_i, N_i, P_i \geq 1$, which verify*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq M_i, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \leq P_i, \quad |g_i(z)| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, there are $\alpha_i, \beta_i, \gamma_i$ there is a number real positive $\alpha_i > 1$ so that

$$\rho = 1 + \sum_{i=1}^{n}\left[(\alpha_i - 1)\left(M_i + N_i + 2\alpha + 2\gamma - 1\right) + \beta_i\left(\lambda_i - 1\right) + \gamma_i\left(P_i + 2\nu - 1\right)\right] > 1.$$

In these conditions the integral operator $\mathcal{C}_n$, defined by (3.0.2) is in the class $\mathcal{N}(\rho)$.

If we consider $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$in Theorem 4.2.2, we obtain the next corollary:

Corollary 4.2.2.1. *Let $f \in \mathcal{SP}(\alpha, \beta)$, $g \in \mathcal{SP}(\gamma, \delta)$, $h \in \mathcal{N}(\lambda)$ and $h \in \mathcal{SP}(\nu, \eta)$, $\alpha, \gamma, \nu, \lambda > 0$, $0 \leq \beta, \delta, \eta < 1$. For any real numbers $M, N, P \geq 1$, which verify*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq M, \quad \left|\frac{zg'(z)}{g(z)}\right| \leq N, \quad \left|\frac{zh'(z)}{h(z)}\right| \leq P, \quad |g(z)| \leq 1,$$

for all $z \in \mathbb{U}$, there is a number real positive α so that

$$\rho = 1 + \alpha\left(M + N + P + 2\alpha + 2\gamma + 2\nu + \lambda - 3\right) > 1.$$

In these conditions the integral operator $\mathcal{C}$, defined by (2.2.10) is in the class $\mathcal{N}(\rho)$.

Theorem 4.2.3. *Let $f_i \in \mathcal{N}(\mu_i)$, $g_i \in \mathcal{SP}(\gamma, \delta)$, $h_i \in \mathcal{N}(\lambda_i)$, $h_i \in \mathcal{SP}(\nu, \eta)$, $k_i \in \mathcal{SP}(\theta, \xi)$ and $k_i \in \mathcal{N}(\sigma_i)$, $\mu_i, \gamma, \nu, \theta, \lambda_i, \sigma_i > 0$, $0 \leq \delta, \eta, \xi < 1$. For any real numbers $M_i, N_i, P_i \geq 1$, which verify*

$$\left|\frac{zg_i'(z)}{g_i(z)}\right| \leq M_i, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \leq N_i, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| \leq P_i, \quad |g_i(z)| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, there are $\alpha_i, \beta_i, \gamma_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 + \sum_{i=1}^{n}\left[(\alpha_i - 1)\left(\mu_i + M_i + 2\gamma - 1\right) + \beta_i\left(N_i + P_i + 2\nu + 4\theta - 2\xi - 1\right) + \gamma_i\left(\lambda_i - \sigma_i\right)\right] > 1.$$

In these conditions the integral operator $\mathcal{G}_n$, defined by (3.0.3) is in the class $\mathcal{N}(\rho)$.

If we consider $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 4.2.3, we obtain the next corollary:

Corollary 4.2.3.1. *Let $f \in \mathcal{N}(\mu)$, $g \in \mathcal{SP}(\gamma,\delta)$, $h \in \mathcal{SP}(\mu,\eta)$, $h \in \mathcal{N}(\lambda)$, $k \in \mathcal{SP}(\theta,\xi)$ and $k \in \mathcal{N}(\sigma)$, $\mu,\gamma,\nu,\theta,\lambda,\sigma > 0$, $0 \le \delta,\eta,\xi < 1$. For any real numbers $M, N, P \ge 1$, which verify*

$$\left|\frac{zg^{'}(z)}{g(z)}\right| \le M, \quad \left|\frac{zh^{'}(z)}{h(z)}\right| \le N, \quad \left|\frac{zk^{'}(z)}{k(z)}\right| \le P, \quad |g(z)| \le 1,$$

for all $z \in \mathbb{U}$, there is a number real positive α so that

$$\rho = 1 + \alpha\left(\mu + M + N + P + 2\gamma + 2\nu + 4\theta - 2\xi + \lambda - \sigma - 1\right) > 1.$$

In these conditions the integral operator $\mathcal{G}$, defined by (2.2.16) is in the class $\mathcal{N}(\rho)$.

Theorem 4.2.4. *Let $f_i \in \mathcal{SP}(\alpha,\beta)$, $g_i \in \mathcal{N}(\lambda_i)$, $h_i \in \mathcal{SP}(\sigma,\eta)$, $k_i \in \mathcal{SP}(\theta,\xi)$, $h_i \in \mathcal{N}(\mu_i)$ and $k_i \in \mathcal{N}(\nu_i)$ $\alpha,\sigma,\theta,\lambda_i,\mu_i,\nu_i > 0$, $0 \le \beta,\eta,\xi < 1$. For any real numbers $M_i, N_i, P_i \ge 1$, which verify*

$$\left|\frac{zf_i^{'}(z)}{f_i(z)}\right| \le M_i, \quad \left|\frac{zh_i^{'}(z)}{h_i(z)}\right| \le N_i, \quad \left|\frac{zk_i^{'}(z)}{k_i(z)}\right| \le P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, there are $\alpha_i,\beta_i,\gamma_i,\delta_i$ real positive numbers and $\alpha_i > 1$ so that

$$\rho = 1 + \sum_{i=1}^{n}\left[(\alpha_i - 1)(M_i + 2\alpha - 1) + \beta_i(\lambda_i - 1) + \gamma_i(N_i + P_i + 2\sigma + 4\theta - 2\xi) + \delta_i(\mu_i - \nu_i)\right] > 1.$$

In these conditions the integral operator $\mathcal{T}_n$, defined by (3.0.4) is in the class $\mathcal{N}(\rho)$.

If we consider $n = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 4.2.4, we obtain the next corollary:

Corollary 4.2.4.1. *Let $f \in \mathcal{SP}(\alpha,\beta)$, $g \in \mathcal{N}(\lambda)$, $h \in \mathcal{SP}(\sigma,\eta)$, $k \in \mathcal{SP}(\theta,\xi)$, $h \in \mathcal{N}(\mu)$ and $k \in \mathcal{N}(\nu)$ $\alpha,\sigma,\theta,\lambda,\mu,\nu > 0$, $0 \le \beta,\eta,\xi < 1$. For any real numbers $M, N, P \ge 1$, which verify*

$$\left|\frac{zf^{'}(z)}{f(z)}\right| \le M, \quad \left|\frac{zh^{'}(z)}{h(z)}\right| \le N, \quad \left|\frac{zk^{'}(z)}{k(z)}\right| \le P,$$

for all $z \in \mathbb{U}$, there is a number real positive α so that

$$\rho = 1 + \alpha\left(M + N + P + 2\alpha + 2\sigma + 4\theta + \lambda + \mu - \nu - 2\xi - 2\right) > 1.$$

In these conditions the integral operator $\mathcal{T}$, defined by (2.2.23) is in the class $\mathcal{N}(\rho)$.

Chapter 5

Conditions for p-valent functions

5.1 Conditions for belonging to the class functions of convex p-valent functions

In this section we present sufficient conditions for belonging to the class of p-valently convex functions $\mathcal{K}_p(\beta)$, $0 \leq \beta \leq p$, for new integral operators with functions in the class of p-valently starlike functions $\mathcal{S}_p^*(\alpha)$, $\alpha < 1$.

Let the integral operator

$$\mathcal{M}_{p,n}(z) = \int_0^z pt^{p-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t^p} \right)^{\alpha_i - 1} \left(\frac{g_i{}'(t)}{pt^{p-1}} \right)^{\beta_i} \left(\frac{g_i(t))}{t^p} \right)^{\gamma_i} \right] dt, \tag{5.1.1}$$

where f_i, g_i are p-valent analytic functions in $\mathbb{U}$ and $\alpha_i, \beta_i, \gamma_i$ are complex numbers.

For $p = 1$ this integral operator becomes $\mathcal{M}_n$, defined by (3.0.1).

Theorem 5.1.1. *Let $f_i, g_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i > 0$ and $\mu_i, \lambda_i, \eta_i < 1$, for all $i = \overline{1,n}$. If $f_i \in \mathcal{S}_p^*(\mu_i)$, $g_i \in \mathcal{K}_p(\lambda_i)$ and $g_i \in \mathcal{S}_p^*(\eta_i)$, then $\mathcal{M}_{p,n} \in \mathcal{K}_p(\rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(1 - \mu_i) + \beta_i (1 - \lambda_i) + \gamma_i (1 - \eta_i) \right].$$

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.1.1, we obtain the next corollary:

Corollary 5.1.1.1. *Let $f, g \in \mathcal{A}$, $\alpha > 0$ and $\mu, \lambda, \eta < 1$. If $f \in \mathcal{S}^*(\mu)$, $g \in \mathcal{K}(\lambda)$ and $g \in \mathcal{S}^*(\eta)$, then the integral operator $\mathcal{M} \in \mathcal{K}(\rho)$*

$$\mathcal{M}(z) = \int_0^z \left[\frac{f(t)}{t} g'(t) \frac{g(t))}{t} \right]^{\alpha} dt, \tag{5.1.2}$$

where

$$\rho = 1 - \alpha (3 - \mu - \lambda - \eta).$$

We consider the integral operator:

$$\mathcal{C}_{p,n}(z) = \int_0^z pt^{p-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t^{p(p-1)}} e^{g_i(t)} \right)^{\alpha_i - 1} \left(\frac{{h_i}'(t)}{pt^{p-1}} \right)^{\beta_i} \left(\frac{h_i(t))}{t^p} \right)^{\gamma_i} \right] dt, \tag{5.1.3}$$

where f_i, g_i, h_i are p-valent analytic functions in $\mathbb{U}$ and $\alpha_i, \beta_i, \gamma_i$ are complex numbers.

For $p = 1$ this integral operator becomes $\mathcal{C}_n$, defined by (3.0.2).

Theorem 5.1.2. *Let $f_i, g_i, h_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$. If $f_i \in \mathcal{S}_p^*(\mu_i)$, $g_i \in \mathcal{S}_p^*(\nu_i)$, $h_i \in \mathcal{K}_p(\lambda_i)$, $h_i \in \mathcal{S}_p^*(\eta_i)$ with $\mu_i, \nu_i, \lambda_i, \eta_i < 1$ and Re $(g_i(z)) \geq 1$, for all $i = \overline{1, n}$, then the integral operator $\mathcal{C}_{p,n} \in \mathcal{K}_p(\rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(3 - \mu_i - \nu_i) + \beta_i (1 - \lambda_i) + \gamma_i (1 - \eta_i) \right].$$

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.1.2, we obtain the next corollary:

Corollary 5.1.2.1. *Let $f, g, h \in \mathcal{A}$, $\alpha > 0$. If $f \in \mathcal{S}^*(\mu)$, $g \in \mathcal{S}^*(\nu)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{S}^*(\eta)$ with $\mu, \nu, \lambda, \eta < 1$ and Re $(g(z)) \geq 1$, then the integral operator $\mathcal{C} \in \mathcal{K}(\rho)$, where*

$$\mathcal{C}(z) = \int_0^z \left[\frac{f(t)}{t} e^{g(t)} h'(t) \frac{h(t))}{t} \right]^{\alpha} dt, \tag{5.1.4}$$

and

$$\rho = 1 - \alpha (5 - \mu - \nu - \lambda - \eta).$$

We consider the integral operator:

$$\mathcal{G}_{p,n}(z) = \int_0^z \prod_{i=1}^{n} \left[pt^{p-1} \left(\frac{{f_i}'(t)}{pt^{p-1}} \frac{e^{g_i(t)}}{t^{p-1}} \right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)} \right)^{\beta_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\gamma_i} \right] dt, \tag{5.1.5}$$

where f_i, g_i, h_i, k_i are p-valent analytic functions in $\mathbb{U}$ and $\alpha_i, \beta_i, \gamma_i$ are complex numbers.

For $p = 1$ this integral operator becomes $\mathcal{G}_n$, defined by (3.0.3).

Theorem 5.1.3. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i > 0$. If $f_i \in \mathcal{K}_p(\mu_i)$, $g_i \in \mathcal{S}_p^*(\nu_i)$, $h_i \in \mathcal{K}_p(\lambda_i)$, $h_i \in \mathcal{S}_p^*(\eta_i)$, $k_i \in \mathcal{K}_p(\sigma_i)$, $k_i \in \mathcal{S}_p^*(\theta_i)$ with $\mu_i, \nu_i, \lambda_i, \eta_i, \sigma_i, \theta_i < 1$ and Re $(g_i(z)) \geq 1$, for all $i = \overline{1, n}$, then the integral operator $\mathcal{G}_{p,n} \in \mathcal{K}_p(\rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(3 - \mu_i - \nu_i) + \beta_i (\theta_i - \eta_i) + \gamma_i (\sigma_i - \lambda_i) \right].$$

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.1.3, we obtain the next corollary:

Corollary 5.1.3.1. *Let $f, g, h, k \in \mathcal{A}$, $\alpha > 0$. If $f \in \mathcal{K}(\mu)$, $g \in \mathcal{S}^*(\nu)$, $h \in \mathcal{K}(\lambda)$, $h \in \mathcal{S}^*(\eta)$, $k \in \mathcal{K}(\sigma)$, $k \in \mathcal{S}^*(\theta)$ with $\mu, \nu, \lambda, \eta, \sigma, \theta < 1$ and Re $(g(z)) \geq 1$, then the integral operator $\mathcal{G} \in \mathcal{K}_p(\rho)$, where*

$$\mathcal{G}(z) = \int_0^z \left[f^{'}(t) e^{g(t)} \frac{h(t)}{k(t)} \frac{h'(t)}{k'(t)} \right]^{\alpha} dt, \tag{5.1.6}$$

and

$$\rho = 1 - \alpha \left(3 + \theta + \sigma - \mu - \nu - \lambda - \eta \right).$$

We consider the integral operator:

$$\mathcal{T}_{p,n}(z) = \int_0^z \prod_{i=1}^{n} \left[p t^{p-1} \left(\frac{f_i(t)}{t^p} \right)^{\alpha_i - 1} \left(\frac{{g_i}'(t)}{p t^{p-1}} \right)^{\beta_i} \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] dt, \tag{5.1.7}$$

where f_i, g_i, h_i, k_i are p-valent analytic functions in $\mathbb{U}$ and $\alpha_i, \beta_i, \gamma_i, \delta_i$ are complex numbers.

For $p = 1$ this integral operator becomes $\mathcal{T}_n$, defined by (3.0.4).

Theorem 5.1.4. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i, \delta_i > 0$. If $f_i \in \mathcal{S}_p^*(\mu_i)$, $g_i \in \mathcal{K}_p(\lambda_i)$, $h_i \in \mathcal{K}_p(\omega_i)$, $h_i \in \mathcal{S}_p^*(\eta_i)$, $k_i \in \mathcal{K}_p(\sigma_i)$, $k_i \in \mathcal{S}_p^*(\theta_i)$ with $\mu_i, \lambda_i, \omega_i, \eta_i, \sigma_i, \theta_i < 1$, for all $i = \overline{1,n}$, then the integral operator $\mathcal{T}_{p,n} \in \mathcal{K}_p(\rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[\left(\alpha_i - 1 \right) \left(1 - \mu_i \right) + \beta_i \left(1 - \lambda_i \right) + \gamma_i \left(\theta_i - \eta_i \right) + \delta_i \left(\sigma_i - \omega_i \right) \right].$$

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 5.1.4, we obtain the next corollary:

Corollary 5.1.4.1. *Let $f, g, h, k \in \mathcal{A}$, $\alpha > 0$. If $f \in \mathcal{S}^*(\mu)$, $g \in \mathcal{K}(\lambda)$, $h \in \mathcal{K}(\omega)$, $h \in \mathcal{S}^*(\eta)$, $k \in \mathcal{K}(\sigma)$, $k \in \mathcal{S}^*(\theta)$, with $\mu, \lambda, \eta, \omega, \sigma, \theta < 1$, then the integral operator $\mathcal{T} \in \mathcal{K}_p(\rho)$, where*

$$\mathcal{T}(z) = \int_0^z \left[\frac{f(t)}{t} g'(t) \frac{h(t)}{k(t)} \frac{h'(t)}{k'(t)} \right]^{\alpha} dt, \tag{5.1.8}$$

and

$$\rho = 1 - \alpha \left(2 + \theta + \sigma - \mu - \lambda - \eta - \omega \right).$$

5.2 Conditions for belonging to the class functions $\mathcal{N}_p(\beta)$

In this paragraph we describe sufficient conditions for belonging to the class of p-valently functions $\mathcal{N}_p(\beta)$, for new integral operators with functions in the classes of p-valent functions $\mathcal{N}_p(\beta)$ and $\mathcal{M}_p(\beta)$, $\beta > 1$.

Theorem 5.2.1. *Let $f_i, g_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i > 0$ and $\mu_i, \lambda_i, \eta_i > 1$, for all $i = \overline{1,n}$. If $f_i \in \mathcal{M}_p(\mu_i)$, $g_i \in \mathcal{N}_p(\lambda_i)$ and $g_i \in \mathcal{M}_p(\eta_i)$, then the integral operator $\mathcal{M}_{p,n} \in \mathcal{N}_p(\rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[\left(\alpha_i - 1 \right) \left(\mu_i - 1 \right) + \beta_i \left(\lambda_i - 1 \right) + \gamma_i \left(\eta_i - 1 \right) \right].$$

Theorem 5.2.2. *Let $f_i, g_i, h_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i > 0$. If $f_i \in \mathcal{M}_p(\mu_i)$, $g_i \in \mathcal{M}_p(\nu_i)$, $h_i \in \mathcal{N}_p(\lambda_i)$, $h_i \in \mathcal{M}_p(\eta_i)$ with $\mu_i, \nu_i, \lambda_i, \eta_i > 1$ and Re $(g_i(z)) \leq 1$, for all $i = \overline{1, n}$, then the integral operator $\mathcal{C}_{p,n} \in \mathcal{N}_p(\rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i + \nu_i - 1) + \beta_i (\lambda_i - 1) + \gamma_i (\eta_i - 1) \right].$$

Theorem 5.2.3. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i > 0$. If $f_i \in \mathcal{N}_p(\mu_i)$, $g_i \in \mathcal{M}_p(\nu_i)$, $h_i \in \mathcal{N}_p(\lambda_i)$, $h_i \in \mathcal{M}_p(\eta_i)$, $k_i \in \mathcal{N}_p(\sigma_i)$, $k_i \in \mathcal{M}_p(\theta_i)$ with $\mu_i, \nu_i, \lambda_i, \eta_i, \sigma_i, \theta_i > 1$ and Re $(g_i(z)) \leq 1$, for all $i = \overline{1, n}$, then the integral operator $\mathcal{G}_{p,n} \in \mathcal{N}_p(\rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i + \nu_i - 1) + \beta_i (\eta_i - \theta_i) + \gamma_i (\lambda_i - \sigma_i) \right].$$

Theorem 5.2.4. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i, , \delta_i > 0$. If $f_i \in \mathcal{M}_p(\mu_i)$, $g_i \in \mathcal{N}_p(\lambda_i)$, $h_i \in \mathcal{N}_p(\omega_i)$, $h_i \in \mathcal{M}_p(\eta_i)$, $k_i \in \mathcal{N}_p(\sigma_i)$, $k_i \in \mathcal{M}_p(\theta_i)$ with $\mu_i, \lambda_i, \eta_i, \omega_i, \sigma_i, \theta_i > 1$, for all $i = \overline{1, n}$, then the integral operator $\mathcal{T}_{p,n} \in \mathcal{N}_p(\rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i - 1) + \beta_i (\lambda_i - 1) + \gamma_i (\eta_i - \theta_i) + \delta_i (\omega_i - \sigma_i) \right].$$

5.3 Conditions for belonging to the class functions $\mathcal{K}_p(a, \alpha)$

In this section we present sufficient conditions for belonging to the class of convex p-valently functions $\mathcal{K}_p(a, \alpha)$, for new integral operators with functions in the class of p-valently starlike functions $\mathcal{S}_p^*(a, \alpha)$, $a \in \mathbb{C}$, $\alpha < 1$.

Theorem 5.3.1. *Let $f_i, g_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i > 0$ and $\mu_i, \lambda_i, \eta_i < 1$, for all $i = \overline{1, n}$. If $f_i \in \mathcal{S}_p^*(a, \mu_i)$, $g_i \in \mathcal{K}_p(a, \lambda_i)$ and $g_i \in \mathcal{S}_p^*(a, \eta_i)$, then the integral operator $\mathcal{M}_{p,n} \in \mathcal{K}_p(a, \rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)\mu_i + \beta_i \lambda_i + \gamma_i \eta_i \right].$$

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.3.1, we obtain the next corollary:

Corollary 5.3.1.1. *Let $f, g \in \mathcal{A}$, $\alpha > 0$ and $\mu, \lambda, \eta < 1$. If $f \in \mathcal{S}^*(a, \mu)$, $g \in \mathcal{K}(a, \lambda)$ and $g \in \mathcal{S}^*(a, \eta)$, then the integral operator $\mathcal{M} \in \mathcal{K}(a, \rho)$, where $\mathcal{M}$ defined by (5.1.2) and*

$$\rho = 1 - \alpha (\mu + \lambda + \eta).$$

Theorem 5.3.2. *Let $f_i, g_i, h_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i > 0$. If $f_i \in \mathcal{S}_p^*(a, \mu_i)$, $g_i \in \mathcal{S}_p^*(a, \nu_i)$, $h_i \in \mathcal{K}_p(a, \lambda_i)$, $h_i \in \mathcal{S}_p^*(a, \eta_i)$ with $\mu_i, \nu_i, \lambda_i, \eta_i < 1$ and Re $(g_i(z)) \geq 1$, for all $i = \overline{1, n}$, then the integral operator $\mathcal{C}_{p,n} \in \mathcal{K}_p(a, \rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n} \left[(\alpha_i - 1)(\mu_i + \nu_i + 1) + \beta_i \lambda_i + \gamma_i \eta_i \right].$$

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.3.2, we obtain the next corollary:

Corollary 5.3.2.1. *Let $f, g, h \in \mathcal{A}$, $\alpha > 0$. If $f \in \mathcal{S}^*(a, \mu)$, $g \in \mathcal{S}^*(a, \nu)$, $h \in \mathcal{K}(a, \lambda)$, $h \in \mathcal{S}^*(a, \eta)$ with $\mu, \nu, \lambda, \eta < 1$ and $Re\,(g)(z)) \geq 1$, then the integral operator $\mathcal{C} \in \mathcal{K}_p(a, \rho)$, where $\mathcal{C}$ defined by (5.1.4) and*

$$\rho = 1 - \alpha\left(\mu + \nu + \lambda + \eta + 1\right).$$

Theorem 5.3.3. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i > 0$. If $f_i \in \mathcal{K}_p(a, \mu_i)$, $g_i \in \mathcal{S}_p^*(a, \nu_i)$, $h_i \in \mathcal{K}_p(a, \lambda_i)$, $h_i \in \mathcal{S}_p^*(a, \eta_i)$, $k_i \in \mathcal{K}_p(a, \sigma_i)$, $k_i \in \mathcal{S}_p^*(a, \theta_i)$ with $\mu_i, \nu_i, \lambda_i, \eta_i, \sigma_i, \theta_i < 1$ and $Re\,(g_i(z)) \geq 1$, for all $i = \overline{1, n}$, then the integral operator $\mathcal{G}_{p,n} \in \mathcal{K}_p(a, \rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n}\left[(\alpha_i - 1)\left(\mu_i + \nu_i + 1\right) + \beta_i\left(\eta_i - \theta_i\right) + \gamma_i\left(\lambda_i - \sigma_i\right)\right].$$

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.3.3, we obtain the next corollary:

Corollary 5.3.3.1. *Let $f, g, h, k \in \mathcal{A}$, $\alpha > 0$. If $f \in \mathcal{K}(a, \mu)$, $g \in \mathcal{S}^*(a, \nu)$, $h \in \mathcal{K}(a, \lambda)$, $h \in \mathcal{S}^*(a, \eta)$, $k \in \mathcal{K}(a, \sigma)$, $k \in \mathcal{S}^*(a, \theta)$ with $\mu, \nu, \lambda, \eta, \sigma, \theta < 1$ and $Re\,(g(z)) \geq 1$, then the integral operator $\mathcal{G} \in \mathcal{K}_p(a, \rho)$, where $\mathcal{G}$ defined by (5.1.6) and*

$$\rho = 1 - \alpha\left(\mu + \nu + \eta + \lambda - \theta - \sigma + 1\right).$$

Theorem 5.3.4. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$, $\alpha_i - 1, \beta_i, \gamma_i, \delta_i > 0$. If $f_i \in \mathcal{S}_p^*(a, \mu_i)$, $g_i \in \mathcal{K}_p(a, \lambda_i)$, $h_i \in \mathcal{K}_p(a, \omega_i)$, $h_i \in \mathcal{S}_p^*(a, \eta_i)$, $k_i \in \mathcal{K}_p(a, \sigma_i)$, $k_i \in \mathcal{S}_p^*(a, \theta_i)$ with $\mu_i, \lambda_i, \omega_i, \eta_i, \sigma_i, \theta_i < 1$, for all $i = \overline{1, n}$, then the integral operator $\mathcal{T}_{p,n} \in \mathcal{K}_p(a, \rho)$, where*

$$\rho = 1 - \sum_{i=1}^{n}\left[(\alpha_i - 1)\,\mu_i + \beta_i\lambda_i + \gamma_i\left(\eta_i - \theta_i\right) + \delta_i\left(\omega_i - \sigma_i\right)\right].$$

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 5.3.4, we obtain the next corollary:

Corollary 5.3.4.1. *Let $f, g, h, k \in \mathcal{A}$, $\alpha > 0$. If $f \in \mathcal{S}^*(a, \mu)$, $g \in \mathcal{K}(a, \lambda)$, $h \in \mathcal{K}(a, \omega)$, $h \in \mathcal{S}^*(a, \eta)$, $k \in \mathcal{K}(a, \sigma)$, $k \in \mathcal{S}^*(a, \theta)$, with $\mu, \lambda, \omega, \eta, \sigma, \theta < 1$, then the integral operator $\mathcal{T} \in \mathcal{K}_p(a, \rho)$, where*

$$\rho = 1 - \alpha\left(\mu + \lambda + \omega + \eta - \theta - \sigma\right).$$

5.4 Conditions for belonging to the class p-valent starlike functions

In this section we present sufficient conditions for belonging to the class of p-valently starlike functions $\mathcal{S}_p^*(\beta)$, $0 \leq \beta \leq p$, for new integral operators with functions in the class of p-valent analytic functions $\mathcal{A}_p$.

Theorem 5.4.1. *Let $f_i, g_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1, n}$. If f_i, g_i satisfies*

$$Re\frac{zf_i'(z)}{f_i(z)} < p + \frac{1}{2\sum_{i=1}^{n}(\alpha_i - 1)}, \quad Re\frac{zg_i'(z)}{g_i(z)} < p + \frac{1}{2\sum_{i=1}^{n}\gamma_i}, \quad Re\left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}\beta_i},$$

then the integral operator $\mathcal{M}_{p,n}$ is p-valently starlike in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.4.1, we obtain the next corollary:

Corollary 5.4.1.1. *Let $f, g \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g satisfies*

$$Re\frac{zf'(z)}{f(z)} < 1 + \frac{1}{2\alpha}, \quad Re\frac{zg'(z)}{g(z)} < 1 + \frac{1}{2\alpha}, \quad Re\left(\frac{zg''(z)}{g'(z)} + 1\right) < 1 - \frac{3}{4\alpha},$$

then the integral operator $\mathcal{M}$ defined by (5.1.2) is starlike in the open unit disk.

Theorem 5.4.2. *Let $f_i, g_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1, n}$. If f_i, g_i satisfies*

$$\left|\frac{zf_i'(z)}{f_i(z)} - p\right| < \frac{p}{2\sum_{i=1}^{n}(\alpha_i - 1)}, \quad \left|\frac{zg_i'(z)}{g_i(z)} - p\right| < \frac{p}{2\sum_{i=1}^{n}\gamma_i}, \quad \left|\frac{zg_i''(z)}{g_i'(z)} - p + 1\right| < \frac{1}{\sum_{i=1}^{n}\beta_i},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}_{p,n}$ is p-valently starlike in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.4.2, we obtain the next corollary:

Corollary 5.4.2.1. *Let $f, g \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{2\alpha}, \quad \left|\frac{zg'(z)}{g(z)} - 1\right| < \frac{1}{2\alpha}, \quad \left|\frac{zg''(z)}{g'(z)}\right| < \frac{1}{\alpha},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$ defined by (5.1.2) is starlike in the open unit disk.

Theorem 5.4.3. *Let $f_i, g_i, h_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1, n}$. If f_i, g_i, h_i satisfies*

$$Re\frac{zf_i'(z)}{f_i(z)} < p + \frac{1}{3\sum_{i=1}^{n}(\alpha_i - 1)}, \quad Re\frac{zg_i'(z)}{g_i(z)} < p + \frac{1}{3\sum_{i=1}^{n}(\alpha_i - 1)},$$

$$Re\left(\frac{zh_i''(z)}{h_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}\beta_i}, \quad Re\frac{zh_i'(z)}{h_i(z)} < p + \frac{1}{3\sum_{i=1}^{n}\gamma_i}, \quad |g_i(z)| \leq 1,$$

then the integral operator $\mathcal{C}_{p,n}$ is p-valently starlike in the open unit disk $\mathbb{U}$.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.4.3, we obtain the next corollary:

Corollary 5.4.3.1. *Let $f, g, h \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h satisfies*

$$Re\frac{zf'(z)}{f(z)} < 1 + \frac{1}{3\alpha}, \quad Re\frac{zg'(z)}{g(z)} < 1 + \frac{1}{3\alpha},$$

$$Re\left(\frac{zh''(z)}{h'(z)} + 1\right) < 1 - \frac{3}{4\alpha}, \quad Re\frac{zh'(z)}{h(z)} < 1 + \frac{1}{3\alpha}, \quad |g(z)| \leq 1,$$

then the integral operator $\mathcal{C}$ defined by (5.1.4) is starlike in the open unit disk.

Theorem 5.4.4. *Let $f_i, g_i, h_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i satisfies*

$$\left|\frac{zf_i'(z)}{f_i(z)} - p\right| < \frac{p}{4\sum_{i=1}^{n}(\alpha_i - 1)}, \quad \left|\frac{zg_i'(z)}{g_i(z)} - p\right| < \frac{p}{4\sum_{i=1}^{n}(\alpha_i - 1)},$$

$$\left|\frac{zh_i'(z)}{h_i(z)} - p\right| < \frac{p}{4\sum_{i=1}^{n}\gamma_i}, \quad \left|\frac{zh_i''(z)}{h_i'(z)} - p + 1\right| < \frac{1}{\sum_{i=1}^{n}\beta_i}, \quad |g_i(z)| \le \frac{1}{4p\sum_{i=1}^{n}(\alpha_i - 1)},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{C}_{p,n}$ is p-valently starlike in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.4.4, we obtain the next corollary:

Corollary 5.4.4.1. *Let $f, g, h \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{4\alpha}, \quad \left|\frac{zg'(z)}{g(z)} - 1\right| < \frac{1}{4\alpha},$$

$$\left|\frac{z'h(z)}{h(z)} - 1\right| < \frac{1}{4\alpha}, \quad \left|\frac{zh''(z)}{h'(z)}\right| < \frac{1}{\alpha}, \quad |g(z)| \le \frac{1}{4\alpha},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{C}$ defined by (5.1.4) is starlike in the open unit disk.

Theorem 5.4.5. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i, k_i satisfies*

$$Re\left(\frac{zf_i''(z)}{f_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}(\alpha_i - 1)}, \; Re\frac{zg_i'(z)}{g_i(z)} < p + \frac{1}{\sum_{i=1}^{n}(\alpha_i - 1)}, \; Re\frac{zh_i'(z)}{h_i(z)} < p + \frac{1}{\sum_{i=1}^{n}\beta_i},$$

$$Re\frac{zk_i'(z)}{k_i(z)} < p + \frac{1}{\sum_{i=1}^{n}\beta_i}, \quad Re\left(\frac{zh_i''(z)}{h_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}\gamma_i},$$

$$Re\left(\frac{zk_i''(z)}{k_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}\gamma_i}, \quad |g_i(z)| \le 1,$$

then the integral operator $\mathcal{G}_{p,n}$ is p-valently starlike in the open unit disk $\mathbb{U}$.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.4.5, we obtain the next corollary:

Corollary 5.4.5.1. *Let $f, g, h, k \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h, k satisfies*

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) < 1 - \frac{3}{4\alpha}, \quad Re\frac{zg'(z)}{g(z)} < 1 + \frac{1}{\alpha}, \quad Re\frac{zh'(z)}{h(z)} < 1 + \frac{1}{\alpha},$$

$$Re\frac{zk'(z)}{k(z)} < 1 + \frac{1}{\alpha}, \quad Re\left(\frac{zh''(z)}{h'(z)} + 1\right) < 1 - \frac{3}{4\alpha}, \quad Re\left(\frac{zk''(z)}{k'(z)} + 1\right) < 1 - \frac{3}{4\alpha}, \quad |g(z)| \le 1,$$

then the integral operator $\mathcal{G}$ defined by (5.1.6) is starlike in the open unit disk.

Theorem 5.4.6. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i, k_i satisfies*

$$\left|\frac{zf_i''(z)}{f_i'(z)} - p + 1\right| < \frac{1}{3\sum_{i=1}^{n}(\alpha_i - 1)}, \ \left|\frac{zg_i'(z)}{g_i(z)} - p\right| < \frac{p}{4\sum_{i=1}^{n}(\alpha_i - 1)}, \ \left|\frac{zh_i'(z)}{h_i(z)} - p\right| < \frac{p}{4\sum_{i=1}^{n}\beta_i},$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - p\right| < \frac{p}{4\sum_{i=1}^{n}\beta_i}, \quad \left|\frac{zh_i''(z)}{h_i'(z)} - p + 1\right| < \frac{1}{3\sum_{i=1}^{n}\gamma_i},$$

$$\left|\frac{zh_i''(z)}{h_i'(z)} - p + 1\right| < \frac{1}{3\sum_{i=1}^{n}\gamma_i}, \quad |g_i(z)| \leq \frac{1}{4p\sum_{i=1}^{n}(\alpha_i - 1)},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{G}_{p,n}$ is p-valently starlike in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.4.6, we obtain the next corollary:

Corollary 5.4.6.1. *Let $f, g, h, k \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h, k satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{3\alpha}, \quad \left|\frac{zg'(z)}{g(z)} - 1\right| < \frac{1}{4\alpha}, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| < \frac{1}{4\alpha},$$

$$\left|\frac{zh''(z)}{h'(z)}\right| < \frac{1}{3\alpha}, \quad \left|\frac{zk''(z)}{k'(z)}\right| < \frac{1}{3\alpha}, \quad |g(z)| \leq \frac{1}{4\alpha},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{G}$ defined by (5.1.6) is starlike in the open unit disk.

Theorem 5.4.7. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i, \delta_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i, k_i satisfies*

$$\mathit{Re}\frac{zf_i'(z)}{f_i(z)} < p + \frac{1}{\sum_{i=1}^{n}(\alpha_i - 1)}, \quad \mathit{Re}\left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}\beta_i},$$

$$\mathit{Re}\frac{zh_i'(z)}{h_i(z)} < p + \frac{1}{\sum_{i=1}^{n}\gamma_i}, \quad \mathit{Re}\frac{zk_i'(z)}{k_i(z)} < p + \frac{1}{\sum_{i=1}^{n}\gamma_i},$$

$$\mathit{Re}\left(\frac{zh_i''(z)}{h_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}\delta_i}, \quad \mathit{Re}\left(\frac{zk_i''(z)}{k_i'(z)} + 1\right) < p - \frac{3}{4\sum_{i=1}^{n}\delta_i},$$

then the integral operator $\mathcal{T}_{p,n}$ is p-valently starlike in the open unit disk $\mathbb{U}$.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 5.4.7, we obtain the next corollary:

Corollary 5.4.7.1. *Let $f, g, h, k \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h, k satisfies*

$$\mathit{Re}\frac{zf'(z)}{f(z)} < 1 + \frac{1}{\alpha}, \quad \mathit{Re}\left(\frac{zg''(z)}{g'(z)} + 1\right) < 1 - \frac{3}{4\alpha}, \quad \mathit{Re}\frac{zh'(z)}{h(z)} < 1 + \frac{1}{\alpha}, \quad \mathit{Re}\frac{zk'(z)}{k(z)} < 1 + \frac{1}{\alpha},$$

$$\mathit{Re}\left(\frac{zh''(z)}{h'(z)} + 1\right) < 1 - \frac{3}{4\alpha}, \quad \mathit{Re}\left(\frac{zk''(z)}{k'(z)} + 1\right) < 1 - \frac{3}{4\alpha},$$

then the integral operator $\mathcal{T}$ defined by (5.1.8) is starlike in the open unit disk.

Theorem 5.4.8. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i, \delta_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i, k_i satisfies*

$$\left|\frac{zf_i'(z)}{f_i(z)} - p\right| < \frac{p}{3\sum_{i=1}^{n}(\alpha_i - 1)}, \quad \left|\frac{zg_i''(z)}{g_i'(z)} - p + 1\right| < \frac{1}{3\sum_{i=1}^{n}\beta_i}, \quad \left|\frac{zh_i'(z)}{h_i(z)} - p\right| < \frac{p}{3\sum_{i=1}^{n}\gamma_i},$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - p\right| < \frac{p}{3\sum_{i=1}^{n}\gamma_i}, \quad \left|\frac{zh_i''(z)}{h_i'(z)} - p + 1\right| < \frac{1}{3\sum_{i=1}^{n}\delta_i}, \quad \left|\frac{zk_i''(z)}{k_i'(z)} - p + 1\right| < \frac{1}{3\sum_{i=1}^{n}\delta_i},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}_{p,n}$ is p-valently starlike in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 5.4.8, we obtain the next corollary:

Corollary 5.4.8.1. *Let $f, g, h, k \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h, k satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{3\alpha}, \quad \left|\frac{zg''(z)}{g'(z)}\right| < \frac{1}{3\alpha}, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| < \frac{1}{3\alpha},$$

$$\left|\frac{zk'(z)}{k(z)} - 1\right| < \frac{1}{3\alpha}, \quad \left|\frac{zh''(z)}{h'(z)}\right| < \frac{1}{3\alpha}, \quad \left|\frac{zh''(z)}{h'(z)}\right| < \frac{1}{3\alpha},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$ defined by (5.1.8) is starlike in the open unit disk.

5.5 Conditions for belonging to the class p-valent close-to-convex functions

In this section we present sufficient conditions of belonging to the class of close-to-convex p-valently functions, for the new integral operators having the functions from the class of p-valent analytic functions.

Theorem 5.5.1. *Let $f_i, g_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i satisfies*

$$\mathit{Re}\frac{zf_i'(z)}{f_i(z)} < p + \frac{a}{2(1+a)(1-b)\sum_{i=1}^{n}(\alpha_i - 1)}, \quad \mathit{Re}\frac{zg_i'(z)}{g_i(z)} < p + \frac{a}{2(1+a)(1-b)\sum_{i=1}^{n}\gamma_i},$$

$$\mathit{Re}\left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) < p + \frac{b}{(1+a)(1-b)\sum_{i=1}^{n}\beta_i},$$

for all $a > 0$, $b \geq 0$, $a + 2b \leq 1$ and $z \in \mathbb{U}$, then the integral operator $\mathcal{M}_{p,n}$ is p-valently close-to-convex in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.5.1, we obtain the next corollary:

Corollary 5.5.1.1. *Let $f, g \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g satisfies*

$$\mathit{Re}\frac{zf'(z)}{f(z)} < 1 + \frac{a}{2(1+a)(1-b)\alpha}, \quad \mathit{Re}\frac{zg'(z)}{g(z)} < 1 + \frac{a}{2(1+a)(1-b)\alpha},$$

$$Re\left(\frac{zg''(z)}{g'(z)}+1\right) < 1 + \frac{b}{(1+a)(1-b)\alpha},$$

for all $a > 0$, $b \geq 0$, $a + 2b \leq 1$ and $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$ defined by (5.1.2) is close-to-convex in the open unit disk.

Theorem 5.5.2. *Let $f_i, g_i, h_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i satisfies*

$$Re\frac{zf_i'(z)}{f_i(z)} < p + \frac{a}{3(1+a)(1-b)\sum_{i=1}^{n}(\alpha_i-1)}, \quad Re\frac{zg_i'(z)}{g_i(z)} < p + \frac{a}{3(1+a)(1-b)\sum_{i=1}^{n}(\alpha_i-1)},$$

$$Re\frac{zh_i'(z)}{h_i(z)} < p + \frac{a}{3(1+a)(1-b)\sum_{i=1}^{n}\gamma_i}, \quad Re\left(\frac{zh_i''(z)}{h_i'(z)}+1\right) < p + \frac{b}{(1+a)(1-b)\sum_{i=1}^{n}\beta_i},$$

$$|g_i(z)| \leq 1,$$

for all $a > 0$, $b \geq 0$, $a + 2b \leq 1$ and $z \in \mathbb{U}$, then the integral operator $\mathcal{C}_{p,n}$ is p-valently close-to-convex in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.5.2, we obtain the next corollary:

Corollary 5.5.2.1. *Let $f, g, h \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h satisfies*

$$Re\frac{zf'(z)}{f(z)} < 1 + \frac{a}{3(1+a)(1-b)\alpha}, \quad Re\frac{zg'(z)}{g(z)} < 1 + \frac{a}{3(1+a)(1-b)\alpha},$$

$$Re\frac{zh'(z)}{h(z)} < 1 + \frac{a}{3(1+a)(1-b)\alpha}, \quad Re\left(\frac{zh''(z)}{h'(z)}+1\right) < 1 + \frac{b}{(1+a)(1-b)\alpha}, \quad |g(z)| \leq 1,$$

for all $a > 0$, $b \geq 0$, $a + 2b \leq 1$ and $z \in \mathbb{U}$, then the integral operator $\mathcal{C}$ defined by (5.1.4) is close-to-convex in the open unit disk.

Theorem 5.5.3. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i, k_i satisfies*

$$Re\left(\frac{zf_i''(z)}{f_i'(z)}+1\right) < p + \frac{b}{(1+a)(1-b)\sum_{i=1}^{n}(\alpha_i-1)}, \; Re\frac{zg_i'(z)}{g_i(z)} < p + \frac{a}{(1+a)(1-b)\sum_{i=1}^{n}(\alpha_i-1)},$$

$$Re\frac{zh_i'(z)}{h_i(z)} < p + \frac{a}{(1+a)(1-b)\sum_{i=1}^{n}\beta_i}, \; Re\frac{zk_i'(z)}{k_i(z)} < p + \frac{a}{(1+a)(1-b)\sum_{i=1}^{n}\beta_i}, \; |g_i(z)| \leq 1,$$

$$Re\left(\frac{zh_i''(z)}{h_i'(z)}+1\right) < p + \frac{b}{(1+a)(1-b)\sum_{i=1}^{n}\gamma_i}, \quad Re\left(\frac{zk_i''(z)}{k_i'(z)}+1\right) < p + \frac{b}{(1+a)(1-b)\sum_{i=1}^{n}\gamma_i},$$

for all $a > 0$, $b \geq 0$, $a + 2b \leq 1$ and $z \in \mathbb{U}$, then the integral operator $\mathcal{G}_{p,n}$ is p-valently close-to-convex in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.5.3, we obtain the next corollary:

Corollary 5.5.3.1. *Let $f, g, h, k \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h, k satisfies*

$$Re\left(\frac{zf''(z)}{f'(z)}+1\right) < 1+\frac{b}{(1+a)(1-b)\alpha}, \quad Re\frac{zg'(z)}{g(z)} < 1+\frac{a}{(1+a)(1-b)\alpha},$$

$$Re\frac{zh'(z)}{h(z)} < 1+\frac{a}{(1+a)(1-b)\alpha}, \quad Re\frac{zk'(z)}{k(z)} < 1+\frac{a}{(1+a)(1-b)\alpha},$$

$$Re\left(\frac{zh''(z)}{h'(z)}+1\right) < 1+\frac{b}{(1+a)(1-b)\alpha}, \quad Re\left(\frac{zk''(z)}{k'(z)}+1\right) < 1+\frac{b}{(1+a)(1-b)\alpha}, \quad |g(z)| \leq 1,$$

for all $a > 0$, $b \geq 0$, $a + 2b \leq 1$ and $z \in \mathbb{U}$, then the integral operator $\mathcal{G}$ defined by (5.1.6) is close-to-convex in the open unit disk.

Theorem 5.5.4. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i, \delta_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i, k_i satisfies*

$$Re\frac{zf_i'(z)}{f_i(z)}+1 < p+\frac{a}{(1+a)(1-b)\sum_{i=1}^{n}(\alpha_i-1)}, \quad Re\left(\frac{zg_i''(z)}{g_i'(z)}+1\right) < p+\frac{b}{(1+a)(1-b)\sum_{i=1}^{n}\beta_i},$$

$$Re\frac{zh_i'(z)}{h_i(z)} < p+\frac{a}{(1+a)(1-b)\sum_{i=1}^{n}\gamma_i}, \quad Re\frac{zk_i'(z)}{k_i(z)} < p+\frac{a}{(1+a)(1-b)\sum_{i=1}^{n}\gamma_i},$$

$$Re\left(\frac{zh_i''(z)}{h_i'(z)}+1\right) < p+\frac{b}{(1+a)(1-b)\sum_{i=1}^{n}\delta_i}, \quad Re\left(\frac{zk_i''(z)}{k_i'(z)}+1\right) < p+\frac{b}{(1+a)(1-b)\sum_{i=1}^{n}\delta_i},$$

for all $a > 0$, $b \geq 0$, $a + 2b \leq 1$ and $z \in \mathbb{U}$, then the integral operator $\mathcal{T}_{p,n}$ is p-valently close-to-convex in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 5.5.4, we obtain the next corollary:

Corollary 5.5.4.1. *Let $f, g, h, k \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h, k satisfies*

$$Re\frac{zf'(z)}{f(z)} < 1+\frac{a}{(1+a)(1-b)\alpha}, \quad Re\left(\frac{zg''(z)}{g'(z)}+1\right) < 1+\frac{b}{(1+a)(1-b)\alpha},$$

$$Re\frac{zh'(z)}{h(z)} < 1+\frac{a}{(1+a)(1-b)\alpha}, \quad Re\frac{zk'(z)}{k(z)} < 1+\frac{a}{(1+a)(1-b)\alpha},$$

$$Re\left(\frac{zh''(z)}{h'(z)}+1\right) < 1+\frac{b}{(1+a)(1-b)\alpha}, \quad Re\left(\frac{zk''(z)}{k'(z)}+1\right) < 1+\frac{b}{(1+a)(1-b)\alpha},$$

for all $a > 0$, $b \geq 0$, $a + 2b \leq 1$ and $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$ defined by (5.1.8) is close-to-convex in the open unit disk.

5.6 Conditions for belonging to the class uniformly p-valent close-to-convex functions

This paragraph contains sufficient conditions for belonging to the class of uniformly p-valent close-to-convex functions, for the new integral operators having the functions from the class of p-valent analytic functions.

Theorem 5.6.1. *Let $f_i, g_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i satisfies*

$$Re\frac{zf_i'(z)}{f_i(z)} < p + \frac{1}{2\sum_{i=1}^{n}(\alpha_i - 1)}, \quad Re\frac{zg_i'(z)}{g_i(z)} < p + \frac{1}{2\sum_{i=1}^{n}\gamma_i}, \quad Re\left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) < p - \frac{2}{3\sum_{i=1}^{n}\beta_i},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}_{p,n}$ is uniformly p-valently close-to-convex in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.6.1, we obtain the next corollary:

Corollary 5.6.1.1. *Let $f, g \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g satisfies*

$$Re\frac{zf'(z)}{f(z)} < 1 + \frac{1}{2\alpha}, \quad Re\frac{zg'(z)}{g(z)} < 1 + \frac{1}{2\alpha}, \quad Re\left(\frac{zg''(z)}{g'(z)} + 1\right) < 1 - \frac{2}{3\alpha},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{M}$ defined by (5.1.2) is uniformly close-to-convex in the open unit disk.

Theorem 5.6.2. *Let $f_i, g_i, h_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i satisfies*

$$Re\frac{zf_i'(z)}{f_i(z)} < p + \frac{1}{3\sum_{i=1}^{n}(\alpha_i - 1)}, \quad Re\frac{zg_i'(z)}{g_i(z)} < p + \frac{1}{3\sum_{i=1}^{n}(\alpha_i - 1)},$$

$$Re\frac{zh_i'(z)}{h_i(z)} < p + \frac{1}{3\sum_{i=1}^{n}\gamma_i}, \quad Re\left(\frac{zh_i''(z)}{h_i'(z)} + 1\right) < p - \frac{2}{3\sum_{i=1}^{n}\beta_i}, \quad |g_i(z)| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{C}_{p,n}$ in Theorem 5.6.1, we obtain the next corollary:.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.6.2, we obtain the next corollary:

Corollary 5.6.2.1. *Let $f, g, h \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h satisfies*

$$Re\frac{zf'(z)}{f(z)} < 1 + \frac{1}{3\alpha}, \quad Re\frac{zg'(z)}{g(z)} < 1 + \frac{1}{3\alpha},$$

$$Re\frac{zh'(z)}{h(z)} < 1 + \frac{1}{3\alpha}, \quad Re\left(\frac{zh''(z)}{h'(z)} + 1\right) < 1 - \frac{2}{3\alpha}, \quad |g(z)| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{C}$ defined by (5.1.4) is uniformly close-to-convex in the open unit disk.

Theorem 5.6.3. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i > 0$ positive real numbers, for all $i = \overline{1,n}$. If f_i, g_i, h_i, k_i satisfies*

$$Re\left(\frac{zf_i''(z)}{f_i'(z)} + 1\right) < p - \frac{2}{3\sum_{i=1}^{n}(\alpha_i - 1)}, \quad Re\frac{zg_i'(z)}{g_i(z)} < p + \frac{1}{\sum_{i=1}^{n}(\alpha_i - 1)},$$

$$Re\frac{zh_i'(z)}{h_i(z)} < p + \frac{1}{\sum_{i=1}^{n}\beta_i}, \quad Re\frac{zk_i'(z)}{k_i(z)} < p + \frac{1}{\sum_{i=1}^{n}\beta_i}, \quad |g_i(z)| \leq 1,$$

$$Re\left(\frac{zh_i''(z)}{h_i'(z)} + 1\right) < p - \frac{2}{3\sum_{i=1}^{n}\gamma_i}, \quad Re\left(\frac{zk_i''(z)}{k_i'(z)} + 1\right) < p - \frac{2}{3\sum_{i=1}^{n}\gamma_i},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{G}_{p,n}$ is uniformly p-valently close-to-convex in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \alpha$ in Theorem 5.6.3, we obtain the next corollary:

Corollary 5.6.3.1. *Let $f, g, h, k \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h, k satisfies*

$$Re\left(\frac{zf''(z)}{f'(z)}+1\right) < 1-\frac{2}{3\alpha}, \quad Re\frac{zg'(z)}{g(z)} < 1+\frac{1}{\alpha}, \quad Re\frac{zh'(z)}{h(z)} < 1+\frac{1}{\alpha}, \quad Re\frac{zk'(z)}{k(z)} < 1+\frac{1}{\alpha},$$

$$Re\left(\frac{zh''(z)}{h'(z)}+1\right) < 1-\frac{2}{3\alpha}, \quad Re\left(\frac{zk''(z)}{k'(z)}+1\right) < 1-\frac{2}{3\alpha}, \quad |g(z)| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{G}$defined by (5.1.6) is uniformly close-to-convex in the open unit disk.

Theorem 5.6.4. *Let $f_i, g_i, h_i, k_i \in \mathcal{A}_p$ and $\alpha_i - 1, \beta_i, \gamma_i, \delta_i > 0$ positive real numbers, for all $i = \overline{1, n}$. If f_i, g_i, h_i, k_i satisfies*

$$Re\frac{zf_i'(z)}{f_i(z)} < p+\frac{1}{\sum_{i=1}^{n}(\alpha_i-1)}, \quad Re\left(\frac{zg_i''(z)}{g_i'(z)}+1\right) < p-\frac{2}{3\sum_{i=1}^{n}\beta_i}, \quad Re\frac{zh_i'(z)}{h_i(z)} < p+\frac{1}{\sum_{i=1}^{n}\gamma_i},$$

$$Re\frac{zk_i'(z)}{k_i(z)} < p+\frac{1}{\sum_{i=1}^{n}\gamma_i}, \quad Re\left(\frac{zh_i''(z)}{h_i'(z)}+1\right) < p-\frac{2}{3\sum_{i=1}^{n}\delta_i}, \quad Re\left(\frac{zk_i''(z)}{k_i'(z)}+1\right) < p-\frac{2}{3\sum_{i=1}^{n}\delta_i},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}_{p,n}$ is uniformly p-valently close-to-convex in the open unit disk.

Letting $n = p = 1$ and $\alpha_i - 1 = \beta_i = \gamma_i = \delta_i = \alpha$ in Theorem 5.6.4, we obtain the next corollary:

Corollary 5.6.4.1. *Let $f, g, h, k \in \mathcal{A}$ and $\alpha > 0$ a positive real number. If f, g, h, k satisfies*

$$Re\frac{zf'(z)}{f(z)} < 1+\frac{1}{\alpha}, \quad Re\left(\frac{zg''(z)}{g'(z)}+1\right) < 1-\frac{1}{3\alpha}, \quad Re\frac{zh'(z)}{h(z)} < 1+\frac{1}{\alpha},$$

$$Re\frac{zk'(z)}{k(z)} < 1+\frac{1}{\alpha}, \quad Re\left(\frac{zh''(z)}{h'(z)}+1\right) < 1-\frac{2}{3\alpha}, \quad Re\left(\frac{zk''(z)}{k'(z)}+1\right) < 1-\frac{2}{3\alpha},$$

for all $z \in \mathbb{U}$, then the integral operator $\mathcal{T}$ defined by (5.1.8) is uniformly close-to-convex in the open unit disk.

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