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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Study of circular functions and applications

PhD Thesis Summary

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Introduction

Morse theory starts with M. Morse's research for differentiable functions on manifolds that recover the topology of the manifold, results that Morse applied in the beginning for studies regarding geodesics and which were utilized after by R. Bott for proving the periodicity theorem for homotopy groups.

Throughout the years, the research has become more complex given the practical applications unraveled by the development of advanced informational processes and graphical modeling using computers.

Morse theory met a natural development based on its recognition as a useful tool for studying the topology of manifolds. Thus, concepts were developed and results were obtained for smooth circular functions, branch named circular Morse theory.

The mathematical object called Reeb graph is a concept with an amazing practical applicability. It was used by the french mathematician G. Reeb in his studies for real functions on a topological space. This field is currently in full expansion due to the fact that is giving researchers a useful way of visualizing surfaces and by simplifying the topology using existing computational power. Similar to Morse theory, the Reeb graph found an expansion for circular maps, main researchers being E.Batista, J.Cost, I.Meza-Sarmiento, U.Bauer, Y. Wang and many more.

This thesis is organized in three chapter, an annexe and a table of contents having 96 references.

First chapter, "Elements of critical point theory", has the purpose of a concise introduction for notions used throughout the thesis. Main points reached are theorems form differential topology such as the immersion theorem, the submersion theorem and the local diffeomorphism theorem, the rank theorem, Morse's lemma, Morse's inequalities, handle decomposition for a surface, types of critical points, homology and cohmology groups and also Lie groups and Grassmann manifolds, all of them used for developing next chapters. Main references used are D. Andrica [4], G. Cicortaş [33], Y. Matsumoto [69], J. Milnor [72], L. Tu [96].

First section contains fundamental notions such as local representation, critical point, regular point, critical set, bifurcation set and also important results like the rank theorem

(1.1.2), the preimage theorem (1.1.4), the local immersion theorem (1.1.5), the local submersion theorem (1.1.6) and the local diffeomorphism theorem (1.1.7).

Second section offers an in depth presentation over basic Morse theory elements: non-degenerate critical point, Morse function, index and Morse's lemma (1.2.1). Also here were introduced three subsections. First one contains relevant theorems for handle decomposition starting from a Morse function, second one is centered on illustrating Morse's inequalities presenting in this process notions like smooth manifold, deformation retraction, chain complex and cochain complex, homology group and Betti number. Here were included the universal coefficients theorem (1.2.7) and the Poincaré duality (1.2.8). Last section presents the definition of Lie groups, translations and offers a list of Lie groups used in Chapter 2 and also presents examples of important Lie homeomorphisms.

Third section contains in its first part, a description of grssmannian notion, together with the particular cases $G(1,n)$, $G(2,n)$. Second section addresses Plücker embedding for $G(k,V)$, eigenvalues and eigenvectors for a linear map, tensor product, wedge product and total decomposition. Last section includes a short description of necessary notions concerning lens spaces and particular cases of homeomorphic lens spaces used in Chapter 2 of this thesis.

Second chapter highlights the study of critical points for a circular function and has the following structure: basic notions, proprieties and computations of $\varphi-$category for particular manifolds. Second part of this chapter contains important concepts, proprieties, theorems and computations for circular Morse-Smale category. Both notions are presented for real Morse functions and also circular Morse functions. This chapter represent a mix between classical elements and original notes presented in mathematical journals. Our main references are Andrica [4], G. Cicortaş [33], P. Church [30], [31], L. Funar [6], C. Pintea [12], [13], [76], G. Rassias [83], [84], V. Sharko [88], F. Tankens [95] and original papers in collaboration with prof. dr. D Andrica and conf. dr. C. Pintea [8], [65].

First section offers a motivation for the φ-category notion of a pair of manifolds (M,N), invariant propriety of $\varphi(M)$ studied by F. Takens, Morse-Smale characteristic for a manifold M following the presentation made by S.Smale, but also the applicability of these notions for a circular Morse function.

The second section contains proprieties of φ-category for a manifold of a real and circular function. Also, it presents inequalities concerning the product of manifolds, applications and finally inequalities that connect all these notions for circular and real maps.

Third section start by giving a wide perspective over the evolution of studies regarding Morse theory and continues with notions such as differential form, 1-form, closed and exact form, Morse form together with relevant examples for these concepts (Example 2.3.1 and 2.3.2). Last part of this section covers the fundamental group for a topological space and the lift of an application.

The next three section are based on original papers written in collaboration. These papers are D. Andrica, A. Lupescu, C. Pintea [8] and A. Lupescu, C. Pintea [65].

Section 2.4 presents computations for circular φ-category for particular manifolds: product and direct sums, submultiplicativity propriety for φ_{S^1} and ends with detailed examples of applications of such inequalities for particular cases given by the special linear group, special orthogonal group, spin group of order n and Grassmann manifold $G_{k,n}$.

Next section contains general elements concerning the real Morse-Smale category, circular Morse-Smale characteristic for a manifold and theorems that offer a direct correspondence between inequalities presented in the prior section and their applicability for Morse-Smale category. Section 2.6 is based on original calculations of submultiplicativity propriety for γ_{S^1}, original results and examples, followed by computations for particular cases of Grassmann manifolds and Lie groups.

The last chapter is centered on the notion of Reeb graph, concept that has been highly developed during past years, given it's applicability for a large scale of fields like computer graphics or computational geometry. As scientific references we used: U. Bauer [18], [19], E.B. Batista, J.C.F. Costa, I.S Meza-Sarmiento [17], B. Fabio, C. Landi [41], M. Kaluba, W. Marzantowicz, N. Silva [58], Y. Matsumoto, O. Saeki [70], L.P. Michalak [71] and V.V. Sharko [90].

In the course of developing this part of our thesis, we presented original elements such as the development of an algorithm used for constructing the Reeb graph for a real function, together with examples incorporated in the paper [63] and the construction of an algorithm for circular functions included in our paper [64].

This last chapter is structured in eight section and follows the general presentation of a Reeb graph for real and circular functions, description of existing algorithms and is based on innovative and original elements.

First section offers a general description, proprieties and relevant examples for the Reeb graph of a real function on the sphere, torus and orientable surface of genus 2. Second sections starts by organizing Reeb graphs in a category where the objects are finite graphs together with maps monotonic on edges and the morphisms are maps that preserve applications between spaces.

Main idea from Section 3.3 is presenting the Reeb graph as a useful tool that facilitates the reconstruction of a surface when given a graph that contains all necessary information for the geometrical representation, along with the recovery of surfaces topology. This section illustrates the Reeb graph as a tree and gathers the information as follows: types of connected components, types of families of paths, the Reeb graph from graph theory perspective, applications and realizations theorems of the Reeb graphs using Morse functions.

Section 3.4 proves the need of stability of a Reeb graph by introducing some metric. For this specific purpose there were defined the following distances: interleaving , functional distortion, bottleneck and edit distance. Theorem 3.4.2 presents the stability of bottleneck distance and Theorem 3.4.3 and Corollary 3.4.1 show a connection between the first three distances mentioned in the paper.

Sections 3.5 and 3.6 are focused on practical applications of Reeb graphs in computational topology. Here we present an original algorithm for constructing a Reeb graph together with examples for applying it and also a survey of existing algorithms, their complexity and specific elements for each one of them. Last two sections contain elements for circular Reeb graph, the construction algorithm for it and the equipped Reeb graph associated to a simple Morse function on an orientable manifold.

This thesis contains also an annexe used for presenting the caption of video slides for the construction algorithm for sphere, torus and orientable surface of genus 2, an in depth analysis of handle decomposition on surfaces together with the geometrical representation of each type of handle and a schematic representation (mind map) of essential notions from each chapter.

I would like to emphasize the importance of Geometry research group that helped immensely in the process of writing this thesis, giving me the opportunity to present my ideas and results, to ask questions that led to the successful completion of my work. First of all, I would like to thank prof. univ. dr. Dorin Andrica for his constant support, availability and time but mostly for instructing and modeling me as a PhD student. Also I would like to thank conf. dr. Paul Blaga for all his kind help, scientific materials that he provided or recommended and also for all his time. I sincerely thank conf. dr. Cornel Pintea and lect. dr. Daniel Văcăreţu for all the observations and suggestions offered while writing scientific papers, that led to successfully finalizing this thesis.

Key words: Morse function, circular Morse function, φ-category, real Morse-Smale characteristic, circular Morse-Smale characteristic, Reeb graph, Morse-Bott function.

1. Critical set and bifurcation set

1.1 Critical set and bifurcation set

Morse theory is an important chapter for critical point theory and offers techniques for recovering the topology of a manifold by analyzing the critical points corresponding to a smooth function over the given manifold. An important reference for investigating Morse theory is John Milnor's book, published in 1960 [72]. For a good understanding of basic concepts we used in this chapter, the book [69] of Y. Matsumoto.

The first section of this chapter presents basic notion of differential topology such as smooth map, local representation, the rank of an application, regular point, critical point, regular set, critical set and bifurcation set, immersion, submersion and diffeomorphism. Also, we presented without proof the rank theorem, pre-image theorem, local immersion, local submersion and local diffeomorphism. We used the references [4], [26], [33], [69], [96].

Definition 1.1.1 *Let M a smooth m-dimensional manifold without boundary. A function $f : M \to \mathbb{R}$ is* ***smooth*** *if for any map (U, φ), the map $f_\varphi = f \circ \varphi^{-1}$ is smooth on $\varphi(U) \subseteq \mathbb{R}^m$.*

The application f_φ is called **the local representation** of f in the chart (U, φ).

Definition 1.1.2 *We call the* ***rank*** *of the function f in p, denoted by $rank_p(f)$, is*

$$rank_p(f) = rank_{\varphi(p)}(f_{\varphi,\psi}) = rank J\left(f_{\varphi,\psi}\right)(\varphi(p)) = dim\ Im\ (Tf)_p.$$

Proposition 1.1.1 *Let $f : M \to N$ be a smooth map. Then the function $rank(f) : M \to \mathbb{Z}$, $p \mapsto rank_p(f)$ is upper semi continuous.*

We say the smooth application $f : M \to N$ has **constant rank** in $p_0 \in M$ if it exists an open neighborhood U_{p_0} around p_0, such that $rank_p(f) = rank_{p_0}(f)$, for all p in U_{p_0}.

Theorem 1.1.2 (rank theorem) *[4] Let $f : M \to N$ be a smooth map with constant rank $k \leq min(m, n)$ in $p_0 \in M$. Then it exists a local representation of f around p_0:*

$$(z^1, \cdots, z^m) \mapsto (z^1, \cdots, z^k, 0, \cdots, 0).$$

Definition 1.1.3 *A point $p \in M$ is called a **regular point** of f if $rank_p(f)$ is maximal, meaning the following relation is true $rank_p(f) = min(m, n)$.*

The set of regular points of an application f is denoted by $R(f)$ and is called the **regular set** of f. If the point $p \in M$ is not regular, it is called **critical**, and the set of such points represents the **critical set** of f, denoted by $C(f)$.

We denote by $B(f) = f\left(C(f)\right)$, **the bifurcation set** of map f.

The bifurcation set has the following proprieties, proven by A.Sard in [87]:

Theorem 1.1.3 *The set $B(f)$, has null Lebesque measure in the manifold N.*

Theorem 1.1.4 (preimage theorem) *[4] Let $f : M \to N$ be a smooth function with $m \geq n$ and $q \notin B(f)$. Then the fiber $f^{-1}(q)$ is a $(m-n)$-dimensional submanifold of M or the void set. Moreover, for any $p \in f^{-1}(q)$ we have $T_p\left(f^{-1}(q)\right) = Ker(df)_p$.*

Definition 1.1.4 *The map $f : M^m \to N^n$ with $m \leq n$ is called an **immersion** in $p \in M$ if $rang_p(f) = m$ is maximal, meaning we have $rank_p(f) = m$.*

Theorem 1.1.5 (local immersion theorem) *[4] Let $f : M^m \to N^n$, with $m \leq n$, a smooth immersion in $p_0 \in M$. Then f has constant rank in p_0 and it exist a local representation around p_0 modeled in the following way*

$$(z^1, \cdots, z^m) \mapsto (z^1, \cdots, z^m, 0, \cdots, 0)\,.$$

Definition 1.1.5 *The map $f : M^m \to N^n$ with $m \geq n$ is a **submersion** in $p \in M$ if $rank_p(f)$ is maximal, meaning we have $rank_p(f) = n$.*

Theorem 1.1.6 (local submersion theorem) *[4] Let $f : M^m \to N^n$, with $m \geq n$, is a submersion in $p_0 \in M$. Then f has constant rank in p_0 if it exists a local representation around p_0 looking like*

$$(z^1, \cdots, z^m) \mapsto (z^1, \cdots, z^n)\,.$$

Definition 1.1.6 *Let $f : M \to N$ a smooth map. The map $Tf : T(M) \to T(N)$, is called the tangent application of f, where*

$$Tf\left([x,(U,\varphi),v]\right)=\left[f(x),d\left(\psi\circ f\circ\varphi^{-1}\right)_{\varphi(x)}(v)\right]$$

for any chart (U,φ), (V,ψ) *around* x, *respectively* $f(x)$.

Theorem 1.1.7 (local dipheomorphism theorem) *[4] Let* $f : M \to N$ *a smooth map such that in* $x \in M$, *the tangent application* $T_xf : T_xM \to T_{f(x)}N$ *is an isomorphism. Then we have the open neighborhoods* U *and* V *around* x *in* M, *respectively* $f(x)$ *in* N, *such that* $f\big|_U : U \to V$ *is a diffeomorphism.*

1.2 Non-degenerate critical points. Morse functions

This section is focused on Morse functions and introduces the following concepts: Morse function, local minimum and local maximum, index, deformation, cellular complex chain, chain morphism, homology group, Betti number, cohomology group and Lie group. A central part of this section covers Morse's lemma and Lie groups, used in the next chapter of this thesis. Also, we have presented without proof universal coefficients theorem, Poincaré duality, and examples of computations for critical points and calculations of cohomology for particular cases of manifolds. For developing this section we used the following references [4], [23], [33], [39], [52], [69], [72] and [94].

A critical point x is called **non-degenerate** for a smooth real function $f : M^m \to \mathbb{R}$ if it exist a chart (U,φ) around p such that the hessian matrix of the local representation f_φ

$$H\left(f_\varphi\right)(\varphi(x))=\left(\frac{\partial^2 f_\varphi}{\partial x_i\partial x_j}(\varphi(x)\right)_{1\leq i,j\leq m}\text{ is invertible,}$$

so we have $(detH(f_\varphi)(\varphi(x)))\neq 0$. Otherwise the critical point x is called **critical degenerate**. The matrix $H\left(f_\varphi\right)(\varphi(x))$ is symmetrical, so it has all eigenvalues real and not zero.

Definition 1.2.1 *A real function* $f : M \to \mathbb{R}$, *is called a* ***Morse function*** *if all its critical points are non-degenerate. Moreover, the Morse function is called* ***simple***, *if every critical value is the result of a critical point.*

Theorem 1.2.1 (Morse's lemma) *[69] Let* $f : M^m \to \mathbb{R}$ *a smooth application and* $p \in C(f)$ *a non-degenerate critical point. Then it exists a local representation* f_φ *of* f *displayed in the following way:*

$$f_\varphi = -x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_m^2 + f(p).$$

Definition 1.2.2 *The number $\lambda \in \{0, 1, \cdots m\}$ is called the* ***index*** *of the critical point p. It represents the number of negative eigenvalues in the hessian matrix of f_φ.*

The number λ does not depend on the local representation f_φ considered, so it represents an invariant of the critical point.

Critical points of index 0 are the **local minimum**, and the ones of index m are the **local maximum**.

1.2.1 Handle decomposition by utilizing Morse functions

For the purpose of presenting results concerning handle decomposition of a differentiable manifolds we used the references [69] şi [72].

Theorem 1.2.2 *Let M be a compact manifold. If $f : M \to \mathbb{R}$ is a Morse function, then f has a finite number of critical points.*

Theorem 1.2.3 *Let M be a closed surface and f a Morse function with two critical points on M. Then the surface is diffeomorphic with the unit sphere S^2. (See Image 1.1)*

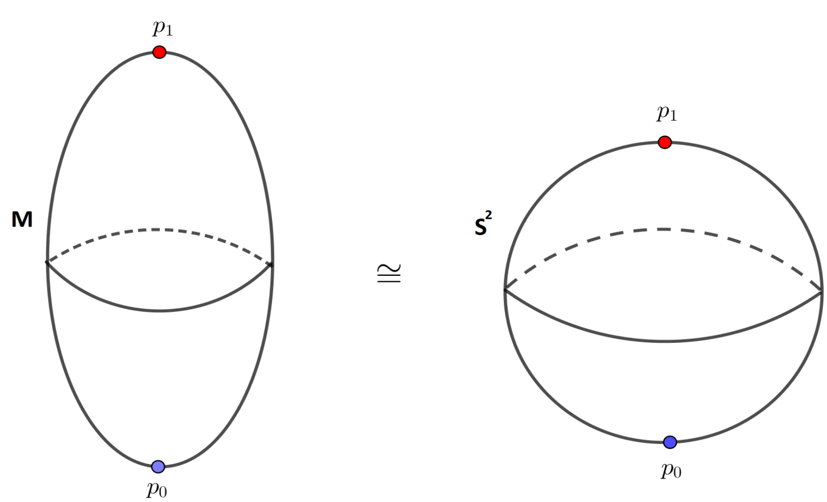


Image 1.1

Theorem 1.2.4 *If on the close surface M we can define a Morse function $f : M \to \mathbb{R}$, then M can be decomposed in a finite reunion of 0, 1 and 2-handles.*

Observation 1.2.1 *Let $f : M \to [a, b]$ be a Morse function without critical values on $[a, b]$. Then the submanifolds M_a and M_b are diffeomorphic.*

1.2.2 Morse's inequalities

For the purpose of outlining Morse's inequalities we will give a general presentation of fundamental concepts following the references [4], [23], [33], [39], [69].

Definition 1.2.3 *[52] It is called a* ***chain complex*** *for X, the semi-exact sequence:*

$$\cdots \to C_{i+1}(X) \stackrel{\partial_{i+1}}{\to} C_i(X) \stackrel{\partial_i}{\to} \cdots\cdots \to C_1(X) \stackrel{\partial_1}{\to} C_0(X) \to \{0\}$$

For a fixed integer i we consider the groups:

$$Z_i(X) = \{Ker\partial_i\} \text{ şi } B_i(X) = \{Im\partial_{i+1}\}$$

The homology group *with integer coefficients of level i is the quotient group:*

$$H_i(X) = Z_i(X)/B_i(X).$$

Theorem 1.2.5 *[69] Given two topological spaces X, Y and $f, g : X \to Y$ continuous maps such that $f \simeq g$, then these maps will induce the same morphism at homology group level:*

$$f_* = g_* : H_q(X) \to H_q(Y)$$

If X is a finite cell complex, its homology group has the following structure:

$$H_q(X) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus T$$

where $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ is its **free part**, and T is the **torsion part**.

The number of copies of $\mathbb{Z}$, represents the rank of the group $H_q(X)$ and it is named **Betti number** of order q for X, so we have $b_q(X) = rankH_q(X), \forall q \in \mathbb{N}$

In this context, we introduce the Euler-Poincaré characteristic of X, as being the alternate sum of Betti numbers, meaning:

$$\chi(X) = b_0 - b_1 + b_2 - \cdots$$

Proposition 1.2.6 *[33] Given the following elements: M^m a smooth compact manifold, $f : M \to \mathbb{R}$ a smooth Morse function and p, q two regular values associated with the function f, we can state the following inequalities:*

1) The weak form of Morse's inequalities: $b_k \leq \mu_k$, $k = 0, \cdots, m$;

2) Morse's inequalities: $b_k - b_{k-1} + \cdots \pm b_0 \leq \mu_k - \mu_{k-1} + \cdots \pm \mu_0$, $k = 0, \cdots, m$.

We consider $C^i(X)$ the set of group morphisms $f : C_i(X) \to \mathbb{Z}$.

Together with the addition operation, $C^i(X)$ has a group structure, named the **i-dimensional cochain group**.

The map $\delta^i : C^i(X) \to C^{i+1}(X)$ that associates to every i-dimensional co-chain f, an $(i+1)$-dimensional chain $f \circ \partial_{i+1}$, it is called **cochain morphism**.

Definition 1.2.4 *[69] It is called a cochain complex for X, the sequence composed from border morphism and cochain groups having the following representation:*

$$\{0\} \to C^0(X) \stackrel{\delta^0}{\to} \cdots \to C^1(X) \stackrel{\delta^1}{\to} \cdots\cdots \to C^{i-1}(X) \stackrel{\delta^{i-1}}{\to} C_i(X) \stackrel{\delta^i}{\to} C_{i+1}(X) \stackrel{\delta^{i+1}}{\to} \cdots$$

For a fixed integer i, we introduce the following groups:

$$Z^i(X) = Ker\delta^i = \{p \in C^i(X) : \delta^i(p) = 0\}$$

*called the **i-dimensional cocyclic group**,*

$$B^i(X) = Im\delta^{i+1} = \{p \in C^i(X) : p = \delta^i i + 1(p'), p' \in C^{i+1}(X)\}$$

*called **i-dimensional coborder group.***

*Therefore, we have the **cohomology group** of level i, as being the quotient group defined by:*

$$H^i(X) = Z^i(X)/B^i(X)$$

It is called **cohomolgy class**, denoted by $[f]$, an element in the group $H^i(X)$. This class induces the morphism:

$$[f] : H_i(X) \to \mathbb{Z}$$

By fixing the set

$$Hom\,(H_i(X), \mathbb{Z}) = \{f : f : H_i(X) \to \mathbb{Z}, f\,morphism\}$$

we can introduce the morphism

$$k : H^i(X) \to Hom\,(H_i(X), \mathbb{Z})$$

Theorem 1.2.7 (Universal coefficients theorem) *[94] The morphism k presented before is a surjective application. Moreover, its kernel represents the torsion part of $H^i(X)$.*

Theorem 1.2.8 (Poincaré duality) *[94] Given the m-dimensional closed and orientable manifold M we have the following isomorphism:*

$$H^i(M) \cong H_{m-i}(M), i = 0, 1, \cdots, m.$$

1.2.3 Lie groups of matrices

Matrix groups represent a connection between algebra and geometry, providing examples and elegant representations for geometrical notions, simplifying their understanding. In this context, the development of techniques for image processing, simplification of geometrical instruments such as rotation matrices or the Reeb graph, have a big contribution to the reduction of processing time for existing algorithms.

Definition 1.2.5 *A smooth manifold together with the group structure, such that the group operation $* \cdot * : M \times M \to M$ and the operation for inverting an element $*^{-1} : M \to M$ are smooth, is called* ***Lie group****.*

Let $(G_1, *), (G_2, \circ)$ two groups and the applications $f : G_1 \to G_2$. The map f is called **group morphism** if:

$$\forall\, x, y \in G_1\colon\ f(x * y) = f(x) \circ f(y).$$

Definition 1.2.6 *[21] Given G a group and $g \in G$ we say that the map $l_g : G \to G$ given by $l_g(h) = gh$ is a* ***left translation****.*

Similarly we have a ***right translation*** *with $g \in G$, $r_g : G \to G$, $r_g(h) = hg$.*

The next two sections are a general presentation for concepts like Grassmann manifolds and lens spaces used in second chapter for computations of $\varphi-$category and circular Morse-Smale category. We have presented briefly definitions like Grassmann manifold, tonsorial product, wedge product, Plücker embedding and proprieties of lens spaces. For the above mentioned concepts we used [24], [51], [61], [68], [81] and [82].

1.3 Grassmann manifolds

1.3.1 General description

The grassmannian notion appeared for the first time in the XIX century, introduced by the german mathematician Julius Plücker in his studies over projective lines in $\mathbb{P}^3$. Called Plücker coordinates, they appeared naturally in algebraic geometry and offer a one to one correspondence between the lines from $\mathbb{P}^3$ and the points on the quadratic from the projective space $\mathbb{P}^5$. The name of grassmannian comes from the german mathematician's name, Hermann Grassmann.

Definition 1.3.1 *Given a vector space V, we call* ***grassmannian****, the set of all k-dimensional linear subspaces of V, meaning:*

$$G(k, V) = \{W \subset V : W \le V, dim(W) = k\}.$$

1.3.2 Plücker's embedding for $G(k,V)$ and the manifold structure of grassmanian

Let $f \in A_r(V), g \in A_s(V)$ two alternating applications. It is called **wedge product** of f and g, the alternate map $f \wedge g$, where

$$f \wedge g = \frac{1}{k!p!}A(f \otimes g),$$

and A is the alternating operator.

$$(f \wedge g)(v_1, \cdots, v_{k+p}) = \frac{1}{k!p!} \sum_{\sigma \in S_{k+p}} (sgn\sigma) \left(f\left(v_\sigma(1), \cdots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+p)}\right)\right).$$

The wedge product is associative, meaning

$$(f \wedge g) \wedge h = f \wedge (g \wedge h),$$

for all alternating maps f, g, h.

Also the operator is anticommutative,

$$f \wedge g = (-1)^{rs} g \wedge f,$$

where $f \in \Lambda^r V$ şi $g \in \Lambda^s V$.

We say that a multivector $\omega \in \Lambda^k V$ has a **total decomposition**, if we can write: $\omega = \omega_1 \wedge \cdots \wedge \omega_k$. Also $v \in V$ is called **divisor of** ω, if it exists $\varphi \in \Lambda^{k-1}V$ such that $\omega = v \wedge \varphi$.

We consider $W \in G(k,V)$ and a basis such that we associate to W the multivector $\lambda = v_1 \wedge \cdots \wedge v_k$, where $W =< v_1, \cdots, v_k >$. It is called the **Plücker embedding**, the map

$$\psi : G(k,V) \to \mathbb{P}\left(\Lambda^k V\right), W =< v_1, \cdots, v_k > \mapsto [\lambda],$$

where $[\lambda]$ is the subspace generated by $v_1 \wedge \cdots \wedge v_k$.

Remark 1.3.1 *The map ψ is an embedding because for $[\omega] = \psi(W)$ we have: $W = \left\{v \in V : v \wedge \omega \in \Lambda^{k+1}V\right\}$. Also $Im\left(\psi\left(G(k,V)\right)\right)$ represents a projection for all vector spaces that have a total decomposition in $\Lambda^k V$.*

The coordinates $\mathbb{P}\left(\wedge^k V\right)$ are called Plücker coordinates for the grassmannian from $G(k,V)$.

For emphasizing the differential manifold structure of the grassmannian we attach to every $[\omega] \in G(k,V)$ a map: $\varphi_\omega : V \to \Lambda^{k+1}V$ such that $\varphi_\omega(v) = v \wedge \omega$. Therefore, we have $\omega \in G(k,V)$ if and only if $rank\left(\varphi_\omega\right) \leq n-k$, and the grassmannian is composed as a finite intersection of projective hypersurfaces, so it has the structure of a smooth manifold.

1.4 Lens space

For presenting the following part we used the reference [24].

Given the spehere $S^{2n-1} = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n : \|z\| = 1\}$, the unit roots of order p, $\epsilon = e^{\frac{2\pi i}{p}}$ and $d_1, \cdots, d_n$ integers coprime to p.

It is called **lens space**, denoted by $L(p, d_1, \cdots d_n)$, the quotient space of S^{2n-1} obtained by Z_p-free action defined by:

$$(z_1, \cdots, z_n) \mapsto \left(e^{\frac{2\pi i d_1}{p}} \cdot z_1, \cdots, e^{\frac{2\pi i d_n}{p}} \cdot z_n\right).$$

Theorem 1.4.1 *[68] The lens spaces $L(p_1; d_1), L(p_2; d_2)$ are homotopy equivalent if and only if*

$$\pm d_1 d_2 \equiv k^2 \ (modp).$$

Moreover, if the two spaces are homeomorphic, then $p_1 = p_2$.

2. Critical points for circular maps

2.1 Motivating the notion of $\varphi-$category

Definition 2.1.1 *Let $\mathcal{F} \subseteq C^{\infty}(M^m, N^n)$ be a family of smooth maps. It is called the $\varphi_{\mathcal{F}}$**-category** of a pair (M, N):*

$$\varphi_{\mathcal{F}}(M, N) = \min\{\mu(f) : f \in \mathcal{F}\},$$

where $\mu(f)$ is the cardinality of $C(f)$, critical set of f.

It is obvious that $\varphi_{\mathcal{F}}(M, N) = 0$ if and only if $\mathcal{F}$ contains immersions, submersions and diffeomorphisms. Also $0 \leq \varphi_{\mathcal{F}}(M, N) \leq \infty$. Next, we will present the necessary proprieties using the references [4], [30], [31], [32], [83], [84], [88] şi [93].

F.Takens studied the invariant $\varphi(M)$ that comes from the particular case $N = \mathbb{R}$ and the family $\mathcal{F}$ given by the algebra of real smooth functions on M, $\mathcal{F}(M) = C^{\infty}(M, \mathbb{R})$. Computations related to this invariant were made in the paper [4].

Let $N = \mathbb{R}$ and $\mathcal{F} = \mathcal{F}_m(M) \subset C^{\infty}(M, \mathbb{R})$, the set of all Morse functions on M. In this case we have $\varphi_{\mathcal{F}}(M, \mathbb{R}) = \gamma(M)$, called the **Morse-Smale characteristic** of a manifold M, an important invariant of M. In the paper [92] S.Smale computed the Morse-Smale characteristic for the particular case of a simple connected manifold of dimension bigger than 5. A general computation for this invariant was not done until today.

Hydrodynamic problems occurred in Nivkov's research, led to the case of $N = S^1$ and also to the family of circular Morse functions on M, $\mathcal{F}_m(M, S^1) \subset C^{\infty}(M, S^1)$. In this case, $\varphi_{\mathcal{F}}(M, S^1)$ is denoted by $\gamma_{S^1}(M)$ and is called the **circular Morse-Smale characteristic** of a manifold M. Therefore we have:

$$\gamma_{S^1}(M) = \min\{\mu(f) : f \in \mathcal{F}_m(M, S^1)\}.$$

2.2 $\varphi-$category and circular $\varphi-$category of a manifold

In this section we consider a smooth manifold M and $f : M \to \mathbb{R}$ a smooth function on M. We introduce the notions of $\varphi-$category of a pair of manifolds (M, N), circular φ-category of a manifold together with relevant proprieties and theorems. The results presented here, follow the work of authors Andrica D. [3], [4], Cicortaş G. [33], Pintea C. [12], [13], [76], Funar L. [5], [6] and Tankens F. [95].

We have the following proprieties:

1. the φ-category is a differential invariant, meaning that if the manifolds M and N are diffeomorphic then we have $\varphi(M) = \varphi(N)$.

2. the φ-category is submultiplicative, so the following inequality is true

$$\varphi(M \times N) \leq \varphi(M) \cdot \varphi(N).$$

Example 2.2.1 We have the following examples:

1.According to the paper [16], over the 2-dimensional torus it can be construct a real smooth function with three critical points: minimum, maximum and a degenerate point, so $\varphi(\mathbb{T}^2) = 3$. It is also possible to find a height function that will immerse the torus in $\mathbb{R}^3$, but this fact is not true for an embedding. Moreover, we have $\varphi(\mathbb{T}^n) = n + 1$, where $\mathbb{T}^n = S^1 \times \cdots \times S^1$ is the n-dimensional torus.

2. The m-dimensional sphere admits a real function with exactly two critical points, a minimum and a maximum and we have $\varphi(S^m) = 2$.

Theorem 2.2.1 *If M is a m-dimensional compact manifold we have the inequality:*

$$\varphi(M) \leq m + 1.$$

The notion of φ-category was extended in the paper [3] moving from the minimum number or critical points of a real valued function on a manifold M, to the minimum number of critical points for functions $M \to N$, leading to the definition of $\varphi-$category of a pair (M, N).

Definition 2.2.1 *Let M and N be two smooth manifolds and $f : M \to N$ a smooth function. We can define the φ-**category** of a pair(M, N) by*

$$\varphi\left(M, N\right) = \min\left\{\mu(f) : f \in C^{\infty}(M, N)\right\}.$$

Theorem 2.2.2 *Let M be a compact m-dimensional manifold and k a natural number such that $2 \leq k \leq m$. Then any map $f \in C^{\infty}(M, \mathbb{R}^k)$ has an infinite number of critical points and we have $\varphi\left(M, \mathbb{R}^k\right) = \infty$.*

Proposition 2.2.3 *For a pair of spheres (S^m, S^1), $m \geq 2$, the minimum number of possible critical points for smooth functions $f : S^m \to S^1$ is 2.*

Definition 2.2.2 *[10] Let M be a smooth manifold and $f : M \to S^1$ a smooth function. We define the **circular** $\varphi-$**category** of M through the relation:*

$$\varphi_{S^1}(M) = \min\left\{\mu(f) : f \in \mathcal{C}^{\infty}\left(M, S^1\right)\right\}.$$

Taking into consideration the theorem 2.2.1 and the above definition we have the following inequalities:

$$\varphi_{S^1}(M) \leq \varphi(M) \leq m + 1.$$

Other results concerning the φ-category of a pairs of surfaces are given in the paper [7]. Families of pairs of differential manifolds having infinite φ-category are emphasized in the papers [77] and [78].

Proposition 2.2.4 *[10] Let M a connected smooth manifold having $Hom\left(\pi(M), \mathbb{Z}\right) = 0$. The we have the propriety*

$$\varphi_{S^1}(M) = \varphi(M).$$

In conclusion, this relation takes places when the fundamental group of M is a torsion group.

2.3 Morse theory. Circular Morse theory

Morse theory offers ways of exploiting the topology of a manifold by analyzing differential functions on a given manifold. In this manner we can extract information about handle decomposition of a manifold and the CW-structure associated to it.

Next, we will present fundamental notions from Morse theory, like 1-form, closed form, exact form, Morse form, circular Morse function, covering maps and lift of a function. Also here we presented examples of closed forms and exact forms. This sections follows the references [42], [43], [69], [72] şi [74] but also [25] which offers an algebraic perspective on Morse functions.

Theorem 2.3.1 *[44] Let* $\omega = \sum_{i=1}^{m} f_i dx_i$ *be an exact smooth form. This form is closed if and only if follows the integrability condition:*

$$\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}, \text{ pentru orice } i, j = 1, \cdots, n.$$

We define the set of **zeros of a 1-form** ω as

$$Z(\omega) = \{p \in M : \omega_p = 0\}.$$

Definition 2.3.1 *Let M be a smooth manifold on which we define the closed 1-form ω. ω is called **Morse form** if around any point p in M, it exist a neighborhood U and a Morse function $f : U \to \mathbb{R}$ for which we have $\omega_{|_U} = df$.*

Similarly to real Morse function theory we have the following concepts

1. A zero of a form ω is called **non-degenerate** if it is a non-degenerate critical point for any function $f : U \to \mathbb{R}$, that satisfies $\omega_{|_U} = df$.
2. **The index** of a non-degenerate zero $p \in Z(\omega)$ is the index of a critical point p of a smooth function $f : U \to \mathbb{R}$. The index is a natural number between 0 and $dim(M)$.
3. We denote by $\mathbf{S}(\omega)$ the set of all zeros of a form ω, to which we associate the cardinality denoted by $\mathbf{m}(\omega)$.

Circular Morse theory appears as a particular case of real Morse theory in studies over 1-forms done by S. Novikov in 1980. The importance of this field is emphasized by the big number of existing papers, giving to this concept a dynamical character and a practical applicability for studies over manifold fibrations on the unit circle or dynamical zeta functions. Among authors that contributed to the development of this field we mention S.Novikov, A. Pajitnov, A. Ranicki, M. Farber.

Definition 2.3.2 *It is called a **circular Morse function**, a smooth function, $f : M \to S^1$, that has only non-degenerate critical points.*

In the study of circular Morse theory we impose conditions on the type of critical points, and therefore, the following notation appeared naturally. The set $C_k(f)$, $k = 0, \ldots, n$, represents the critical set of index k, and $\mu_k(f)$ is the cardinality of this set.

So $C(f)$ contains the set of critical points of f, and its cardinality is given by:

$$\mu(f) = \mu_0(f) + \cdots \mu_n(f)$$

The real topological space $\mathbb{R}$, together with the universal covering map of the circle:

$$exp : \mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$$

defines the covering of the unit circle S^1.

Proposition 2.3.2 *Let X be a compact connected subset of $\mathbb{R}^p$, a smooth map $f : (X, x_0) \to (S^1, 1)$ and $t_0 \in \mathbb{R}$. Then it exists a unique continuous map $\widetilde{f} : (X, x_0) \to (\mathbb{R}, t_0)$ for which we have: $f = exp \circ \widetilde{f} = e^{2\pi i \widetilde{f}}$.*

The map $\widetilde{f}$ from the previous proposition is called **the lift of a map** f.

The next two sections offer an estimation about the minimum number of critical points for a circular function on products of manifolds, using the minimum number of critical points for circular functions, defined on each of the named manifolds. Similar computations were done for real functions in the original paper [8], having the purpose of establishing the same results for circular Morse functions.

As a starting point for this analysis we used Takens' inequality proven in his thesis [95], stating that:

$$\mathrm{cat}(M) \leq \varphi(M) \leq \dim(M) + 1,$$

where $\mathrm{cat}(M)$ is the *Lusternik-Schnirelmann* category, or shortly *LS-category*. We remember that the category of a space X represents the smallest natural number such that $X = \bigcup_{i=0}^{n} U_i$, where U_i are open and contractible sets in X.

2.4 Computations regarding circular $\varphi-$category of particular manifolds

2.4.1 Results concerning φ-category of products and direct sums

If $k, l, m_1, \ldots, m_k \geq 2$, are integers, then the following relations are true:

1. $\varphi_{S^1}(S^{m_1} \times \cdots \times S^{m_k}) = \varphi(S^{m_1} \times \cdots \times S^{m_k}) = k + 1$;
2. $\varphi_{S^1}(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = \varphi(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) \leq m_1 + m_2 + \cdots + m_k + 1$;
3. $\varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = \varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = 5$, where $L(r, s)$ is the lens space 3 of type (r,s);
4. $\varphi_{S^1}(\mathbb{RP}^k \times S^l) = \varphi(\mathbb{RP}^k \times S^l) \leq k + 2$.

The proof for relations

$$\varphi(S^{m_1} \times \cdots \times S^{m_k}) = k + 1$$
$$\varphi(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = 5$$

was done by C. Gavrilă [47] (Proposition 4.6, Example 4.7) and the estimation

$$\varphi(\mathbb{RP}^k \times S^l) \le k + 2$$

comes from [47] (Proposition 4.19). An immediate consequence of Proposition 2.2.4 is:

Corollary 2.4.1 *If $M_1^n, \ldots, M_r^n, n \ge 3$, are connected manifolds with fundamental torsion group, we have $\varphi_{S^1}(M_1\# \cdots \#M_r) = \varphi(M_1\# \cdots \#M_r)$. Particularly the following is true: $\varphi_{S^1}(r\mathbb{RP}^n) = \varphi(r\mathbb{RP}^n)$, where $r\mathbb{RP}^n$ represents the connected sum $\mathbb{RP}^n\# \cdots \#\mathbb{RP}^n$ of r copies of $\mathbb{RP}^n$.*

The following result is mentioned in the monograph [38, p. 221].

Lemma 2.4.1 *If M are N are closed manifolds, then the following inequalities occur*

$$\varphi(M\#N) \le \max\{\varphi(M), \varphi(N)\}.$$

Particularly we have : $\varphi(X\#X) \le \varphi(X)$ for any closed manifold X.

2.4.2 The submultiplicativity property for φ_{S^1} [65]

Let M, N be two manifolds and $f : M \to G$ and $g : N \to G$ two maps with values in the Lie group $(G, \cdot)$. We define the operation "$\odot$" for f and g, by:

$$f \odot g : M \times N \to G,\ (f \odot g)(x, y) = f(x)g(y).$$

Proposition 2.4.2 *[45] Let M, N be two smooth manifolds with $\dim(M) = m$, $\dim(N) = n$ and a Lie group $(G, \cdot)$ with dimension $\dim(G) \le min(m, n)$. For two smooth maps $A : M \to G$ and $B : N \to G$ we have the following inclusion of critical sets:*

$$C(A \odot B) \subseteq C(A) \times C(B).$$

Corollary 2.4.2 *For two manifolds M, N the following inequality is true:*

$$\varphi_{S^1}(M \times N) \le \varphi_{S^1}(M)\varphi_{S^1}(N).$$

Moreover, if $\chi(M), \chi(N) \ne 0$, then $\varphi_{S^1}(M \times N) \ge 1$.

Example 2.4.1 Let M a smooth n-dimensional manifold and $SO(n)$ and $Spin(n)$ special orthogonal group, respectively spin group of order n. If $n \geq 2$, then the following inequalities are true:

1. $\varphi_{S^1}(M \times S^n) \leq 2\varphi_{S^1}(M)$;
   If $\chi(M) \neq 0$, then $\varphi_{S^1}(M \times S^n) \geq 1$;
2. $\varphi_{S^1}(M \times SO(n)) \leq 2^{n-1}\varphi_{S^1}(M)$;
3. $\varphi_{S^1}(M \times Spin(n)) \leq 2^n\varphi_{S^1}(M)$;
4. If $n \geq 3$, $m \geq 3$ and $1 \leq k \leq n-1$, $1 \leq p \leq m-1$ then
$$\varphi_{S^1}(G_{k,n} \times M) \leq \binom{n+k}{k}\varphi_{S^1}(M),$$

where $G_{k,n}$ represents the Grassmann manifold of all k-dimensional subspaces of $\mathbb{R}^{n+k}$.

2.5 Circular Morse-Smale characteristic

In this section, we consider M a smooth n-dimensional manifold without boundary and $f : M \to \mathbb{R}$ a Morse function on M. Then:

$$\mu(f) = \mu_0(f) + \mu_1(f) + \cdots + \mu_n(f)$$

where $\mu_k(f)$ represents the number of critical points of index k.

Definition 2.5.1 *We denoted by $\gamma(M)$, the minimum number of possible critical points for all Morse functions. The number $\gamma(M)$ is called **real Morse-Smale characteristic** of a manifold M and so we have:*

$$\gamma(M) = min\left\{\mu(f), f : M \to \mathbb{R}\right\},$$

that corresponds to the case $N = \mathbb{R}$ mentioned in Section 2.1.

We emphasize the number

$$\gamma_i(M) = min\left\{\mu_i(f), f : M \to \mathbb{R}\right\}.$$

Considering the presented elements, it is obvious that for any compact manifold we have $\gamma_0(M) = \gamma_n(M) = 1$. Also, the following relation takes place:

$$\gamma(M) \geq \gamma_0(M) + \gamma_1(M) + \cdots + \gamma_n(M).$$

Moreover, for smooth non-compact manifolds without border, the Morse-Smale characteristic is 0.

Following the references [1], [2], [4], [9], [10] and [11] we present results that highlight the invariant property of the numbers $\gamma(M)$ şi $\gamma_i(M)$.

We consider the smooth manifolds M, N, the diffeomorphism ψ and smooth maps f and g such that the following diagram is commutative:

$$\begin{tikzcd} N \arrow[rr, "\psi"] \arrow[dr, "g"'] & & M \arrow[dl, "f"] \\ & \mathbb{R} & \end{tikzcd}$$

Proposition 2.5.1 *1. The critical set of f is equal to the image of critical set of g through the diffeomorphism ψ, namely:*

$$C(f) = \psi\left(C(g)\right)$$

2. The Morse indexes of critical points associated to the maps f and g throught the diffeomorphism ψ are equal.

Our next theorem highlights the invariant character of γ and γ_i.

Theorem 2.5.2 *Let M, N be two differential diffeomorphic manifolds. Then for any natural number $i \in \{0, 1, \cdots, n\}$ we have: $\gamma(M) = \gamma(N)$ and $\gamma_i(M) = \gamma_i(N)$.*

Paper [4] shows the following result for computing the Morse-Smale category.

Theorem 2.5.3 *Let M^m, N^n be two smooth manifolds and γ, γ_i circular Morse-Smale characteristics presented before. Then the following statements are true:*

1. $\gamma_i(M) = \gamma_{m-i}(M)$, *for all* $i \in \{0, 1, \cdots, m\}$;
2. $\gamma(M \times N) \leq \gamma(M)\gamma(N)$;
3. $\gamma_i\left(M \times N\right) \leq \sum_{j+k=i} \gamma_j(M)\gamma_k(N)$, *for all* $i \in \{0, 1, \cdots, m+n\}$.

Similarly we can define the circular Morse-Smale category corresponding to circular Morse functions on a smooth manifold M as the number

$$\gamma_{S^1}(M) = min\left\{\mu(f) : f : M \to S^1\right\}.$$

Proposition 2.5.4 *[11] If $\widetilde{M}$ is a k-covering of M, then the following inequality takes place*

$$\gamma_{S^1}\left(\widetilde{M}\right) \leq k \cdot \gamma_{S^1}(M).$$

An alternate form of Morse inequalities for circular Morse function is given in the paper [66]. An estimation of the number of non-degenerate critical points for a circular Morse function is obtained in [67].

2.6 Calculations of circular Morse-Smale category

2.6.1 The submultiplicativity property for γ_{S^1} [65]

In Pajitnov's book [74] we have the following description for a function f that lifts to a real-valued Morse function F on $\widetilde{M}$. Let M be a smooth closed manifold, $f : M \to S^1$ a circular Morse function and $p : \widetilde{M} \to M$ the infinite cyclic covering induced by f through the universal covering $exp : \mathbb{R} \to S^1$, where $\exp(t) = e^{2\pi i t}$. Therefore we have:

$$f \circ p = \exp \circ F. \tag{2.1}$$

Proposition 2.6.1 *Let f, g be two circular Morse functions that follow* (2.1). *Then $f \odot g$ is also a Morse function, and the third diagram which shows the lift of the function $f \odot g$ to $F + G$ is commutative.*

$$\begin{tikzcd} \widetilde{M} \arrow[r, "F"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "\exp"] \\ M \arrow[r, "f"'] & S^1 \end{tikzcd} \qquad \begin{tikzcd} \widetilde{N} \arrow[r, "G"] \arrow[d, "q"'] & \mathbb{R} \arrow[d, "\exp"] \\ N \arrow[r, "g"'] & S^1 \end{tikzcd} \qquad \begin{tikzcd} \widetilde{M} \times \widetilde{N} \arrow[r, "F+G"] \arrow[d, "p \times q"'] & \mathbb{R} \arrow[d, "\exp"] \\ M \times N \arrow[r, "f \odot g"'] & S^1 \end{tikzcd}$$

Corollary 2.6.1 *Let M, N be two manifolds of dimension m, respectively n. We have:*

$$\gamma_{S^1}(M \times N) \leq \gamma_{S^1}(M)\gamma_{S^1}(N).$$

Moreover, if $\chi(M), \chi(N) \neq 0$, then $\gamma_{S_1}(M \times N) \geq 1$.

Theorem 2.6.2 *[10] The circular Morse-Smale characteristic of a closed surface $\Sigma \neq \mathbb{RP}^2$ is given by $\gamma_{S^1}(\Sigma) = |\chi(\Sigma)|$, where $\chi(\Sigma)$ is the Euler-Poincaré characteristic of the surface Σ.*

Proposition 2.6.3 *[8] By $U(n)$ and $SU(n)$ we denoted the unitary group and special unitary group. We have the following results:*

1. $n \leq \varphi(U(n)) \leq \gamma(U(n)) \leq 2^n$;
2. $n-1 \leq \varphi_{S^1}(SU(n)) = \varphi(SU(n)) \leq \gamma(SU(n)) = \gamma_{S^1}(SU(n)) \leq 2^{n-1}$.

Remark 2.6.1 *The unitary group is diffeomorphic to $SU(n) \times S^1$. So*

$$0 = \varphi_{S^1}(U(n)) < n \leq \varphi(U(n)).$$

Considering the notations and computations given, we will present relations and inequalities for special categories of manifolds and groups.

2.6.2 Particular categories of Grassmann manifolds

Proposition 2.6.4 *[10] Let M be a smooth connected manifold. If M has the lifting property $Hom(\pi(M);\mathbb{Z}) = 0$, then $\varphi_{S^1}(M) = \varphi(M)$ and $\gamma_{S^1}(M) = \gamma(M)$. Also $\varphi_{S^1}(M) = \varphi(M)$ and $\gamma_{S^1}(M) = \gamma(M)$, when the fundamental group of M is a torsion group.*

Proposition 2.6.5 *If $n \geq 2$ is an integer, then*

1. $\varphi_{S^1}(S^n) = \varphi(S^n) = \gamma_{S^1}(S^n) = \gamma(S^n) = 2$;
2. $$\begin{aligned} \varphi_{S^1}(\mathbb{RP}^n) &= \varphi(\mathbb{RP}^n) = \gamma_{S^1}(\mathbb{RP}^n) = \gamma(\mathbb{RP}^n) = \mathrm{cat}(\mathbb{RP}^n) = \\ \varphi_{S^1}(\mathbb{CP}^n) &= \varphi(\mathbb{CP}^n) = \gamma_{S^1}(\mathbb{CP}^n) = \gamma(\mathbb{CP}^n) = \mathrm{cat}(\mathbb{CP}^n) = n+1, \end{aligned}$$

where $\mathrm{cat}(\mathbb{CP}^n)$ represents Lusternik-Schnirelmann category of the complex projective space $\mathbb{CP}^n$.

To be observed that the equalities $\varphi_{S^1}(\mathbb{RP}^n) = \varphi(\mathbb{RP}^n) = \mathrm{cat}(\mathbb{RP}^n) = n+1$ are proven similarly in [10] by using the fundamental group structure of $\mathbb{RP}^n$, together with the Morse function

$$F_n : \mathbb{RP}^n \longrightarrow \mathbb{R},\ F_n([x_1, \ldots, x_{n+1}]) = \frac{x_1^2 + 2x_2^2 + \cdots + nx_n^2 + (n+1)x_{n+1}^2}{x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2},$$

whose critical points are found in the set $C(F_n) = \{[1, 0, \ldots, 0], [0, 1, \ldots, 0], \ldots, [0, 0, \ldots, 1]\}$, and the inequalities mentioned before $\varphi(\mathbb{RP}^n) \geq \mathrm{cat}(\mathbb{RP}^n) \geq n+1$.

Proposition 2.6.6 *If $n \geq 3$ and $1 \leq k \leq n-1$, then*

$$\varphi_{S^1}(G_{k,n}) = \varphi(G_{k,n}) \leq \gamma(G_{k,n}) = \gamma_{S^1}(G_{k,n}) \leq \begin{pmatrix} n+k \\ k \end{pmatrix},$$

where $G_{k,n}$ represents the Grassmann manifold of all k-dimensional subspaces of the space $\mathbb{R}^{n+k}$.

Corollary 2.6.2 *If $n = 1$ or $k = 1$ or ($n = 2$ and $k = 2p-1$ for an arbitrary p) or ($n = 2p-1$ and $k = 2$), then*

$$n\,k \leq \varphi_{S^1}(G_{k,n}) = \varphi(G_{k,n}) \leq \gamma_{S^1}(G_{k,n}) = \gamma(G_{k,n}) \leq \begin{pmatrix} n+k \\ k \end{pmatrix}.$$

2.6.3 Particular categories of Lie groups

Proposition 2.6.7 *If $n \geq 3$, then the following relations are true:*

$$\varphi_{S^1}(SO(n)) = \varphi(SO(n)) \leq \gamma(SO(n)) = \gamma_{S^1}(SO(n)) \leq 2^{n-1}.$$

Corollary 2.6.3 $9 \leq \varphi(SO(5)) = \varphi_{S^1}(SO(5)) \leq \gamma_{S^1}(SO(5)) = \gamma(SO(5)) \leq 16.$

Proposition 2.6.8 *The following inequalities are true:*

1. $n \leq \varphi(U(n)) \leq \gamma(U(n)) \leq 2^n$;
2. $n-1 \leq \varphi_{S^1}(SU(n)) = \varphi(SU(n)) \leq \gamma(SU(n)) = \gamma_{S^1}(SU(n)) \leq 2^{n-1}$.

Remark 2.6.2 *The inequality $\varphi(U(n)) \leq \varphi_{S^1}(U(n))$, can be strict because the unitary group is diffeomorphic but not isomorphic with the product $SU(n) \times S^1$ [69, p. 103] and Proposition 2.6.4 cannot be applied, because the fundamental group for $U(n)$ is $\mathbb{Z}$.*

3. The Reeb graph of an application

The Reeb graph has its origins in the evolution of level sets of a real-valued function, defined on a differential manifold. It was introduced by the french mathematicians Georges Henri Reeb (12.XI.1920 - 6.XI.1993) in his paper [85] as a mathematical tool used while studying real-valued Morse functions. Afterwards, it gain a bigger utility, being used in fields like geometrical design assisted by mathematical software, computer graphics, computational geometry and geometric thermodynamics.

Together with the evolution of technology a necessity of detection and processing images has risen and the Reeb graph answers to this needs by storing only the necessary information, reducing in this way the processing time of algorithms.

3.1 Basic notions and examples

Definition 3.1.1 *Let X be a topological space and $f : X \to \mathbb{R}$ a real valued function. We define on X, the equivalence relation "$\sim$": $x \sim y$ if and only if x and are placed in the same connected component of the level set $f^{-1}(a)$, $a \in \mathbb{R}$.*

Notations: Let M be a smooth manifold and $f : M \to \mathbb{R}$ a real smooth function. We consider the following:

$M_c = f^{-1}(c)$, the set of level c of function f and M_c^s a connected component of this set;

$M^{(c,c')} = \{x \in M \mid c < f(x) < c'\}$, a section defined on the interval (c, c');

$M^c = \{x \in M \mid f(x) \leq c\}$, the set of level c. Obviously the following relation takes place

$$M = \bigcup_{-\infty<c<\infty} M_c.$$

Definition 3.1.2 *Let X be a topological space, $f : X \to \mathbb{R}$ a real valued function and "$\sim$" the equivalence relation mentioned in 3.1.1. It is called the **Reeb graph** of map f, the space $X/\sim$ together with the quotient topology.*

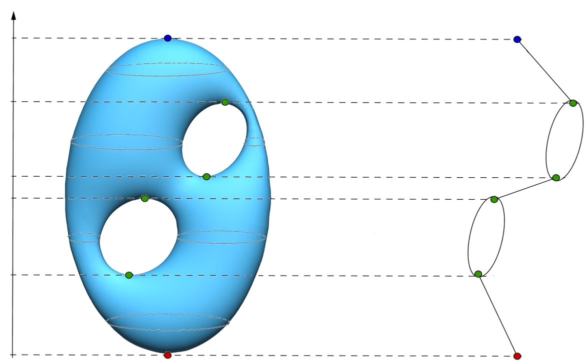
Notation: In the references used, the Reeb graph of a f has been denoted by $\mathcal{R}(f)$ or **(X,f)**. We will use the notation $\mathcal{R}(f)$.

Let $\pi : M \to \mathcal{R}(f)$ be the canonical projection on the manifold M of the Reeb graph of a map $f : M \to \mathbb{R}$.

3.1.1 Exemple

Beside the cases of a Reeb graph defined by the height function on the sphere S^2 and torus T^2, we also present the following example:

3. The Reeb graph of a height function on the orientable surface of genus 2:



3.2 The Reeb graph category

The Reeb graphs form a category denoted by **Reeb**, where the **objects** are finite graphs with real valued functions that are strictly monotonic on edges, and the **morphisms** are functions preserving maps between the underlying spaces. More precisely [19], an object of the category **Reeb** is a finite graph, seen as a topological space X, together with a real valued function that is strictly monotonic on its edges. In what follows, we use the notation $\mathcal{R}(f)$ for such an object. A morphism between $\mathcal{R}(f)$ and $\mathcal{R}(g)$, is a function preserving maps $\varphi : X \to Y$, meaning we have $f = g \circ \varphi$.

The vertices of a Reeb graph represent classess of critical points of f. Moreover, f induces the application $\widetilde{f} : \mathcal{R}(f) \to \mathbb{R}$ such that $f = \widetilde{f} \circ \pi$. In the paper [90], the function $\widetilde{f}$ induces and orientation on the edges of a Reeb graph. For a finite orientable graph we introduce the notions of **inner degree**, respectively **outer degree** denoted by $deg_{in}(v)$, $deg_{ext}(v)$, representing the number of edges that go in, respectively come out of a vertex v. Therefore, **the degree of a vertex v** is defined by:

$$deg(v) = deg_{in}(v) + deg_{ext}(v)$$

Definition 3.2.1 *[90] We say the graph Γ has a **good orientation** if is equiped with an orientation given by the continuous function $g : \Gamma \to \mathbb{R}$, such that g is strictly monotonic on edges and has extrema only in edges of degree 1.*

3.3 Proprieties and representations of the Reeb graph on an application

In the following section, we will present important result for a Reeb graph, following papers [19] and [58]. The way of defining connected components and types of paths, were also used in the presentation of the algorithm from paper [63].

Definition 3.3.1 *Let c be a critical value of f. We say that a connected component M_c^s is **essential** if contains at least one critical point, meaning*

$$M_c^s \cap C(f) \neq \varnothing.$$

This component will be denoted by M_c^{es}.

A graph is called **connected** if any two vertices are connected by a path. A connected graph without cycles will be called from now on a **tree**.

Proposition 3.3.1 *Let M be a smooth, compact manifold and $f : M \to \mathbb{R}$ a smooth function with isolated critical points. If c is a critical value, we consider the set:*

$$A = M_c^{es} \cap C(f) = \{a_1, a_2, \ldots, a_n\}$$

of critical points in the essential component associated to c. Then we have the following properties:

1. *Any path-connected component associated to a level set is a connected component.*
2. *For all a_i, a_j from the closure of the set $(M_c^{es} \setminus A)$, where $a_i \neq a_j$ and any $i, j \in \{1, 2, \ldots, n\}$ can be connected by a path $\gamma : I \to M_c^{es}$ such that $\gamma(0) = a_i, \gamma(1) = a_j$ where γ is an homeomorphism.*
3. *It exists $K \subset M_c^{es}$ a closed subspace homeomorphic to a tree for which A is the set contained the vertices of the tree.*

Corollary 3.3.1 *For all a_i, a_j from A critical points, it exist an arc (regarded as the composition of arcs that was described before) that connects them.*

3.3.1 Types of connected components

The connected component C from $M^{(c,c')}$ is of type:

(I) or **vertex component** if contains a critical point;

(II) in all other cases;
(IIa) (or**edge component**) if $f\left(cl(C)\right) \cap C(f) = \{c, c'\}$.
The above relation is equivalent to:
The type II component is of type IIa (or edge component) if we have the following relations:

$$M_c^{es} \cap cl(C) \neq \varnothing \text{ and } M_{c'}^{es} \cap cl(C) \neq \varnothing.$$

Proposition 3.3.2 *For an edge-component C we have the diffeomorphism:*

$$(M_a \cap C) \times (c, c') \sim M^{(c,c')} \cap C$$

Lemma 3.3.3 *A type II component intersect only one other component from M_q, for $q \in (c, c')$.*

Corollary 3.3.2 *$M^{(c,c')}$ regarded as a manifold, has a finite number of connected components.*

Proposition 3.3.4 *It exist a one to one correspondence between type (IIa) components of the manifold M and the edges of the graph $\mathcal{R}(f)$.*

Let "$\sim_{es}$" be the equivalence relation on M defined as:

$$x \sim_{es} y \in M \text{ if and only if } x, y \in M_c^{es},\ c \in C(f).$$

We denote by $M_{es} = M/_{\sim_{es}}$ the quotient space of the mentioned equivalence relation, by $\pi_{es} : M \to M_{es}$ canonical projection and $f_{es} : M_{es} \to \mathbb{R}$ the function defined by $f_{es}\left([x]\right) = f(x)$. Therefore we have $f_{es} \circ \pi_{es} = f$.

Lemma 3.3.5 *The Reeb graph for an essential function f_{es}, denoted by $\mathcal{R}_{es}(f)$ is the same as the Reeb graph $\mathcal{R}(f)$ of the function f.*

3.3.2 Types of families of paths

Definition 3.3.2 ***1)*** *Let $c, c' \in V_{cr}(f)$ and γ_1, γ_2 two paths with*

$$f\left(\gamma_1(0)\right) = f\left(\gamma_2(0)\right) = c',\ f\left(\gamma_1(1)\right) = f\left(\gamma_2(1)\right) = c.$$

A map $H : I \times I \to M$ is called ***homotopy*** *relative to essential components γ_1 and γ_2 if:*

(i) H is an homotopy between γ_1 and γ_2;

(ii) $H(0,s) \in M_{c'}^{es}$ and $H(1,s) \in M_c^{es}$, for all $s \in I$.

A path $\epsilon : I \to M$ is a ***vertex-path*** *if it is contained in an essential component of a critical level.*

A path γ is ***contractible relative to an essential component*** *if it exists a homotopy relative to a connected component from γ to a vertex-path.*

2) *For a type IIa component C of $M^{(c,c')}$, we say thatγ is an* ***edge-path*** *if:*

(i) $\gamma(0) \in M_{c'}^{es}$ and $\gamma(1) \in M_c^{es}$;

(ii) $\gamma((0,1)) \subset C$

The edge-path is decreasing if $f(\gamma(t)) < f(\gamma(t'))$, for $t > t'$.

Definition 3.3.3 *For a type IIa component, we say that $\gamma : I \to M$ is an* ***extended edge-path*** *of C if:*

1) $\gamma(0) \in M_{c'}^{es}$ and $\gamma(1) \in M_c^{es}$;

2) it exists $t_0, t_1 \in I, t_0 < t_1$ such that:

- if $f(\gamma(t)) = c' \Rightarrow t \leq t_0$;

- if $f(\gamma(t)) = c \Rightarrow t \geq t_1$.

Moreover, we have $\gamma(t) \cap M_d^{es} = \varnothing$ for $d \neq c, c'$.

Theorem 3.3.6 *It exists a one to one correspondence between homotopy classes relative to the essential components of edge-paths and the edges of the Reeb graph $\mathcal{R}(f)$. Also, every homotopy class described, contains a decreasing path that connects two critical points.*

3.3.3 The Reeb graph from graph theory perspective

For emphasizing the graph structure of a Reeb graph we use the following two sets:

$\mathbf{V(f)}$ represents the set of homotopy classes relative to essential component of vertex-paths;

$\mathbf{E(f)}$ represents the set of homotopy classes relative to essential component of extend edge-paths.

In this way we obtain the finite graph $\Gamma(f) = \{V(f), E(f)\}$.

Theorem 3.3.7 *Between $\Gamma(f)$ and the Reeb graph $\mathcal{R}(f)$ of f it exists a simplicial homeomorphism.*

Proposition 3.3.8 *The graph $\Gamma(f) \subset M$ is homotopy equivalent to the graph $\mathcal{R}(f)$. So we have the relation:*

$$\beta_1(\Gamma) = |E| - |V| + 1.$$

In the following subsection, we pursue the reasoning provided in the paper [71] that allows us to present realization theorems for the Reeb graph. We will point out the notion of Reeb number, maximization of number of loops in a Reeb graph and a way of bounding the degree of vertices associated to the graph.

3.3.4 Applications. Interpretations [58], [71]

The paper [58] published in 2014 presents the following situation: For a real-valued C^1 function on a smooth manifold M, that has isolated critical points, the following particularization for the domain can be made:

1) If M is the n-dimensional unit sphere S^n with $n \geq 2$ or the real projective space or the n-dimensional complex space $\mathbb{RP}^n$, respectively $\mathbb{CP}^n$, then $\mathcal{R}(f)$ is a tree.

2) If M is the n-dimensional torus $\mathbb{T}^n$, then the Reeb graph is a tree or homotopy equivalent to a circle.

3) If M is an orientable surface Σ_g of genus g, then the Reeb graph contains at most $2g$ cycles.

4) If M is an non-orientable surface M_g of genus g, then the Reeb graph contains at most g cycles.

Lemma 3.3.9 *Let $[p]$ a point on an edge of the Reeb graph $\mathcal{R}(f)$. If $\mathcal{R}(f) \setminus \{[p]\}$ is path-connected then $M \setminus \pi^{-1}\{[p]\}$ is also path-connected.*

Corollary 3.3.3 *Let $[p_1], \cdots, [p_r]$ points on edges of the Reeb graph $\mathcal{R}(f)$. If $\mathcal{R}(f) \setminus \{[p_1], \cdots, [p_r]\}$ is connected, then $M \setminus \pi^{-1}\{[p_1], \cdots, [p_r]\}$ is also connected.*

Theorem 3.3.10 *If Σ_g is an orientable surface of genus g, then the number of cycles in $\mathcal{R}(f)$ is smaller or equal then g.*

Let $f : M \to \mathbb{R}$ be a C^1 function on a closed smooth manifold. If f has exactly three critical points, then $\mathcal{R}(f)$ is a tree with two edges.

Definition 3.3.4 *It is called the **Reeb number** $R(M)$ of a manifold, the number of loops of the Reeb graph relatively to all real valued functions on M with finite number of critical points. We have the relation*

$$R(M) = \max\{\beta_1(R(f)) : f : M \to \mathbb{R} \textit{has finite number of critical points}\}.$$

Lemma 3.3.11 *[36] Let $f : \Sigma \to \mathbb{R}$ be a simple Morse function on the closed surface Σ.*
If $\Sigma = \Sigma_g$, then $\beta_1(\mathcal{R}(f)) = g$.
If $\Sigma = S_g$, then $\beta_1(\mathcal{R}(f)) \leq \left[\frac{g}{2}\right]$, where $[x]$ is the floor of x.

Proposition 3.3.12 *Let $f : \Sigma \to \mathbb{R}$ be a simple Morse function and Σ a closed surface. To the critical point p, we associate the vertex v from the Reeb graph $\mathcal{R}(f)$. Then the following proprieties are true:*
(a) ind(p) = 0 or 2 if and only if deg(v) =1;
(b) ind(p) = 1 if and only if deg(v) = $\begin{cases} 3, \text{if } \Sigma \text{ is orientable} \\ 2 \text{ or } 3, \text{if } \Sigma \text{ is non-orientable} \end{cases}$

The next three results contained in [71] offer results over the way in which the Reeb graph of a simple Morse function maximizes the number of loops of the graph.

Lemma 3.3.13 *Let M be a smooth closed manifold and $f : M \to \mathbb{R}$ a function with finite number of critical points on M. Then it exist a simple Morse function $g : M \to \mathbb{R}$ such that*

$$\beta_1(\mathcal{R}(g)) \geq \beta_1(\mathcal{R}(f))$$

We mention that for a Morse function, the simple Morse function can be obtained without changing the critical points of f or their indexes, and g is different from f only in the neighborhood of these points.

Corollary 3.3.4 *The following relation takes place*
$\mathcal{R}(M) = \max\{\beta_1(\mathcal{R}(f)) : f : M \to \mathbb{R} \text{ simple Morse function}\}$.

Corollary 3.3.5 *If a surface Σ has the Euler characteristic $\chi(\Sigma) = 2 - k$, then*

$$\mathcal{R}(\Sigma) = \frac{k}{2}.$$

3.3.5 Realization theorems of a Reeb graph using Morse functions

The foundation for realization theorems for a Reeb graph using a Morse function has been laid by S.Sharko in his paper [90], using construction techniques given by F. Takens in [95]. Next, we will follow the steps from the paper [71] for establishing the conditions under which the Reeb graph is isomorphic to a graphΓ.

Theorem 3.3.14 *Let Γ be a finite graph equipped with a good orientation. Then it exists a n-dimensională closed manifold M, $(n \geq 2)$ and a Morse function $f : M \to \mathbb{R}$ such that the Reeb graph $\mathcal{R}(f)$ is isomorphic to Γ.*

We consider the case of $n = 2$:

Proposition 3.3.15 *Let Γ be a graph equipped with a good orientation and $g = \beta_1(\Gamma)$. Then it exists a closed surface Σ and a function $f : \Sigma \to \mathbb{R}$ having finite number of critical points, such that the Reeb graph of f is isomorphic to Γ. The surface Σ is considered orientable of genus g or non-orientable of genus $2g$. If the graph Γ is a tree then the surface is diffeomorphic to S^2.*

The following theorem describes the particular case of Γ_0, meaning the graph with two vertices and one edge. This graph is the only graph that does not appear on another surface except the sphere. Moreover, we present a necessary and sufficient condition under which the Reeb graph is isomorphic to Γ.

Theorem 3.3.16 *Let $\Gamma \neq \Gamma_0$ be a finite graph with good orientation and Σ a closed surface. Then it exist a function $f : \Sigma \to \mathbb{R}$ with finite number of critical points such that its Reeb graph $\mathcal{R}(f)$ is isomorphic to Γ if and only if $\beta_1(\Gamma) \leq \mathcal{R}(\Sigma)$. If $\Gamma = \Gamma_0$ then this graph its realized only for $\Sigma = S^2$.*

Theorem 3.3.17 *Let $\Gamma \neq \Gamma_0$ be a finite graph with good orientation Δ_2 the number of vertices of degree 2 of Γ and let Σ a closed surface of genus g (orientable or non-orientable). Then it exists a Morse function $f : \Sigma \to \mathbb{R}$ such that its Reeb graph $\mathcal{R}(f)$ is isomorphic to Γ if and only if*

(i) $g \geq \beta_1(\Gamma) + \Delta_2$, when Σ is orientable;

(ii) $g \geq 2\beta_1(\Gamma) + \Delta_2$, when Σ is non-orientable.

Proposition 3.3.18 *Let Σ a closed surface and Γ an orientable graph without loops such that $\beta_1(\Gamma) \leq \mathcal{R}(\Sigma)$. Then it exists a function $f : \Sigma \to \mathbb{R}$ with a finite number of critical values such that f is an homeomorphism that keeps the orientation on Γ. If Γ has a vertex w for which $deg_{in}(w)$ and $deg_{out}(w)$ are different from zero, then $\mathcal{R}(f)$ and Γ are isomorphic.*

The Reeb graph associated to the function f represents an useful tool, especially in the case of surfaces, when their reconstruction is made starting from the Reeb graph [70]. Also, one of the most important qualities of a Reeb graph, is the fact that allows the creations of a structure that incorporates information about the shape and geometrical representation and also its topological interpretation.

3.4 Distances between Reeb graphs

Research for Reeb graphs has known a powerful development that led to the necessity of introducing metrics between the objects considered. Practical applications that use Reeb graphs are imposing the analysis of differences between existing graphs. Also, the actual collection of data, prone to errors, yields questions about the stability of this graph. Modern approaches in literature are based on two types of reasoning: developing heuristics for improving graphs resistant to perturbations and a more theoretical approach by developing distances and pseudo-distances.

The following distances are highlighted in literature: interleaving distance, functional distortion, bottleneck and edit distance. In this section the references used are: [18], [19], [28] şi [41].

3.4.1 Interleaving distance

Let $\mathcal{R}(f)$ be the Reeb graph of a function and we consider

$$\mathcal{R}(f)_\epsilon := \mathcal{R}(f) \times [-\epsilon, \epsilon].$$

We define the $\epsilon-$**smoothing** of $\mathcal{R}(f)$ as the Reeb graph of perturbed function

$$f_\epsilon : \mathcal{R}(f)_\epsilon \to \mathbb{R},\ (x,t) \mapsto f(x) + t.$$

Therefore $\epsilon-$smoothing is given by the quotient space $\mathcal{R}(f)_\epsilon/_\sim$ and is denoted by $\mathcal{U}_\epsilon(\mathcal{R}(f))$.

The following relation is ture:

$$\mathcal{U}_\epsilon\left(\mathcal{U}_\epsilon(\mathcal{R}(f))\right) = \mathcal{U}_{2\epsilon}(\mathcal{R}(f)).$$

Definition 3.4.1 *Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ two real functions. It is called* ϵ***-interleaving*** *of the graphs $\mathcal{R}(f)$, $\mathcal{R}(g)$, a pair of maps that preserve applications*

$$\varphi : X \to \mathcal{U}_\epsilon\left(\mathcal{R}(f)\right) \text{ şi } \psi : Y \to \mathcal{U}_\epsilon\left(\mathcal{R}(g)\right)$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{R}(f) \arrow[r, "i"] \arrow[d, "\varphi"] & U_\epsilon(\mathcal{R}(f)) \arrow[d, "\varphi_\epsilon"] \\
U_\epsilon(\mathcal{R}(g)) \arrow[r, "i_\epsilon"] & U_{2\epsilon}(\mathcal{R}(g))
\end{tikzcd}$$

where:
$i : \mathcal{R}(f) \to \mathcal{U}_\epsilon\left(\mathcal{R}(f)\right),\ x \mapsto [x, 0]$
$\varphi_\epsilon : \mathcal{U}_\epsilon\left(\mathcal{R}(f)\right) \to \mathcal{U}_{2\epsilon}\left(\mathcal{R}(g)\right),\ [x, t] \mapsto [\varphi(x), t]$
In a similar way we can define the concepts for ψ.

Definition 3.4.2 *The maps* $\varphi : \mathcal{R}(f) \to \mathcal{U}_\epsilon\left(\mathcal{R}(g)\right)$, $\psi : \mathcal{R}(g) \to \mathcal{U}_\epsilon\left(\mathcal{R}(f)\right)$ *are called* ϵ***-interleaving****, if* φ *and* ψ *preserve applications and the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{R}(f) \arrow[r, "i"] \arrow[dr, "\varphi"] & U_\epsilon(\mathcal{R}(f)) \arrow[r, "i_\epsilon"] \arrow[dr, "\varphi_\epsilon"] & U_{2\epsilon}(\mathcal{R}(f)) \\
\mathcal{R}(g) \arrow[r, "i"] \arrow[ur, dashed, "\psi"] & U_\epsilon(\mathcal{R}(g)) \arrow[r, "i_\epsilon"] \arrow[ur, dashed, "\psi_\epsilon"] & U_{2\epsilon}(\mathcal{R}(g))
\end{tikzcd}$$

Definition 3.4.3 *Given two Reeb graphs* $\mathcal{R}(f), \mathcal{R}(g)$, *we define the* ***interleaving distance*** *as*

$$d_I\left(\mathcal{R}(f), \mathcal{R}(g)\right) = inf\left\{\epsilon : \exists\ o\ \epsilon - interleaving\ between\ \mathcal{R}(f), \mathcal{R}(g)\right\} \tag{3.1}$$

3.4.2 Functional distorsion distance

Let π be a path between x and y in $\mathcal{R}(f) \in$ **Reeb**. It is called the **the height of path** π the number

$$h(\pi) = \underset{x\in\pi}{max} f(x) - \underset{x\in\pi}{min} f(x) \tag{3.2}$$

With this number we define the distance:

$$d_f(x,y) = \underset{\pi:x\rightsquigarrow y}{min}\ h(\pi) \tag{3.3}$$

.

Definition 3.4.4 *Let $\mathcal{R}(f), \mathcal{R}(f)$ be two Reeb graphs and $\Phi : X \to Y$, $\Psi : Y \to X$ two functions. We consider*

$$C(\Phi, \Psi) = \{(x,y) \in X \times Y \mid \Phi(x) = y\ sau\ \Psi(y) = x\} \tag{3.4}$$

and

$$D(\Phi, \Psi) = \underset{(x,y),(x',y')\in C(\Phi,\Psi)}{sup} \frac{1}{2} \left| d_f(x,x') - d_g(y,y') \right|. \tag{3.5}$$

We introduce the **functional distortion distance** by

$$d_{FD} = \underset{\Phi,\Psi}{inf}\ max \left\{ D(\Phi,\Psi), \|f - g\circ\Phi\|_\infty, \|g - f\circ\Psi\|_\infty \right\}. \tag{3.6}$$

The following theorem presents the relation between the metric interleaving distance and functional distortion distance and is a very important result for the study of convergence proprieties.

Theorem 3.4.1 *[19] Let $f : M^m \to \mathbb{R}$, $g : N^n \to \mathbb{R}$ be two smooth functions and their Reeb graphs $\mathcal{R}(f)$, $\mathcal{R}(g)$. The following inequalities are true:*

$$d_I(f,g) \le d_{FD}(f,g) \le 3d_I(f,g).$$

3.4.3 Bottleneck distance

Bottleneck distance represents globally a pseudo-metric, but locally, in a small enough neighborhood, is as efficient as any metric for differentiating between two Reeb graphs. For presenting this type of distance it is necessary to introduce the concepts: Morse type function, extended filtration, extended persistence diagram and the cost of a diagram. In order to present the elements stated above we used the reference [28].

Definition 3.4.5 *Let X be a topological space and $f : X \to \mathbb{R}$ a continuous function. The function f is called* ***Morse type function*** *if we have the following proprieties:*

a) it exists a finite set of critical values $C(f)$, such that for any open interval (a_i, a_{i+1}) it exists a compact and connected space Y_i and an homeomorphism $\mu_i : Y_i \times (a_i, a_{i+1}) \to X^{(a_i,a_{i+1})}$ such that $f\big|_{X^{(a_i,a_{i+1})}} = \pi_2 \circ \mu_i^{-1}$, for any natural number i;

b) μ_i is extended to a continuous map $\overline{\mu_i} : Y_i \times [a_i, a_{i+1}] \to X^{[a_i,a_{i+1}]}$ considering that for particular cases of $i = 0$ and $i = n$: μ_0 is extended to $\overline{\mu_0} : Y_0 \times (-\infty, a_0] \to X^{(-\infty,a_0]}$, respectively μ_n is extended to $\overline{\mu_n} : Y_n \times [a_n, \infty) \to X^{[a_n,\infty)}$;

c) every level set $f^{-1}(x)$ has a finitely generated homology.

Let f be the function described before. The family of level sets of f, $\left\{X^{(-\infty,a]}\right\}_{a\in\mathbb{R}}$ defines a **filtration**, meaning we have $X^{(-\infty,a]} \subseteq X^{(-\infty,b]}$, for all $a \leq b$.

We introduce the sets $\mathbb{R}^{op} = \{\widetilde{x} : x \in \mathbb{R}\}$ equipped with the order $\widetilde{x} \leq \widetilde{y} \Leftrightarrow x \geq y$ and $\mathbb{R}_{Ext} = \mathbb{R} \cup \{+\infty\} \cup \mathbb{R}^{op}$.

We define the **extended filtration** of f relative to $\mathbb{R}_{Ext}$ by: $F_a = X^{(-\infty,a]}$, for $a \in \mathbb{R}$, $F_\infty = X \equiv (X, \varnothing)$ and $F_{\widetilde{a}} = \left(X, X^{[\widetilde{a},\infty)}\right)$, for $\widetilde{a} \in \mathbb{R}^{op}$. F_a is called the ordinary part of this filtration, and $F_{\widetilde{a}}$ the relative part. Applying the homology functor H_* on the filtration, we obtain a module of extended persistence . For Morse type functions this module is decomposed as direct sum of semi-open interval modules. We define **extended persistence diagram** as the representation of any interbal from this module as a point in the extended plane, having coordinated given by the interval's borders.

We denote the extended persistence diagram by $\mathbf{Dg(f)}$ and we separate the filtration parts as it follows:

a) the point $p = (x, y)$ is called **ordinary** point if $x, y \in \mathbb{R}$ and p is above the diagonal $\Delta = \{(x, x) : x \in \mathbb{R}\}$;

b) the point $p = (x, y)$ is called **relative** point if $x, y \in \mathbb{R}^{op}$ and p is under the diagonal Δ;

c) the point $p = (x, y)$ is called **extended** if $x \in \mathbb{R}$ and $y \in \mathbb{R}^{op}$ can be place anywhere in space, including on the diagonal Δ. We denote by $Ext^+(f)$, the extended points on Δ, and by $Ext^-(f)$ the extended points that are not on Δ.

The persistence diagram can be written as:

$$Dg(f) = Ord(f) \cup Rel(f) \cup Ext^+(f) \cup Ext^-(f).$$

Let D, D' two persistence diagrams. We say the subset $\Gamma \subseteq D \times D'$ is a **partial match** if for any point $p \in D$ it exists at most one point $p' \in D'$ such that $(p, p') \in \Gamma$. Moreover, Γ identifies only points of the same type and same homological dimension.

The **cost** of Γ is defined by: $cost(\Gamma) = max\left\{max\delta_D(p), max\delta_{D'}(p')\right\}$, where $\delta_D(p) = \|p - p'\|_\infty$ if p is associated to p' from D' and $\delta_D(p) = d_\infty(p, \Delta)$, if p has no correspondent in D'.

Definition 3.4.6 *We name **bottleneck distance** between two persistence diagrams D and D', the number:*

$$d_B(D, D') = inf_\Gamma cost(\Gamma)$$

where Γ goes through all partial matchings between D and D'.

Theorem 3.4.2 (Stability theorem of bottleneck distance) *[35]*

For any Morse type function $f, g : X \to \mathbb{R}$ we have

$$d_B\left(Dg(f), Dg(g)\right) \leq \|f - g\|_\infty$$

Similarly, we define the bottleneck distance between the Reeb graph $\mathcal{R}(f)$ and $\mathcal{R}(g)$ by

$$d_B\left(\mathcal{R}(f), \mathcal{R}(g)\right) = d_B\left(Dg(f), Dg(g)\right).$$

Theorem 3.4.3 *[18] Let $\mathcal{R}(f)$, $\mathcal{R}(g)$ the Reeb graphs corresponding to $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. We have the following inequalities:*

1. $d_B\left(Dg_0(f), Dg_0(g)\right) \leq d_{FD}(f, g)$;
2. $d_B\left(ExDg_1(f), ExDg_1(g)\right) \leq 3d_{FD}(f, g)$.

where $ExDg0$ is the persistence diagram of order 0 and $ExDg1$ represents the first persistence diagram associated to a map.

Taking into consideration the result from 3.4.1 and 3.4.3 we can state the following corollary:

Corollary 3.4.1 *The following inequalities are true:*

1. $d_B\left(Dg_0(f), Dg_0(g)\right) \leq 3d_I(f, g)$;
2. $d_B\left(ExDg_1(f), ExDg_1(g)\right) \leq 9d_I(f, g)$.

3.4.4 The edit distance

The edit distance was introduce for the first time in the paper [86] and represents a useful tool for measuring differences between two graphs. Some practical applications of this distance are pattern recognition, handwriting recognition, fingerprint identification in databases and machine learning. The edit distance is a combinatorial distance that offers the best estimation to this day for the differences between two graphs.

For presenting the next section we used the references [41], [46] and [86].

The edit distance is introduced on a graph of which vertices are labeled using the correspondence $l_f : V(\Gamma_f) \to \mathbb{R}$, where l_f is the restriction of f to the critical set $C(f)$. This type of graph is denoted by $(\mathcal{R}(f), l_f)$ and for any vertex of degree 3 we have at least two adjacent vertices v_1 şi v_2 that follow the relation:

$$l_f(v_1) < l_f(v) < l_f(v_2).$$

Definition 3.4.7 *We say that two labeled graphs $(\mathcal{R}(f), l_f)$ and $(\mathcal{R}(g), l_g)$ are isomorphic if it exist an isomorphism $\varphi : C(f) \to C(g)$ that preserves the label of vertices and edges.*

According to [41] we have for types of elementary deformations illustrated as follows:

a) **Deformation B** (birth) that introduces new vertices between two existing vertices.

$$c(T) = \frac{|l_g(u_1) - l_g(u_2)|}{2}$$

b) **Deformation D** (death) that deletes vertices between two existing vertices

$$c(T) = \frac{|l_f(u_1) - l_f(u_2)|}{2}$$

c) **Deformation R** (relabeling) that changes the order of two existing vertices.

$$c(T) = \max |l_f(v) - l_g(v)|$$

d) **Deformations** K_i that transforms a graph in a symmetrical graph or reorganizes the structure of a given graph.

$$c(T) = \max \{|l_f(u_1) - l_g(u_1)|, |l_f(u_2) - l_g(u_2)|\}$$

Proposition 3.4.4 *[41] Let M be a connected, closed, orietables surface of genus g and $f : M \to \mathbb{R}$ a simple Morse function. If we have the graph $(\Gamma, l) = T(\Gamma_f, l_f)$ obtained by composing elementary deformations, then it exist a Morse function $g : M \to \mathbb{R}$ for which $(\Gamma_g, l_g) \cong (\Gamma, l)$.*

It is called **deformation** of the graph (Γ_f, l_f) an ordered sequence $T = (T_1, T_2, \cdots, T_n)$ made from elementary transformations that has a recursive actions as it follows : T_1 represents an elementary transformation for which (Γ_f, l_f), T_2 represents an elementary transformation for which $T_1(\Gamma_f, l_f)$, $\cdots$, respectively T_n represents an elementary transformation for which $T_{n-1}T_{n-2}\cdots T_1(\Gamma_f, l_f)$.

We will denote this by $\mathcal{T}\left((\Gamma_f, l_f), (\Gamma_g, l_g)\right) = \{T = (T_1, \cdots, T_n : T(\Gamma_f, l_f)\} \cong (\Gamma_g, l_g)$.

Definition 3.4.8 *It is called* ***edit distance****, the pseudo-metrics defined by:*

$$d_E\left((\Gamma_f, l_f), (\Gamma_g, l_g)\right) = \underset{T\in\mathcal{T}\left((\Gamma_f, l_f), (\Gamma_g, l_g)\right)}{inf} c(T).$$

The following result shows the stability of the edit distance.

Theorem 3.4.5 *[41] For any Morse type simple function $f, g : M \to \mathbb{R}$ we have*

$$d_E\left((\Gamma_f, l_f), (\Gamma_g, l_g)\right) \leq \underset{p\in M}{max} |f(p) - g(p)|$$

Considering all types of distances presented for real valued Morse functions f, g on a surface M, we have the following connections between them:

i) $d_E\left((\Gamma_f, l_f), (\Gamma_g, l_g)\right) \geq d_B(D_f, D_g)$, where D_f and D_g are persistence diagrams for f and g;

ii) $d_E\left((\Gamma_f, l_f), (\Gamma_g, l_g)\right) \geq d_{FR}(\mathcal{R}(f), \mathcal{R}(g))$, where $\mathcal{R}(f)$ and $\mathcal{R}(g)$ represent the Reeb graph of functions f and g.

3.5 The Reeb graph in computation topology

Classification of components for a Reeb graph led to a better understanding of the evolution of level curves on a manifold and also to the necessary parameters for the realization of a representation sketch. The main purpose of this section is to present an algorithm of construction for a Reeb graph associated to a real valued function on a compact surface without border in $\mathbb{R}^3$. The results from this section follow our paper [63], work cited in [57].

3.5.1 Description of the algorithm

1. Identify critical points;

2. Arrange critical points in a vector $v[i]$ in the same order that they appear when tracing along the surface from absolute minimum to absolute maximum;

3. Every entry in the vector has the following form: $v[i] = (P_i;\ type)$, where type can be:

   - **a** for minimum, with the special type a_A for the absolute minimum;
   - b for a saddle point with the special types b_i for the saddle that comes after the absolute minimum and b_p the saddle that comes after a minimum (note: categ.sp). If the saddle does not come after any type of minimum, the type will be obtained by alternating the order from the last saddle that falls in the categ.sp.
   - **c** for maximum, with the special type c_A for the absolute maximum;

4. Pseudocode algorithm:

   Entry data: X= surface, f= real valued function

   Define a boolean function ver.type(P,Q) that compares the main type of two given points;

   Example: ver.type(a,a_A)=true; ver.type (c,c_A)=true, ver.type(b_p, b_i)= true, ver.type(a,c)=false;

   Define a function point(v[i]) that returns the point P_i;

   First result: $v[i] = (P_i;\ type)$, $n = lenght\,(v[i])$;

   Let i = natural number;

   Initialize i=1;

   While i < n

   if type(v[i])=a

   if ver.type(point(v[i]), point(v[i+1])) is false

   then draw element I;

   i=i+1;

   else draw element II;

   i=i+2;

   if type(v[i])=$\mathbf{b_i}$ and type(v[i+1])=$\mathbf{b_p}$

   then draw element III;

   i=i+1;

   if type(v[i])=$\mathbf{b_p}$ and type(v[i+1])=$\mathbf{b_i}$

then draw element I;

i=i+1;

if type(v[i])=b and v[i+1]=c

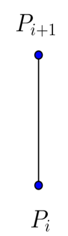
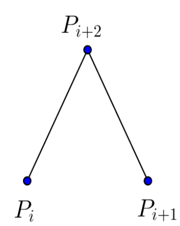
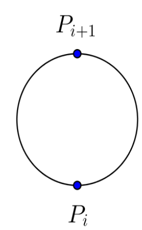
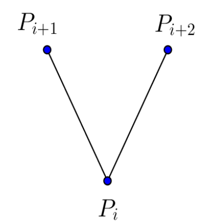
if ver.type(point(v[i+1]), point(v[i+2])) is false

then draw element I;

i=i+1;

else draw element IV;

i=i+2;

Type of component	I	II	III	IV
Representation				

3.6 Survey on the existing algorithms for a Reeb graph of real valued functions

The survey on the complexity of existing algorithms , presented in this section, is based on out paper [64].

The Reeb graph's importance has led to an expansion of algorithms and research towards the simplification and summarization of techniques. Given the fact that the volume of data needed to be computed has increased drastically, the complexity of algorithms is among one of the first concerns when constructing a representation technique for Reeb graphs. In this section we will present a selection of the most common algorithms together with their complexity.

In 2003 Carr, Snoeyink and Axen [27] proposed an algorithm for computing contour trees using a technique of joining an splitting the trees and forming a final contour tree. Given that v is the number of vertices, e the number of edges, n the number of supernodes and $f(n)$ the slow-growing inverse Ackerman function the algorithm provides a complexity of type

$$O\left(vlg(v) + e + nf(n)\right)$$

Another popular method was introduced by Kunii and Shinagawa in the paper [60]. The algorithm works by keeping a record of the triangles that have common points with the level set while tracing the complexity in order of function values. For 2-dimensional manifold the initial complexity was $O(n^2)$ with a generalization with the parameters v-the number of vertices and n the size of vertices, edges and triangles, providing a running time of $O\left(vn\right)$. Paper [?] offers an improvement for the special case of 2-dimensional manifolds by obtaining the complexity $O\left(mln(v)\right)$. Also, using a more complex technique Doraiswamy and Natarajan [40] managed to obtain the complexity $O\left(mlg(m)(lg(lg(m)))^3\right)$.

The paper [75] provided an algorithm for 3 dimensional manifold considering i_l to be the number of independent loops of the graph and obtaining a complexity of $O\left(mlg(m) + i_l m\right)$.

A different approach was offered by J.Cheng [29] by eliminating the old technique of swiping the contour trees and replacing it with a surjection from the manifold to the Reeb graph. And finally the more recent approach of Hajij and Rosen [49] considers an algorithm that requires an easy implementation and takes up a the least memory storage compared to existing algorithms.

3.7 The Reeb graph for circular functions

Definition 3.7.1 *Let M be a smooth n-dimension manifold and $f, g : M \to \mathbb{R}$ smooth functions. We say that f and g are* ***topological equivalent*** *if it exists an homeomorphism $h : M \to M$ and $k : \mathbb{R} \to \mathbb{R}$ such that the following relation is ture:*

$$f = k \circ g \circ h^{-1}$$

$$\begin{tikzcd} M \arrow[r, "h^{-1}"] \arrow[rrr, bend right, "f"'] & M \arrow[r, "g"] & \mathbb{R} \arrow[r, "k"] & \mathbb{R} \end{tikzcd}$$

Following the work [17], we will further present some useful notations and helping concepts that will help with introducing the realization theorem a the Reeb graph for a simple circular Morse-Bott function on S^2.

A submanifold $N \subset M$ is called **critical non-degenerate** for the function $f : M \to \mathbb{R}$ if for any point $p \in N$, the matrix $(Hf)_p\big|_N$ is invertible. A point $p \in M$ is called **critical** for f, if the rank of the application $df(p)$ is not maximal. A real value b is called **critical** for f if $f^{-1}(b)$ has at least one critical point of f.

A function $f : M \to \mathbb{R}$ is called **simple**, if it exists a unique connected component that contains critical points at that critical level. Moreover, a critical fiber is called **reducible**, if all the fibers from its neighborhood are homeomorphic to it.

Definition 3.7.2 *We say that a smooth submanifold $S \subset C(f)$ where $f : M \to \mathbb{R}$ is smooth, is a smooth critical **non-degenerate** submanifold if*

i) S has no boundary;

ii) S is compact and connected;

iii) for any point $p \in S$, we have $T_p(S) = ker\left((Hf)_p\right)$

Considering the above elements we introduce the following central notion for this section:

Definition 3.7.3 *It is called a **Morse-Bott** function (shortly **MB** function), a function f that has in its critical set $C(f)$ only isolated points or non-degenerate critical submanifolds.*

It is obvious that any round function is a Morse-Bott function. We remember that a smooth function is called **rotund** if its critical set is a reunion of non-degenerate critical circles.

Proposition 3.7.1 *The critical set of the Morse-Bott function $f : M^2 \to \mathbb{R}$ can be classified as it follows:*

1. *critical circles, meaning points that can be found in a critical submanifold homeomorphic to S^1;*
2. *isolated critical extrema (minimum or maximum);*
3. *saddle points, meaning isolated critical points of index 1.*

Next we will present important notions and relevant examples for constructing the Reeb Morse-Bott graph for simple circular Morse-Bott functions. Considering the paper [17], we will expose the classification problem for simple Morse-Bott functions from S^2 to S^1.

Definition 3.7.4 *Let M^2 an orientable closed surface. A circular function $f : M \to S^1$ is called **circular Morse-Bott function** if for any point $x \in M$, exists a neighborhood V*

of $f(x)$ and a diffeomorphism $\phi : V \to \mathbb{R}$, such that for any $U = f^{-1}(V)$, the application $\phi \circ \left(f\big|_U\right)$ is a real Morse-Bott function.

Given a simple Morse-Bott function $f : S^2 \to S^1$ we can depict the following conclusions:

i) f is not a regular map;

ii) the non-degenerate critical submanifolds with finite number of points are homeomorphic to S^1.

So the critical set $C(f)$ can be devided into three main section: singular circles, extrema and saddle points.

The classification problem for Morse functions, studied by Arnold in [14], [15] and Sharko in [89], [90] (see also the monograph [88]) using the Reeb graph can be extend, using the generalized Reeb graph for simple circular Morse-Bott functions. Similarly to the definition 3.1.2 given for real valued functions, we can highlight the graph structure of the quotient space $S^2/\sim$ associated to the simple Morse-Bott function $f : S^2 \to S^1$, as it follows:

i) for a critical value $c \in S^1$, the **vertices** of the graph are connected components of the level curves $f^{-1}(c)$;

ii) for a regular value $c \in S^1$, the **edges** of the graph are connected components associated to the level $f^{-1}(c)$.

A similar representation associated to this types of connected component for a Reeb graph of a real valued function [63] is presented in the paper [17] considering four types of existing vertices.

Definition 3.7.5 *The graph defined on the quotient space $S^2/\sim$ together with the classification of edges and vertices is called the* ***Reeb MB graph*** *associated to the circular Morse-Bott function $f : S^2 \to S^1$.*

The Reeb graph of a simple Morse-Bott function $f : S^2 \to S^1$ is a tree, because the Euler-Poincaré characteristic of the graph is equal to 1.

Considering that the labeling of elements of a Reeb graph is dependent on the chosen orientation for the surface and also on the choice of the basis point for pointing out the concept of isomorphism between two MB Reeb graphs. Therefore we call an **isomorphism** between the Reeb graphs associated to the functions $f, g : S^2 \to S^1$, a one to one map between the vertex-sets of the two graphs following the condition that the images of two adjacent vertices from the graph induced by f are adjacent on the graph induced by g.

The following sketch of the algorithms id taken from our paper [64].

3.7.1 Description of algorithm

In order to give a better understanding to circular Morse-Bott functions and the applicability of this concept for practical purposes we will describe the algorithm and present three examples using the pre-defined components: extrema, saddle point, regular point with critical values as image and regular point with singular circle as image. The basic steps are the following:

11) identifying the critical points;

2) choosing an orientation for tracing the surface;

3) arranging the points into a vector $v[i] = (P_i, type)$;

4) identifying the type to each corresponding critical point:

m = minimum, M = maximum with pre-defined component of type I;

s = saddle point with component type II;

prc = regular point having image a critical value with component III;

prs = regular point having image a singular circle with component IV;

Remark: For components of type IV the Reeb graph will always start and end with symbol of singular circle $\circ$, therefor it will switch the order with the next point appearing on the circle used for making the radial projection. This component will also mark 3 points on the Reeb graph, the one corresponding to the singular circle $\circ$ and another two for each branch. The two double points will be marked as prs_{aux}.

3.8 The Reeb graph associated to a Morse function on an orientable surface with boundary

The research of critical points for smooth functions on closed manifolds and their classification represents a very popular theme in differentiable topology and other fields of mathematics. In 1934 Morse found a canonical representation of functions in the neighborhood of non-degenerate critical points as second degree polynomials, followed later by other mathematicians like Bolsinov and Fomenko [22] that introduced concepts like atom, f-atom and frame equivalences using fibers. This section uses as its main reference the paper [54] but also the works [53], [59] şi [80].

Let M, N be two compact smooth manifolds and $f : M \to \mathbb{R}$, respectively $g : N \to \mathbb{R}$ smooth functions. f şi g are called **layer equivalent** if it exists an homeomorphism $\lambda : M \to N$, that takes every connected component from the level set of f in connected component from the level set of g, preserving the direction in which the functions increase. By restricting f to the set $f^{-1}(c-\epsilon, c+\epsilon)$, where c is a critical value of f and $\epsilon > 0$ small enough such that the segment line $[c-\epsilon, c+\epsilon]$ does not contain other critical value beside c, we obtain the layered equivalence classes called **atoms** or **f-atoms**. If λ preservers the orientation of f and g they are called **$\mathcal{O}$-equivalent**, and the class of $\mathcal{O}$-equivalence of the pair $\left(U, f\big|_U\right)$ is called **$\mathcal{O}$-atom** for the orientable surface.

Considering the definition of topological equivalent functions, introduced in Section 3.7.1, we say that two functions f and g are called **topological $\mathcal{O}$-equivalent** if the homeomorphism h preserves orientation on the surface M. The papers [55], [79], [80], [89] investigate topological properties of the Reeb graph like a special equivalence class called m-equivalences that have a large applicability for fields like dynamic systems.

Definition 3.8.1 *[54] Let M^2 be a smooth surface with boundary and $f : M \to \mathbb{R}$ Morse function. f is called a* ***m-function*** *if all its critical points are interior, the restriction on the border f_∂ of the function f is also Morse and any critical level of f does not contain critical points of f_∂.*

The authors B. Hladysh and A. Prishlyak define in the paper [54] the class of $\Omega(M)$ functions as simple smooth functions $f : M \to \mathbb{R}$ on an orientable, smooth, compact, connected and without border that satisfies the following properties:

a) if $p_0 \in C(f)$ is not on the border ∂M, then its a critical non-degenerate point for f;

b) if $p_0 \in C(f)$ is on the border ∂M, then its a critical non-degenerate point fo f but also for its restriction $f\big|_{\partial M}$ of f on the border;

c) if $p_0 \in C(f\big|_{\partial M})$ then $p_0 \in C(f)$.

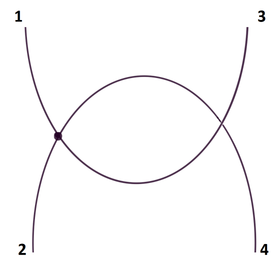
Theorem 3.8.1 *[54] For a smooth, orientable, compact surface M^2, we have the following:*

a) for an arbitrary function $f \in \Omega(M)$, it exists $g : M \to \mathbb{R}$ a m-function, such that f and g are topological equivalent;

b) for $g : M \to \mathbb{R}$ a arbitrary m-function, it exist a function $f \in \Omega(M)$ such that f and g are topological equivalent.

The components of level lines for a function $f \in \Omega(M)$ are called **layers**. These layers are divided in type I layers if they correspond to connected components homeomorphic to line segment and of type II components if they come from components homeomorphic to S^1. Similarly, the edges of a graph are classified as type I or II.

Definition 3.8.2 *[54] Vertices of degree 3 and 4 of a graph Γ_f incidents from an edge of type I are called* ***type Y*** *edges, respectively* ***tip X***. *In the image below we can observe the notation given for a type X vertex.*



Therefore, we call an **equipped Reeb graph** associated to the function $f \in \Omega(M)$, the graph Γ_f equipped with the division of edges in types, the orientation and cyclic ordering of vertices of type Y respectively X.

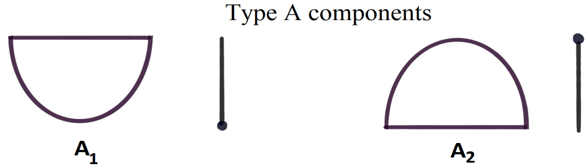
The vertices v of the graph γ_f can be classified according to their degree as it follows:

- minimum and maximum if $deg(v) = 1$;
- points on the border if $deg(v) = 2$;
- saddle point (interior or on the border, vertex of type Y) if $deg(v) = 3$;
- interior saddle point (vertex of type X) if $deg(v) = 4$.

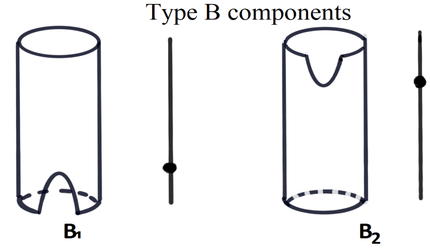
According to the paper [54] there are 7 possible atoms and 13 simple $\mathcal{O}$-atoms, and their classification depends on the index of the critical point and its belonging to the boundary of the surface. We will present a survey of such atoms together with their Reeb graph and the way gluing a component affect the invariants genus **(g)**, number of border connected components ∂ and **(c)** number of connected components of the surface M_t.

If $p_0 \in \partial M$ we have three atoms of type A, B, C with 6 corresponding $\mathcal{O}$-atoms A_1, A_2, B_1, B_2, C_1 and C_2 with the following representation

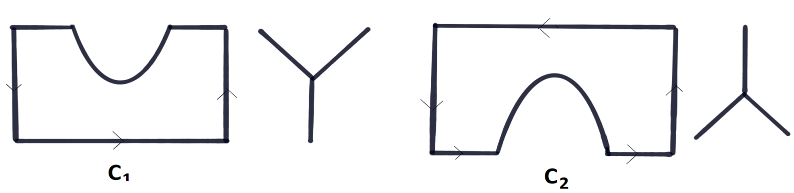
Type A components



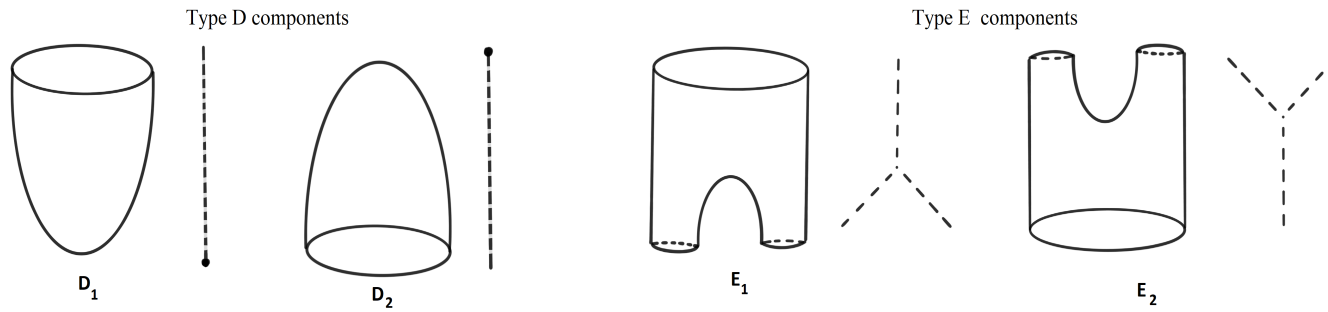
Type B components



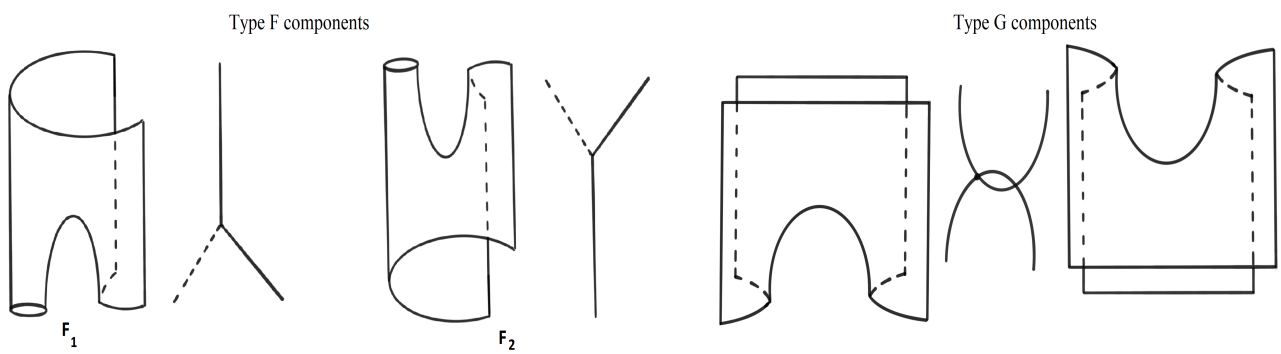
Type C components



If $p_0 \notin \partial M$ the atoms do not intersect the border and we have two types of atoms: D, E and 4 $\mathcal{O}$-atoms D_1, D_2, E_1 and E_2 represented like



If $p_0 \notin \partial M$ and the atoms intersect the border and we have two types of atoms: F, G and 3 $\mathcal{O}$-atoms F_1, F_2 şi $G = G_1 = G_2$ with the representation



The orientation of atoms is given by the orientation of the space. Otherwise, we will choose the trigonometric orientation for circles and inferior semicircles and clockwise for for circles and superior semicircles.

Definition 3.8.3 *If two equipped Reeb graphs Γ_f and Γ_g associated to the functions f and g are called* ***equivalent*** *by the isomorphism $\varphi : \Gamma_f \to \Gamma_g$ if the map φ:*

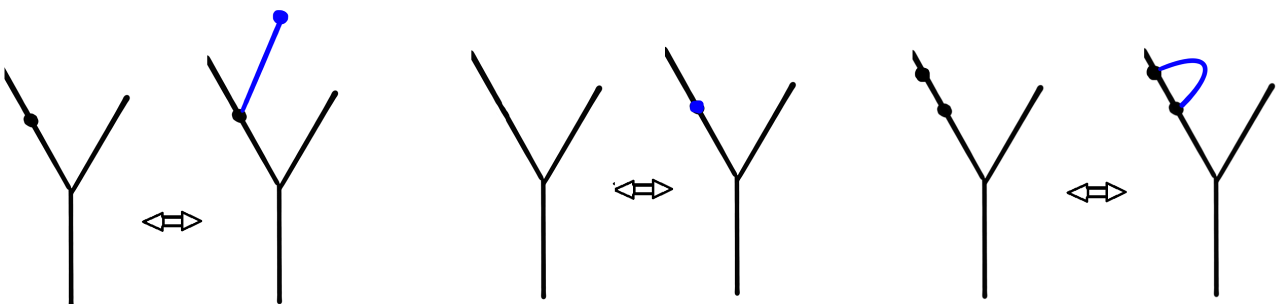
- *preserves the division of edges;*
- *preserves the order of cycles on edges for every vertex of type X and Y;*
- *preserves the division of vertices.*

Theorem 3.8.2 *[54]*

Given two compact manifolds with border M, N and the functions $f \in \Omega(M)$, $g \in \Omega(N)$, we say that $f$and g are ***$\mathcal{O}$-equivalent*** *if and only if the associated Reeb graphs, Γ_f, Γ_g are equivalent. Moreover, if the isomorphism φ preserves the labeling of vertices, the graph is called* ***equivalent ordered****.*

The pair of numbers (i, j) is called the **vertex index** for the vertex v of an orientable graph, where i represents the interior degree and j the exterior one. The authors of the paper [54] have introduced the notion $\Gamma_n^{\leq 4}$ for the graphs with n vertices and $i + j \leq 4$. Therefore the graph will have vertices of the following types $(1, 1), (1, 2), (2, 1), (2, 2)$ and at least one vertex of type $(0, 1)$ and $(1, 0)$. For the graph presented before we have the following possible operations:

- $\mathbf{O_1}$: addition of a vertex and the vertex corresponding to it;
- $\mathbf{O_2}$: division of an edge by an interior point that becomes the new vertex;
- $\mathbf{O_3}$: division of an edge without the addition of new vertices.



$\mathbf{O_1}$ $\mathbf{O_2}$ $\mathbf{O_3}$

Any operation on the graph $\gamma \in \Gamma_n^{\leq 4}$ is composed by a finite sequences of operations for the list $O_1, O_2, O_2, O_1^{-1}, O_2^{-1}$ şi O_3^{-1}.

Any graph $\gamma \in \Gamma_n^{\leq 4}$ can be obtained from the graph $\overline{\gamma} \in \Gamma_2^{\leq 4}$ (see the below image) by a finite number of operations from the above list. Therefore, it exist 57 $\mathcal{O}$-non-equivalent $m-$functions presented in detail in the paper [54].

$\overline{\gamma}$

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