BABEȘ BOLYAI UNIVERSITY CLUJ-NAPOCA

Faculty of Mathematics and Computer Science

RODRIGUES FORMULAS AND APPLICATIONS

PhD Thesis Abstract

Scientific supervisor:
Prof. Univ. Dr. ANDRICA DORIN

PhD student:
CHENDER OANA-LILIANA
(BROAINA)

CLUJ-NAPOCA

2020

Contents

Keywords: matrix function, exponential map, Lie group, Lie algebra, general linear real group $\mathbf{GL}(n, \mathbb{R})$, special Euclidean group $\mathbf{SE}(n)$, special orthogonal group $\mathbf{SO}(n)$, Schwerdtfeger formula, Cayley transform, Rodrigues formula.

Introduction

Geometry is a domain in which groups were used systematically, and the study of Lie groups was founded in 1884 by the Norwegian mathematician Sophus Lie. The main examples of Lie groups being general linear groups $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{GL}(n, \mathbb{C})$.

Lie groups are used in many domanis of mathematics and modern physics. They are applied in engineering where they are used as configuration space of mechanical systems, and in physics where they are used as symmetry groups associated with conservation laws. Many of these applications are essentially based on the use of exponential maps.

In Lie theory, the exponential map is an essential tool, because it makes the connection between an element in Lie algebra and the corresponding element in the Lie group. Although the existence of an exponential map is guaranteed for any Lie group, finding an explicit formula is a difficult problem. However, for some Lie groups of small dimension there are explicit formulas for the exponential map, the most known being the Rodrigues formula for the exponential of the rotation group $\mathbf{SO}(3, \mathbb{R})$.

Applying the Hamilton-Cayley theorem, the exponential map becomes a polynomial of X, thus the problem of determining a formula for the exponential map, known as the Rodrigues problem, given in formula (3.3), is reduced to the problem of finding the coefficients $a_0(X), \ldots, a_{n-1}(X)$.

The purpose of this paper is to generalize this problem and to determine the Rodrigues coefficients. Thus, the thesis brings out original contributions related to the Rodrigues problem for matrix functions, presenting new methods for computing the Rodrigues coefficients, based on the trace and on the Hermite interpolation polynomial.

A new way to determine Rodrigues coefficients is the trace method presented in Theorem 3.1. The idea is to reduce the problem to a linear system, having as unknowns the Rodrigues coefficients. The Hermite interpolation polynomial method is based on the fact that the problem of determining the Rodrigues coefficients when we know the spectrum of the matrix X is equivalent to find the algebraic form of the Hermite polynomial. The Rodrigues type formulas for the Cayley transform of the groups $\mathbf{SO}(n)$ and $\mathbf{SE}(n)$ are studied in the last chapter.

This thesis develops the ideas above presented and it is organized into four chapters as follows.

The first chapter, entitled *Matrix Functions*, consists in six sections in which are presented the notions and the fundamental results which are necessary in our presentation.

The notion of matrix function plays an important role in many domains of mathematics with numerous applications in science and engineering, especially in control theory and in the theory of differential equations. In the first section are presented the Jordan canonical form (subsection 1.1.1) and the definition of a matrix function using the Jordan canonical form (Definition 1.3). The second section is dedicated to the polynomial matrix functions, where it is highlighted the role of the characteristic polynomial and of the minimal polynomial. We introduce the anulator polynomial and the minimal polynomial and we present some of their properties. We also introduce the notion of function defined on the spectrum of a matrix. Matrix functions are also defined using the Hermite interpolation (Definition 1.6), and the examples presented have the role of bringing clarifications. Definition of the matrix functions using the Cauchy's integral formula is presented in Section 1.3, the definition of a matrix function as a power series is illustrated in Section 1.4 and these definitions are accompanied by examples. Section 1.5 is devoted to bring out the connection between these definitions, Theorem 1.6 presenting the equivalence of the given definitions for the matrix function. The last section presents Schwerdtfeger's formula (1.17) and an extension, as well as the implementation in MATHEMATICA of the computation of Frobenius covariates and of the obtained formula. Among the references used we mention O.L. Chender (Broaina) [13], G.H. Golub, C.F. Van Loan [22], N.J. Higham [24], R.A. Horn, Ch.R. Johnson [28], [29], [30], P. Lancaster, M. Tismenetsky [35], R.F. Rinehart [58].

Chapter 2, *Matrix Lie Groups. The Exponential Application*, is structured in five sections in which are presented results regarding the matrix Lie groups, the exponential map and the surjectivity problem. In the first section are highlighted known results regarding the exponential map for square matrices with real or complex entries as well as the proofs of the main properties (Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.4, Lemma 2.5). Section 2.2 is dedicated to the general linear real group $\mathbf{GL}(n,\mathbb{R})$. There are also presented the special linear group $\mathbf{SL}(n,\mathbb{R})$, the orthogonal group $\mathbf{O}(n)$, the special orthogonal group $\mathbf{SO}(n)$, and their Lie algebras, highlighting some properties of the exponential maps. In Section 2.3 it is defined the special Euclidean group $\mathbf{SE}(n)$ of affine functions induced by the orthogonal transformations, also called rigid motions, and the corresponding Lie algebra. The groups $\mathbf{SE}(2)$ and $\mathbf{SE}(3)$ play a fundamental role in robotics, dynamics and in the motion interpolation process. In Section 2.4 are presented the complex linear group $\mathbf{GL}(n,\mathbb{C})$ and its subgroup $\mathbf{SL}(n,\mathbb{C})$, group of units $\mathbf{U}(n)$ and its subgroup $\mathbf{SU}(n)$ (Definition 2.2). These are all Lie groups, and their corresponding Lie algebras and the exponential maps are well defined. The last section is deading to the surjectivity problem of the exponential map. Theorem 2.1, Theorem 2.2, and Theorem 2.3 illustrate the problem of determining the image of the exponential map for the matrix Lie groups. The groups $\mathbf{GL}(n,\mathbb{R}), n \geq 2$, and $\mathbf{SL}(n,\mathbb{R}), n \geq 2$ are not exponential. In Theorems 2.4, 2.6, 2.7 are presented and proved the results regarding the surjectivity of the exponential map. Among the references used in the elaboration of this chapter we mention D. Andrica and L. Mare [5], D. Andrica, R.-A. Rohan [6], H.L. Lai [34], L. Mare [38], S. Mondal [42], M. Moskowitz, M. Wüstner [43], M. Nishikawa [44], [45], [46], [47], [48], [49], S. Rădulescu, D. Andrica [57], M. Wüstner [65], [66], [67].

Chapter 3 is suggestively titled *Rodrigues Formulas for Matrix Functions. Methods for Determination of Rodrigues Coefficients*. This chapter is organized in six sections. The first section introduces the Rodrigues problem for matrix functions and presents the Rodrigues coefficients, as well as their invariance in relation to the matrix conjugation operation. In the second section is presented a new method of determining Rodrigues coefficients, result based on the paper D. Andrica, R.-A. Rohan [7]. In Theorem 3.1 we present, if the eigenvalues of the matrix are pairwise distinct, a direct method of determining the general Rodrigues coefficients reducing the Rodrigues problem to the system (3.7). Then, Theorem 3.2 gives explicit formulas in terms of fundamental symmetric polynomials of the eigenvalues of the matrix. These formulas allow us to consider the degenerate cases (that is the situations when the eigenvalues have multiplicities) and to obtain formulas for the coefficients. Section 3.3 illustrates the particular cases $n = 2, 3, 4$ for which the computation are effectively presented. In Section 3.4 are studied the possible degenerated cases. Sections 3.5 and 3.6 are dedicated to the Hermite interpolation polynomial method and to the special case of the exponential map for the special orthogonal group. The special orthogonal group $\mathbf{SO}(n)$ has important applications in mechanics, its elements being also called the rotation matrices. After presenting the classical cases $n = 2, 3$, the Rodrigues formula is given in the cases $n = 4$ and $n = 5$, taking into account all possible situations. The MATHEMATICA program was used to perform the computation. The main reference used in this chapter is our paper D. Andrica, O.L. Chender (Broaina) [4]. Other references are D. Andrica, I.N. Casu [2], D. Andrica, R.-A. Rohan [7], T. Bröcker, T. tom Dieck [12], C. Chevalley [14], O. Furdui [17], J. Gallier, D. Xu [19], S. Kida, E. Trimandalawati, S. Ogawa [31], M.-J. Kim, M.-S. Kim, A. Shin [32], [33], B. Jütler [36], [37], J.E. Marsden și T.S. Raţiu [41], F.C. Park, B. Ravani [51], [52], L.I. Piscoran [53], V. Pop, O. Furdui [55], E.J. Putzer [56], R.-A. Rohan [59], F. Warner [62], R. Vein, P. Dale [63], M. Wüstner [64].

Chapter 4 is entitled *The Cayley Transform and the Rodrigues Type Formulas*. In the first section we present the Cayley transform of the group $\mathbf{SO}(n)$ and we show that this map is well defined. Theorem 4.1 shows that the Cayley transform is bijective and its inverse is given. In Section 4.2 we define the Cayley transform type for the special Euclidean group $\mathbf{SE}(n)$ in connection with the Cayley transform of $\mathbf{SO}(n)$. Section 4.3 is devoted to the generalization of this notion and some properties are presented. The Rodrigues formulas for the Cayley transform are obtained in Section 4.4. For the group $\mathbf{SO}(n)$ these formulas are given in the special cases $n = 2, 3, 4$. For the group $\mathbf{SE}(n)$ the cases $n = 2$ and $n = 3$ are treated. The presentation follows our work [3]. Among the references used in this chapter we mention R.-A. Rohan [60].

This paper does not exhaust the subject. It brings a contribution in the field and

opens new horizons for knowledge.

In the elaboration of this paper I enjoyed the support and collaboration provided by specialists with exceptional qualities both professional and human, whom I want to thank.

I would like to thank the scientific supervisor, Prof. Dr. Andrica Dorin for the patience with which he guided and supported me during the years of doctoral studies and to express my gratitude and respect for the recommendations and indications I had in order to develop this paper.

Honestly thanks to the guidance commission formed by Conf. Dr. Blaga Paul, Conf. Dr. Pintea Cornel and Lect. Dr. Văcărețu Daniel.

Finally, I would like to thank my family for their support, for their trust and for accepting all the sacrifices required by my involvement in the activities related to the preparation and elaboration of this thesis.

Chapter 1

Matrix Functions

The concept of matrix function plays an important role in many domains of mathematics with numerous applications in science and engineering, especially in control theory and in the theory of the differential equations in which $\exp(tA)$ has an important role. An example is given by the *nuclear magnetic resonance* described by the Solomon equations

$$dM/dt = -RM, M(0) = I,$$

where $M(t)$ is the matrix of intensities and R is the matrix of symmetrical relaxation.

Given a scalar function $f : D \to \mathbb{R}$ we define the matrix $f(A) \in M_n(\mathbb{C})$, formally replacing x with A. For example, for $f(x) = \frac{x+1}{x-1}, x \neq 1$, we have $f(A) = (A+I)(A-I)^{-1}$ if $1 \notin \sigma(A)$, where we denoted by $\sigma(A)$ the set of eigenvalues of A, i.e. the *spectrum* of A.

Similarly, the scalar functions defined by a series of powers generate matrix functions. If

$$f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

then

$$f(A) = \log(1+A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \dots.$$

It can be shown that this series converges if and only if $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of the matrix A.

Numerous series of powers have infinite convergence radius. For example, we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

and this generates the matrix function

$$\cos A = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \dots$$

Again it can be shown that this makes sense for any matrix $A \in M_n(\mathbb{C})$.

This direct approach to defining a matrix function is sufficient for a wide range of functions, but does not provide a general definition. It also does not necessarily provide a correct way to numerically evaluate the matrix $f(A)$. In this chapter we will consider alternative definitions for the notion of matrix functions.

In the six sections are presented the notions and the fundamental results which are necessary in our presentation. In the first section are presented the Jordan canonical form (subsection 1.1.1) and the definition of a matrix function using the Jordan canonical form (Definition 1.3). The second section is dedicated to the polynomial matrix functions, where it is highlighted the role of the characteristic polynomial and of the minimal polynomial. We introduce the anulator polynomial and the minimal polynomial and we present some of their properties. We also introduce the notion of function defined on the spectrum of a matrix. Matrix functions are also defined using the Hermite interpolation (Definition 1.6), and the examples presented have the role of bringing clarifications. Definition of the matrix functions using the Cauchy's integral formula is presented in Section 1.3,the definition of a matrix function as a power series is illustrated in Section 1.4 and these definitions are accompanied by examples. Section 1.5 is devoted to bring out the connection between these definitions, Theorem 1.6 is presenting the equivalence of the given definitions for the matrix function. The last section presents the Schwerdtfeger's formula (1.17) and an extension, as well as the implementation in MATHEMATICA of the computation of the Frobenius covariates and of the obtained formula. Among the references used we mention O.L. Chender (Broaina) [13], G.H. Golub, C.F. Van Loan [22], N.J. Higham [24], R.A. Horn, Ch.R. Johnson [28], [29], [30], P. Lancaster, M. Tismenetsky [35], R.F. Rinehart [58].

1.1 Definitions for $f(A)$

A matrix function can be defined in different ways, the following three being the most useful for the developments in this paper.

1.1.1 The Jordan canonical form

Many problems involving a matrix A can be easily solved if the matrix is diagonalizable. But not every square matrix is diagonalizable over $\mathbb{C}$ or over $\mathbb{R}$. However, using similarity transformations any square matrix can be brought to a matrix that is "almost diagonal" in a certain sense. This almost diagonal matrix is known as the *Jordan canonical form* and is important both theoretically and for practical applications.

Definition 1.1. *The matrix*

$$J_k(\lambda_k) = \begin{pmatrix} \lambda_k & 1 & \dots & 0 \\ 0 & \lambda_k & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_k \end{pmatrix} \in M_n(\mathbb{C}) \tag{1.1}$$

it is called Jordan block of dimension n_k with eigenvalue λ_k. The scalar λ_k appears n_k times on the main diagonal and $+1$ appears $(n_k - 1)$ times on the superdiagonal. All other entries are 0.

Definition 1.2. *A vector x different from zero is called **generalized eigenvector** of rank k of A associated with eigenvalue λ if we have*

$$(A - \lambda I_n)^k x = O_n \text{ and } (A - \lambda I_n)^{k-1} x \neq O_n.$$

1.1.2 The definition of a matrix function using Jordan canonical form

Definition 1.3. *Let f be defined on a neighborhood of the spectrum of $A \in M_n(\mathbb{C})$. If A has the Jordan canonical form J, then*

$$f(A) = Xf(J)X^{-1} = X\mathrm{diag}(f(J_k(\lambda_k)))X^{-1} \tag{1.2}$$

where

$$f(J_k) = f(J_k(\lambda_k)) = \begin{pmatrix} f(\lambda_k) & f'(\lambda_k) & \dots & \frac{f^{(n_k-1)}(\lambda_k)}{(n_k-1)!} \\ & f(\lambda_k) & \ddots & \vdots \\ & & \ddots & f'(\lambda_k) \\ & & & f(\lambda_k) \end{pmatrix} \tag{1.3}$$

The right member of the relation (1.2) is independent of the choice of X and J.

1.2 Polynomial matrix functions

Two important polynomials are associated with a quadratic matrix: the characteristic polynomial and the minimal polynomial. These polynomials play a special role in solving various matrix theory problems.

Definition 1.4. *A polynomial ψ is called **anulator polynomial** of square matrix $A \in M_n(\mathbb{C})$ if*

$$\psi(A) = O_n. \tag{1.4}$$

*An anulator polynomial ψ_A which is monic and of minimum degree is called the **minimal polynomial** of A.*

The minimal polynomial is unique. From the Hamilton-Cayley theorem, the characteristic polynomial p_A is an anulator polynomial of A, but this is not generally the minimal polynomial of A.

The following simple properties take place and we present them without proof.

Lemma 1.1. *Any anulator polynomial of a matrix is divisible by the minimal polynomial.*

Lemma 1.2. *The minimal polynomial of the Jordan block of order* m *with eigenvalue* λ *is* $(t-\lambda)^m$.

Lemma 1.3. *Let be* $A \in M_n(\mathbb{C})$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_s$ *the distinct eigenvalues of* A. *Then the minimal polynomial of* A *is*

$$\psi(\lambda) = \prod_{i=1}^{s} (\lambda - \lambda_i)^{n_i} \tag{1.5}$$

where n_i *is the size of the largest Jordan block in which* λ_i *appears.*

Theorem 1.1. *[28, p. 86, Theorem 2.4.2] Let* $A \in M_n(\mathbb{C})$ *be a square matrix and* $p_A(\lambda) = \det(\lambda I_n - A)$ *its characteristic polynomial. Then* $p_A(A) = O_n$.

Any polynomial p with complex coefficients,

$$p(t) = a_0 + a_1 t + \ldots + a_{m-1} t^{m-1} + a_m t^m, a_m \neq 0 \tag{1.6}$$

determines a matrix polynomial by simply replacing t with A in (1.6)

$$p(A) = a_m A^m + a_{m-1} A^{m-1} + \ldots + a_0 I_n \tag{1.7}$$

More generally, for a function f defined on an open disk containing the spectrum of A, we can define the matrix function $f(A)$ by the following theorem.

Theorem 1.2. *[22, p. 565, Theorem 11.2.3] If* f *is defined by*

$$f(t) = \sum_{i=0}^{\infty} a_i t^i$$

on an open disk containing $\sigma(A)$, *then*

$$f(A) = \sum_{i=0}^{\infty} a_i A^i.$$

Definition 1.5. *The values* $f^{(j)}(\lambda_i), i = 1, \ldots, s, j = 0, \ldots, n_i - 1$ *are called the values of the function* f *and its derivatives on the spectrum of* A. *If these values exist we say that* f *is* ***defined on the spectrum of*** A.

We notice that the minimal polynomial ψ_A takes the value zero on the spectrum of A.

Theorem 1.3. *[24, p. 5, Theorem 1.3] For polynomials* p *and* q *and* $A \in M_n(\mathbb{C})$ *we have* $p(A) = q(A)$ *if and only if* p *and* q *take the same values on the spectrum of* A.

1.2.1 Matrix functions defined using Hermite interpolation

Definition 1.6. *Let f be defined on the spectrum of $A \in M_n(\mathbb{C})$. Then $f(A) = r(A)$, where r is the Hermite interpolation polynomial that satisfies the interpolation conditions*

$$r^{(j)}(\lambda_i) = f^{(j)}(\lambda_j), i = 1, \ldots, s, j = 0, \ldots, n_i - 1,$$

where $\lambda_1, \ldots, \lambda_s$ are the distinct eigenvalues of A with the multiplicities $n_1, \ldots n_s$.

We observe that the polynomial r depends on A due to the values of the function f on the spectrum of A.

We will now mention two important properties of the matrix functions discussed in [35, p. 310, Theorem 1, Theorem 2].

Lemma 1.4. *[35, p. 310, Theorem 2] If $A, B, X \in M_n(\mathbb{C})$, where $B = XAX^{-1}$ and f is defined on the spectrum of A, then*

$$f(B) = Xf(A)X^{-1}. \tag{1.8}$$

Lemma 1.5. *[35, p. 310, Theorem 1] If $A \in M_n(\mathbb{C})$ is a matrix in blocks on the diagonal*

$$A = \operatorname{diag}(A_1, A_2, \ldots, A_s)$$

where $A_1, A_2, \ldots, A_s$ are square matrices, then

$$f(A) = \operatorname{diag}(f(A_1), f(A_2), \ldots, f(A_s)). \tag{1.9}$$

1.3 Matrix functions defined using Cauchy's integral formula

The Cauchy's integral formula is an elegant result of complex analysis that states that under certain conditions, the value of a function can be determined using an integral. Given a function $f(z)$ we can determine the value $f(a)$ through

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a} dz, \tag{1.10}$$

where Γ is a simple closed curve around a and f is analytic on and inside Γ. This formula extends to the case of the matrices.

Definition 1.7. *Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ a analytic function. Let $A \in M_n(\mathbb{C})$ be diagonalizable so that all eigenvalues of A are in Ω. We define $f(A) \in M_n(\mathbb{R})$*

through

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI_n - A)^{-1} dz, \tag{1.11}$$

where $(zI_n - A)^{-1}$ *is the resolvente of* A *in* z *and* $\Gamma \subset \Omega$ *is a simple closed curve around the spectrum* $\sigma(A)$*, oriented in the opposite trigonometric direction.*

Computation of the integrals in $f(A)$ it is difficult to evaluate especially for $n \geq 3$.

Theorem 1.4. *[30, p. 427, Theorem 6.2.28] Let* $A \in M_n(\mathbb{C})$ *be a diagonalizable matrix and* f *an analytic function on a domain that contains the eigenvalues of* A*. Then*

$$f(A) = Xf(\Lambda)X^{-1},$$

where $A = X\Lambda X^{-1}$*, with* $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *and* f *is defined by the Cauchy's integral formula.*

In conclusion the theorem above says that $f(A)$ is similar to the matrix $f(\Lambda)$.

1.4 Matrix functions defined as power series

The following result allows us to define $f(A)$ if f has a development in power series.

Theorem 1.5. *[22, p. 565, Theorem 11.2.3] If the function* f *is given by*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

on an open disk containing the eigenvalues of A*, then*

$$f(A) = \sum_{k=0}^{\infty} c_k A^k.$$

1.5 The equivalence of definitions for the matrix function

If $A \in M_n(\mathbb{C})$ and f is an analytic function on a domain that contains the spectrum of A, we saw that there are three ways to define the matrix $f(A)$.

R.F. Rinehart [58] showed that the three definitions are equivalent.

Theorem 1.6. *Let be* $A \in M_n(\mathbb{C})$*. Let* f *be an analytical function defined on a domain containing the spectrum of* A*. We denote by*

1. $f_J(A)$ *the matrix* $f(A)$ *obtained using the definition with the Jordan canonical form;*

2. *$f_H(A)$ the matrix $f(A)$ obtained using the definition with the Hermite's interpolation polynomial;*

3. *$f_C(A)$ the matrix $f(A)$ obtained using the definition with the Cauchy's integral formula.*

Then

$$f_J(A) = f_H(A) = f_C(A). \tag{1.12}$$

To prove this theorem we need some preliminary results. We will first consider the value of f on a diagonal matrix.

Lemma 1.6. *[30, p. 385] Let be $A \in M_n(\mathbb{C}) = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and f an analytical function defined on a domain containing the spectrum of A. Then*

$$f_C(A) = f_H(A) = f_J(A) = \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n)). \tag{1.13}$$

Then it is shown that the three definitions interact with the similarities in the same way.

Lemma 1.7. *[30, p. 412, Theorem 6.2.9(c)] Let be $A \in M_n(\mathbb{C})$ and f an analytical function defined on a domain containing the spectrum of A. Then the following relations holds*

$$f_J(XAX^{-1}) = Xf_J(A)X^{-1} \tag{1.14}$$

$$f_H(XAX^{-1}) = Xf_H(A)X^{-1} \tag{1.15}$$

$$f_C(XAX^{-1}) = Xf_C(A)X^{-1} \tag{1.16}$$

for any nonsingular matrix $X \in M_n(\mathbb{C})$.

We can generalize the idea to evaluate matrix functions using the relation $f(XAX^{-1}) = Xf(A)X^{-1}$. Then it is proved that $f_J(A) = f_H(A) = f_C(A)$ when A it is diagonalizable [30, p. 407].

Lemma 1.8. *[30, p. 408] Let be $A \in M_n(\mathbb{C})$ and $\epsilon > 0$. Then there is a matrix $A_\epsilon \in M_n(\mathbb{C})$ such that*

$$\|A_\epsilon - A\| \leq \epsilon,$$

where A_ϵ has distinct eigenvalues and is therefore diagonalizable.

The continuity properties of f_H and f_C are established, to show that $f_H(A) = f_C(A)$ in the nondiagonalizable case.

Lemma 1.9. *[30, p. 396, Theorem 6.1.28] [30, p. 427, Theorem 6.2.28] Let be $A \in M_n(\mathbb{C})$ and f an analytical function defined on a domain containing the spectrum of A. Then the matrix functions f_H and f_C are continuous in A.*

1.6 The Schwerdtfeger formula and an extension

During recent years, there has been a considerably amount of research concerning the functions of matrices as they provide solutions to systems of linear differential equations. For example, the solution of $y' = Ay$ with $y(0) = x$ is given by $y(t) = \exp(tA)$, where $\exp(tA)$ is the exponential of the matrix tA defined in Section 1.4. Similary, we would like to find explicit forms for $\cos(tA), \sin(tA)$, and in general for $f(tA)$. Schwerdtfeger proves that for any holomorphic function f and for any matrix $A \in M_n(\mathbb{C})$ the formula (1.17) holds, where μ is the number of distinct eigenvalues λ_j of A, r_j is the multiplicity of λ_j and A_j are the Frobenious covariants of A. Here Γ_j is a smooth closed curve around the complex number λ_j. In this section we use formula (1.17) to derive formula 1.19 and to study some matrix functions.

We follow the results and the presentation of our paper [13].

The function f defined on the positive real numbers $f : \mathbb{R}^+ \to \mathbb{C}$, such that f is integrable on $[0, T]$ for all $T > 0$, and there exist constants $\alpha \in \mathbb{R}$ and $M > 0$ such that

$$|f(t)| \leq Me^{\alpha t} \text{ for } t \geq 0$$

is called **exponentially bounded**.

In 1961, Schwerdtfeger (see [29]) proved that for any holomorphic function f and for any matrix $A \in M_n(\mathbb{C})$ the following formula holds

$$f(A) = \sum_{j=1}^{\mu} A_j \sum_{k=0}^{r_j-1} \frac{1}{k!} f^{(k)}(\lambda_j)(A - \lambda_j I_n)^k, \tag{1.17}$$

where μ is the number of distinct eigenvalues λ_j of A, r_j is the multiplicity of λ_j and

$$A_j = \frac{1}{2\pi i} \int_{\Gamma_j} (sI_n - A)^{-1} ds = \frac{1}{2\pi i} \int_{\Gamma_j} \mathfrak{L}[e^{tA}](s) ds \tag{1.18}$$

are the Frobenius covariants of A. Here Γ_j is a smooth closed curve around the complex number λ_j and $\mathfrak{L}$ denote the Laplace transform.

To compute the Frobenius covariants we use the Penrose generalized inverse matrix.

Theorem 1.7. *(Penrose generalized inverse matrix) If $A \in M_n(\mathbb{C})$, then there exists a unique matrix $X \in M_n(\mathbb{C})$ that satisfies the following equations:*

(i) $AXA = A$;

(ii) $XAX = X$;

(iii) $(AX)^* = AX$;

(iv) $(XA)^* = XA$,

where B^* is the conjugate transpose of B.

The matrix X is said to be the **generalized inverse** of A, and it is denoted by $A^\dagger$.

Theorem 1.8. *(Penrose) If $A \in M_n(\mathbb{C})$ and $\lambda_j \in \sigma(A)$, then*

$$A_j = (F_{\lambda_j} E_{\lambda_j})^\dagger,$$

where $E_{\lambda_j} = (I_n - (A - \lambda_j I_n)^{r_j})^\dagger (A - \lambda_j I_n)^{r_j}$ *and* $F_{\lambda_j} = (I_n - (A - \lambda_j I_n)^{r_j})((A - \lambda_j I_n)^{r_j})^\dagger$.

Now, we present the extended Schwerdtfeger formula which follows our paper [13].

Theorem 1.9. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be holomorphic, where D is an open and connected set. If $A \in M_n(\mathbb{C})$ such that $\sigma(A) \subset D$ and $t \in \mathbb{R}^*$, then*

$$f(tA) = \sum_{j=1}^{\mu} A_j \sum_{k=0}^{r_j - 1} \frac{1}{k!} f^{(k)}(t\lambda_j) t^k (A - \lambda_j I_n)^k, \tag{1.19}$$

where μ denotes the number of different eigenvalues λ_j of A, r_j is the multiplicity of λ_j, and

$$A_j := \frac{1}{2\pi i} \int_{\Gamma_j} (sI_n - A)^{-1} ds = \frac{1}{2\pi i} \int_{\Gamma_j} \mathfrak{L}[e^{tA}](s) ds \tag{1.20}$$

are the Frobenius covariants of A. Here Γ_j is a closed smooth curve enclosing λ_j only.

Proof. Applying formula (1.17) for matrix tA we have

$$f(tA) = \sum_{j=1}^{n} (tA)_j \sum_{k=0}^{r_j - 1} \frac{1}{k!} f^{(k)}(t\lambda_j)(tA - t\lambda_j I_n)^k = \sum_{j=1}^{\mu} A_j \sum_{k=0}^{r_j - 1} \frac{1}{k!} f^{(k)}(t\lambda_j) t^k (A - \lambda_j I_n)^k,$$

where we have used the property $\sigma(tA) = t\sigma(A)$ and Theorem 1.8. □

1.6.1 The MATHEMATICA implementation

In the first code we have the MATHEMATICA implementation for the Frobenious covariants of a given matrix A defined by formula (1.18) for a given $\lambda_j \in \sigma(A)$.

The second code shows the MATHEMATICA implementation for the formula (1.19). Notice that the line 21 can be adjusted to use different holomorphic functions.

Chapter 2

Matrix Lie Groups. The Exponential Map

This chapter is structured in five sections in which are presented results regarding the matrix Lie groups, the exponential map and the surjectivity problem. In the first section are highlighted known results regarding the exponential map for square matrices with real or complex entries as well as the proofs of the main properties (Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.4, Lemma 2.5). Section 2.2 is dedicated to the general linear real group $\mathbf{GL}(n, \mathbb{R})$. There are also presented the special linear group $\mathbf{SL}(n, \mathbb{R})$, the orthogonal group $\mathbf{O}(n)$, the special orthogonal group $\mathbf{SO}(n)$, and their Lie algebras, highlighting some properties of the exponential maps. In Section 2.3 it is defined the special Euclidean group $\mathbf{SE}(n)$ of affine functions induced by the orthogonal transformations, also called rigid motions, and the corresponding Lie algebra. In Section 2.4 are presented the complex linear group $\mathbf{GL}(n, \mathbb{C})$ and its subgroup $\mathbf{SL}(n, \mathbb{C})$, group of units $\mathbf{U}(n)$ and its subgroup $\mathbf{SU}(n)$ (Definition 2.2). The last section is deading to the surjectivity problem of the exponential map. Theorem 2.1, Theorem 2.2, and Theorem 2.3 illustrate the problem of determining the image of the exponential map for the matrix Lie groups. In Theorems 2.4, 2.6, 2.7 are presented and proved the results regarding the surjectivity of the exponential map. Among the references used in the elaboration of this chapter we mention D. Andrica and L. Mare [5], D. Andrica, R.-A. Rohan [6], H.L. Lai [34], L. Mare [38], S. Mondal [42], M. Moskowitz, M. Wüstner [43], M. Nishikawa [44], [45], [46], [47], [48], [49], S. Rădulescu, D. Andrica [57], M. Wüstner [65], [66], [67].

2.1 The exponential map

Given a matrix $A = (a_{ij}) \in M_n(K)$ where $K = \mathbb{R}$ or $K = \mathbb{C}$, we define the exponential map of A, denoted by e^A, or $\exp A$, as the matrix defined by the formal series

$$e^A = \sum_{p=0}^{\infty} \frac{1}{p!} A^p,$$

considering $A^0 = I_n$. The following lemma shows that the series from the above definition is absolutely convergent.

Lemma 2.1. *For $A = (a_{ij}) \in M_n(K)$ where $K = \mathbb{R}$ or $K = \mathbb{C}$, we define the number $\mu = \max\left\{\left|a_{ij}^{(p)}\right| : 1 \le i, j \le n\right\}$, where $A^p = \left(a_{ij}^{(p)}\right)$. Then the following inequalities occur $\left|a_{ij}^{(p)}\right| \le (n\mu)^p$ for any $1 \le i, j \le n$. As a consequence, for any i, j, with $1 \le i, j \le n$, the series $\sum_{p=0}^{\infty} \frac{a_{ij}^{(p)}}{p!}$ is absolutely convergent, and thus the exponential map of the matrix A is well definied.*

With a proof similar to that of Lema 2.1 we obtain the following result: If $A \in M_n(\mathbb{C})$, the series $\sum_{p=0}^{\infty} \frac{t^p}{p!} A^p$, where $t \in \mathbb{R}$, converges evenly over any compact interval. Moreover, the function $t \mapsto e^{tA}$ it is differentiable and the following relation occurs

$$\frac{d}{dt} e^{tA} = A e^{tA}.$$

A fundamental property of the exponential map shows that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix A, then the eigenvalues of the exponential map e^A are $e^{\lambda_1}, \dots, e^{\lambda_n}$. In order to prove this property we need the folowing results.

Lemma 2.2. *Let be $A, U \in M_n(K)$ where $K = \mathbb{R}$ or $K = \mathbb{C}$. Assume that the matrix U is invertible. Then the following relation occur*

$$e^{UAU^{-1}} = U e^A U^{-1}.$$

Lemma 2.3. *Given a matrix $A \in M_n(\mathbb{C})$, there exists an invertible matrix P and a upper triangular matrix T such that*

$$A = PT^{-1}P.$$

Remark 2.1. If E is a Hermitian space, the proof of Lemma 2.3 can be easily adapted to prove that there exists an orthonormal basis $\{u_1, \dots, u_n\}$ against which the matrix of the map f is upper triangular. In other words, there exists a *unitary* matrix U and a upper triangular matrix T such that $A = UTU^{-1}$, result known as *Schur's Lemma.* Using this result we can get to the fact that if A is a Hermitian matrix, then there exists an unitary matrix U and a diagonal matrix D with real entries such that $A = UDU^*$.

Lemma 2.4. *Let be the square matrix $A \in M_n(\mathbb{C})$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then $e^{\lambda_1}, \dots, e^{\lambda_n}$ are the eigenvalues of the matrix e^A. More, if u is an eigenvector of matrix A corresponding to λ_i, then u is an eigenvector of matrix e^A of the eigenvalue e^{λ_i}.*

As an imediate consequence we show the relation holds

$$\det(e^A) = e^{\operatorname{tr}(A)},$$

where tr(A) is the trace of the matrix A, i.e. the sum $a_{11} + \cdots + a_{nn}$ of the entries on the main diagonal, so the sum of the eigenvalues of A. Result that the matrix e^A is always invertible.

Lemma 2.5. *For any matrix $A, B \in M_n(\mathbb{C})$ which commutes, i.e. $AB = BA$, we have*

$$e^{A+B} = e^A e^B. \tag{2.1}$$

Using Lemma 2.4 and the fact that the matrices A și $-A$ commutes, we have $e^A e^{-A} = e^{A+(-A)} = e^{0_n} = I_n$, which shows that the matrix inverse e^A is e^{-A}, i.e. we have the relation

$$(e^A)^{-1} = e^{-A}. \tag{2.2}$$

Remark 2.2. 1. We can prove the formula (2.2) noting that for any $t \in \mathbb{R}$, we have

$$\frac{d}{dt}(e^{tA}e^{-tA}) = Ae^{tA}e^{-tA} + e^{tA}(-Ae^{-tA}) = (A - A)e^{tA}e^{-tA} = O_n,$$

so the function $g(t) = e^{tA}e^{-tA}$ is constant on $\mathbb{R}$, and we have $g(t) = g(0) = I_n$. We consider $t = 1$ in relation $e^{tA}e^{-tA} = I_n$ and we obtain the formula (2.2).

2. An alternative proof for Lemma 2.5 can be obtained as follows. For any $t \in \mathbb{R}$, we have

$$\begin{aligned}\frac{d}{dt}e^{t(A+B)}e^{-tA}e^{-tB} &= (A+B)e^{t(A+B)}e^{-tA}e^{-tB} - e^{t(A+B)}Ae^{-tA}e^{-tB}\\ &- e^{t(A+B)}e^{-tA}Be^{-tB} = (A+B-(A+B))e^{t(A+B)}e^{-tA}e^{-tB} = O_n,\end{aligned}$$

because immediately result that if A and B commutes, then the matrices $A, e^{t(A+B)}$ and B, e^{-tA} and $B, e^{t(A+B)}$ commutes.
Results that the function $h(t) = e^{t(A+B)}e^{-tA}e^{-tB}$ is constant on $\mathbb{R}$, so $h(t) = h(0) = I_n$. Thus, $e^{-tA}e^{-tB} = (e^{t(A+B)})^{-1} = e^{-t(A+B)}$ and for $t = -1$ we obtain the desired formula.

3. In the case in which the matrices A and B does not commute, the Baker-Cambell-Hausdorff formula holds

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[B,[B,A]]+\dots},$$

where $[X, Y] = XY - YX$ is the matrix commutator X and Y.

2.2 The general linear real group $\mathbf{GL}(n, \mathbb{R})$

The set of the invertible square matrices of order n, with real entries, forms a group under multiplication, denoted by $\mathbf{GL}(n, \mathbb{R})$. The subset of matrices from the group $\mathbf{GL}(n, \mathbb{R})$ with the value of determinant equal to $+1$ is a subgroup of $\mathbf{GL}(n, \mathbb{R})$, denoted by $\mathbf{SL}(n, \mathbb{R})$. It is easy to verify that the set of orthogonal square matrices of order n, is a subgroup of $\mathbf{GL}(n, \mathbb{R})$, denoted by $\mathbf{O}(n)$. The subset of the group $\mathbf{O}(n)$ formed from matrices which have the value of the determinant equal with $+1$ is a subgroup of $\mathbf{O}(n)$, denoted by $\mathbf{SO}(n)$. The matrices from the group $\mathbf{SO}(n)$ are also called the *rotation matrices*. We show that the set of square matrices of order n, with real entries and which have trace null forms a vector space together with the classical operation of addition and multiplication with scalars. The same can be shown immediately that the set of antisymmetric matrices forms a vector space.

The group $\mathbf{GL}(n, \mathbb{R})$ is called the *general linear real group* and his subgroup $\mathbf{SL}(n, \mathbb{R})$ is called the *special linear group*. The group $\mathbf{O}(n)$ of the orthogonal matrices is called the *orthogonal group* and his subgroup $\mathbf{SO}(n)$ is called the *special orthogonal group* (or the *rotation group*). The vector space of the square matrices, of dimension n, with real entries and with null trace is denoted by $\mathfrak{sl}(n, \mathbb{R})$ and the vector space of antisymmetric square matrices, of dimension n is denoted by $\mathfrak{so}(n)$.

For the notations $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{so}(n)$ we need some further explanation. The groups $\mathbf{GL}(n, \mathbb{R})$, $\mathbf{SL}(n, \mathbb{R})$, $\mathbf{O}(n)$ and $\mathbf{SO}(n)$ are also topological groups, which means that there are topological spaces (seen as subspaces of $\mathbb{R}^{n^2}$), and the multiplication and the inverse are continuous operations. This groups are *Lie groups*. The real vector spaces $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{so}(n)$ are *Lie algebras*. The structure of Lie algebra is given by *Lie bracket*, which in this case is the usual commutator of matrices defined by $[A, B] = AB - BA$.

In fact, the Lie algebra of a Lie group is the tangent space to the unit element of the group seen as differentiable manifold, i.e. the space of all vectors tangents to the unit element (in this case the identity matrix I_n). In a sense, the Lie algebra is a linearization of the Lie group.

In general, let G be a Lie grup with corresponding Lie algebra $\mathfrak{g}$. It's known that the exponential map $\exp : \mathfrak{g} \to G$ is defined by $\exp(X) = \gamma_X(1)$, where $X \in \mathfrak{g}$ and γ_X is the subgroup of G with a parameter corresponding to X. We recall the following properties of the exponential map:

1) $\gamma x(t) = \exp(tX)$, for any $t \in \mathbb{R}$ and any $X \in \mathfrak{g}$;

2) $\exp(sX)\exp(tX) = \exp(s + t)X$, for any $s, t \in \mathbb{R}$ and any $X \in \mathfrak{g}$;

3) $\exp(-tX) = (\exp tX)^{-1}$, for any $t \in \mathbb{R}$ and any $X \in \mathfrak{g}$;

4) $\exp : \mathfrak{g} \to G$ is a smooth application which is a local diffeomorphism in $0 \in \mathfrak{g}$ and $\exp(0) = e$, where e is the neutral element of the group G;

5) the image $\exp(\mathfrak{g})$ of the exponential map generates the connected component G_e of the unity $e \in G$;

6) if $f : G_1 \to G_2$ is a morphism of Lie groups and $f_* : L(\mathfrak{g}_1) \to L(\mathfrak{g}_2)$ is the morphism of Lie algebras induced by f, then the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{g}_1 & \xrightarrow{f_*} & \mathfrak{g}_2 \\ {\scriptstyle \exp_1}\downarrow & & \downarrow{\scriptstyle \exp_2} \\ G_1 & \xrightarrow[f]{} & G_2 \end{array}$$

we have the relation $f \circ \exp_1 = \exp_2 \circ f_*$.

The exponential map allows a parametrization of the Lie group elements with the simpler objects of the Lie algebra. The exponential map on various Lie groups has multiple applications in mechanics [1], image processing [18] and in describing other processes from real world [19], [20].

The Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ of group $\mathbf{GL}(n, \mathbb{R})$ consists in the set of square matrices of order n, with real elements. It can be easilly shown that the map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathbf{GL}(n, \mathbb{R})$ is defined by the formula $\exp(A) = e^A$. This is correctly defined because we saw that we have $\det(e^A) = e^{\mathrm{tr}(A)} \neq 0$. Moreover, from the property 6) from above, results that the exponential maps $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$, $\exp : \mathfrak{sl}(n, \mathbb{R}) \to \mathbf{SL}(n, \mathbb{R})$ and $\exp : \mathfrak{o}(n) \to \mathbf{O}(n)$ are restriction of $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathbf{GL}(n, \mathbb{R})$.

The map $\exp : \mathfrak{o}(n) \to \mathbf{O}(n)$ is correctly defined because from the property $\exp\, {}^tA = {}^t\exp A$, results that we have

$${}^t(\exp A) \exp A = (\exp\, {}^tA) \exp A = \exp({}^tA + A) = \exp O_n = I_n,$$

i.e. $\exp A$ is a rotation matrix if A is an antisymmetric matrix.

Furthermore, for the map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$ we have $\det \exp A = e^{\mathrm{tr}(A)} = 1$, because the entries on the main diagonal of A are all zero, so $\mathrm{tr}(A) = 0$.

For the exponential map $\exp : \mathfrak{sl}(n, \mathbb{R}) \to \mathbf{SL}(n, \mathbb{R})$, obviously we have $\det \exp A = e^{\mathrm{tr}(A)} = 1$, because again $\mathrm{tr}(A) = 0$.

The references used in the presentation of matrix groups are A. Baker [10], M.L. Curtis [15], F.R. Gantmacher [21].

2.3 The special Euclidean group $\mathbf{SE}(n)$

In this section we present the group $\mathbf{SE}(n)$ of affine maps induced by orthogonal transformations, also called rigid motions, and the corresponding Lie algebra. The groups $\mathbf{SE}(2)$ and $\mathbf{SE}(3)$ plays a fundamental role in robotics, dynamics and in the process of interpolation of movement.

First we recall the usual method of representing the affine maps of the space $\mathbb{R}^n$ in terms given by square matrices of size $n + 1$.

Definition 2.1. *The set of affine maps ρ of the space $\mathbb{R}^n$, defined by $\rho(X) = RX + U$, where R is a rotation matrix, i.e. $R \in \mathbf{SO}(n)$ and U is a vector from $\mathbb{R}^n$, forms a group* *in relation to the composition operation, called the* ***group of direct affine isometries*** *(or rigid movements) or the* ***special Euclidean group****. This is denoted by* $\mathbf{SE}(n)$.

Each rigid motion can be represented by a square matrix of size $n+1$ decomposed into blocks like

$$\begin{pmatrix} R & U \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix},$$

if and only if $\rho(X) = RX + U$.

The vector space of square matrices of order $n+1$, with real entries, decomposed into blocks like

$$A = \begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix},$$

where Ω is an antisymmetric matrix and U is a vector in space $\mathbb{R}^n$ is denoted by $\mathfrak{se}(n)$.

The group $\mathbf{SE}(n)$ is a Lie group, and its corresponding Lie algebra is $\mathfrak{se}(n)$.

We will show that the exponential map $\exp : \mathfrak{se}(n) \to \mathbf{SE}(n)$ is correctly defined. First we prove the following lemma.

Lemma 2.6. *Given a square matrix of order* $n+1$*, defined in blocks like*

$$A = \begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix},$$

where Ω *is an antisymmetric matrix and* U *is a vector in space* $\mathbb{R}^n$*, the following relation holds*

$$A^k = \begin{pmatrix} \Omega^k & \Omega^{k-1}U \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

unde $\Omega^0 = I_n$.

As a consequence we have

$$\exp A = \begin{pmatrix} \exp\Omega & VU \\ 0 & 1 \end{pmatrix}, \tag{2.4}$$

where

$$V = I_n + \sum_{k=1}^{\infty} \frac{1}{(k+1)!}\Omega^k = \int_0^1 \exp t\Omega dt \tag{2.5}$$

2.4 The groups $\mathbf{GL}(n,\mathbb{C})$, $\mathbf{SL}(n,\mathbb{C})$, $\mathbf{U}(n)$ and $\mathbf{SU}(n)$

The set of square matrices, of the order n, with complex and invertible elements form a group in relation to multiplication, denoted by $\mathbf{GL}(n,\mathbb{C})$. Its subset consists of those matrices whose determinant has value $+1$ is a subgroup of $\mathbf{GL}(n,\mathbb{C})$, denoted by $\mathbf{SL}(n,\mathbb{C})$. It is easy to verify that the subset of unitary matrices, i.e. those that verify the relation $A\,{}^t\bar{A} = I_n$, where $\bar{A}$ is conjugated of the matrix A, forms a subgroup denoted by $\mathbf{U}(n)$. The subset of the group $\mathbf{U}(n)$ which contains matrices with the value of the determinant equal to $+1$ is a subgroup of $\mathbf{U}(n)$, denoted by $\mathbf{SU}(n)$. We can verify that the set of square matrices, of the order n, with complex entries that have zero trace form a real vector space together with the addition and multiplication operation with real scalars. Similarly, we have the same property for symmetric Hermitian matrices and for symmetric hermitian matrices with zero trace.

Definition 2.2. *The group* $\mathbf{GL}(n,\mathbb{C})$ *is called the* ***general linear complex group*** *and his subgroup* $\mathbf{SL}(n,\mathbb{C})$ *is called the* ***special linear complex group****. The group of unitary matrices* $\mathbf{U}(n)$ *is called the* ***unitary group*** *and his subgroup* $\mathbf{SU}(n)$ *is called the* ***special unitary group****.*

The real vector space of square matrices, of dimension n, with complex entries that have zero trace is denoted by $\mathfrak{sl}(n,\mathbb{C})$, the space of the antisymmetric Hermitian matrices is denoted by $\mathfrak{u}(n)$ and the real vector space defined by the intersection $\mathfrak{u}(n)\cap\mathfrak{sl}(n,\mathbb{C})$ is denoted by $\mathfrak{su}(n)$.

Remark 2.3. 1. As in the real case, the groups $\mathbf{GL}(n,\mathbb{C})$, $\mathbf{SL}(n,\mathbb{C})$, $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are topological groups (seen as subspaces of $\mathbb{R}^{2n^2}$), in fact real smooth manifolds. They possess a *Lie* group structure. The real vector spaces $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ are Lie algebras associated with groups $\mathbf{SL}(n,\mathbb{C})$, $\mathbf{U}(n)$ and $\mathbf{SU}(n)$. The structure of Lie algebra is given by the Lie bracket, which is defined by the usual commutator of matrices

$$[A,B] = AB - BA.$$

2. It is also possible to define complex Lie groups, which means that they are topological groups and smooth complex manifolds. It is proved that the groups $\mathbf{GL}(n,\mathbb{C})$ and $\mathbf{SL}(n,\mathbb{C})$ are complex manifolds while groups $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ do not have this property.

The properties of the exponential map play an important role in the study of complex Lie groups. As in the real case, the exponential maps for these groups are the restrictions of the standard map $\exp:\mathfrak{gl}(n,\mathbb{C})\to\mathbf{GL}(n,\mathbb{C})$, discussed in Section 2.1, where $\mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$.

Because we have $\det(\exp A) = e^{\mathrm{tr}A}$, results that the map $\exp:\mathfrak{sl}(n,\mathbb{C})\to\mathbf{SL}(n,\mathbb{C})$ is correctly defined.

If $A\in M_n(\mathbb{C})$, we note $A^* = {}^t\bar{A}$. Let's show now that the map $\exp:\mathfrak{u}(n)\to\mathbf{U}(n)$ is correctly defined. From relation $(\exp A)^* = \exp A^*$, if $A\in\mathfrak{u}(n)$, we have

$$(\exp A)^* = \exp A^* = \exp(-A),$$

we get
$$(\exp A)^*(\exp A) = \exp(-A)\exp A = \exp(-A+A) = I_n,$$

so $\exp A \in \mathbf{U}(n)$.

It is obvious that the map $\exp : \mathfrak{su}(n) \to \mathbf{SU}(n)$ is well defined.

2.5 The surjectivity problem of the exponential map

From property 5) in Section 2.2 of exponential map of a Lie group it follows that the following two problems are of particular importance.

Problem 1. *Find the conditions for the Lie group G so that the exponential map is surjective.*

Problem 2. *In the case in which is not surjective, determine the image $E(G) = \exp(\mathfrak{g})$.*

J. Dixmier first posed the problem of determining the image of the exponential map for solvable Lie groups that are simply connected. Only in a few special situations we have $G = E(G)$, and groups with this property are called *exponential* Lie groups. A monograph dedicated to exponential Lie groups is [64]. Compact and connected Lie groups are exponential [12]. Many of the rank 1 Lie groups that have a free center and are simply connected have the same property [11], [25]. The problem is very complicated in the case of semi-simple groups of rank ≥ 2 and in the case of mixed groups.

These issues are of special interest being studied by several authors, among whom we mention N.J. Higham [23], K.H. Hofmann, A. Mukhergea [26], K.H. Hofmann, W.A.F. Rupert [27], M. Wüstner [65], [66].

In this section we will discuss these problems for the matrix groups reviewed in the previous sections.

2.5.1 The group $\mathbf{GL}(n, \mathbb{R}), n \geq 2$, is not exponential

For $X \in M_n(\mathbb{R}) = \mathfrak{gl}(n, \mathbb{R})$ result that $\mathrm{tr}X \in \mathbb{R}$ and so $\det(\exp(X)) > 0$. Thus, $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathbf{GL}^+(n, \mathbb{R})$, where $\mathbf{GL}^+(n, \mathbb{R})$ is the subgroup of $\mathbf{GL}(n, \mathbb{R})$ defined by non-singular matrices with the strictly positive determinant. Therefore, we have the inclusion $\exp(M_n(\mathbb{R})) \subseteq \mathbf{GL}^+(n, \mathbb{R})$. This is normal if we consider the property 4) of the exponential map in general, mentioned in Section 2.2. So $\mathbf{GL}(n, \mathbb{R})$ is not exponential.

For $n = 1$ we have the identifications $\exp(M_1(\mathbb{R})) = \mathbb{R}$ and $\mathbf{GL}^+(1, \mathbb{R}) = (0, +\infty)$, so in this case $\exp : \mathbb{R} \to (0, +\infty)$ is surjective.

The determination of the set $E(\mathbf{GL}(n, \mathbb{R}))$ is interesting if $n \geq 2$.

The result that completely solves this problem was initially obtained by M. Nishikawa in [44], starting from the paper [50], and reproved independent by D. Andrica and L. Mare [5].

Theorem 2.1. *We consider the matrix $A \in \mathbf{GL}^+(n, \mathbb{R}), n \geq 2$. Then $A \in E(\mathbf{GL}(n, \mathbb{R}))$ if and only if the blocks corresponding to the negative eigenvalues of its Jordan decomposition appear with even multiplicity.*

For example, for the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix},$$

we have $A \in \mathbf{GL}^+(2, \mathbb{R})$, but $A \notin E(\mathbf{GL}(2, \mathbb{R}))$.

Another characterization for the set $E(\mathbf{GL}(n, \mathbb{R}))$ is contained in the following result (see the papers [39] and [40]).

Theorem 2.2. *We have $A \in E(\mathbf{GL}(n, \mathbb{R}))$ if and only if the equation $X^2 = A$ has solutions in $\mathbf{GL}(n, \mathbb{R})$.*

2.5.2 The group $\mathbf{SL}(n, \mathbb{R}), n \geq 2$, is not exponential

The group $\mathbf{SL}(n, \mathbb{R})$ is connected, but is not compact. We will show that the exponential map $\exp : \mathfrak{sl}(2, \mathbb{R}) \to \mathbf{SL}(2, \mathbb{R})$ is not surjective. If

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}),$$

then is observed that

$$X^2 = (a^2 + bc)I_2 = -\det(X)I_2.$$

When $a^2 + bc = 0$ we have $X^p = O_2$ for any $p \geq 2$, so

$$\exp X = I_2 + X.$$

If $a^2 + bc < 0$, let be $\omega > 0$ such that $\omega^2 = -(a^2 + bc)$. Then

$$\exp X = (\cos \omega)I_2 + \frac{\sin \omega}{\omega} X.$$

If $a^2 + bc > 0$, let be $\omega = \sqrt{a^2 + bc}$. Then

$$\exp X = (\operatorname{ch} \omega)I_2 + \frac{\operatorname{sh} \omega}{\omega} X.$$

This matrix function is not surjective. Indeed, we have $\operatorname{tr}(\exp X) = 2\cos \omega$ if $a^2 + bc < 0$, $\operatorname{tr}(\exp X) = 2\operatorname{ch} \omega$ if $a^2 + bc > 0$ and $\operatorname{tr}(\exp X) = 2$ if $a^2 + bc = 0$.

Therefore, for any matrix with zero trace the following relation holds

$$\operatorname{tr}(\exp X) \geq -2,$$

so any matrix A with the determinant equal to 1 and whose trace has a value less than -2 is not exponential of a matrix X with zero trace.

A result of the form of theorem 2.2 it also takes place for the group $\mathbf{SL}(n, \mathbb{R})$ (see

[39]).

Theorem 2.3. *We have* $A \in E(\mathbf{SL}(n,\mathbb{R}))$ *if and only if the equation* $X^2 = A$ *has solutions in* $\mathbf{SL}(n,\mathbb{R})$.

2.5.3 The group SO(n) is exponential

This property results from the fact that $SO(n)$ is compact and connected (see [12], [59]). We further present an elementary proof.

Theorem 2.4. *The special orthogonal group* $\mathbf{SO}(n)$ *is exponential.*

For any square antisymmetric matrix, of the order 3, with real entries

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

we consider the number $\theta = \sqrt{a^2+b^2+c^2}$ and the matrix

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}.$$

Takes place the following result known as the *Rodrigues formula*(1840).

Theorem 2.5. *With the above notations, the exponential map* $\exp : \mathfrak{so}(3) \to \mathbf{SO}(3)$ *is given by*

$$\exp A = (\cos\theta)I_3 + \frac{\sin\theta}{\theta}A + \frac{(1-\cos\theta)}{\theta^2}B.$$

Equivalently, we can write the formula in the form

$$\exp A = I_3 + \frac{\sin\theta}{\theta}A + \frac{(1-\cos\theta)}{\theta^2}A^2,$$

if $\theta \neq 0$.

The surjectivity of the exponential map on the group $\mathbf{SO(n)}$ is an important property that implies the existence of a local inverse function, denoted by ln, $\ln : \mathbf{SO}(n) \to \mathfrak{so}(n)$ which has interesting applications. In the paper of J.Gallier, D.Xu [19] is mentioned that the functions exp and ln for the group $\mathbf{SO(n)}$ can be used in motion interpolation (see M.-J. Kim, M.-S. Shin [32], [33] and F.C. Park, B. Ravani [51],[52]). Motion interpolation and rational motions were also studied by B. Jüttler [36], [37]. Also, the surjectivity of the exponential map of the group $\mathbf{SO(n)}$ gives us the possibility to describe the rotations of Euclidean space $\mathbb{R}^n$ (see R.-A. Rohan [59]). The connection with non-commutative differential geometry is given by the paper of L.I. Piscoran [53].

2.5.4 The group $\mathbf{SE}(n), n \geq 2$, is exponential

Theorem 2.6. *The exponential map* $\exp : \mathfrak{se}(n) \to \mathbf{SE}(n)$ *is surjective.*

In the case $n = 3$, being given an antisymmetric matrix

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

let be $\theta = \sqrt{a^2 + b^2 + c^2}$. It is easy to show that if $\theta = 0$, then

$$\exp A = \begin{pmatrix} I_3 & U \\ 0 & 1 \end{pmatrix}.$$

If $\theta \neq 0$, using that $\Omega^3 = -\theta^2 \Omega$, we obtain

$$\exp \Omega = I_3 + \frac{\sin \theta}{\theta} \Omega + \frac{1 - \cos \theta}{\theta^2} \Omega^2$$

and

$$V = I_3 + \frac{1 - \cos \theta}{\theta^2} \Omega + \frac{\theta - \sin \theta}{\theta^3} \Omega^2.$$

2.5.5 The groups $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are exponential

Theorem 2.7. *The exponential maps*

$$\exp : \mathfrak{u}(n) \to \mathbf{U}(n) \text{ and } \exp : \mathfrak{su}(n) \to \mathbf{SU}(n)$$

are surjective.

The following result shows that any positively defined Hermitian matrix A has the form $\exp B$, where B it is a uniquely determined Hermitian matrix.

Theorem 2.8. *For any Hermitian matrix B, the matrix $\exp B$ is a positively defined Hermitian matrix. For any positively defined Hermitian matrix A, there is a unique Hermitian matrix determined by B so the relation $A = \exp B$ holds.*

Chapter 3

Rodrigues Formulas for Matrix Functions. Methods for Determination of Rodrigues Coefficients

This chapter is organized in six sections. The first section introduces the Rodrigues problem for matrix functions and presents the Rodrigues coefficients. In Theorem 3.1 we present, if the eigenvalues of the matrix are pairwise distinct, a direct method to determine the general Rodrigues coefficients reducing the Rodrigues problem to the system (3.7). Then, Theorem 3.2 gives explicit formulas in terms of fundamental symmetric polynomials of the eigenvalues of the matrix. These formulas allow us to consider the degenerate cases (that is the situations when the eigenvalues have multiplicities) and to obtain formulas for the coefficients. Section 3.3 illustrates the particular cases $n = 2, 3, 4$ for which the computation are effectively presented. In Section 3.4 are studied the possible degenerated cases. Sections 3.5 and 3.6 are dedicated to the Hermite interpolation polynomial method and to the special case of the exponential map for the special orthogonal group. The special orthogonal group $\mathbf{SO}(n)$ has important applications in mechanics, its elements being also called the rotation matrices. After presenting the classical cases $n = 2, 3$, the Rodrigues formula is given in the cases $n = 4$ and $n = 5$, taking into account all possible situations. The MATHEMATICA program was used to perform the computation. The main reference used in this chapter is our paper D. Andrica, O.L. Chender (Broaina) [4]. Other references are D. Andrica, I.N. Casu [2], D. Andrica, R.-A. Rohan [7], T. Bröcker, T. tom Dieck [12], C. Chevalley [14], O. Furdui [17], J. Gallier, D. Xu [19], S. Kida, E. Trimandalawati, S. Ogawa [31], M.-J. Kim, M.-S. Kim, A. Shin [32], [33], B. Jütler [36], [37], J.E. Marsden și T.S. Raţiu [41], F.C. Park, B. Ravani [51], [52], L.I. Piscoran [53], V. Pop, O. Furdui [55], E.J. Putzer [56], R.-A. Rohan [59], F. Warner [62], R. Vein, P. Dale [63], M. Wüstner [64].

3.1 The Rodrigues problem for matrix functions

We saw that the exponential map $\exp : gl(n,\mathbb{R}) = M_n(\mathbb{R}) \to \mathbf{GL}(n,\mathbb{R})$ is defined by (see C. Chevalley [14], J.E. Marsden and T.S. Raţiu [41], or F. Warner [62])

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{3.1}$$

According to the well-known Hamilton-Cayley theorem, it follows that every power X^k, $k \geq n$, is a linear combination of powers X^0, X^1,..., X^{n-1}, hence we can write

$$\exp(X) = \sum_{k=0}^{n-1} a_k(X) X^k, \tag{3.2}$$

where the real coefficients $a_0(X), \dots, a_{n-1}(X)$ are uniquely defined and depend on the matrix X. From this formula, it follows that $\exp(X)$ is a polynomial of X with coefficients functions of X. The problem to find a formula for $\exp(X)$ is reduced to the problem to determine the coefficients $a_0(X), \dots, a_{n-1}(X)$. We will call this general problem, the *Rodrigues problem*, and the numbers $a_0(X), \dots, a_{n-1}(X)$ *Rodrigues coeffiicients* of the exponential map with respect to the matrix $X \in M_n(\mathbb{R})$.

The origin of this problem is the classical Rodriques formula obtained in 1840 for the special orthogonal group $\mathbf{SO(3)}$:

$$\exp(X) = I_3 + \frac{\sin\theta}{\theta} X + \frac{1-\cos\theta}{\theta^2} X^2,$$

where $\sqrt{2}\theta = \|X\|$ and we denoted by $\|X\|$ Frobenius norm of the matrix X (see Theorem 2.5). From the many arguments pointing out the importance of this formula we mention here the study of the rigid body rotation in $\mathbb{R}^3$, and the parametrization of the rotations in $\mathbb{R}^3$.

The general idea of construction of such kind of matrix function is to consider an analytic function $f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m + \cdots$, such that the induced series $\tilde{f}(X) = \alpha_0 I_n + \alpha_1 X + \cdots + \alpha_m X^m + \cdots$ are convergent in an open subset of $M_n(\mathbb{R})$. Then, via Hamilton-Cayley-Frobenius theorem we can write a reduced form for this matrix $\tilde{f}(X)$:

$$\tilde{f}(X) = \sum_{k=0}^{n-1} a_k^{(f)}(X) X^k. \tag{3.3}$$

We call the above relation, the *Rodrigues formula* with respect to $\tilde{f}$. The numbers $a_0^{(f)}(X), \dots, a_{n-1}^{(f)}(X)$ are the *Rodrigues coefficients* of the map $\tilde{f}$ with respect to the matrix $X \in M_n(\mathbb{R})$. Clearly, the real coefficients $a_0^{(f)}(X), \dots, a_{n-1}^{(f)}(X)$ are uniquely defined, they depend on the matrix X, and $\tilde{f}(X)$ is a polynomial of X.

An important property of the Rodrigues coefficients is the invariance under the matrix conjugacy and we have:

Proposition 3.1. *For every invertible matrix U the following relations hold*

$$a_k^{(f)}(UXU^{-1}) = a_k^{(f)}(X), k = 0, \ldots, n-1. \tag{3.4}$$

3.2 The trace method in determination of Rodrigues coefficients

In this section we will present a new way to determine the general Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ introduced in (3.3). Following the paper [7], our main idea consists in the reduction of relation (3.3) to a linear system with the unknowns $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$. Concrete applications in obtaining the Rodrigues formula for the Lorentz group were given in the paper [8].

In this respect we multiply both sides of (3.3) by the matrix power X^j, $j = 0, \ldots, n-1$ and we obtain the matrix relations

$$X^j \tilde{f}(X) = \sum_{k=0}^{n-1} a_k^{(f)} X^{k+j}, \; j = 0, \ldots, n-1, \tag{3.5}$$

where $a_k^{(f)} = a_k^{(f)}(X)$, $k = 0, \ldots, n-1$. Now, considering the matrix trace in the both sides of (3.5) we obtain the linear system

$$\sum_{k=0}^{n-1} \operatorname{tr}(X^{k+j}) a_k^{(f)} = \operatorname{tr}(X^j \tilde{f}(X)), \; j = 0, \ldots, n-1, \tag{3.6}$$

where the coefficients are functions of the matrix X. Now, assume that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of matrix X. Then, it is well-known that the matrix X^{k+j} has the eigenvalues $\lambda_1^{k+j}, \ldots, \lambda_n^{k+j}$, and the matrix $X^j \tilde{f}(X)$ has the eigenvalues $\lambda_1^j f(\lambda_1), \ldots, \lambda_n^j f(\lambda_n)$ (see [29]).

Indeed, the function $f_j : \mathbb{C} \to \mathbb{C}$, $f_j(z) = z^j f(z)$ is analytic, hence the eigenvalues of the matrix $f_j(X)$ are $f_j(\lambda_1), \ldots, f_j(\lambda_n)$ and we have $f_j(\lambda_s) = \lambda_s^j f(\lambda_s), s = 1, \ldots, n$.

According to the considerations above, the system (3.6) is equivalent to

$$\sum_{k=0}^{n-1} \left(\sum_{s=1}^{n} \lambda_s^{k+j} \right) a_k^{(f)} = \sum_{s=1}^{n} \lambda_s^j f(\lambda_s), j = 0, \ldots, n-1. \tag{3.7}$$

From the system (3.7) we obtain the following result concerning the solution to the general Rodrigues problem with respect to the function f (see [4]).

Theorem 3.1. *1) The Rodrigues coefficients in formula (3.3) are solutions to the system (3.7).*

2) If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct, then the Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are perfectly determined by the system (3.7) and *they are given by the formulas*

$$a_k^{(f)}(X) = \frac{V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)}{V_n(\lambda_1, \ldots, \lambda_n)}, k = 0, \ldots, n-1, \tag{3.8}$$

where $V_n(\lambda_1, \ldots, \lambda_n)$ *is the Vandermonde determinant of order* n*, and* $V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)$ *is the determinant of order* n *obtained from* $V_n(\lambda_1, \ldots, \lambda_n)$ *by replacing the line* $k+1$ *by* $f(\lambda_1), \ldots, f(\lambda_n)$.

3) If the eigenvalues $\lambda_1, \ldots, \lambda_n$ *of the matrix* X *are pairwise distinct, then the Rodrigues coefficients* $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ *are linear combinations of* $f(\lambda_1), \ldots, f(\lambda_n)$ *having the coefficients rational functions of* $\lambda_1, \ldots, \lambda_n$*, i.e. we have*

$$a_k^{(f)}(X) = b_k^{(1)}(X)f(\lambda_1) + \ldots + b_k^{(n)}(X)f(\lambda_n), k = 0, \ldots, n-1, \tag{3.9}$$

where $b_k^{(1)}, \ldots, b_k^{(n)} \in \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$.

Expanding the determinant $V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)$ in Theorem 3.1 2) with respect to the line $k+1$ it follows

$$a_k^{(f)}(X) = \frac{1}{V_n} \sum_{j=1}^{n} (-1)^{k+j+1} LV_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) f(\lambda_j), \tag{3.10}$$

where $LV_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n)f(\lambda_j)$ is the $(k+1)$ lacunary Vandermonde determinant in the variables $\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n$, i.e. the determinant obtained from $V_n(\lambda_1, \ldots, \lambda_n)$ by cutting out the row $k+1$ and the column j. Applying the well-known formula (see the reference [63])

$$LV_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) = s_{n-k-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) V_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n),$$

where s_l is the l-th symmetric polynomial in the $n-1$ variabiles $\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n$, where λ_j is missing, we obtain the following result which completely solves the general problem in the case when the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct.

Theorem 3.2. *For every* $k = 0, \ldots, n-1$*, the following formulas hold*

$$a_k^{(f)}(X) = \sum_{j=1}^{n} (-1)^{k+j+1} \frac{V_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) s_{n-k-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n)}{V_n(\lambda_1, \ldots, \lambda_n)} f(\lambda_j), \tag{3.11}$$

where s_l *denotes the* l*-th symmetric polynomial, and* $\widehat{\lambda_j}$ *means that in the Vandermonde determinant* V_{n-1} *the variable* λ_j *is omitted.*

3.3 The particular cases $n = 2, 3, 4$

Clearly, when $X = O_n$, we have $\tilde{f}(X) = \alpha_0 I_n$, and in this situation $a_0^{(f)}(X) = \alpha_0, a_1^{(f)}(X) = \cdots = a_{n-1}^{(f)}(X) = 0$.

In this section we assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct. We follow the presentation from the paper [4].

3.3.1 The case $n = 2$

General Rodrigues formula

$$\tilde{f}(X) = \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} f(\lambda_1) - \frac{\lambda_1}{\lambda_2 - \lambda_1} f(\lambda_2)\right) I_2 + \left(-\frac{1}{\lambda_2 - \lambda_1} f(\lambda_1) + \frac{1}{\lambda_2 - \lambda_1} f(\lambda_2)\right) X. \quad (3.12)$$

This formula appears [55, Theorem 4.7, page 194] and in paper [17].

3.3.2 The case $n = 3$

The corresponding general Rodrigues formula

$$\tilde{f}(X) = \left(\frac{\lambda_2\lambda_3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} f(\lambda_1) - \frac{\lambda_1\lambda_3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} f(\lambda_2) + \frac{\lambda_1\lambda_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} f(\lambda_3)\right) I_3 +$$

$$\left(-\frac{\lambda_2 + \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} f(\lambda_1) + \frac{\lambda_3 + \lambda_1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} f(\lambda_2) - \frac{\lambda_1 + \lambda_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} f(\lambda_3)\right) X +$$

$$\left(\frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} f(\lambda_1) - \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} f(\lambda_2) + \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} f(\lambda_3)\right) X^2.$$

3.3.3 The case $n = 4$

$$a_0^{(f)}(X) = \frac{\lambda_2\lambda_3\lambda_4}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} f(\lambda_1) - \frac{\lambda_1\lambda_3\lambda_4}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} f(\lambda_2) +$$

$$\frac{\lambda_1\lambda_2\lambda_4}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)} f(\lambda_3) - \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} f(\lambda_4),$$

$$a_1^{(f)}(X) = -\frac{\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} f(\lambda_1) + \frac{\lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} f(\lambda_2) -$$

$$\frac{\lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)} f(\lambda_3) + \frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} f(\lambda_4),$$

$$a_2^{(f)}(X) = \frac{\lambda_2+\lambda_3+\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} f(\lambda_1) - \frac{\lambda_1+\lambda_3+\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} f(\lambda_2) +$$

$$\frac{\lambda_1+\lambda_2+\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_3) - \frac{\lambda_1+\lambda_2+\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_4),$$

$$a_3^{(f)}(X) = -\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} f(\lambda_1) + \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} f(\lambda_2) -$$

$$\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_3) + \frac{1}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_4),$$

and the corresponding general Rodrigues formula but we do not write it here because of the space reason.

3.4 The degenerate cases $n = 2, 3, 4$

In this section we show how to obtain the general Rodrigues coefficients when the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the matrix X are not distinct, when $n = 2, 3, 4$. We follow the presentation from the paper [4].

3.4.1 The case $n = 2$

Assume that $\lambda_1 = \lambda_2$. Then the corresponding general Rodrigues coefficients can be obtained from the formulas in subsection 3.3.1 for $\lambda_2 \to \lambda_1$. Using the formula of the derivative of a functional determinant we get

$$a_0^{(f)}(X) = \begin{vmatrix} f(\lambda_1) & f'(\lambda_1) \\ \lambda_1 & 1 \end{vmatrix} = f(\lambda_1) - \lambda_1 f'(\lambda_1)$$

$$a_1^{(f)}(X) = \begin{vmatrix} 1 & 0 \\ f(\lambda_1) & f'(\lambda_1) \end{vmatrix} = f'(\lambda_1).$$

This formula appear in [55, Theorem 4.8, page 194] and in paper [17].

3.4.2 The case $n = 3$

In this case we have to consider the folowing two possibilities, if we dont take into account the permutations of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

The case $\lambda_1 = \lambda_2 \neq \lambda_3$

The corresponding general Rodrigues coefficients can be obtained from the formulas in subsection 3.3.2 for $\lambda_2 \to \lambda_1$. Using again the formula of the derivative of a functional determinant we get

$$a_0^{(f)}(X) = \frac{\lambda_3^2 - 2\lambda_1\lambda_3}{(\lambda_3-\lambda_1)^2} f(\lambda_1) - \frac{\lambda_1\lambda_3}{\lambda_3-\lambda_1} f'(\lambda_1) + \frac{\lambda_1^2}{(\lambda_3-\lambda_1)^2} f(\lambda_3)$$

$$a_1^{(f)}(X) = \frac{2\lambda_1}{(\lambda_3-\lambda_1)^2} f(\lambda_1) + \frac{\lambda_3^2}{(\lambda_3-\lambda_1)^2} f'(\lambda_1) + \frac{2\lambda_1}{(\lambda_3-\lambda_1)^2} f(\lambda_3)$$

$$a_2^{(f)}(X) = -\frac{1}{(\lambda_3-\lambda_1)^2} f(\lambda_1) - \frac{1}{\lambda_3-\lambda_1} f'(\lambda_1) + \frac{1}{(\lambda_3-\lambda_1)^2} f(\lambda_3).$$

The case $\lambda_1 = \lambda_2 = \lambda_3$

We use the formulas obtained in 3.4.2 for $\lambda_3 \to \lambda_1$, and we obtain

$$a_0^{(f)}(X) = f(\lambda_1) - \lambda_1 f'(\lambda_1) + \frac{1}{2}\lambda_1^2 f''(\lambda_1), a_1^{(f)}(X) = f'(\lambda_1) - \lambda_1 f''(\lambda_1), a_2^{(f)}(X) = \frac{1}{2} f''(\lambda_1)$$

and the corresponding Rodrigues formula.

3.4.3 The case $n = 4$

In this case we have to consider the following four possibilities, without taking into account the permutations of the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

The case $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$

The general Rodrigues coefficients can be obtained from the formulas in subsection 3.3.3 for $\lambda_2 \to \lambda_1$ and using the formula of the derivative of a functional determinant. We get

$$a_0^{(f)}(X) = \frac{\lambda_3\lambda_4(3\lambda_1^2 + \lambda_3\lambda_4 - 2\lambda_1(\lambda_3+\lambda_4))}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2} f(\lambda_1) - \frac{\lambda_1\lambda_3\lambda_4}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} f'(\lambda_1)$$

$$+\frac{\lambda_1^2\lambda_4}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_3)} f(\lambda_3) - \frac{\lambda_1^2\lambda_3}{(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)} f(\lambda_4),$$

$$a_1^{(f)}(X) = \frac{-\lambda_1(3\lambda_1(\lambda_3+\lambda_4) - 2(\lambda_3^2 + \lambda_3\lambda_4 + \lambda_4^2))}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2} f(\lambda_1) + \frac{\lambda_3\lambda_4 + \lambda_1(\lambda_3+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} f'(\lambda_1)$$

$$-\frac{\lambda_1(\lambda_1+2\lambda_4)}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_3)} f(\lambda_3) + \frac{\lambda_1(\lambda_1+2\lambda_3)}{(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)} f(\lambda_4),$$

$$a_2^{(f)}(X) = \frac{3\lambda_1^2 - \lambda_3^2 - \lambda_3\lambda_4 - \lambda_4^2}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2} f(\lambda_1) - \frac{(\lambda_1+\lambda_3+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} f'(\lambda_1)$$

$$+\frac{2\lambda_1+\lambda_4}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_3)}f(\lambda_3)-\frac{2\lambda_1+\lambda_3}{(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)}f(\lambda_4),$$

$$a_3^{(f)}(X)=\frac{-2\lambda_1+\lambda_3+\lambda_4}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2}f(\lambda_1)+\frac{1}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}f'(\lambda_1)$$

$$-\frac{1}{(\lambda_4-\lambda_3)(\lambda_3-\lambda_1)^2}f(\lambda_3)+\frac{1}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_1)^2}f(\lambda_4).$$

The case $\lambda_1=\lambda_2=\lambda_3\neq\lambda_4$

We use the formulas in the case 3.4.3 for $\lambda_3\to\lambda_1$, and obtain

$$a_0^{(f)}(X)=\frac{\lambda_1^3+(\lambda_4-\lambda_1)^3}{(\lambda_4-\lambda_1)^3}f(\lambda_1)+\frac{\lambda_1\lambda_4(2\lambda_1-\lambda_4)}{(\lambda_4-\lambda_1)^2}f'(\lambda_1)+\frac{\lambda_1^2\lambda_4}{2(\lambda_4-\lambda_1)}f''(\lambda_1)-\frac{\lambda_1^3}{(\lambda_4-\lambda_1)^3}f(\lambda_4),$$

$$a_1^{(f)}(X)=\frac{-3\lambda_1^2}{(\lambda_4-\lambda_1)^3}f(\lambda_1)+\frac{-2\lambda_1^2-2\lambda_1\lambda_4+\lambda_4^2}{(\lambda_4-\lambda_1)^2}f'(\lambda_1)-\frac{\lambda_1(\lambda_1+2\lambda_4)}{2(\lambda_4-\lambda_1)}f''(\lambda_1)+\frac{3\lambda_1^2}{(\lambda_4-\lambda_1)^3}f(\lambda_4),$$

$$a_2^{(f)}(X)=\frac{3\lambda_1}{(\lambda_4-\lambda_1)^3}f(\lambda_1)+\frac{3\lambda_1}{(\lambda_4-\lambda_1)^2}f'(\lambda_1)+\frac{2\lambda_1+\lambda_4}{2(\lambda_4-\lambda_1)}f''(\lambda_1)-\frac{3\lambda_1}{(\lambda_4-\lambda_1)^3}f(\lambda_4),$$

$$a_3^{(f)}(X)==\frac{-1}{(\lambda_4-\lambda_1)^3}f(\lambda_1)-\frac{1}{(\lambda_4-\lambda_1)^2}f'(\lambda_1)-\frac{1}{2(\lambda_4-\lambda_1)}f''(\lambda_1)+\frac{1}{(\lambda_4-\lambda_1)^3}f(\lambda_4).$$

The case $\lambda_1=\lambda_2=\lambda_3=\lambda_4$

We use the formulas obtained in the previous case for $\lambda_4\to\lambda_1$, and we get

$$a_0^{(f)}(X)=2f(\lambda_1)-2\lambda_1f'(\lambda_1)+\lambda_1^2f''(\lambda_1)-\frac{\lambda_1^3}{3}f'''(\lambda_1),$$

$$a_1^{(f)}(X)=2f'(\lambda_1)-2\lambda_1f''(\lambda_1)+\lambda_1^2f'''(\lambda_1),$$

$$a_2^{(f)}(X)=f''(\lambda_1)-\lambda_1f'''(\lambda_1),$$

$$a_3^{(f)}(X)=\frac{1}{3}f'''(\lambda_1).$$

The case $\lambda_1=\lambda_2,\lambda_3=\lambda_4$ **și** $\lambda_2\neq\lambda_4$

The general Rodrigues coefficients can be obtained from the formulas in subsection 3.4.3 for $\lambda_4\to\lambda_3$. We obtain

$$a_0^{(f)}(X) = \frac{\lambda_3^2(-3\lambda_1+\lambda_3)}{(\lambda_3-\lambda_1)^3}f(\lambda_1) - \frac{\lambda_1\lambda_3^2}{(\lambda_3-\lambda_1)^2}f'(\lambda_1) - \frac{\lambda_1^2(\lambda_1-3\lambda_3)}{(\lambda_3-\lambda_1)^3}f(\lambda_3) - \frac{\lambda_1^2\lambda_3}{(\lambda_3-\lambda_1)^2}f'(\lambda_3),$$

$$a_1^{(f)}(X) == \frac{6\lambda_1\lambda_3}{(\lambda_3-\lambda_1)^3}f(\lambda_1) + \frac{\lambda_3(2\lambda_1+\lambda_3)}{(\lambda_3-\lambda_1)^2}f'(\lambda_1) - \frac{6\lambda_1\lambda_3}{(\lambda_3-\lambda_1)^3}f(\lambda_3) + \frac{\lambda_1(\lambda_1+2\lambda_3)}{(\lambda_3-\lambda_1)^2}f'(\lambda_3),$$

$$a_2^{(f)}(X) = \frac{-3(\lambda_1+\lambda_3)}{(\lambda_3-\lambda_1)^3}f(\lambda_1) - \frac{\lambda_1+2\lambda_3}{(\lambda_3-\lambda_1)^2}f'(\lambda_1) + \frac{3(\lambda_1+\lambda_3)}{(\lambda_3-\lambda_1)^3}f(\lambda_3) - \frac{2\lambda_1+\lambda_3}{(\lambda_3-\lambda_1)^2}f'(\lambda_3),$$

$$a_3^{(f)}(X) = \frac{2}{(\lambda_3-\lambda_1)^3}f(\lambda_1) + \frac{1}{(\lambda_3-\lambda_1)^2}f'(\lambda_1) - \frac{2}{(\lambda_3-\lambda_1)^3}f(\lambda_3) + \frac{1}{(\lambda_3-\lambda_1)^2}f'(\lambda_3).$$

3.5 The Hermite interpolating polynomial method

Assume that the function f is defined on spectrum of matrix $X \in M_n(\mathbb{C})$. Considering Theorem 1.6 we have $\tilde{f}(X) = r(X)$, where r is the Hermite interpolating polynomial that satisfies the conditions

$$r^{(j)}(\lambda_i) = f^{(j)}(\lambda_i), i = 1, \ldots, s, j = 0, \ldots, n_i - 1$$

where $\lambda_1, \ldots, \lambda_s$ are distinct eigenvalues of X with multiplicities $n_1, \ldots, n_s$ and $n_1 + \cdots + n_s = n$.

In this case the Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_n^{(f)}(X)$ of the map $\tilde{f}$ for the matrix X are the coefficients of the Hermite polynomial defined by the above interpolation conditions. This is defined by

$$r(t) = \sum_{i=1}^{s}\left[\left(\sum_{j=0}^{n_i-1}\frac{1}{j!}\Phi_i^{(j)}(\lambda_i)(t-\lambda_i)^j\right)\prod_{j\neq i}(t-\lambda_j)^{n_j}\right] \tag{3.13}$$

where $\Phi_i(t) = f(t)/\prod_{j\neq i}(t-\lambda_j)^{n_j}$.

If the eigenvalues of the matrix X are pairwise distinct, then the Hermite polynomial r is reduced to Lagrange interpolating polynomial with conditions $r(\lambda_i) = f(\lambda_i), i = 1, \ldots, n$,

$$r(t) = \sum_{i=1}^{n} f(\lambda_i)l_i(t), \tag{3.14}$$

where l_i are the Lagrange fundamental polynomials defined by

$$l_i(t) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{t-\lambda_j}{\lambda_i - \lambda_j}, i = 1, \ldots, n. \tag{3.15}$$

3.5.1 The complexity of the Rodrigues problem

The determination of the algebraic form of the Hermite polynomial given by (3.13) is a problem equivalent to the problem of determining the Rodrigues coefficients of the map $\tilde{f}$ when the eigenvalues of the matrix X are known. How the effective determination of the spectrum of the matrix X involves solving an algebraic equation of degree n, we can say that the Rodrigues problem has greater complexity than the problem of explicitly determining the coefficients of the Hermite polynomial for the general context. This is a complicated problem if n and the multiplicities $n_1, \ldots, n_s$ are greater (see [31]).

On the other hand, the Frobenius covariants X_j are polynomials in X, so we have $X_j = p_j(X), j = 1, \ldots, \mu$. Developing $(X - \lambda_j I_n)^k$ in Schwerdtfeger formula (1.17) we obtain

$$\sum_{k=0}^{n-1} a_k^{(f)}(X)X^k = \sum_{j=1}^{\mu} p_j(X) \sum_{k=0}^{m_j-1} \frac{1}{k!} f^{(k)}(\lambda_j) \sum_{s=0}^{k} (-1)^s C_k^s \lambda_j^s X^{k-s}.$$

Identifying the coefficient of X^k in this relation, we obtain the Rodrigues coefficients $a_k^{(f)}(X)$, for $k = 0, \ldots, n-1$. This approach provides another image of the complexity of the Rodrigues problem by reducing it to the determination of the polynomials $p_j, j = 1, \ldots, \mu$.

If the eigenvalues of the matrix X are pairwise distinct, the formulas (3.8) and (3.11) give the explicit form for the coefficients of the Lagrange polynomial that satisfies the above interpolation conditions.

We further illustrate this method for the degenerate case presented for $n = 4$ in subsection 3.4.3 corresponding to the situation $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$. The interpolation conditions are $r(\lambda_1) = f(\lambda_1), r'(\lambda_1) = f'(\lambda_1), r(\lambda_3) = f(\lambda_3), r(\lambda_4) = f(\lambda_4)$.

The Hermite's interpolation polynomial is

$$\begin{aligned} r(t) = & \left\{ \frac{f(\lambda_1)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} + \left[\left(\frac{1}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)} + \frac{1}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)^2} \right) f(\lambda_1) + \right.\right. \\ & \left.\left. \frac{f'(\lambda_1)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} \right] (t-\lambda_1) \right\} (t-\lambda_3)(t-\lambda_4) - \frac{f(\lambda_3)}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)}(t-\lambda_1)^2(t-\lambda_4) \\ & + \frac{f(\lambda_4)}{(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)}(t-\lambda_1)^2(t-\lambda_3). \end{aligned}$$

By calculating the coefficients of the polynomial r we find the formulas corresponding to this case in the subsection 3.4.3.

3.5.2 Solving the Rodrigues problem for eigenvalues with double multiplicity

In this subsection, we consider that the function f is defined on the spectrum of the matrix $X \in M_{2s}(\mathbb{C})$ and distinct eigenvalues $\lambda_1, \dots, \lambda_s$ of X have double multiplicity, that is, we have $n_1 = \dots = n_s = 2$. In this case the Hermite interpolation polynomial r satisfies the conditions

$$r(\lambda_i) = f(\lambda_i), r'(\lambda_i) = f'(\lambda_i), i = 1, \dots, s$$

and the formula (3.13) becomes

$$r(t) = \sum_{i=1}^{s} \left[f(\lambda_i) \left(1 - 2l_i'(\lambda_i)(t - \lambda_i)\right) + f'(\lambda_i)(t - \lambda_i) \right] l_i^2(t), \tag{3.16}$$

where l_i are the fundamental Lagrange polynomials defined in (3.15).

We notice that the formula(3.16) it can be written in the form

$$r(t) = \sum_{i=1}^{s} (A_i t + B_i) r_i(t), \tag{3.17}$$

where

$$A_i = \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{s} (\lambda_i - \lambda_j)^2} \left[f'(\lambda_i) - 2f(\lambda_i) l_i'(\lambda_i) \right],$$

$$B_i = \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{s} (\lambda_i - \lambda_j)^2} \left[f(\lambda_i) \left(1 + 2\lambda_i l_i'(\lambda_i)\right) - \lambda_i f'(\lambda_i) \right]$$

and r_i is the polynomial $\prod\limits_{\substack{j=1 \\ j \neq i}}^{s} (t - \lambda_j)^2, i = 1, \dots, s$.

On the other hand we have $l_i(\lambda_i) = 1$ and

$$\frac{l_i'(t)}{l_i(t)} = \sum_{\substack{j=1 \\ j \neq i}}^{s} \frac{1}{t - \lambda_j}, i = 1, \dots, s,$$

so we get

$$l_i'(t) = \sum_{\substack{j=1 \\ j \neq i}}^{s} \frac{1}{\lambda_i - \lambda_j}, i = 1, \dots, s. \tag{3.18}$$

To obtain the algebraic form of the polynomial r_i notice that we can write

$$r_i(t) = \prod_{\substack{j=1\\j\neq i}}^{s}(t-\lambda_j)^2 = \prod_{\substack{j=1\\j\neq i}}^{s}(t-\lambda_j)(t-\lambda_j) = t^{2s-2}-\sigma_{i,1}t^{2s-1}+\sigma_{i,2}t^{2s-2}-\cdots+\sigma_{i,2s-2}, \quad (3.19)$$

where $\sigma_{i,k}(\lambda_1,\ldots,\lambda_s) = s_k(\lambda_1,\lambda_1,\ldots,\widehat{\lambda_i},\widehat{\lambda_i},\ldots,\lambda_s,\lambda_s)$ is the symmetric polynomial of order k in $2s-2$ variable $\lambda_1,\lambda_1,\ldots,\widehat{\lambda_i},\widehat{\lambda_i},\ldots,\lambda_s,\lambda_s$, of which λ_i is missing, for all $k = 1,\ldots,2s-2$.

Combining formulas (3.17) and (3.19) we obtain

$$\begin{aligned} r(t) &= \sum_{i=1}^{s}(A_it+B_i)(t^{2s-2}-\sigma_{i,1}t^{2s-1}+\sigma_{i,2}t^{2s-2}-\cdots+\sigma_{i,2s-2}) \\ &= \left(\sum_{i=1}^{s}A_i\right)t^{2s-1}+\sum_{i=1}^{s}(-A_i\sigma_{i,1}+B_i)t^{2s-2}+\cdots \\ &+\sum_{i=1}^{s}(A_i\sigma_{i,2}-B_i\sigma_{i,1})t^{2s-3}+\cdots+\sum_{i=1}^{s}(A_i\sigma_{i,2s-2}-B_i\sigma_{i,2s-3})t+ \\ &+\sum_{i=1}^{s}B_i\sigma_{i,2s-2}. \end{aligned}$$

Thus we obtain the following result which completely solves the Rodrigues general problem if the eigenvalues $\lambda_1,\ldots,\lambda_s$ are distinct and have double multiplicity.

Theorem 3.3. *For any $k = 0,1,\ldots,n-1$, we have*

$$\begin{aligned} a_k^{(f)}(X) = (-1)^{k+1}\sum_{i=1}^{s}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{s}(\lambda_i-\lambda_j)^2}&\left\{\left[f'(\lambda_i)-2f(\lambda_i)\sum_{\substack{j=1\\j\neq i}}^{s}\frac{1}{\lambda_i-\lambda_j}\right]\sigma_{i,2s-k-1}\right. \\ &\left.-\left[f(\lambda_i)\left(1+2\lambda_i\sum_{\substack{j=1\\j\neq i}}^{s}\frac{1}{\lambda_i-\lambda_j}\right)-\lambda_if'(\lambda_i)\right]\sigma_{i,2s-k-2}\right\} \end{aligned} \quad (3.20)$$

Corollary 3.1. *If the eigenvalues $\lambda_1,\ldots,\lambda_s$ of the matrix $X \in M_n(\mathbb{C}), n = 2s$, are pairwise distinct and have double multiplicity, than the Rodrigues coefficients $a_0^{(f)}(X),\ldots,a_{n-1}^{(f)}(X)$ are linear combinations of $f(\lambda_1),\ldots,f(\lambda_s),f'(\lambda_1),\ldots,f'(\lambda_s)$ having the coefficients rational functions of $\lambda_1,\ldots,\lambda_s$, that is, we have*

$$a_k^{(f)}(X) = b_k^{(1)}(X)f(\lambda_1)+\cdots+b_k^{(s)}(X)f(\lambda_s)+c_k^{(1)}(X)f'(\lambda_1)+\cdots+c_k^{(s)}f'(\lambda_s),$$

where $b_k^{(1)},\ldots,b_k^{(s)},c_k^{(1)},\ldots,c_k^{(s)} \in \mathbb{Q}[\lambda_1,\ldots,\lambda_s], k = 0,\ldots,n-1$.

Next we present the formulas (3.19) to determine the coefficient $a_0^{(f)}(X)$ in the case $n = 4$ presented in subsection 3.4.3 for $\lambda_1 = \lambda_3, \lambda_2 = \lambda_4$ and $\lambda_1 \neq \lambda_2$. In this situation we have

$$\sigma_{1,1}(\lambda_1, \lambda_2) = \sigma_1(\widehat{\lambda_1}, \widehat{\lambda_1}, \lambda_2, \lambda_2) = 2\lambda_2, \sigma_{1,2}(\lambda_1, \lambda_2) = \sigma_2(\widehat{\lambda_1}, \widehat{\lambda_1}, \lambda_2, \lambda_2) = \lambda_2^2,$$

$$\sigma_{2,1}(\lambda_1, \lambda_2) = \sigma_1(\lambda_1, \lambda_1, \widehat{\lambda_2}, \widehat{\lambda_2}) = 2\lambda_1, \sigma_{2,2}(\lambda_1, \lambda_2) = \sigma_2(\lambda_1, \lambda_1, \widehat{\lambda_2}, \widehat{\lambda_2}) = \lambda_1^2.$$

Applying the formula(3.20) we find the coefficient $a_0^{(f)}(X)$ in the form

$$\begin{aligned} a_0^{(f)}(X) = & \frac{1}{(\lambda_1 - \lambda_2)^2} \left[f(\lambda_1) \left(\lambda_2^2 + \frac{2\lambda_1\lambda_2^2}{\lambda_1 - \lambda_2} \right) - \lambda_1\lambda_2^2 f'(\lambda_1) + \right. \\ & \left. + f(\lambda_2) \left(\lambda_1^2 + \frac{2\lambda_1^2\lambda_2}{\lambda_2 - \lambda_1} \right) - \lambda_1^2\lambda_2 f'(\lambda_2) \right] = \\ & = \frac{\lambda_2^2(-3\lambda_1 + \lambda_2)}{(\lambda_2 - \lambda_1)^3} f(\lambda_1) - \frac{\lambda_1\lambda_2^2}{(\lambda_2 - \lambda_1)^2} f'(\lambda_1) + \\ & + \frac{-\lambda_1^2(\lambda_1 - 3\lambda_2)}{(\lambda_2 - \lambda_1)^3} f(\lambda_2) - \frac{\lambda_1^2\lambda_2}{(\lambda_2 - \lambda_1)^2} f'(\lambda_2). \end{aligned}$$

3.5.3 The determinant formula for Rodrigues coefficients in the case of eigenvalues with double multiplicity

The formulas (3.20) can be written in a compact and uniform form by using conveniently chosen determinants. The starting point is the formula (3.8) applicated for the function f defined on the spectrum of the matrix $X \in M_{2s}(\mathbb{C})$ which we assume is analytical. We consider the distinct eigenvalues $\lambda_1, \lambda_1', \lambda_2, \lambda_2', \ldots, \lambda_s, \lambda_s'$ of the matrix X and we make successively $\lambda_1' \to \lambda_1, \lambda_2' \to \lambda_2, \ldots, \lambda_s' \to \lambda_s$. Using the derivative formula of a functional determinant and l'Hospital's rule we obtain the following result.

Theorem 3.4. *If the eigenvalues $\lambda_1, \ldots, \lambda_s$ of the matrix X are pairwise distinct, then the Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are given by*

$$a_k^{(f)}(X) = \frac{1}{\prod\limits_{1 \leq i < j \leq s} (\lambda_j - \lambda_i)^2} \det U_{n,k}^{(f,f')}(\lambda_1, \ldots, \lambda_s) \tag{3.21}$$

where $U_{n,k}^{(f,f')}(\lambda_1, \ldots, \lambda_s)$ is the $n \times n$ matrix defined in $n \times 2$ blocks

$$U_{n,k}^{(f,f')}(\lambda_1, \ldots, \lambda_s) = \left(\left[U_k^{(f,f')}(\lambda_1) \right] \ldots \left[U_k^{(f,f')}(\lambda_s) \right] \right)$$

and the block $U_k^{(f,f')}$ *is given by*

$$U_k^{(f,f')}(\lambda_j) = \begin{pmatrix} 1 & 0 \\ \lambda_j & 1 \\ \vdots & \vdots \\ f(\lambda_j) & f'(\lambda_j) \\ \vdots & \vdots \\ \lambda_j^{n-1} & (n-1)\lambda_j^{n-2} \end{pmatrix}, j = 1, \ldots, s. \tag{3.22}$$

The entries $f(\lambda_j), f'(\lambda_j)$ *are found on the line* $k+1$*, and the entries situated on the second column are obtained by derivation in relation to* λ_j *of the corresponding entries on the first column.*

To illustrate the formula (3.21), we consider $n = 6$, the analytical function f is defined on the spectrum of the matrix $X \in M_6(\mathbb{C})$ with distinct eigenvalues each with double multiplicity $\lambda_1, \lambda_2, \lambda_3$. The blocks (3.22) defined by the 6×2 matrices which give the Rodrigues coefficient $a_1^{(f)}(X)$ from formula (3.21) are

$$U_1^{(f,f')}(\lambda_1) = \begin{pmatrix} 1 & 0 \\ f(\lambda_1) & f'(\lambda_1) \\ \lambda_1^2 & 2\lambda_1 \\ \lambda_1^3 & 3\lambda_1^2 \\ \lambda_1^4 & 4\lambda_1^3 \\ \lambda_1^5 & 5\lambda_1^4 \end{pmatrix}, U_1^{(f,f')}(\lambda_2) = \begin{pmatrix} 1 & 0 \\ f(\lambda_2) & f'(\lambda_2) \\ \lambda_2^2 & 2\lambda_2 \\ \lambda_2^3 & 3\lambda_2^2 \\ \lambda_2^4 & 4\lambda_2^3 \\ \lambda_2^5 & 5\lambda_2^4 \end{pmatrix}, U_1^{(f,f')}(\lambda_3) = \begin{pmatrix} 1 & 0 \\ f(\lambda_3) & f'(\lambda_3) \\ \lambda_3^2 & 2\lambda_3 \\ \lambda_3^3 & 3\lambda_3^2 \\ \lambda_3^4 & 4\lambda_3^3 \\ \lambda_3^5 & 5\lambda_3^4 \end{pmatrix}$$

From formula (3.21) we obtain

$$a_1^{(f)}(X) = \frac{1}{(\lambda_2-\lambda_1)^2(\lambda_3-\lambda_1)^2(\lambda_3-\lambda_2)^2} \det\left(\left[U_1^{(f,f')}(\lambda_1)\right]\left[U_1^{(f,f')}(\lambda_2)\right]\left[U_1^{(f,f')}(\lambda_3)\right]\right).$$

3.6 The exponential map of the special orthogonal group SO(n)

We saw in Section 2.2 that the set of the real $n \times n$ orthogonal matrices forms a Lie group under multiplication, denoted by $\mathbf{O}(n)$. The subset of $\mathbf{O}(n)$ consisting of those matrices having the determinant equal to $+1$ is a subgroup, denoted by $\mathbf{SO}(n)$ and called the *special orthogonal group* of the Euclidean space $\mathbb{R}^n$. $\mathbf{SO}(n)$ is an important group used in Mechanics (see the famous book of V.I.Arnold [9]) and in other research directions. Due to geometric reasons, the matrices in $\mathbf{SO}(n)$ are also called *rotation matrices.*

The Lie algebra $\mathfrak{so}(n)$ of $\mathbf{SO(n)}$ consists in all skewsymmetric matrices in $M_n(\mathbb{R})$, and the Lie bracket is the standard matrices commutator defined by $[A, B] = AB - BA$. The exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$ is defined by the restriction $\exp|_{\mathfrak{so}(n)}$ of the exponential map $\exp : \mathfrak{gl}(n,\mathbb{R}) \to \mathbf{GL}(n,\mathbb{R})$ (see Section 1.1).

In what follows we apply the results obtained in Sections 3.2−3.4 to get the Rodrigues formulas for the exponential map on the special orthogonal group $\mathbf{SO(n)}$. The matrices from the Lie algebra $\mathfrak{so}(n)$ have two essential properties which simplify the computation of the Rodrigues coefficients:

- If n is odd, then they are singular, i.e. they have one eigenvalue equal to 0 (possible with a multiplicity);
- The non-zero eigenvalues are purely imaginary and, of course, conjugated.

Some particular cases have been studied in the papers [54] and [61].

3.6.1 The classical cases $n = 2, 3$

When $n = 2$, a skew-symmetric matrix $X \neq O_2$ can be written as

$$X = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*,$$

having the eigenvalues $\lambda_1 = ai$, $\lambda_2 = -ai$.

From the formulas derived in subsection 3.3.1 we immediately obtain

$$a_0 = \frac{\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}}{\lambda_2 - \lambda_1} = \frac{1}{2}\left(e^{ai} + e^{-ai}\right) = \cos a,$$

$$a_1 = \frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} = \frac{e^{ai} - e^{-ai}}{2ai} = \frac{\sin a}{a},$$

and then the corresponding Rodrigues formula is

$$\exp(X) = (\cos a) I_2 + \frac{\sin a}{a} X.$$

When $n = 3$, a real skew-symmetric matrix X is of the form

$$X = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

having the characteristic polynomial

$$p_X(t) = t^3 + (a^2 + b^2 + c^2)t = t^3 + \theta^2 t,$$

where $\theta = \sqrt{a^2 + b^2 + c^2}$. The eigenvalues of X are $\lambda_1 = \theta i$, $\lambda_2 = -\theta i$, $\lambda_3 = 0$. It is clear that $X = O_3$ if and only if $\theta = 00$, hence it suffices to consider only the situation $\theta \neq 0$.

Because $\theta \neq 0$, using the formulas obtained in subsection 3.3.1, it follows that

$$a_0 = 1,\ a_1 = \frac{\sin\theta}{\theta},\ a_2 = \frac{1-\cos\theta}{\theta^2},$$

giving the well-known classical formula due to Rodrigues

$$\exp(X) = I_3 + \frac{\sin\theta}{\theta}X + \frac{1-\cos\theta}{\theta^2}X^2.$$

also trated in the Section 2.3.

3.6.2 The case $n = 4$

The general form of a matrix $X \in so(4)$ is

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix},$$

and the corresponding characteristic polynomial is given by

$$p_X(t) = t^4 + (a^2+b^2+c^2+d^2+e^2+f^2)t^2 + (af - be + cd)^2.$$

Let $\lambda_{1,2} = \pm\alpha i$, $\lambda_{3,4} = \pm\beta i$ be the eigenvalues of the matrix X, where $\alpha,\ \beta \in \mathbb{R}$. It is clear that the real numbers α and β can be effectively determined in terms of a, b, c, d, e, f by solving the equation $p_X(t) = 0$.

We consider the following three cases

Case 1. If $|\alpha| \neq |\beta|$, $\alpha,\ \beta \in \mathbb{R}^*$, then using the formulas in subsection 3.3.3, after simple computations we obtain the Rodrigues coefficients

$$a_0 = \frac{\beta^2\cos\alpha - \alpha^2\cos\beta}{\beta^2-\alpha^2}, a_1 = \frac{\beta^3\sin\alpha - \alpha^3\sin\beta}{\alpha\beta(\beta^2-\alpha^2)},$$

$$a_2 = \frac{\cos\alpha - \cos\beta}{\beta^2-\alpha^2}, a_3 = \frac{\beta\sin\alpha - \alpha\sin\beta}{\alpha\beta(\beta^2-\alpha^2)}.$$

In this case it follows the corresponding Rodrigues formula in the form

$$\begin{aligned}\exp(X) = {} & \frac{\beta^2\cos\alpha - \alpha^2\cos\beta}{\beta^2-\alpha^2}I_4 + \frac{\beta^3\sin\alpha - \alpha^3\sin\beta}{\alpha\beta(\beta^2-\alpha^2)}X \\ & + \frac{\cos\alpha - \cos\beta}{\beta^2-\alpha^2}X^2 + \frac{\beta\sin\alpha - \alpha\sin\beta}{\alpha\beta(\beta^2-\alpha^2)}X^3.\end{aligned} \tag{3.23}$$

This formula was obtained by D. Andrica și R.-A. Rohan [7] using Putzer's method (see [56]).

Case 2. If $\alpha \neq 0$ and $\beta = 0$, then we will use the formulas in subsection 3.4.3 when

$\lambda_1 \neq \lambda_2 \neq \lambda_3 = \lambda_4$, and obtain

$$a_0 = 1, a_1 = 1, a_2 = \frac{1-\cos\alpha}{\alpha^2}, a_3 = \frac{\alpha - \sin\alpha}{\alpha^3}. \tag{3.24}$$

Therefore, the corresponding Rodrigues formula to this case is

$$\exp(X) = I_4 + X + \frac{1-\cos\alpha}{\alpha^2}X^2 + \frac{\alpha-\sin\alpha}{\alpha^3}X^3. \tag{3.25}$$

Case 3. If $\alpha = \beta \neq 0$, then we will use the formulas in subsection 3.4.3 for $\lambda_1 = \lambda_3, \lambda_2 = \lambda_4, \lambda_1 \neq \lambda_2$, and after simple computations we get

$$a_0 = \frac{\alpha\sin\alpha + 2\cos\alpha}{2}, a_1 = \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha}, a_2 = \frac{\sin\alpha}{2\alpha}, a_3 = \frac{\sin\alpha - \alpha\cos\alpha}{2\alpha^3}, \tag{3.26}$$

hence the Rodrigues formula is

$$\exp(X) = \frac{\alpha\sin\alpha + 2\cos\alpha}{2}I_4 + \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha}X \quad + \frac{\sin\alpha}{2\alpha}X^2 + \frac{\sin\alpha - \alpha\cos\alpha}{2\alpha^3}X^3. \tag{3.27}$$

Note that in the paper [7] the formulas (3.25), (3.26), (3.27) are derived by using so-called Putzers method (see [56]).

3.6.3 The case $n = 5$

The approach of this case was not made in the paper [4] and we present in detail in this subsection.

The general form of a matrix $X \in so(5)$ is

$$X = \begin{pmatrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ -b & -e & 0 & h & j \\ -c & -f & -h & 0 & k \\ -d & -g & -j & -k & 0 \end{pmatrix},$$

with the corresponding characteristic polynomial given by

$$\begin{aligned} p_X(t) = \det(tI_5 - X) = t^5 + (a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+j^2+k^2)t^3 + \\ (c^2e^2 + d^2e^2 - 2bcef + b^2f^2 + d^2f^2 - 2bdeg - 2cdfg + b^2g^2 + c^2g^2 + 2aceh \\ - 2abfh + a^2h^2 + d^2h^2 + g^2h^2 + 2adej - 2abgj - 2cdhj - 2fghj + a^2j^2 + c^2j^2 \\ + f^2j^2 + 2adfk - 2acgk + 2bdhk + 2eghk - 2bcjk - 2efjk + a^2k^2 + b^2k^2 + e^2k^2)t \end{aligned}$$

Let $\lambda_{1,2} = \pm\alpha i$, $\lambda_{3,4} = \pm\beta i$, $\lambda_5 = 0$ be the eigenvalues of the matrix X, where $\alpha, \beta \in \mathbb{R}$. Solving the ecuation $p_X(t) = 0$, we obtain α and β as functions of $a, b, c, d, e, f, g, h, j, k$.

To determine the Rodrigues coefficients we consider the following three cases.

Case 1. If $|\alpha| \neq |\beta|$, α, $\beta \in \mathbb{R}^*$, then using formula (3.8) we have

$$V_5(\alpha i, -\alpha i, \beta i, -\beta i, 0) = -4\alpha^3\beta^3(\beta^2 - \alpha^2)^2$$

and we obtain

$$a_0(X) = \frac{V_{5,0}(\alpha i, -\alpha i, \beta i, -\beta i, 0)}{V_5(\alpha i, -\alpha i, \beta i, -\beta i, 0)} = \frac{4\alpha^3\beta^3(\beta^2 - \alpha^2)^2}{4\alpha^3\beta^3(\beta^2 - \alpha^2)^2} = 1$$

$$a_1(X) = \frac{V_{5,1}(\alpha i, -\alpha i, \beta i, -\beta i, 0)}{V_5(\alpha i, -\alpha i, \beta i, -\beta i, 0)} = \frac{\beta^3 \sin\alpha - \alpha^3 \sin\beta}{\alpha\beta(\beta^2 - \alpha^2)}$$

$$a_2(X) = \frac{V_{5,2}(\alpha i, -\alpha i, \beta i, -\beta i, 0)}{V_5(\alpha i, -\alpha i, \beta i, -\beta i, 0)} = \frac{\beta^4 \cos\alpha - \alpha^4 \cos\beta + \alpha^4 - \beta^4}{\alpha^2\beta^2(\beta^2 - \alpha^2)}$$

$$a_3(X) = \frac{V_{5,3}(\alpha i, -\alpha i, \beta i, -\beta i, 0)}{V_5(\alpha i, -\alpha i, \beta i, -\beta i, 0)} = \frac{\beta \sin\alpha - \alpha \sin\beta}{\alpha\beta(\beta^2 - \alpha^2)}$$

$$a_4(X) = \frac{V_{5,4}(\alpha i, -\alpha i, \beta i, -\beta i, 0)}{V_5(\alpha i, -\alpha i, \beta i, -\beta i, 0)} = \frac{\beta^2 \cos\alpha - \alpha^2 \cos\beta + \alpha^2 - \beta^2}{\alpha^2\beta^2(\beta^2 - \alpha^2)}.$$

The corresponding Rodrigues formula to this case is

$$\exp(X) = I_5 + \frac{\beta^3 \sin\alpha - \alpha^3 \sin\beta}{\alpha\beta(\beta^2 - \alpha^2)}X + \frac{\beta^4 \cos\alpha - \alpha^4 \cos\beta + \alpha^4 - \beta^4}{\alpha^2\beta^2(\beta^2 - \alpha^2)}X^2 + \frac{\beta \sin\alpha - \alpha \sin\beta}{\alpha\beta(\beta^2 - \alpha^2)}X^3 + \frac{\beta^2 \cos\alpha - \alpha^2 \cos\beta + \alpha^2 - \beta^2}{\alpha^2\beta^2(\beta^2 - \alpha^2)}X^4.$$

Case 2. If $\alpha = \beta$, $\alpha, \beta \in \mathbb{R}^*$, the eigenvalues of the matrix X are given by $\lambda_1 = \lambda_3 = \alpha i, \lambda_2 = \lambda_4 = -\alpha i, \lambda_5 = 0$. Therefore, we have double multiplicity for $\alpha i, \alpha i, -\alpha i, -\alpha i$ and simple for 0.

Consider the real numbers ϵ, ϵ' such that $\alpha i, \alpha i + \epsilon, -\alpha i, -\alpha i + \epsilon', 0$ are distinct. In this situation we have

$$V_5(\alpha i, \alpha i + \epsilon, -\alpha i, -\alpha i + \epsilon', 0) = -2\alpha^3 i\epsilon(2\alpha i + \epsilon)(\alpha i + \epsilon)\epsilon'(2\alpha i + \epsilon - \epsilon')(\alpha i - \epsilon')(2\alpha i - \epsilon')$$

Applying Theorem 3.1. 1) we get

$$a_k^{(f)}(X_{\epsilon,\epsilon'}) = \frac{V_{5,k}^{(f)}(\alpha i, \alpha i + \epsilon, -\alpha i, -\alpha i + \epsilon', 0)}{V_5(\alpha i, \alpha i + \epsilon, -\alpha i, -\alpha i + \epsilon', 0)} = -\frac{V_{5,k}^{(f)}(\alpha i, \alpha i + \epsilon, -\alpha i, -\alpha i + \epsilon', 0)}{2\alpha^3 i(2\alpha i + \epsilon)(\alpha i + \epsilon)\epsilon'(2\alpha i + \epsilon - \epsilon')(\alpha i - \epsilon')(2\alpha i - \epsilon')\epsilon'}, \; k = 0, 1, 2, 3, 4$$

where $f(z) = e^z$ and $X_{\epsilon,\epsilon'}$ is the matrix with distinct eigenvalues $\alpha i, \alpha i + \epsilon, -\alpha i, -\alpha i + \epsilon', 0$.

For $\epsilon \to 0$, result

$$a_k^{(f)}(X_{0,\epsilon'}) = \frac{\det\left(\left[U_k^{(f,f')}(\alpha i)\right]\left[U_k^{(f)}(-\alpha i)\right]\left[U_k^{(f)}(-\alpha i+\epsilon')\right]\left[U_k^{(f)}(0)\right]\right)}{4\alpha^5 i(2\alpha i-\epsilon')(\alpha i-\epsilon')(2\alpha i-\epsilon')}, \tag{3.28}$$

where $U_k^{(f,f')}(z)$ is the 5×2 block given by

$$U_k^{(f,f')}(z) = \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ e^z & e^z \\ \vdots & \vdots \\ z^4 & 4z^3 \end{pmatrix}$$

obtained from formula (3.22), and $U_k^{(f)}(z)$ is the 5×1 block defined by

$$U_k^{(f)}(z) = \begin{pmatrix} 1 \\ \vdots \\ e^z \\ \vdots \\ z^4 \end{pmatrix}.$$

For $\epsilon' \to 0$, we obtain

$$a_k^{(f)}(X_{0,0}) = \frac{1}{16\alpha^8}\det\left(\left[U_k^{(f,f')}(\alpha i)\right]\left[U_k^{(f,f')}(-\alpha i)\right]\left[U_k^{(f)}(0)\right]\right), \tag{3.29}$$

where $X_{0,0} = X$ și $k = 0,1,2,3,4$.

By explicitly writing the formulas (3.29) we find

$$a_0(X) = 1,\ a_1(X) = \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha},\ a_2(X) = -\frac{\alpha\sin\alpha + 4\cos\alpha}{2\alpha^2},$$

$$a_3(X) = \frac{\sin\alpha - \alpha\cos\alpha}{2\alpha^3},\ a_4(X) = -\frac{\alpha\sin\alpha + 2\cos\alpha}{2\alpha^4},$$

where the determinants were computed using MATHEMATICA.

The corresponding Rodrigues formula to this case is

$$\exp(X) = I_5 + \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha}X - \frac{\alpha\sin\alpha + 4\cos\alpha}{2\alpha^2}X^2 + \\ + \frac{\sin\alpha - \alpha\cos\alpha}{2\alpha^3}X^3 - \frac{\alpha\sin\alpha + 2\cos\alpha}{2\alpha^4}X^4.$$

Case 3. If $\beta = 0$ and $\alpha \in \mathbb{R}^*$, the eigenvalues of the matrix X are given by $\lambda_1 = \alpha i, \lambda_2 = -\alpha i, \lambda_3 = \lambda_4 = \lambda_5 = 0$.

Consider the distinct and nonzero real numbers ϵ, ϵ'. Then $\alpha i, -\alpha i, 0, \epsilon, \epsilon'$ are distinct and we have

$$V_5(\alpha i, -\alpha i, 0, \epsilon, \epsilon') = -2\alpha^3 i(\epsilon^2 + \alpha^2)(\epsilon'^2 + \alpha^2)\epsilon\epsilon'(\epsilon' - \epsilon).$$

Applying Theorem 3.1. 1), it follows

$$\begin{aligned} a_k^{(f)}(X_{\epsilon,\epsilon'}) &= \frac{V_{5,k}^{(f)}(\alpha i, -\alpha i, 0, \epsilon, \epsilon')}{V_5(\alpha i, -\alpha i, 0, \epsilon, \epsilon')} \\ &= -\frac{V_{5,k}^{(f)}(\alpha i, -\alpha i, 0, \epsilon, \epsilon')}{-2\alpha^3 i(\epsilon^2 + \alpha^2)(\epsilon'^2 + \alpha^2)\epsilon\epsilon'(\epsilon' - \epsilon)}, \; k = 0, 1, 2, 3, 4, \end{aligned} \tag{3.30}$$

where $f(z) = e^z$ and $X_{\epsilon,\epsilon'}$ is the matrix with distinct eigenvalues $\alpha i, -\alpha i, 0, \epsilon, \epsilon'$. For $\epsilon \to 0$, we obtain

$$a_k^{(f)}(X_{0,\epsilon'}) = -\frac{\det\left(\left[U_k^{(f)}(\alpha i)\right]\left[U_k^{(f)}(-\alpha i)\right]\left[U_k^{(f,f')}(0)\right]\left[U_k^{(f)}(\epsilon')\right]\right)}{2\alpha^5 i(\epsilon'^2 + \alpha^2)(\epsilon')^2}, \tag{3.31}$$

where $U_k^{(f)}(z)$ and $U_k^{(f,f')}(z)$ are the 5×1 block and the 5×2 block defined in Case 2. For $\epsilon' \to 0$, from formula (3.31) it follows

$$a_k^{(f)}(X_{0,0}) = -\frac{\det\left(\left[U_k^{(f)}(\alpha i)\right]\left[U_k^{(f)}(-\alpha i)\right]\left[U_k^{(f,f',f'')}(0)\right]\right)}{4\alpha^7 i}, \tag{3.32}$$

where $U_k^{(f,f',f'')}(z)$ is the 5×3 block given by

$$U_k^{(f,f',f'')}(z) = \begin{pmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ e^z & e^z & e^z \\ \vdots & \vdots & \vdots \\ z^4 & 4z^3 & 12z^2 \end{pmatrix}$$

where the second column is the derivative of the first column, and the third column is the second derivative of the first column. Clearly, we have $X_{0,0} = X$, so the formulas (3.32) give the Rodrigues coefficients in this case.

Writing the formulas explicitly (3.32) we find

$$a_0(X) = 1, \; a_1(X) = 0, \; a_2(X) = 0,$$

$$a_3(X) = -\frac{\sin\alpha}{\alpha^3}, \; a_4(X) = \frac{\cos\alpha - 1}{\alpha^4}.$$

The corresponding Rodrigues formula to this case is

$$\exp(X) = I_5 - \frac{\sin\alpha}{\alpha^3}X^3 + \frac{\cos\alpha - 1}{\alpha^4}X^4.$$

Chapter 4

The Cayley Transform and the Rodrigues Type Formulas

In this chapter we present in first section the Cayley transform of the group $\mathbf{SO}(n)$. In Section 4.2 we define the Cayley transform type for the special Euclidean group $\mathbf{SE}(n)$ in connection with the Cayley transform of $\mathbf{SO}(n)$. Section 4.3 is devoted to the generalization of this notion and some properties are presented. The Rodrigues formulas for the Cayley transform are obtained in Section 4.4. For the group $\mathbf{SO}(n)$ these formulas are given in the special cases $n = 2, 3, 4$. For the group $\mathbf{SE}(n)$ the cases $n = 2$ and $n = 3$ are treated. The presentation follows our work [3]. Among the references used in this chapter we mention R.-A. Rohan [60].

4.1 The classical Cayley transform of the group $\mathbf{SO}(n)$

As we have already mentioned in the previous section, the matrices of the $\mathbf{SO}(n)$ group describe the rotations as movements in the space $\mathbb{R}^n$. If the matrix A belongs to the Lie Algebra $\mathfrak{so}(n)$ of the Lie group $\mathbf{SO}(n)$, then the matrix $I_n - A$ is invertible.

Indeed, the eigenvalues $\lambda_1, ..., \lambda_n$ of the matrix A are 0 or purely imaginary, so the eigenvalues of the matrix $I_n - A$ are $1 - \lambda_1, ..., 1 - \lambda_n$. They are clearly different from 0, therefore we have $\det(I_n - A) = (1 - \lambda_1)...(1 - \lambda_n) \neq 0$, so $I_n - A$ is invertible.

The map $\mathrm{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n)$, defined by

$$\mathrm{Cay(A) = (I_n + A)(I_n - A)^{-1}}$$

is called the *Cayley transform* of the group $\mathbf{SO}(n)$.

Denote by $\sum$ the subset of the group $\mathbf{SO}(n)$ containing the matrices with eigenvalue -1. Clearly, we have $R \in \sum$ if and only if the matrix $I_n + R$ is singular.

Theorem 4.1. *The map* $\mathrm{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n) \setminus \Sigma$ *is bijective and its inverse is* $\mathrm{Cay}^{-1} : \mathbf{SO}(n) \setminus \sum \to \mathfrak{so}(n)$, *where* $\mathrm{Cay}^{-1}(R) = (R + I_n)^{-1}(R - I_n)$.

4.2 The Cayley transform of the special Euclidean group $\mathbf{SE}(n)$

In this section we will define the Cayley transform for the special Euclidean group $\mathbf{SE}(n)$. By analogy with the special orthogonal group $\mathbf{SO}(n)$, we define the map $\mathrm{Cay_{n+1}} : \mathfrak{se}(n) \to \mathbf{SE}(n)$, where

$$\mathrm{Cay}_{n+1}(S) = (I_{n+1} + S)(I_{n+1} - S)^{-1}$$

we will call this map the *Cayley transform* of the group $\mathbf{SE}(n)$.

The connection between the maps $\mathrm{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n)$ și $\mathrm{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$ is given by formula

$$\mathrm{Cay}_{n+1}(S) = \begin{pmatrix} \mathrm{Cay}(A) & (R + I_n)u \\ 0 & 1 \end{pmatrix}.$$

4.3 The generalized Cayley transform

We say that the matrix $A \in M(n, \mathbb{K})$ is *orthogonal* if $AA^* = \mathrm{id}$, where $A^* = \bar{A}^t$ is the transpose conjugate, and $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, where $\mathbb{H}$ is the quaternion algebra. Thus a matrix can be identified with an $\mathbb{K}$ liniar map $\mathbb{K}^n \to \mathbb{K}^n$ which invariates the product $\langle v, w \rangle = v^* w$. Denote by $\mathbf{O}(n, \mathbb{K})$ the Lie group of orthogonal matrices. Depending on $\mathbb{K}$ this group corresponds to orthogonal group $\mathbf{O}(n)$, group of units $\mathbf{U}(n)$ or symplectic group $\mathbf{Sp}(n)$, as the $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$.

Let $A \in \mathbf{O}(n, \mathbb{K})$ be an orthogonal matrix, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Definition 4.1. *Denote by $\Omega(A) \subset M(n, \mathbb{K})$ the open set of matrices X with the property that $A + X$ is invertible. The Cayley map centered in A is the function*

$$c_A : \Omega(A) \to \Omega(A^*)$$

given by

$$c_A(X) = (I - A^* X)(A + X)^{-1}.$$

The classical Cayley map correspond with $A = I_n$. As we will see in the next proposition, the map c_A is well defined and is invertible with $c_A^{-1} = c_{A^*}$.

Proposition 4.1. *For $X \in \Omega(A)$ the following properties holds:*

1. *$c_A(X) = (A + X)^{-1}(I - XA^*)$;*

2. *the matrix inverse $A^* + c_A(X)$ is $\frac{1}{2}(A + X)$;*

3. $c_A(X) \in \Omega(A^*)$;

4. c_A *is a diffeomorphism with* $c_A^{-1} = c_{A^*}$.

We will also need the following interesting property, which is easy to prove.

Proposition 4.2. *For* $X \in \Omega(A)$ *the properties holds:*

1. $X^* \in \Omega(A^*)$ *și* $c_{A^*}(X^*) = c_A(X)^*$;

2. for any matrix $U \in \boldsymbol{O}(n, \mathbb{K})$ *we have* $UXU^* \in \Omega(UAU^*)$ *and*

$$c_{UAU^*}(UXU^*) = Uc_A(X)U^*;$$

3. if the matrix X *is invertible, then* $X^{-1} \in \Omega(A^*)$ *because*

$$(A^* + X^{-1})^{-1} = A(A + X)^{-1}X.$$

Moreover, we have $c_{A^*}(X^{-1}) = -Ac_A(X)A$.

4.4 Rodrigues type formulas for the Cayley transform

4.4.1 Computations for the group SO(n) in small dimension

The formulas obtained in this subsection are taken from our paper [3]. Obviously the Cayley transform is obtained from the analytical function

$$f(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots, |z| < 1,$$

so we can apply the results from the Sections 3.2-3.4. Because the inverse of the matrix $I_n - A$ can be written in the form

$$(I_n - A)^{-1} = I_n + A + A^2 + ...$$

for a sufficiently small neighborhood of O_n, from the Hamilton-Cayley theorem it follows that the Cayley transform of A can be written in the polynomial form

$$\mathrm{Cay}(A) = b_0(A)I_n + b_1(A)A + ... + b_{n-1}(A)A^{n-1} \tag{4.1}$$

where the coefficients $b_0, ..., b_{n-1}$ are uniquely determined and depend on the matrix A. We will call these numbers, as in the general case, the **Rodrigues coefficients** of A with respect to the Cayley transform.

The cases $n = 2, 3$

We will continue by the presentation of the particular cases $n = 2$ and $n = 3$. We saw that, in general, for $A = O_n$, we have $\text{Cay}(A) = I_n$, so $b_0(O_n) = 1, b_1(O_n) = ... = b_{n-1}(O_n) = 0$.

In the case $n = 2$, we consider the antisymmetric matrix $A \neq O_2$, where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*,$$

with eigenvalues $\lambda_1 = ai, \lambda_2 = -ai$. From the formulas derived in subsection 3.3.1 we obtain

$$b_0 = \frac{1-a^2}{1+a^2} \quad \text{și} \quad b_1 = \frac{1}{1+a^2},$$

thus, the Rodrigues type formula for the Cayley transform is

$$\text{Cay}(A) = \frac{1-a^2}{1+a^2} I_2 + \frac{2}{1+a^2} A. \tag{4.2}$$

For $n = 3$ any real antisymmetric matrix X is of the form

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

with the characteristic polynomial $p_A(t) = t^3 + \theta^2 t$, unde $\theta = \sqrt{a^2 + b^2 + c^2}$. The eigenvalues of the matrix A are $\lambda_1 = \theta i, \lambda_2 = -\theta i, \lambda_3 = 0$. We have $A = O_3$ if and only if $\theta = 0$, so it is enough to consider only the situation in which $\theta \neq 0$. Using the formulas obtained in subsection 3.3.2, it follows

$$b_0 = 1, b_1 = \frac{2}{1+\theta^2}, b_2 = \frac{2}{1+\theta^2}.$$

and the Rodrigues type formula for the Cayley transform of the group $\mathbf{SO}(3)$ este

$$\text{Cay}(A) = I_3 + \frac{2}{1+\theta^2} A + \frac{2}{1+\theta^2} A^2. \tag{4.3}$$

Formula (4.3) offers the possibility to obtain another form for the inverse of Cayley transform. Indeed, let be $R \in \mathbf{SO}(3)$ such that

$$R = I_3 + \frac{2}{1+\theta^2} A + \frac{2}{1+\theta^2} A^2,$$

where A A is an antisymmetric matrix. Considering the matrix transpose in both sides of the above relation and taking into account that ${}^t A = -A$, we obtain

$$R -^t R = \frac{4}{1+\theta^2} A. \tag{4.4}$$

On the other hand, we have

$$\mathrm{tr}(R) = 3 - \frac{4\theta^2}{1+\theta^2} = -1 + \frac{4}{1+\theta^2},$$

and by replacing in the relation (4.4), we get the formula

$$\mathrm{Cay}^{-1}(R) = \frac{1}{1+\mathrm{tr}(R)}(R -^t R). \tag{4.5}$$

Formula (4.5) makes sense for rotations $R \in \mathbf{SO}(3)$ for which $1 + \mathrm{tr}(R) \neq 0$. If R is a rotation of angle α, then we have $\mathrm{tr}(R) = 1 + 2\cos\alpha$, so the application Cay^{-1} is not defined for the rotations of angle $\alpha = \pm\pi$. Because in the domain where is defined, the application Cay is bijective, it follows that the antisymmetric matrices from $\mathfrak{so}(3)$ can be used as coordinates for rotations. Considering the Lie algebra isomorphism "$\widehat{\ }$" between $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\cdot,\cdot])$, where " $\times$ " denote the vector product, defined by $v \in \mathbb{R}^3 \to \widehat{v} \in \mathfrak{so}(3)$, where v is considered a column

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and $\widehat{v}$ is defined by

$$\widehat{v} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

by composing the applications

$$\mathbb{R}^3 \xrightarrow{\widehat{}} \mathfrak{so}(3) \xrightarrow{Cay} \mathbf{SO}(3)$$

we get a vectorial parameterization of rotations from $\mathbf{SO}(3)$. This is very useful in solving mechanical problems.

The case $n = 4$

As in subsection 3.6.2, for a skew-symmetric matrix $A \in so(4)$, let $\lambda_{1,2} = \pm\alpha i$, $\lambda_{3,4} = \pm\beta i$ be the eigenvalues of the matrix A, where $\alpha, \beta \in \mathbb{R}$. We consider the following three situations.

1. If $|\alpha| \neq |\beta|$, α, $\beta \in \mathbb{R}^*$, then using the formulas in subsection 3.3.3, we obtain

$$b_0 = \frac{1+\alpha^2+\beta^2-\alpha^2\beta^2}{(1+\alpha^2)(1+\beta^2)}, b_1 = \frac{2(1+\alpha^2+\beta^2)}{(1+\alpha^2)(1+\beta^2)}$$

$$b_2 = \frac{2}{(1+\alpha^2)(1+\beta^2)}, b_3 = \frac{2}{(1+\alpha^2)(1+\beta^2)},$$

and the corresponding Rodrigues formula

$$\mathrm{Cay}(A) = \frac{1+\alpha^2+\beta^2-\alpha^2\beta^2}{(1+\alpha^2)(1+\beta^2)} I_4 + \frac{2(1+\alpha^2+\beta^2)}{(1+\alpha^2)(1+\beta^2)} A +$$

$$\frac{2}{(1+\alpha^2)(1+\beta^2)} A^2 + \frac{2}{(1+\alpha^2)(1+\beta^2)} A^3.$$

2. If $\alpha \neq 0$ and $\beta = 0$, then we will use the formulas in subsection 3.4.3 when $\lambda_1 \neq \lambda_2 \neq \lambda_3 = \lambda_4$, and we obtain

$$b_0 = 1,\ b_1 = 2,\ b_2 = \frac{2}{1+\alpha^2},\ b_3 = \frac{2}{1+\alpha^2}.$$

The Rodrigues formula in this case is

$$\mathrm{Cay}(A) = I_4 + 2A + \frac{2}{(1+\alpha^2)} A^2 + \frac{2}{(1+\alpha^2)} A^3.$$

3. If $\alpha = \beta \neq 0$, then we will use the formulas in subsection 3.4.3 for $\lambda_1 = \lambda_3, \lambda_2 = \lambda_4, \lambda_1 \neq \lambda_2$, and after simple computations we get

$$b_0 = \frac{1+2\alpha^2-\alpha^4}{(1+\alpha^2)^2},\ b_1 = \frac{2(2\alpha^2+1)}{(1+\alpha^2)^2},\ b_2 = \frac{2}{(1+\alpha^2)^2},\ b_3 = \frac{2}{(1+\alpha^2)^2},$$

and the corresponding Rodrigues formula

$$\mathrm{Cay}(A) = \frac{1+2\alpha^2-\alpha^4}{(1+\alpha^2)^2} I_4 + \frac{2(2\alpha^2+1)}{(1+\alpha^2)^2} A + \frac{2}{(1+\alpha^2)^2} A^2 + \frac{2}{(1+\alpha^2)^2} A^3.$$

4.4.2 Computations for the group $\mathbf{SE}(n)$ in small dimension

As for the classical map $\mathrm{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n)$, we can obtain the effective Rodrigues formulas for the map $\mathrm{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$, for small values of n. Using the observation in section 5.1 of the paper R.-A. Rohan [60], we obtain this for a matrix $S \in \mathfrak{se}(n)$ defined in blocks as above. Its characteristic polynomial p_S satisfies the relation $p_S(t) = tp_A(t)$. The Rodrigues formula for the map $\mathrm{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$ has the form

$$\mathrm{Cay}_{n+1}(S) = c_0 I_{n+1} + c_1 S + \ldots + c_n S^n,$$

where the coefficients $c_0, c_1, \ldots, c_n$ depend on the matrix A.

For $n = 2$, we consider the antisimetric matrix $A \neq O_2$, where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*.$$

Using the above observation, it follows that the matrix $S \in \mathfrak{se}(2)$ has the eigenvalues $\lambda_1 = ai, \lambda_2 = -ai, \lambda_3 = 0$, and the corresponding Rodrigues formula has the form

$$\mathrm{Cay}_3(S) = c_0 I_3 + c_1 S + c_2 S^2.$$

We obtained a result analogous to that in the Theorem 3.1, which is reduced to the system

$$\begin{cases} S_0c_0 + S_1c_1 + S_2c_2 = 1 + \frac{1+\lambda_1}{1-\lambda_1} + \frac{1+\lambda_2}{1-\lambda_2} \\ S_1c_0 + S_2c_1 + S_3c_2 = 1 + \frac{1+\lambda_1}{1-\lambda_1} + \frac{1+\lambda_2}{1-\lambda_2} \\ S_2c_0 + S_3c_1 + S_4c_2 = \lambda_1^2\frac{1+\lambda_1}{1-\lambda_1} + \lambda_2^2\frac{1+\lambda_2}{1-\lambda_2} \end{cases}$$

which has the solution

$$c_0 = 1, c_1 = \frac{1}{1+a^2}, c_2 = \frac{1}{1+a^2}$$

So the Rodrigues formula for the map Cay_3 is

$$\mathrm{Cay}_3 = I_3 + \frac{1}{1+a^2}S + \frac{1}{1+a^2}S^2. \tag{4.6}$$

For $n = 3$, we consider the antisimetric matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

with the characteristic polynomial $p_A(t) = t^3 + \theta^2 t$, where $\theta = \sqrt{a^2+b^2+c^2}$. The matrix $S \in \mathfrak{se}(3)$ has the characteristic polynomial $p_S(t) = tp_A(t) = t^4 + \theta^2t^2$, with eigenvalues $\lambda_1 = \theta i, \lambda_2 = -\theta i, \lambda_3 = 0, \lambda_4 = 0$. The Rodrigues formula has the form

$$\mathrm{Cay}_4(S) = c_0I_4 + c_1S + c_2S^2 + c_cS^3.$$

After a similar computation, we obtain the formula

$$\mathrm{Cay}_4(S) = I_3 + 2S + \frac{2}{1+\theta^2}S^2 + \frac{2}{1+\theta^2}S^3. \tag{4.7}$$

As for the Cayley transform of the group $\mathbf{SO}(n)$, denote by Σ_{n+1} the set of matrices from $\mathbf{SE}(n)$ which have the eigenvalue -1. Obviously, we have $M \in \mathbf{SE}(n)$ if and only if the matrix $I_{n+1} + M$ is singular. A proof similar to that of Theorem 3.1 leads us to

Theorem 4.2. *The map* $\mathrm{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n) \setminus \Sigma_{n+1}$ *is bijective and its inverse is given by*

$$\mathrm{Cay}_{n+1}(M) = \begin{pmatrix} \mathrm{Cay}^{-1}M & (R+I_n)^{-1}\boldsymbol{t} \\ 0 & 0 \end{pmatrix}$$

where the matrix M *is defined in blocks by*

$$S = \begin{pmatrix} R & \boldsymbol{t} \\ 0 & 1 \end{pmatrix}.$$

Bibliography

[1] R. Abraham, J.E. Marsden, *Foundations of Mechanics*, Addison Wesley, 2nd Edition, 1978.

[2] D. Andrica, I.N. Casu, *Grupuri Lie, aplicația exponențială și mecanica geometrică*, Presa Universitară Clujeană, 2008.

[3] D. Andrica, O.L. Chender (Broaina), *Rodrigues formula for the Cayley transform of groups $SO(n)$ and $SE(n)$*, Studia Univ. Babeş-Bolyai-Mathematica, Vol 60(2015), No. 1, 31-38.

[4] D. Andrica, O.L. Chender (Broaina), *A New Way to Compute the Rodrigues Coefficients of Functions of the Lie Groups of Matrices*, in "Essays in Mathematics and its Applications" in the Honor of Vladimir Arnold, Springer, 2016, 1-24.

[5] D. Andrica, L. Mare, *The image of the exponential mapping on the linear group $GL(n, \mathbb{R})$*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 1989.

[6] D. Andrica, R.-A. Rohan, *The image of the exponential map and some applications*, Proc. 8th Joint Conference on Mathematics and Computer Science MaCS, Komarno, Slovakia, July 14-17, 2010, 3-14.

[7] D. Andrica, R.-A. Rohan, *Computing the Rodrigues coefficients of the exponential map of the Lie groups of matrices*, Balkan Journal of Geometry and Applications, Vol.18, 2013, No.2, 1-10.

[8] D. Andrica, R.-A. Rohan, *A new way to derive the Rodrigues formula for the Lorentz group*, Carpathian Journal of Mathematics, 30, 2014, No.1, 23-29.

[9] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics 60, Second Edition, Springer-Verlag, 1998.

[10] A. Baker, *Matrix Groups. An Introduction to Lie Group Theory*, SUMS, Springer, 2002.

[11] N. Bourbaki, *Elements of Mathematics. Lie Groupsand Lie Algebra*, Chapters 1-3, Springer, 1989.

[12] T. Bröcker, T. tom Dieck, *Representations of compact Lie groups*, Springer-Verlag, GTM, vol. 98, New York, 1985.

[13] O.L. Chender (Broaina), *Schwerdtfeger formula for matrix functions*, The 16th International Conference on Applied Mathematics and Computer Science, July 3 rd to 6 th, 2019, Annals of the Tiberiu Popoviciu Seminar, Vol. 16, 2018, în curs de publicare.

[14] C. Chevalley, *Theory of Lie groups I*, Princeton Mathematical Series, No.8, Princeton University Press, 1946.

[15] M.L. Curtis, *Matrix groups*, Universitext, Springer Verlag, 2nd Edition, 1984.

[16] D. Djokovic, *On the exponential map in classical Lie groups*, Journal of Algebra, **64**:76-88, 1980.

[17] O. Furdui, *Computing exponential and trigonometric functions of matrices in $M_2(\mathbb{C})$*, Gazeta Matematică, Seria A, Anul XXXVI, Nr. 1-2/2018, 1-13.

[18] J.H. Gallier, *Geometric Methods and Application, for Computer Science and Engineering*, TAM, Vol. 38, Springer, 2011.

[19] J. Gallier, D. Xu, *Computing exponentials of skew-symmetric matrices and logarithms of orthogonal matrices*, International Journal of Robotics and Automation, Vol. 17, No. 4, 2002, 2-11.

[20] J. Gallier, *A Concrete Introduction to Classical Lie Groups Via the Exponential Map*, University of Pennsylvannia, 2002.

[21] F.R. Gantmacher, *The Theory of Matrices*, Vol. 1, AMS, Chelsea, 1977.

[22] G.H. Golub, C.F. Van Loan *Matrix Computations*, The Johns Hopkins University Press, 1996.

[23] N.J. Higham, *The scaling and squaring method of the matrix exponential revised*, SIAM Journal on Matrix Analisis and Applications, **26**:1179-1193, 2005.

[24] N.J. Higham, *Functions of Matrices Theory and Computation*, SIAM Society for Industrial and Applied Mathematics , Philadelphia, PA. USA, 2008.

[25] K.H. Hofmann, *A memo on the exponential function and regular points*, Arch. Math. 59, 1992, 24-37.

[26] K.H. Hofmann, A. Mukhergea, *On the density of the image of the exponential function*, Math. Ann. 234, 1978, 263-273.

[27] K.H. Hofmann, W.A.F. Rupert, *Lie groups and subsemigroups with surjective exponential function*, Memoirs of the Amer. Math. Soc 618, 1997.

[28] R.A. Horn, Ch.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[29] R.A. Horn, Ch.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.

[30] R.A. Horn, Ch.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[31] S. Kida, E. Trimandalawati, S. Ogawa, *Matrix Expression of Hermite Interpolation Polynomials*, Computer Math. Applic., Vol. **33** (1997), No. 11, 11-13.

[32] M.-J. Kim, M.-S. Kim, A. Shin, *A general construction scheme for unit quaternion curves with simple high order derivatives*, Computer Graphics Proceedings, Annual Conference Series, ACM, 1995, 369-376.

[33] M.-J. Kim, M.-S. Kim, A. Shin, *A compact differential formulafor the first derivative of a unit quaternion curve*, Journal of Visualization and Computer Animation, **7**, 1996, 43-57.

[34] H.L. Lai, *Surjectivity of exponential mapping on semisimple Lie groups*, J. Math. Soc. Japan, **29**, 1977, 303-3325.

[35] P. Lancaster, M. Tismenetsky *The Theory of Matrices*, Second Edition With Application, Academis Press, 1984.

[36] B. Jütler, *Visualization of moving objects using dual quaternion curves*, Computers and Graphics, **18**(3)(1994), 315-326.

[37] B. Jütler, *An osculating motion with second order contact for spacial Euclidean motions*, Mechanics and Machine Theory, **32**(7)(1997), 843-853.

[38] L. Mare, *The image of exponential map on a few classes of Lie groups*, Babeş-Bolyai University, Faculty of Mathematics, Seminar on Geometry, Preprint No. 2, 1991, 71-78.

[39] L. Mare, *A topological property of the exponential*, Proc. Symposium in Geometry on the occasion of the 190th aniversary of Janos Bolyaiand of the 60th aniversary of Marian Țarină, Babeş-Bolyai University, Cluj-Napoca and Târgu Mureș, August 31-September 1, 1992, Preprint No. 2, 1993, (P. Enghiș and D. Andrica, Eds.), 127-132.

[40] L. Mare, *Asupra unei note din Gazeta Matematică* , Lucrările Seminarului "Didactica Matematicii", vol. 5 (1989), 165-172.

[41] J.E. Marsden, T.S. Raţiu, *Introduction to Mechanics and Symmetry*, TAM, vol. 17, Springer-Verlag, 1994.

[42] S. Mondal, *Surjectivity of Maps Induced on Matrices by Polynomials and Entire Functions*, The Mathematical Association of America, 124, March 2017, 260-264.

[43] M. Moskowitz, M. Wüstner, *Exponentiality of certain real solvable Lie groups*, Cand. Math. Bull., 41, 1998, 386-373.

[44] M. Nishikawa, *Exponential image in the real general linear group*, Bull. Fukuoka Univ. Ed. III, 26, 1976, 35-44.

[45] M. Nishikawa, *Exponential image in the real special linear group*, Bull. Fukuoka Univ. Ed. III, 28, 1978, 1-6.

[46] M. Nishikawa, *Exponential image in the Lorentz group* $O(2,1)$, Bull. Fukuoka Univ. Ed. III, 29, 1979, 9-19.

[47] M. Nishikawa, *Exponential image in the group* $O(n,1)$*, for* $n = 3,4$, Bull. Fukuoka Univ. Ed. III, 31, 1981, 1-11(1982).

[48] M. Nishikawa, *On the exponential map of the group* $O(p,q)_0$, Mem. Fac. Sco. Kyushu Univ. Ser. A, 37, 1983, No. 1, 63-69.

[49] M. Nishikawa, *Exponential image in the group* $O(3,3)$, Bull. Fukuoka Univ. Ed. III, 34, 1984, 13-18(1985).

[50] T. Nôno, *On the singularity of general linear groups*, J. Sci. Hiroshima Univ. Ser. A, 20, 1956/1957, 115-123.

[51] F.C. Park, B. Ravani, *Bézier curves on Riemmanian manifolds and Lie groups with kinematic applications*, Mechanism, Synthesis and Analysis, **70**(1994), 15-21.

[52] F.C. Park, B. Ravani, *Smooth invariant interpolation of rotations*, ACM Transactions of Graphics, **16**(1997), 277-295.

[53] L.I. Piscoran, *Noncommutative differential direct Lie derivative and the algebra of special Euclidean group* $SE(2)$, Creative Math.and Inf., **17**(2008),No.3, 493-498.

[54] T.A. Politi, *A formula for the exponential of a real skewsymmetric matrix of order 4*, BIT, 2001, Vol. 41, No. 4, 842-845.

[55] V. Pop, O. Furdui, *Squares Matrices of Order 2. Theory, Applications and Problems*, Springer, 2017.

[56] E.J. Putzer, *Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients*, Amer. Math. Monthly, 73, 1966, 2-7.

[57] S. Rădulescu, D. Andrica, *Some remarks on the equation* $f(X) = A$ *in* $M_n(\mathbb{C})$ *and applications*, Itinerant seminar on functional equations, approximation and convexity , Cluj-Napoca, 1988, 281-285.

[58] R.F. Rinehart, *The equivalence of Definitions of a Matrix Function*, American Mathematical Monthly, Vol. 62, 1955, 395-414.

[59] R.-A. Rohan, *The exponential map and the Euclidean isometries*, Acta Universitatis Apulensis, **31**(2012), 199-204.

[60] R.-A. Rohan, *Some remarks on the exponential map on the group $SO(n)$ and $SE(n)$*, In Proc. of the XIV Int. Conference Geometry, Integrability and Quantization, 8-13 June, Varna, Bulgaria, 2012, I.M. Mladenov and A. Yoshioka, Eds., Sofia, 2013, 160-175.

[61] F. Silva Leite, P. Crouch, *Closed forms for the exponential mapping on matrix Lie groups based on Putzer's method*, Journal of Mathematical Physics, Vol. 40, No. 7, 1999, 3561-3568.

[62] F. Warner, *Foundations of Differential Manifolds and Lie Groups*, GTM, No. 94, Springer-Verlag, 1983.

[63] R. Vein, P. Dale, *Determinants and Their Applications in Mathematical Physics*, Springer, 1999.

[64] M. Wüstner, *Lie groups with surjective exponential function*, Shaker Verlag, Berichte aus der Mathematik, Aachen 2001.

[65] M. Wüstner, *On the exponential function of real splittable and real semisimple Lie groups*, Beitr. Alg. Geometrie 39, 1998, 37-46.

[66] M. Wüstner, *On the surjectivity of the exponential function of solvable Lie groups*, Math. Nachr. 192, 1998, 255-266.

[67] M. Wüstner, *Product decomposition of solvable Lie groups*, Man. Math.91, 1996, 179-194.