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Duality and Optimality Conditions in Vector Optimization

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Introduction

In mathematical areas an important place is reserved for the optimization theory. The main idea is to provide an overview and deliver new results referring to the duality in the vector optimization. There are a lot of papers in this field, like [80], D.T. Luc [87], R.I. Boţ, S.-M. Grad and G. Wanka [25], C. Zălinescu [151], T. Antczak [4], G. Cristescu and L. Lupşa [43], R.I. Boţ [15], R.I. Boţ, S.-M. Grad and G. Wanka [22], R.I. Boţ and G. Wanka [28, 29], R. Cambini and L. Carosi [33], R.R. Egudo [49, 50], R.R. Egudo, T. Weir and B. Mond [51], M. Ehrgott [52], A. Göpfert [60], D. Inoan [75], D. Inoan and J. Kolumbán [76], M.A. Islam [74], H. Kawasaki [82], K.M. Miettinen [91], N. Popovici [112], Y. Sawaragi, H. Nakayama and T. Tanino [116], T. Tanino and H. Kuk [126], T. Tanino and Y. Sawaragi [127], G. Wanka and R.I. Boţ [129–131], X.M. Yang, K.L. Teo and X.Q. Yang [149] and M. Zeleny [152].

To a general vector optimization problem are attached different vector duals and one can establish connections between these problems referring to their solutions. Also, there exists other types of problems that can be attached to a general vector optimization problem, like the approximated ones. When to a minimum vector optimization problem are attached maximum vector optimization problems (or duals) and are formulated duality results we refer to vector duality. The duality results that are usually provided refers to the weak duality, strong duality and converse duality. For the strong duality and the converse duality, one needs to use some supplementary conditions, called regularity conditions (see, for example R.I. Boţ [14], R.I. Boţ, S.-M. Grad and G. Wanka [22, 23, 25], E.R. Csetnek [44], J. Jahn [77], D.T. Luc [87], B.S. Mordukhovich [103, 104], H. Nakayamma [105], R.T. Rockafellar [114], C. Tammer, A. Göpfert [122], T. Tanino [125], C. Zălinescu [150]). There are a lot of regularity conditions that can be used, but here we work especially with the classical one that involves continuity, the one that works for Frechét spaces, the one for finite dimensional case and the closedness type one (cf. R.I. Boţ, S.-M. Grad and G. Wanka [25]).

The solutions used in the duality results can be the optimal solutions, efficient solutions, weakly efficient solutions and properly efficient solutions (in our case, in the sense of linear scalarization) (see, for example, R.I. Boţ, S.-M. Grad and G. Wanka [25], A.M. Geoffrion [57], A. Guerraggio, E. Molho and A. Zaffaroni [69], I. Kaliszewski [81]). The different types of solutions that can be considered to a vector optimization problem give different vector duals to the primal. Moreover, for a primal vector optimization problem one can attach different duals and study the optimality conditions between these problems, too (see, for example R.I. Boţ, S.-M. Grad and G. Wanka [25], A. Ben-Israel, A. Ben-Tal and S. Zlobec [8]).

We work especially in infinite dimensional settings, but in the end we present also some results in the finite dimensional case, where we consider some constraint qualifications in the formulations of the results.

In this thesis for vector duality we present different directions and interpretations. To a primal vector optimization problem one can attach different vector duals by using also different vector perturbation functions, like Lagrange, Fenchel or Fenchel-Lagrange vector perturbation functions. We include the general duality concepts and results presented by a lot of authors like, R.T. Rockafellar [113], C. Zălinescu [151], R.I. Boţ, S.-M. Grad and G. Wanka [14, 25], J. Jahn [77–80], C.R. Chen and S.J. Li [36], W. Breckner and I. Kolumbán [31,32]. A very important part of this thesis refers to the Wolfe and the Mond-Weir vector duality, subject studied also by a lot of authors (see, for example R.I. Boţ and S.-M. Grad [20,21]). For Wolfe vector duality concepts and results we can refer to B.D. Craven [41], M. Schechter [119] and P. Wolfe [148] and in the case of Mond-Weir vector duality to B. Monde [98], B. Mond and M.A. Hanson [99], B. Mond and S. Zlobec [102], T. Weir [135–140], T. Weir and B. Mond [100, 101, 141–144], T. Weir, B. Mond and B.D. Craven [145,146]. Some characterizations for the optimization problems referring to their solutions (efficient, properly efficient in the sense of linear scalarization, weakly efficient, quasi-efficient) and some duality results are presented, too. We also give comparisons between the image sets of different vector duals attached to the same vector optimization problem, delivering either inclusion relations between them, or counterexamples that prove that in general neither of them is a subset of the other. Also, we study the optimality conditions between the primal vector optimization problem and the vector duals attached to it obtained by using different scalarizations (linear scalarization, maximum($-$linear) scalarization, set scalarization, (semi)norm scalarization, oriented distance scalarization).

Here we include the author's own results obtained in joint works and addressing different problems in vector optimization. The theoretical statements are accompanied by a proof and are followed by examples. The results are partially included in the following papers: S.-M. Grad and E.-L. Pop [66–68], E.-L. Pop [107], E.-L. Pop and D.I. Duca [108–111].

The thesis is split into five chapters, preceded by an introduction and succeeded by some references.

The **first chapter** presents the most important notions and results from the specialized literature (cf. [14, 25, 39, 46, 54, 77, 80, 103–105, 113, 114, 151]) used in the following chapters and in the author's own results. We work especially with the basic notions and results from convex analysis (cf. J.M. Borwein and A.S. Lewis [13], J.-B. Hiriart-Urruty and C. Lamaréchal [73], D.T. Luc [87], S.K. Mishra, S. Wang and K.K. Lai [97], C. Tammer and A. Göpfert [122], R.T. Rockafellar [113], T. Tanino [125],

P. Weidner [134], C. Zălinescu [151], for example). In the first section we introduce some special subsets of the vector spaces, like the convex sets; we give definitions for cones, interior notions and generalized interior notions for convex sets that are used especially in the formulation of the regularity conditions, partial orderings, separation theorems and useful properties. The next part contains the notions and results concerning the extended real and vector functions. We remember the definitions for the convex function, indicator function and support functional of a set, the domain and the epigraph of a function, the conjugate function, the Young-Fenchel inequality and the subdifferential. In the second part of this chapter we present the conjugate theory for a scalar optimization problem and we give some connections between a vector optimization problem and some dual problems attached to it and obtained by using the vector perturbation approach. We work with an unconstrained vector optimization problem having a composition with a linear continuous mapping as objective function, to which properly efficient solutions in the sense of linear scalarization and weakly efficient solutions are considered. Due to R.I. Boţ, S.-M. Grad and G. Wanka [25] we particularize the case of Fenchel duality for this unconstrained vector optimization problem where the linear continuous mapping is considered also invertible and the duals obtained are only equivalent formulations of the ones already known. Also, some connections between properly efficient solutions for the primal vector optimization problem and efficient solutions for Wolfe and Mond-Weir vector dual problems attached, follows, as a particular case, too.

In the **second chapter** we attach to a general vector optimization problem two new vector duals by means of perturbation theory. These vector duals are constructed with the help of the recent Wolfe and Mond-Weir scalar duals for general optimization problems proposed by R.I. Boţ and S.-M. Grad [21], by exploiting an idea due to W. Breckner and I. Kolumbán [31, 32]. Then we give some duality results (see [66]). We particularize the initial primal vector optimization problem to be a constrained vector optimization problem (a vector optimization problem with geometric and cone constraints) and then an unconstrained vector optimization problem (a vector optimization problem having a composition with a linear continuous mapping as objective function) and from the general case we obtain vector dual problems of Wolfe type and Mond-Weir type for it by using different vector perturbation functions (Lagrange, Fenchel, Fenchel-Lagrange). Then we formulate some comparisons between duals. In the second part of this chapter we introduce the Wolfe type and Mond-Weir type vector duals with respect to the weakly efficient solutions.

The author's contributions are presented in the following theorems: 2.1.4, 2.1.6, 2.1.7, 2.1.8, 2.1.16, 2.1.17, 2.1.24, 2.1.25, 2.1.29, 2.1.30, 2.1.37 and 2.1.38; remarks: 2.1.3, 2.1.5, 2.1.9, 2.1.10, 2.1.11, 2.1.12, 2.1.14, 2.1.19, 2.1.20, 2.1.21, 2.1.27, 2.1.28, 2.1.31, 2.1.32 and 2.1.43; lemma's: 2.1.2; propositions: 2.1.13, 2.1.22 and 2.1.33; and examples: 2.1.15, 2.1.23, 2.1.39 and 2.1.40; and some of these can be found in [66].

Chapter three present some duality results with respect to the quasi-efficient solutions. This subject was also studied by J.M. Borwein and A.S. Lewis [12], R.I. Boţ and E.R. Csetnek [16], R.I. Boţ, E.R. Csetnek and A. Moldovan [17], R.I. Boţ, E.R. Csetnek and G. Wanka [18], E.R. Csetnek [44], B.D. Craven [42]. From the general case by considering the primal vector optimization problem to be constrained and unconstrained we obtain different vector dual problems by using the vector perturbation functions and also some preliminary results, where some separation theorems are required (see, for example J.M. Borwein, A.S. Lewis [12], R.I. Boţ, E.R. Csetnek and A. Moldovan [17], R.I. Boţ, E.R. Csetnek and G. Wanka [18], C. Gerth and P. Weidner [62]). The duality results that are delivered refers to weak, strong and converse duality and also to some connections between the image sets of different vector duals attached to the same vector optimization problem. In the second part of this chapter we make some remarks in the case when the relative interior is considered instead of quasi interior and consequently the corresponding solutions, due to J.M. Borwein, R. Goebel [11].

The author's contributions are presented in the following theorems: 3.1.11, 3.1.13, 3.1.17, 3.1.19, 3.1.21, 3.1.22, 3.1.24, 3.1.25, 3.1.26, 3.1.32 and 3.1.33; remarks: 3.1.4, 3.1.7, 3.1.9, 3.1.18, 3.1.20, 3.1.23, 3.1.29 and 3.1.31; lemma's: 3.1.8 and 3.1.12; propositions: 3.1.10, 3.1.28 and 3.1.30; corollaries: 3.1.14 and 3.1.34; definitions: 3.1.3, 3.1.6, 3.1.15 and 3.1.16; and examples: 3.1.5; and some of these can be found in [68].

In the **fourth chapter** we turn our attention to the formulation of the optimality conditions between a general vector optimization problem with geometric and cone constraints and some vector duals attached to it and obtained by using different scalarizations (like linear scalarization, maximum($-$linear) scalarization, set scalarization, (semi)norm scalarization, oriented distance scalarization). Here are used the scalarization functions and the set of the scalarization functions in the construction of the duals (see, for example R.I. Boţ, S.-M. Grad and G. Wanka [22, 25], E. Carrizosa and J. Fliege [35], J. Fliege [55], S. Helbig [72], J, Jahn [78, 80], P.Q. Khanh [83], D.T. Luc [87], E. Miglierina and E. Molho [93], C. Tammer and K. Winkler [123], P. Weidner [134]).

The author's contributions are presented in the following theorems: 4.1.11, 4.1.14, 4.1.17, 4.1.22, 4.1.32 and 4.1.36; and some of these can be found in [67].

Chapter five present some applications. First we study the connections between the optimal solutions and saddle points for the Lagrangian of an approximated optimization problem attach to

$$\begin{cases} \min & f(x) \\ \text{s. t.} & x \in X \\ & g(x) \leqq 0 \\ & h(x) = 0, \end{cases}$$

where X is a subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}$, $g = (g_1, ..., g_m) : X \to \mathbb{R}^m$ and $h = (h_1, ..., h_q) : X \to \mathbb{R}^q$ are three functions, and the optimization problem. Here we present some results referring to the optimal solutions and saddle points for the Lagrangian of the primal optimization problem and the $(0, 1) - \eta$ approximated optimization problem and also some connections between these optimal solutions and saddle points for the Lagrangian (cf. [108–110]). We close here by extending our problems to the vector case and we deliver some connections between the efficient solutions and the saddle points for the Lagrangian of these problems (cf. [111]). Another application can be given in set-valued. For example, by considering the notion of relative interior we can formulate new set relations with the help of a convex cone introduced by D. Kuroiwa [124]. Then by using the idea of A. Grad [64, 65] we can deliver duality results and optimality conditions for a set-valued optimization problem (see [107]).

The author's contributions are presented in the following theorems: 5.2.4, 5.2.8, 5.2.12 and 5.2.14; remarks: 5.1.5; and lemma's: 5.2.10; and some of these can be found in [109, 111].

Key Words
vector optimization problem, dual vector optimization problem, perturbation function, conjugate function, (convex) subdifferential, regularity condition, vector duality, weak/strong/converse duality, Wolfe duality, Mond-Weir duality, efficient solution, properly efficient solution, weakly efficient solution, Lagrange vector perturbation function, Fenchel-Lagrange vector perturbation function, Fenchel vector perturbation function, quasi-interior, quasi-minimal element, quasi-efficient solution, relative interior, general scalarization, linear scalarization, maximum($-$linear) scalarization, set scalarization, (semi)norm scalarization, oriented distance scalarization, optimality condition, optimal solution, saddle point for the Lagrangian, optimization problem, $(0,1)-\eta$ approximated optimization problem, dual.

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Chapter 1

Vector optimization problems: an overview

1.1 Preliminaries

We commence by presenting the basic notions and results from the convex analysis and the ones concerning the extended real and vector functions (cf. J.M. Borwein, A.S. Lewis [12], R.I. Boţ [14], R.I. Boţ, S.-M. Grad and G. Wanka [25], P. Daniele, S. Giuffré, G. Idone, A. Maugeri [46], R.I. Boţ, S.-M. Grad and G. Wanka [23], J. Jahn [80], R.T. Rockafellar [113], C. Zălinescu [151]).

1.1.1 Sets and interiors of sets

Convex sets, affine sets and cones

Let X be a vector space. A set $U \subseteq X$ is called *convex* if $(1-\lambda)x + \lambda y \in U$ for all $x, y \in U$ and all $\lambda \in [0, 1]$.

A *cone* $K \subseteq X$ is a nonempty set which fulfills $\lambda K \subseteq K$ for all $\lambda \geq 0$. A *convex cone* is a cone which is a convex set. A cone $K \subseteq X$ is called *nontrivial* if $K \neq \{0\}$ and $K \neq X$ and *pointed* if $K \cap (-K) = \{0\}$.

On X we consider the partial ordering "$\leqq_K$" induced by the convex cone $K \subseteq X$, defined by $x \leqq_K y \Leftrightarrow y - x \in K$ when $x, y \in X$. The notation $x \leq_K y$ is used to write more compactly that $x \leqq_K y$ and $x \neq y$, where $x, y \in X$. A convex cone which induces a partial ordering on X is called *ordering cone.* If $K \neq \{0\}$, a greatest element with respect to "$\leqq_K$" which does not belong to X denoted by ∞_K is attached to X, and let $X^\bullet = X \cup \{\infty_K\}$.
For $U \subseteq X$ a nonempty set we consider the

- *linear hull* of U $\operatorname{lin} U = \{\sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, x_i \in U, \lambda_i \in \mathbb{R}, i = 1, ..., n\}$;
- *convex hull* of U $\operatorname{co} U = \{\sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, x_i \in U, \lambda_i \geq 0, i = 1, ..., n, \sum_{i=1}^{n} \lambda_i = 1\}$;
- *conical hull* of U $\operatorname{cone} U = \{\lambda x : \lambda \geq 0, x \in U\}$.

Also, we denote by $\operatorname{int} U$, $\operatorname{cl} U$ and $\dim U$ the *interior*, the *closure* and the *dimension* of the set U. For $U \subseteq X \times Y$, where X and Y are nontrivial real vector spaces, the *projection function* of U on X, $\operatorname{Pr}_X : X \times Y \to X$ is defined by $\operatorname{Pr}_X(U) = \{x \in X : \exists y \in Y \text{ such that } (x, y) \in U\}$. The *identity function* on X is a special linear mapping, $\operatorname{id}_X : X \to X$ defined by $\operatorname{id}_X(x) = x$ for all $x \in X$.

Let now X be a topological vector space and X^* its topological dual spaces endowed with the corresponding weak* topology and denote by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$.

The *dual cone* of K is $K^* = \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \ \forall x \in K\}$, the *quasi interior of the dual cone* of K is given by $K^{*0} = \{x^* \in X^* : \langle x^*, x\rangle > 0 \ \forall x \in K \setminus \{0\}\}$ and the *normal cone* associated with a set $U \subseteq X$ is defined by $N_U(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \text{ for all } y \in U\}$.

By $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we denote the *extended real space* and it has the the same operations as $\mathbb{R}$ and also some new ones.

Interiors of sets

In what follows we remember some generalized interiors of sets and connections between them that are useful especially in the formulation of the regularity conditions.

For X a nontrivial vector space and $U \subseteq X$ a set we have the *algebraic interior* of U $\operatorname{core} U = \{x \in X : \text{ for every } y \in X \ \exists \delta > 0 \text{ such that } x + \lambda y \in U \forall \lambda \in [0, \delta]\}$.

Some topological notions of generalized interiors of $U \subseteq X$, where X is a Hausdorff locally convex space and X^* its topological dual space endowed with the weak* topology, follow.

- The *quasi interior* of U is the set $\operatorname{qi} U = \{x \in U : \operatorname{cl}(\operatorname{cone}(U - x)) = X\}$;
- The *quasi relative interior* of U is the set $\operatorname{qri} U = \{x \in U : \operatorname{cl}(\operatorname{cone}(U - x)) \text{ is a linear subspace}\}$;
- The *strong quasi relative interior* of U is the set $\operatorname{sqri} U = \{x \in U : \operatorname{cone}(U - x) \text{ is a closed linear subspace}\}$.

In the case when $X = \mathbb{R}^n$, $n \in \mathbb{N}$ and $U \subseteq \mathbb{R}^n$ a set, we have the *relative interior* of U defined by $\operatorname{ri} U = \{x \in \operatorname{aff} U : \exists \varepsilon > 0 \text{ such that } B(x, \varepsilon) \cap \operatorname{aff} U \subseteq U\}$, where $B(x, \varepsilon)$ is the closed ball centered at x with radius ε and $\operatorname{aff} U$ is the affine hull of U.

Separation theorems

We introduce a separation theorem for convex sets by mean of quasi-relative interior.

Theorem 1.1.17 (*[18, Theorem 2.7]) Let U be a nonempty convex set of a separated locally convex space X and $\overline{x} \in U$. If $\overline{x} \notin \operatorname{qri} U$, then there exists an $x^* \in X^* \setminus \{0\}$ such that $\langle x^*, x\rangle \leq \langle x^*, \overline{x}\rangle$ for all $x \in U$.*

1.1.2 Some functions and their properties

Extended real functions

Let X be a locally convex space, X^* its topological dual space endowed with the corresponding weak* topology and $U \subseteq X$ a nonempty set.

The *indicator function* $\delta_U : X \to \overline{\mathbb{R}}$ is defined by 0 if $x \in U$ and $+\infty$, otherwise and the *support function* $\sigma_U : X^* \to \overline{\mathbb{R}}$ by $\sigma_U(x^*) = \sup\{\langle x^*, x\rangle : x \in U\}$. For $U \subseteq X$ a convex absorbing set the *gauge* (or *Minkowski function*) associated to it $\gamma_U : X \to \overline{\mathbb{R}}$ is defined by $\gamma_U(x) = \inf\{\lambda \geq 0 : x \in \lambda U\}$.

The function $f : X \to \overline{\mathbb{R}}$ is called *convex* if for all $x, y \in X$ and for all $\lambda \in [0, 1]$ one has $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. The function $f : X \to \overline{\mathbb{R}}$ is called *concave* if $(-f)$ is convex. We denote by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We recall that f is *lower semicontinuous* if and only if $\operatorname{epi} f$ is a closed set. The function f is *proper* if $f(x) > -\infty$ for all $x \in X$ and $\operatorname{dom} f \neq \emptyset$.

Let now $x \in X$ an arbitrary point such that $f(x) \in \mathbb{R}$. The set $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}$ is called the (*convex*) *subdifferential* of f at x. Its elements are called *subgradients* of f at x. If $\partial f(x) \neq \emptyset$ then the function f is called *subdifferentiable* at x. If $f(x) \notin \mathbb{R}$ then by convention we establish $\partial f(x) = \emptyset$.

The function $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup\limits_{x\in X}\{\langle x^*, x\rangle - f(x)\}$ is called the (*Fenchel*) *conjugate function of* f and $f^*_U : X^* \to \overline{\mathbb{R}}$ defined by $f^*_U(x^*) = (f + \delta_U)^*(x^*) = \sup_{x\in U}\{\langle x^*, x\rangle - f(x)\}$ is called the *conjugate function of f with respect to the nonempty set $U \subseteq X$*. Between a function and its conjugate there is the *Young-Fenchel inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$. This inequality is fulfilled as an equality if and only if $x^* \in \partial f(x)$.

Let now X and Y be two topological vector spaces. For a linear continuous mapping $A : X \to Y$ we have its *adjoint operator* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$.

Definition 1.1.24 *Let X be a vector space partially ordered by the convex cone K, $U \subseteq X$ and $f : X \to \overline{\mathbb{R}}$ a given function.*

(a) If $f(x) \leq f(y)$ for all $x, y \in U$ such that $x \leqq_K y$, the function f is called $K-$increasing on U.

(b) If $f(x) < f(y)$ for all $x, y \in U$ such that $x \leq_K y$, the function f is called

strongly $K-$increasing on U.

(c) *If f is $K-$increasing on U,* $\operatorname{core} K \neq \emptyset$ *and for all $x, y \in U$ fulfilling $x <_K y$ follows $f(x) < f(y)$, the function f is called strictly $K-$increasing on U.*

(d) *If $X = U$ these classes of functions are called $K-$increasing, strongly $K-$increa sing and strictly $K-$increasing.*

Extended vector functions

Let X be a vector space and let V be locally convex space partially ordered by a nonempty convex cone K and $V^\bullet = V \cup \{\pm\infty_K\}$.

For the vector function $F : X \to V^\bullet$ we have $\operatorname{dom} F = \{x \in X : F(x) \in V\}$ its *domain* and $\operatorname{epi}_K F = \{(x, v) \in X \times V : F(x) \leqq_K v\}$ its $K-$*epigraph.* The function F is called *proper* if $\operatorname{dom} F$ is nonempty; $K-$*convex* if $F(\lambda x + (1-\lambda)y) \leqq_K \lambda F(x) + (1-\lambda)F(y)$ for all $x, y \in X$ and all $\lambda \in [0, 1]$; $K - \operatorname{epi}-$*closed* if K is closed and its $K-$epigraph is closed and $K-$*semicontinuous* if for every $x \in X$, each neighborhood W of zero in V and for any $b \in V$ satisfying $b \leqq_K F(x)$, there exists a neighborhood U of x in X such that $F(U) \subseteq b + W + K \cup \{+\infty_K\}$.

For $v^* \in K^*$ the function $(v^*F) : X \to \overline{\mathbb{R}}$ is defined by $(v^*F)(x) = \langle v^*, F(x)\rangle$ for all $x \in X$.

1.2 Optimization problems

In this section we present the conjugate theory for a scalar optimization problem and we consider the vector optimization problem having the composition with a linear continuous mapping in the objective function and give some duality results referring to properly efficient solutions in the sense of linear scalarization and weakly efficient solutions. Then, for the unconstrained vector optimization problem we introduce the Wolfe and Mond-Weir type vector duals and obtain weak and strong duality results (like in [21, 139, 140, 143]). This part presents a particular case for the Fenchel vector duality where the linear and continuous mapping is also invertible. The vector duals introduced here are only equivalent formulations of the ones already known (see [25]).

1.2.1 Scalar optimization problems

In this part we present some duality results for duals by using the perturbation theory.

1.2.2 Vector optimization problems

In this section to an unconstrained vector optimization problem having the composition with a linear continuous and invertible mapping in the objective function we attach vector dual problems (that are equivalent formulations of the ones introduced in [25]) and we deliver duality results.

Chapter 2

Vector duality of Wolfe type and Mond-Weir type

The Wolfe and the Mond-Weir duality approaches from this chapter were originally considered for scalar constrained optimization problems (see, for example R.I. Boţ and S.-M. Grad [21]) and were quickly generalized for vector optimization problems, too. For a constrained vector minimization problem the approach was employed by T.Q. Chien [37], where the involved functions were considered quasidifferentiable.

2.1 Wolfe type and Mond-Weir type vector duality

We introduce new Wolfe type and Mond-Weir type vector duals achieved via the approach from W. Breckner and I. Kolumbán [31,32] and J. Jahn [80] to a general vector minimization problem. We compare these new vector duals with the vector duals from R.I. Boţ and S.-M. Grad [20] and we deliver weak and strong duality statements for them. Then we consider particular primal vector optimization problems and for different vector perturbation functions we obtain new Wolfe and Mond-Weir type vector duals to these vector problems. We compare the image sets of the different vector duals attached to the same vector optimization problem, delivering either inclusion relations between them, or counterexamples that prove that in general neither of them is a subset of the other.

Some of the results were obtained by the author in joint work with dr. S.-M. Grad [66].

2.1.1 General duality results

Let X, Y and V be separated locally convex vector spaces, with V partially ordered by the nonempty pointed convex cone $K \subseteq V$. Let $F : X \to V^\bullet$ be a proper and $K-$convex function and consider the general vector optimization problem

$$(PVG) \qquad \operatorname*{Min}_{x \in X} F(x).$$

For this vector optimization problem we remember the solution concepts.

(i) An element $\overline{x} \in X$ is called *efficient solution* to the vector optimization problem (PVG) if $\overline{x} \in \operatorname{dom} F$ and for all $x \in \operatorname{dom} F$ from $F(x) \leqq_K F(\overline{x})$ follows $F(x) = F(\overline{x})$.

(ii) An element $\overline{x} \in X$ is called *properly efficient solution* to the vector optimization problem (PVG) if there exists $v^* \in K^{*0}$ such that $(v^*F)(\overline{x}) \leq (v^*F)(x)$ for all $x \in X$.

Using the $K-$convex vector perturbation function $\Phi : X \times Y \to V^\bullet$ which fulfills $0 \in \operatorname{Pr}_Y(\operatorname{dom} \Phi)$ and $\Phi(x, 0) = F(x)$ for all $x \in X$, the primal vector optimization problem introduced above can be reformulated as

$$(PVG) \qquad \operatorname*{Min}_{x \in X} \Phi(x, 0).$$

To (PVG) we attach two vector dual problems. To construct them, we used the scalar Wolfe type and Mond-Weir type duals introduced by R.I. Boţ and S.-M. Grad in [21], exploiting moreover the vector duality approach from J. Jahn [80] and W. Breckner and I. Kolumbán [31, 32].

The *Wolfe type vector dual* to (PVG) we consider is

$$(DVG^W) \qquad \operatorname*{Max}_{(v^*, y^*, v, u, y) \in \mathcal{B}_G^W} h_G^W(v^*, y^*, v, u, y)$$

where

$$\begin{aligned} \mathcal{B}_G^W = \{(v^*, y^*, v, u, y) \in K^{*0} \times Y^* \times V \times X \times Y : (0, y^*) \in \partial(v^*\Phi)(u, y), \\ \langle v^*, v \rangle \leq -(v^*\Phi)^*(0, y^*)\} \end{aligned}$$

and

$$h_G^W(v^*, y^*, v, u, y) = v,$$

while the *Mond-Weir type vector dual* one is

$$(DVG^M) \qquad \operatorname*{Max}_{(v^*, y^*, v, u) \in \mathcal{B}_G^M} h_G^M(v^*, y^*, v, u)$$

where

$$\begin{aligned} \mathcal{B}_G^M = \{(v^*, y^*, v, u) \in K^{*0} \times Y^* \times V \times X : (0, y^*) \in \partial(v^*\Phi)(u, 0), \\ \langle v^*, v \rangle \leq \langle v^*, \Phi(u, 0) \rangle\} \end{aligned}$$

and

$$h_G^M(v^*, y^*, v, u) = v.$$

Lemma 2.1.2 (*S.-M. Grad and E.-L. Pop [66]*) *One has* $h_G^M(\mathcal{B}_G^M) \subseteq h_G^W(\mathcal{B}_G^W)$.

Remark 2.1.3 (*S.-M. Grad and E.-L. Pop [66]*) *The sets* $h_G^M(\mathcal{B}_G^M)$ *and* $h_G^W(\mathcal{B}_G^W)$ *do not coincide in general. A situation like this will be given later in Example 2.1.15.*

Now we investigate the connections between the duals to (PVG) considered here and other Wolfe and Mond-Weir type vector duals introduced by R.I. Boţ and S.-M. Grad in [20] to it, which are

$$(DVG_W) \qquad \underset{(v^*,y^*,u,y,r)\in\mathcal{B}_W^G}{\text{Max}} h_W^G(v^*, y^*, u, y, r)$$

where

$$\mathcal{B}_W^G = \{(v^*, y^*, u, y, r) \in K^{*0} \times Y^* \times X \times Y \times (K \setminus \{0\}) : (0, y^*) \in \partial(v^*\Phi)(u, y)\}$$

and

$$h_W^G(v^*, y^*, u, y, r) = \Phi(u, y) - \frac{\langle y^*, y\rangle}{\langle v^*, r\rangle} r$$

and, respectively,

$$(DVG_M) \qquad \underset{(v^*,y^*,u)\in\mathcal{B}_M^G}{\text{Max}} h_M^G(v^*, y^*, u)$$

where

$$\mathcal{B}_M^G = \{(v^*, y^*, u) \in K^{*0} \times Y^* \times X : (0, y^*) \in \partial(v^*\Phi)(u, 0)\}$$

and

$$h_M^G(v^*, y^*, u) = \Phi(u, 0).$$

Unlike these vector duals, the ones we introduced above do not have the objective function of (PVG) as objective functions. The newly introduced vector duals inherit all the constraints of the vector duals from R.I. Boţ and S.-M. Grad [20], having an additional one which involves the vector that acts as an objective function and the corresponding dual problem of the scalarized primal. Moreover, the image sets of the vector duals introduced here are larger than the ones of their counterparts of R.I. Boţ and S.-M. Grad [20], as one can see below.

Theorem 2.1.4 (*S.-M. Grad and E.-L. Pop [66]*) *One has* $h_W^G(\mathcal{B}_W^G) \subseteq h_G^W(\mathcal{B}_G^W)$ *and* $h_M^G(\mathcal{B}_M^G) \subseteq h_G^M(\mathcal{B}_G^M)$.

Remark 2.1.5 (*S.-M. Grad and E.-L. Pop [66]*) *The inclusions proven in Theorem 2.1.4 are in general strict, as the situation depicted in Example 2.1.15 shows.*

For the newly introduced dual problems there is weak duality.

Theorem 2.1.6 (*S.-M. Grad and E.-L. Pop [66]*) *There are no $x \in X$ and $(v^*, y^*, v, u, y) \in \mathcal{B}_G^W$ such that $F(x) \leq_K h_G^W(v^*, y^*, v, u, y)$.*

Theorem 2.1.7 (*S.-M. Grad and E.-L. Pop [66]*) *There are no $x \in X$ and $(v^*, y^*, v, u) \in \mathcal{B}_G^M$ such that $F(x) \leq_K h_G^M(v^*, y^*, v, u)$.*

In order to give strong duality statements concerning the vector optimization problem (PVG) and its two newly introduced vector dual problems, we consider the following regularity conditions (cf. R.I. Boţ [14], R.I. Boţ and S.-M. Grad [20], R.I. Boţ, S.-M. Grad and G. Wanka [25]). We begin with a classical one involving continuity

$$(RCV^1) \quad \Big| \quad \exists x' \in X \text{ such that } (x', 0) \in \operatorname{dom}\Phi \text{ and } \Phi(x', \cdot) \text{ is continuous at } 0;$$

followed by one that works for X and Y Frechét spaces

$$(RCV^2) \quad \left| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } \Phi \text{ is } K\text{-lower semicontinuous and} \\ 0 \in \operatorname{sqri}(\Pr_Y(\operatorname{dom}\Phi)); \end{array} \right.$$

then in finite dimensional case

$$(RCV^3) \quad \Big| \quad \dim(\operatorname{lin}(\Pr_Y(\operatorname{dom}\Phi))) < +\infty \text{ and } 0 \in \operatorname{ri}(\Pr_Y(\operatorname{dom}\Phi));$$

and the closedness type regularity condition

$$(RCV^4) \quad \left| \begin{array}{l} \Phi \text{ is } K\text{-lower semicontinuous and } \Pr_{X^* \times \mathbb{R}}(\operatorname{epi}(v^*\Phi)^*) \text{ is closed in} \\ \text{the topology } w(X^*, X) \times \mathbb{R}, \text{ for all } v^* \in K^{*0}. \end{array} \right.$$

Theorem 2.1.8 (*S.-M. Grad and E.-L. Pop [66]*) *Assume that one of the regularity conditions (RCV^i), $i \in \{1, 2, 3, 4\}$, is fulfilled. If $\overline{x} \in X$ is a properly efficient solution to (PVG), then there are the efficient solutions $(\overline{v}^*, \overline{y}^*, \overline{v}^*, \overline{u}, \overline{y})$ to (DVG^W) and $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$ to (DVG^M) such that $F(\overline{x}) = h_G^W(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}, \overline{y}) = h_G^M(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$.*

Remark 2.1.9 (*S.-M. Grad and E.-L. Pop [66]*) *For the strong duality statement we can also use the regularity condition mentioned by R.I. Boţ, S.-M. Grad and G. Wanka in [25, Remark 4.3.2]: for all $v^* \in K^{*0}$ the problem $\inf_{x \in X} \langle v^*, F(x) \rangle$ is normal.*

Remark 2.1.10 (*S.-M. Grad and E.-L. Pop [66]*) *In case $V = \mathbb{R}$ and $K = \mathbb{R}_+$, identifying $V^\bullet$ with $\mathbb{R} \cup \{+\infty\}$ and $\infty_{\mathbb{R}_+}$ with $+\infty$, for the proper and convex function $F : X \to \mathbb{R} \cup \{+\infty\}$, we rediscover the Wolfe and Mond-Weir type scalar duality scheme from R.I. Boţ, S.-M. Grad and G. Wanka [25], as the problem (PVG) becomes the general scalar optimization problem (PG) and the vector duals (DVG^W) and (DVG^M) turn out to coincide with the scalar Wolfe and Mond-Weir type duals to (PG) introduced in that paper, i.e. (DG_W) and (DG_M), respectively.*

2.1.2 Duality results for particular classes of problems

To some special instances of the vector optimization problem (PVG), constrained and unconstrained vector optimization problems, we attach vector duals which are special cases of the vector duals (DVG^W) and (DVG^M), obtained by using different perturbation vector functions.

Constrained vector optimization problems

We use the same framework as in Section 2.1, with Y partially ordered by the nonempty convex cone $C \subseteq Y$, and we consider the nonempty convex set $S \subseteq X$, the proper $K-$convex function $f : X \to V^\bullet$ and the proper $C-$convex function $g : X \to Y^\bullet$ fulfilling $\operatorname{dom} f \cap S \cap g^{-1}(C) \neq \emptyset$. The primal vector optimization problem with geometric and cone constraints we work with is

$$(PV_C) \qquad \underset{x \in \mathcal{A}}{\operatorname{Min}} f(x),$$

where

$$\mathcal{A} = \big\{x \in S : g(x) \in -C\big\}.$$

To it we attach different pairs of vector duals obtained by making use of some vector perturbation functions. Consider the Lagrange vector type perturbation function $\Phi^V_{C_L} : X \times Y \to V^\bullet$ given by

$$\Phi^V_{C_L}(x, y) = \begin{cases} f(x), & x \in S, g(x) \in y - C, \\ \infty_K, & \text{otherwise.} \end{cases}$$

Thus we obtain the *Lagrange vector dual of Wolfe type*

$$(DV^W_{C_L}) \qquad \underset{(v^*,y^*,v,u) \in \mathcal{B}^W_{C_L}}{\operatorname{Max}} h^W_{C_L}(v^*, y^*, v, u)$$

where

$$\begin{aligned} \mathcal{B}^W_{C_L} = \{(v^*, y^*, v, u) \in K^{*0} \times C^* \times V \times S : \langle v^*, v - f(u)\rangle \leq -(y^*g)(u), \\ 0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\} \end{aligned}$$

and

$$h^W_{C_L}(v^*, y^*, v, u) = v$$

and the *Lagrange vector dual of Mond-Weir type*

$$(DV^M_{C_L}) \qquad \underset{(v^*,y^*,v,u) \in \mathcal{B}^M_{C_L}}{\operatorname{Max}} h^M_{C_L}(v^*, y^*, v, u)$$

where

$$\begin{aligned} \mathcal{B}^M_{C_L} = \{(v^*, y^*, v, u) \in K^{*0} \times C^* \times V \times S : (y^*g)(u) \geq 0, g(u) \in -C, \\ \langle v^*, v\rangle \leq (v^*f)(u), 0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\} \end{aligned}$$

and

$$h^M_{C_L}(v^*, y^*, v, u) = v.$$

Note that in the constraints of this dual one can replace $(y^*g)(u) \geq 0$ by $(y^*g)(u) = 0$ without altering anything since $g(u) \in -C$ and $y^* \in C^*$. Like in R.I. Boţ and S.-M. Grad [20,21], R.I. Boţ, S.-M. Grad and G. Wanka [25], from the Lagrange vector dual of Mond-Weir type we remove the constraint $g(u) \in -C$, obtaining a new vector dual to (PV_C)

$$(DV_{C_L}^{MW}) \qquad \operatorname*{Max}_{(v^*,y^*,v,u)\in \mathcal{B}_{C_L}^{MW}} h_{C_L}^{MW}(v^*,y^*,v,u)$$

where

$$\begin{aligned}\mathcal{B}_{C_L}^{MW} = \{&(v^*,y^*,v,u) \in K^{*0} \times C^* \times V \times S : (y^*g)(u) \geq 0, \langle v^*, v\rangle \leq (v^*f)(u),\\ &0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\}\end{aligned}$$

and

$$h_{C_L}^{MW}(v^*,y^*,v,u) = v.$$

Remark 2.1.11 (*S.-M. Grad and E.-L. Pop [66]*) *Assume that f is a $K-$convex vector function and g is a $C-$convex vector function. Using that S is a convex set one can verify that the vector perturbation function $\Phi_V^{C_L}$ is $K-$convex. We denote $\Delta_{X^3} = \{(x,x,x) : x \in X\}$. If one of the following conditions (see R.I. Boţ, S.-M. Grad and G. Wanka [25])*

(i) f and g are continuous at a point in $\operatorname{dom} f \cap \operatorname{dom} g \cap S$;

(ii) $\operatorname{dom} f \cap \operatorname{int}(S) \cap \operatorname{dom} g \neq \emptyset$ and f or g is continuous at a point in $\operatorname{dom} f \cap \operatorname{dom} g$;

(iii) X is a Fréchet space, S is closed, f is $K-$lower semicontinuous, g is $C-$lower semicontinuous and $0 \in \operatorname{sqri}(\operatorname{dom} f \times S \times \operatorname{dom} g - \Delta_{X^3})$;

(iv) $\dim(\operatorname{lin}(\operatorname{dom} f \times S \times \operatorname{dom} g - \Delta_{X^3})) < +\infty$ and $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(S) \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$; is satisfied, then, for all $v^* \in K^{*0}$ and all $y^* \in C^*$, it holds

$$\partial((v^*f) + (y^*g) + \delta_S)(x) = \partial(v^*f)(x) + \partial(y^*g)(x) + N_S(x) \ \forall x \in X.$$

Consequently, when one of these situations occurs the constraint involving the subdifferential in $(DV_{C_L}^W)$, $(DV_{C_L}^M)$ and $(DV_{C_L}^{MW})$ can be modified correspondingly.

Remark 2.1.12 (*S.-M. Grad and E.-L. Pop [66]*) *A vector dual similar to $(DV_{C_L}^W)$, but with respect to weakly efficient solutions, was introduced in T.Q. Chien [37], under quasidifferentiability hypotheses for the functions involved. Later, it was mentioned also in T. Weir, B. Mond and B.D. Craven [145], where the functions were taken differentiable.*

Let us investigate now the image sets of these vector duals.

Proposition 2.1.13 (*S.-M. Grad and E.-L. Pop [66]*) *One has $h_{C_L}^M(\mathcal{B}_{C_L}^M) \subseteq h_{C_L}^{MW}(\mathcal{B}_{C_L}^{MW})$ and $h_{C_L}^M(\mathcal{B}_{C_L}^M) \subseteq h_{C_L}^W(\mathcal{B}_{C_L}^W)$.*

Remark 2.1.14 (*S.-M. Grad and E.-L. Pop [66]*) *The inclusions in Proposition 2.1.13 are in general strict, as the following example shows.*

Example 2.1.15 (*S.-M. Grad and E.-L. Pop [66]*) *Let* $X = \mathbb{R}$, $Y = \mathbb{R}$, $V = \mathbb{R}^2$, $C = \mathbb{R}_+$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, x)^T$ *and* $g : X \to \mathbb{R} \cup \{+\infty\}$,

$$g(x) = \begin{cases} x, & \text{if } x > 0 \\ 1, & \text{if } x = 0 \\ +\infty, & \text{if } x < 0. \end{cases}$$

Then $g(x) \neq 0$ *for all* $x \in \mathbb{R}$ *and to obtain* $(y^*g)(u) = 0$ *for some feasible* $u \geq 0$ *it is binding to have* $y^* = 0$. *Since when* $u > 0$ *and* $v^* = (v_1^*, v_2^*)^T$ *the subdifferential of the function* $(v^*f + 0g + \delta_S)(\cdot) = (v_1^* + v_2^*)(\cdot) + \delta_{\mathbb{R}_+}(\cdot)$ *is the set* $\{v_1^* + v_2^*\}$, *the only eligible element for* $\mathcal{B}^M_{C_L}$ *would be* $u = 0$, *as* $g(u) = +\infty$ *when* $u < 0$. *But* $g(0) = 1 \notin -C$, *thus* $\mathcal{B}^M_{C_L} = \emptyset$. *Moreover, considering the Lagrange type vector dual to* (PV_C) *obtained from* (DV^G_W) *(cf. R.I. Boţ and S.-M. Grad [20]), we see that its objective function takes as values only vectors with equal entries.*

On the other hand, for $v^* = (1/2, 1/2)^T$ *we have* $0 \in \partial((v^*f) + (y^*g) + \delta_S)(0) = (-\infty, 1]$, $(y^*g)(u) = 0$ *and for* $v = (0, -1)$ *we obtain that* $\langle v^*, v\rangle - (v^*f)(u) = -1/2 < 0$. *Thus* $((1/2, 1/2)^T, 0, (0, -1), 0) \in \mathcal{B}^{MW}_{C_L}$ *and* $((1/2, 1/2)^T,\ 0, (0, -1), 0) \in \mathcal{B}^W_{C_L}$. *Therefore* $(0, -1) \in h^{MW}_{C_L}(\mathcal{B}^{MW}_{C_L}) \cap h^W_{C_L}(\mathcal{B}^W_{C_L})$.

Consequently, $h^W_{C_L}(\mathcal{B}^W_{C_L}) \neq h^M_{C_L}(\mathcal{B}^M_{C_L})$ *and* $h^{MW}_{C_L}(\mathcal{B}^{MW}_{C_L}) \neq h^M_{C_L}(\mathcal{B}^M_{C_L})$ *and, in general,* $h^M_G(\mathcal{B}^M_G) \neq h^W_G(\mathcal{B}^W_G)$ *and* $h^W_G(\mathcal{B}^W_G) \neq h^G_W(\mathcal{B}^G_W)$.

Concerning possible inclusion relations that could exist between the Lagrange vector dual of Wolfe type and the Lagrange vector dual of "M-W" type, we know only that the image of the first one is not a subset of the one of the second dual.

In order to achieve strong duality for the vector duals of Lagrange type we attached to (PV_C), we need the fulfillment of some sufficient conditions. Particularizing (RCV^i), $i \in \{1, 2, 3, 4\}$ one get $(RCV^i_{C_L})$, $i \in \{1, 2, 3, 4\}$, where for example $(RCV^1_{C_L})$ is

$$(RCV^1_{C_L}) \quad \Big| \quad \exists x' \in \operatorname{dom} f \cap S \text{ such that } g(x') \in -\operatorname{int}(C).$$

Particularizing the results from the general case, we obtain the following duality statements.

Theorem 2.1.16 (*S.-M. Grad and E.-L. Pop [66]*) (*weak and strong duality for* (PV_C) *and* $(DV^W_{C_L})$)

(*a*) *There are no* $x \in \mathcal{A}$ *and* $(v^*, y^*, v, u) \in \mathcal{B}^W_{C_L}$ *such that* $f(x) \leq_K h^W_{C_L}(v^*, y^*, v, u)$.

(*b*) *If* $\overline{x} \in \mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV^i_{C_L})$, $i \in \{1, 2, 3, 4\}$ *is fulfilled, then there exists* $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}) \in \mathcal{B}^W_{C_L}$ *efficient solution to* $(DV^W_{C_L})$ *such that* $f(\overline{x}) = h^W_{C_L}(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$.

Theorem 2.1.17 (*S.-M. Grad and E.-L. Pop [66]*) (*weak and strong duality for* (PV_C) *and* $(DV^M_{C_L})$)

(*a*) *There are no* $x \in \mathcal{A}$ *and* $(v^*, y^*, v, u) \in \mathcal{B}^M_{C_L}$ *such that* $f(x) \leq_K h^M_{C_L}(v^*, y^*, v, u)$.

(b) If $\overline{x} \in \mathcal{A}$ is a properly efficient solution to (PV_C) and one of the regularity conditions $(RCV^i_{C_L})$, $i \in \{1, 2, 3, 4\}$, is fulfilled, then there exists $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}) \in \mathcal{B}^M_{C_L}$ efficient solution to $(DV^M_{C_L})$ such that $f(\overline{x}) = h^M_{C_L}(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$.

Analogously one can prove similar duality assertions for (PV_C) and $(DV^{MW}_{C_L})$, too.

Remark 2.1.19 (*S.-M. Grad and E.-L. Pop [66]*) *The regularity condition in Theorems 2.1.16 (b) and 2.1.17 (b) can be replaced by any condition which guarantees the stability of the optimization problem* $\inf_{x\in\mathcal{A}}(\overline{v}^* f)(x)$ *with respect to its Lagrange dual.*

Remark 2.1.20 (*S.-M. Grad and E.-L. Pop [66]*) *If $V = \mathbb{R}$ and $K = \mathbb{R}_+$, then the duals $(DV^W_{C_L})$, $(DV^M_{C_L})$ and $(DV^{MW}_{C_L})$ are nothing else than the scalar Lagrange dual problems of Wolfe and Mond-Weir type corresponding to (PV_C), considered by R.I. Boţ and S.-M. Grad in [21].*

Another vector perturbation function we consider is the Fenchel-Lagrange type vector perturbation function $\Phi^V_{FL} : X \times X \times Y \to V^\bullet$ given by

$$\Phi^V_{C_{FL}}(x, t, y) = \begin{cases} f(x + t), & x \in S, g(x) \in y - C \\ \infty_K, & \text{otherwise.} \end{cases}$$

The *Fenchel-Lagrange vector dual of Wolfe type* to (PV_C) is

$$(DV^W_{C_{FL}}) \qquad \underset{(v^*,t^*,y^*,v,u,t)\in\mathcal{B}^W_{C_{FL}}}{\operatorname{Max}} h^W_{C_{FL}}(v^*, t^*, y^*, v, u, t)$$

where

$$\begin{aligned} \mathcal{B}^W_{C_{FL}} = \{&(v^*, t^*, y^*, v, u, t) \in K^{*0} \times X^* \times C^* \times V \times S \times X : \langle v^*, v\rangle \leq \langle t^*, u\rangle \\ &-(v^* f)^*(t^*) - (y^* g)(u), 0 \in \partial((v^* f)(u + t) \cap (-\partial((y^* g) + \delta_S)(u))\} \end{aligned}$$

and

$$h^W_{C_{FL}}(v^*, t^*, y^*, v, u, t) = v$$

and *the Fenchel-Lagrange vector dual of Mond-Weir type* is

$$(DV^M_{C_{FL}}) \qquad \underset{(v^*,y^*,v,u)\in\mathcal{B}^M_{C_{FL}}}{\operatorname{Max}} h^M_{C_{FL}}(v^*, y^*, v, u)$$

where

$$\begin{aligned} \mathcal{B}^M_{C_{FL}} = \{&(v^*, y^*, v, u) \in K^{*0} \times C^* \times V \times S : (y^* g)(u) \geq 0, g(u) \in -C, \\ &\langle v^*, v\rangle \leq (v^* f)(u), 0 \in \partial(v^* f)(u) + \partial((y^* g) + \delta_S)(u)\} \end{aligned}$$

and

$$h^M_{C_{FL}}(v^*, y^*, v, u) = v.$$

Note that in its constraints one can replace $(y^* g)(u) \geq 0$ by $(y^* g)(u) = 0$ without altering anything since $g(u) \in -C$ and $y^* \in C^*$. Like in the other case, removing the constraint $g(u) \in -C$, one obtains another vector dual to (PV_C)

$$(DV_{C_{FL}}^{MW}) \qquad \underset{(v^*,y^*,v,u)\in\mathcal{B}_{C_{FL}}^{MW}}{\text{Max}} h_{C_{FL}}^{MW}(v^*,y^*,v,u)$$

where

$$\mathcal{B}_{C_{FL}}^{MW} = \begin{array}{r}\{(v^*,y^*,v,u) \in K^{*0} \times C^* \times V \times S : (y^*g)(u) \geq 0,\\ \langle v^*, v\rangle \leq (v^*f)(u), 0 \in \partial(v^*f)(u) + \partial((y^*g)+\delta_S)(u)\}\end{array}$$

and

$$h_{C_{FL}}^{MW}(v^*,y^*,v,u) = v.$$

Remark 2.1.21 *Like in Remark 2.1.11 we can formulate some conditions for separating the functions that appear together in the subdifferentials from the constraint sets of the Fenchel-Lagrange vector duals to* (PV_C) *(see R.I. Boţ, S.-M. Grad and G. Wanka [25, Section 3.5]).*

Using the way $(DV_{C_{FL}}^{M})$ is constructed and applying Lemma 2.1.2, one gets the following inclusions.

Proposition 2.1.22 (*S.-M. Grad and E.-L. Pop [66]*) *One has* $h_{C_{FL}}^{M}(\mathcal{B}_{C_{FL}}^{M}) \subseteq h_{C_{FL}}^{MW}(\mathcal{B}_{C_{FL}}^{MW})$ *and* $h_{C_{FL}}^{M}(\mathcal{B}_{C_{FL}}^{M}) \subseteq h_{C_{FL}}^{W}(\mathcal{B}_{C_{FL}}^{W})$.

The question if similar inclusions are valid also for the Lagrange vector dual of Wolfe type to (PV_C) has a negative answer, as the following examples show.

Example 2.1.23 (*S.-M. Grad and E.-L. Pop [66]*) *Let* $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $V = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, $f : \mathbb{R} \to (\mathbb{R}^2)^\bullet$,

$$f(x) = \begin{cases} (1,1)^T x, & \text{if } x \leq 0,\\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,}\end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}^2$, $g(x) = (x, 1-x)^T$.

Like in R.I. Boţ and S.-M. Grad [21, Example 2] one can show that $\mathcal{B}_{C_{FL}}^{MW} = \emptyset$ *and on the other hand that* $((1/2,1/2)^T, 0, (0,0), (0,0)^T, 0, 0) \in \mathcal{B}_{C_{FL}}^{W}$ *and* $(0,0)^T \in h_{C_{FL}}^{W}(\mathcal{B}_{C_{FL}}^{W})$. *Consequently,* $h_{C_{FL}}^{W}(\mathcal{B}_{C_{FL}}^{W}) \nsubseteq h_{C_{FL}}^{MW}(\mathcal{B}_{C_{FL}}^{MW})$.

In order to guarantee strong duality, one can particularize the regularity conditions (RCV^i), $i \in \{1,2,3,4\}$. They become $(RCV_{C_{FL}}^i)$, $i \in \{1,2,3,4\}$, where for instance $(RCV_{C_{FL}}^1)$ is

$$(RCV_{C_{FL}}^1) \quad \Big| \quad \exists x' \in \operatorname{dom} f \cap S \text{ such that } f \text{ is continuous at } x' \text{ and } g(x') \in -\operatorname{int}(C)$$

and the others can be analogously obtained (see R.I. Boţ and S.-M. Grad [21]).

From the general case we obtain the following weak and strong duality statements.

Theorem 2.1.24 (*S.-M. Grad and E.-L. Pop [66]*) (*weak and strong duality for* (PV_C) *and* $(DV^W_{C_{FL}})$)

(*a*) *There are no* $x \in \mathcal{A}$ *and* $(v^*, t^*, y^*, v, u, t) \in \mathcal{B}^W_{C_{FL}}$ *such that* $f(x) \leq_K h^W_{C_{FL}}(v^*, t^*, y^*, v, u, t)$.

(*b*) *If* $\overline{x} \in \mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV^i_{C_{FL}})$, $i \in \{1, 2, 3, 4\}$, *is fulfilled, then there exists* $(\overline{v}^*, \overline{t}^*, \overline{y}^*, \overline{v}, \overline{u}, \overline{t}) \in \mathcal{B}^W_{C_{FL}}$ *efficient solution to* $(DV^W_{C_{FL}})$ *such that* $f(\overline{x}) = h^W_{C_{FL}}(\overline{v}^*, \overline{t}^*, \overline{y}^*, \overline{v}, \overline{u}, \overline{t})$.

Theorem 2.1.25 (*S.-M. Grad and E.-L. Pop [66]*) (*weak and strong duality for* (PV_C) *and* $(DV^M_{C_{FL}})$)

(*a*) *There are no* $x \in \mathcal{A}$ *and* $(v^*, y^*, v, u) \in \mathcal{B}^M_{C_{FL}}$ *such that* $f(x) \leq_K h^M_{C_{FL}}(v^*, y^*, v, u)$.

(*b*) *If* $\overline{x} \in \mathcal{A}$ *is a properly efficient solution to* (PV_C) *and one of the regularity conditions* $(RCV^i_{C_{FL}})$, $i \in \{1, 2, 3, 4\}$, *is fulfilled, then there exists* $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}) \in \mathcal{B}^M_{C_{FL}}$ *efficient solution to* $(DV^M_{C_{FL}})$ *such that* $f(\overline{x}) = h^M_{C_{FL}}(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u})$.

Analogously one can prove the following duality statements for (PV_C) and $(DV^{MW}_{C_{FL}})$.

Remark 2.1.27 (*S.-M. Grad and E.-L. Pop [66]*) *The regularity condition in Theorems 2.1.24 (b) and 2.1.25 (b) can be replaced by any condition which guarantees the stability of the optimization problem* $\inf_{x \in \mathcal{A}} (\overline{v}^* f)(x)$ *with respect to its Fenchel-Lagrange dual.*

Remark 2.1.28 (*S.-M. Grad and E.-L. Pop [66]*) *If* $V = \mathbb{R}$ *and* $K = \mathbb{R}_+$, *then the duals* $(DV^W_{C_{FL}})$, $(DV^M_{C_{FL}})$ *and* $(DV^{MW}_{C_{FL}})$ *are nothing else than the scalar Fenchel-Lagrange dual problems of Wolfe and Mond-Weir type corresponding to* (PV_C) *considered by R.I. Boţ and S.-M. Grad in [21], respectively.*

Unconstrained vector optimization problems

Using the same framework as in Section 2.1, we consider the proper $K-$convex vector functions $f : X \to V^\bullet$ and $h : Y \to V^\bullet$ and $A : X \to Y$ a linear continuous mapping such that $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) \neq \emptyset$. The primal unconstrained vector optimization problem

$$(PV_A) \qquad \operatorname*{Min}_{x \in X} [f(x) + h(Ax)]$$

is a special case of (PVG) where $F = f + h \circ A$ and we consider the vector perturbation function $\Phi^V_A : X \times Y \to V^\bullet$ defined by

$$\Phi^V_A(x, y) = f(x) + h(Ax + y).$$

The vector duals to (PV_A) are

$$(DV^W_A) \qquad \operatorname*{Max}_{(v^*, y^*, v, u, y) \in \mathcal{B}^W_A} h^W_A(v^*, y^*, v, u, y)$$

where

$$\begin{array}{rcl}\mathcal{B}_A^W & = & \{(v^*, y^*, v, u, y) \in K^{*0} \times Y^* \times V \times X \times Y : y^* \in (A^*)^{-1}(-\partial(v^* f)(u)) \\ & & \cap \partial(v^* h)^*(Au + y) \text{ and } \langle v^*, v\rangle \leq -(v^* f)^*(-A^* y^*) + (v^* h)^*(y^*)\}\end{array}$$

and

$$h_A^W(v^*, y^*, v, u, y) = v$$

and, respectively,

$$(DV_A^M) \qquad \operatorname*{Max}_{(v^*, v, u) \in \mathcal{B}_A^M} h_A^M(v^*, v, u)$$

where

$$\begin{array}{rcl}\mathcal{B}_A^M & = & \{(v^*, v, u) \in K^{*0} \times V \times X : 0 \in (A^*)^{-1}(-\partial(v^* f)(u)) - \partial(v^* h)(Au) \\ & & \text{and } \langle v^*, v\rangle \leq \langle v^*, f(u) + h(Au)\rangle\}\end{array}$$

and

$$h_A^M(v^*, v, u) = v.$$

For the primal vector problem (PV_A) and Wolfe type and Mond-Weir type vector duals to (PV_A), (DV_A^W) and (DV_A^M), respectively, the weak and strong duality statements follow from the general case.

Theorem 2.1.29 (*S.-M. Grad and E.-L. Pop [66]*) (*weak duality for* (PV_A) *and* (DV_A^W), (PV_A) *and* (DV_A^M))

(*a*) *There are no* $x \in X$ *and* $(v^*, y^*, v, u, y) \in \mathcal{B}_A^W$ *such that* $f(x) + h(Ax) \leq_K h_A^W(v^*, y^*, v, u, y)$.

(*b*) *There are no* $x \in X$ *and* $(v^*, v, u) \in \mathcal{B}_A^M$ *such that* $f(x)+h(Ax) \leq_K h_A^M(v^*, v, u)$.

In formulation of strong duality are need it convexity assumptions which guarantee the $K-$convexity of the vector perturbation function and the regularity conditions obtained by particularizing the classical ones from R.I. Boţ, S.-M. Grad and G. Wanka [25], namely (RC_i^A), $i \in \{1, 2, 3, 4\}$, where for example (RC_1^A) is

$$(RC_1^A) \quad \Big| \quad \exists x' \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) \text{ such that } h \text{ is continuous at } Ax'.$$

Theorem 2.1.30 (*S.-M. Grad and E.-L. Pop [66]*) (*strong duality for* (PV_A) *and* (DV_A^W), *respectively* (PV_A) *and* (DV_A^M))

Assume that f *and* h *are* $K-$*convex vector functions and one of the regularity conditions* $(RC_i^A), i \in \{1, 2, 3, 4\}$, *is fulfilled. If* $\overline{u}$ *is a proper efficient solution to* (PV_A), *then there exist* $\overline{v}^* \in K^{*0}$, $\overline{y}^* \in Y^*$ *and* $\overline{v} \in V$ *such that* $(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}, 0)$ *is an efficient solution to* (DV_A^W), $(\overline{v}^*, \overline{v}, \overline{u})$ *is an efficient solution to* (DV_A^M) *and* $f(\overline{u}) + h(A\overline{u}) = h_A^W(\overline{v}^*, \overline{y}^*, \overline{v}, \overline{u}, 0) = h_A^M(\overline{v}^*, \overline{v}, \overline{u})$.

Remark 2.1.31 (*S.-M. Grad and E.-L. Pop [66]*) *In case* $V = \mathbb{R}$ *and* $K = \mathbb{R}_+$, *taking the functions* $f : X \to \overline{\mathbb{R}}$ *and* $h : Y \to \overline{\mathbb{R}}$ *proper we rediscover the Wolfe and*

Mond-Weir duality schemes for unconstrained scalar optimization problems from R.I. Boţ and S.-M. Grad [21].

Back to (PV_C), seeing it as an unconstrained vector optimization problem, we can attach to it two vector dual problems generated by (DVG^W) and (DVG^M) by considering the Fenchel type vector perturbation function $\Phi^V_{C_F} : X \times Y \to V^\bullet$ given by

$$\Phi^V_{C_F}(x, y) = \begin{cases} f(x + y), & x \in \mathcal{A}, \\ \infty_K, & \text{otherwise.} \end{cases}$$

The first dual obtained is the *Fenchel vector dual of Wolfe type*

$$(DV^W_{C_F}) \qquad \underset{(v^*,y^*,v,u,y)\in\mathcal{B}^W_{C_F}}{\mathrm{Max}} h^W_{C_F}(v^*, y^*, v, u, y)$$

where

$$\begin{aligned} \mathcal{B}^W_{C_F} = \{&(v^*, y^*, v, u, y) \in K^{*0} \times Y^* \times V \times X \times X : \langle v^*, v\rangle \le \langle y^*, u\rangle - \\ &(v^* f)^*(y^*), y^* \in \partial(v^* f)(u + y) \cap (-N_{\mathcal{A}}(u))\} \end{aligned}$$

and

$$h^W_{C_F}(v^*, y^*, v, u, y) = v;$$

and the second dual obtained is the *Fenchel vector dual of Mond-Weir type* given by

$$(DV^M_{C_F}) \qquad \underset{(v^*,v,u)\in\mathcal{B}^M_{C_F}}{\mathrm{Max}} h^M_{C_F}(v^*, v, u)$$

where

$$\mathcal{B}^M_{C_F} = \{(v^*, v, u) \in K^{*0} \times V \times X : \langle v^*, v\rangle \le (v^* f)(u), 0 \in \partial(v^* f)(u) + N_{\mathcal{A}}(u)\}$$

and

$$h^M_{C_F}(v^*, v, u) = v.$$

Remark 2.1.32 (*S.-M. Grad and E.-L. Pop [66]*) *In the definition of Fenchel vector dual of Mond-Weir type $(DV^M_{C_F})$, the condition $g(u) \in -C$ does not appear explicitly. Thus we will not consider another vector dual of "M-W" type vector dual problem to (PV_C) in this case.*

From Lemma 2.1.2 one can derive the following statement.

Proposition 2.1.33 (*S.-M. Grad and E.-L. Pop [66]*) *One has $h^M_{C_F}(\mathcal{B}^M_{C_F}) \subseteq h^W_{C_F}(\mathcal{B}^W_{C_F})$.*

The regularity conditions (RCV^i), $i = 1, ..., 4$ can be formulated in this case, too, and from the general case one can quickly obtain the weak and strong duality theorems.

2.1.3 Comparisons between duals

In what follows we compare the image sets of some of the Wolfe type and Mond-Weir type vector duals to (PV_C) with respect to the Lagrange, Fenchel and Fenchel-Lagrange type vector perturbation functions.

Theorem 2.1.37 (*S.-M. Grad and E.-L. Pop [66]*) *One has* $h^M_{C_{FL}}(\mathcal{B}^M_{C_{FL}}) \subseteq h^M_{C_L}(\mathcal{B}^M_{C_L})$ *and* $h^M_{C_{FL}}(\mathcal{B}^M_{C_{FL}}) \subseteq h^M_{C_F}(\mathcal{B}^M_{C_F})$.

Concerning the "M-W" vector duals, one can easily prove the following statement.

Theorem 2.1.38 (*S.-M. Grad and E.-L. Pop [66]*) *One has* $h^{MW}_{C_{FL}}(\mathcal{B}^{MW}_{C_{FL}}) \subseteq h^{MW}_{C_L}(\mathcal{B}^{MW}_{C_L})$.

However, the question if similar inclusions are valid also for the Wolfe type vector duals to (PV_C) has, like in the scalar case (see R.I. Boţ and S.-M. Grad [21]), a negative answer, as the following examples show.

Example 2.1.39 (*S.-M. Grad and E.-L. Pop [66]*) *Let* $X = \mathbb{R}$, $Y = \mathbb{R}$, $V = \mathbb{R}^2$, $C = \mathbb{R}_+$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}$, $f : \mathbb{R} \to (\mathbb{R}^2)^\bullet$ *and* $g : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} (1,1)^T x, & \text{if } x > 0, \\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} -x, & \text{if } x \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for all $v^* = (v_1^*, v_2^*)^T \in \operatorname{int}(\mathbb{R}^2_+)$ *and* $y^* \geq 0$ *one has*

$$\partial((v^* f) + (y^* g) + \delta_S)(u) = \partial(v^* f)(u) = \begin{cases} \{v_1^* + v_2^*\}, & \text{if } u > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Consequently, $\mathcal{B}^W_{C_L} = \emptyset$. *On the other hand it can be shown that* $((1/2, 1/2)^T, 1, 1, (0,0)^T, 0,$ $\mathcal{B}^W_{C_{FL}}$, *thus* $(0,0)^T \in h^W_{C_{FL}}(\mathcal{B}^W_{C_{FL}})$. *Consequently,* $h^W_{C_{FL}}(\mathcal{B}^W_{C_{FL}}) \not\subseteq h^W_{C_L}(\mathcal{B}^W_{C_L})$.

Example 2.1.40 (*S.-M. Grad and E.-L. Pop [66]*) *Let* $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $V = \mathbb{R}^2$, $C = \mathbb{R}_+$, $K = \mathbb{R}^2_+$,

$$S = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \begin{array}{ll} 3 \leq x_2 \leq 4, & \text{if } x_1 = 0 \\ 1 \leq x_2 \leq 4, & \text{if } x_1 \in (0, 2] \end{array} \right\},$$

$$f : \mathbb{R}^2 \to (\mathbb{R}^2)^\bullet, \ f(x_1, x_2) = \begin{cases} (1,1)^T x_2, & \text{if } x_1 \leq 0, \\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,} \end{cases}$$

and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, x_2) = 0$.

We have that $((1/2, 1/2)^T, y^*, (3,3)^T, (0,3)) \in \mathcal{B}^W_{C_L}$ *and* $(3,3)^T \in h^W_{C_L}(\mathcal{B}^W_{C_L})$, *but* $(3,3)^T \notin h^W_{C_{FL}}(\mathcal{B}^W_{C_{FL}})$. *Consequently,* $h^W_{C_L}(\mathcal{B}^W_{C_L}) \not\subseteq h^W_{C_{FL}}(\mathcal{B}^W_{C_{FL}})$.

Remark 2.1.43 (*S.-M. Grad and E.-L. Pop [66]*) *From the examples given above we can construct situations which demonstrate that in general no inclusion holds between the image sets of* $(DV^W_{C_L})$ *and* $(DV^W_{C_F})$.

2.2 Wolfe type and Mond-Weir type vector duality with respect to weakly efficient solutions

2.2.1 General duality results

Here we introduce new Wolfe type and Mond-Weir type vector duals and establish duality results between the general vector optimization problem with respect to weakly efficient solutions and these new duals.

2.2.2 Duality results for particular classes of problems

We particularize the original vector optimization problem with respect to weakly efficient solutions to be constrained and then unconstrained and we construct new Wolfe type and Mond-Weir type vector duals and formulate duality results.

Chapter 3

Vector duality with respect to quasi-minimality

3.1 Convex vector optimization problems with respect to quasi-minimality

We define and characterize the quasi-minimal elements of a set with respect to a convex cone. Then we attach to a general vector optimization problem a dual vector optimization problem with respect to quasi-efficient solutions and establish new duality results. By considering particular cases of the primal vector optimization problem we derive vector dual problems attached to it with respect to quasi-efficient solutions and we obtain weak, strong and converse duality statements.

Some of the results were obtained by the author in joint work with dr. S.-M. Grad [68].

Some preliminaries and notions related with the quasi interior of a cone follows (see, for example [12, 16–18, 25]). Let X be a separated locally convex space.

Remark 3.1.2 *Let $K \subseteq X$ be a convex cone.*

(a) *If K is also pointed, then $0 \notin \operatorname{qi} K$.*

(b) *One has $\operatorname{qi} K + K = \operatorname{qi} K$.*

(c) *The set $\operatorname{qi} K \cup \{0\}$ is a cone, too.*

(d) *If K is also closed, then $\operatorname{qi} K^* = \{x^* \in K^* : \langle x^*, x\rangle > 0 \ \forall x \in K \setminus \{0\}\}$, a set usually denoted by K^{*0} and known as the quasi interior of the dual cone of K.*

Let $K \subseteq X$ be convex cone. When $\operatorname{qi} K \neq \emptyset$ we denote $x <_K y$ if $y - x \in \operatorname{qi} K$, extending the notation usually considered in the literature for the case $\operatorname{int} K \neq \emptyset$.

Definition 3.1.3 *Let the space X be partially ordered by the convex cone K, a nonempty set $U \subseteq X$ and $f : X \to \overline{\mathbb{R}}$ a given function. If f is $K-$increasing on U,*

$\operatorname{qi} K \neq \emptyset$ *and for all* $x, y \in U$ *fulfilling* $x <_K y$ *follows* $f(x) < f(y)$ *the function* f *is called strictly* $K-$*increasing on* U.

Remark 3.1.4 *In Definition 3.1.3 we extend the notion of strictly* $K-$*increasing on* U *functions given so far in the literature for the case* $\operatorname{int} K \neq \emptyset$ (*or* $\operatorname{core} K \neq \emptyset$).

Let us illustrate this definition with the following example (see [25]).

Example 3.1.5 *Let* $x^* \in X^*$. *If* $x^* \in K^*$, *then for all* $x_1, x_2 \in X$ *such that* $x_1 \leqq_K x_2$ *we have that* $\langle x^*, x_2 - x_1 \rangle \geq 0$. *Therefore* $\langle x^*, x_1 \rangle \leq \langle x^*, x_2 \rangle$ *and this means that the elements of* K^* *are actually* $K-$*increasing functions on* X.

If $x^* \in K^{*0}$, *then for all* $x_1, x_2 \in X$ *such that* $x_1 \leq_K x_2$ *it holds* $\langle x^*, x_2 - x_1 \rangle > 0$. *This means by definition that the elements of* K^{*0} *are strongly* $K-$*increasing functions on* X.

If $K \subseteq X$ *is a convex closed cone,* $\operatorname{qi} K \neq \emptyset$, *then via Remark 3.1.2* (*d*) $\operatorname{qi} K = \{x \in X : \langle x^*, x \rangle > 0 \ \forall x^* \in K^* \setminus \{0\}\}$ *and thus every* $x^* \in K^* \setminus \{0\}$ *is strictly* $K-$*increasing on* X.

3.1.1 Quasi-minimal elements

We introduce and characterize the quasi-minimal elements of a set.

Let V be a separated locally convex vector space partially ordered by the pointed convex cone $K \subseteq V$ with a nonempty quasi interior, and $U \subseteq V$ a nonempty convex set.

Definition 3.1.6 *An element* $\overline{x} \in U$ *is said to be a quasi-minimal element of* U (*regarding the partial ordering induced by* K) *if* $(\overline{x} - \operatorname{qi} K) \cap U = \emptyset$.

Remark 3.1.7 *Quasi-minimal elements were also considered in works like [65, 71, 128], being usually called quasi-weakly minimal elements. We opted for the simpler name presented in Definition 3.1.6, even if it is used in the literature also for other types of minimal elements* (*see, for instance, [86]*). *However, if the conjecture presented below, namely that* $U + \operatorname{qi} K = \operatorname{qi}(U + K)$ *always holds, turns out to be valid, we believe that the quasi-minimal elements should be actually called weakly minimal. Note also that in [5, 6, 71] one can find quasi-relative minimal elements.*

We denote by $\operatorname{QMin}(U, K)$ the set of all quasi-minimal elements of the set U (regarding the partial ordering induced by K).

Recall that an element $\overline{x} \in U$ is said to be a *minimal element* of U (regarding the partial ordering induced by K) if there is no $x \in U$ satisfying $x \leq_K \overline{x}$.

The relation $(\overline{x} - \operatorname{qi} K) \cap U = \emptyset$ in Definition 3.1.6 can be equivalently rewritten as $(U - \overline{x}) \cap (-\operatorname{qi} K) = \emptyset$. Whenever the cone K is nontrivial we notice that if we consider as ordering cone $\widehat{K} = \operatorname{qi} K \cup \{0\}$, then $\overline{x} \in \operatorname{QMin}(U, K)$ if and only if $(\overline{x} - \widehat{K}) \cap U = \{\overline{x}\}$.

If $K \neq V$, any minimal element of U is also quasi-minimal since $(\overline{x} - K) \cap U = \{\overline{x}\}$ implies via Remark 3.1.2 (*a*) that $(\overline{x} - \operatorname{qi} K) \cap U = \emptyset$. If $K = V$ then $\operatorname{QMin}(U, K) = \emptyset$.

Note that in case $\operatorname{core} K \neq \emptyset$ (or $\operatorname{int} K \neq \emptyset$) the following investigations rediscover results from [25, Section 2.4.2, Section 2.4.4 and Section 4.3.4], thus they can be seen as generalizations of the latter.

Lemma 3.1.8 (*S.-M. Grad and E.-L. Pop [68]*) *It holds* $\operatorname{QMin}(U, K) \subseteq \operatorname{QMin}(U + K, K)$.

Remark 3.1.9 (*S.-M. Grad and E.-L. Pop [68]*) *In Definition 3.1.6 and Lemma 3.1.8 is not necessary to assume that U is convex.*

Proposition 3.1.10 (*S.-M. Grad and E.-L. Pop [68]*) *One has that* $\operatorname{qi}(U + \operatorname{qi} K) = U + \operatorname{qi} K \subseteq \operatorname{qi}(U + K)$.

In what follows we suppose that it holds $U + \operatorname{qi} K = \operatorname{qi}(U + K)$ and we maintain this additional hypothesis for their counterparts in the rest of the section.

Next we formulate some necessary and sufficient characterizations via linear scalarizations of the quasi-minimal elements of the set U with respect to K.

Theorem 3.1.11 (*S.-M. Grad and E.-L. Pop [68]*) *If* $\overline{x} \in \operatorname{QMin}(U, K)$ *then there exists* $x^* \in K^* \setminus \{0\}$ *such that* $\langle x^*, \overline{x}\rangle \leq \langle x^*, x\rangle$, *for all* $x \in U$.

Lemma 3.1.12 (*S.-M. Grad and E.-L. Pop [68]*) *Let a function* $f : V \to \overline{\mathbb{R}}$ *which is strictly $K-$increasing on U. If there is an element* $\overline{x} \in U$ *fulfilling* $f(\overline{x}) \leq f(x)$ *for all* $x \in U$, *then* $\overline{x} \in \operatorname{QMin}(U, K)$.

Further let K be also closed. The following theorem is a straightforward conclusion of Lemma 3.1.12 and Example 3.1.5.

Theorem 3.1.13 (*S.-M. Grad and E.-L. Pop [68]*) *If there exist* $x^* \in K^* \setminus \{0\}$ *and* $\overline{x} \in U$ *such that for all* $x \in U$ *it holds* $\langle x^*, \overline{x}\rangle \leq \langle x^*, x\rangle$, *then* $\overline{x} \in \operatorname{QMin}(U, K)$.

From Theorem 3.1.11 and Theorem 3.1.13 we obtain an equivalent characterization via linear scalarization for the quasi-minimal elements of U with respect to K.

Corollary 3.1.14 (*S.-M. Grad and E.-L. Pop [68]*) *Let* $x \in U$. *Then* $\overline{x} \in \operatorname{QMin}(U, K)$ *if and only if there exists* $x^* \in K^* \setminus \{0\}$ *satisfying* $\langle x^*, \overline{x}\rangle \leq \langle x^*, x\rangle$ *for all* $x \in U$.

3.1.2 General duality results

Here we introduce a vector dual problem with respect to quasi-efficient solutions to a general vector optimization problem and establishing the corresponding weak, strong and converse duality results.

We consider the vector optimization problem

$$(PVG_q) \qquad \operatorname*{QMin}_{x \in X} F(x),$$

where $F : X \to V^\bullet$ is a proper and K-convex function with $\operatorname{dom} F = \{x \in X : F(x) \neq \emptyset\}$ and we detect the quasi-minimal elements of $F(\operatorname{dom} F)$ with respect to

K. We also assume that $F(\operatorname{dom} F) + \operatorname{qi} K = \operatorname{qi}(F(\operatorname{dom} F) + K)$ and that K is a closed convex cone.

Definition 3.1.15 *An element $\overline{x} \in X$ is called quasi-efficient solution to the vector optimization problem (PVG_q) if $\overline{x} \in \operatorname{dom} F$ and $F(\overline{x}) \in \operatorname{QMin}(F(\operatorname{dom} F), K)$.*

We mention that the problems where the quasi-efficient solutions of the vector optimization problems can play an important role because the ordering cones of the image spaces have empty interiors, but nonempty quasi interiors, can be found for instance in finance mathematics (see [1, 63]).

Using the vector perturbation function $\Phi : X \times Y \to V^{\bullet}$ which fulfills $0 \in \Pr_Y(\operatorname{dom} \Phi)$ and $\Phi(x, 0) = F(x)$ for all $x \in X$, the primal vector optimization problem introduced above can be reformulated as

$$(PVG_q) \qquad \operatorname*{QMin}_{x \in X} \Phi(x, 0).$$

To (PVG_q) we attach the following vector dual problem with respect to quasi-efficient solutions

$$(DVG_q) \qquad \operatorname*{QMax}_{(v^*, y^*, v) \in \mathcal{B}_q^G} h_q^G(v^*, y^*, v)$$

where

$$\mathcal{B}_q^G = \{(v^*, y^*, v) \in (K^* \setminus \{0\}) \times Y^* \times V : \langle v^*, v \rangle \leq -(v^*\Phi)^*(0, -y^*)\}$$

and

$$h_q^G(v^*, y^*, v) = v.$$

Definition 3.1.16 *An element $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}_q^G$ is called quasi-efficient solution to the vector dual optimization problem (DVG_q) if $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \operatorname{dom} h_q^G$ and $h_q^G(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v} \in \operatorname{QMax}(h_q^G(\operatorname{dom}\, h_q^G), K)$.*

Next we formulate the weak and strong duality theorems.

Theorem 3.1.17 (*S.-M. Grad and E.-L. Pop [68]*) *There are no $x \in X$ and $(v^*, y^*, v) \in \mathcal{B}_q^G$ such that $F(x) <_K h_q^G(v^*, y^*, v)$.*

Remark 3.1.18 (*S.-M. Grad and E.-L. Pop [68]*) *F needs not be K-convex and K closed in order to formulate the vector dual problem and for proving the weak duality statement.*

For the strong duality we consider the following regularity conditions (cf. [25]). First, a classical condition

$$(RCV^1) \quad \big| \quad \exists x' \in X \text{ such that } (x', 0) \in \operatorname{dom} \Phi \text{ and } \Phi(x', \cdot) \text{ is continuous at } 0;$$

then the most general one that works when X and Y are Fréchet spaces

(RCV^2) | X and Y are Fréchet spaces, Φ is $K-$lower semicontinuous and $0 \in \operatorname{sqri}(\operatorname{Pr}_Y(\operatorname{dom}\Phi))$;

followed by the one good in finite dimensional cases

(RCV^3) | $\dim(\operatorname{lin}(\operatorname{Pr}_Y(\operatorname{dom}\Phi))) < +\infty$ and $0 \in \operatorname{ri}(\operatorname{Pr}_Y(\operatorname{dom}\Phi))$;

and the closedness type regularity condition

(RCV^4) | Φ is $K-$lower semicontinuous and $\operatorname{Pr}_{X^*\times\mathbb{R}}(\operatorname{epi}(v^*\Phi)^*)$ is closed in the topology $w(X^*, X) \times \mathbb{R}$ for all $v^* \in K^* \setminus \{0\}$.

Theorem 3.1.19 (*S.-M. Grad and E.-L. Pop [68]*) *Assume that one of the regularity conditions* (RCV^i), $i \in \{1, \ldots, 4\}$, *is fulfilled. If* $\overline{x} \in X$ *is a quasi-efficient solution to* (PVG_q), *then there exists* $(\overline{v}^*, \overline{y}^*, \overline{v})$ *a quasi-efficient solution to* (DVG_q) *such that* $F(\overline{x}) = h_q^G(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v}$.

Remark 3.1.20 (*S.-M. Grad and E.-L. Pop [68]*) *Instead of the mentioned regularity conditions, for achieving strong duality it is enough to assume that for all* $\overline{v}^* \in K^* \setminus \{0\}$ *the scalar optimization problem* $\inf_{x\in X}(\overline{v}^*\Phi)(x, 0)$ *is stable.*

Next, we give a preliminary result for the converse duality statement, followed by the mentioned assertion itself.

Theorem 3.1.21 (*S.-M. Grad and E.-L. Pop [68]*) *Assume that one of the regularity conditions* (RCV^i), $i \in \{1, \ldots, 4\}$, *is fulfilled. Then* $V \setminus \operatorname{cl}(F(\operatorname{dom} F) + K) \subseteq \operatorname{core}(h_q^G(\mathcal{B}_q^G))$.

Theorem 3.1.22 (*S.-M. Grad and E.-L. Pop [68]*) *Assume that one of the regularity conditions* (RCV^i), $i \in \{1, \ldots, 4\}$, *is fulfilled and that the set* $F(\operatorname{dom} F) + K$ *is closed. Then for every quasi-efficient solution* $(\overline{v}^*, \overline{y}^*, \overline{v})$ *to* (DVG_q) *one has that* $\overline{v}$ *is a quasi-minimal element of the set* $F(\operatorname{dom} F) + K$.

Remark 3.1.23 (*S.-M. Grad and E.-L. Pop [68]*) *In Theorem 3.1.21 and Theorem 3.1.22, the regularity conditions* (RCV^i), $i \in \{1, \ldots, 4\}$, *can be replaced with the weaker assumption that for all* $\overline{v}^* \in K^* \setminus \{0\}$ *the problem* $\inf_{x\in X}\langle \overline{v}^*, F(x)\rangle$ *is normal (see [25, Theorem 4.3.3]).*

3.1.3 Duality results for particular classes of problems

In what follows we consider constrained and unconstrained vector optimization problems as special cases of the general vector optimization problem and derive for them vector dual problems with respect to quasi-efficient solutions, followed by weak, strong and converse duality statements.

Constrained vector optimization problems

Let us consider the same framework as in the previous section. Let also Y be partially ordered by the nonempty convex cone $C \subseteq Y$. Moreover, we consider the nonempty convex set $S \subseteq X$, the proper K-convex function $f : X \to V^\bullet$ and the proper $C-$convex function $g : X \to Y^\bullet$ fulfilling $\operatorname{dom} f \cap S \cap g^{-1}(C) \neq \emptyset$. Assume again that $f(\operatorname{dom} f \cap \mathcal{A}) + \operatorname{qi} K = \operatorname{qi}(f(\operatorname{dom} f \cap \mathcal{A}) + K)$ and K is a closed convex cone. The primal vector optimization problem with geometric and cone constraints that we work with is

$$(PV_q^C) \qquad \operatorname*{QMin}_{x \in \mathcal{A}} f(x),$$

where

$$\mathcal{A} = \{x \in S : g(x) \in -C\},$$

which is a special case of (PVG_q). We construct different vector dual problems to (PV_q^C) with respect to quasi-efficient solutions, by considering different vector perturbation functions. Then we formulate weak, strong and converse duality.

First we consider the Lagrange vector type perturbation function $\Phi_{C_L}^V$ given in Chapter 2 and from (DVG_q) we obtain the *Lagrange type vector dual problem to* (PV_q^C) with respect to quasi-efficient solutions

$$(DV_q^{C_L}) \qquad \operatorname*{QMax}_{(v^*,y^*,v) \in \mathcal{B}_q^{C_L}} h_q^{C_L}(v^*, y^*, v),$$

where

$$\mathcal{B}_q^{C_L} = \left\{ (v^*, y^*, v) \in (K^* \setminus \{0\}) \times C^* \times V : \ \langle v^*, v\rangle \leq \inf_{u \in S} \{(v^* f)(u) + (y^* g)(u)\} \right\}$$

and

$$h_q^{C_L}(v^*, y^*, v) = v.$$

For the strong duality we need some regularity conditions obtained by particularizing (RCV^i), $i \in \{1,2,3,4\}$. These becomes $(RCV_{C_L}^i)$, $i \in \{1,2,3,4\}$, and for example $(RCV_{C_L}^1)$ is

$$(RCV_{C_L}^1) \ \big| \ \exists x' \in \operatorname{dom} f \cap S \text{ such that } g(x') \in -\operatorname{int} C.$$

Then the weak, strong and converse duality results follow.

Theorem 3.1.24 (*S.-M. Grad and E.-L. Pop [68]*)

(a) There are no $x \in X$ and $(v^*, y^*, v) \in \mathcal{B}_q^{C_L}$ such that $f(x) <_K h_q^{C_L}(v^*, y^*, v)$.

(b) Assume that one of the regularity conditions $(RCV_{C_L}^i)$, $i \in \{1,2,3,4\}$, is fulfilled. If $\overline{x} \in X$ is a quasi-efficient solution to (PV_q^C), then there exists $(\overline{v}^*, \overline{y}^*, \overline{v})$ a quasi-efficient solution to $(DV_q^{C_L})$ such that $f(\overline{x}) = h_q^{C_L}(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v}$.

(c) Assume that one of the regularity conditions $(RCV_{C_L}^i)$, $i \in \{1,2,3,4\}$, is fulfilled, and the set $f(\operatorname{dom} f \cap \mathcal{A}) + K$ is closed. Then for every quasi-efficient solution $(\overline{v}^*, \overline{y}^*, \overline{v})$ to $(DVG_q^{C_L})$ one has that $\overline{v}$ is a quasi-minimal element of the set $f(\operatorname{dom} f \cap \mathcal{A}) + K$.

Another vector perturbation function we consider is the Fenchel-Lagrange type vector perturbation function Φ_{FL}^{V} given also in Chapter 2 and from (DVG_q) we obtain the *Fenchel-Lagrange type vector dual problem to* (PV_q^C) with respect to quasi-efficient solutions

$$(DV_q^{C_{FL}}) \qquad \underset{(v^*,t^*,y^*,v)\in\mathcal{B}_q^{C_{FL}}}{\text{QMax}} h_q^{C_{FL}}(v^*,t^*,y^*,v),$$

where

$$\mathcal{B}_q^{C_{FL}} = \{(v^*,t^*,y^*,v)\in(K^*\setminus\{0\})\times X^*\times C^*\times V:\ \langle v^*,v\rangle \le -(v^*f)^*(t^*)- (y^*g)_S^*(-t^*)\}$$

and

$$h_q^{C_{FL}}(v^*,t^*,y^*,v)=v.$$

Particularizing (RCV^i), $i\in\{1,2,3,4\}$ in this case, one get $(RCV_{C_{FL}}^i)$, $i\in\{1,2,3,4\}$, where for example $(RCV_{C_{FL}}^1)$ is

$$(RCV_{C_{FL}}^1)\ \Big|\ \exists x'\in \operatorname{dom} f\cap S \text{ such that } f \text{ is continuous at } x' \text{ and } g(x')\in -\operatorname{int} C.$$

Then the weak, strong and converse duality results follow from the general case.

Theorem 3.1.25 (*S.-M. Grad and E.-L. Pop [68]*)
(*a*) *There are no* $x\in X$ *and* $(v^*,t^*,y^*,v)\in\mathcal{B}_q^{C_{FL}}$ *such that* $f(x)<_K h_q^{C_{FL}}(v^*,t^*,y^*,v)$.

(*b*) *Assume that one of the regularity conditions* $(RCV_{C_{FL}}^i)$, $i\in\{1,\dots,4\}$, *is fulfilled. If* $\overline{x}\in X$ *is a quasi-efficient solution to* (PV_q^C), *then there exists* $(\overline{v}^*,\overline{t}^*,\overline{y}^*,\overline{v})$ *a quasi-efficient solution to* $(DV_q^{C_{FL}})$ *such that* $f(\overline{x})=h_q^{C_{FL}}(\overline{v}^*,\overline{t}^*,\overline{y}^*,\overline{v})=\overline{v}$.

(*c*) *Assume that one of the regularity conditions* $(RCV_{C_{FL}}^i)$, $i\in\{1,\dots,4\}$, *is fulfilled and the set* $f(\operatorname{dom} f\cap\mathcal{A})+K$ *is closed. Then for every quasi-efficient solution* $(\overline{v}^*,\overline{t}^*,\overline{y}^*,\overline{v})$ *to* $(DVG_q^{C_{FL}})$ *one has that* $\overline{v}$ *is a quasi-minimal element of the set* $f(\operatorname{dom} f\cap\mathcal{A})+K$.

Unconstrained vector optimization problems

In the same framework, we consider the proper K-convex vector functions $f:X\to V^\bullet$ and $h:Y\to V^\bullet$ and $A:X\to Y$ a linear continuous mapping such that $\operatorname{dom} f\cap A^{-1}(\operatorname{dom} h)\neq\emptyset$. Assume again that $\operatorname{dom} f\cap A^{-1}(\operatorname{dom} h)+\operatorname{qi} K=\operatorname{qi}(\operatorname{dom} f\cap A^{-1}(\operatorname{dom} h)+K)$ and K is a closed convex cone. The primal unconstrained vector optimization problem

$$(PV_q^A) \qquad \underset{x\in X}{\text{QMin}}[f(x)+h(Ax)]$$

is a special case of (PVG_q) where $F = f + h \circ A$.

We consider the vector perturbation function $\Phi_q^A : X \times Y \to V^\bullet$ defined by $\Phi_q^A(x, y) = f(x) + h(Ax + y)$. Using this perturbation function we obtain the vector dual to (PV_q^A) given by

$$(DV_q^A) \qquad \underset{(v^*, y^*, v) \in \mathcal{B}_q^A}{\mathrm{QMax}} \; h_q^A(v^*, y^*, v)$$

where

$$\mathcal{B}_q^A = \{(v^*, y^*, v) \in (K^* \setminus \{0\}) \times Y^* \times V : \langle v^*, v\rangle \le -(v^* f)^*(-A^* y^*) + (v^* h)^*(y^*)\}$$

and

$$h_q^A(v^*, y^*, v) = v.$$

For the primal vector optimization problem (PV_q^A) and the vector dual (DV_q^A) we have the weak, strong and converse duality statements, that follow from the general case. To guarantee strong duality we use the regularity conditions obtained from (RCV^i), $i \in \{1, 2, 3, 4\}$. For instance (RCV_A^1) is

$$(RCV_A^1) \quad \Big| \quad \exists x' \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) \text{ such that } h \text{ is continuous at } Ax'.$$

Theorem 3.1.26 (*S.-M. Grad and E.-L. Pop [68]*)

(a) There are no $x \in X$ and $(v^*, y^*, v) \in \mathcal{B}_q^A$ such that $f(x) + h(Ax) <_K h_q^A(v^*, y^*, v)$.

(b) Assume that one of the regularity conditions (RCV_A^i), $i \in \{1, \ldots, 4\}$, is fulfilled. If $\overline{x} \in X$ is a quasi-efficient solution to (PV_q^A), then there exists $(\overline{v}^*, \overline{y}^*, \overline{v})$ a quasi-efficient solution to (DV_q^A) such that $f(\overline{x}) + h(A\overline{x}) = h_q^A(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v}$.

(c) Assume that one of the regularity conditions (RCV_A^i), $i \in \{1, \ldots, 4\}$, is fulfilled and the set $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) + K$ is closed. Then for every quasi-efficient solution $(\overline{v}^*, \overline{y}^*, \overline{v})$ to (DVG_q^A) one has that $\overline{v}$ is a quasi-minimal element of the set $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} h) + K$.

Back to (PV_q^C), seeing it as an unconstrained vector optimization problem, we can attach to it a vector dual problem generated by (DVG_q) by considering the Fenchel type vector perturbation function $\Phi_{C_F}^V$ given in Chapter 2. We assume again that $f(\operatorname{dom} f \cap \mathcal{A}) + \operatorname{qi} K = \operatorname{qi}(f(\operatorname{dom} f \cap \mathcal{A}) + K)$ and K is a closed convex cone.

Thus from (DVG_q) we obtain the *Fenchel type vector dual problem to* (PV_q^C) with respect to quasi-efficient solutions

$$(DV_q^{C_F}) \qquad \underset{(v^*, t^*, v) \in \mathcal{B}_q^{C_F}}{\mathrm{QMax}} \; h_q^{C_F}(v^*, t^*, v),$$

where

$$\mathcal{B}_q^{C_F} = \{(v^*, t^*, v) \in (K^* \setminus \{0\}) \times X^* \times V : \; \langle v^*, v\rangle \le -(v^* f)^*(t^*) - \sigma_{\mathcal{A}}(-t^*)\}$$

and

$$h_q^{C_F}(v^*, t^*, v) = v.$$

Like in Theorem 3.1.26 one can quickly obtain the weak, strong and converse duality statements for (PV_q^C) and $(DV_q^{C_F})$, too.

3.1.4 Comparisons between duals

In this section we compare the image sets of some of the vector duals attached to (PV_q^C) via the Lagrange, Fenchel and Fenchel-Lagrange type vector perturbation functions, respectively.

Proposition 3.1.28 (*S.-M. Grad and E.-L. Pop [68]*) *One has that* $h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}) \subseteq h_q^{C_L}(\mathcal{B}_q^{C_L})$.

Remark 3.1.29 (*S.-M. Grad and E.-L. Pop [68]*) *A situation when the inclusion in Proposition 3.1.28 is not fulfilled as equality can be found in [28, Example 2.2].*

Proposition 3.1.30 (*S.-M. Grad and E.-L. Pop [68]*) *One has that* $h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}) \subseteq h_q^{C_F}(\mathcal{B}_q^{C_F})$.

Remark 3.1.31 (*S.-M. Grad and E.-L. Pop [68]*) *A situation when the inclusion in Proposition 3.1.30 is not fulfilled as equality can be found in [28, Example 2.1].*

Under certain hypotheses, the image sets of the vector duals attached to (PV_q^C) in the previous section coincide.

Theorem 3.1.32 (*S.-M. Grad and E.-L. Pop [68]*) *If one of the following conditions*

(*a*) *there exists* $x' \in \operatorname{dom} f \cap S \cap \operatorname{dom} g$ *such that* f *is continuous at* x'*;*

(*b*) *for* X *and* Z *Fréchet spaces,* S *closed and* g $C-$*epi closed one has* $0 \in \operatorname{sqri}((\operatorname{dom} g \cap S) - \operatorname{dom} f)$*;*

(*c*) *if* $\operatorname{lin}((\operatorname{dom} g \cap S) - \operatorname{dom} f) < +\infty$ *one has* $0 \in \operatorname{ri}((\operatorname{dom} g \cap S) - \operatorname{dom} f)$*;*

is fulfilled, then $h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}) = h_q^{C_L}(\mathcal{B}_q^{C_L})$.

Theorem 3.1.33 (*S.-M. Grad and E.-L. Pop [68]*) *If one of the following conditions*

(*a*) *there exists* $x' \in \operatorname{dom} f \cap S \cap \operatorname{dom} g$ *such that* $g(x') \in -\operatorname{int} C$*;*

(*b*) *for* X *and* Z *Fréchet spaces,* S *closed and* g $C-$*epi closed one has* $0 \in \operatorname{sqri}(g(\operatorname{dom} g \cap S) + C)$*;*

(*c*) *if* $\operatorname{lin}(g(\operatorname{dom} g \cap S) + C) < +\infty$ *one has* $0 \in \operatorname{ri}(g(\operatorname{dom} g \cap S) + C)$*;*

is fulfilled, then $h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}) = h_q^{C_F}(\mathcal{B}_q^{C_F})$.

To guarantee the coincidence of the image sets of the vector duals with respect to quasi-efficient solutions we attached to (PV_q^C) one can combine the last two theorems, or, taking advantage of Proposition 3.1.28, Proposition 3.1.30 and Theorem 3.1.25, can formulate the following conclusion.

Corollary 3.1.34 (*S.-M. Grad and E.-L. Pop [68]*) *If one of the regularity conditions*

$(RCV^i_{C_{FL}})$, $i \in \{1, \dots, 4\}$, *is fulfilled, then*

$$h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}) = h_q^{C_F}(\mathcal{B}_q^{C_F}) = h_q^{C_L}(\mathcal{B}_q^{C_L}).$$

If additionally, $f(\mathrm{dom}(f \cap \mathcal{A})) + K$ *is closed, then one has*

$$\mathrm{QMin}(f(\mathrm{dom}\, f \cap \mathcal{A}), K) \subseteq \mathrm{QMax}(h_q^{C_{FL}}(\mathcal{B}_q^{C_{FL}}), K) = \mathrm{QMax}(h_q^{C_L}(\mathcal{B}_q^{C_L}), K)$$

$$= \mathrm{QMax}(h_q^{C_F}(\mathcal{B}_q^{C_F}), K) \subseteq \mathrm{QMin}(f(\mathrm{dom}\, f \cap \mathcal{A}) + K, K).$$

3.2 Some remarks for vector optimization problems with respect to relative interior

Unfortunately, an analogous theory like in the previous section cannot be developed with respect to the quasi-relative interior of the cone K. Even in $\mathbb{R}^n$, where $\mathrm{qri}\, K = \mathrm{ri}\, K$, we have only some of the similar results, but nothing to assure the existence of the duality results.

Chapter 4

Optimality conditions for general vector optimization problems

4.1 Optimality conditions for a constrained vector optimization problem via different scalarizations

In literature are presented different scalarization methods that use linear functions, norms and other constructions (see, for example [28, 35, 55, 58, 72, 78, 83, 87, 88, 95, 115, 118, 121–123, 130, 132, 134]). To a vector optimization problem are attached vector duals by using different scalarizations (cf. [22, 25, 70, 83]). We work with strongly $K-$ or strictly $K-$increasing functions. The functions considered are called scalaration functions and correspondingly one has the set of the scalarization functions.

In this chapter we formulate the optimality conditions for the primal vector optimization problem with geometric and cone constraints to which properly efficient solutions are defined and for the vector dual problems constructed by using the linear scalarization, maximum($-$linear) scalarization, set scalarization, (semi)norm scalarization and oriented distance scalarization to which are attached efficient or weakly efficient solutions (cf. [40,94]) to which are attach efficient solutions or weakly efficient solutions.

Most of the results presented here were obtained by the author in joint work with dr. S.-M. Grad [67].

Let X, Y and V be Hausdorff locally convex vector spaces and assume that Y is partially ordered by the convex cone $C \subseteq Y$, while V is partially ordered by the nontrivial pointed convex cone $K \subseteq V$. Further, let $S \subseteq X$ be a nonempty convex set, $f : X \to V^\bullet = V \cup \{+\infty_K\}$ a proper and $K-$convex function and $g : X \to Y^\bullet = Y \cup \{+\infty_C\}$ a proper and $C-$convex function such that $\operatorname{dom} f \cap S \cap g^{-1}(C) \neq \emptyset$. The primal vector optimization problem with geometric and cone constraints is

$$(PV_C) \qquad \operatorname*{Min}_{x \in \mathcal{A}} f(x),$$

where

$$\mathcal{A} = \big\{x \in S : g(x) \in -C\big\}.$$

Definition 4.1.1 *Let X be a vector space partially ordered by a convex cone K and $U \subseteq X$ a nonempty set with respect to the partial ordering "$\leqq_K$" induced by K. An element $\overline{x} \in U$ is called*

(*a*) *minimal element of U (regarding the partial ordering induced by K) if there is no $x \in U$ satisfying $x \leq_K \overline{x}$.*

(*b*) *weakly minimal element of U (regarding the partial ordering induced by K) if $(\overline{x} - \operatorname{int} K) \cap U = \emptyset$.*

Remark 4.1.2 In Definition 4.1.1 (*b*) the weakly minimal elements can be considered also with respect to the algebraic interior (core) or quasi-interior (qi) instead of the interior (int), by making only the corresponding replacement. The hypotheses of each subsection will establish the type of weakly minimal element used (for example when $\operatorname{int} K \neq \emptyset$ we work with exactly with Definition 4.1.1 (*b*)).

4.1.1 General scalarization

The general scalarization is using cone-monotone functions (cf. [22, 25]). Let $\mathcal{S}$ be an arbitrary set of proper and convex functions $s : V \cup \{+\infty_K\} \to \overline{\mathbb{R}}$ fulfilling $s(+\infty_K) = +\infty$, $f(\operatorname{dom} f \cap \mathcal{A}) + K \subseteq \operatorname{dom} s$ and moreover s is strongly $K-$increasing on the set $f(\operatorname{dom} f \cap \mathcal{A}) + K$. The elements of the set $\mathcal{S}$ are called *scalarization functions.*

Definition 4.1.3 *An element $\overline{x} \in \mathcal{A}$ is called $\mathcal{S}-$properly efficient solution to (PV_C) if $\overline{x} \in \operatorname{dom} f$ and there exists an $s \in \mathcal{S}$ such that $s(f(\overline{x})) \leq s(f(x))$ for all $x \in \mathcal{A}$.*

In [25, Section 4.4] were given results for the primal vector optimization problem with geometric and constraints (PV_C) and the dual vector optimization problem attach to it. More exactly we refer to weak and strong duality and the optimality conditions. For strong duality, it is used the regularity condition

$$(RCV_{C_{FL}}) \quad \big| \quad \exists x' \in \operatorname{dom} f \cap S \text{ such that } f \text{ is continuous at } x' \text{ and } g(x') \in -\operatorname{int}(C).$$

Let $\operatorname{qi} K \neq \emptyset$, $\mathcal{T}$ be an arbitrary set of proper and convex functions $s : V^\bullet \to \overline{\mathbb{R}}$ fulfilling $s(+\infty_K) = +\infty$, $f(\operatorname{dom} f \cap \mathcal{A}) + K \subseteq \operatorname{dom} s$ and s is strictly $K-$increasing on the set $f(\operatorname{dom} f \cap \mathcal{A}) + K$.

Definition 4.1.7 *An element $\overline{x} \in \mathcal{A}$ is called $\mathcal{T}-$properly efficient solution to (PV_C) if $\overline{x} \in \operatorname{dom} f$ and there exists an $s \in \mathcal{T}$ such that $s(f(\overline{x})) \leq s(f(x))$ for all $x \in \mathcal{A}$.*

A vector dual problem to (PV_C) with respect to the set of the scalarization functions $\mathcal{T}$ was introduced by replacing $\mathcal{S}$ with $\mathcal{T}$ in the definition of the dual vector optimization problem given with respect to the set of the scalarization function $\mathcal{S}$ and were obtained the duality results and the optimality conditions.

From the general scalarization some particular cases follows: linear, maximum(-linear), set, (semi)norm, quadratic, oriented distance scalarizations (see, for example, [19, 22, 25, 78, 80, 83, 92, 93]).

4.1.2 Linear scalarization

We consider the following set of scalarization functions

$$\mathcal{S}_l = \{s_{v^*} : V^\bullet \to \overline{\mathbb{R}} : \quad v^* \in K^{*0}, s_{v^*}(v) = \langle v^*, v\rangle \ \forall v \in V^\bullet\}.$$

For $s_{v^*} \in \mathcal{S}_l$ it holds $s_{v^*}(+\infty_K) = +\infty$, because $\langle v^*, \infty_K\rangle = +\infty \ \forall v^* \in K^*$. Obviously, for all $v^* \in K^{*0}$, $f(\operatorname{dom} f \cap \mathcal{A}) + K \subseteq V = \operatorname{dom} s_{v^*}$ and s_{v^*} is strongly $K-$increasing, linear and continuous. Next, we notice that for all $k^* \in K^*$ one has $s_{v^*}(k^*) = \delta_{\{v^*\}}(k^*)$.

An element $\overline{x} \in \mathcal{A}$ is called $\mathcal{S}_l-$*properly efficient solution* to (PV_C) if $\overline{x} \in \operatorname{dom} f$ and there exists an $s_{v^*} \in \mathcal{S}_l$ such that $s_{v^*}(f(\overline{x})) \leq s_{v^*}(f(x))$ for all $x \in \mathcal{A}$.

The dual vector optimization problem to (PV_C) we investigate is

$$(DV^{C_{S_l}}) \qquad \operatorname*{Max}_{(v^*,y^*,z^*,v)\in\mathcal{B}^{C_{S_l}}} h^{C_{S_l}}(v^*, y^*, z^*, v)$$

where

$$\begin{aligned}\mathcal{B}^{C_{S_l}} = \{&(v^*, y^*, z^*, v) \in K^{*0} \times X^* \times C^* \times V : \langle v^*, v\rangle \leq -(v^*f)^*(y^*)- \\ &(z^*g)^*_{\mathcal{S}}(-y^*)\}\end{aligned}$$

and

$$h^{C_{S_l}}(v^*, y^*, z^*, v) = v.$$

This vector dual problem is the Fenchel-Lagrange type dual problem $(DV^{C_{FL}})$ in [25, Section 4.3]. The weak and strong duality statements follows from the general case.

Proposition 4.1.11 (*Weak and strong duality for (PV_C) and $(DV^{C_{S_l}})$*)

(a) There are no $x \in \mathcal{A}$ and $(v^*, y^*, z^*, v) \in \mathcal{B}^{C_{S_l}}$ such that $f(x) \leq_K h^{C_{S_l}}(v^*, y^*, z^*, v)$.

(b) Assume that the regularity condition $(RCV_{C_{FL}})$ is fulfilled. If $\overline{x} \in \mathcal{A}$ is an $\mathcal{S}_l-$properly efficient solution to (PV_C), then there exists $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C_{S_l}}$ an efficient solution to $(DV^{C_{S_l}})$, such that $f(\overline{x}) = h^{C_{S_l}}(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) = \overline{v}$.

Theorem 4.1.12 (*S.-M. Grad and E.-L. Pop [67]*) (*Optimality conditions for (PV_C) and $(DV^{C_{S_l}})$*)

(a) Let $\overline{x} \in \mathcal{A}$ be an $\mathcal{S}_l-$properly efficient solution to (PV_C) and the regularity condition $(RCV_{C_{FL}})$ be fulfilled. Then there exists $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C_{\mathcal{S}_l}}$ an efficient solution to $(DV^{C_{\mathcal{S}_l}})$ such that

(i) $f(\overline{x}) = \overline{v}$;

(ii) $\overline{v}^* = \overline{k}^*$;

(iii) $(\overline{v}^* f)^*(\overline{y}^*) + (\overline{v}^* f)(\overline{x}) = \langle \overline{y}^*, \overline{x} \rangle$;

(iv) $(\overline{z}^* g)^*_S(-\overline{y}^*) = -\langle \overline{y}^*, \overline{x} \rangle$;

(v) $(\overline{z}^* g)(\overline{x}) = 0$.

(b) Assume that $\overline{x} \in \mathcal{A}$ and $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C_{\mathcal{S}_l}}$ fulfill the relations $(i) - (v)$. Then $\overline{x}$ is an $\mathcal{S}_l-$properly efficient solution to (PV_C) and $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v})$ is an efficient solution to the dual problem $(DV^{C_{\mathcal{S}_l}})$.

If $\operatorname{qi} K \neq \emptyset$ let us now consider as set of scalarization functions

$$\mathcal{T}_l = \{ s_{v^*} : V^\bullet \to \overline{\mathbb{R}} : \quad v^* \in K^* \setminus \{0\}, s_{v^*}(v) = \langle v^*, v \rangle \ \forall v \in V^\bullet \}$$

and it yields that every scalarization function $s_{v^*} \in \mathcal{T}$ is strictly $K-$increasing, linear and continuous, while the domain is $f(\operatorname{dom} f \cap \mathcal{A}) + K$. Moreover we assume further that $V + \operatorname{qi} K = \operatorname{qi}(V + K)$ with $V = f(\operatorname{dom} f \cap \mathcal{A})$.

An element $\overline{x} \in \mathcal{A}$ is called $\mathcal{T}_l-$*properly efficient solution* to (PV_C) if $\overline{x} \in \operatorname{dom} f$ and there exists $s_{v^*} \in \mathcal{T}_l$ such that $s_{v^*}(f(\overline{x})) \leq s_{v^*}(f(x))$ for all $x \in \mathcal{A}$.

The dual introduced with respect to the set of scalarization functions $\mathcal{T}_l$ is

$$(DV^{C_{\mathcal{T}_l}}) \qquad \underset{(v^*, y^*, z^*, v) \in \mathcal{B}^{C_{\mathcal{T}_l}}}{\operatorname{WMax}} h^{C_{\mathcal{T}_l}}(v^*, y^*, z^*, v)$$

where

$$\mathcal{B}^{C_{\mathcal{T}_l}} = \{ (v^*, y^*, z^*, v) \in (K^* \setminus \{0\}) \times X^* \times C^* \times V : \langle v^*, v \rangle \leq -(v^* f)^*(y^*) - (z^* g)^*_S(-y^*) \}$$

and

$$h^{C_{\mathcal{T}_l}}(v^*, y^*, z^*, v) = v.$$

For this vector dual problem the weak and strong duality and the optimality conditions follows similarly like in the case of vector dual problem with respect to the set of scalarization functions $\mathcal{S}_l$.

Proposition 4.1.14 (*Weak and strong duality for (PV_C) and $(DV^{C_{\mathcal{T}_l}})$*)

(a) There are no $x \in \mathcal{A}$ and $(v^*, y^*, z^*, v) \in \mathcal{B}^{C_{\mathcal{T}_l}}$ such that $f(x) \leq_K h^{C_{\mathcal{T}_l}}(v^*, y^*, z^*, v)$.

(b) Assume that the regularity condition $(RCV_{C_{FL}})$ is fulfilled. If $\overline{x} \in \mathcal{A}$ is a $\mathcal{T}_l-$properly efficient solution to (PV_C), then there exists $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C_{\mathcal{T}_l}}$ a weakly efficient solution to $(DV^{C_{\mathcal{T}_l}})$, such that $f(\overline{x}) = h^{C_{\mathcal{S}_l}}(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) = \overline{v}$.

Theorem 4.1.15 (*S.-M. Grad and E.-L. Pop [67]*) (*Optimality conditions for* (PV_C) *and* $(DV^{C\mathcal{T}_l})$)

(a) Let $\overline{x} \in \mathcal{A}$ *be a* $\mathcal{T}_l-$*properly efficient solution to* (PV_C) *and the regularity condition* $(RCV_{C_{FL}})$ *be fulfilled. Then there exists* $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C_{S_l}}$ *a weakly efficient solution to* $(DV^{C\mathcal{T}_l})$ *such that the conditions* $(i)-(v)$ *from Theorem 4.1.11 are fulfilled.*

(b) Assume that $\overline{x} \in \mathcal{A}$ *and* $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C\mathcal{T}_l}$ *fulfill the relations* $(i)-(v)$*. Then* $\overline{x}$ *is a* $\mathcal{T}_l-$*properly efficient solution to* (PV_C) *and* $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v})$ *is a weakly efficient solution to the dual problem* $(DV^{C\mathcal{T}_l})$*.*

Remark 4.1.16 *Note that* $(i)-(iii)$ *in Theorem 4.1.15 do not coincide with their counterparts in Theorem 4.1.12, as here* $v^* \in K^* \setminus \{0\}$*, while there* $v^* \in K^{*0}$*.*

4.1.3 Maximum(-linear) scalarization

One of the scalarizations met especially in the applications of vector optimization for V finite dimensional is the so-called Tchebyshev (maximum) scalarization. We work here with a general scalarization function defined by combining a weighted maximum scalarization function (cf. [80, 126]) with a linear function. Maximum(-linear) scalarization was investigated by K. Mitani and H. Nakayama in [95] (see, also [19, 22, 25]).

Let $V = \mathbb{R}^k$, $V^\bullet = \mathbb{R}^k \cup \{+\infty_{\mathbb{R}^k_+}\} = (\mathbb{R}^k)^\bullet$, $K = \mathbb{R}^k_+$ and $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., k$, be proper and convex functions such that $\bigcap_{i=i}^k \operatorname{dom} f_i \cap S \cap g^{-1}(-C) \neq \emptyset$. We define $f : X \to (\mathbb{R}^k)^\bullet$ as being

$$f(x) = \begin{cases} (f_1(x), ..., f_k(x))^T, & \text{if } x \in \bigcap\limits_{i=1}^{k} \operatorname{dom} f_i, \\ +\infty_{\mathbb{R}^k_+}, & \text{otherwise.} \end{cases}$$

Let $\eta \geq 0$, $w = (w_1, ..., w_k)^T \in \operatorname{int}(\mathbb{R}^k_+)$ and $a = (a_1, ..., a_k)^T \in \mathbb{R}^k$. We consider the scalarization function

$$s_{w,a}(y) = \max_{j=1,\dots,k} \{w_j(y_j - a_j)\} + \eta \sum_{j=1}^{k} w_j y_j, \; y = (y_1, ..., y_k)^T \in \mathbb{R}^k,$$

with $s_{w,a}(+\infty_{\mathbb{R}^k_+}) = +\infty$. For all $w \in \operatorname{int}(\mathbb{R}^k_+)$ and $a \in \mathbb{R}^k$, the function just introduced is convex and strictly $\mathbb{R}^k_+-$increasing and fulfills $f(\cap_{i=1}^k \operatorname{dom} f_i \cap \mathcal{A}) + \mathbb{R}^k_+ \subseteq \mathbb{R}^k$. Then we introduce the set of scalarization functions

$$\mathcal{T}_{ml} = \{s_{w,a} : (\mathbb{R}^k)^\bullet \to \overline{\mathbb{R}} : \quad (w, a) \in \operatorname{int}(\mathbb{R}^k_+) \times \mathbb{R}^k\}.$$

An element $\overline{x} \in \mathcal{A}$ is called $\mathcal{T}_{ml}-$*properly efficient solution* to (PV_C) if there exist $w \in \operatorname{int}(\mathbb{R}^k_+)$ and $a \in \mathbb{R}^k$ such that $\max_{j=1,\dots,k}\{w_j(f_j(\overline{x}) - a_j)\} + \eta \sum_{j=1}^k w_j f_j(\overline{x}) \leq \max_{j=1,\dots,k}\{w_j(f_j(x) - a_j)\} + \eta \sum_{j=1}^k w_j f_j(x)$ for all $x \in \mathcal{A}$.

For $w = (w_1, ..., w_k)^T \in \operatorname{int}(\mathbb{R}^k_+)$, $a = (a_1, ..., a_k)^T \in \mathbb{R}^k$ fixed and $k^* = (k^*_1, ..., k^*_k)^T \in \mathbb{R}^k$, the conjugate function of $s_{w,a} \in \mathcal{T}_{ml}$ is

$$s^*_{w,a}(k^*) = \begin{cases} (k^* - \eta w)^T a, & \text{if } \eta w \leqq k^* \text{ and } \sum\limits_{j=1}^{k} \frac{k^*_j}{w_j} = k\eta + 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

The corresponding vector dual problem to (PV_C) with respect to the set of scalarization functions $\mathcal{T}_{ml}$ is

$$(DV^{C\mathcal{T}_{ml}}) \qquad \underset{(w,a,y^*,k^*,z^*,v)\in\mathcal{B}^{C\mathcal{T}_{ml}}}{\operatorname{WMax}} h^{C\mathcal{T}_{ml}}(w,a,y^*,k^*,z^*,v)$$

where

$$\begin{aligned} \mathcal{B}^{C\mathcal{T}_{ml}} = \Big\{ & (w,a,y^*,k^*,z^*,v) \in \operatorname{int}(\mathbb{R}^k_+) \times \mathbb{R}^k \times X^* \times \mathbb{R}^k_+ \times C^* \times \mathbb{R}^k : \\ & \eta w \leqq k^*,\ \sum_{j=1}^{k} \frac{k^*_j}{w_j} = k\eta + 1,\ \max_{j=1,\dots,k}\{w_j(v_j - a_j)\} + \eta \sum_{j=1}^{k} w_j v_j \\ & \leq -(k^* - \eta w)^T a - \left(\sum_{j=1}^{k} k^*_j f_j\right)^* (y^*) - (z^*g)^*_S(-y^*) \Big\} \end{aligned}$$

and

$$h^{C\mathcal{T}_{ml}}(w,a,y^*,k^*,z^*,v) = v.$$

The weak and strong duality statements follow from the general case.

Proposition 4.1.17 (*[25, Theorem 4.4.8, Theorem 4.4.9]*) (*Weak and strong duality for (PV_C) and $(DV^{C\mathcal{T}_{ml}})$*)

(*a*) *There are no $x \in \mathcal{A}$ and $(w,a,y^*,k^*,z^*,v) \in \mathcal{B}^{C\mathcal{T}_{ml}}$ such that $f_i(x) < h_i^{C\mathcal{T}_{ml}}(w,a,y^*,\ k^*,z^*,v)$.*

(*b*) *Assume that the regularity condition $(RCV_{C_{FL}})$ is fulfilled. If $\overline{x} \in \mathcal{A}$ is a $\mathcal{T}_{ml}-$properly efficient solution to (PV_C), then there exists $(\overline{w},\overline{a},\overline{y}^*,\overline{k}^*,\overline{z}^*,\overline{v})$ a weakly efficient solution to $(DV^{C\mathcal{T}_{ml}})$, such that $f_i(\overline{x}) = h_i^{C\mathcal{T}_{ml}}(\overline{w}^*,\overline{a},\overline{y}^*,\overline{k}^*,\overline{z}^*,\overline{v}) = \overline{v}_i$, $i = 1,...,k$.*

Theorem 4.1.18 (*S.-M. Grad and E.-L. Pop [67]*) (*Optimality conditions for (PV_C) and $(DV^{C\mathcal{T}_{ml}})$*)

(*a*) *Let $\overline{x} \in \mathcal{A}$ be a $\mathcal{T}_{ml}-$properly efficient solution to (PV_C) and the regularity condition $(RCV_{C_{FL}})$ be fulfilled. Then there exists $(\overline{w},\overline{a},\overline{y}^*,\overline{k}^*,\overline{z}^*,\overline{v}) \in \mathcal{B}^{C\mathcal{T}_{ml}}$ a weakly efficient solution to $(DV^{C\mathcal{T}_{ml}})$ with $\overline{w} = (\overline{w}_1,...,\overline{w}_k)^T$ and $\overline{k}^* = (\overline{k}^*_1,...,\overline{k}^*_k)^T$ such that*

(*i*) $f(\overline{x}) = \overline{v}$;

(*ii*) $\max\limits_{j=1,\dots,k}\{\overline{w}_j(f_j(\overline{x}) - \overline{a}_j)\} + \eta \sum\limits_{j=1}^{k} \overline{w}_j f_j(\overline{x}) + (\overline{k}^* - \eta\overline{w})^T\overline{a} + \delta_{-\mathbb{R}^k_+}(\eta\overline{w} - \overline{k}^*) = \left(\overline{k}^{*T} f\right)(\overline{x})$;

(ii') $\sum\limits_{j=1}^{k} \frac{\overline{k}_j^*}{\overline{w}_j} = k\eta + 1$;

(iii) $\left(\overline{k}^{*T} f\right)^* (\overline{y}^*) + \left(\overline{k}^{*T} f\right)(\overline{x}) = \langle \overline{y}^*, \overline{x} \rangle$;

(iv) $(\overline{z}^* g)_S^*(-\overline{y}^*) = -\langle \overline{y}^*, \overline{x} \rangle$;

(v) $(\overline{z}^* g)(\overline{x}) = 0$.

(b) Assume that $\overline{x} \in \mathcal{A}$ and $(\overline{w}, \overline{a}^*, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C\mathcal{T}_{ml}}$ fulfill the relations (i) − (v). Then $\overline{x}$ is a $\mathcal{T}_{ml}-$properly efficient solution to (PV_C) and $(\overline{w}^*, \overline{a}^*, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{v})$ is a weakly efficient solution to the dual problem $(DV^{C\mathcal{T}_{ml}})$.

In case $\eta = 0$, $w_j = 1$ and $a_j = 0$ for all $j = 1, ..., k$, the set of the scalarization functions $\mathcal{T}_m$ is given by

$$\mathcal{T}_m = \left\{ s : (\mathbb{R}^k)^\bullet \to \overline{\mathbb{R}} : \quad s(y) = \max_{j=1,\dots,k} y_j \ \forall y \in \mathbb{R}^k, s(\infty_{\mathbb{R}^k_+}) = +\infty \right\}.$$

An element $\overline{x} \in \mathcal{A}$ is called $\mathcal{T}_m-$*properly efficient solution* to (PV_C) if $\max_{j=1,\dots,k}\{f_j(\overline{x})\} \leq \max_{j=1,\dots,k}\{f_j(x)\}$ for all $x \in \mathcal{A}$.

Correspondingly, we obtained the vector dual problem to (PV_C) with respect to the set of scalarization functions $\mathcal{T}_m$ and the weak and strong duality theorems on one hand and the optimality conditions on the other hand.

4.1.4 Set scalarization

Here we include those scalarization approaches for which the scalarization functions are defined by means of some given sets. We consider a quite general scalarization function in connection to the one due to C. Gerth and P. Weidner (cf. [62]). This scalarization function was investigated also in [22, 25, 122, 123, 134].

Let $\operatorname{core} K \neq \emptyset$ and consider the nonempty convex set $E \subsetneqq V$ which satisfies $\operatorname{cl}(E) + \operatorname{int}(K) \subseteq \operatorname{core}(E)$. For all $\mu \in \operatorname{core}(K)$ define $s_\mu : V^\bullet \to \overline{\mathbb{R}}$ by

$$s_\mu(v) = \inf\{t \in \mathbb{R} : v \in t\mu - \operatorname{cl}(E)\}$$

the scalarization function. Then $s_\mu(+\infty_K) = +\infty$. For $\mu \in \operatorname{core}(K)$ the function s_μ is convex, strictly $K-$increasing and takes only real values on V and thus $f(\operatorname{dom} f \cap \mathcal{A}) + K \subseteq V = \operatorname{dom} s_\mu$. If $\operatorname{core} K \neq \emptyset$ we consider the set of scalarization functions

$$\mathcal{T}_s = \{s_\mu : V^\bullet \to \overline{\mathbb{R}}, \ \mu \in \operatorname{core}(K)\}.$$

An element $\overline{x} \in \mathcal{A}$ is called $\mathcal{T}_s-$*properly efficient solution* to (PV_C) if there exists $\mu \in \operatorname{core} K$ such that $s_\mu(f(\overline{x}) \leq s_\mu(f(x))$ for all $x \in \mathcal{A}$.

When $\mu \in \operatorname{core}(K)$ is fixed, the conjugate function of the scalarization function $s_\mu^* : V^* \to \overline{\mathbb{R}}$ is defined by

$$s_\mu^*(k^*) = \begin{cases} \sigma_{-\operatorname{cl}(E)}(k^*), & \text{if } \langle k^*, \mu \rangle = 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

The vector dual problem attached to (PV_C) via the set of scalarization functions is given by

$$(DV^{C\mathcal{T}_s}) \qquad \underset{(\mu,y^*,k^*,z^*,v)\in\mathcal{B}^{C\mathcal{T}_s}}{\operatorname{WMax}} h^{C\mathcal{T}_s}(\mu, y^*, k^*, z^*, v)$$

where

$$\mathcal{B}^{C\mathcal{T}_s} = \Big\{(\mu, y^*, k^*, z^*, v) \in \operatorname{core}(K) \times X^* \times K^* \times C^* \times V : \langle k^*, \mu \rangle = 1, \inf\{t \in \mathbb{R} : v \in t\mu - \operatorname{cl}(E)\} \leq -\sigma_{-\operatorname{cl}(E)}(k^*) - (k^* f)^*(y^*) - (z^* g)_S^*(-y^*)\Big\}$$

and

$$h^{C\mathcal{T}_s}(\mu, y^*, k^*, z^*, v) = v.$$

The weak and strong duality statements follow from the general case and then the optimality conditions.

Proposition 4.1.22 (*[25, Theorem 4.4.10, Theorem 4.4.11]*) (*Weak and strong duality for* (PV_C) *and* $(DV^{C\mathcal{T}_s})$)

(a) There are no $x \in \mathcal{A}$ *and* $(\mu, y^*, k^*, z^*, v) \in \mathcal{B}^{C\mathcal{T}_s}$ *such that* $f(x) <_K h^{C\mathcal{T}_s}(\mu, y^*, k^*, z^*, v)$.

(b) Assume that the regularity condition $(RCV_{C_{FL}})$ *is fulfilled. If* $\overline{x} \in \mathcal{A}$ *is a* $\mathcal{T}_s-$*properly efficient solution to* (PV_C), *then there exists* $(\overline{\mu}, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{v})$ *a weakly efficient solution to* $(DV^{C\mathcal{T}_s})$, *such that* $f(\overline{x}) = h^{C\mathcal{T}_s}(\overline{\mu}^*, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{v}) = \overline{v}$.

Theorem 4.1.23 (*S.-M. Grad and E.-L. Pop [67]*) (*Optimality conditions for* (PV_C) *and* $(DV^{C\mathcal{T}_s})$)

(a) Let $\overline{x} \in \mathcal{A}$ *be a* $\mathcal{T}_s-$*properly efficient solution to* (PV_C) *and the regularity condition* $(RCV_{C_{FL}})$ *be fulfilled. Then there exists* $(\overline{\mu}, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C\mathcal{T}_s}$ *a weakly efficient solution to* $(DV^{C\mathcal{T}_s})$ *such that*

(i) $f(\overline{x}) = \overline{v}$;

(ii) $\inf\{t \in \mathbb{R} : f(\overline{x}) \in t\mu - \operatorname{cl}(E)\} + \sigma_{-\operatorname{cl}(E)}(\overline{k}^*) = (\overline{k}^* f)(\overline{x})$;

(ii′) $\langle \overline{k}^*, \overline{\mu} \rangle = 1$;

(iii) $(\overline{k}^* f)^*(\overline{y}^*) + (\overline{k}^* f)(\overline{x}) = \langle \overline{y}^*, \overline{x} \rangle$;

(iv) $(\overline{z}^* g)_S^*(-\overline{y}^*) = -\langle \overline{y}^*, \overline{x} \rangle$;

(v) $(\overline{z}^* g)(\overline{x}) = 0.$

(b) Assume that $\overline{x} \in \mathcal{A}$ *and* $(\overline{\mu}, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}^{C\mathcal{T}_s}$ *fulfill the relations (i) – (v). Then* $\overline{x}$ *is a* $\mathcal{T}_s-$*properly efficient solution to* (PV_C) *and* $(\overline{\mu}^*, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{v})$ *is a weakly efficient solution to the dual problem* $(DV^{C\mathcal{T}_s})$.

Also, can be consider the set scalarization with a conical set and the one with sets generated by norms (cf. [22]) and the optimality conditions follow.

4.1.5 (Semi)Norm Scalarization

In this part we have as starting point the fact that in some circumstances (semi)norms on V turn out to be strongly$K-$increasing functions (see, for example [80, 118, 147]). This type of scalarization functions has been used for location problems in [133] and for goal programming in [35]. In what follows we investigate the scalarization functions based on strongly $K-$increasing gauges (cf. [22, 25]).

Let us consider $b \in V$ such that $f(\operatorname{dom} f \cap \mathcal{A}) \subseteq b + K$, $E \subseteq V$ a convex set with $0 \in \operatorname{int}(E)$ and its Minkovski gauge γ_E is strongly $K-$increasing on K. Since $0 \in \operatorname{int}(E)$ we have $\gamma_E(v) \in \mathbb{R}$ for all $v \in V$. For $a \in b - K$ define $s_a : V^\bullet \to \overline{\mathbb{R}}$ by

$$s_a(v) = \begin{cases} \gamma_E(v - a), & \text{if } v \in b + K, \\ +\infty, & \text{otherwise,} \end{cases}$$

with $s_a(+\infty_K) = +\infty$. For $a \in b - K$ fixed, the function s_a is convex with $f(\operatorname{dom} f \cap \mathcal{A}) \subseteq b + K = \operatorname{dom} s_a$ and moreover s_a is strongly $K-$increasing on $f(\operatorname{dom} f \cap \mathcal{A})$. We consider the following family of scalarization functions

$$\mathcal{S}_g = \{s_a : V^\bullet \to \overline{\mathbb{R}} : \ a \in b - K\}.$$

An element $\overline{x} \in \mathcal{A}$ is called $\mathcal{S}_g-$*properly efficient solution* to (PV_C) if there exists $a \in b - K$ such that $s_a(f(\overline{x})) \leq s_a(f(x))$ for all $x \in \mathcal{A}$.

For $a \in b - K$ fixed and $k^* \in V^*$, the conjugate function $s_a^* : V^* \to \overline{\mathbb{R}}$ is

$$s_a^*(k^*) = \langle k^*, a\rangle + \min_{\substack{w^* \in -K^* \\ \sigma_E(k^* - w^*) \leq 1}} \langle w^*, b - a\rangle,$$

where σ_E defines the dual gauge to γ_E and if γ_E is a norm it turns out to be the dual norm.

The vector dual problem attached to (PV_C) via the (semi)norm scalarization is given by

$$(DV^{C\mathcal{S}_g}) \qquad \operatorname*{Max}_{(a, y^*, k^*, z^*, w^*, v) \in \mathcal{B}^{C\mathcal{S}_g}} h^{C\mathcal{S}_g}(a, y^*, k^*, z^*, w^*, v)$$

where

$$\mathcal{B}^{C s_g} = \Big\{(a, y^*, k^*, z^*, w^*, v) \in (b-K) \times X^* \times K^* \times C^* \times (-K^*) \times (b+K) : \sigma_E(k^* - w^*) \leq 1,\ \gamma_E(v-a) \leq \langle w^*, a-b\rangle - \langle k^*, a\rangle - (k^* f)^*(y^*) - (z^* g)_S^*(-y^*)\Big\}$$

and

$$h^{C s_g}(a, y^*, k^*, z^*, w^*, v) = v.$$

The weak and strong duality statements follow from the general case and then the optimality conditions.

Proposition 4.1.32 (*[25, Theorem 4.4.12, Theorem 4.4.13]*) (*Weak and strong duality for* (PV_C) *and* $(DV^{C s_g})$)

(*a*) *There are no* $x \in \mathcal{A}$ *and* $(a, y^*, k^*, z^*, w^*, v) \in \mathcal{B}^{C s_g}$ *such that* $f(x) <_K h^{C s_g}(a, y^*, k^*, z^*, w^*, v)$.

(*b*) *Assume that the regularity condition* $(RCV_{C_{FL}})$ *is fulfilled. If* $\overline{x} \in \mathcal{A}$ *is an* $\mathcal{S}_g-$*properly efficient solution to* (PV_C), *then there exists* $(\overline{a}, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{w}^*, \overline{v})$ *an efficient solution to* $(DV^{C s_g})$, *such that* $f(\overline{x}) = h^{C s_g}(\overline{a}^*, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{w}^*, \overline{v}) = \overline{v}$.

Theorem 4.1.33 (*S.-M. Grad and E.-L. Pop [67]*) (*Optimality conditions for* (PV_C) *and* $(DV^{C s_g})$)

(*a*) *Let* $\overline{x} \in \mathcal{A}$ *be an* $\mathcal{S}_g-$*properly efficient solution to* (PV_C) *and the regularity condition* $(RCV_{C_{FL}})$ *be fulfilled. Then there exists* $(\overline{a}, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{w}^*, \overline{v}) \in \mathcal{B}^{C s_g}$ *an efficient solution to* $(DV^{C s_g})$ *such that*

(*i*) $f(\overline{x}) = \overline{v}$;

(*ii*) $\gamma_E(f(\overline{x}) - \overline{a}) + \langle \overline{w}^*, b - \overline{a}\rangle + \langle \overline{k}^*, \overline{a}\rangle = (\overline{k}^* f)(\overline{x})$;

(*ii*′) $\sigma_E(\overline{k}^* - \overline{w}^*) \leq 1$;

(*ii*″) $\langle \overline{w}^*, b - \overline{a}\rangle = \min\limits_{\substack{w^* \in -K^* \\ \sigma_E(k^* - w^*) \leq 1}} \langle w^*, b - \overline{a}\rangle$;

(*iii*) $(\overline{k}^* f)^*(\overline{y}^*) + (\overline{k}^* f)(\overline{x}) = \langle \overline{y}^*, \overline{x}\rangle$;

(*iv*) $(\overline{z}^* g)_S^*(-\overline{y}^*) = -\langle \overline{y}^*, \overline{x}\rangle$;

(*v*) $(\overline{z}^* g)(\overline{x}) = 0$.

(*b*) *Assume that* $\overline{x} \in \mathcal{A}$ *and* $(\overline{a}, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{w}^*, \overline{v}) \in \mathcal{B}^{C s_g}$ *fulfill the relations* (*i*)$-$(*v*). *Then* $\overline{x}$ *is a* $\mathcal{S}_g-$*properly efficient solution to* (PV_C) *and* $(\overline{a}^*, \overline{y}^*, \overline{k}^*, \overline{z}^*, \overline{w}^*, \overline{v})$ *is an efficient solution to the dual problem* $(DV^{C s_g})$.

Remark 4.1.35 *The duality approach described in this part can be considered in the particular case when γ_E is a norm with the unit ball E, too. Conditions which ensure that a norm is strongly $K-$increasing on a given set were investigated in [78, 80, 147]*

4.1.6 Oriented distance scalarization

This scalarization has not been consider for conjugate vector duality, because of the difficulty to compute the conjugate of the scalarization function. In [40] it was finally computed in case $V = \mathbb{R}^k$, therefore we can consider this scalarization within our framework, too.

For the set $U \subseteq \mathbb{R}^n$ let $\Delta_U : \mathbb{R}^n \to \mathbb{R}^n$ given by $\Delta_U(x) = \sup\limits_{x^* \in \mathbb{R}^n} [\langle x^*, x\rangle - \sigma_U(x^*) : ||x^*|| = 1]$ and the set $\mathcal{B} = \{x^* \in \mathbb{R}^n : ||x^*|| = 1\}$. So, Δ_U can be equivalently written as $\Delta_U(x) = \sup_{x^* \in \mathcal{B}}[\langle x^*, x\rangle - \sigma_U(x^*)]$. This function is finite and convex on the whole space $\mathbb{R}^n$ and coincides with the distance function $d_U(x) = \sup[\langle x^*, x\rangle - \sigma_U(x^*) : ||x^*|| \leq 1]$ outside K (cf. [94]). Moreover, Δ_{-K} is strictly $K-$increasing on $\mathbb{R}^n_+$.

We consider the scalarization function $s_d : (\mathbb{R}^k)^\bullet \to (\overline{\mathbb{R}})^\bullet$ given by

$$s_d(y) = \begin{cases} \Delta_{-K}(y), & \text{if } y \in \mathbb{R}^k \\ +\infty, & \text{else.} \end{cases}$$

This function is proper, convex and strictly $K-$increasing. The set of the scalarization functions is given in this case by

$$\mathcal{T}_d = \{\Delta_{-K} : \mathbb{R}^n \to \mathbb{R}^n\}.$$

An element $\overline{x} \in \mathcal{A}$ is a $\mathcal{T}_d-$*properly efficient solution to* (PV_C) if $\Delta_{-K}(f(\overline{x})) \leq \Delta_{-K}(f(x))$ for all $x \in \mathcal{A}$.

The conjugate function of $s_d \in \mathcal{T}_d$ is (cf. [40])

$$s_d^*(k^*) = \Delta_{-K}^*(k^*) = \inf\left\{\sum_{j=1}^{l} \delta_{K^*}(x_j^*) : \ 1 \leq l \leq n+2, \ k^* = \sum_{j=1}^{l} x_j^*, \ \sum_{j=1}^{l} ||x_j^*|| = 1\right\}$$

and for writing the dual we use the following formula

$$\sup_{x^* \in U} [\langle x^*, v\rangle - \sigma_{-K}(x^*)] \leq -(k^* f)^*(y^*) - (z^* g)_S^*(-y^*),$$

when $1 \leq l \leq n+2, \ k^* = \sum_{j=1}^{l} x_j^*, \ \sum_{j=1}^{l} ||x_j^*|| = 1$ and $x_j^* \in K^*$.

The dual vector problem to (PV_C) with respect to the set of scalarization functions $\mathcal{T}_d$ is

$$(DV^{\mathcal{T}_d}) \qquad \underset{(y^*,k^*,x^*,z^*,v)\in\mathcal{B}^{\mathcal{T}_d}}{\text{WMax}} h^{\mathcal{T}_d}(y^*,k^*,x^*,z^*,v)$$

where

$$\begin{aligned}\mathcal{B}^{\mathcal{T}_d} = \Big\{&(y^*,k^*,x^*,z^*,v)\in X^*\times K^*\times(K^*)^l\times C^*\times\mathbb{R}^n : \Delta_{-K}(v)\leq -(k^*f)^*(y^*)\\ &-(z^*g)^*_S(-y^*),\ \text{for } 1\leq l\leq n+2,\ k^*=\sum_{j=1}^l x_j^*,\ \sum_{j=1}^l \|x_j^*\|=1\\ &\text{and } x^*=(x_1^*,...x_l^*)\Big\}\end{aligned}$$

and

$$h^{\mathcal{T}_d}(y^*,k^*,x^*,z^*,v)=v.$$

For this dual we have weak and strong duality and also optimality conditions.

Proposition 4.1.36 (*Weak and strong duality for* (PV_C) *and* $(DV^{\mathcal{T}_d})$)

(*a*) *There are no* $x\in\mathcal{A}$ *and* $(y^*,k^*,x^*,z^*,v)\in\mathcal{B}^{\mathcal{T}_d}$ *such that* $f(x)<h^{\mathcal{T}_d}(y^*,k^*,x^*,z^*,v)$.

(*b*) *Assume that the regularity condition* $(RCV_{C_{FL}})$ *is fulfilled. If* $\overline{x}\in\mathcal{A}$ *is a* $\mathcal{T}_d-$*properly efficient solution to* (PV_C), *then there exists* $(\overline{y}^*,\overline{k}^*,\overline{x}^*,\overline{z}^*,\overline{v})$ *a weakly efficient solution to* $(DV^{\mathcal{T}_d})$, *such that* $f(\overline{x})=h^{\mathcal{T}_d}(\overline{y}^*,\overline{k^*},\overline{x}^*,\overline{z}^*,\overline{v})=\overline{v}$.

Theorem 4.1.37 (*S.-M. Grad and E.-L. Pop [67]*) (*Optimality conditions for* (PV_C) *and* $(DV^{\mathcal{T}_d})$)

(*a*) *Let* $\overline{x}\in\mathcal{A}$ *be a* $\mathcal{T}_d-$*properly efficient solution to* (PV_C) *and the regularity condition* $(RCV_{C_{FL}})$ *be fulfilled. Then there exists* $(\overline{y}^*,\overline{k^*},\overline{x}^*,\overline{z}^*,\overline{v})\in\mathcal{B}^{\mathcal{T}_d}$ *a weakly efficient solution to* $(DV^{\mathcal{T}_d})$ *such that*

(*i*) $f(\overline{x})=\overline{v}$;

(*ii*) $\Delta_{-K}(f(\overline{x}))=(\overline{k}^*f)(\overline{x})$;

(*ii'*) $\overline{k}^*=\sum_{j=1}^l\overline{x}_j^*$ *and* $\sum_{j=1}^l\|\overline{x}_j^*\|=1$;

(*iii*) $(\overline{k}^*f)^*(\overline{y}^*)+(\overline{k}^*f)(\overline{x})=\langle\overline{y}^*,\overline{x}\rangle$;

(*iv*) $(\overline{z}^*g)^*_S(-\overline{y}^*)=-\langle\overline{y}^*,\overline{x}\rangle$;

(*v*) $(\overline{z}^*g)(\overline{x})=0$.

(*b*) *Assume that* $\overline{x}\in\mathcal{A}$ *and* $(\overline{y}^*,\overline{k^*},\overline{x}^*,\overline{z}^*,\overline{v})\in\mathcal{B}^{\mathcal{T}_d}$ *fulfill the relations* $(i)-(v)$. *Then* $\overline{x}$ *is a* $\mathcal{T}_d-$*properly efficient solution to* (PV_C) *and* $(\overline{y}^*,\overline{k^*},\overline{x}^*,\overline{z}^*,\overline{v})$ *is a weakly efficient solution to the dual problem* $(DV^{\mathcal{T}_d})$.

Chapter 5

Some applications

In this chapter, to an optimization problem we attach the $(0, 1) - \eta$ approximated optimization problem and deliver connections between the optimal solutions and the saddle points for the Lagrangian of these two problems (see E.-L. Pop and D.I. Duca [108–110]). Starting from [47], similar discusions were given by H. Boncea and D.I. Duca in [10] and L. Cioban and D.I. Duca in [38].

In the second part of this chapter the problems considered were extended to the vector case and some connections were obtained for the efficient solutions and saddle points for the Lagrangian of these problems (see E.-L. Pop and D.I. Duca [111]).

5.1 Connections between the optimization problems and their first order approximated optimization problems

In this part to an optimization problem (P_v) we attach other optimization problems, problems whose solution give us information about optimal solutions of the initial problem. The problems attached are the first order approximated optimization problem (AP_v) and the dual one. The connections studied refers to the saddle points for the Lagrangian and optimal solutions of these two problems. There are also some examples presented.

5.1.1 Optimization problems and first order approximated optimization problems

Here we attach to the optimization problem its first order approximated optimization problem and establish the connections between these two problems, referring to the optimal solutions and saddle points for the Lagrangian of each one (see [109]).

5.1.2 Connections between the optimal solutions and saddle points for the Lagrangian of Problem (P_v) and of Problem (AP_v)

Here we present some connections between the feasible solutions of Problem (P_v) and of Problem (AP_v). Moreover, we investigate the connections between the optimal solutions of both Problem (P_v) and Problem (AP_v) and also the connections between their saddle points for the Lagrangian (see [109]).

5.1.3 Connections between optimization problems and their duals

For solving the optimization problem (P_v), we can attach to it the dual one and we obtain information about optimal solutions and saddle points for the Lagrangian of the initial problem (see [110]).

5.2 Connections between the vector optimization problems, their efficient solutions and saddle points for the Lagrangian

Considering the vector case for the optimization problem (P_v) and the $(0,1)-\eta$ approximated optimization problem, we study the connections between the efficient solutions and saddle point for the Lagrangian of these two problems (due to D.I. Duca [47]).

We consider the vector optimization problem

$$(P_v) \qquad \min f(x)$$

$$\begin{aligned} \text{s. t.} \quad & x \in X \\ & g(x) \leqq 0 \\ & h(x) = 0, \end{aligned}$$

where X is a subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^q$ are three functions. Let $\mathcal{F}(P_v) := \{x \in X : g(x) \leqq 0,\ h(x) = 0\}$ denote the set of all feasible solutions for Problem (P_v).

Definition 5.2.1 *We say that $\overline{x} \in \mathcal{F}(P_v)$ is an efficient solution for Problem (P_v) if there is no $x \in \mathcal{F}(P_v)$ such that $f(x) \leq f(\overline{x})$.*

Theorem 5.2.2 *Let X be subset of $\mathbb{R}^n$, $\overline{x}$ be an interior point of X and $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^q$ three differentiable functions at $\overline{x}$ and let $\overline{x}$ be an efficient solution for problem (P_v).*

(a) (Fritz-John Theorem) Then there exists $(\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^q \setminus \{(0,0,0)\}$ such that

$$(5.1) \qquad [\nabla f(\overline{x})]^T(\overline{u}) + [\nabla g(\overline{x})]^T(\overline{v}) + [\nabla h(\overline{x})]^T(\overline{w}) = 0,$$

$$(5.2) \qquad \langle \overline{v}, g(\overline{x}) \rangle = 0$$

are fulfilled.

(b) (Karush-Kuhn-Tucker Theorem) If a suitable constraint qualification for Problem (P_v) is fulfilled at $\overline{x}$, then there exists $(\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^q$ with $\overline{u} \neq \emptyset$ such that (5.1) and (5.2) are fulfilled.

If $\overline{x}$ is an efficient solution for Problem (P_v) and f, g and h are differentiable at $\overline{x}$, then there exist $(\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^q \setminus \{(0,0,0)\}$ for which we have (5.1) and (5.2) fulfilled (cf. Fritz-John Theorem). It follows that the efficient solutions for Problem (P_v), $\overline{x}$, can be found among the components $\overline{x}$ of the solutions $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ of the system (5.1) – (5.2). If this system has only solutions with $\overline{u} = 0$, then $(\overline{x}, 0, \overline{v}, \overline{w})$ remains the solution of the system (5.1) – (5.2) for every function f; in this case the Fritz-John Theorem is not useful. The hypotheses which are added and assure the existence of one of the solutions $(\overline{x}, \overline{u}, \overline{v}, \overline{w})$ with $\overline{u} \neq 0$ are called *constraint qualifications*. In the literature there exist many types of constraint qualifications (see [89, 96]): Slater, Karlin, Kuhn-Tucker, Arrow-Hurwicz-Uzawa, strict, reverse convex and others. In what follows, by a *suitable constraint qualification*, we mean one of the above constraint qualifications.

Let $L_{P_v} : X \times \mathbb{R}^m_+ \times \mathbb{R}^q \to \mathbb{R}^p$ defined by

$$L_{P_v}(x, v, w) = f(x) + \sum_{i=1}^{m} g_i(x) v_i e + \sum_{k=1}^{q} h_k(x) w_k e,$$

for all $(x, v, w) \in X \times \mathbb{R}^m_+ \times \mathbb{R}^q$, where $e = (1, 1, .., 1) \in \mathbb{R}^p$, denote the *vector Lagrangian of Problem (P_v)*.

Let now $u \in \mathbb{R}^p_+ \setminus \{0\}$. Then the function $L^u_{P_v} : X \times \mathbb{R}^m_+ \times \mathbb{R}^q \to \mathbb{R}$ defined by

$$L^u_{P_v}(x, v, w) = \sum_{j=1}^{p} u_j f_j(x) + \sum_{i=1}^{m} v_i g_i(x) + \sum_{k=1}^{q} w_k h_k(x),$$

for all $(x, v, w) \in X \times \mathbb{R}^m_+ \times \mathbb{R}^q$, denote the *scalar Lagrangian of Problem (P_v)*.

Definition 5.2.3 *(i) A point $(\overline{x}, \overline{v}, \overline{w}) \in X \times \mathbb{R}^m_+ \times \mathbb{R}^q$ is called saddle point for the vector Lagrangian L_{P_v} of Problem (P_v) if* *(a) there is no* $(v,w)\in\mathbb{R}_+^m\times\mathbb{R}^q$ *such that* $L_{P_v}(\overline{x},\overline{v},\overline{w})\leq L_{P_v}(\overline{x},v,w)$*;*

(b) there is no $x\in X$ *such that* $L_{P_v}(\overline{x},\overline{v},\overline{w})\geq L_{P_v}(x,\overline{v},\overline{w})$*.*

(ii) Let $\overline{u}\in\mathbb{R}_+^p\setminus\{0\}$*. A point* $(\overline{x},\overline{v},\overline{w})\in X\times\mathbb{R}_+^m\times\mathbb{R}^q$ *is called saddle point for the scalar Lagrangian* $L_{P_v}^{\overline{u}}$ *of Problem* (P_v) *if*

$$L_{P_v}^{\overline{u}}(\overline{x},v,w)\leqq L_{P_v}^{\overline{u}}(\overline{x},\overline{v},\overline{w})\leqq L_{P_v}^{\overline{u}}(x,\overline{v},\overline{w}),$$

for all $(x,v,w)\in X\times\mathbb{R}_+^m\times\mathbb{R}^q$*.*

For the $(0,1)-\eta$ approximated vector optimization problem

$$(AP_v)\qquad\qquad \min f(x)$$

$$\begin{aligned}\text{s. t.}\quad & x\in X\\ & g(\overline{x})+[\nabla g(\overline{x})]\,(\eta(x,\overline{x}))\leqq 0\\ & h(\overline{x})+[\nabla h(\overline{x})]\,(\eta(x,\overline{x}))=0,\end{aligned}$$

we denote by $\mathcal{F}(AP_v):=\{x\in X: g(\overline{x})+[\nabla g(\overline{x})]\,(\eta(x,\overline{x}))\leqq 0, h(\overline{x})+[\nabla h(\overline{x})]\,(\eta(x,\overline{x}))=0\}$ the set of all feasible solutions for Problem (AP_v).

The *vector Lagrangian of Problem* (AP_v) and the *scalar Lagrangian of Problem* (AP_v) can be defined analogously with the ones for Problem (P_v).

Next, we give some useful notions that we use in what follows.

Definition 5.1.4 (*cf. [96]*) *Let* X *be a nonempty subset of* $\mathbb{R}^n$*,* $\overline{x}$ *be an interior point of* X*,* $f:X\to\mathbb{R}$ *be a differentiable function at* $\overline{x}$ *and* $\eta:X\times X\to\mathbb{R}^n$ *be a function. We say that the function* f *is*

(i) invex at $\overline{x}$ *with respect to (w.r.t.)* η *if*

$$f(x)-f(\overline{x})\geqq\langle\nabla f(\overline{x}),\eta(x,\overline{x})\rangle\,,\ \textit{for all } x\in X.$$

(ii) incave at $\overline{x}$ *with respect to (w.r.t.)* η *if* $(-f)$ *is invex at* $\overline{x}$ *w.r.t.* η*.*

(iii) avex at $\overline{x}$ *with respect to (w.r.t.)* η *if* f *is both invex and incave at* $\overline{x}$ *w.r.t.* η*, or equivalently*

$$f(x)-f(\overline{x})=[\nabla f(\overline{x})]\,(\eta(x,\overline{x}))\,,\ \textit{for all } x\in X.$$

(iv) avex on X *with respect to (w.r.t.)* η *if* X *is open,* f *is differentiable on* X *and avex at every* $\overline{x}\in X$ *w.r.t.* η*.*

For $f=(f_1,...,f_p):X\to\mathbb{R}^p$ a vector function, we say that f is invex (respectively incave, avex) if each component function is invex (respectively incave, avex).

Remark 5.1.5 (*E.-L. Pop and D.I. Duca [109]*) *In general, there exists no unique function* η *such that the function* f *is invex at the point* $\overline{x}\in X$ *w.r.t.* η*.*

Next, we present for the vector optimization problem (P_v) a result referring to the connections between the efficient solutions and the saddle points for the Lagrangian (due to [89]).

Theorem 5.2.4 (*E.-L. Pop and D.I. Duca [111]*) *Let X be a subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^q$ functions. If $(\overline{x}, \overline{v}, \overline{w}) \in X \times \mathbb{R}^m_+ \times \mathbb{R}^q$ is a saddle point for the vector Lagrangian L_{P_v} of Problem (P_v), then $\overline{x}$ is an efficient solution for Problem (P_v).*

Theorem 5.2.8 (*E.-L. Pop and D.I. Duca [111]*) *Let X be a subset of $\mathbb{R}^n$, $\overline{x}$ be an interior point of X, $\eta : X \times X \to X$ be a function, $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^q$ other functions. Assume that:*

(i) the functions f, g and h are differentiable at $\overline{x}$;

(ii) the functions f and g are invex at $\overline{x}$ w.r.t. η;

(iii) the function h is avex at $\overline{x}$ w.r.t. η;

(iv) a suitable constraint qualification for Problem (P_v) is fulfilled at $\overline{x}$.

If $\overline{x}$ is an efficient solution for Problem (P_v), then there exists a point $(\overline{u}, \overline{v}, \overline{w}) \in (\mathbb{R}^p_+ \setminus \{0\}) \times \mathbb{R}^m_+ \times \mathbb{R}^q$ such that $(\overline{x}, \overline{v}, \overline{w})$ is a saddle point for the scalar Lagrangian $L^{\overline{u}}_{P_v}$ of Problem (P_v).

Now we refer to the connections between the feasible solutions of Problem (P_v) and of Problem (AP_v) (cf. [111]).

Lemma 5.2.10 (*E.-L. Pop and D.I. Duca [111]*) *Let X be a subset of $\mathbb{R}^n$, $\overline{x}$ be an interior point of X,$\eta : X \times X \to X$ be a function, $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^q$. If:*

(i) the function g is differentiable at $\overline{x}$ and incave at $\overline{x}$ w.r.t. η;

(ii) the function h is differentiable at $\overline{x}$ and avex at $\overline{x}$ w.r.t. η,

then $\mathcal{F}(AP_v) \subseteq \mathcal{F}(P_v)$.

Theorem 5.2.12 (*E.-L. Pop and D.I. Duca [111]*) *Let X be a subset of $\mathbb{R}^n$, $\overline{x}$ be an interior point of X, $\eta : X \times X \to X$, $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^q$ functions. Assume that:*

(i) the functions g and h are differentiable at $\overline{x}$;

(ii) the function g is invex at $\overline{x}$ w.r.t. η;

(iii) the function h is avex at $\overline{x}$ w.r.t. η;

(iv) $\eta(\overline{x}, \overline{x}) = 0$.

(a) If $(\overline{x}, \overline{v}, \overline{w})$ is a saddle point for the vector Lagrangian L_{AP_v} of Problem (AP_v), then $(\overline{x}, \overline{v}, \overline{w})$ is a saddle point for the vector Lagrangian L_{P_v} of Problem (P_v).

(b) If $(\overline{x}, \overline{v}, \overline{w})$ is a saddle point for the scalar Lagrangian $L^{\overline{u}}_{AP_v}$ of Problem (AP_v), then $\overline{x}$ is an efficient solution for Problem (P_v).

Theorem 5.2.14 (*E.-L. Pop and D.I. Duca [111]*) *Let X be a subset of $\mathbb{R}^n$, $\overline{x}$ be an interior point of X, $\eta : X \times X \to X$ be a function, and $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^q$ other functions. Assume that:*

(i) the function f is differentiable at $\overline{x}$;

(ii) the function g is differentiable at $\overline{x}$ and incave at $\overline{x}$ w.r.t. η;
(iii) the function h is differentiable at $\overline{x}$ and avex at $\overline{x}$ w.r.t. η;
(iv) $\overline{x} \in \mathcal{F}(AP_v)$.
If $\overline{x}$ is an efficient solution for Problem (P_v), then $\overline{x}$ is an efficient solution for Problem (AP_v).

Other new connections between the efficient solutions and the saddle points for the Lagrangian of the vector optimization problems (P_v) and $(0,1)-\eta$ approximated vector optimization problem (AP_v) can be formulated, also.

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