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Identities in rings

Summary of the Ph.D. Thesis

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Contents

Introduction

In this work we will study decompositions of the elements of rings as sums whom terms satisfy various identities. Particularly, we are interested in sums with terms idempotent, nilpotent or periodic elements. These types of decompositions were considered first in operators theory. For example in [27], [43], and [46] there is a study of decompositions of operators classes in sum of idempotent elements. The study of such decompositions in the general context of rings with identity began in 1977, with the work [39]. W.K. Nicholson introduced in it the clean properties for elements in rings and rings. The class of clean rings is included in the class of exchange rings. Moreover, an abelian ring is clean if and only if it is exchange, [39]. An element is clean if it is the sum of an idempotent element and an invertible element in that ring, and a ring is clean if all its elements are clean.

There was a real interest in studying properties which are particular cases or generalizations of clean property. For example there were studied w-clean rings, a generalization of clean rings, introduced and investigated in [4], in 2006. A ring is weakly clean if each of its elements is the sum or the difference of an invertible element and an idempotent element. Then in 2013 Diesl introduced and studied in [24] nil-clean elements and nil-clean rings, a particular clean rings case, the invertible in the definition of the clean element being replaced with a nilpotent element. Not least there were studied generalizations of nil-clean rings, the commutative weakly nil-clean rings, by Danchev and McGovern, in [22], and weakly nil-clean rings by Breaz, Danchev and Zhou in [11]. These rings were obtained replacing the sum of a nilpotent and an idempotent with the sum or the difference of a nilpotent and an idempotent.

The main aim of this thesis is to study properties associated with decompositions of rings elements in sums of elements satisfying various identities. Particularly, we will study also rings with all elements that have properties like the ones described before.

The First Chapter is made out of three Sections. Each of them has the role to prepare the work necessary in the following three chapters. In The First Section there are presented properties of clean and nil-clean decompositions, [55]. Next there are presented results related to decompositions that use tripotents, [54], [55]. In The Third Section "Matrices decompositions" we will specify results about decompositions consisted of idempotent matrices and nilpotent matrices that prepare Chapter 4, "m-nil-clean companion matrices".

In the next three Chapters, we will present the original results that we have obtained while developing this thesis. There are taken from the articles [21], [10], respectively [20].

In Chapter 2, named "Weakly nil-clean index and uniquely weakly nil-clean rings", based on [21], we introduce the weakly nil-clean index. This counts how many weakly nil-clean decompositions there are. A fundamental Proposition is the one which states that the weakly nil-clean index of a ring is 1 if and only if the ring is abelian (Proposition 2.1 or [21, Proposition 2.11]). As a consequence of this Proposition, we obtain that a ring is uniquely weakly nil-clean if and only that ring is abelian weakly nil-clean. Moreover by the characterisation of rings with weakly nil-clean index 1 (respectively 2) and the characterization of rings with ni-clean index 1 (respectively 2), obtained by Basnet and Bhattacharyya in [5], it is obtained the following statement: a ring has weakly nil-clean index 1 (respectively 2) if and only if it has nil-clean index 1 (respectively 2). We will also give computation example for weakly nil-clean index of variouse matrix rings, in The Second Section.

The aim of Chapter 3, named "Weakly tripotent rings", is to study rings R with the property that for each element $x \in R$, at least one of the elements x or $x + 1$ is tripotent. Weakly tripotent rings are a generalization of tripotent rings. In the first part of this Chapter there are studied these rings in general and there are considered links between their class and classes of rings considered in [54]. The

Chapter ends with a study of commutative weakly tripotent rings. Particularly, in Theorem 3.2 we give a Structural Theorem for these rings: a commutative ring is weakly tripotent if and only if it can be embedded in a at most there rings product of at most three rings, of which at most one is a tripotent ring of characteristic 3, at most one is a weakly tripotent ring, with only trivial idempotents, and with 3 invertible, respectively at most one is a direct product of Boolean rings. Weakly tripotent rings, with 3 invertible and with only trivial idempotents are characterized in Corollary 3.1.24.

Chapter 4, based on [20] presents the characterization of companion matrices, over fields of positive characteristic, which are m-nil-clean, that is they have a decomposition as the sum of m idempotent matrices and a nilpotent matrix. This is the main result, Theorem 4.2.6. To prove this we will have to analyze carefully matrices classes which are similar to companion matrices of type $\operatorname{diag}(1,\ldots,1,0,\ldots,0)+C'$, where C' is a convenient companion matrix.

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Key words: weakly nil-clean rings, uniquely weakly nil-clean rings, abelian rings, tripotent element, Boolean ring, companion matrix, idempotent, nilpotent, m-nil-clean

Chapter 1

Decompositions in rings

In this chapter we will mention notions and results known in specialty literature, which we consider useful for the addressed studies in the next chapters. In The First Section, "Clean and nil-clean properties" we will present results connected with The Second Chapter, "Weakly nil-clean index and uniquely weakly nil-clean rings". More precisely, we will give definitions and properties for clean, weakly clean, nil-clean, weakly nil-clean, and uniquely weakly nil-clean elements in rings and rings. We will also mention the connection between uniquely weakly nil clean rings and abelian weakly nil-clean rings, properties of clean index, and nil-clean index.

The Second Section, "Decompositions related to tripotents", has as a main subject presentation of specialty literature, that are connected to rings which elements admit decompositions that use tripotent elements and they are related to The Third Chapter, "Weakly tripotent rings". More precisely, we will give structure theorems for rings in which every element is the sum of two idempotents that commute, rings in which every element is the sum of a nilpotent and a tripotent that commute, and also rings in which every element is the sum of a nilpotent a tripotent that commute.

In The Third Section, "Matrix decompositions" we will present results about decompositions of matrices depending on idempotent or nilpotent matrices that prepare results from Chapter 4, "m-nil-clean companion matrices".

1.1 The clean and the nil-clean properties

In this work all rings are associative, with identity. Let R be a ring. An element w of R is called *nilpotent* if there exists a strictly positive integer n such that $w^n = 0$. An element e of R is *idempotent* if $e^2 = e$. An element a of R is *tripotent* if $a^3 = a$. We will denote by $\mathrm{Nil}(R)$ the set of nilpotent elements of R, by $\mathrm{Id}(R)$ the set of idempotent elements of R, and by $U(R)$ the set of invertible elements of R.

Definition. 1.1.1. Let R be a ring and $a \in R$. We say that a is *clean* if there exist u, an invertible element of R, and e an idempotent element of R such that $a = u + e$. A ring is clean if all its elements are clean.

Clean rings appeared in the context of modules with finite exchange property, this being linked with some direct decompositions as direct sums of modules.

Definition. 1.1.3. We say that R is an *exchange ring* if ${}_RR$ has finite exchange property.

The exchange property is symmetric, that is R is exchange if and only if R_R has finite exchange property. We will encounter abelian rings. Let us see their definition.

Definition. 1.1.4. A ring is said to be *abelian ring* if all its idempotents are central.

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The next Theorem is a linking one between clean and exchange rings.

Theorem. 1.1.5. *[40, Theorem 1.1] Any clean ring is exchange; the converse being true if the ring is abelian.*

Next, we will mention weakly clean rings' definition. These rings generalize clean rings and they were studied in [4].

Definition. 1.1.21. Let R be a ring and $a \in R$. Then a is weakly clean if there exist u, an invertible element of R and e, an idempotent of R, such that $a = u + e$ or $a = u - e$. A ring is clean if all its elements are weakly clean.

Next there is a property related to weakly clean rings.

Lemma. 1.1.23.*[4, Theorem 1.7] Let $\{R_\alpha\}$ be a rings family. Then the direct product $R = \prod R_\alpha$ is weakly clean if and only if each R_α is weakly clean, and at most one R_α is not clean.*

Diesl introduced and investigated nil-clean rings in [24].

Definition. 1.1.24. Let R be a ring and $a \in R$. Then a is *nil-clean* if there exist b a nilpotent element of R and e an idempotent element of R such that $a = b + e$. A ring is nil-clean if all its elements are nil-clean.

Next we will mention properties for nil-clean rings.

Lemma. 1.1.25.*[24, Proposition 3.4] Any nil-clean ring is clean.*

Proposition. 1.1.27.*[24, Proposition 3.13] Any finite direct product of nil-clean rings is nil-clean.*

Definition. 1.1.28. An ideal of a ring is *nil* if all its elements are nilpotent.

Proposition. 1.1.29.*[24, Proposition 3.15] Let R be a ring and I a nil ideal of R. Then R is nil-clean if and only if R/I is nil-clean.*

Danchev and McGovern generalize commutative nil-clean rings and Breaz, Danchev, and Zhou generalize nil-clean rings and they obtain commutative weakly nil-clean rings, respectively weakly nil-clean rings, in the same way as Ahn and Anderson have generalized clean rings and obtained weakly clean rings.

Definition. 1.1.39. Let R be a ring and $a \in R$. Then a is weakly nil-clean if there exist b a nilpotent element of R and e an idempotent element of R, such that $a = b+e$ or $a = b - e$. A ring is weakly nil-clean if all its elements are weakly nil-clean.

The next property of weakly nil-clean rings is similar to the case when a finite direct product of rings is weakly clean.

Proposition. 1.1.46*[11, Proposition 3] Let $R = \prod_{i\in I} R_i$ be a direct product of rings with $|I| \geq 2$. Assume I is finite. Then R is weakly nil-clean if and only if there exists $k \in I$ such that R_k is weakly nil-clean, and R_j is nil-clean, for every $j \neq k$.*

The next aim is to mention a structure Theorem for weakly nil-clean rings.

Theorem. 1.1.48.*[11, Theorem 5] The next statements are equivalent for a ring R.*

1. *R is a weakly nil-clean ring.*

2. *There exist R_1, a nil-clean ring and R_2, the zero ring or an indecomposable weakly nil-clean ring, with $3 \in J(R_2)$, such that $R \cong R_1 \times R_2$.*

In what follows we give a property for weakly nil-clean rings.

Corollary. 1.1.50.*[11, Corollary 7] Any weakly nil-clean ring is clean.*

Definition. 1.1.52. Let R be a ring and $a \in R$. If there exists a unique idempotent $e \in R$ such that $a - e$ or $a + e$ is nilpotent in R, then a is uniquely weakly nil-clean. A ring is uniquely weakly nil-clean if all its elements are uniquely weakly nil-clean.

The Theorem which follows next states that the ring property of being abelian weakly nil-clean is equivalent to the property of the same ring of being uniquely weakly nil-clean. This result is also a structure Theorem.

Theorem. 1.1.53.*[11, Theorem 12] The following statements are true for a ring R :*

1. *R is weakly nil-clean abelian.*

2. *There exist R_1, an abelian ring, with $J(R_1)$ nil, $R_1/J(R_1)$ Boolean and R_2 the zero ring or with the property $R_2/J(R_2) \cong \mathbb{F}_3$, with $J(R_2)$ nil, such that $R \cong R_1 \times R_2$.*

3. *R is abelian, $J(R)$ is nil and $R/J(R)$ is isomorphic to a Boolean ring, with $\mathbb{F}_3$, or with a direct product of these to types of rings.*

4. *R is uniquely weakly nil-clean.*

Corollary. 1.1.54.*[11, Corollary 15] A ring R is uniquely nil-clean if and only if R is uniquely weakly nil-clean and $2 \in J(R)$.*

1.1.1 The clean index

Also on clean framework, it has been introduced the clean index of a ring.

Notation. 1.1.55. *Let R be a ring and $a \in R$. We have the following notation:* $\mathcal{E}(a) = \{e \in R \mid e^2 = e,\ a - e \in U(R)\}$.

Definition. 1.1.56. *The clean index* of a ring R, denoted by $c(R)$, is defined as $c(R) = \sup\{|\mathcal{E}(a)| :\ a \in R\}$.

The next Theorem characterizes rings with clean index one.

Theorem. 1.1.59.*[37, Theorem 5] Let R be a ring. Then $c(R) = 1$ if and only if R is abelian and for every $e \in Id(R)$, u and v in $U(R)$, $e \neq 0$, we have $e \neq u + v$.*

The next aim is to characterize the rings with clean index two.

Definition. 1.1.63 A ring is called *elemental* if all its idempotents are trivial, and there exist $u, v \in U(R)$ such that $1_R = u + v$.

Theorem. 1.1.64*[37, Theorem 12] $c(R) = 2$ if and only if one of the following statements takes place:*

1. *R is an elemental ring.*
2. *$R = A \times B$, with A an elemental ring and $c(B) = 1$.*
3. *There exist the rings A and B, with $c(A) = c(B) = 1$ the bimodule ${}_AM_B$, with $|M| = 2$, such that $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.*

1.1.2 The nil-clean index

Taking ideas from the clean index, in [5] nil-clean index has been introduced.

Notation. 1.1.66. *Let R be a ring and $a \in R$. We have the following notation:* $\eta(a) = \{e \in R \mid e^2 = e$ *şi* $a - e \in \mathrm{Nil}(R)\}$.

Definition. 1.1.67. We define the *nil-clean index* of a ring R in this way: $nc(R) = \sup\{|\eta(a)| : \ a \in R\}$.

Here is the connection between clean index and nil-clean index of the same ring.

Lemma. 1.1.70. *[5, lemma 2.8] Let R be a ring. Then $c(R) \geq nc(R)$.*

In what it follows we will characterize rings with clean index one.

Theorem. 1.1.71. *[5, Theorem 3.2] $nc(R) = 1$ if and only if R is an abelian ring.*

Next, we will give the index nil-clean two rings characterization.

Theorem. 1.1.72. *[5, Theorem 4.1] $nc(R) = 2$ if and only if there exist the rings A and B, with $nc(A) = nc(B) = 1$ and the bimodule ${}_AM_B$, with $|M| = 2$, such that $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.*

1.2 Decompositions related to tripotents

In the following result, the equivalence of the first three statements appears in [30], and the statement 4. is added in [54].

Theorem. 1.2.5. *[30, Theorem 1] şi [54, Proposition 2.2] The following statements are equivalent:*

1. *Each element of R is the sum of two commuting idempotents.*

2. *R is a commutative ring and every element is the sum of two idempotents.*

3. *$x^3 = x$ for every $x \in R$.*

4. *Each element of R is the difference of two commuting idempotents.*

Next we present a characterization for rings in which every element is the sum or difference of two commuting idempotents.

Theorem. 1.2.10.*[54, Theorem 4.4] The following statements are equivalent for the ring R :*

1. *Every element is the sum or difference of two idempotents that commute.*

2. *R has one of the following types:*

   (a) *$R/J(R)$ is Boolean with $J(R) = 0$ or $J(R) = \{0, 2\}$.*

   (b) *R is a subdirect product of $\mathbb{F}_3's$.*

   (c) *There exist the ring R_1, with $R_1/J(R_1)$ Boolean and with $J(R_1) = 0$ or $J(R_1) = \{0, 2\}$ and the ring R_2, a subdirect product of $\mathbb{F}_3's$, such that $R \cong R_1 \times R_2$.*

Moreover, 1. *with additional condition* $2 \in J(R)$ *is equivalent to* 2.(*a*)

The following result is a characterization of rings in which every element is the sum of a nilpotent and two tripotents that commute.

Theorem. 1.2.12.*[55, Theorem 2.11] The following statements are equivalent for the ring R :*

1. *Each element is the sum of a nilpotent and two tripotents that commute.*

2. *There exist the ring A, the zero ring or with the property $A/J(A)$ is Boolean, with $J(A)$ nil, and the ring B, the zero or with the property $B/J(B)$ is a subdirect product of $\mathbb{F}_3's$, with $J(B)$ nil, and the ring C, the zero ring or with property $C/J(C)$ is a subdirect product of $\mathbb{F}_5's$, with $J(C)$ nil, such that $R \cong A \times B \times C$.*

3. *$J(R)$ is nil and $x^5 = x$ for every $x \in R/J(R)$.*

4. *$a^5 - a$ is nilpotent for every $a \in R$.*

1.3 Matrices decompositions

In studying various rings which satisfy various generalizations of clean properties introduced by Nicholson, it has become useful also the research direction which has aim the writing of matrices depending on special matrices which satisfy particular identities.

We will refer first to matrices decompositions depending on idempotents.

Theorem. 1.3.1.*[49, Theorem 1.1] Let $\mathbb{F}$ be a field and $n \geq 1$. The following statements are equivalent.*

1. *Any matrix in $\mathbb{M}_n(\mathbb{F})$ is the sum of three idempotents.*

2. *Any invertible matrix in $\mathbb{M}_n(\mathbb{F})$ is the sum of three idempotents.*

3. *$\mathbb{F} \cong \mathbb{F}_2$ or $\mathbb{F} \cong \mathbb{F}_3$.*

[28] answers the question "when a matrix over a division ring of characteristic 0 is the sum of idempotents?". Next we have two results from this reference.

Theorem. 1.3.8.*[28, Theorem 1] Let $\mathbb{F}$ be a field of characteristic 0 and $M \in \mathbb{M}_n(\mathbb{F})$. Then M is the sum of idempotents if and only if $\mathrm{tr}(M) = k \cdot 1_{\mathbb{F}}$, $k \in \mathbb{Z}$ and $k \geq \mathrm{rang}(M)$.*

We refer to matrices decompositions as sums of nilpotent matrices.

We need a definition for the following result.

Definition. 1.3.19. Let k be a positive integer. An element of a ring is (k-)nilgood if it is the the sum of (k) nilpotents.

Theorem. 1.3.20.*[8, Theorem 2] The following statements are equivalent for the matrix A over a commutative ring*

1. *A is 3-nilgood.*

2. *A is nilgood.*

3. The trace of A is nilpotent.

We will refer to clean and nil-clean matrices decompositions. It is known by [25] that matrix rings over clean rings are clean. Diesel has asked if in [24] if there exists a similar result for nil-clean rings. More precisely, are matrix rings $M_n(\mathbb{F}_2)$ nil-clean? The answer is given in the next Theorem.

Theorem. 1.3.24. *[9, Theorem 3] Let K be a field/ The following statements are equivalent:*

1. *$K \simeq \mathbb{F}_2$.*
2. *For every positive integer n, the matrix ring $\mathbb{M}_n(K)$ is nil-clean.*
3. *There exists a positive integer n, such that the matrix ring $\mathbb{M}_n(K)$ is nil-clean.*

We continue with a Lemma that is a useful tool in [47], and we will use in Chapter 4 a similar useful result in characterizing m-nil-clean companion matrices.

Lemma. 1.3.30. *[47, Lemma 2.1] Let $\mathbb{F}$ be a field and q a monic polynomial over $\mathbb{F}$. Then there exists a monic polynomial Q such that the companion matrix $C_q \in \mathbb{M}_n(\mathbb{F})$ is similar to the matrix $C_Q + \mathrm{diag}(1, 0, 1, 0, \dots, 1, 0)$, if n is even, and with the matrix $C_Q + \mathrm{diag}(1, 0, 1, 0, \dots, 1, 0, 1)$, if n is odd.*

Definition. 1.3.33. An element of a ring R is unipotent if it is the sum of 1_R and a nilpotent of R.

Also in Chapter 4 we will use the following characterization of nil-clean companion matrices given by Breaz and Modoi in [12]:

Theorem. 1.3.34. *[12, Theorem 5] Let $\mathbb{F}$ be a field, of positive characteristic p. Let $C = C_{c_0,c_1,\dots,c_{n-1}} \in M_n(\mathbb{F})$ a companion matrix. The following statements are equivalent*

1. *C is nil-clean.*
2. *One of the following conditions is true:*

(a) *C is nilpotent (i.e.* $c_0 = \ldots = c_{n-1} = 0$*);*

(b) *C is unipotent (i.e.* $c_i = (-1)^i \binom{n}{n-i}$ *for every* $i \in \{0, \ldots, n-1\}$);

(c) *there exists a integer* $k \in \{1, \ldots, p\}$ *such that* $-c_{n-1} = k \cdot 1$ *and* $n > k$.

As a consequence of this fact the following result takes place:

Corollary. 1.3.35.*[12, Corollary 8] Let* $n \geq 3$ *be a positive integer. The following statements are equivalent for the field* $\mathbb{F}$*:*

1. $\mathbb{F} \cong \mathbb{F}_p$ *for a prime number* $p < n$*;*

2. *any companion matrix* $C \in M_n(\mathbb{F})$ *is nil-clean.*

Chapter 2

Weakly nil-clean index and uniquely weakly nil-clean rings

2.1 Results on weakly nil-clean index and uniquely weakly nil-clean rings

In this Chapter we will introduce and study weakly nil-clean index associated with a ring. We will present the basic properties of this index. A fundamental result of this part is Proposition 2.1.9, where we state that the weakly nil-clean index 1 rings are exactly the abelian ones. Because of this result we obtain that a ring is uniquely weakly nil-clean if and only if it is abelian weakly nil-clean. In The Second Section, we will compute the weakly nil-clean index of various matrix rings.

Definition. 2.1.1. Let R be a ring and $a \in R$. We define the set

$$\alpha(a) = \{e \in R : e^2 = e \text{ and } a - e \text{ or } a + e \text{ is a nilpotent}\}.$$

Definition. 2.1.2. For an element $a \in R$ the *weakly nil-clean index* of a, abbreviated as $wnc(a)$, is defined to be the cardinality of the set $\alpha(a)$.

Definition. 2.1.3. We define the *weakly nil-clean index* of a ring R as it follows:

$$wnc(R) = \sup\{|\alpha(a)| : a \in R\}.$$

In what follows next we will give some properties of this index.

Lema 2.1.1. 2.1.6. *If R is a ring with a subring S, then $wnc(R) \geq wnc(S)$.*

Lema 2.1.2. 2.1.7. *If R is a ring with a nil-ideal I, then $wnc(R/I) \leq wnc(R)$.*

We continue with a linking Lemma between the weakly nil-clean index and the nil-clean index of the same ring.

Lemma. 2.1.8. *For any ring R we have the inequality $wnc(R) \geq nc(R)$.*

Proposition. 2.1.9. *Let R be a ring. Then $wnc(R) = 1$ if and only if R is abelian.*

According to this Proposition, it follows the equivalence 1. $\Leftrightarrow$ 2. from the next Theorem.

Theorem. 2.1.10. *The following are equivalent for a ring R:*

1. *R is uniquely weakly nil-clean;*
2. *R is abelian weakly nil-clean;*
3. *$R \cong R_1 \times R_2$, where R_1 is either 0 or an abelian nil-clean ring and R_2 is either 0 or a local weakly nil-clean ring such that $J(R_2)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$.*

2.2 Weakly nil-clean index for matrix rings

In this Section, we compute the weakly nil-clean index for various matrix rings.

Example. 2.2.2. Let p be a prime number. Then $wnc(\mathbb{T}_2(\mathbb{F}_p)) = \mathrm{nc}(\mathbb{T}_2(\mathbb{F}_p)) = p$.

Example. 2.2.3. Let p be a prime number. Then $\mathrm{wnc}(\mathbb{T}_3(\mathbb{F}_p)) = \mathrm{nc}(\mathbb{T}_3(\mathbb{F}_p)) = p^2$.

Example. 2.2.5. $wnc(\mathbb{M}_2(\mathbb{F}_3)) = 5$.

Besides being a computation example of weakly nil-clean index of a matrix ring, the next Proposition gives the characterization of rings with weakly nil-clean index 2. By [5] we get that it is the same characterization as the one of rings with nil-clean index 2.

Proposition. 2.2.7. *Let R be a ring. The following statements are equivalent:*

1. *$wnc(R) = 2$;*
2. *There exist the abelian rings A and B and the bimodule ${}_AM_B$, with $|M| = 2$, such that $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.*

Chapter 3

Weakly tripotent rings

Record that an element a of a ring R is called tripotent if $a^3 = a$. A ring is called tripotent if all its elements are tripotent. A ring is weakly tripotent if for every element x of it, it is true that at least one element of x or $1 + x$ is tripotent.

There are major differences between the class of tripotent rings and the class of weakly tripotent rings. The first ones are always commutative. The second type exists also in a noncommutative kind. A direct product of tripotent rings is a tripotent ring, but a direct product of weakly tripotent rings need not to be weakly tripotent. Under this circumstance (of a direct product of rings), the difference is of the same type as the one described in the case of weakly nil-clean rings. We are going to establish also other properties for weakly tripotent rings and we will determine the characteristic of such rings. We will obtain a structure Theorem for weakly tripotent rings with only trivial idempotents and with 3 an invertible element. Moreover we will obtain a structure Theorem for commutative case. This is the central result of this Chapter.

The results in this Chapter have been published in [10].

3.1 General weakly tripotent rings

Definition. 3.1.1. A ring is weakly tripotent, if for every element x of it, it is true that at least one of the elements x or $1+x$ are tripotent.

Lemma. 3.1.2. *Let R be a weakly tripotent ring. Then*

(1) *every subring of R is weakly tripotent;*

(2) *every homomorphic image of R is weakly tripotent;*

(3) $24 = 0$, *hence R has a decomposition $R = R_1 \times R_2$ such that R_1 and R_2 are weakly tripotent rings, $8R_1 = 0$, and $3R_2 = 0$;*

For the case of R of characteristic 3, it is easy to be seen that the identity $(1+x)^3 = 1+x$ implies $x^3 = x$. By this, weakly tripotent rings of characteristic 3 are tripotent. So we can restrict our study for rings of characteristic 2^k, $k \in \{1, 2, 3\}$.

Corollary. 3.1.5. *The class of weakly tripotent rings is a proper subclass of rings with the property that all its elements are sums of an idempotent and a tripotent element, that commute.*

A direct product of weakly tripotent rings need not to be weakly tripotent.

Proposition. 3.1.11. *A direct product $\prod_{i\in I} R_i$ of weakly tripotent rings is weakly tripotent if and only if there exists $i_0 \in I$ such that R_{i_0} is weakly tripotent and for all $i \in I \setminus \{i_0\}$ the rings R_i are tripotent.*

Theorem. 3.1.12. *[35, 12.3] (Birkhoff's Theorem) Any nonzero ring can be represented as a subdirect product of subdirectly irreducible rings.*

The following result was proved in [54, Theorem 3.6] by using the fact that if R is a ring such that every element of R is a sum of an idempotent and a tripotent that commute then R is strongly nil-clean, hence $R/J(R)$ is Boolean and $J(R)$ is nil, [32, Theorem 5.6]. We present, for the reader's convenience, a proof which uses the general strategy described in [35, Section 12].

Theorem. *[54, Theorem 3.6] Let R be a ring of characteristic 2^k. The following are equivalent:*

(1) *every element of R is a sum of an idempotent and a tripotent that commute;*

(2) $x^4 = x^6$ *for every* $x \in R$*;*

(3) $R/J(R)$ *is Boolean and* $U(R)$ *is a group of exponent* 2*;*

(4) $R/J(R)$ *is Boolean, and for every* $x \in J(R)$ *we have* $x^2 = 2x$*.*

Consequently, if R satisfies the above conditions then $J(R)$ is the set of all nilpotent elements of R.

In the following corollary we have a characterization of weakly tripotent rings without nontrivial idempotents. This will be useful in order to describe commutative weakly tripotent rings, since in this case every subdirectly irreducible ring is without nontrivial idempotents.

Corollary. *The following are equivalent for a ring R without nontrivial idempotents such that $3 \in U(R)$:*

(1) *R is weakly tripotent;*

(2) *R is a local ring, such that $R/J(R) \cong \mathbb{F}_2$, and $x^2 = 1$, for every $x \in U(R)$;*

(3) *R has the property that $R/J(R) \cong \mathbb{F}_2$ and for every $x \in J(R)$ we have $x^2 = 2x$.*

Consequently, every weakly tripotent ring without nontrivial idempotents is commutative.

3.2 Commutative weakly tripotent rings

The aim of this section is to give a characterization of commutative weakly tripotent rings. So we need the following Lemma:

Lemma. 3.2.1. *Let R be a commutative weakly tripotent ring of characteristic 2^k ($k \leq 3$), and let N be the nil radical of R. Then for every idempotent e in R one of the following properties is true:*

(a) $en = 0$ *for all* $n \in N$;

(b) $en = n$ *for all* $n \in N$.

We have the following characterization for commutative weakly tripotent rings.

Theorem. 3.2.2. *A commutative ring* R *is weakly tripotent if and only if* $R = R' \times R''$ *such that:*

(1) R'' *is a tripotent ring of characteristic* 3 *or* $R'' = 0$;

(2) $R' = 0$ *or* $3 \in U(R')$ *and* R' *can be embedded as a subring of a direct product* $R_0 \times \left(\prod_{i \in I} R_i\right)$ *such that* R_0 *is a weakly tripotent ring without nontrivial idempotents, and all* R_i *are Boolean rings.*

Chapter 4

m-nil-clean companion matrices

Let $m \geq 2$ be an integer. An element of a ring R is m-nil-clean if it can be written as a sum of m idempotents from R and a nilpotent of R. In section 3 from [1] there have been investigated rings with the property that every element is m-nil-clean. The motivation of studying m-nil-clean companion matrices is the fact that every matrix that is similar to a direct some of companion matrices, in the fact that same as nil-clean matrix, the m-nil-clean matrices can be written in function of idempotents and nilpotents, and in the fact that a matrix similar to a nilpotent (idempotent) is nilpotent (idempotent).

As a consequence of Theorem 3 from [12] if the dimension of the $n \times n$ matrix over the field $\mathbb{F}$, of positive characteristic p, is greater than that characteristic, then the matrix is nil-clean. That is why in the studying characterization of m-nil-clean companion matrices we will consider $n \leq p$. This study will be materialized as a central result, and it will be done taking into account the trace of the matrix, the dimension of the matrix, and the p characteristic of the field from where the entries of the matrix are.

As a useful tool, similarly as in [47] and in [7], we will use the similarity of a companion matrices with the sum of a diagonal matrix with entries $0's$ and $1's$ and a companion matrix.

The results in this Chapter are taken from the article [20].

4.1 A useful tool

The following Lemma will be useful when there are proved results about 2-nil-clean companion matrices.

Lemma. 4.1.1 *Let $\mathbb{F}$ be a field. For every companion matrix $C_q \in M_n(\mathbb{F})$ and every $k \in \{1, \ldots, n\}$ there exists a companion matrix $C_{q'}$ such that C_q and* $\operatorname{diag}(\underbrace{1, \ldots, 1}_{k\text{-times}}, 0, \ldots, 0) + C_{q'}$ *are similar.*

4.2 A characterization of m-nil-clean companion matrices

Definition. 4.2.1. Let $m \geq 1$ an integer number. A matrix is *m-nil-clean* if it is the sum of m idempotent matrices and a nilpotent matrices.

Let $\mathbb{F}$ a field of positive characteristic p. As a consequence of Theorem 1.3.34., if $n > p$ then every $n \times n$ companion matrix over $\mathbb{F}$ is nil-clean. Therefore we will assume that $n \leq p$.

Secondly, for $n = 1$ the only nilpotent of $M_n(\mathbb{F})$ is (0) and the only idempotents of $M_n(\mathbb{F})$ are (0) and (1). Therefore $C \in M_1(\mathbb{F})$ is $m-$nil-clean if and only if $C \in \{(0), (1), (2), \ldots, (m)\}$. Hence we will not refer to the case $n = 1$, so we will assume $n > 1$ from now on.

Lemma. 4.2.2. *Let $m \geq 2$ be an integer. Let $\mathbb{F}$ be a field of positive characteristic, p. Let $A \in \mathbb{M}_n(\mathbb{F})$ be a (not necessarily companion) matrix, for which there exists the decomposition $A = E_1 + E_2 + \cdots + E_m + N$, with $k_i = \operatorname{rang}(E_i)$, E_i idempotent, $i \in \{1, 2, \ldots, m\}$, and N is nilpotent. Then $\operatorname{tr}(A) = c \cdot 1$, where c is the sum of $k_1, k_2, \ldots, k_m$ modulo p, so the sum of m natural numbers less or equal to n modulo p.*

Lemma. 4.2.4. *Let $m \geq 2$ be an integer. Let $\mathbb{F}$ be a field of positive characteristic, p and $1 < n \leq p$. If $-c_{n-1} = c \cdot 1$ and $c \in \{m, m+1, \ldots, mn-1\}$ then $C = C_{c_0, c_1, \ldots, c_{n-1}} \in \mathbb{M}_n(\mathbb{F})$ is $m-$nil-clean.*

The next Corollary is of the Theorem 1.3.34:

Corollary. 4.2.5 *Let $\mathbb{F}$ be a field of characteristic 2. Let $C = C_{c_0,c_1,\ldots,c_{n-1}} \in M_n(\mathbb{F})$ a companion matrix. Then C is nil-clean if and only if $-c_{n-1} \in \{0, 1\}$.*

Having the case $p = 2$ proved (C being nil-clean it will be also m-nil-clean) we can assume that p is odd.

Theorem. 4.2.6. *Let $m \geq 2$ be an integer. Let $\mathbb{F}$ be a field of positive odd characteristic p, and $1 < n \leq p$. Let $C = C_{c_0,c_1,\ldots,c_{n-1}}$ be a companion matrix. Let c be the nonnegative integer such that $-c_{n-1} = c \cdot 1$.*

The following hold:

1. *if $c = 0$ and $mn - 1 < p$ then C is $m-$nil-clean if and only if C is nilpotent or $C - (m-1)I_n$ is unipotent;*

2. *if $c = t$, $1 \leq t \leq m$ then C is $t-$nil-clean ($1-$nil-clean is just nil-clean), so is $m-$nil-clean;*

3. *if $c \in \{m, m+1, \ldots, mn-2, mn-1\}$ then C is $m-$nil-clean,*

4. *if $mn - 2 \geq p$ then C is $m-$nil-clean;*

5. *Assume $mn - 2 < p$,*

   (a) *if $c = mn$ and $p = mn - 1$ then C is nil-clean, so is $m-$nil-clean;*

   (b) *if $c = mn$ and $p = mn$ then C is $m-$nil-clean if and only if C is nilpotent or $C - (m-1)I_n$ is a unipotent matrix;*

   (c) *if $c = mn$, $p > mn$ then C is $m-$nil-clean if and only if $C - (m-1)I_n$ is a unipotent matrix.*

   (d) *if $c > mn$ then C is not $m-$nil-clean.*

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