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Mixed Boundary Value Problems for Nonlinear Systems in Fluid Mechanics and Porous Media

Ph.D. Thesis - Summary

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Contents

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Introduction

The purpose of the present thesis is to study boundary value problems of mixed type (Dirichlet-Neumann, Dirichlet-Robin) for various elliptic systems in fluid mechanics and the theory of porous media. In order to do so, other types of boundary value problems (BVPs) are also examined, concerning Dirichlet, Neumann and Robin boundary conditions.

Let $\mathfrak{D} \subset \mathbb{R}^n, n \geq 2$ be a Lipschitz domain, wherein a viscous incompressible fluid is located and let Γ denote the boundary, $\Gamma := \partial\mathfrak{D}$. For a given constant $\alpha > 0$, *the normalized Brinkman system* consists of the following equations

$$\triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \quad \text{in } \mathfrak{D} \tag{0.0.1}$$

where $\mathbf{u}$ is *the velocity field* and π is *the pressure field* of the fluid flow under consideration. Moreover, $\mathbf{f}$ is called *the driving force*, which acts upon the fluid flow. For $\alpha = 0$ the system (0.0.1) reduces to *the normalized Stokes system* consisting of the following equations

$$\triangle \mathbf{u} - \nabla \pi = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \quad \text{in } \mathfrak{D}. \tag{0.0.2}$$

For problems in which the inertia of the fluid is not negligible, one has to consider *the Darcy-Forchheimer-Brinkman system*, given by

$$\triangle \mathbf{u} - \alpha \mathbf{u} - \kappa |\mathbf{u}| \mathbf{u} - \beta (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \pi = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \quad \text{in } \mathfrak{D}. \tag{0.0.3}$$

Note that the physical properties of the fluid and the properties of the porous medium in which it is located are described by the parameters $\alpha, \kappa, \beta > 0$ (see Chapter 11. for a discussion of the numerical results related to these systems and the description in [87, p. 17]).

When the nonlinear term does not appear in the expression of (0.0.3), i.e., the constant $\beta = 0$, we say that we consider *the semilinear Darcy-Forchheimer-Brinkman system.* Similarly, in the case $\kappa = 0$, we highlight the absence of the semilinear term by calling the system *the nonlinear Darcy-Forchheimer-Brinkman system.* Another special case of system (0.0.3) is the well-known *Navier-Stokes system* obtained when $\alpha = 0$ and $\kappa = 0$, i.e.,

$$\triangle \mathbf{u} - \beta (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \pi = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \quad \text{in } \mathfrak{D}. \tag{0.0.4}$$

Moreover, the boundary conditions associated to the above systems can be of Dirichlet, Neumann or Robin type over the whole boundary

$$\gamma^+ \mathbf{u} = \mathbf{h}, \quad \mathbf{t}_\alpha^+(\mathbf{u}, \pi) = \mathbf{g}, \quad \mathbf{t}_\alpha^+(\mathbf{u}, \pi) + \lambda \gamma^+ \mathbf{u} = \mathbf{l} \ \text{ on } \Gamma, \tag{0.0.5}$$

where the trace operator γ^+, the conormal derivative operator $\mathbf{t}_\alpha^+$ and a matrix value function λ are given in the sequel. Let us point out that the "sense" in which these operators are considered, plays a fundamental role in the understanding of the various boundary value problems throughout this thesis. On the other hand, in order to formulate mixed boundary value problems, the boundary Γ of the Lipschitz domain $\mathfrak{D}$ needs to be decomposed into two parts Γ_D and Γ_N. The precise formulations are given in Chapter 3 and Chapter 4, respectively, depending on the regularity of the boundary data considered.

Before we begin with a brief historical overview of the scientific literature related to such boundary value problems, let us mention the importance of these boundary problems regarding special mixed type boundary conditions. Many engineering problems nowadays deal from the mathematical point of view with such boundary value problems (see, e.g., [5, 6, 11, 77, 85, 86, 7]). The flow of a fluid in a pipe is modeled often by the Navier-Stokes equations and Dirichlet type conditions are imposed on the pipe walls, whereas the inlet of the fluid is given by Dirichlet or Neumann boundary conditions depending on the known data, while the outlet is mostly simulated by Neumann boundary conditions based on the pressure of the fluid. Depending on the surface tension between the fluid and the pipe wall, an additional sliding parameter can be imposed and leads to Robin boundary conditions. Other interesting practical applications, which are directly related to the problems that are analyzed in this thesis, can be consulted in [46], [76], [109], [31, 87].

Various different methods have been employed in order to study boundary value problems in fluid mechanics, such as the variational approach [61], methods of potential theory and boundary domain integral methods based on the parametrix (or the Levi function). We begin by mentioning the work of Fabes, Kenig and Verchota [30], which reduces the study of the Dirichlet and Neumann problems for the Stokes system to the analysis of some related boundary integral equations (BIEs). They proved the well-posedness for the Regularity and Neumann problems when the boundary data belong to L^p-based spaces, with p near 2. Interpolating the weak maximum result obtained by Dahlberg and Kenig in [24] with the result in [30], Shen obtained in [101] the well-posedness result of the Dirichlet problem for the Stokes system with data in L^p whenever $2 < p < \infty$. Mitrea and Wright [86] obtained several well-posedness results for a large spectrum of boundary problems for the Stokes system in Lipschitz domains, with boundary data in various spaces.

Regarding the Laplace system, I. Mitrea and M. Mitrea provided sharp well-posedness results in [83] for the Poisson problem with mixed Dirichlet-Neumann boundary conditions on bounded Lipschitz domains in $\mathbb{R}^3$, where the boundaries satisfy a special geometric condition (related to the notion of creased Lipschitz domain), and where the data belongs to Sobolev and Besov spaces. Pipher and Verchota [90] and Dahlberg and Kenig [22] constructed the Green function for the biharmonic system with Dirichlet-Neumann boundary conditions in Lipschitz domains in three dimensional Euclidean settings. Taylor, Ott and Brown in [106] studied mixed Dirichlet-Neumann problem in L^p-based Sobolev spaces for the Laplace equation in a bounded Lipschitz domain in $\mathbb{R}^n$ with general decomposition of the boundary. Costabel and Stephan in [21] analyzed mixed boundary value problems for the Laplacian in polygonal domains by using a boundary integral approach. Precup [93] obtained existence and localization results of positive nontrivial solutions for semilinear elliptic variational systems based on the Laplace equation.

Moreover, let us mention that similar well-posedness results were obtain for the Lamé system by Dahlberg and Kenig in [24]. Using the well-posedness result for the mixed Dirichlet-Neumann boundary value problem for the Lamé system obtained by Brown and Mitrea [12], Brown et al. [13] provided well-posedness results for the mixed Dirichlet-Neumann problem for the Stokes system on creased Lipschitz domains in $\mathbb{R}^3$, by reducing its study to that of a family of Fredholm operators related to the the Lamé system and to some useful Rellich-type estimates.

The solvability of the mixed *Dirichlet-Robin* problem for the Brinkman system in a creased domain with boundary data in L^2-based spaces has been provided by Kohr, Lanza de Cristoforis and Wendland in [51, Theorem 6.1]. Ott, Kim and Brown [88] constructed the Green function for the linear Stokes system in a Lipschitz domain in $\mathbb{R}^2$, by imposing some conditions on the decomposition of the boundary. Also, we mention that the Laplace system on a Lipschitz domain with a general decomposition of the boundary has been treated in [106]. Boundary integral equations for a mixed boundary value problem for the biharmonic equation has been developed by Cakoni, Hsiao and Wendland in [15] (see also [23]).

Combined methods, such as boundary integral methods and fixed point theorems have been successfully employed in the analysis of boundary value problems for linear elliptic systems with nonlinear boundary conditions and for nonlinear elliptic systems. Recently, Kohr, Lanza de Cristoforis and Wendland [50] applied boundary integral methods to get existence results of nonlinear Neumann-transmission type problems for the Stokes and Brinkman systems on Lipschitz domains and with boundary data in various L^p, Sobolev, or Besov spaces. Layer potential techniques were employed to the Stokes and Brinkman systems in [53] in order to analyze Poisson problems for semilinear Brinkman systems on Lipschitz domains in $\mathbb{R}^n$ with Dirichlet or Robin boundary conditions in various Sobolev and Besov spaces. An integral potential method for transmission problems with Lipschitz interface in $\mathbb{R}^3$ for the Stokes and Darcy-Forchheimer-Brinkman systems and boundary data in weighted Sobolev spaces has been proven in [49].

In [18, 19], Chkadua, Mikhailov and Natroshvili analysed direct segregated systems of boundary-domain integral equations, which are equivalent to mixed Dirichlet-Neumann problem for a scalar second-order divergent elliptic partial differential equation (PDE) with variable coefficients in interior and exterior domains in $\mathbb{R}^3$ (see also [17] for the mixed problems with cracks and [80] for united boundary-domain integral equations). Moreover, Kohr, Mikhailov and Wendland [56] obtained well-posedness of transmission problems in L^p-based weighted Sobolev spaces for the Stokes and Navier-Stokes systems with anisotropic L^∞ strongly elliptic coefficients, located in complementary Lipschitz domains of $\mathbb{R}^n$, $(n \geq 3)$, by employing a variational approach.

The thesis is structured into three parts. The goal of the first part is to study mixed boundary value problems for the Brinkman and Darcy-Forchheimer-Brinkman systems on a Lipschitz domain in the Euclidean setting. To this end, we require some well-posedness results for some of the main boundary value problems, such as the Dirichlet, Neumann or Robin boundary problems for the mentioned systems. In the second part, we extend the previous results to the setting of compact Riemannian manifolds. We analyze the well-posedness results for mixed Dirichlet-Neumann boundary value problems for the Stokes system, the Oseen system and the Navier-Stokes system in Lipschitz domains on compact Riemannian manifolds. Moreover, we obtain also existence and uniqueness results for the transmission problem for the Oseen system and the Brinkman system. The last part is concerned with numerical results related to the well-posedness results obtained in the previous parts. We study the special case of the lid-driven cavity problem for a rectangular cavity in two dimensions and analyze the dependence of the main physical parameters for the fluid flow.

Part I is divided into four chapters related to boundary value problems in Euclidean settings.

- **Chapter 1** is an introduction of the main definitions, notations, spaces and operators that are needed throughout this thesis. We begin with the definitions of the Lipschitz domain, the creased Lipschitz domain and the dissection of the boundary needed for mixed boundary value problems. We continue with a brief presentation of the Lebesgue, Sobolev and also Besov spaces defined on $\mathbb{R}^n$, on Lipschitz domains, on Lipschitz boundaries and on subsets of Lipschitz boundaries or admissible patches. Next, we introduce the trace operators and the conormal derivative operators related to the Brinkman and Stokes systems. Let us remark that the equivalence theorem between the non-tangential and the Gagliardo trance operator, as well as the equivalence between the non-tangential, canonical and classical conormal derivative operators are obtained in [41, Theorems 2.5 and 2.13].

- **Chapter 2** begins with the introduction of the fundamental solution for the Brinkman system, which enables one to define the main layer potential operators, such as the Newtonian, the single- and the double-layer operators. We investigate the mapping properties of these layer potential operators in various spaces, the jump relations over the boundary of the Lipschitz domain and the Green representation formula for the solution for the Brinkman system, as they were obtained in [41, Section 3].

- **Chapter 3** is concerned with boundary value problems of mixed Dirichlet-Neumann type for the linear Brinkman system and the semilinear Darcy-Forchheimer-Brinkman system on creased Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, when the boundary data belong to L^p-based Sobolev spaces with p in a neighborhood of 2, as they have been obtained by us in [41, Section 5 and 6]. In order to derive the well-posedness results for the two systems under consideration, we require the well-posedness result for the Dirichlet and Neumann boundary problems for the Brinkman system with boundary data in L^p-based spaces (for related results, we refer to [100, Theorem 5.5], [86, Corollary 9.1.5, Theorems 9.1.4, 9.2.2 and 9.2.5] and [85, Theorem 7.1]). Special attention is given to the "meaning" of the trace operator and the conormal derivative operator, i.e., we mention the cases when the operators are considered in the non-tangential sense or in the Gagliadro and canonical sense (we refer to Sections 1.5 and 1.6 of Chapter 1, or to [41, Sections 2.1 and 2.2]).

- In **Chapter 4** we focus our attention on the weak solution of boundary value problems for the Brinkman system on Lipschitz domains located in $\mathbb{R}^2$. Thus, we are able to consider a variational formulation for the direct boundary integral equations derived from the mixt boundary value problems. The main sources for this chapter are [37] and [40], but we refer the reader also to [61], [76]. Moreover, the assumption that the boundary data belong to some fractional Sobolev spaces enables us to study the mixed Dirichlet-Robin boundary value problems for the nonlinear Darcy-Forchheimer-Brinkman system (cf., e.g., [37, Theorem 2.9], see also [40, Theorem 3.2] for the Robin problem).

Part II is structured into two chapters and present boundary value problems in fluid mechanics on compact Riemannian manifolds. We study mixed Dirichlet-Neumann boundary value problems for the Stokes, Oseen and Navier-Stokes systems on an infinitely smooth, compact, boundaryless Riemannian manifold M of dimension $m \geq 2$. However, we are restricted to the dimension $m = 2$ or 3, whenever the Navier-Stokes system is involved, due to some compactness embedding results that are required in our analysis.

The main elliptic operator that appears in the structure of the Stokes system on a compact Riemannian manifold is

$$L := \mathrm{Def}^*\mathrm{Def} = -\Delta + d\delta - 2\mathrm{Ric}, \tag{0.0.6}$$

where Def is the deformation operator, $\Delta := -(d\delta + \delta d)$ is the Hodge Laplacian, which is defined in terms of the exterior derivative operator d and the exterior co-derivative operator δ and where Ric is the Ricci tensor of M. Then *the "incompressible" Stokes* system on a smooth, compact Riemannian manifold is given by the following linear PDE system

$$L\mathbf{u} + d\pi = 0, \quad \delta\mathbf{u} = 0 \text{ in } \mathfrak{D}, \tag{0.0.7}$$

where the unknowns $\mathbf{u}$ and π can be considered as the velocity and pressure fields of a fluid flow in a Lipschitz domain $\mathfrak{D} \subset M$. Similarly, for $\alpha > 0$, *the Brinkman system* is described by the following equations

$$L\mathbf{u} + \alpha\mathbf{u} + d\pi = 0, \quad \delta\mathbf{u} = 0, \text{ in } \mathfrak{D}. \tag{0.0.8}$$

For a fixed divergence-free vector field ω, *the non-homogeneous Oseen system* consists of the following equations

$$L\mathbf{u} + \nabla_\omega\mathbf{u} + d\pi = \mathbf{f}, \quad \delta\mathbf{u} = 0, \text{ in } \mathfrak{D}, \tag{0.0.9}$$

where ∇ is the Levi-Civita connection on M (for further details of these operators we refer to Chapter 5). For $\beta > 0$, the nonlinear system

$$L\mathbf{u} + \beta\nabla_{\mathbf{u}}\mathbf{u} + d\pi = \mathbf{f}, \quad \delta\mathbf{u} = 0, \text{ in } \mathfrak{D}, \tag{0.0.10}$$

is called *the Navier-Stokes system.* Here $\nabla_{\mathbf{u}}\mathbf{u}$ is the covariant derivative of $\mathbf{u}$ with respect to $\mathbf{u}$. As in the Euclidean setting, the physical properties of a fluid flow modeled by the system (0.0.10) are described by the constants α and β.

The study of fluid flows on compact, smooth Riemannian manifolds plays an important role in the analysis of the fundamental equations of meteorology and oceanography as pointed out in [109, 71] (see also [108, 26]). Also, other types of flow equations, e.g., the Stokes system or the Darcy-Forchheimer-Brinkman system, can be considered over compact surfaces (e.g., on the sphere S^2) and model flow of water or other viscous fluids, passing through porous rocks or porous soil (cf, e.g., [55]). Mixed boundary conditions describe in a most intuitive way the behavior of a shallow ocean, where the shore are represented by homogeneous Dirichlet conditions, the inlet streams by Dirichlet or Neumann conditions and the output flow by Neumann boundary conditions, since they are normally described by pressure outlets.

Boundary integral methods have also been used for the study of boundary value problems for elliptic systems on compact Riemannian manifolds. Note that the Dirichlet problem for the Stokes system on arbitary Lipschitz domains, with boundary data in L^2 based spaces has been studied by Mitrea and Taylor [85]. Moreover, Dindós and Mitrea [25] studied the well-posedness of the Poisson problem for the Stokes and Navier-Stokes systems on C^1 and Lipschitz domains on compact Riemannian manifolds by using boundary integral methods.

Mitrea and Taylor [85] and Dindos and Mitrea [25] have used the theory of pseudodifferential operators in order to show the existence of the fundamental solution for the Stokes system on compact Riemannian manifolds. One of the main assumptions needed in order to construct the fundamental solution for the Stokes system, is the assumption that the manifold lacks nontrivial Killing fields (see Definition 5.1.3), which guarantees that the deformation operator Def given in (5.1.14) is invertible. The assumption that the Riemannian manifold has no nontrivial Killing fields imposes no restrictions, since the manifold can be altered in order to satisfy this condition. A proof of this fact can be found at the beginning or Section 3 in [85].

An alternative technique to that of Mitrea and Taylor [85] has been developed by Kohr, Pintea and Wendland [57, Section 3] (see also [58]) in order to obtain the fundamental solution in the general case of Agmon-Douglis-Niremberg elliptic operators on compact Riemannian manifolds.

In their recent work, Kohr and Wendland [63] have developed the potentials theory for the Stokes system with non-smooth coefficients of class L^∞ on compact Riemannian manifolds, starting from a variational method. In the particular case of the smooth coefficients, the authors found many of the results that have been previously obtained by Mitrea, Taylor [85]. Kohr and Wendland [62, Theorem 7.9] have obtained well-posedness results on compact Riemannian manifolds for the nonhomogeneous Poisson problem of mixed type for the Brinkman system with nonsmooth coefficient, when the solution belongs to some L^p-based Sobolev spaces with p in a neighborhood of 2. Kohr and Wendland [63] have recently obtained the equivalence between some transmission problems for the Stokes system with nonsmooth coefficient in complementary Lipschitz domains on compact Riemannian manifolds, by employing the remarkable Nečas-Babuška-Brezzi technique and by proving a well-posedness result of their mixed variational counterparts.

Let us mention, that the study in this second part is based on a powerful approach based on layer potential theory and indirect variational methods in order to obtain well-posedness results for various boundary value problems.

This part of the thesis is structured into two chapters as follows:

- **Chapter 5** begins with the main geometrical definitions and concepts that are needed in the study of boundary value problems on compact Riemannian manifolds. An important part is the definitions of the Levi-Civita connection ∇, the deformation operator Def and the second order elliptic differential operator L (see also (0.0.7)). Next, we introduce the fundamental solution for the Stokes system and the associated layer potential operators. The chapter ends with some original results regarding the invertibility of the single-layer potential operator and the hypersingular potential operator obtained in [38, Theorem 4.2], as well as some compactness properties of the double layer potential operators (cf. [38, Theorem 4.3]) related to a part of the boundary decomposition.

- **Chapter 6** is concerned with boundary value problems for the Stokes, Oseen and Navier-Stokes systems on compact Riemannian manifolds. In this summary, we study boundary value problems with mixed Dirichlet-Neumann boundary conditions for the Stokes (cf. [38, Theorem 4.1] and for the Oseen system [39]. Based on the well-posedness result for the Oseen system, we continue with the analysis of the nonlinear Navier-Stokes system by employing a fixed point theorem and the well-posedness result for the Oseen system. These results are included in our paper [39].

Part III contains the last chapter of this thesis. The purpose of this part is to give some numerical examples and to describe the physical behavior of a fluid flow modeled by boundary value problems for the nonlinear Darcy-Forchheimer-Brinkman system, for which a well-posedness result has been obtained in Chapter 4. Therefore, the analysis described in Chapter 4 represents a mathematical background of the practical problem under consideration. The problem that is analyzed in Chapter 7 consists of a square cavity filled with a fluid with three rigid walls on which non-slip boundary conditions are imposed and a tangentially moving lid with unit velocity. This problem is known in literature as the *lid driven cavity flow problem*, denoted by short *lid problem* in this thesis.

First, we give a brief description of the two numerical methods used to analyze some boundary value problems in this thesis: the central-difference method and the boundary element method. Then we focus our attention on a special two-dimensional problem, which consists out of a rectangular cavity filled with a porous domain in which the fluid is driven by the movement of the upper wall.

The lid problem has been the subject of many physical, theoretical and numerical studies since this problem has simple geometry and connects the relevant physical aspects to mathematical models and computational methods. The work of U. Ghia, K. Ghia and Shin [31] uses a strongly implicit coupled multigrid method and became a benchmark reference over the years.

Certainly, by far the most used numerical technique is the finite difference method, due to the special geometry of the square cavity under consideration, which makes this method ideal for this problem ([14], [36], [99], [98]). In their work Erturk, Corke and Gokcol [28] computed steady solutions for the lid problem in the case of the Navier-Stokes system with a Reynolds number up to $Re = 20000$. Though many numerical studies were made, very few experimental studies related to the lid problems are available in literature and we mentions those of Koseff and Street [67], [66].

Boundary element methods have been widely used to solve different engeneering problems, related to fluid mechanics and other areas of interest. Let us mention relevant books by Brebbia and Telles [9], Brebbia and Wroble [89] and Katsikadelis [48], related to boundary element methods, as well as to the Dual Reciprocity Boundary Element Method (DRBEM).

The outline of the chapter is the following:

- **Chapter 7** presents the above mentioned numerical methods used in order to study some particular problems related to mixed boundary value problems in fluid mechanics. We start with a brief description of the non-dimensional form of the Navier-Stokes and continuity equations (see, e.g., [44]) and the stream function-vorticity formulation (cf., e.g., [60]) for the nonlinear Darcy-Forchheimer-Brinkman system, which simplifies the numerical treatment of the equations in two dimensions. Also, we discuss the relations of the physical properties related to the fluid and the porous domain [87]. In order to validate the results obtained by the two numerical methods employed, central difference (CD) method and Dual Reciprocity Boundary Element Method (DRBEM), we consider that the fluid motion is governed by the Navier-Stokes system. In addition, we compare the results obtained with both methods with classical results in literature. Next, we describe some numerical results for the lid problem for the Darcy-Forchheimer-Brinkman system in two dimensions. Dirichlet and mixed Robin-Dirichlet boundary conditions are considered, described by the physical meaning of a sliding parameter (see, e.g., [44], [43]). The content of this chapter is based on our results obtained in [40, Section 4] and [37, Section 3].

The original results in the thesis have been included in the following papers:

- **R. Gutt**, M. Kohr, S.E. Mikhailov, W.L. Wendland, *On the mixed problem for the semilinear Darcy-Forchheimer-Brinkman PDE system in Besov spaces on creased Lipschitz domains*, Mathematical Methods in the Applied Sciences, 40 (18), 7780-7829, 2017, (**ISI**), DOI: 10.1002/mma.4562.
- **R. Gutt**, M. Kohr, C. Pintea, W.L. Wendland, *On the transmission problem for the Oseen and Brinkman systems on Lipschitz domains in compact Riemannian manifolds*, Mathematische Nachrichten, 289 (4), 2015, (**ISI**), DOI: 10.1002/mana.201400365
- **R. Gutt**, T. Groşan, *On the lid-driven problem in a porous cavity. A theoretical and numerical approach*, Applied Mathematics and Computation, 266:1070-1082, 2015, (**ISI**), DOI: 10.1016/j.amc.2015.06.038.
- **R. Gutt**, *Mixed boundary value problems for the Stokes system on compact Riemannian manifolds*, Mathematica Cluj, 60 (83):152-165 2018.
- **R. Gutt**, *BIE and BEM approach for the mixed Dirichlet-Robin boundary value problem for the nonlinear Darcy-Forchheimer-Brinkman system*, submitted, arXiv:1810.09543.
- **R. Gutt**, *Mixed boundary value problems for the Navier-Stokes system on compact Riemannian manifolds*, submitted.

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Part I

Mixed boundary value problems for the Brinkman and Darcy-Forchheimer-Brinkman systems in Euclidean setting

Chapter 1

Geometric concepts and functional settings

In this chapter we provide the main definitions, notations and fundamental properties that are needed in order to define and treat boundary value problems for the Stokes, Brinkman and Darcy-Forchheimer-Brinkman systems in the Euclidean setting that are studied in the first part of this thesis.

We begin with the geometrical concepts and definitions of a bounded Lipschitz domain and a bounded creased Lipschitz domain in $\mathbb{R}^n$, where the boundary value problems are considered. Note that the property of creased Lipschitz domain is needed when we consider boundary value problems whose solution should have high regularity [11], whereas lower regularity problems can be considered in bounded Lipschitz domains. Next, we recall the definitions of Sobolev, Bessel-potential and Besov spaces by following the presentations of [110], [113], [8, Sections 2-3]. Special attention is given to embedding theorems and interpolation results which are essential for the forthcoming study.

The next sections contain known results but also original results obtained in [41, Section 1.1], which refer to the two different types of trace operators, i.e., the nontangential and the Gagliardo trace operators, as well as the connections between them. The equivalence between these operators has been obtained in [41, Theorem 2.5]. Moreover, we introduce the Stokes and Brinkman operators by following [50, Section 2.2], and then we define the corresponding conormal derivative operators. The differences between the canonical, the generalized and the nontangential conormal derivative operators are explained in the next subsection, with emphasis on the cases in which they are equivalent (Theorem 1.3.3). A part of these results constitute original work obtained by joint work with M. Kohr, S. E. Mikhailov and W. L. Wendland in [41, Section 1.2].

1.1 Lipschitz domains in $\mathbb{R}^n$

The purpose of this section is to introduce the concept of a bounded Lipschitz domain, where we consider the main boundary value problems of this thesis. We also recall the definition of a creased Lipschitz domain as in, e.g., [13, 83], which plays a fundamental role for boundary value problems whose solution should have high regularity.

1.1.1 Bounded Lipschitz domains in $\mathbb{R}^n$

First, we recall the definition of a bounded Lipschitz domain and describe its main characteristics (cf., e.g., [83, Definition 2.1]). Any point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ can be written in the simplified form $x = (x', x_n)$, where $x' := (x_1, x'') \in \mathbb{R}^{n-1}$ and $x'' := (x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-2}$.

Definition 1.1.1. [25] Let $\mathfrak{D} \subset \mathbb{R}^n$ $(n \geq 2)$ be an open, connected and bounded set and let $\Gamma = \partial\mathfrak{D}$. We say that $\mathfrak{D}$ is a *Lipschitz domain*, if one can find $M > 0$ such that for each $x \in \Gamma$ there exists a coordinate system in $\mathbb{R}^n$ (isometric to the canonical one), $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, a radius $r > 0$, a cylinder $\mathcal{C}_r(x) := \{(y', y_n) : |y' - x'| < r, |y_n - x_n| < 2Mr\}$, and a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\|\nabla\phi\|_{L^\infty(\mathbb{R}^{n-1})} \leq M$, such that

$$\begin{aligned} \mathcal{C}_r(x) \cap \mathfrak{D} &= \{(y', y_n) : y_n > \phi(y')\} \cap \mathcal{C}_r(x), \\ \mathcal{C}_r(x) \cap \Gamma &= \{(y', y_n) : y_n = \phi(y')\} \cap \mathcal{C}_r(x). \end{aligned} \tag{1.1.1}$$

In the sequel, let $\mathfrak{D} \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded Lipschitz domain with connected boundary $\Gamma = \partial\mathfrak{D}$, and set $\mathfrak{D}_+ := \mathfrak{D}$ and $\mathfrak{D}_- := \mathbb{R}^n \setminus \overline{\mathfrak{D}}$.

Let $\kappa = \kappa(\Gamma) > 1$ be a fixed sufficiently large constant (see, e.g., [33]). Then the sets

$$\mathfrak{C}_\pm(x) := \{y \in \mathfrak{D}_\pm : \operatorname{dist}(x, y) < \kappa \operatorname{dist}(\mathrm{y}, \Gamma),\ \mathrm{x} \in \Gamma\}, \tag{1.1.2}$$

are *non-tangential approach cones* located in $\mathfrak{D}_+$ and $\mathfrak{D}_-$, respectively (see, e.g., [86]). Note that these concepts will be needed also in the definition of the non-tangential trace operators.

1.1.2 Dissection of the boundary of a Lipschitz domain

One of the main problems of interest of this thesis is the mixed boundary value problem for some elliptic systems. Next, we define the notion of *a dissection of the boundary* related to such boundary value problems.

Since we are working with this notion only in $\mathbb{R}^2$, we state the definition of a boundary dissection for a two-dimensional Lipschitz domain, since we analyze this particular case in Chapter 4.

Definition 1.1.2. Consider a bounded Lipschitz domain $\mathfrak{D} := \mathfrak{D}_+ \subset \mathbb{R}^2$ with connected boundary Γ, which is partitioned into nonempty subsets Γ_D, Λ and Γ_N, such that $\Gamma = \Gamma_D \cup \Lambda \cup \Gamma_N$. Moreover, we assume that Γ_D and Γ_N are disjoint, relatively open subset of Γ, having Λ as their common boundary points. For each $x \in \Lambda$, we require that there exist a coordinate system (x', x''), a coordinate cylinder $\mathcal{C}_r(x)$ centered in x, a Lipschitz function ϕ and a constant M_1 such that

$$\begin{aligned} \mathcal{C}_r(x) \cap \Gamma_D &= \{(y', y'') : y' > M_1,\ y'' = \phi(y')\} \cap \mathcal{C}_r(x), \\ \mathcal{C}_r(x) \cap \Gamma_N &= \{(y', y'') : y' < M_1,\ y'' = \phi(y')\} \cap \mathcal{C}_r(x), \end{aligned} \tag{1.1.3}$$

(see also [88], [62] in the case of a two-dimensional Lipschitz domain with a special decomposition of the boundary into two parts, one of them being an Ahlfors regular set). We say that Γ_D and Γ_N determine a *dissection* of the boundary Γ.

1.1.3 Creased Lipschitz domains in $\mathbb{R}^n$

We begin with the definition of special patches on the boundary Γ, in order to define the notion of creased a Lipschitz domain (cf. [83, Section 2]). Note that the condition (1.1.5) below for a creased Lipschitz domain plays an essential role when we deal with mixed boundary value problems whose solutions should have high regularity and accordingly for Dirichlet data in Sobolev spaces $H^1_p(S, \mathbb{R}^n)$. R. M. Brown discusses this issue in [11] for the mixed problem for the Laplace equation.

Let us now introduce the notion of a bounded creased Lipschitz domain (cf. [83, Definition 2.3], [41, Definition 6.2]).

Definition 1.1.3. Assume that $\mathfrak{D} \subset \mathbb{R}^n$ is a bounded Lipschitz domain with connected boundary Γ, and that $\Gamma_D, \Gamma_N \subset \Gamma$ are two non-empty, disjoint admissible patches such that $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \partial\Gamma_D = \partial\Gamma_N$ and $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \Gamma$. Then $\mathfrak{D}$ is *creased* if (*a*) There exist $m \in \mathbb{N}$, $a > 0$ and $z_i \in \Gamma$, $i = 1, \dots, m$, such that $\partial\mathfrak{D} \subset \cup_{i=1}^m B_a(z_i)$, where $B_a(z_i)$ is the ball of radius a and center at z_i.

(*b*) For any point z_i, $i = 1, \dots, m$, there exist a coordinate system $\{x_1, \dots, x_n\}$ with origin at z_i and a Lipschitz function ϕ_i from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ such that the set $\mathfrak{D}_i := \{(x', x_n) \in \mathbb{R}^n : x_n > \phi_i(x')\}$, whose boundary Γ_i admits the decomposition $\Gamma_i = \overline{\Gamma_{D_i}} \cup \overline{\Gamma_{N_i}}$, is a creased graph Lipschitz domain in the sense of Definition 6.2 in [41], and

$$\mathfrak{D} \cap B_{2a}(z_i) = \mathfrak{D}_i \cap B_{2a}(z_i),\ \Gamma_D \cap B_{2a}(z_i) = \Gamma_{D_i} \cap B_{2a}(z_i),\ \Gamma_N \cap B_{2a}(z_i) = \Gamma_{N_i} \cap B_{2a}(z_i). \quad (1.1.4)$$

The geometric meaning of the above definition is that the admissible patches Γ_D and Γ_N are separated by a Lipschitz interface ($\overline{\Gamma_D} \cap \overline{\Gamma_N}$ is a *crease* or *collision* manifold for $\mathfrak{D}$) and that Γ_D and Γ_N meet at an angle which is strictly less than π (cf., e.g., [11, 83]).

A main property of a (bounded or graph) creased Lipschitz domain is the existence of a function $\boldsymbol{\varphi} \in C^\infty(\overline{\mathfrak{D}})$ and of a constant $\delta > 0$ such that

$$\boldsymbol{\varphi} \cdot \boldsymbol{\nu} > \delta \text{ a.e. on } \Gamma_N, \quad \boldsymbol{\varphi} \cdot \boldsymbol{\nu} < -\delta \text{ a.e. on } \Gamma_D, \quad (1.1.5)$$

i.e., the scalar product $\boldsymbol{\varphi} \cdot \boldsymbol{\nu}$, between $\boldsymbol{\varphi}$ and the unit normal $\boldsymbol{\nu}$, changes the sign when moving from Γ_D to Γ_N (cf., e.g., [12, (1.122)]). For further details cencerning the geometric properties of Lipschitz domains we refer to [76], [41], [25].

1.2 Sobolev and Besov spaces in $\mathbb{R}^n$ and in Lipschitz domains

In this section we give a brief review of some basic notation and definitions related to the Sobolev, Bessel-potential, Sobolev-Slobodeckij and Besov spaces, with emphasis on the relations between these spaces. The main sources used in the preparation of this chapters are [2], [4], [8], [94], [76], [113].

1.2.1 Sobolev and Besov spaces in $\mathbb{R}^n$

Let $k \in \mathbb{N}_0$ and $p, p' \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then *the Sobolev space* $W_p^k(\mathbb{R}^n)$ is defined by (see , e.g., [94, Section 7.1]

$$W_p^k(\mathbb{R}^n) := \left\{ f \in L^p(\mathbb{R}^n) \ : \ \|f\|_{W_p^k(\mathbb{R}^n)} := \sum_{\nu \le k} \|\partial^\nu f\|_{L^p(\mathbb{R}^n)} < \infty \right\}. \quad (1.2.1)$$

The Sobolev space $W_p^{-k}(\mathbb{R}^n)$ is defined as the dual of $W_{p'}^k(\mathbb{R}^n)$. It is well-known that $C_0^\infty(\mathbb{R}^n)$ is dense in $W_p^k(\mathbb{R}^n)$, and moreover $W_p^k(\mathbb{R}^n)$ can be equivalently defined as the completion of the space of smooth functions with compact support with respect to the norm $\|\cdot\|_{W_p^s(\mathbb{R}^n)}$ given in (1.2.1).

For $s \in \mathbb{R}$, *the L^p-based Bessel potential* spaces $H_p^s(\mathbb{R}^n)$ and $H_p^s(\mathbb{R}^n, \mathbb{R}^n)$ are defined by

$$H_p^s(\mathbb{R}^n) := \{f \in \mathscr{S}'(\mathbb{R}^n) : (\mathbb{I} - \triangle)^{\frac{s}{2}} f \in L^p(\mathbb{R}^n)\} = \{f : J^s f \in L^p(\mathbb{R}^n)\}, \quad (1.2.2)$$

$$H_p^s(\mathbb{R}^n, \mathbb{R}^n) := \left\{ \mathbf{f} = (f_1, f_2, \dots, f_n) : f_i \in H_p^s(\mathbb{R}^n),\ j = 1, \dots, n \right\}, \quad (1.2.3)$$

where $J^s : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is *the Bessel potential operator of order s* defined by (see, e.g., [76, Chapter 3])

$$J^s f := \mathcal{F}^{-1}(\rho^s \mathcal{F} f), \quad \rho^s = (1 + |\xi|^2)^{\frac{s}{2}}. \quad (1.2.4)$$

Note that $H_p^s(\mathbb{R}^n)$ is a Banach space with respect to the norm (see, e.g., [4])

$$\|f\|_{H_p^s(\mathbb{R}^n)} = \|J^s f\|_{L^p(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(\rho^s \mathcal{F} f)\|_{L^p(\mathbb{R}^n)}. \quad (1.2.5)$$

1.2.2 Sobolev and Besov spaces in Lipschitz domains

Next, we define the Sobolev and Bessel-potential spaces on Lipschitz domains. To this end, let $\mathfrak{D} \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain, and set $\mathfrak{D}_+ := \mathfrak{D}$ and $\mathfrak{D}_- := \mathbb{R}^n \setminus \overline{\mathfrak{D}}$.

Let $\mathcal{D}(\mathfrak{D}_\pm) := C_0^\infty(\mathfrak{D}_\pm)$ be the space of infinitely differentiable functions with compact support in $\mathfrak{D}_\pm$, equipped with the inductive limit topology. The space $\mathcal{D}'(\mathfrak{D}_\pm)$ is the space of distributions defined as the topological dual of $\mathcal{D}(\mathfrak{D}_\pm)$. Throughout this thesis the notation $(\cdot)|_X$ means the restriction operator to the set $X \subset \mathbb{R}^n$. This operator is often denoted by r_X.

The Bessel potential spaces $H_p^s(\mathfrak{D}_\pm)$ and $\widetilde{H}_p^s(\mathfrak{D}_\pm)$ are defined by

$$H_p^s(\mathfrak{D}_\pm) := \{f \in \mathcal{D}'(\mathfrak{D}_\pm) : \exists\, F \in H_p^s(\mathbb{R}^n) \text{ such that } F|_{\mathfrak{D}_\pm} = f\}, \tag{1.2.6}$$

$$\widetilde{H}_p^s(\mathfrak{D}_\pm) := \left\{f \in H_p^s(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\mathfrak{D}_\pm}\right\}. \tag{1.2.7}$$

The Bessel potential spaces $H_p^s(\mathfrak{D}_\pm, \mathbb{R}^n)$ and $\widetilde{H}_p^s(\mathfrak{D}_\pm, \mathbb{R}^n)$ are defined as the spaces of vector-valued functions (distributions) whose components belong to the spaces $H_p^s(\mathfrak{D}_\pm)$ and $\widetilde{H}_p^s(\mathfrak{D}_\pm)$, respectively (see, e.g., [76]).

For any $s \in \mathbb{R}$, $C^\infty(\overline{\mathfrak{D}_\pm})$ is dense in $H_p^s(\mathfrak{D}_\pm)$ and the following duality relations hold (cf., e.g., [47, Proposition 2.9], [29, (1.9)], [84, (4.14)])

$$\left(H_p^s(\mathfrak{D}_\pm)\right)' = \widetilde{H}_{p'}^{-s}(\mathfrak{D}_\pm), \quad H_{p'}^{-s}(\mathfrak{D}_\pm) = \left(\widetilde{H}_p^s(\mathfrak{D}_\pm)\right)'. \tag{1.2.8}$$

Here and further on, for $p \in (1, \infty)$ given, p' denots the conjugate exponent given by

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{1.2.9}$$

Similar to (1.2.6) and (1.2.7), for $s \in \mathbb{R}$ and $p, q \in (1, \infty)$ the Besov spaces $B_{p,q}^s(\mathfrak{D}_\pm)$ and $B_{p,q}^s(\mathfrak{D}_\pm, \mathbb{R}^n)$ are defined by

$$B_{p,q}^s(\mathfrak{D}_\pm) := \{f \in \mathcal{D}'(\mathfrak{D}_\pm) : \exists\, F \in B_{p,q}^s(\mathbb{R}^n) \text{ such that } F|_{\mathfrak{D}_\pm} = f\}, \tag{1.2.10}$$

$$B_{p,q}^s(\mathfrak{D}_\pm, \mathbb{R}^n) := \{\mathbf{f} = (f_1, f_2, \ldots, f_n) : f_i \in B_{p,q}^s(\mathfrak{D}_\pm), j = 1, \ldots, n\}, \tag{1.2.11}$$

$$\widetilde{B}_{p,q}^s(\mathfrak{D}_\pm, \mathbb{R}^n) := \left\{\tilde{\mathbf{f}} \in B_{p,q}^s(\mathbb{R}^n, \mathbb{R}^n) : \operatorname{supp} \tilde{\mathbf{f}} \subseteq \overline{\mathfrak{D}_\pm}\right\}. \tag{1.2.12}$$

1.2.3 Sobolev and Besov spaces on Lipschitz boundaries

The main sources in the preparation of this part are [46, Section 4.3], [113].

For $s \in (0, 1)$, we can define the space $H_p^s(\Gamma)$ as the completion of the space

$$C_s^0 = \left\{f \in C^0(\Gamma) : \|f\|_{H_p^s(\Gamma)} < \infty\right\} \tag{1.2.13}$$

with respect to the norm

$$\|f\|_{H_p^s(\Gamma)} := \left\{\|f\|_{L^p(\Gamma)}^p + \int_\Gamma \int_\Gamma \frac{|f(x) - f(y)|^p}{|x - y|^{N-1+ps}} d\sigma_x d\sigma_y\right\}^{\frac{1}{p}}. \tag{1.2.14}$$

For $p \in (1, \infty)$ and $s \in (-1, 0)$, we have $H_{p'}^{-s}(\Gamma) = \left(H_p^s(\Gamma)\right)'$. Also, note that $H_p^0(\Gamma) = L^p(\Gamma)$.

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ be the outward unit normal to $\mathfrak{D}$, which is defined almost everywhere with respect to the surface measure $d\sigma$ on Γ.

Let $p \in (1, \infty)$, $q \in (1, \infty]$ and $s \in (0, 1]$. In the sequel, we need the following spaces.

$$L_{\boldsymbol{\nu}}^p(\Gamma, \mathbb{R}^n) := \left\{\mathbf{v} \in L^p(\Gamma, \mathbb{R}^n) : \int_{\partial\Omega} \mathbf{v} \cdot \boldsymbol{\nu} d\sigma = 0\right\},$$

$$
\begin{aligned}
H^s_{p;\boldsymbol{\nu}}(\Gamma,\mathbb{R}^n) &:= \left\{\mathbf{v}\in H^s_p(\Gamma,\mathbb{R}^n) : \int_\Gamma \mathbf{v}\cdot\boldsymbol{\nu}d\sigma = 0\right\},\\
B^s_{p,q;\boldsymbol{\nu}}(\Gamma,\mathbb{R}^n) &:= \left\{\mathbf{v}\in B^s_{p,q}(\Gamma,\mathbb{R}^n) : \int_\Gamma \mathbf{v}\cdot\boldsymbol{\nu}d\sigma = 0\right\}. \qquad (1.2.15)
\end{aligned}
$$

Moreover, we also need the following subspaces.

$$
\begin{aligned}
H^s_{p;\mathrm{div}}(\mathfrak{D},\mathbb{R}^n) &:= \left\{\mathbf{v}\in H^s_p(\mathfrak{D},\mathbb{R}^n) : \mathrm{div}\ \mathbf{v} = 0\right\},\\
B^s_{p,q;\mathrm{div}}(\mathfrak{D},\mathbb{R}^n) &:= \left\{\mathbf{v}\in B^s_{p,q}(\mathfrak{D},\mathbb{R}^n) : \mathrm{div}\ \mathbf{v} = 0\right\}. \qquad (1.2.16)
\end{aligned}
$$

Finally, let us introduce some special subspaces of locally integrable functions. Let Ω be an open set (in particular $\Omega\in\{\mathbb{R}^n,\mathfrak{D},\Gamma\}$). The spaces of locally integrable functions $L^p_{\mathrm{loc}}(\Omega)$,$H^s_{p;\mathrm{loc}}(\Omega)$ and $B^s_{p,q;\mathrm{loc}}(\Omega)$ are defined as

$$
\begin{aligned}
L^p_{\mathrm{loc}}(\Omega,\mathbb{R}^n) &:= \left\{\mathbf{v}\in L^p(\Omega,\mathbb{R}^n) : \mathbf{v}|_K\in L^p(K,\mathbb{R}^n), \forall K\subset\Omega, K \text{ is compact}\right\},\\
H^s_{p;\mathrm{loc}}(\Omega,\mathbb{R}^n) &:= \left\{\mathbf{v}\in H^s_p(\Omega,\mathbb{R}^n) : \mathbf{v}|_K\in H^s_p(K,\mathbb{R}^n), \forall K\subset\Omega, K \text{ is compact}\right\},\\
B^s_{p,q;\mathrm{loc}}(\Omega,\mathbb{R}^n) &:= \left\{\mathbf{v}\in B^s_{p,q}(\Omega,\mathbb{R}^n) : \mathbf{v}|_K\in B^s_{p,q}(K,\mathbb{R}^n), \forall K\subset\Omega, K \text{ is compact}\right\}. \qquad (1.2.17)
\end{aligned}
$$

1.3 The Trace operators

As in the previous section, let us consider $\mathfrak{D}\subset\mathbb{R}^n$ $(n\geq 3)$ be a bounded Lipschitz domain with connected boundary Γ, and set $\mathfrak{D}_+ := \mathfrak{D}$ and $\mathfrak{D}_- := \mathbb{R}^n\setminus\overline{\mathfrak{D}}$. Let $\kappa=\kappa(\Gamma)>1$ be a fixed sufficiently large constant. Then *the non-tangential maximal operator* of an arbitrary function $u:\mathfrak{D}_\pm\to\mathbb{R}$ is defined by (see, e.g., [41, Section 2.1], [50])

$$M(u)(x) := \{\sup|u(y)| : y\in\mathfrak{C}_\pm(x),\ x\in\Gamma\}, \qquad (1.3.1)$$

where $\mathfrak{C}_\pm$ are non-tangential approach cones given in (1.1.2) located in $\mathfrak{D}_+$ and $\mathfrak{D}_-$, respectively (see, e.g., [86]). Moreover,

$$u^\pm_{\mathrm{nt}}(x) := \lim_{\mathfrak{C}_\pm\ni y\to x} u(y) \qquad (1.3.2)$$

are the *non-tangential limits* of u with respect to $\mathfrak{D}_\pm$ at $x\in\Gamma$. Note that if $M(u)\in L^p(\Gamma)$ for one choice of κ, then this property holds for an arbitrary choice of κ (see, e.g., [78, p 63.]). We will use the notation $\mathfrak{C}_\pm$ instead of $\mathfrak{C}_{\kappa;\pm}$.

A useful result for the problems we are going to investigate in this thesis is the Gagliardo Trace Lemma that we mention below (see [20], [47, Proposition 3.3], [81, Theorem 2.3, Lemma 2.6], [79], [86, Theorem 2.5.2]).

Lemma 1.3.1. *Let $p\in(1,\infty)$ and $s\in(0,1)$ be given. Then there exist linear and continuous Gagliardo trace operators $\gamma^\pm : H_p^{s+\frac{1}{p}}(\mathfrak{D}_\pm)\to B^s_{p,p}(\Gamma)$ such that $\gamma^\pm g = g|_\Gamma$ for any $g\in C^\infty(\overline{\mathfrak{D}}_\pm)$. These operators are surjective and have non-unique, linear and continuous right inverse operators*

$$(\gamma^\pm)^{-1} : B^s_{p,p}(\Gamma)\to H_p^{s+\frac{1}{p}}(\mathfrak{D}_\pm). \qquad (1.3.3)$$

It is immediate that Lemma 1.3.1 remains valid also for vector-valued and matrix valued functions. The result below is a version of Lemma 1.3.1 for Besov spaces defined on Lipschitz boundaries (see, e.g., [86, Theorem 2.5.2]).

Lemma 1.3.2. *Let $p,q\in(1,\infty)$ and $s\in(0,1)$ be given. Then there exist linear and continuous Gagliardo trace operators $\gamma^\pm : B_{p,q}^{s+\frac{1}{p}}(\mathfrak{D}_\pm)\to B^s_{p,q}(\Gamma)$ such that $\gamma^\pm g = g|_\Gamma$ for any $g\in C^\infty(\overline{\mathfrak{D}}_\pm)$. These operators are surjective and have non-unique, linear and continuous right inverse operators $(\gamma^\pm)^{-1} : B^s_{p,q}(\Gamma)\to B_{p,q}^{s+\frac{1}{p}}(\mathfrak{D}_\pm)$.*

The following result answers the natural question when the nontangential trace operator coincides with the Gagliardo trace operator [41, Theorem 2.5]

Theorem 1.3.3. *Assume that $\mathfrak{D}$ is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $p, q \in (1, \infty)$.*

(i) If there exists a constant $s > 0$ with the property that $\mathbf{u} \in B_{p,q}^{s+\frac{1}{p}}(\mathfrak{D}, \mathbb{R}^n)$ and the pointwise nontangential trace $\mathbf{u}_{\rm nt}^+$ exists almost everywhere on Γ, then the trace of Gagliardo type $\gamma^+\mathbf{u}$ is defined on Γ and $\gamma^+\mathbf{u} = \mathbf{u}_{\rm nt}^+$. If $s \in (0,1)$, then $\gamma^+\mathbf{u} = \mathbf{u}_{\rm nt}^+ \in B_{p,q}^s(\Gamma, \mathbb{R}^n)$.

(ii) Moreover, if $\mathbf{u}_{\rm nt}^+ \in H_p^{s+\frac{1}{p}}(\Gamma, \mathbb{R}^n)$ for $s \in (0,1]$, then $\gamma^+\mathbf{u} \in H_p^s(\Gamma, \mathbb{R}^n)$.

1.4 The Stokes and the Brinkman operators

Next, we define the Brinkman operator which plays a key role throughout this thesis. Recall that $\mathscr{S}(\mathbb{R}^n, \mathbb{R}^n)$ is the space of Schwartz functions and $\mathscr{S}'(\mathbb{R}^n, \mathbb{R}^n)$ stands for the space of tempered distributions.

Let $\alpha > 0$ be a given constant. Let $\mathbb{B}_\alpha$ denote *the Brinkman operator* defined such as (see, e.g., [50, Section 2.2])

$$\mathbb{B}_\alpha := \begin{pmatrix} (\Delta - \alpha\mathbb{I}) & -\nabla \\ \operatorname{div} & 0 \end{pmatrix} : \mathscr{S}(\mathbb{R}^n, \mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n, \mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \tag{1.4.1}$$

and the associated operator $\mathcal{L}_\alpha$ defined by

$$\mathcal{L}_\alpha(\mathbf{u}, \pi) := (\Delta - \alpha\mathbb{I})\mathbf{u} - \nabla\pi : \mathscr{S}(\mathbb{R}^n, \mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n, \mathbb{R}^n). \tag{1.4.2}$$

For $\alpha = 0$, one obtains *the Stokes operator*. The operators $\mathbb{B}_\alpha$ and $\mathbb{B}_0$ defined in (1.4.1) are Agmon-Douglis-Nirenberg elliptic operators (cf., e.g., [50, Lemma 2.1], [46, p. 330-331]), Consequently, for $\alpha > 0$, $p, q \in (1, \infty)$ and $s \in (0, 1)$, these operators extend to linear and bounded operators on Sobolev (Bessel potential) spaces, as follows

$$\mathbb{B}_\alpha : H_p^{s+\frac{1}{p}}(\mathbb{R}^n, \mathbb{R}^n) \times H_p^{s+\frac{1}{p}-1}(\mathbb{R}^n) \to H_p^{s+\frac{1}{p}-2}(\mathbb{R}^n, \mathbb{R}^n) \times H_p^{s+\frac{1}{p}-1}(\mathbb{R}^n), \tag{1.4.3}$$

$$\mathcal{L}_\alpha : H_p^{s+\frac{1}{p}}(\mathbb{R}^n, \mathbb{R}^n) \times H_p^{s+\frac{1}{p}-1}(\mathbb{R}^n) \to H_p^{s+\frac{1}{p}-2}(\mathbb{R}^n, \mathbb{R}^n), \tag{1.4.4}$$

or with similar definitions on Besov spaces.

Another operator that plays a main role in the definition of various boundary value problems of Neumann, mixed, or transmission type, is the conormal derivative operator. In order to define this operator, we introduce the following spaces. For $s \in \mathbb{R}$, $p, q \in (1, \infty)$ and $t \geq -1/p'$, where p' is the conjugate exponent of p. Then we consider the following spaces (cf. [81, Definition 3.3])

$$\mathfrak{H}_{p,\rm div}^{s+\frac{1}{p},t}(\mathfrak{D}, \mathcal{L}_\alpha) := \Big\{(\mathbf{u}, \pi) \in H_p^{s+\frac{1}{p}}(\mathfrak{D}, \mathbb{R}^n) \times H_p^{s+\frac{1}{p}-1}(\mathfrak{D}) : \mathcal{L}_\alpha(\mathbf{u}, \pi) = \tilde{\mathbf{f}}|_{\mathfrak{D}},\ \tilde{\mathbf{f}} \in \widetilde{H}_p^t(\mathfrak{D}, \mathbb{R}^n) \\ \text{and } \operatorname{div}\mathbf{u} = 0 \text{ in } \mathfrak{D}\Big\}.$$

Also let

$$\mathfrak{B}_{p,q,\rm div}^{s+\frac{1}{p},t}(\mathfrak{D}, \mathcal{L}_\alpha) := \Big\{(\mathbf{u}, \pi) \in B_{p,q}^{s+\frac{1}{p}}(\mathfrak{D}, \mathbb{R}^n) \times B_{p,q}^{s+\frac{1}{p}-1}(\mathfrak{D}) : \mathcal{L}_\alpha(\mathbf{u}, \pi) = \tilde{\mathbf{f}}|_{\mathfrak{D}},\ \tilde{\mathbf{f}} \in \widetilde{B}_{p,q}^t(\mathfrak{D}, \mathbb{R}^n) \\ \text{and } \operatorname{div}\mathbf{u} = 0 \text{ in } \mathfrak{D}\Big\},$$

where $\mathcal{L}_\alpha(\mathbf{u}, \pi)$ is defined in (1.4.2).

Similar to Theorem 2.16 in [81], for $p, q \in (1, \infty)$ we introduce the operator $\mathring{E}_\pm$ of extension of functions from $H^t_p(\mathfrak{D}_\pm)$ by zero on $\mathbb{R}^n \setminus \mathfrak{D}_\pm$. Also for $0 \leq t < \frac{1}{p}$, let $\widetilde{E}_\pm$ be the operator defined on $H^t_p(\mathfrak{D}_\pm)$ as $\widetilde{E}_\pm := \mathring{E}_{\pm},$. If $-\frac{1}{p'} < t < 0$, then $\widetilde{E}_\pm$ is defined as

$$\langle \widetilde{E}_\pm h, v\rangle_{\mathfrak{D}_\pm} := \langle h, \widetilde{E}_\pm v\rangle_{\mathfrak{D}_\pm} = \langle h, \mathring{E}_\pm v\rangle_{\mathfrak{D}_\pm},\ h \in H^t_p(\mathfrak{D}_\pm),\ v \in H^{-t}_{p'}(\mathfrak{D}_\pm), \tag{1.4.5}$$

where $h \in H^t_p(\mathfrak{D}_\pm)$, $v \in H^{-t}_{p'}(\mathfrak{D}_\pm)$ or $h \in B^t_{p,q}(\mathfrak{D}_\pm)$, $v \in B^{-t}_{p',q'}(\mathfrak{D}_\pm)$, respectively.

Then, for $-1/p' < t < 1/p$, the operators

$$\widetilde{E}_\pm : H^t_p(\mathfrak{D}_\pm) \to \widetilde{H}^t_p(\mathfrak{D}_\pm), \quad \widetilde{E}_\pm : B^t_{p,q}(\mathfrak{D}_\pm) \to \widetilde{B}^t_{p,q}(\mathfrak{D}_\pm) \tag{1.4.6}$$

are bounded linear extension operators. These properties extend also to the corresponding spaces of vector valued functions or distributions.

As in the case of the corresponding Definition 3.6 in [81], we can introduce the *canonical extension operator* $\tilde{\mathcal{L}}_\alpha$ as follows (cf. Definition 2.8 in [41]).

Definition 1.4.1. Let $p, q \in (1, \infty)$, $s \in \mathbb{R}$ and $t \geq -1/p'$. The operator $\tilde{\mathcal{L}}_\alpha$, which maps

(*i*) functions $(\mathbf{u}, \pi) \in \mathfrak{H}^{s+\frac{1}{p},t}_{p,\mathrm{div}}(\mathfrak{D}, \mathcal{L}_\alpha)$ to the extension of the distribution $\mathcal{L}_\alpha(\mathbf{u}, \pi) \in H^t_p(\mathfrak{D}, \mathbb{R}^n)$ to $\widetilde{H}^t_p(\mathfrak{D}, \mathbb{R}^n)$

(*ii*) functions $(\mathbf{u}, \pi) \in \mathfrak{B}^{s+\frac{1}{p},t}_{p,q,\mathrm{div}}(\mathfrak{D}; \mathcal{L}_\alpha)$ to the extension of the distribution $\mathcal{L}_\alpha(\mathbf{u}, \pi) \in B^t_{p,q}(\mathfrak{D}, \mathbb{R}^n)$ to $\widetilde{B}^t_{p,q}(\mathfrak{D}, \mathbb{R}^n)$,

is called *the canonical extension operator.*

1.5 The conormal derivative operators for the Brinkman system

In this section, we introduce the conormal derivative operators that are used all along this thesis, i.e., *the classical, the nontangential, the canonical and the generalized derivative operators* and describe the relations between them.

To this end, let $\mathfrak{D} \subset \mathbb{R}^n$ be a bounded Lipschitz domain with the boundary Γ.

1.5.1 The classical conormal derivative operator

If $(\mathbf{u}, \pi) \in C^1(\overline{\mathfrak{D}}_\pm, \mathbb{R}^n) \times C^0(\overline{\mathfrak{D}}_\pm)$, such that div $\mathbf{u} = 0$ in $\mathfrak{D}_\pm$, then the *classical derivative operator* (or the traction field) for the Stokes (or Brinkman) operator is defined by the well-known formula (i.e., the constitutive equation of viscous incompressible fluid)

$$\mathbf{t}^{\mathrm{c}\pm}_\alpha(\mathbf{u}, \pi) := \gamma_\pm \boldsymbol{\sigma}(\mathbf{u}, \pi)\boldsymbol{\nu}, \tag{1.5.1}$$

where

$$\boldsymbol{\sigma}(\mathbf{u}, \pi) := (-\pi\mathbb{I} + 2\mathbb{E}(\mathbf{u})) \tag{1.5.2}$$

is the stress tensor, and $\boldsymbol{\nu} = \boldsymbol{\nu}^+$ is the outward unit normal to $\mathfrak{D}_+$, defined almost everywhere on Γ. For any function $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$, the following Green identity holds

$$\pm \langle \mathbf{t}^{\mathrm{c}\pm}_\alpha(\mathbf{u}, \pi), \boldsymbol{\varphi}\rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{u}), \mathbb{E}(\boldsymbol{\varphi})\rangle_{\mathfrak{D}_\pm} + \alpha\langle \mathbf{u}, \boldsymbol{\varphi}\rangle_{\mathfrak{D}_\pm} - \langle \pi, \mathrm{div}\ \boldsymbol{\varphi}\rangle_{\mathfrak{D}_\pm} + \langle \mathcal{L}_\alpha(\mathbf{u}, \pi), \boldsymbol{\varphi}\rangle_{\mathfrak{D}_\pm}. \tag{1.5.3}$$

Formula (1.5.3) follows by an integration by parts argument.

1.5.2 The nontangential conormal derivative operator

If in a given point of the boundary Γ there exist the non-tangential traces $\mathbf{t}^{\pm}_{\text{nt}}(\mathbf{u},\pi)$, and if the outward unit normal $\boldsymbol{\nu}$ can be defined in that point, then the non-tangential conormal derivatives are defined at this point as

$$\mathbf{t}^{\pm}_{\text{nt}}(\mathbf{u},\pi) := \boldsymbol{\sigma}^{\pm}_{\text{nt}}\,\boldsymbol{\nu}. \tag{1.5.4}$$

1.5.3 The generalized derivative operator

Formula (1.5.3) suggests the following definition of *the generalized conormal derivative* in the setting of Besov spaces, (cf., Lemma 3.2 in [20], Lemma 2.2 in [53], Definition 3.8, Theorem 3.9 in [81], Definition 6.5, Theorem 6.6 in [82], Proposition 10.2.1 in [86]).

Definition 1.5.1. Let $\alpha \geq 0$, $s \in (0,1)$, $p,q \in (1,\infty)$. Then the *generalized conormal derivative operator* $\mathbf{t}^{+}_{\alpha}$ is defined on any $(\mathbf{u},\pi) \in \mathfrak{H}^{s+\frac{1}{p}}_{p,\text{div}}(\mathfrak{D},\mathcal{L}_\alpha)$ or $(\mathbf{u},\pi) \in \mathfrak{B}^{s+\frac{1}{p}}_{p,q,\text{div}}(\mathfrak{D},\mathcal{L}_\alpha)$, in the weak sense by the formula

$$\begin{aligned}\pm\left\langle \mathbf{t}^{+}_{\alpha}(\mathbf{u},p;\mathbf{f}),\varphi\right\rangle &= 2\langle \widetilde{E}_{+}\mathbb{E}(\mathbf{u}),\mathbb{E}((\gamma^{+})^{-1}\varphi)\rangle_{\mathfrak{D}} + \alpha\langle \widetilde{E}_{+}\mathbf{u},(\gamma^{+})^{-1}\varphi\rangle_{\mathfrak{D}} \\ &- \langle \widetilde{E}_{+}\pi,\text{div }(\gamma^{+})^{-1}\varphi\rangle_{\mathfrak{D}} + \left\langle \tilde{\mathbf{f}},(\gamma^{+})^{-1}\varphi\right\rangle_{\mathfrak{D}}, \forall\varphi \in B^{1-s}_{p',p'}(\Gamma,\mathbb{R}^n) \text{ or } \varphi \in B^{1-s}_{p',q'}(\Gamma,\mathbb{R}^n).\end{aligned} \tag{1.5.5}$$

Lemma 1.5.2. *Under the assumption of Definition 1.5.1, the generalized conormal derivative operators*

$$\mathbf{t}^{+}_{\alpha} : \mathfrak{H}^{s+\frac{1}{p}}_{p,\text{div}}(\mathfrak{D},\mathcal{L}_\alpha) \to B^{s-1}_{p,p}(\Gamma,\mathbb{R}^n),\ \ \mathbf{t}^{+}_{\alpha} : \mathfrak{B}^{s+\frac{1}{p}}_{p,q,\text{div}}(\mathfrak{D},\mathcal{L}_\alpha) \to B^{s-1}_{p,q}(\Gamma,\mathbb{R}^n),$$

are linear and bounded. Moreover, the following first Green identity holds

$$\langle \mathbf{t}^{+}_{\alpha}(\mathbf{u},\pi),\gamma^{+}\mathbf{w}\rangle_{\Gamma} = 2\left\langle \widetilde{E}_{+}\mathbb{E}(\mathbf{u}),\mathbb{E}(\boldsymbol{w})\right\rangle_{\mathfrak{D}} + \alpha\left\langle \widetilde{E}_{+}\mathbf{u},\mathbf{w}\right\rangle_{\mathfrak{D}} - \left\langle \widetilde{E}_{+}\pi,\text{div }\mathbf{w}\right\rangle_{\mathfrak{D}} + \left\langle \tilde{\mathbf{f}},\mathbf{w}\right\rangle_{\mathfrak{D}} \tag{1.5.6}$$

for all $(\boldsymbol{u},\pi) \in \mathfrak{H}^{s+\frac{1}{p},-\frac{1}{p'}}_{p,\text{div}}(\mathfrak{D},\mathcal{L}_\alpha)$, $\boldsymbol{w} \in H^{1+\frac{1}{p'}-s}_{p'}(\mathfrak{D},\mathbb{R}^n)$ *and all* $(\boldsymbol{u},\pi) \in \mathfrak{B}^{s+\frac{1}{p},-\frac{1}{p'}}_{p,q,\text{div}}(\mathfrak{D},\mathcal{L}_\alpha)$, $\boldsymbol{w} \in B^{1+\frac{1}{p'}-s}_{p',q'}(\mathfrak{D},\mathbb{R}^n)$.

Note that the *canonical* conormal derivative operators introduced in Definition 1.5.3 are different from the *generalized* conormal derivative operator given in the next lemma, as described below (cf. [53, Lemma 2.2], [81, Definition 3.1, Theorem 3.2], [82, Definition 5.2, Theorem 5.3].)

1.5.4 The canonical derivative operator

The generalized conormal derivative operator defined by formula (1.5.5) is related to an extension of the Brinkman operator from the domain $\mathfrak{D}$ to the domain boundary, where $\mathcal{L}_\alpha$ has not nessasary a high regulatity (cf. [81]). Since the extensions of the operator $\mathcal{L}_\alpha$ from $\mathfrak{D}$ to $\mathbb{R}^n$ are non-unique, for some regularities of the corresponding spaces, it appears that the generalized conormal derivative operator is non-unique and non-linear unless a linear relation between the PDE solution and the extension of its right hand side is imposed. Thus, for boundary value problems whose solution should have high regularity, we need to revise the problem setting in order to make the conormal derivative operator insensitive to this non-uniqueness, i.e., we make use of the uniqueness of the extension operator given in (1.4.6) for $1/p' < t < 1/p$, as we have done in [41].

Having in view formula (1.5.3) in the classical case, we now introduce *the canonical conormal derivative* in the setting of Besov spaces by following [41, Definition 2.10], [20, Lemma 3.2], [53, Lemma 2.2], [81, Definition 3.8, Theorem 3.9], [82, Definition 6.5, Theorem 6.6], [86, Proposition 10.2.1]).

Definition 1.5.3. Let $\alpha \geq 0$, $s \in (0,1)$, $p, q \in (1,\infty)$. Then the *canonical* conormal derivative $\mathbf{t}_\alpha^+(\mathbf{u},\pi)$ is defined for any $(\mathbf{u},\pi) \in \mathfrak{H}_{p,\mathrm{div}}^{s+\frac{1}{p},-\frac{1}{p'}}(\mathfrak{D},\mathcal{L}_\alpha)$, or for any $(\mathbf{u},\pi) \in \mathfrak{B}_{p,q,\mathrm{div}}^{s+\frac{1}{p},-\frac{1}{p'}}(\mathfrak{D},\mathcal{L}_\alpha)$, in the weak sense, by the formula

$$\langle \mathbf{t}_\alpha^+(\mathbf{u},\pi), \boldsymbol{\varphi}\rangle_\Gamma := 2\left\langle \widetilde{E}_+\mathbb{E}(\mathbf{u}), \mathbb{E}((\gamma^+)^{-1}\boldsymbol{\varphi})\right\rangle_{\mathfrak{D}} + \alpha\langle \widetilde{E}_+\mathbf{u}, (\gamma^+)^{-1}\boldsymbol{\varphi}\rangle_{\mathfrak{D}} - \left\langle \widetilde{E}_+\pi, \mathrm{div}((\gamma^+)^{-1}\boldsymbol{\varphi})\right\rangle_{\mathfrak{D}} \tag{1.5.7}$$

$$+\langle \tilde{\mathcal{L}}_\alpha(\mathbf{u},\pi), (\gamma^+)^{-1}\boldsymbol{\varphi}\rangle_{\mathfrak{D}}, \quad \forall\ \boldsymbol{\varphi} \in B_{p',p'}^{1-s}(\Gamma,\mathbb{R}^n), \text{ or } \forall\ \boldsymbol{\varphi} \in B_{p',q'}^{1-s}(\Gamma,\mathbb{R}^n), \text{ respectively.} \tag{1.5.8}$$

Having in view Definition 1.5.3, we obtain the following Green identities based on our result [41, Lemma 2.11].

Lemma 1.5.4. *Under the assumption of Definition 1.5.3, the conormal derivative operators*

$$\mathbf{t}_\alpha^+ : \mathfrak{H}_{p,\mathrm{div}}^{s+\frac{1}{p},-\frac{1}{p'}}(\mathfrak{D},\mathcal{L}_\alpha) \to B_{p,p}^{s-1}(\Gamma,\mathbb{R}^n),\ \mathbf{t}_\alpha^+ : \mathfrak{B}_{p,q,\mathrm{div}}^{s+\frac{1}{p},-\frac{1}{p'}}(\mathfrak{D},\mathcal{L}_\alpha) \to B_{p,q}^{s-1}(\Gamma,\mathbb{R}^n),$$

are linear and bounded. Moreover, the following first Green formula holds

$$\langle \mathbf{t}_\alpha^+(\mathbf{u},\pi), \gamma^+\mathbf{w}\rangle_\Gamma = 2\left\langle \widetilde{E}_+\mathbb{E}(\mathbf{u}), \mathbb{E}(\mathbf{w})\right\rangle_{\mathfrak{D}} + \alpha\left\langle \widetilde{E}_+\mathbf{u}, \mathbf{w}\right\rangle_{\mathfrak{D}} - \left\langle \widetilde{E}_+\pi, \mathrm{div}\ \mathbf{w}\right\rangle_{\mathfrak{D}} + \left\langle \widetilde{E}_+\mathcal{L}_\alpha, \mathbf{w}\right\rangle_{\mathfrak{D}} \tag{1.5.9}$$

for all $(\boldsymbol{u},\pi) \in \mathfrak{H}_{p,\mathrm{div}}^{s+\frac{1}{p},-\frac{1}{p'}}(\mathfrak{D},\mathcal{L}_\alpha)$, $\boldsymbol{w} \in H_{p'}^{1+\frac{1}{p'}-s}(\mathfrak{D},\mathbb{R}^n)$ and all $(\boldsymbol{u},\pi) \in \mathfrak{B}_{p,q,\mathrm{div}}^{s+\frac{1}{p},-\frac{1}{p'}}(\mathfrak{D},\mathcal{L}_\alpha)$, $\boldsymbol{w} \in B_{p',q'}^{1+\frac{1}{p'}-s}(\mathfrak{D},\mathbb{R}^n)$ and the second Green formula holds

$$\pm\left(\langle \mathbf{t}_\alpha^\pm(\mathbf{u},\pi), \gamma^+\mathbf{v}\rangle_\Gamma - \langle \mathbf{t}_\alpha^\pm(\mathbf{v},q), \gamma^+\mathbf{u}\rangle_\Gamma\right) = \left\langle \tilde{\mathcal{L}}_\alpha(\mathbf{u},\pi), \mathbf{v}\right\rangle_{\mathfrak{D}_\pm} - \left\langle \tilde{\mathcal{L}}_\alpha(\mathbf{v},q), \mathbf{u}\right\rangle_{\mathfrak{D}_\pm} \tag{1.5.10}$$

for all $(\boldsymbol{u},\pi) \in \mathfrak{H}_{p,\mathrm{div}}^{s+\frac{1}{p},-\frac{1}{p'}}(\mathfrak{D}_\pm,\mathcal{L}_\alpha)$, $(\boldsymbol{v},q) \in \mathfrak{H}_{p',\mathrm{div}}^{1+\frac{1}{p'}-s,-\frac{1}{p}}(\mathfrak{D}_\pm,\mathbb{R}^n)$, and for all $(\boldsymbol{u},\pi) \in \mathfrak{B}_{p,q,\mathrm{div}}^{s+\frac{1}{p},-\frac{1}{p'}}(\mathfrak{D}_\pm,\mathcal{L}_\alpha)$, $(\boldsymbol{v},q) \in \mathfrak{B}_{p',q'}^{1+\frac{1}{p'}-s,-\frac{1}{p}}(\mathfrak{D}_\pm,\mathbb{R}^n)$, respectively.

The next result shows the equivalence between canonical, non-tangential conormal derivatives and classical conormal derivative. This result has been obtained in Theorem 2.13 of [41].

Theorem 1.5.5. *Assume that $\mathfrak{D}_+$ is a bounded domain with Lipschitz boundary Γ in $\mathbb{R}^n, n \geq 2$ and let $\mathfrak{D}_- = \mathbb{R}^n \backslash \overline{\mathfrak{D}_+}$. Let $\alpha \geq 0$, and $p, q \in (1,\infty)$. Then the following assertions hold.*

(i) If $s > 1$ and $(\mathbf{u},\pi) \in B_{p,q,\mathrm{div}}^{s+\frac{1}{p}}(\mathfrak{D}_\pm,\mathbb{R}^n) \times B_{p,q}^{s-1+\frac{1}{p}}(\mathfrak{D}_\pm)$, then $\mathbf{t}_\alpha^{\mathrm{c}+}(\mathbf{u},\pi)$ and $\mathbf{t}_\alpha^+(\mathbf{u},\pi)$ are well defined and are equal $\mathbf{t}_\alpha^+(\mathbf{u},\pi) = \mathbf{t}_\alpha^{\mathrm{c}+}(\mathbf{u},\pi) \in L^p(\Gamma,\mathbb{R}^n)$.

Moreover, if the non-tangential trace of the stress tensor, $\boldsymbol{\sigma}_{\mathrm{nt}}^+(\mathbf{u},\pi)$, exists almost everywhere on Γ, then the non-tangential conormal derivative also exists a.e. on Γ and are equal $\mathbf{t}_{\mathrm{nt},\alpha}^+(\mathbf{u},\pi) = \mathbf{t}_\alpha^+(\mathbf{u},\pi) = \mathbf{t}_\alpha^{\mathrm{c}+}(\mathbf{u},\pi) \in L^p(\Gamma,\mathbb{R}^n)$.

(ii) Let $0 < s \leq 1$ and $(\mathbf{u},\pi) \in \mathfrak{B}_{p,q,\mathrm{div}}^{s+\frac{1}{p},t}(\mathfrak{D}_\pm,\mathcal{L}_\alpha)$, for some $t > -\frac{1}{p'}$. If the non-tangential maximal function $M(\boldsymbol{\sigma}(\mathbf{u},\pi))$ and the non-tangential trace of the stress tensor, $\boldsymbol{\sigma}_{\mathrm{nt}}^+(\mathbf{u},\pi)$, exist almost everywhere on Γ and belong to $L^p(\Gamma,\mathbb{R}^{n\times n})$, then $\mathbf{t}_\alpha^+(\mathbf{u},\pi) = \mathbf{t}_{\mathrm{nt},\alpha}^+(\mathbf{u},\pi) \in L^p(\Gamma,\mathbb{R}^n)$.

Chapter 2

Layer potential theory for Stokes and Brinkman systems on Lipschitz domains in $\mathbb{R}^n$

Layer potential theory is a powerful tool in the study of elliptic boundary value problems (see, e.g., [4], [20], [46], [60], [76], [86], [111]). In this section we recall the definition and some properties of the potential operators for the Stokes and the Brinkman systems, i.e., the Newtonian potential operator and the single- and double-layer potential operators.

The potential theory is based on the existence of a fundamental solution for the system under consideration. Therefore, we begin by introducing the fundamental solution for the Brinkman operator following the main ideas in [50], but also for the Stokes system. Many properties of the potential operators are based on the expression of the fundamental velocity tensor and the fundamental stress tensor, as is pointed out throughout this section. Next, we introduce the Newtonian potential operator which defines a solution for the system under consideration driven by an acting force. The mapping properties of the Newtonian layer potential that appear in this thesis have been published in [41].

The following section is concerned with layer potential operators and their main properties. First, we introduce the Brinkman single-layer potential operator and discuss its mapping properties. Special attention is devoted to the nontangential maximal operator. Second, the double-layer potential operator is introduced, following an analogous structure as for the single-layer operator. Afterward, we consider the jump relation across the boundary for the layer potentials operators pointing out the difference between the nontangential and canonical approach. We end this section with some useful invertibility results.

Note that the results presented in this section represent a collection of known results, but also many new results obtained by joint work with M. Kohr, S. E. Mikhailov and W. L. Wendland in [41].

2.1 The fundamental solution of the Stokes and Brinkman systems in $\mathbb{R}^n$

In this section we present the fundamental solutions for the two considered systems, i.e., the fundamental solutions for the Stokes system and the Brinkman system in the n-dimensional Euclidean space $\mathbb{R}^n$. These fundamental solutions play a key role in defining the layer operators, that have an essential role in the development of the potential theory for these systems. The main sources used in the preparation of this chapter are [50] and [51].

The fundamental solution for the Brinkman system in $\mathbb{R}^n$

Let $\mathcal{G}^\alpha(\mathbf{x},\mathbf{y}) \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \otimes \mathbb{R}^n)$ and $\Pi^\alpha(\mathbf{x},\mathbf{y}) \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ denote *the fundamental velocity tensor* and *the fundamental pressure vector* for the Brinkman system in $\mathbb{R}^n$ $(n \geq 2)$. The pair $(\mathcal{G}^\alpha(\mathbf{x},\mathbf{y}), \Pi^\alpha(\mathbf{x},\mathbf{y}))$ is the fundamental solution of the Stokes system and satisfies the equation

$$(\triangle_{\mathbf{x}} - \alpha\mathbb{I})\mathcal{G}^\alpha(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\Pi(\mathbf{x},\mathbf{y}) = \delta_{\mathbf{y}}(\mathbf{x})\mathbb{I}, \quad \mathrm{div}_{\mathbf{x}}\mathcal{G}^\alpha(\mathbf{x},\mathbf{y}) = 0, \tag{2.1.1}$$

where $\delta_{\mathbf{y}}$ is the Dirac distribution with mass in $\mathbf{y} \in \mathbb{R}^n$, while the subscript $\mathbf{x}$ added to an differential operator refers to the action of that operator with respect to the variable $\mathbf{x}$.

The components of the fundamental velocity tensor and the fundamental pressure vector are given by (see, e.g., [75, (3.6)], [60, Section 3.2.1])

$$\mathcal{G}^\alpha_{jk}(\mathbf{x},\mathbf{y}) = \frac{1}{\omega_n}\left\{\frac{\delta_{jk}}{|\mathbf{y}-\mathbf{x}|^{n-2}}A_1(\alpha|\mathbf{y}-\mathbf{x}|) + \frac{x_j x_k}{|\mathbf{y}-\mathbf{x}|^n}A_2(\alpha|\mathbf{y}-\mathbf{x}|)\right\}, \quad \Pi_k(x) = \frac{1}{\omega_n}\frac{x_k}{|\mathbf{y}-\mathbf{x}|^n} \tag{2.1.2}$$

where $A_1(z)$ and $A_2(z)$ are defined by

$$A_1(z) := \frac{\left(\frac{z}{2}\right)^{\frac{n}{2}-1}K_{\frac{n}{2}-1}(z)}{\Gamma\left(\frac{n}{2}\right)} + 2\frac{\left(\frac{z}{2}\right)^{\frac{n}{2}}K_{\frac{n}{2}}(z)}{\Gamma\left(\frac{n}{2}\right)z^2} - \frac{1}{z^2}, \; A_2(z) := \frac{n}{z^2} - 4\frac{\left(\frac{z}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(z)}{\Gamma\left(\frac{n}{2}\right)z^2}, \tag{2.1.3}$$

$K_\varkappa$ is the Bessel function of the second kind and order $\varkappa \geq 0$, Γ is the Gamma function, and ω_n is the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. Equations (2.1.2) and (2.1.3) show that the fundamental velocity tensor is symmetric, i.e., $(\mathcal{G}^\alpha(\mathbf{x},\mathbf{y}))^\top = \mathcal{G}^\alpha(\mathbf{y},\mathbf{x})$.

In the sequel, we use the repeated index summation convention in order to simplify the notation. *The fundamental stress tensor* $\mathbf{S}^\alpha(\cdot,\cdot)$ has the components

$$S^\alpha_{ij\ell}(\mathbf{x},\mathbf{y}) = -\Pi_j(\mathbf{x},\mathbf{y})\delta_{i\ell} + \frac{\partial\mathcal{G}^\alpha_{ij}(\mathbf{x},\mathbf{y})}{\partial x_\ell} + \frac{\partial\mathcal{G}^\alpha_{\ell j}(\mathbf{x},\mathbf{y})}{\partial x_i}, \tag{2.1.4}$$

where δ_{jk} is the Kronecker symbol, $\Pi_j(\mathbf{x},\mathbf{y})$ are the components of Π, and $\mathcal{G}^\alpha_{ij}(\mathbf{x},\mathbf{y})$ are the components of $\mathcal{G}^\alpha(\mathbf{x},\mathbf{y})$ (e.g., [50, Section 2.3]).

2.2 Mapping properties of the Newtonian potential for the Brinkman system

By $*$ we denote the convolution product. Therefore the velocity and pressure Newtonian potential operators associated to the Brinkman system are given by

$$\left(\mathbf{N}_{\alpha;\mathbb{R}^n}\boldsymbol{\varphi}\right)(\mathbf{x}) := -\left(\mathcal{G}^\alpha * \boldsymbol{\varphi}\right)(\mathbf{x}) = -\langle\mathcal{G}^\alpha(\mathbf{x},\cdot),\boldsymbol{\varphi}\rangle_{\mathbb{R}^n}, \tag{2.2.1}$$

$$\left(\mathcal{Q}_{\alpha;\mathbb{R}^n}\boldsymbol{\varphi}\right)(\mathbf{x}) = \left(\mathcal{Q}_{\mathbb{R}^n}\boldsymbol{\varphi}\right)(\mathbf{x}) := -\left(\mathbf{\Pi} * \boldsymbol{\varphi}\right)(\mathbf{x}) = -\langle\mathbf{\Pi}(\mathbf{x},\cdot),\boldsymbol{\varphi}\rangle_{\mathbb{R}^n}, \tag{2.2.2}$$

where $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)$ and the fundamental tensor $\mathcal{G}^\alpha$ is presented through its components in (2.1.2).

By using the expression (2.2.1), we have the following property (cf. [41, Lemma 3.1] and also Theorem 3.10 in [75]).

Lemma 2.2.1. *Let $\alpha > 0$. Then for all $p,q \in (1,\infty)$ and $s \in \mathbb{R}$ the following operators*

$$\mathbf{N}_{\alpha;\mathbb{R}^n} : H^s_p(\mathbb{R}^n,\mathbb{R}^n) \to H^{s+2}_p(\mathbb{R}^n,\mathbb{R}^n), \tag{2.2.3}$$

$$\mathbf{N}_{\alpha;\mathbb{R}^n} : B^s_{p,q}(\mathbb{R}^n,\mathbb{R}^n) \to B^{s+2}_{p,q}(\mathbb{R}^n,\mathbb{R}^n), \tag{2.2.4}$$

$$\mathcal{Q}_{\mathbb{R}^n} : H^s_p(\mathbb{R}^n,\mathbb{R}^n) \to H^{s+1}_{p,\mathrm{loc}}(\mathbb{R}^n), \tag{2.2.5}$$

$$\mathcal{Q}_{\mathbb{R}^n} : B^s_{p,q}(\mathbb{R}^n,\mathbb{R}^n) \to B^{s+1}_{p,q,\mathrm{loc}}(\mathbb{R}^n), \tag{2.2.6}$$

are linear and continuous.

Consequently, we obtain the following result.

Corollary 2.2.2. *Assume that* $\mathfrak{D}_+ \subset \mathbb{R}^n$ $(n \geq 2)$ *is a Lipschitz domain and let* $\mathfrak{D}_- = \mathbb{R}^n \setminus \overline{\mathfrak{D}_+}$. *Let* $\alpha > 0$, $p \in (1, \infty)$, *and* $p^* = \max\{2, p\}$. *Then the Newtonian velocity and pressure potentials satisfy the Brinkman system and the following operators*

$$(\mathbf{N}_{\alpha;\mathfrak{D}_+}, \mathcal{Q}_{\mathfrak{D}_+}) : L^p(\mathfrak{D}_+, \mathbb{R}^n) \to \mathfrak{H}^{2,0}_{p,\mathrm{div}}(\mathfrak{D}_+, \mathcal{L}_\alpha), \tag{2.2.7}$$

$$(\mathbf{N}_{\alpha;\mathfrak{D}_-}, \mathcal{Q}_{\mathfrak{D}_-}) : L^p(\mathfrak{D}_-, \mathbb{R}^n) \to \mathfrak{H}^{2,0}_{p,\mathrm{div,loc}}(\overline{\mathfrak{D}}_-, \mathcal{L}_\alpha), \tag{2.2.8}$$

$$(\mathbf{N}_{\alpha;\mathfrak{D}_+}, \mathcal{Q}_{\mathfrak{D}_+}) : L^p(\mathfrak{D}_+, \mathbb{R}^n) \to \mathfrak{B}^{2,0}_{p,p^*,\mathrm{div}}(\mathfrak{D}_+, \mathcal{L}_\alpha), \tag{2.2.9}$$

$$(\mathbf{N}_{\alpha;\mathfrak{D}_-}, \mathcal{Q}_{\mathfrak{D}_-}) : L^p(\mathfrak{D}_-, \mathbb{R}^n) \to \mathfrak{B}^{2,0}_{p,p^*,\mathrm{div,loc}}(\overline{\mathfrak{D}}_-, \mathcal{L}_\alpha), \tag{2.2.10}$$

are continuous.

2.3 Mapping properties of the Brinkman layer potentials in Sobolev and Besov spaces

In this section we introduce the single- and double layer operators for the Brinkman system and give the main properties which are needed in the study of the boundary problems in the following sections. First, we are concerned with the mapping properties of the layer potentials defined on L^p-based Sobolev and Besov spaces, with p in a neighbourhood of 2. These properties play a key role in the analysis of boundary problems on creased, bounded Lipschitz domains, treated in Chapter 3. Then, we will extend these properties to Sobolev and Besov spaces with an index $s \in (0, 1)$. Such an extension is very useful in the analysis of some boundary problems, which is developed in Chapter 4. Special attention is given to the jump relations across the boundary for the single- and double-layer potential operators, with emphasis on the difference between the non-tangential and Gagliardo traces and the non-tangential and canonical conormal derivative operators. Moreover, we give the representation formula of the velocity and pressure field in terms of layer potentials, by employing both nontangential and canonical approaches. We end this section with some useful inveribility results for layer potentials. The results of this section are original contributions obtained in [41].

2.3.1 The Brinkman single-layer potential and related mapping properties

From now on, in this chapter we consider the following assumption, unless stated otherwise.

Assumption 2.3.1. Let $\mathfrak{D}_+ \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with connected boundary Γ and let $\mathfrak{D}_- := \mathbb{R}^n \setminus \overline{\mathfrak{D}}_+$.

Let $\alpha > 0$, $s \in [0, 1]$ and $p \in (1, \infty)$. For a given density $\mathbf{g} \in H^{s-1}_p(\Gamma, \mathbb{R}^n)$, the *Brinkman velocity single-layer potential*, $\mathbf{V}_\alpha \mathbf{g}$, and the corresponding scalar *pressure potential*, $Q^s_\alpha \mathbf{g}$, are given by

$$(\mathbf{V}_\alpha \mathbf{g})(\mathbf{x}) := \langle \mathcal{G}^\alpha(\mathbf{x}, \cdot)|_\Gamma, \mathbf{g} \rangle_\Gamma, \ (Q^s_\alpha \mathbf{g})(\mathbf{x}) := \langle \Pi(\mathbf{x}, \cdot)|_\Gamma, \mathbf{g} \rangle_\Gamma, \ \mathbf{x} \in \mathbb{R}^n \setminus \Gamma. \tag{2.3.1}$$

By (2.1.1), the pair $(\mathbf{V}_\alpha \mathbf{g}, Q^s_\alpha \mathbf{g})$ satisfies the homogeneous Brinkman system in $\mathfrak{D}_\pm$,

$$(\triangle - \alpha \mathbb{I}) \mathbf{V}_\alpha \mathbf{g} - \nabla Q^s_\alpha \mathbf{g} = 0, \ \ \mathrm{div} \mathbf{V}_\alpha \mathbf{g} = 0 \ \text{ in } \ \mathbb{R}^n \setminus \Gamma. \tag{2.3.2}$$

The single-layer potential can be similarly defined on Besov spaces whenever $\mathbf{g} \in B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n)$, where $s \in (0, 1)$ and $p, q \in (1, \infty)$. In the case of the Stokes system, i.e., for $\alpha = 0$, we use the notations $\mathbf{V}\mathbf{g}$ and $Q^s \mathbf{g}$, i.e.,

$$\mathbf{V}_0 \mathbf{g} \equiv \mathbf{V}\mathbf{g}, \quad Q^s_0 \mathbf{g} \equiv Q^s \mathbf{g}. \tag{2.3.3}$$

Next, we describe the action of the nontangential maximal operator on the single-layer potential $\mathbf{V}_\alpha \mathbf{g}$, when $\mathbf{g} \in L^p(\Gamma, \mathbb{R}^n)$ and $\mathbf{g} \in H_p^{-1}(\Gamma, \mathbb{R}^n)$ respectively. Moreover, we show that the nontangential limits exist almost everywhere for the single-layer potential in both cases. We follow the arguments developed for Lemma 3.4 in our paper [41].

Lemma 2.3.2. *Under Assumption 2.3.1 and for $\alpha \geq 0$ and $p \in (1, \infty)$, we have:*

(*i*) *There exist some constants $C_i = C_i(\Gamma, p, \alpha) > 0$, $i = 1, \ldots, 2$, such that*

$$\|M(\nabla \mathbf{V}_\alpha \mathbf{g})\|_{L^p(\Gamma)} + \|M(\mathbf{V}_\alpha \mathbf{g})\|_{L^p(\Gamma)} + \|M(Q^s \mathbf{g})\|_{L^p(\Gamma)} \leq C_1 \|\mathbf{g}\|_{L^p(\Gamma, \mathbb{R}^n)}, \tag{2.3.4}$$

for all $\mathbf{g} \in L^p(\Gamma, \mathbb{R}^n)$ and

$$\|M(\mathbf{V}_\alpha \mathbf{g})\|_{L^p(\Gamma)} \leq C_2 \|\mathbf{g}\|_{H_p^{-1}(\Gamma, \mathbb{R}^n)}, \tag{2.3.5}$$

for all $\mathbf{g} \in H_p^{-1}(\Gamma, \mathbb{R}^n)$.

(*ii*) *For any $\mathbf{g} \in L^p(\Gamma, \mathbb{R}^n)$, there exist the nontangential limits of $\mathbf{V}_\alpha \mathbf{g}$, $\nabla \mathbf{V}_\alpha \mathbf{g}$ and $Q^s \mathbf{g}$ almost everywhere on Γ and*

$$\|(\mathbf{V}_\alpha \mathbf{g})^{\pm}_{\mathrm{nt}}\|_{L^p(\Gamma, \mathbb{R}^n)},\ \|(\nabla \mathbf{V}_\alpha \mathbf{g})^{\pm}_{\mathrm{nt}}\|_{L^p(\Gamma, \mathbb{R}^n)},\ \|(Q^s \mathbf{g})^{\pm}_{\mathrm{nt}}\|_{L^p(\Gamma, \mathbb{R}^n)} \leq C_1 \|\mathbf{g}\|_{L^p(\Gamma, \mathbb{R}^n)}. \tag{2.3.6}$$

(*iii*) *For any $\mathbf{g} \in H_p^{-1}(\Gamma, \mathbb{R}^n)$, there exist the nontangential limits of $\mathbf{V}_\alpha \mathbf{g}$ at almost all points of Γ, and*

$$\|(\mathbf{V}_\alpha \mathbf{g})^{\pm}_{\mathrm{nt}}\|_{L^p(\Gamma)} \leq C_2 \|\mathbf{g}\|_{H_p^{-1}(\Gamma, \mathbb{R}^n)}. \tag{2.3.7}$$

The mapping properties of the Stokes single-layer potentials in Bessel-potential and Besov spaces on bounded Lipschitz domains, are well known and we refer to, e.g., [30], [46], [86, Theorem 10.5.3], [85, Theorem 3.1, Proposition 3.3].

The next theorem collects the main properties of the Brinkman single-layer velocity and pressure potential operators. We have obtained them in [41, Theorem 3.5]. Let us mention that some of these properties have been also obtained in [25, Proposition 3.4], [49, Lemma 3.4], [50, Lemma 3.1], [85, Theorem 3.1], [100, Theorems 3.4 and 3.5]).

Theorem 2.3.3. *Let Assumption 2.3.1 hold. Let $p, q \in (1, \infty)$, $\alpha > 0$, and $p^* := \max\{2, p\}$. Let $t \geq -\frac{1}{p'}$ be arbitrary, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then the following statements hold.*

(*i*) *Then the operators*

$$\mathbf{V}_\alpha|_{\mathfrak{D}_+} : L^p(\Gamma, \mathbb{R}^n) \to B^{1+\frac{1}{p}}_{p,p^*;\mathrm{div}}(\mathfrak{D}_+, \mathbb{R}^n),\ \mathcal{Q}^s|_{\mathfrak{D}_+} : L^p(\Gamma, \mathbb{R}^n) \to B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D}_+), \tag{2.3.8}$$

$$\left(\mathbf{V}_\alpha|_{\mathfrak{D}_+}, \mathcal{Q}^s|_{\mathfrak{D}_+}\right) : L^p(\Gamma, \mathbb{R}^n) \to \mathfrak{B}^{1+\frac{1}{p},t}_{p,p^*;\mathrm{div}}(\mathfrak{D}_+, \mathcal{L}_\alpha), \tag{2.3.9}$$

$$\mathbf{V}_\alpha|_{\mathfrak{D}_+} : H_p^{-1}(\Gamma, \mathbb{R}^n) \to B^{\frac{1}{p}}_{p,p^*;\mathrm{div}}(\mathfrak{D}_+, \mathbb{R}^n),\ \mathcal{Q}^s|_{\mathfrak{D}_+} : H_p^{-1}(\Gamma, \mathbb{R}^n) \to B^{-1+\frac{1}{p}}_{p,p^*}(\mathfrak{D}_+), \tag{2.3.10}$$

$$\left(\mathbf{V}_\alpha|_{\mathfrak{D}_+}, \mathcal{Q}^s|_{\mathfrak{D}_+}\right) : H_p^{-1}(\Gamma, \mathbb{R}^n) \to \mathfrak{B}^{\frac{1}{p},t}_{p,p^*;\mathrm{div}}(\mathfrak{D}_+, \mathcal{L}_\alpha), \tag{2.3.11}$$

are linear and continuous.

(*ii*) *For any $s \in (0, 1)$, the following operators*

$$\mathbf{V}_\alpha : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}}_{p,q;\mathrm{div}}(\mathbb{R}^n, \mathbb{R}^n),\ Q^s : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}-1}_{p,q;\mathrm{loc}}(\mathbb{R}^n), \tag{2.3.12}$$

$$\mathbf{V}_\alpha|_{\mathfrak{D}_+} : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}}_{p,q;\mathrm{div}}(\mathfrak{D}_+, \mathbb{R}^n),\ (Q^s)|_{\mathfrak{D}_+} : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}-1}_{p,q}(\mathfrak{D}_+), \tag{2.3.13}$$

$$\left(\mathbf{V}_\alpha|_{\mathfrak{D}_+}, Q^s|_{\mathfrak{D}_+}\right) : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to \mathfrak{B}^{s+\frac{1}{p},t}_{p,q,\mathrm{div}}(\mathfrak{D}_+, \mathcal{L}_\alpha), \tag{2.3.14}$$

$$\mathbf{V}_\alpha|_{\mathfrak{D}_-} : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}}_{p,q;\mathrm{div}}(\mathfrak{D}_-, \mathbb{R}^n),\ Q^s|_{\mathfrak{D}_-} : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}-1}_{p,q;\mathrm{loc}}(\overline{\mathfrak{D}}_-), \tag{2.3.15}$$

$$\left(\mathbf{V}_\alpha|_{\mathfrak{D}_-}, Q^s|_{\mathfrak{D}_-}\right) : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to \mathfrak{B}^{s+\frac{1}{p},t}_{p,q;\mathrm{div};\mathrm{loc}}(\overline{\mathfrak{D}}_-, \mathcal{L}_\alpha), \tag{2.3.16}$$

are linear and continuous.

2.3.2 The Brinkman double-layer potential and related mapping properties

For a given density $\mathbf{h} \in H_p^s(\Gamma, \mathbb{R}^n)$, *the velocity and pressure double-layer potentials*, $\mathbf{W}_\alpha \mathbf{h}$ and $Q_\alpha^d \mathbf{h}$, are defined by the integral representations

$$(\mathbf{W}_\alpha \mathbf{h})_j(\mathbf{x}) := \int_\Gamma S_{ij\ell}^\alpha(\mathbf{x}, \mathbf{y}) \nu_\ell(\mathbf{y}) h_i(\mathbf{y}) d\sigma_y, \quad \forall\, \mathbf{x} \in \mathbb{R}^n \setminus \Gamma, \tag{2.3.17}$$

$$(Q_\alpha^d \mathbf{h})(\mathbf{x}) := \int_\Gamma \Lambda_{j\ell}^\alpha(\mathbf{x}, \mathbf{y}) \nu_\ell(\mathbf{y}) h_j(\mathbf{y}) d\sigma_y, \quad \forall\, \mathbf{x} \in \mathbb{R}^n \setminus \Gamma, \tag{2.3.18}$$

where ν_ℓ, $\ell = 1, \ldots, n$, are the components of the outward unit normal $\boldsymbol{\nu}$ to $\mathfrak{D}$, which is defined almost everywhere on Γ.

Similar definitions for the double-layer velocity potential and the corresponding pressure potential apply in the case of Besov spaces for $\mathbf{h} \in B_{p,q}^s(\Gamma, \mathbb{R}^n)$, where $s \in (0,1)$ and $p, q \in (1, \infty)$.

For $\alpha = 0$, we use the notations $\mathbf{Wh}$ and $Q^d \mathbf{h}$ for the corresponding double-layer potentials, i.e.,

$$\mathbf{W}_0 \mathbf{h} \equiv \mathbf{Wh}, \quad Q_0^d \mathbf{h} \equiv Q^d \mathbf{h}. \tag{2.3.19}$$

In view of equations (2.3.17) and (2.3.18), the pair $(\mathbf{W}_\alpha^s \mathbf{h}, Q_\alpha^d \mathbf{h})$ satisfies the Brinkman system in each of the domains $\mathfrak{D}_+$ and $\mathfrak{D}_-$, respectively, i.e.,

$$(\triangle - \alpha \mathbb{I}) \mathbf{W}_\alpha \mathbf{h} - \nabla Q_\Gamma^s \mathbf{h} = 0, \;\; \mathrm{div} \mathbf{W}_{\alpha;\Gamma} \mathbf{h} = 0 \;\text{ in }\; \mathbb{R}^n \setminus \Gamma. \tag{2.3.20}$$

The direct value of the double-layer potential $\mathbf{W}_\alpha \mathbf{h}$ on the boundary is defined by the Cauchy principal value as

$$(\mathbf{K}_\alpha \mathbf{h})_k(\mathbf{x}) := \text{p.v.} \int_\Gamma S_{jk\ell}^\alpha(\mathbf{y}, \mathbf{x}) \nu_\ell(\mathbf{y}) h_j(\mathbf{x}) d\sigma_{\mathbf{y}} \tag{2.3.21}$$

$$= \lim_{\varepsilon \to 0} \int_{\Gamma \setminus B_\varepsilon(\mathbf{x})} S_{jk\ell}^\alpha(\mathbf{y}, \mathbf{x}) \nu_\ell(y) h_j(\mathbf{x}) d\sigma_{\mathbf{y}} \text{ a.e. } \;\mathbf{x} \in \Gamma. \tag{2.3.22}$$

where $B_\varepsilon(\mathbf{x})$ denotes an open ball centered in $\mathbf{x}$ with radius ε (see, e.g., Hsiao-Wendland, [46], or Mitrea-Wright, [86]).

Similar to the properties of the single layer potential given by Lemma , the following lemma describes the action of the nontangential maximal operator on the double-layer potential operator and states that the nontangential limits exist almost everywhere for the double-layer potential with a density $\mathbf{h}$ in $L^p(\Gamma, \mathbb{R}^n)$ and $H_p^1(\Gamma, \mathbb{R}^n)$ respectively (cf. [41, Lemma 3.4]).

Lemma 2.3.4. *Let Assumption 2.3.1 be satisfied. Let $\alpha \geq 0$ and $p \in (1, \infty)$ be given constants. Then we have*

(*i*) *There exist some constants $C_i = C_i(\Gamma, p, \alpha) > 0$, $i = 1, 2$, such that*

$$\|M(\nabla \mathbf{W}_\alpha \mathbf{h})\|_{L^p(\Gamma)} + \|M(\mathbf{W}_\alpha \mathbf{h})\|_{L^p(\Gamma)} + \|M(Q_\alpha^d \mathbf{h})\|_{L^p(\Gamma)} \leq C_1 \|\mathbf{h}\|_{H_p^1(\Gamma, \mathbb{R}^n)}, \tag{2.3.23}$$

for all $\mathbf{h} \in H_p^1(\Gamma, \mathbb{R}^n)$ and

$$\|M(\mathbf{W}_\alpha \mathbf{h})\|_{L^p(\Gamma)} \leq C_2 \|\mathbf{h}\|_{L^p(\Gamma, \mathbb{R}^n)}. \tag{2.3.24}$$

for all $\mathbf{h} \in L^p(\Gamma, \mathbb{R}^n)$.

(*ii*) *For any $\mathbf{h} \in L^p(\Gamma, \mathbb{R}^n)$, there exists the nontangential limits of $\mathbf{W}_\alpha \mathbf{h}$ at almost everywhere on Γ and*

$$\|(\mathbf{W}_\alpha \mathbf{g})_{\mathrm{nt}}^{\pm}\|_{L^p(\Gamma, \mathbb{R}^n)} \leq C_1 \|\mathbf{h}\|_{L^p(\Gamma, \mathbb{R}^n)}. \tag{2.3.25}$$

(*iii*) *Let* $\mathbf{h} \in H_p^1(\Gamma, \mathbb{R}^n)$*. Then there exist the nontangential limits of* $\mathbf{W}_\alpha \mathbf{h}$*,* $\nabla \mathbf{W}_\alpha \mathbf{h}$ *and* $Q_\alpha^d \mathbf{h}$ *at almost all points of* Γ *and*

$$\|(\mathbf{W}_\alpha \mathbf{h})^\pm_{\rm nt}\|_{L^p(\Gamma,\mathbb{R}^n)},\ \|(\nabla \mathbf{W}_\alpha \mathbf{h})^\pm_{\rm nt}\|_{L^p(\Gamma,\mathbb{R}^n)},\ \|(Q_\alpha^d \mathbf{h})^\pm_{\rm nt}\|_{L^p(\Gamma,\mathbb{R}^n)} \le C_2 \|\mathbf{h}\|_{H_p^1(\Gamma,\mathbb{R}^n)}. \quad (2.3.26)$$

For the mapping properties of the double-layer potential operators for the *Stokes* system (i.e., for $\alpha = 0$) on bounded Lipschitz domains we refer, e.g., [30], [46], [86, Theorem 10.5.3], [85, Theorem 3.1, Proposition 3.3].

The main properties of double-layer potential operators are summarized below. We have proved them in [41, Theorem 3.5] (some of them are obtained in [25, Proposition 3.4], [49, Lemma 3.4], [50, Lemma 3.1], [85, Theorem 3.1], [100, Theorem 3.4 and 3.5]).

Theorem 2.3.5. *Let Assumption 2.3.1 hold. Let* $p, q \in (1, \infty)$*,* $\alpha > 0$*, and* $p^* := \max\{2, p\}$*. Let* $t \ge -\frac{1}{p'}$ *be arbitrary, where* $\frac{1}{p} + \frac{1}{p'} = 1$*. Then the following statements hold.*

(*i*) *The following operators are linear and continuous,*

$$\mathbf{W}_\alpha|_{\mathfrak{D}_+} : H_p^1(\Gamma, \mathbb{R}^n) \to B^{1+\frac{1}{p}}_{p,p^*;{\rm div}}(\mathfrak{D}_+, \mathbb{R}^n),\ Q_\alpha^d\big|_{\mathfrak{D}_+} : H_p^1(\Gamma, \mathbb{R}^n) \to B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D}_+), \quad (2.3.27)$$

$$\left(\mathbf{W}_\alpha|_{\mathfrak{D}_+}, Q_\alpha^d|_{\mathfrak{D}_+}\right) : H_p^1(\Gamma, \mathbb{R}^n) \to \mathfrak{B}^{1+\frac{1}{p},t}_{p,p^*;{\rm div}}(\mathfrak{D}_+, \mathcal{L}_\alpha). \quad (2.3.28)$$

$$\mathbf{W}_\alpha|_{\mathfrak{D}_+} : L^p(\Gamma, \mathbb{R}^n) \to B^{\frac{1}{p}}_{p,p^*;{\rm div}}(\mathfrak{D}_+, \mathbb{R}^n),\ Q_\alpha^d\big|_{\mathfrak{D}_+} : L^p(\Gamma, \mathbb{R}^n) \to B^{\frac{1}{p}-1}_{p,p^*}(\mathfrak{D}_+), \quad (2.3.29)$$

$$\left(\mathbf{W}_\alpha|_{\mathfrak{D}_+}, Q_\alpha^d|_{\mathfrak{D}_+}\right) : L^p(\Gamma, \mathbb{R}^n) \to \mathfrak{B}^{\frac{1}{p},t}_{p,p^*;{\rm div}}(\mathfrak{D}_+, \mathcal{L}_\alpha). \quad (2.3.30)$$

(*ii*) *For any* $s \in (0, 1)$ *the following operators*

$$\mathbf{W}_\alpha|_{\mathfrak{D}_+} : B^s_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}}_{p,q;{\rm div}}(\mathfrak{D}_+, \mathbb{R}^n),\ Q_\alpha^d|_{\mathfrak{D}_+} : B^s_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}-1}_{p,q}(\mathfrak{D}_+), \quad (2.3.31)$$

$$\left(\mathbf{W}_\alpha|_{\mathfrak{D}_+}, Q_\alpha^d|_{\mathfrak{D}_+}\right) : B^s_{p,q}(\Gamma, \mathbb{R}^n) \to \mathfrak{B}^{s+\frac{1}{p},t}_{p,q;{\rm div}}(\mathfrak{D}_+, \mathcal{L}_\alpha), \quad (2.3.32)$$

$$\mathbf{W}_\alpha|_{\mathfrak{D}_-} : B^s_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}}_{p,q;{\rm div;loc}}(\overline{\mathfrak{D}}_-, \mathbb{R}^n),\ Q^d|_{\mathfrak{D}_-} : B^s_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s+\frac{1}{p}-1}_{p,q;{\rm loc}}(\overline{\mathfrak{D}}_-), \quad (2.3.33)$$

$$\left(\mathbf{W}_\alpha|_{\mathfrak{D}_-}, Q_\alpha^d|_{\mathfrak{D}_-}\right) : B^s_{p,q}(\Gamma, \mathbb{R}^n) \to \mathfrak{B}^{s+\frac{1}{p},t}_{p,q,{\rm div;loc}}(\overline{\mathfrak{D}}_-, \mathcal{L}_\alpha). \quad (2.3.34)$$

are linear and continuous.

2.4 Properties of Brinkman layer potential operators in Sobolev and Besov spaces

In the next section, we obtain the jump relations satisfied by the layer potentials for the Brinkman system in the case when the trace operator and the conormal derivative operator are considered in the non-tangential sense, as well as in the Gagliardo or canonical sense. Moreover, we give special attention to the situations in which these notions are equivalent.

2.4.1 Jump relations of the single- and double-layer potentials

We focus our attention on the jump relations across a Lipschitz boundary for the Brinkman layer potentials in Sobolev spaces, but also in Besov spaces. For the case of the Stokes layer potentials we refer the reader to, e.g., [30], [46], [86, Theorem 10.5.3], [85, Theorem 3.1, Proposition 3.3]. We have obtained the result below in [41, Theorem 3.5].

First, we focus our attention to the the non-tangential case, cf. [41, Theorem 3.5] (moreover we mention the following results [25, Proposition 3.4], [49, Lemma 3.4], [50, Lemma 3.1], [85, Theorem 3.1], [100, Theorem 3.4 and 3.5]).

Theorem 2.4.1. *Assume that $\mathfrak{D}_+$ is a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 2)$ with connected boundary Γ and $\mathfrak{D}_- = \mathbb{R}^n \setminus \overline{\mathfrak{D}_+}$. Let $p, q \in (1, \infty)$, $\alpha > 0$, and $p^* := \max\{2, p\}$. Let $t \geq -\frac{1}{p'}$ be arbitrary, where $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\mathbf{h} \in H^1_p(\Gamma, \mathbb{R}^n)$ and $\mathbf{g} \in L^p(\Gamma, \mathbb{R}^n)$. Then the following relations hold almost everywhere on Γ,*

$$(\mathbf{V}_\alpha \mathbf{g})^+_{\mathrm{nt}} = (\mathbf{V}_\alpha \mathbf{g})^-_{\mathrm{nt}} =: \mathcal{V}_\alpha \mathbf{g}, \quad \forall\ \mathbf{g} \in H^{-1}_p(\Gamma, \mathbb{R}^n); \tag{2.4.1}$$

$$\frac{1}{2}\mathbf{h} + (\mathbf{W}_\alpha \mathbf{h})^+_{\mathrm{nt}} = -\frac{1}{2}\mathbf{h} + (\mathbf{W}_\alpha \mathbf{h})^-_{\mathrm{nt}} =: \mathbf{K}_\alpha \mathbf{h}, \quad \forall\ \mathbf{h} \in L^p(\Gamma, \mathbb{R}^n); \tag{2.4.2}$$

$$-\frac{1}{2}\mathbf{g} + \mathbf{t}^+_{\mathrm{nt}}(\mathbf{V}_\alpha \mathbf{g}, Q^s \mathbf{g}) = \frac{1}{2}\mathbf{g} + \mathbf{t}^-_{\mathrm{nt}}(\mathbf{V}_\alpha \mathbf{g}, Q^s \mathbf{g}) =: \mathbf{K}^*_\alpha \mathbf{g}, \quad \forall\ \mathbf{g} \in L^p(\Gamma, \mathbb{R}^n); \tag{2.4.3}$$

$$\mathbf{t}^+_{\mathrm{nt}}(\mathbf{W}_\alpha \mathbf{h}, Q^d_\alpha \mathbf{h}) = \mathbf{t}^-_{\mathrm{nt}}(\mathbf{W}_\alpha \mathbf{h}, Q^d_\alpha \mathbf{h}) =: \mathbf{D}_\alpha \mathbf{h}, \quad \forall\ \mathbf{h} \in H^1_p(\Gamma, \mathbb{R}^n); \tag{2.4.4}$$

where $\mathbf{K}^*_\alpha$ is the transpose of $\mathbf{K}_{\alpha;\Gamma}$. Moreover, the following integral operators

$$\mathcal{V}_\alpha : L^p(\Gamma, \mathbb{R}^n) \to H^1_p(\Gamma, \mathbb{R}^n),\ \mathbf{K}_\alpha : H^1_p(\Gamma, \mathbb{R}^n) \to H^1_p(\Gamma, \mathbb{R}^n), \tag{2.4.5}$$

$$\mathcal{V}_\alpha : H^{-1}_p(\Gamma, \mathbb{R}^n) \to L^p(\Gamma, \mathbb{R}^n),\ \mathbf{K}_\alpha : L^p(\Gamma, \mathbb{R}^n) \to L^p(\Gamma, \mathbb{R}^n), \tag{2.4.6}$$

$$\mathbf{K}^*_\alpha : L^p(\Gamma, \mathbb{R}^n) \to L^p(\Gamma, \mathbb{R}^n),\ \mathbf{D}_\alpha : H^1_p(\Gamma, \mathbb{R}^n) \to L^p(\Gamma, \mathbb{R}^n). \tag{2.4.7}$$

are linear and continuous.

The next theorem is a version of Theorem 2.4.1 stated such that the jump realtions are considered in the Gagliardo and canonical sense (cf. [41, Theorem 3.5]).

Theorem 2.4.2. *Let Assumption 2.3.1 hold. Let $p, q \in (1, \infty)$, $\alpha > 0$, and $p^* := \max\{2, p\}$. Let $t \geq -\frac{1}{p'}$ be arbitrary, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for $\mathbf{h} \in B^s_{p,q}(\Gamma, \mathbb{R}^n)$ and $\mathbf{g} \in B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n)$, $s \in (0, 1)$, the following relations hold almost everywhere on Γ,*

$$\gamma^+(\mathbf{V}_\alpha \mathbf{g}) = \gamma^-(\mathbf{V}_\alpha \mathbf{g}) =: \mathcal{V}_\alpha \mathbf{g}, \tag{2.4.8}$$

$$\frac{1}{2}\mathbf{h} + \gamma^+(\mathbf{W}_\alpha \mathbf{h}) = -\frac{1}{2}\mathbf{h} + \gamma^-(\mathbf{W}_\alpha \mathbf{h}) =: \mathbf{K}_\alpha \mathbf{h}, \tag{2.4.9}$$

$$-\frac{1}{2}\mathbf{g} + \mathbf{t}^+_\alpha(\mathbf{V}_\alpha \mathbf{g}, Q^s \mathbf{g}) = \frac{1}{2}\mathbf{g} + \mathbf{t}^-_\alpha(\mathbf{V}_\alpha \mathbf{g}, Q^s \mathbf{g}) =: \mathbf{K}^*_\alpha \mathbf{g}, \tag{2.4.10}$$

$$\mathbf{t}^+_\alpha(\mathbf{W}_\alpha \mathbf{h}, Q^d_\alpha \mathbf{h}) = \mathbf{t}^-_\alpha(\mathbf{W}_\alpha \mathbf{h}, Q^d_\alpha \mathbf{h}) =: \mathbf{D}_\alpha \mathbf{h}. \tag{2.4.11}$$

In addition, the following operators

$$\mathcal{V}_\alpha : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to B^s_{p,q}(\Gamma, \mathbb{R}^n),\ \mathbf{K}_\alpha : B^s_{p,q}(\Gamma, \mathbb{R}^n) \to B^s_{p,q}(\Gamma, \mathbb{R}^n), \tag{2.4.12}$$

$$\mathbf{K}^*_\alpha : B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n),\ \mathbf{D}_\alpha : B^s_{p,q}(\Gamma, \mathbb{R}^n) \to B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n). \tag{2.4.13}$$

are linear and continuous.

Now we give the answer to the natural question whether the Gagliardo trace and the non-tangential trace operators applied to layer-potentials are the same. We have proved the next result in [41, Lemma 3.6].

Lemma 2.4.3. *Let Assumption 2.3.1 be satisfied. Then the following properties hold.*

(i) If $p \in (1, \infty)$, $\alpha \in (0, \infty)$, $\mathbf{g} \in L^p(\Gamma, \mathbb{R}^n)$ and $\mathbf{h} \in H^1_p(\Gamma, \mathbb{R}^n)$, then the following equalities hold

$$\gamma^\pm(\mathbf{V}_\alpha \mathbf{g}) = (\mathbf{V}_\alpha \mathbf{g})^\pm_{\mathrm{nt}} \in H^1_{p;\boldsymbol{\nu}}(\Gamma, \mathbb{R}^n), \tag{2.4.14}$$

$$\gamma^\pm(\mathbf{W}_\alpha \mathbf{h}) = (\mathbf{W}_\alpha \mathbf{h})^\pm_{\mathrm{nt}} \in H^1_{p;\boldsymbol{\nu}}(\Gamma, \mathbb{R}^n), \tag{2.4.15}$$

$$\mathbf{t}^\pm_\alpha(\mathbf{V}_\alpha \mathbf{g}, Q^s \mathbf{g}) = \mathbf{t}^\pm_{\mathrm{nt}}(\mathbf{V}_\alpha \mathbf{g}, Q^s \mathbf{g}) \in L^p(\Gamma, \mathbb{R}^n), \tag{2.4.16}$$

$$\mathbf{t}^\pm_\alpha(\mathbf{W}_\alpha \mathbf{h}, Q^d_\alpha \mathbf{h}) = \mathbf{t}^\pm_{\mathrm{nt}}(\mathbf{W}_\alpha \mathbf{h}, Q^d_\alpha \mathbf{h}) \in L^p(\Gamma, \mathbb{R}^n), \tag{2.4.17}$$

with the corresponding norm estimates.

(ii) *If* $p, q \in (1, \infty)$, $s \in (0, 1)$, $\alpha \in (0, \infty)$, $\mathbf{g} \in B^{s-1}_{p,q}(\Gamma, \mathbb{R}^n)$ *and* $\mathbf{h} \in B^s_{p,q}(\Gamma, \mathbb{R}^n)$, *then the following equalities hold*

$$\gamma^{\pm}(\mathbf{V}_\alpha \mathbf{g}) = (\mathbf{V}_\alpha \mathbf{g})^{\pm}_{\mathrm{nt}} \in B^s_{p,q;\boldsymbol{\nu}}(\Gamma, \mathbb{R}^n), \tag{2.4.18}$$

$$\gamma^{\pm}(\mathbf{W}_\alpha \mathbf{h}) = (\mathbf{W}_\alpha \mathbf{h})^{\pm}_{\mathrm{nt}} \in B^s_{p,q;\boldsymbol{\nu}}(\Gamma, \mathbb{R}^n), \tag{2.4.19}$$

with the corresponding norm estimates.

We will further need the following integral representation formula for the solution of the homogeneous Brinkman system, in terms of the Gagliardo trace operator and the canonical conormal derivative operator (see, e.g., [41, Lemma 3.7]).

Lemma 2.4.4. *Let Assumption 2.3.1 be satisfied. Let* $\alpha \in (0, \infty)$, $p, q \in (1, \infty)$ *and* $s \in (0, 1)$. *If* $(\mathbf{u}, \pi) \in H_p^{s+\frac{1}{p}}(\mathfrak{D}_+, \mathbb{R}^n) \times H_p^{s-1-\frac{1}{p}}(\mathfrak{D}_+)$, *or* $(\mathbf{u}, \pi) \in B_{p,q}^{s+\frac{1}{p}}(\mathfrak{D}_+, \mathbb{R}^n) \times B_{p,q}^{s-1-\frac{1}{p}}(\mathfrak{D}_+)$ *and the the pair* $(\mathbf{u}, \pi)$ *satisfies the Brinkman system*

$$\triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{0}, \quad \mathrm{div}\ \mathbf{u} = 0 \quad \textit{in} \ \ \mathfrak{D}_+ , \tag{2.4.20}$$

then

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_\alpha \left(\mathbf{t}^+_\alpha(\mathbf{u}, \pi) \right)(\mathbf{x}) - \mathbf{W}_\alpha \left(\gamma^+ \mathbf{u} \right)(\mathbf{x}), \ \forall\ \mathbf{x} \in \mathfrak{D}_+, \tag{2.4.21}$$

$$\pi(\mathbf{x}) = Q^s \left(\mathbf{t}^+_\alpha(\mathbf{u}, \pi) \right)(\mathbf{x}) - Q^d_\alpha \left(\gamma^+ \mathbf{u} \right)(\mathbf{x}), \ \forall\ \mathbf{x} \in \mathfrak{D}_+. \tag{2.4.22}$$

Next, we obtain the analogous integral representation formula (the third Green identity) (2.4.21) in terms of the non-tangential trace and conormal derivative (cf., [41, Lemma 3.8]).

Lemma 2.4.5. *Let Assumption 2.3.1 hold. Let* $\alpha > 0$ *and* $p \in (1, \infty)$. *If* $M(\mathbf{u}), M(\nabla \mathbf{u}), M(\pi) \in L^p(\Gamma)$, *then there exist the non-tangential limits of* $\mathbf{u}$, $\nabla \mathbf{u}$ *and* π *almost everywhere on* Γ, *and that the pair* $(\mathbf{u}, \pi)$ *satisfies the homogeneous Brinkman system*

$$\triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{0}, \quad \mathrm{div}\ \mathbf{u} = 0 \quad \textit{in} \ \ \mathfrak{D}_+. \tag{2.4.23}$$

Then $\mathbf{u}$ *satisfies also the following formula*

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_\alpha \left(\mathbf{t}^+_{\mathrm{nt}}(\mathbf{u}, \pi) \right)(\mathbf{x}) - \mathbf{W}_\alpha \left(\mathbf{u}^+_{\mathrm{nt}} \right)(\mathbf{x}), \quad \forall\ \mathbf{x} \in \mathfrak{D}_+. \tag{2.4.24}$$

2.4.2 Invertibility properties of the layer potential operators

Throughout this thesis, we need some invertibility results regarding the double-layer potential operators. Note that, most of the next results are based on general properties of Fredholm operators and have been obtained in [41, Section 4.]. Recall that the L^2-based Sobolev (Bessel-potential) space $H^s_2(\mathfrak{D}, \mathbb{R}^n)$ is denoted for simplicity by $H^s(\mathfrak{D}, \mathbb{R}^n)$.

To simplify the notations for the next theorems and in the following section for the isomorphism properties for certain operators on L^p-based Sobolev spaces, we introduce the following two intervals as in [41, Relations (166) and (167)],

$$\mathcal{R}_0(n, \varepsilon) = \left(\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon \right) \cap (1, +\infty), \quad \mathcal{R}_1(n, \varepsilon) = \begin{cases} (2 - \varepsilon, +\infty) & \text{if } \ n = 3, \\ \left(2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon \right) & \text{if } \ n > 3 \end{cases}. \tag{2.4.25}$$

These sets are particular cases of the following set

$$\mathcal{R}_\theta(n, \varepsilon) = \begin{cases} (2 - \varepsilon, +\infty) & \text{if } \ n = 3 \text{ and } \theta = 1, \\ \left(\frac{2(n-1)}{n+1-2\theta} - \varepsilon, \frac{2(n-1)}{n-1-2\theta} + \varepsilon \right) \cap (1, +\infty) & \text{if } \ n > 3 \text{ and } 0 \le \theta \le 1 \end{cases}. \tag{2.4.26}$$

Lemma 4.2 in [41] and [41, Lemmas A.1 and B.1(ii)] imply by interpolation the following assertion (cf. [41, Collorary 4.3]).

Corollary 2.4.6. *Let Assumption 2.3.1 hold. Then for* $\alpha \in (0,\infty)$*, there exists* $\varepsilon = \varepsilon(\Gamma) > 0$ *such that for any* $p \in \mathcal{R}_s(n,\varepsilon)$ *and* $p' \in \mathcal{R}_{1-s}(n,\varepsilon)$ *(cf. (2.4.26)), the operators*

$$\frac{1}{2}\mathbb{I} + \mathbf{K}_\alpha : H^s_{p'}(\Gamma,\mathbb{R}^n) \to H^s_{p'}(\Gamma,\mathbb{R}^n), \quad s \in [0,1], \tag{2.4.27}$$

$$\frac{1}{2}\mathbb{I} + \mathbf{K}^*_\alpha : H^{-s}_{p}(\Gamma,\mathbb{R}^n) \to H^{-s}_{p}(\Gamma,\mathbb{R}^n), \quad s \in [0,1], \tag{2.4.28}$$

$$\frac{1}{2}\mathbb{I} + \mathbf{K}_\alpha : B^s_{p',q}(\Gamma,\mathbb{R}^n) \to B^s_{p',q}(\Gamma,\mathbb{R}^n), \quad s \in (0,1),\ q \in (1,\infty), \tag{2.4.29}$$

$$\frac{1}{2}\mathbb{I} + \mathbf{K}^*_\alpha : B^{-s}_{p,q}(\Gamma,\mathbb{R}^n) \to B^{-s}_{p,q}(\Gamma,\mathbb{R}^n), \quad s \in (0,1),\ q \in (1,\infty). \tag{2.4.30}$$

are isomorphisms. If $\mathfrak{D}_+$ *is of class* C^1*, then the properties hold for all* $p, p' \in (1,\infty)$*.*

Lemma 4.4 in [41] and [41, Lemmas A.1 and B.1(ii)] imply by interpolation the following assertion as in [40, Lemma 4.5].

Corollary 2.4.7. *Let Assumption 2.3.1 hold. Then for* $\alpha \in (0,\infty)$*, there exists* $\varepsilon = \varepsilon(\Gamma) > 0$ *such that for any* $p \in \mathcal{R}_s(n,\varepsilon)$ *and* $p' \in \mathcal{R}_{1-s}(n,\varepsilon)$*, cf. (2.4.26), the following operators are isomorphisms,*

$$-\frac{1}{2}\mathbb{I} + \mathbf{K}_\alpha : H^s_{p';\nu}(\Gamma,\mathbb{R}^n) \to H^s_{p';\nu}(\Gamma,\mathbb{R}^n), \quad s \in [0,1], \tag{2.4.31}$$

$$-\frac{1}{2}\mathbb{I} + \mathbf{K}^*_\alpha : H^{-s}_{p}(\Gamma,\mathbb{R}^n)/\mathbb{R}\boldsymbol{\nu} \to H^{-s}_{p}(\Gamma,\mathbb{R}^n)/\mathbb{R}\boldsymbol{\nu}, \quad s \in [0,1], \tag{2.4.32}$$

$$-\frac{1}{2}\mathbb{I} + \mathbf{K}_\alpha : B^s_{p',q;\nu}(\Gamma,\mathbb{R}^n) \to B^s_{p',q;\nu}(\Gamma,\mathbb{R}^n), \quad s \in (0,1),\ q \in (1,\infty), \tag{2.4.33}$$

$$-\frac{1}{2}\mathbb{I} + \mathbf{K}^*_\alpha : B^{-s}_{p,q}(\Gamma,\mathbb{R}^n)/\mathbb{R}\boldsymbol{\nu} \to B^{-s}_{p,q}(\Gamma,\mathbb{R}^n)/\mathbb{R}\boldsymbol{\nu}, \quad s \in (0,1),\ q \in (1,\infty). \tag{2.4.34}$$

If $\mathfrak{D}_+$ *is of class* C^1*, then the properties is valid for any* $p, p' \in (1,\infty)$*.*

In the case $\alpha = 0$, the statement of the lemma below has been obtained in [86, Theorem 9.1.4, Corollary 9.1.5] (see also [85, Theorem 6.1]).

Finally, by an interpolation argument based on [41, Lemmas A.1 and B.1] and Lemma 4.6 in [41], we obtain the following property (see, e.g., [41, Corollary 4.7]).

Corollary 2.4.8. *Let Assumption 2.3.1 hold. Then for* $\alpha \in (0,\infty)$ *and* $p \in \mathcal{R}_s(n,\epsilon)$*, there exists* $\varepsilon = \varepsilon(\Gamma) > 0$ *such that the following operators*

$$\mathcal{V}_\alpha : H^{-s}_{p}(\Gamma,\mathbb{R}^n)/\mathbb{R}\boldsymbol{\nu} \to H^{1-s}_{p;\nu}(\Gamma,\mathbb{R}^n), \quad s \in [0,1], \tag{2.4.35}$$

$$\mathcal{V}_\alpha : B^{-s}_{p,q}(\Gamma,\mathbb{R}^n)/\mathbb{R}\boldsymbol{\nu} \to B^{1-s}_{p,q;\nu}(\Gamma,\mathbb{R}^n), \quad s \in (0,1),\ q \in (1,\infty). \tag{2.4.36}$$

are isomorphisms. If, in addition, $\mathfrak{D}_+$ *is of class* C^1*, then the property holds for any* $p \in (1,\infty)$*.*

Chapter 3

Mixed problems for the Brinkman and Darcy-Forchheimer-Brinkman systems in Besov spaces on bounded creased Lipschitz domains in $\mathbb{R}^n, n \geq 3$

This chapter is concerned with boundary problems of mixed Dirichlet-Neumann type for the linear Brinkman system and the semilinear Darcy-Forchheimer-Brinkman system on creased Lipschitz domains in $\mathbb{R}^n$, where $n \geq 3$, by assuming that the boundary data belong to L^p-based Sobolev spaces with p in a neighborhood of 2, as they have been obtained by joint work with M. Kohr, S. E. Mikhailov and W. L. Wendland in [41, Section 5 and 6]. In order to derive the well-posedness results for the two systems under consideration, we require the well-posedness result for the Dirichlet and Neumann boundary problems for the Brinkman system with boundary data in L^p- based spaces (for related results, we refer the reader to [100, Theorem 5.5], [86, Corollary 9.1.5, Theorem 9.1.4, 9.2.2 and 9.2.5] and [85, Theorem 7.1]). Special attention is given to the special cases related to different type of trace and conormal derivative operators, i.e., to the cases when the operators are considered in the non-tangential or in the Gagliardo trace sense (we refer to Sections 1.5 and 1.6 of Chapter 1, as well as to [41, Sections 2.1 and 2.2]). Moreover, we emphasize the requirement of the special creased Lipschitz domain for mixed type boundary problems, since the boundary data under consideration have high regularity.

Brown et al. in [13, Theorem 1.1] have obtained the solvability result for the mixed Dirichlet-Neumann problem for the Stokes system with boundary data in L^2-based spaces on creased Lipschitz domains in $\mathbb{R}^n$ ($n \geq 3$), by reducing such a boundary problem to the analysis of a boundary integral equations. Moreover, the authors in [51, Theorem 6.1] have proved the well-posedness of the mixed *Dirichlet-Robin* problem for the Brinkman system in a creased Lipschitz domain with boundary data in L^p-based spaces, with p in some neighbourhood of 2. Having in view the results in [13], we show in this section the well-posedness of the mixed *Dirichlet-Neumann* boundary problem for the Brinkman system in L^2-based Bessel potential spaces defined on a bounded, creased Lipschitz domain $\mathfrak{D}$ in $\mathbb{R}^n$ ($n \geq 3$) and finally extending them to L^p-based spaces for some p in the neighborhood of 2 by using the complex interpolation theory and embedding results of Sobolev spaces.

3.1 Boundary value problems for the Brinkman system on Lipschitz domains in $\mathbb{R}^n$

In this section we analyze boundary problems of Dirichlet, Neumann and mixed Dirichlet-Neumann type for the Brinkman system. Recall that by a creased Lipschitz domain we mean a domain satisfying the conditions in Definition 1.1.3. The boundary problems of this section are analyzed for both cases when the boundary condition is considered in the non-tangential and Gagliadro sense.

3.1.1 Mixed Dirichlet-Neumann problem for the Brinkman system with data in L^2-based spaces

We begin by stating the analog version of Assumption 2.3.1 for mixed boundary problems.

Assumption 3.1.1. Let $\mathfrak{D} \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded, *creased* Lipschitz domain with connected boundary Γ, which is decomposed in two disjoint admissible patches Γ_D and Γ_N (see Definition 1.1.3).

This means that Γ_D and Γ_N do not meet tangentially and moreover that, Γ_D and Γ_N are separated by a Lipschitz interface where the angle between them is less than π. Also, let $(\cdot)|_{\Gamma_D}$, $(\cdot)|_{\Gamma_N}$ denote the operator of restriction from $H^s_p(\Gamma, \mathbb{R}^n)$ to $H^s_p(\Gamma_D, \mathbb{R}^n)$ and $H^s_p(\Gamma_N, \mathbb{R}^n)$, respectively.

In this section we show the well-posedness of the mixed *Dirichlet-Neumann* boundary problem for the Brinkman system,

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{0}, \ \text{div } \mathbf{u} = 0 \ \text{ in } \ \mathfrak{D}, \\ \mathbf{u}^+_{\text{nt}}|_{\Gamma_D} = \mathbf{h}_0, \\ \mathbf{t}^+_{\text{nt}}(\mathbf{u}, \pi)|_{\Gamma_N} = \mathbf{g}_0, \end{cases} \tag{3.1.1}$$

where the boundary conditions are considered in the nontangential case as well as the counterpart problem

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{0}, \ \text{div } \mathbf{u} = 0 \ \text{ in } \ \mathfrak{D}, \\ \gamma^+ \mathbf{u}|_{\Gamma_D} = \mathbf{h}_0, \\ \mathbf{t}^+_{\alpha}(\mathbf{u}, \pi)|_{\Gamma_N} = \mathbf{g}_0, \end{cases} \tag{3.1.2}$$

where the trace is taken in the Gagliardo sense and the conormal derivative in the canonical sense.

For the first problem (3.1.1), we prove that for $\mathbf{h}_0 \in H^1_p(\Gamma_D, \mathbb{R}^n)$ and $\mathbf{g}_0 \in L^p(\Gamma_N, \mathbb{R}^n)$ given and for some range of p, there exists an unique L^p-solution of the mixed problem (3.1.3), i.e., an unique pair $(\mathbf{u}, \pi)$ such that $\mathbf{u}$ and π satisfy the Brinkman system in $\mathfrak{D}_+$, there exist the non-tangential limits of $\mathbf{u}$, $\nabla \mathbf{u}$ and π almost everywhere of Γ, $M(\mathbf{u}), M(\nabla \mathbf{u}), M(\pi) \in L^p(\Gamma)$, and the boundary conditions in (3.1.3) are satisfied in the sense of non-tangential limit almost everywhere on Γ_D and Γ_N, respectively. In addition, we show that $(\mathbf{u}, \pi) \in B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D}, \mathbb{R}^n) \times B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D})$.

In the case of problem (3.1.2), where the boundary conditions are understood in the Gagliardo and canonical sense, we will show that for $\mathbf{h}_0 \in H^1_p(\Gamma_D, \mathbb{R}^n)$ and $\mathbf{g}_0 \in L^p(\Gamma_N, \mathbb{R}^n)$ given and for some range of p, there exists a unique L^p-solution $(\mathbf{u}, \pi) \in B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D}, \mathbb{R}^n) \times B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D})$ of the mixed problem (3.1.2), which satisfy $M(\mathbf{u}), M(\nabla \mathbf{u}), M(\pi) \in L^p(\Gamma)$.

Let us mention that, for a bounded, creased Lipschity domain, Brown [11] have proven that the mixed problem for the Laplace equation has an unique solution, and moreover, that it's gradient belongs to $L^2(\Gamma)$ when the Dirichlet datum belongs to $H^1_2(\Gamma_D) = H^1(\Gamma_D)$ and the

Neumann datum to $L^2(\Gamma_N)$. For the same class of domains, well-posedness results in a range of L^p-based spaces have been obtained in [83].

In order to show the following result obtained in [41, Theorem 6.4], we use the main ideas of the proof of [51, Theorem 6.1].

Theorem 3.1.2. *Let $\alpha \in (0, \infty)$. Then under Assumption 3.1.1, the mixed Dirichlet-Neumann boundary problem for the Brinkman system with given data $(\mathbf{h}_0, \mathbf{g}_0)$ in the space $H_2^1(\Gamma_D, \mathbb{R}^n) \times L^2(\Gamma_N, \mathbb{R}^n)$ given by*

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{0} \quad in \quad \mathfrak{D}, \\ \text{div } \mathbf{u} = 0 \quad in \quad \mathfrak{D}, \\ \left(\mathbf{u}_{\text{nt}}^{+}\right)|_{\Gamma_D} = \mathbf{h} \\ \left(\mathbf{t}_{\text{nt}}^{+}(\mathbf{u}, \pi)\right)|_{\Gamma_N} = \mathbf{g}, \\ M(\nabla \mathbf{u}),\ M(\mathbf{u}),\ M(\pi) \in L^2(\Gamma), \end{cases} \tag{3.1.3}$$

has a unique solution $(\mathbf{u}, \pi)$, which satisfies the boundary conditions in the sense of non-tangential limit almost everywhere on Γ_D and Γ_N, respectively. Moreover, $(\mathbf{u}, \pi)$ belongs to the space $H_2^{\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n) \times H_2^{\frac{1}{2}}(\mathfrak{D})$ and there exist some constants C_M and C depending only on Γ_D, Γ_N, α and n such that

$$\|M(\nabla \mathbf{u})\|_{L_2(\Gamma)} + \|M(\mathbf{u})\|_{L_2(\Gamma)} + \|M(\pi)\|_{L_2(\Gamma)} \leq C_M \left(\|\mathbf{h}_0\|_{H_2^1(\Gamma_D, \mathbb{R}^n)} + \|\mathbf{g}_0\|_{L_2(\Gamma_N, \mathbb{R}^n)}\right), \tag{3.1.4}$$

$$\|\mathbf{u}\|_{H_2^{\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n)} + \|\pi\|_{H_2^{\frac{1}{2}}(\mathfrak{D})} \leq C \left(\|\mathbf{h}_0\|_{H_2^1(\Gamma_D, \mathbb{R}^n)} + \|\mathbf{g}_0\|_{L_2(\Gamma_N, \mathbb{R}^n)}\right). \tag{3.1.5}$$

In the next subsection, we extend the results established in Theorem 3.1.2, to L^p-based spaces with p in some neighbourhood of 2, for the mixed boundary problem (3.1.3), with data $(\mathbf{h}_0, \mathbf{g}_0) \in H_p^1(\Gamma_D, \mathbb{R}^n) \times L^p(\Gamma_N, \mathbb{R}^n)$.

Moreover, we show the the solution of the mixed problem (3.1.1) belongs to $B_{p,p^*}^{1+\frac{1}{p}}(\mathfrak{D}, \mathbb{R}^n) \times B_{p,p^*}^{\frac{1}{p}}(\mathfrak{D})$, where $p^* = \max\{2, p\}$.

In the sequel, we need the following space defined for a subset $S_0 \subset \Gamma$

$$\widetilde{H}_p^0(S_0, \mathbb{R}^n) := \left\{\mathbf{\Phi} \in L^p(\Gamma, \mathbb{R}^n) : \text{supp } \mathbf{\Phi} \subseteq \overline{S_0}\right\}. \tag{3.1.6}$$

3.1.2 The Neumann-to-Dirichlet operator for the Brinkman system

Inspired by the work [83], where the authors study the mixed Dirichlet-Neumann boundary problem for the Laplace equation in a creased Lipschitz domain, we introduce the Neumann-to-Dirichlet operator $\Upsilon_{\text{nt};\alpha}$, which associates to the datum $\mathbf{g} \in L^p(\Gamma, \mathbb{R}^n)$, the restriction of the non-tangential trace $\mathbf{u}_{\text{nt}}^{+}$ to the patch Γ_D, where $(\mathbf{u}, \pi)$ is the unique L^p*-solution* of the Neumann problem with the non-tangential conormal derivative $\mathbf{g}$. Thus, $(\mathbf{u}, \pi)$ satisfies the Neumann condition almost everywhere on Γ in the sense of nontangential limit, and the conditions $M(\mathbf{u}), M(\nabla \mathbf{u}), M(\pi) \in L^p(\Gamma)$, and

$$\Upsilon_{\text{nt};\alpha} \mathbf{g} = \mathbf{u}_{\text{nt}}^{+}|_{\Gamma_D}. \tag{3.1.7}$$

Similarly, we consider Υ_α, which associates to $\mathbf{g} \in L^p(\Gamma, \mathbb{R}^n)$, the datum $\gamma^+ \mathbf{u}$ to the patch Γ_D, where $(\mathbf{u}, \pi)$ is the unique solution of the Neumann problem with $\mathbf{f} = \mathbf{0}$ and the canonical conormal derivative $\mathbf{g}$, i.e.,

$$\Upsilon_\alpha \mathbf{g} = \gamma^+ \mathbf{u}|_{\Gamma_D}. \tag{3.1.8}$$

The key idea here, is the property that the invertibility of each of the Neumann-to-Dirichlet operator $\Upsilon_{\text{nt};\alpha}$ and Υ_α on L^p-based Sobolev spaces leads to the extension of the well-posedness result in Theorem 3.1.2 on such spaces. An intermediary step to obtain this property is given by the following result, obtained in [41, Lemma 6.5]. Recall that $\mathcal{R}_0(n,\varepsilon)$ is the set defined in (2.4.25).

Lemma 3.1.3. *Let $\alpha \in (0,\infty)$. Then under Assumption 3.1.1, there exists $\varepsilon = \varepsilon(\Gamma) > 0$ such that for any $p \in \mathcal{R}_0(n,\varepsilon)$ the following properties hold.*

(*i*) *The operators $\Upsilon_{\mathrm{nt};\alpha}$ and Υ_α coincide, are continuous and have the expression*

$$\Upsilon_{\mathrm{nt};\alpha} = \Upsilon_\alpha = \left(\mathcal{V}_\alpha \circ \left(\frac{1}{2}\mathbb{I} + \mathbf{K}_\alpha^*\right)^{-1}\right)\Bigg|_{\Gamma_D}. \tag{3.1.9}$$

(*ii*) *The mixed Dirichlet-Neumann boundary problem* (3.1.1) *with data* $(\mathbf{h}_0,\mathbf{g}_0) \in H_p^1(\Gamma_D,\mathbb{R}^n) \times L^p(\Gamma_N,\mathbb{R}^n)$ *has a unique solution* $(\mathbf{u},\pi)$ *such that there exist the non-tangential limits of* $\mathbf{u}$, $\nabla\mathbf{u}$ *and* π *almost everywhere on* Γ, $M(\boldsymbol{u}), M(\nabla\boldsymbol{u}), M(\pi) \in L^p(\Gamma)$ *and* $\mathbf{u}$ *and* π *satisfy the Dirichlet and Neumann boundary conditions almost everywhere as a non-tangential limit, if and only if the operator*

$$\Upsilon_{\mathrm{nt};\alpha} : \widetilde{H}_p^0(\Gamma_D,\mathbb{R}^n) \to H_p^1(\Gamma_D,\mathbb{R}^n) \tag{3.1.10}$$

is an isomorphism.

(*iii*) *Problem* (3.1.2) *with data* $(\mathbf{h}_0,\mathbf{g}_0) \in H_p^1(\Gamma_D,\mathbb{R}^n) \times L^p(\Gamma_N,\mathbb{R}^n)$ *has a unique solution* $(\boldsymbol{u},\pi) \in B_{p,p^*}^{1+\frac{1}{p}}(\mathfrak{D},\mathbb{R}^n) \times B_{p,p^*}^{\frac{1}{p}}(\mathfrak{D})$ *if and only if the operator*

$$\Upsilon_\alpha : \widetilde{H}_p^0(\Gamma_D,\mathbb{R}^n) \to H_p^1(\Gamma_D,\mathbb{R}^n) \tag{3.1.11}$$

is an isomorphism.

In addition, when the solution $(\mathbf{u},\pi)$ exists, then it belongs to the space $B_{p,p^*}^{1+\frac{1}{p}}(\mathfrak{D},\mathbb{R}^n) \times B_{p,p^*}^{\frac{1}{p}}(\mathfrak{D})$ and there exist some constants $C_M \equiv C_M(\alpha,p,\Gamma_D,\Gamma_N,n) > 0$, $C \equiv C(\alpha,p,\Gamma_D,\Gamma_N,n) > 0$ and $C' \equiv C'(\alpha,p,\Gamma_D,\Gamma_N,n) > 0$ such that

$$\|M(\nabla\mathbf{u})\|_{L^p(\Gamma)} + \|M(\mathbf{u})\|_{L^p(\Gamma)} + \|M(\pi)\|_{L^p(\Gamma)} \le C_M\left(\|\mathbf{h}_0\|_{H_p^1(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}_0\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right), \tag{3.1.12}$$

$$\|\mathbf{u}\|_{B_{p,p^*}^{1+\frac{1}{p}}(\mathfrak{D},\mathbb{R}^n)} + \|\pi\|_{B_{p,p^*}^{\frac{1}{p}}(\mathfrak{D})} \le C\left(\|\mathbf{h}_0\|_{H_p^1(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}_0\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right), \quad p^* = \max\{2,p\}, \tag{3.1.13}$$

$$\|\gamma^+\mathbf{u}\|_{H_p^1(\Gamma,\mathbb{R}^n)} + \|\mathbf{t}^+(\mathbf{u},\pi)\|_{L^p(\Gamma,\mathbb{R}^n))} \le C'\left(\|\mathbf{h}_0\|_{H_p^1(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}_0\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right). \tag{3.1.14}$$

3.1.3 Mixed Dirichlet-Neumann problem for the Brinkman system with data in L^p-based spaces

Now, we are able to formulate the main result of the section, which refers to the well-posedness for the mixed Dirichlet-Neumann problem (3.1.3) with boundary data in L^p-based Bessel potential spaces and with p in a neighborhood of 2, when the boundary conditions are considered in the non-tangential sense. The proof of this result is a consequence of Theorem 3.1.2 and Lemma 3.1.3 and has been obtained in our paper [41, Theorem 6.6(i)].

Theorem 3.1.4. *Let $\alpha \in (0,\infty)$ and let Assumption 3.1.1 be satisfied. Then there exists a number $\varepsilon > 0$ such that for any $p \in (2-\varepsilon, 2+\varepsilon)$ and for all given data $(\mathbf{h},\mathbf{g}) \in H^1_p(\Gamma_D,\mathbb{R}^n) \times L^p(\Gamma_N,\mathbb{R}^n)$ the mixed type boundary problem for the Brinkman system*

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{0} \quad in \ \mathfrak{D}, \\ \operatorname{div} \mathbf{u} = 0 \quad in \ \mathfrak{D}, \\ \left(\mathbf{u}^+_{\mathrm{nt}}\right)|_{\Gamma_D} = \mathbf{h} \in H^1_p(\Gamma_D,\mathbb{R}^n) \\ \left(\mathbf{t}^+_{\mathrm{nt}}(\mathbf{u},\pi)\right)|_{\Gamma_N} = \mathbf{g} \in L^p(\Gamma_N,\mathbb{R}^n), \\ M(\nabla \mathbf{u}),\ M(\mathbf{u}),\ M(\pi) \in L^p(\Gamma), \end{cases} \tag{3.1.15}$$

where the Dirichlet and Neumann boundary conditions are satisfied in the sense of non-tangential limit almost everywhere on Γ_D and Γ_N, respectively, has a unique solution $(\mathbf{u},\pi)$. Moreover, $(\mathbf{u},\pi) \in B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D},\mathbb{R}^n) \times B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D})$, and there exist some constants $C_j \equiv C_j(\alpha,p,\Gamma_D,\Gamma_N,n) > 0$, $j = 1,\ldots,3$ such that

$$\|M(\nabla \mathbf{u})\|_{L^p(\Gamma)} + \|M(\mathbf{u})\|_{L^p(\Gamma)} + \|M(\pi)\|_{L^p(\Gamma)} \le C_1 \left(\|\mathbf{h}\|_{H^1_p(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right), \tag{3.1.16}$$

$$\|\mathbf{u}\|_{B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D},\mathbb{R}^n)} + \|\pi\|_{B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D})} \le C_2 \left(\|\mathbf{h}\|_{H^1_p(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right), \tag{3.1.17}$$

$$\|\gamma^+ \mathbf{u}\|_{H^1_p(\Gamma,\mathbb{R}^n)} + \|\mathbf{t}^+_\alpha(\mathbf{u},\pi)\|_{L^p(\Gamma,\mathbb{R}^n))} \le C_3 \left(\|\mathbf{h}\|_{H^1_p(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right). \tag{3.1.18}$$

The next theorem is the counterpart of Theorem 3.1.4, when the trace operator is considered in the Gagliardo sense and the conormal derivative in the canonical sense. This result refers to the second part of Theorem 6.6 in our work [41].

Theorem 3.1.5. *Let Assumption 3.1.1 hold. Then for any $\alpha \in (0,\infty)$, there exists a number $\varepsilon > 0$ such that for any $p \in (2-\varepsilon, 2+\varepsilon)$ and for all given data $(\mathbf{h},\mathbf{g}) \in H^1_p(\Gamma_D,\mathbb{R}^n) \times L^p(\Gamma_N,\mathbb{R}^n)$ the mixed Dirichlet-Neumann boundary problem given by*

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{0} \quad in \ \mathfrak{D}, \\ \operatorname{div} \mathbf{u} = 0 \quad in \ \mathfrak{D}, \\ (\gamma^+ \mathbf{u})|_{\Gamma_D} = \mathbf{h} \in H^1_p(\Gamma_D,\mathbb{R}^n) \\ (\mathbf{t}^+_\alpha(\mathbf{u},\pi))|_{\Gamma_N} = \mathbf{g} \in L^p(\Gamma_N,\mathbb{R}^n), \end{cases} \tag{3.1.19}$$

has a unique solution $(\mathbf{u},\pi)$ with boundary conditions taken in the Gagliadro sense. Moreover, $(\mathbf{u},\pi) \in B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D},\mathbb{R}^n) \times B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D})$, and there exist some constants $C_j \equiv C_j(\alpha,p,\Gamma_D,\Gamma_N,n) > 0$, $j = 1,2,3$ such that

$$\|M(\nabla \mathbf{u})\|_{L^p(\Gamma)} + \|M(\mathbf{u})\|_{L^p(\Gamma)} + \|M(\pi)\|_{L^p(\Gamma)} \le C_1 \left(\|\mathbf{h}\|_{H^1_p(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right), \tag{3.1.20}$$

$$\|\mathbf{u}\|_{B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D},\mathbb{R}^n)} + \|\pi\|_{B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D})} \le C_2 \left(\|\mathbf{h}\|_{H^1_p(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right), \tag{3.1.21}$$

$$\|\gamma^+ \mathbf{u}\|_{H^1_p(\Gamma,\mathbb{R}^n)} + \|\mathbf{t}^+_\alpha(\mathbf{u},\pi)\|_{L^p(\Gamma,\mathbb{R}^n))} \le C_3 \left(\|\mathbf{h}\|_{H^1_p(\Gamma_D,\mathbb{R}^n)} + \|\mathbf{g}\|_{L^p(\Gamma_N,\mathbb{R}^n)}\right). \tag{3.1.22}$$

3.1.4 Poisson problem of mixed Dirichlet and Neumann type for the Brinkman system with data in L^p-based spaces

Using Theorem 3.1.4, we prove the well-posedness of the following Poisson problem of mixed Dirichlet-Neumann type for the Brinkman system in a creased Lipschitz domain $\mathfrak{D}$, with data in some L^p-based spaces,

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{f} \in L^p(\mathfrak{D},\mathbb{R}^3), \\ \operatorname{div} \mathbf{u} = 0 \ \text{ in } \ \mathfrak{D} \\ (\gamma^+ \mathbf{u})|_{\Gamma_D} = \mathbf{h}_0 \in H^1_p(\Gamma_D,\mathbb{R}^3) \\ (\mathbf{t}^+_\alpha(\mathbf{u},\pi))|_{\Gamma_N} = \mathbf{g}_0 \in L^p(\Gamma_N,\mathbb{R}^3), \end{cases} \tag{3.1.23}$$

where the trace and the conormal derivative are considered in Gagliardo and canonical sense. First, we mention the following definition.

Definition 3.1.6. By a *solution* of the boundary problem (3.1.23) we mean a pair $(\mathbf{u}, \pi) \in B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D}_+, \mathbb{R}^n) \times B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D}_+)$, where $p^* = \max\{2, p\}$, which satisfies the non-homogeneous Brinkman system in $\mathfrak{D}_+$, the Dirichlet condition on Γ_D in the Gagliardo trace sense, and the Neumann condition on Γ_N in the canonical sense described in Definition 1.5.3.

Theorem 3.1.7. *Let Assumption 3.1.1 hold. Then for any $\alpha \in (0, \infty)$, there exists a number $\varepsilon > 0$ such that for any $p \in (2-\varepsilon, 2+\varepsilon)$ and for all given data $(\mathbf{f}, \mathbf{h}_0, \mathbf{g}_0) \in L^p(\mathfrak{D}, \mathbb{R}^n) \times H^1_p(\Gamma_D, \mathbb{R}^3) \times L^p(\Gamma_N, \mathbb{R}^3)$ the boundary problem (3.1.23) has a unique solution*

$$(\mathbf{u}, \pi) \in B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D}, \mathbb{R}^3) \times B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D}), \tag{3.1.24}$$

where $p^* = \max\{2, p\}$. The solution satisfies the conditions

$$\gamma^+\mathbf{u} \in H^1_p(\Gamma, \mathbb{R}^3), \quad \mathbf{t}^+_\alpha(\mathbf{u}, \pi) \in L^p(\Gamma, \mathbb{R}^3), \tag{3.1.25}$$

and there exists a linear, continuous operator

$$\mathcal{A}_p : L^p(\mathfrak{D}, \mathbb{R}^3) \times H^1_p(\Gamma_D, \mathbb{R}^3) \times L^p(\Gamma_N, \mathbb{R}^3) \to B^{1+\frac{1}{p}}_{p,p^*}(\mathfrak{D}, \mathbb{R}^3) \times B^{\frac{1}{p}}_{p,p^*}(\mathfrak{D})$$

delivering this solution, which means that $\mathcal{A}_p(\mathbf{f}, \mathbf{h}_0, \mathbf{g}_0) = (\mathbf{u}, \pi)$.

3.2 Semilinear Problems for the Darcy-Forchheimer-Brinkman System

In this section we provide a well-posedness result for the mixed Dirichlet-Neumann boundary problem for the semilinear Darcy-Forchheimer-Brinkman system.

3.2.1 Mixed Dirichlet-Neumann problem for the semilinear Darcy-Forchheimer-Brinkman system in Besov spaces

Next, we study the mixed Dirichlet-Neumann problem for the semilinear Darcy-Forchheimer-Brinkman system

$$\begin{cases} \triangle\mathbf{u} - \alpha\mathbf{u} - \beta|\mathbf{u}|\mathbf{u} - \nabla\pi = \mathbf{0} \ \text{ in } \ \mathfrak{D}, \\ \text{div } \mathbf{u} = 0 \ \text{ in } \ \mathfrak{D}. \end{cases} \tag{3.2.1}$$

This semilinear system plays an major role in fluid mechanics, since it describes the flow in porous media saturated with incompressible Newtonian fluids [87, p. 17]. The constants $\alpha, \beta > 0$ are related to the physical properties of such a porous medium and describe the convection of the fluid flow.

The analysis of this problem is restricted to the three-dimensional setting, due to the fact our arguments are based on some embedding results.

In the sequel, we extend the solvability result obtained in [51, Theorem 7.1] for (3.2.1) with the given data in L^2-based Sobolev spaces, to the case of L^p-based Bessel potential spaces, when the given boundary data $(\mathbf{h}_0, \mathbf{g}_0)$ belong to the space $H^1_p(\Gamma_D, \mathbb{R}^3) \times L^p(\Gamma_N, \mathbb{R}^3)$, with $p \in (2-\varepsilon, 2+\varepsilon)$ and a constant $\varepsilon > 0$ as in Theorem 3.1.7, and this data is sufficiently small. The next theorem has been obtained in collaboration with M. Kohr, S. Mikhailov and W. Wendland in [41, Theorem 7.1].

Theorem 3.2.1. *Let Assumption 3.1.1 hold. Then for all $\alpha, \beta \in (0, \infty)$, there exists $\varepsilon > 0$ such that for any $p \in (2-\varepsilon, 2+\varepsilon)$ there exist two constants $\zeta_p \equiv \zeta_p(\mathfrak{D}, \alpha, \beta, p) > 0$ and $\eta_p \equiv \eta_p(\mathfrak{D}, \alpha, \beta, p) > 0$ with the property that for all given data $(\mathbf{h}_0, \mathbf{g}_0) \in H_p^1(\Gamma_D, \mathbb{R}^3) \times L^p(\Gamma_N, \mathbb{R}^3)$ satisfying the condition*

$$\|\mathbf{h}_0\|_{H_p^1(\Gamma_D,\mathbb{R}^3)} + \|\mathbf{g}_0\|_{L^p(\Gamma_N,\mathbb{R}^3)} \leq \zeta_p, \tag{3.2.2}$$

the mixed boundary problem for the semilinear Darcy-Forchheimer-Brinkman system

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \beta |\mathbf{u}|\mathbf{u} - \nabla \pi = \mathbf{0} & \textit{in } \mathfrak{D}, \\ \operatorname{div} \mathbf{u} = 0 & \textit{in } \mathfrak{D}, \\ (\gamma^+ \mathbf{u})\,|_{\Gamma_D} = \mathbf{h}_0 & \textit{on } \Gamma_D \\ (\mathbf{t}_\alpha^+(\mathbf{u}, \pi))\,|_{\Gamma_N} = \mathbf{g}_0 & \textit{on } \Gamma_N \end{cases} \tag{3.2.3}$$

has a unique solution $(\mathbf{u}, \pi) \in B_{p,p^*}^{1+\frac{1}{p}}(\mathfrak{D}, \mathbb{R}^3) \times B_{p,p^*}^{\frac{1}{p}}(\mathfrak{D})$, which satisfies

$$\|\mathbf{u}\|_{B_{p,p^*}^{1+\frac{1}{p}}(\mathfrak{D},\mathbb{R}^3)} \leq \eta_p, \tag{3.2.4}$$

and the relations $\gamma^+ \mathbf{u} \in H_p^1(\Gamma, \mathbb{R}^3)$, $\mathbf{t}_\alpha^+(\mathbf{u}, \pi) \in L^p(\Gamma, \mathbb{R}^3)$. Moreover, the solution depends continuously on the given data, which means that there exists $C \equiv C(\mathfrak{D}, \alpha, \beta, p) > 0$ such that

$$\|\mathbf{u}\|_{B_{p,p^*}^{1+\frac{1}{p}}(\mathfrak{D},\mathbb{R}^3)} + \|\pi\|_{B_{p,p^*}^{\frac{1}{p}}(\mathfrak{D})} \leq C \left(\|\mathbf{h}_0\|_{H_p^1(\Gamma_D,\mathbb{R}^3)} + \|\mathbf{g}_0\|_{L^p(\Gamma_N,\mathbb{R}^3)} \right). \tag{3.2.5}$$

Chapter 4

Variational and potential approach for mixed problems for the Brinkman and the Darcy-Forchheimer-Brinkman systems in $\mathbb{R}^2$

In this chapter, we focus our attention to the weak solution of boundary problems for the Brinkman system on Lipschitz domains of the two dimensional Euclidean space $\mathbb{R}^2$. We consider a variational formulation for the related boundary integral equations derived from the boundary value problems under consideration. As in the previous chapter, we consider the Dirichlet, Neumann and also the Robin problem for the Brinkman system. The main sources used in the preparation of this chapter are the papers [37] and [40], based on [61], [76]. In addition, we also analyze the mixed Dirichlet-Robin boundary problem for the nonlinear Darcy-Forchheimer-Brinkman system (cf., e.g., [37, Theorem 2.9], see also [40, Theorem 3.2] for the Robin problem).

We begin with a short presentation of some valuable results related to elliptic boundary problems in two dimensional Euclidean setting. Many authors have considered boundary problems related to fluid flow problems in two-dimensions. Two-dimensional problems are of great interest, since many oceanographic and meteorological problems can be reduced to the study of such problems [26], [71]. Moreover, great efforts have been made in order to study exterior and interior fluid flows around or inside a cylinder or another irregular domain in two dimensions [64], [92]. In [45], Hsiao and Kress studied the two-dimensional exterior Dirichlet problem for the Stokes system by reducing the problem to a system of Fredholm integral equations of the second kind. Exterior two-dimensional Stokes flow problems in multiply connected domains were studied by Power [91] by employing a completed double-layer boundary integral method. An extention of the main results in [30], obtained in the flat, Euclidean setting to the case of Lipschitz domains in Riemannian domains is obtained by Mitrea and Taylor in [85], where a detailed treatment of the Stokes layer potential is presented.

A main reference throughout this chapter is the paper published by Kohr and Wendland [61], in which they analyze direct boundary equations for the Dirichlet, the Neumann, and the mixed boundary problem of the Stokes system in $\mathbb{R}^3$ on Lipschitz boundaries, by employing variational formulations. Many coerciveness properties described in the sequel for the Brinkman system are based on the coerciveness properties proved in [61]. Based on the well-posedness of the Dirichlet problem we construct the existence and uniqueness results of the mixed Dirichlet-Robin problem for the nonlinear Darcy-Forchheimer-Brinkman system. This result constitutes the theoretical foundation of the mixed boundary problems which are studied numerically in Chapter 7.

4.1 Variational formulation for boundary value problems for the Brinkman system

Everywhere in the section, we assume $n = 2$ and, in addition, we consider the following:

Assumption 4.1.1. Let $\mathfrak{D} \subset \mathbb{R}^2$ denote a bounded Lipschitz domain with connected boundary $\Gamma = \partial\mathfrak{D}$.

Following the main outline of the previous section, we begin with the analysis of the mixt Dirichlet-Neumann problem for the Brinkman system as in Section 3 by employing a variational approach, based on that in [37].

4.1.1 Variational formulation of the mixed Dirichlet-Neumann problem for the Brinkman system

Similarly to the previous subsections, we consider the following:

Assumption 4.1.2. Recall that $\mathfrak{D} \subset \mathbb{R}^2$ satisfies Assumption 4.1.1, i.e., $\mathfrak{D}$ is a bounded Lipschitz domain with connected boundary $\Gamma = \partial\mathfrak{D}$. In addition, let Γ_D, Γ_N be relatively open, disjoint, non-empty subsets of Γ such that $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ (see Definition 1.1.2).

Then, we consider the mixed Dirichlet-Neumann boundary problem for the Brinkman system

$$\begin{cases} \Delta \mathbf{u} - \alpha\mathbf{u} - \nabla\pi = 0 \text{ in } \mathfrak{D}, \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \mathfrak{D}, \\ (\gamma^+ \, \mathbf{u})|_{\Gamma_D} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2), \\ (\mathbf{t}_\alpha^+(\mathbf{u}, \pi))_{\Gamma_N} = \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R}^2), \end{cases} \tag{4.1.1}$$

where $(\cdot)|_{\Gamma_D}$ denotes the restriction operator from the Sobolev space $H^{\frac{1}{2}}(\Gamma, \mathbb{R}^2)$ to $H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2)$, and $(\cdot)|_{\Gamma_N}$ is the restriction operator from $H^{-\frac{1}{2}}(\Gamma, \mathbb{R}^2)$ to $H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R}^2)$.

Before we state an equivalence theorem between the mixed problem (4.1.1) and a system of boundary integral equations, let us make first some observations. We reformulate the boundary problem (4.1.1) as a system of boundary integral equations, inspired by the main ideas in [61] as we have done in [37] (see also [76, Theorem 7.9]). Starting with the Green representation of a weak solution (see, e.g., [46] and [86] for $\alpha = 0$)

$$\mathbf{u}(x) = \mathbf{V}_\alpha(\mathbf{t}_\alpha^+(\mathbf{u}, \pi)) - \mathbf{W}_\alpha(\gamma^+\mathbf{u}), \quad \pi(x) = Q_\alpha^s(\mathbf{t}_\alpha(\mathbf{u}, \pi)) - Q_\alpha^d(\gamma^+ \, \mathbf{u}), \tag{4.1.2}$$

and taking the traces from inside of $\mathfrak{D}$, we obtain the following equation

$$\mathcal{V}_\alpha(\mathbf{t}_\alpha^+(\mathbf{u}, \pi)) - \left(\frac{1}{2}\mathbb{I} + \mathbf{K}_\alpha\right)\gamma^+\mathbf{u} = 0. \tag{4.1.3}$$

Now, by applying the conormal derivative operator in (4.1.2) and by using the corresponding jump formulas (Theorem 2.4.2), we obtain the equation

$$\left(-\frac{1}{2}\mathbb{I} + \mathbf{K}_\alpha^*\right)(\mathbf{t}_\alpha^+(\mathbf{u}, \pi)) - \mathbf{D}_\alpha\gamma^+\mathbf{u} = 0. \tag{4.1.4}$$

The definition of the spaces $H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2)$ and $H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R}^2)$ imply that there exist $\mathbf{h}^* \in H^{\frac{1}{2}}(\Gamma, \mathbb{R}^2)$ and $\mathbf{g}^* \in H^{-\frac{1}{2}}(\Gamma, \mathbb{R}^2)$ such that $\mathbf{h}^*|_{\Gamma_D} = \mathbf{h}$ and $\mathbf{g}^*|_{\Gamma_N} = \mathbf{g}$. Therefore, the trace and conormal derivative $\gamma^+\mathbf{u}$ and $t_\alpha^+(\mathbf{u}, \pi)$ for the mixed problem (4.1.1) can be written as

$$\left(\gamma^+\mathbf{u}\right)|_\Gamma = \varphi_N + \mathbf{h}^*, \quad \left(\mathbf{t}_\alpha^+(\mathbf{u}, \pi)\right)|_\Gamma = \psi_D + \mathbf{g}^*. \tag{4.1.5}$$

with some unknowns $\varphi_N \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \mathbb{R}^2)$ and $\psi_D \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \mathbb{R}^2)$.

According to the condition $\langle \gamma^+ \mathbf{u}, \boldsymbol{\nu} \rangle = 0$ that should be satisfied by the trace field $\gamma^+\mathbf{u}$ on Γ, we choose an extension $\mathbf{h}_D^*$ of the Dirichlet datum $\mathbf{h}$, such that the following orthogonality condition holds $\langle \mathbf{h}_D^*, \boldsymbol{\nu} \rangle = 0$, i.e.,

$$\mathbf{h}_D^* \in H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\Gamma, \mathbb{R}^2). \tag{4.1.6}$$

Therefore, the trace of $\mathbf{u}$ on Γ can be written as

$$\left(\gamma^+ \mathbf{u}\right)|_{\Gamma} = \varphi_N + \mathbf{h}_D^*, \tag{4.1.7}$$

and by the continuity equation and the flux-divergence theorem, we deduce that the desired density φ_N satisfies also the orthogonality condition

$$\langle \varphi_N, \boldsymbol{\nu} \rangle = 0, \text{ i.e., } \varphi \in \widetilde{H}_{\boldsymbol{\nu}}^{\frac{1}{2}}(\Gamma, \mathbb{R}^2). \tag{4.1.8}$$

Before we begin with the analysis of the boundary equations (4.1.3), (4.1.4) related to the mixed Dirichlet-Neumann problem for the Brinkman system, we introduce for clarity the following notations. Since we are working with restrictions of the boundary, let us denote by

$$\mathcal{V}_\alpha^D \psi := \mathcal{V}_\alpha \psi|_{\Gamma_D}, \quad \mathbf{K}_\alpha^{*N} \psi := \mathbf{K}_\alpha^* \varphi|_{\Gamma_N}, \quad \psi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \mathbb{R}^2), \tag{4.1.9}$$

$$\mathbf{K}_\alpha^D \varphi := \mathbf{K}_\alpha \varphi|_{\Gamma_D}, \quad \mathbf{D}_\alpha^N \varphi := \mathbf{D}_\alpha \varphi|_{\Gamma_N}, \quad \varphi \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \mathbb{R}^2). \tag{4.1.10}$$

By restricting equation (4.1.3) to Γ_D and equation (4.1.4) to Γ_N, we obtain the system of boundary equations with the unknowns $\psi_D \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \mathbb{R}^2)$ and $\varphi_N \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \mathbb{R}^2)$ in the form

$$\begin{cases} \mathcal{V}_\alpha^D \psi_D - \mathbf{K}_\alpha^D \varphi_N = f_1, & x \in \Gamma_D \\ \mathbf{K}_\alpha^{*N} \psi_D - \mathbf{D}_\alpha^N \varphi_N = f_2, & x \in \Gamma_N \end{cases} \tag{4.1.11}$$

where $(f_1, f_2) \in H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2) \times H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R}^2)$ are given by

$$f_1 = \frac{1}{2}\mathbf{h}_D^* + \mathbf{K}_\alpha^D \mathbf{h}_D^* - \mathcal{V}_\alpha^D \mathbf{g}^*, \quad f_2 = \mathbf{D}_\alpha^N \mathbf{h}_D^* + \frac{1}{2}\mathbf{g}^* - \mathbf{K}_\alpha^{*N} \mathbf{g}^*. \tag{4.1.12}$$

Having the above arguments, we state an equivalence theorem between the mixed Dirichlet-Neumann boundary problem for the Brinkman system (4.1.1) and the system of boundary integral equations (4.1.11). We also refer to Theorem 7.9 in [76].

Theorem 4.1.3. *Let Assumption 4.1.2 hold and let* $\alpha \in (0, \infty)$. *Let* $\mathbf{h} \in H^{\frac{1}{2}}(\Gamma, \mathbb{R}^2)$ *and* $\mathbf{g} \in H^{-\frac{1}{2}}(\Gamma, \mathbb{R}^2)$. *Also, let* $\mathbf{g}^*$ *and* $\mathbf{h}_D^*$ *be given by* (4.1.5) *and* (4.1.6), *respectively. Then the following statements hold.*

(i) *If* $(\mathbf{u}, \pi) \in H^1(\mathfrak{D}, \mathbb{R}^2) \times L^2(\mathfrak{D})$ *is a solution of* (4.1.1), *then* (ψ_D, φ_N) *given by*

$$\psi_D = \mathbf{t}_\alpha^+(\mathbf{u}, \pi) - \mathbf{g}^*, \quad \varphi_N = \gamma^+ \mathbf{u} - \mathbf{h}_D^*, \tag{4.1.13}$$

is a solution of (4.1.11). *Moreover, the solution* $(\mathbf{u}, \pi)$ *can be represented*

$$\mathbf{u} = \mathbf{V}_\alpha(\psi_D + \mathbf{g}^*) - \mathbf{W}_\alpha(\varphi_N + \mathbf{h}_D^*), \quad \pi = Q_\alpha^s(\psi_D + \mathbf{g}^*) - Q_\alpha^d(\varphi_N + \mathbf{h}_D^*), \tag{4.1.14}$$

(ii) *If,* $(\varphi_N, \psi_D) \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \mathbb{R}^2) \times \widetilde{H}^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2)$ *is a solution of the boundary integral equations* (4.1.11), *then the formula* (4.1.14) *defines a solution of* (4.1.1).

For simplicity, we introduce the notation $\mathcal{H}$ for the product space

$$\mathcal{H} := \widetilde{H}_{\boldsymbol{\nu}}^{\frac{1}{2}}(\Gamma_N, \mathbb{R}^2) \times \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \mathbb{R}^2) \subset H^{\frac{1}{2}}(\Gamma, \mathbb{R}^2) \times H^{-\frac{1}{2}}(\Gamma, \mathbb{R}^2). \tag{4.1.15}$$

Let us define the bilinear form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ as

$$a((\varphi_N, \psi_D); (\varphi, \psi)) := \langle \mathcal{V}_\alpha^D \psi_D, \psi \rangle - \langle \mathbf{K}_\alpha^D \varphi_N, \psi \rangle + \langle \mathbf{K}_\alpha^{*N} \psi_D, \varphi \rangle + \langle -\mathbf{D}_\alpha^N \varphi_N, \varphi \rangle. \tag{4.1.16}$$

Then we consider the following variational problem for the mixed Dirichlet-Neumann boundary problem for the Brinkman system (4.1.1). Find $(\varphi_N, \psi_D) \in \mathcal{H}$ such that the following equation is satisfied (see [61] for $\alpha = 0$)

$$a((\varphi_N, \psi_D); (\varphi, \psi)) = l(\varphi, \psi), \qquad \forall (\varphi, \psi) \in \mathcal{H}, \tag{4.1.17}$$

where

$$l(\varphi, \psi) = \left\langle \frac{1}{2}\mathbf{h}_D^* + \mathbf{K}_\alpha^D \mathbf{h}_D^* - \mathcal{V}_\alpha^D \mathbf{g}^*, \psi \right\rangle + \left\langle \mathbf{D}_\alpha^N \mathbf{h}_D^* + \frac{1}{2}\mathbf{g}^* - \mathbf{K}_\alpha^{*N} \mathbf{g}^*, \varphi \right\rangle. \tag{4.1.18}$$

Before we attend to study the system of integral equations (4.1.11), we analyze the coerciveness properties of the associated single and double-layer potential operators.

Theorem 4.1.4. *Under Assumption 4.1.2 and for $\alpha \in (0, \infty)$, the following assertions hold.*

(i) The single-layer integral operator

$$\mathcal{V}_\alpha^D : \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \mathbb{R}^2) \to H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2), \tag{4.1.19}$$

satisfies the coerciveness inequality

$$\langle \mathcal{V}_\alpha^D \psi, \psi \rangle \geq c_V \|\psi\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \mathbb{R}^2)}, \quad \forall\ \psi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \mathbb{R}^2). \tag{4.1.20}$$

(ii) The hypersingular integral operator

$$\boldsymbol{D}_\alpha^N : \widetilde{H}_{\boldsymbol{\nu}}^{\frac{1}{2}}(\Gamma_N, \mathbb{R}^2) \to H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R}^2)/\mathbb{R}\boldsymbol{\nu}, \tag{4.1.21}$$

satisfies the coerciveness inequality

$$\langle -\boldsymbol{D}_\alpha^N \varphi, \varphi \rangle \geq c_D \|\varphi\|_{\widetilde{H}_{\boldsymbol{\nu}}^{\frac{1}{2}}(\Gamma_N, \mathbb{R}^2)}, \quad \forall\ \varphi \in \widetilde{H}_{\boldsymbol{\nu}}^{\frac{1}{2}}(\Gamma_N, \mathbb{R}^2). \tag{4.1.22}$$

Next we show the following well-posedness result for the variational problem (4.1.17).

Theorem 4.1.5. *Let Assumption 4.1.2 hold and let $\alpha \in (0, \infty)$. Then the variational problem* (4.1.17) *has a unique solution.*

4.2 The Poisson problem with mixed boundary conditions for the Brinkman system

The following subsection is devoted to the extension of the solvability result for the mixed Dirichlet-Neumann problem for the homogeneous Brinkman system to the related Poisson problem using a constructive approach regarding the Newtonian layer potential ([37, Theorem 3]). Afterward, we focus our attention to the Dirichlet-Robin boundary problem for the Brinkman system, which plays a main role in the analysis of the nonlinear systems in the last part of this section. The desired well-posednes result is based on the Fredholm property of the related operator and has been obtained in our work [37, Theorem 4].

4.2.1 The Poisson problem of mixed type for the Brinkman system with mixed Dirichlet and Neumann boundary conditions

In this section we show the well-posedness result of the weak solution for the mixed boundary problem of Dirichlet-Neumann type for the Brinkman system in L^2-based Sobolev spaces defined on a bounded Lipschtz domain $\mathfrak{D}$ in $\mathbb{R}^2$ with connected boundary. The well-posedness result for this problem is based on the well-posedness property obtained in the previous section, and follows similar arguments as in [53], where the authors have studied the Poisson problem for the Stokes and Brinkman system (see also [65, Section 4]).

For simplicity of notation, let us define the solution space $\mathcal{X}$, the space of given boundary data $\mathcal{B}$ and the space $\mathcal{Y}$ for the mixed boundary problem for the Brinkman system as

$$\mathcal{X} := H^1(\mathfrak{D}, \mathbb{R}^2) \times L^2(\mathfrak{D}), \quad \mathcal{B} := H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2) \times H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R}^2), \quad \mathcal{Y} := \widetilde{H}^{-1}(\mathfrak{D}, \mathbb{R}^2) \times \mathcal{B}. \tag{4.2.1}$$

Theorem 4.2.1. *Under Assumption 4.1.2 and for* $\alpha \in (0, \infty)$ *and for all given data* $(\mathbf{h}, \mathbf{g}) \in \mathcal{B}$, *the mixed boundary problem of Dirichlet-Neumann type for the Brinkman system*

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{f}|_{\mathfrak{D}}, \\ \operatorname{div} \mathbf{u} = 0 \quad in \ \mathfrak{D} \\ (\gamma^+ \mathbf{u})\,|_{\Gamma_D} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2) \\ (\mathbf{t}_\alpha^+(\mathbf{u}, \pi))\,|_{\Gamma_N} = \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R}^2), \end{cases} \tag{4.2.2}$$

has a unique solution $(\mathbf{u}, p) \in \mathcal{X}$. *Moreover, there exists a linear continuous operator* $\mathcal{A}_\alpha : \mathcal{Y} \to \mathcal{X}$ *delivering the solution, and, hence, a constant* $C \equiv C(\alpha, \Gamma_D, \Gamma_N) > 0$ *such that*

$$\|\mathbf{u}\|_{H^1(\mathfrak{D}, \mathbb{R}^2)} + \|\pi\|_{L^2(\mathfrak{D})} \leq C \left(\|\mathbf{f}\|_{\widetilde{H}^{-1}(\mathfrak{D}, \mathbb{R}^2)} + \|\mathbf{h}\|_{H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2)} + \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R}^2)} \right).$$

4.2.2 The Poisson problem for the Brinkman system with mixed Dirichlet and Robin boundary conditions

Next, we are concerned with the mixed Dirichlet-Robin boundary problem for the Brinkman system. Let us consider now that the boundary Γ is partitioned in two non-overlapping parts Γ_D and Γ_R such that $\overline{\Gamma}_D \cup \overline{\Gamma}_R = \Gamma$ in analogy with Definition 1.1.2, i.e., now we have $\mathcal{B} = H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2) \times H^{-\frac{1}{2}}(\Gamma_R, \mathbb{R}^2)$. Then the mixed Dirichlet-Robin boundary problem for the Brinkman system is

$$\begin{cases} \Delta \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = \mathbf{f}|_{\mathfrak{D}}, \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \mathfrak{D} \\ (\gamma^+ \mathbf{u})|_{\Gamma_D} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2), \\ (\mathbf{t}_\alpha^+(\mathbf{u}, \pi)|_{\Gamma_R} + (\lambda \gamma^+ \mathbf{u})|_{\Gamma_R} = \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma_R, \mathbb{R}^2). \end{cases} \tag{4.2.3}$$

where $(\cdot)|_{\Gamma_R}$ denotes the restriction operator from the space $H^{-\frac{1}{2}}(\Gamma, \mathbb{R}^2)$ defined on the entire boundary to the corresponding one defined on Γ_R, and $\lambda \in L^\infty(\Gamma_R, \mathbb{R}^2 \otimes \mathbb{R}^2)$ is a symmetric matrix valued function, such that (as in [52, Theorem 4.1])

$$\langle \lambda \mathbf{v}, \mathbf{v} \rangle_{\Gamma_R} \geq 0, \ \forall \mathbf{v} \in L^2(\Gamma_R, \mathbb{R}^2). \tag{4.2.4}$$

Theorem 4.2.2. *Let Assumption 4.1.2 be satisfied. Let* $\alpha \in (0, \infty)$ *and let* $\lambda \in L^\infty(\Gamma, \mathbb{R}^2 \otimes \mathbb{R}^2)$ *be a symmetric matrix function with property* (4.2.4). *Then the problem* (4.2.3) *has a unique solution, which satisfies an estimate*

$$\|\mathbf{u}\|_{H^1(\mathfrak{D}, \mathbb{R}^2)} + \|\pi\|_{L^2(\mathfrak{D}, \mathbb{R}^2)} \leq c(\|\mathbf{f}\|_{\widetilde{H}^{-1}(\mathfrak{D}, \mathbb{R}^2)} + \|\mathbf{h}\|_{H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2)} + \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\Gamma_R, \mathbb{R}^2)}). \tag{4.2.5}$$

4.3 Boundary value problems for the nonlinear Darcy-Forchheimer-Brinkman system in $\mathbb{R}^2$

This section is devoted to the analysis of boundary value problems for the nonlinear Darcy-Forchheimer-Brinkman system for a Lipschitz domain in $\mathbb{R}^2$. For the sake of completion, we begin with the Dirichlet problem and state the well-posedness result as a particular case of related boundary problems throughout literature. Then we provide a constructive proof for the Neumann problem following the main ideas in [51, Theorem 4.1] and [34, Theorem 4.1]. An existence and uniqueness results for the Robin problem is stated as in our work [34, Theorem 4.2].

Finally, we arrive at the main result of this section, which is the mixed Dirichlet-Robin boundary problem for the nonlinear Darcy-Forchheimer-Brinkman system as an original result published in [37]. This problem represents the theoretical foundation for the mixed boundary problems which are studied numerically in the last part.

4.3.1 Boundary value problems of mixed Dirichlet-Robin type for the nonlinear Darcy-Forchheimer-Brinkman system

Going further, we obtain a similar existence and uniqueness result as in [51, Theorem 7.1] for the weak solution of the mixed Dirichlet-Robin problem (4.3.2), with the given data $(\mathbf{h}, \mathbf{g}) \in \mathcal{B}$. The Darcy-Forchheimer-Brinkman system with Robin boundary conditions in Lipschitz domains in Euclidean settings has been investigated in [52] (see also [54] and [65] for transmission problems).

Theorem 4.3.1. *Let Assumption 4.1.2 be satisfied. Let $\alpha, \beta > 0$ be given constants and $\lambda \in L^\infty(\Gamma, \mathbb{R}^2 \otimes \mathbb{R}^2)$ is a symmetric matrix valued function with property* (4.2.4)*. Then there exist two constants $C_j \equiv C_j(\mathfrak{D}, \alpha, \beta) > 0$, $j = 1, 2$, with the property that for all data $(\mathbf{f}, \mathbf{h}, \mathbf{g}) \in \mathcal{Y}$ satisfying the condition*

$$\|\mathbf{f}\|_{\widetilde{H}^{-1}(\mathfrak{D},\mathbb{R}^2)} + \|\mathbf{h}\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R}^2)} + \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\Gamma_R,\mathbb{R}^2)} \leq C_1, \tag{4.3.1}$$

the mixed Dirichlet-Robin problem for the nonlinear Darcy-Forchheimer-Brinkman system

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - \beta(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \pi = \mathbf{f}, \\ \text{div } \mathbf{u} = 0 \quad \text{in } \mathfrak{D}, \\ (\gamma^+ \mathbf{u})\,|_{\Gamma_D} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}^2) \\ (\mathbf{t}_\alpha^+(\mathbf{u}, \pi))\,|_{\Gamma_R} + \lambda\,(\gamma^+ \mathbf{u})\,|_{\Gamma_R} = \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma_R, \mathbb{R}^2) \end{cases} \tag{4.3.2}$$

has a unique solution $(\mathbf{u}, \pi) \in \mathcal{X}$, with the property $\|\mathbf{u}\|_{H^1(\mathfrak{D},\mathbb{R}^2)} \leq C_2$.

Part II

Mixed boundary value problems for the Stokes and Navier-Stokes system on compact Riemannian manifolds

Chapter 5

Preliminary results related to boundary value problems on compact Riemannian manifolds

This chapter is devoted to preliminary results related to the study of boundary value problems on compact Riemannian manifolds. Therefore, for the sake of completion, we include the definition of the smooth, compact Riemannian manifold without boundary, the Riemannian metric, the tangent and the cotangent spaces as well as the spaces of vector functions and one forms, based on [110], [85] and [113, Chapter 8]. Afterwards, we introduce the notion of a Lipschitz domain, the main Sobolev spaces, the trace and the conormal derivative operator needed in the sequel. Let us also mention at this point, that an important condition that ensures the invertibility of the deformation operator is the condition that the only Killing vector field is the trivial one (see Definition 5.1.17).

Next, following the main outline as for the first part of this thesis, we introduce in the second part of this chapter the fundamental solution for the Stokes system and the associated layer potential operators. In the final part of this chapter, we introduce some original results regarding the invertibility of the single-layer potential operator and the hypersingular potential operator related to mixed boundary problems [38, Theorem 4.2] and some compactness properties of the double layer potential operators [38, Theorem 4.3] associated to a part of the boundary decomposition.

5.1 Functional Settings and related results to compact Riemannian manifolds

In this section, we begin with a brief introduction about manifolds and the related results needed in the sequel. Therefore, we introduce the main geometrical definitions and concept related to compact Riemannian manifolds, such as the definitions of a smooth, compact Riemannian manifold without boundary, the Riemannian metric, the tangent and the cotangent bundles as well as the spaces of vector functions and one forms. The presentation of these concepts is based on the books [110], [85] and [113, Chapter 5].

In addition, we introduce the notion of a Lipschitz domain, the Sobolev spaces on compact Riemannian manifolds, the trace and the conormal derivative operator needed in the sequel. An important part is the definition of the Levi-Civita connection, deformation operator Def and the second order elliptic differential operator L, which is the main elliptic operator in the definition of the Stokes system on compact Riemannian manifolds. Note that an important condition that ensures the invertibility of the deformation operator is the condition that the manifold does not have any nontrivial Killing vector fields (see Definition 5.1.17).

5.1.1 Geometric concepts related to compact Riemannian manifolds

First, we recall the definition of *a compact, smooth Riemannian manifold* of dimension $m \geq 2$ without boundary $(M, \langle\cdot,\cdot\rangle)$ by following [113, Chapter 5], [69, Chapter 1], [70] and [112].

A Riemannian metric on a smooth manifold M is a tensor of $(0,2)-$type g that is symmetric, i.e., $g(X,Y) = g(Y,X)$ and positive definite, i.e., $g(X,X) > 0$ for all $X \neq 0$. Working in a coordinate frame, the Riemannian metric has the form

$$g : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U), \quad g = \sum_{j,k} g_{jk} dx_j \otimes dx_k = g_{jk} dx_j \otimes dx_k{}^1. \tag{5.1.1}$$

Therefore, the Riemannian determines a scalar product on each tangent space T_pM, such that for each $p \in M$, the cotangent space can be naturally identified with the tangent space and the cotangent bundle T^*M with the tangent bundle TM. Moreover, the space of differential one forms $\Lambda^1 TM$ is being identified with the space $\mathfrak{X}(M)$ of smooth vector fields via the isometry $\partial_j \to g_{jl} dx_l$ (lowering index), or its inverse $dx_j \to g^{jl}\partial_l$ (raising index), where (g^{jl}) denotes the inverse of (g_{jl}), i.e., $g^{jl} g_{lk} = \delta_{jk}$. Let $g = \det(g_{jk})$. Then the volume element in M, $dvol$ is given by the metric tensor of M. Therefore, in local coordinates we have $dvol = \sqrt{g}dx_1 \ldots dx_m$.

Therefore, the gradient operator grad : $C^\infty(M) \to \mathfrak{X}(M)$ becomes *the exterior derivative operator*

$$d : C^\infty(M) \to C^\infty(M, \Lambda^1 TM), \text{ given by } d = \partial_j dx^j, \tag{5.1.2}$$

and the divergence operator $-\text{div} : \mathfrak{X}(M) \to C^\infty(M)$ becomes *the exterior co-derivative operator*

$$\delta : C^\infty(M, \Lambda^1 TM) \to C^\infty(M), \ \delta = d^*. \tag{5.1.3}$$

For further details about differential geometry on Riemannian manifolds, we refer to [113, Chapter 5], [69, Chapter 1], [70] and [112].

Lipschitz domains on compact Riemannian manifolds

Let us recall for clarity the Definition 1.1.1 of the Lipschitz domain the Euclidean setting, which constitutes the basis for the definition of a Lipschitz domain on compact smooth Riemannian manifolds. Having the Definition 1.1.1, we give now the definition of a Lipschitz domain on compact Riemannian manifolds based on the Definition 3.5 in [32].

Definition 5.1.1. Let M be a compact boundaryless topological manifold of dimension n equipped with a smooth atlas A. A Lipschitz domain on M is an open set $\mathfrak{D} \subset M$ relative to A, if for every $\mathbf{x}_0 \in \Gamma$, there exists a local chart $(U, \varphi) \in A$ with $\mathbf{x}_0 \in U$ such that $\varphi(U \cap \mathfrak{D}) \subset \mathbb{R}^n$ is a Lipschitz domain in $\mathbb{R}^n$.

The following definition gives the notion of a dissection of the boundary into two parts Γ_D and Γ_N, which is used in order to formulate mixed boundary problems on compact Riemannian manifolds.

Definition 5.1.2. Let $\mathfrak{D} \subset M$ be a bounded Lipschitz domain with connected boundary $\Gamma = \partial\mathfrak{D}$. A dissection of the boundary is a decomposition into two adjacent, nonoverlapping parts Γ_D, Γ_N with the following properties

$$\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N, \ \partial\Gamma_D = \partial\Gamma_N = \overline{\Gamma}_D \cap \overline{\Gamma}_N, \text{ and meas } \Gamma_D > 0, \text{ meas } \Gamma_N > 0. \tag{5.1.4}$$

[1] In the sequal we use Einstein summation convention.

5.1.2 Sobolev spaces on compact Riemannian manifolds

Let $s \in \mathbb{R}$. The Sobolev space $H^s(M)$ is defined as the L^2-based space on M, which can be obtained by lifting the Sobolev (or Bessel potential) space

$$H^s(\mathbb{R}^m) := \{(\mathbb{I} - \triangle)^{-s/2} f : f \in L^2(\mathbb{R}^m)\} \tag{5.1.5}$$

via a partition of unity on M and pullback on corresponding local charts. Note that the spaces $H^s(M)$ and $H^{-s}(M)$ are dual to each other.

Let $\mathfrak{D} := \mathfrak{D}_+ \subset M$ be a Lipschitz domain and assume that $M \setminus \overline{\mathfrak{D}} := \mathfrak{D}_-$ is connected and non-empty.

For $s > 0$, the L^2-based Sobolev spaces of functions on $\mathfrak{D}$ are defined as

$$H^s(\mathfrak{D}) := \{f|_{\mathfrak{D}} : f \in H^s(M)\},\ \widetilde{H}^s(\mathfrak{D}) := \{f \in H^s(M) : \mathrm{supp} f \subseteq \overline{\mathfrak{D}}\}, \tag{5.1.6}$$

$$\widetilde{H}^s(\mathfrak{D})|_{\mathfrak{D}} := \{f|_{\mathfrak{D}} : f \in \widetilde{H}^s(\mathfrak{D})\}. \tag{5.1.7}$$

Moreover, $H^{-s}(\mathfrak{D})$ is the dual of the space $\widetilde{H}^s(\mathfrak{D})$. Note that for any $s \in \mathbb{R}$ (see [47, Proposition 2.9], [84, (4.14)])

$$(H^s(\mathfrak{D}))' = \widetilde{H}^{-s}(\mathfrak{D}),\ H^{-s}(\mathfrak{D}) = (\widetilde{H}^s(\mathfrak{D}))'. \tag{5.1.8}$$

The L^2-based Sobolev spaces of one forms on $\mathfrak{D}$ are given by

$$H^s(\mathfrak{D}, \Lambda^1 TM) := H^s(\mathfrak{D}) \otimes \Lambda^1 TM|_{\mathfrak{D}},\ \widetilde{H}^s(\mathfrak{D}, \Lambda^1 TM) = \widetilde{H}^s(\mathfrak{D}) \otimes \Lambda^1 TM, \tag{5.1.9}$$

$$\widetilde{H}^s(\mathfrak{D}, \Lambda^1 TM)|_{\mathfrak{D}} = \widetilde{H}^s(\mathfrak{D})|_{\mathfrak{D}} \otimes \Lambda^1 TM|_{\mathfrak{D}}, \tag{5.1.10}$$

where the symbol $\otimes$ denotes the tensor product (see [105, Chapter 4, Section 3]).

The L^2-based boundary Sobolev spaces: For $s \in [0, 1]$, $H^s(\Gamma)$ and $H^s(\Gamma, \Lambda^1 TM)$ denote the boundary Sobolev spaces of functions and one forms, respectively. For $s \in [-1, 0)$, the space $H^s(\Gamma)$ is the space of distributions defined on $H^{-s}(\Gamma)$, i.e., $H^s(\Gamma) = H^{-s}(\Gamma)$. For more details we refer to [110].

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ be the outward unit normal to Γ, which is defined almost everywhere with respect to the surface measure $d\sigma$ on Γ. Throughout the following, we are working with the closed subspace $H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\Gamma, \Lambda^1 TM)$ of $H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ and the quotient space $H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)/\mathbb{R}\boldsymbol{\nu}$ of $H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)$, defined by

$$H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\Gamma, \Lambda^1 TM) := \left\{\mathbf{f} \in H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM) : \langle \boldsymbol{\nu}, \mathbf{f} \rangle_\Gamma = 0\right\}, \tag{5.1.11}$$

$$H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)/\mathbb{R}\boldsymbol{\nu} := \left\{[\mathbf{g}] = \mathbf{g} + \mathbb{R}\boldsymbol{\nu} \text{ where } \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)\right\}. \tag{5.1.12}$$

Note that $H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)/\mathbb{R}\boldsymbol{\nu} = (H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\Gamma, \Lambda^1 TM))^*$, (cf., e.g., [86, 5.118]).

Finally, let the space

$$H^1_\delta(\mathfrak{D}_\pm, \Lambda^1 TM) := \left\{\mathbf{u} \in H^1(\mathfrak{D}_\pm, \Lambda^1 TM) : \delta\mathbf{u} = 0 \text{ in } \mathfrak{D}_\pm\right\}, \tag{5.1.13}$$

denote the space of divergence free vector fields (on forms) on $\mathfrak{D}_\pm$.

5.1.3 The deformation operator on Sobolev spaces

A *Levi-Civita connection* on M is an affine connection which is compatible with the Riemannian metric g and is torsion-free. A key result in the setting of Riemannian geometry asserts that for a given Riemannian manifold (M, g) there exists a unique Levi-Civita connection ∇, defined by the torsion-free condition (cf., e.g.,[107, Proposition 11.1, Chapter 1 §11]).

The symmetric part of ∇X, called the deformation of X, is denoted by Def X. Therefore,

$$(\text{Def } X)(Y,Z) = \frac{1}{2}\{\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle\}, \quad \forall\, Y,Z \in \mathfrak{X}(M). \tag{5.1.14}$$

The tensor field Def X can be globally represented in the equivalent form $\frac{1}{2}\mathcal{L}_X g$, where $\mathcal{L}_X g$ is the Lie derivative of g in the direction of X (see, e.g., [107]). Denoting by S^2T^*M the set of symmetric tensor fields of type $(0,2)$, we have

$$\text{Def} : \mathfrak{X}(M) \to C^\infty(M, S^2T^*M). \tag{5.1.15}$$

The adjoint of Def is defined by $\text{Def}^* w = -\text{div} w$, $w \in S^2T^*M$ (see, e.g., [107]). The operator (5.1.15) admits a linear and bounded extension

$$\text{Def} : H^1(M, \Lambda^1 TM) \to H^{-1}(M, S^2T^*M). \tag{5.1.16}$$

Definition 5.1.3. A vector field $X \in \mathfrak{X}(M)$ which satisfies the equation

$$\text{Def } X = 0 \;\text{ on }\; M, \tag{5.1.17}$$

is called a *Killing field*.

Throughout this chapter we assume that the only Killing vector field is the trivial one. Altering M away from $\overline{\mathfrak{D}}$, this condition can be realized (see, e.g., [85], [25]).

5.1.4 The Stokes and Oseen operators on a compact Riemannian manifold

In the sequel, let M be a compact, smooth Riemannian manifold without boundary and let $\mathfrak{D}_+$ stand for a Lipschitz domain on M and $\mathfrak{D}_- = M \backslash \overline{\mathfrak{D}}$. Let us consider the second-order elliptic differential operator

$$L : \mathfrak{X}(M) \to \mathfrak{X}(M), \quad L := 2\text{Def}^*\text{Def} = -\triangle + d\delta - 2\text{Ric}, \tag{5.1.18}$$

where $\triangle := -(d\delta + \delta d)$ is the Hodge Laplacian and Ric is the Ricci tensor (see, e.g., [25, (2.6)]). For any $s \in (0,1)$, the operator (5.1.18) extends to a bounded linear operator (see, e.g., [68, p. 177])

$$L = 2\text{Def}^*\text{Def} : H^{s+\frac{1}{2}}(M, \Lambda^1 TM) \to H^{s-\frac{3}{2}}(M, \Lambda^1 TM). \tag{5.1.19}$$

The *Oseen operator* is a perturbation of order one of the Stokes operator, which is defined by

$$\begin{gathered} B_\omega : H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D}) \to H^{-1}(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D}), \\ B_\omega := \begin{pmatrix} L & d \\ \delta & 0 \end{pmatrix} + \begin{pmatrix} \nabla_\omega & 0 \\ 0 & 0 \end{pmatrix}, \end{gathered} \tag{5.1.20}$$

where $\omega \in H^1(\mathfrak{D}, \Lambda^1 TM)$ is a divergence free vector field, i.e.,

$$\delta\omega = 0 \text{ in } \mathfrak{D}. \tag{5.1.21}$$

Let us remark that for $\omega = 0$, we obtain the *Stokes operator* B_0. Throughout this part of the thesis, we assume that the manifold M is low dimensional, i.e., $\dim(M) \in \{2,3\}$, whenever we deal with the Oseen operator. Hence, the following embeddings are continuous (see, e.g., [53])

$$H^1(\mathfrak{D}, \Lambda^1 TM) \cdot L^2(\mathfrak{D}, \Lambda^1 TM \otimes \Lambda^1 TM) \hookrightarrow L^{\frac{3}{2}}(\mathfrak{D}, \Lambda^1 TM) \hookrightarrow \widetilde{H}^{-1}(\mathfrak{D}, \Lambda^1 TM). \tag{5.1.22}$$

Note that (5.1.22) and the inclusion $\widetilde{H}^{-1}(\mathfrak{D}, \Lambda^1 TM) \hookrightarrow H^{-1}(\mathfrak{D}, \Lambda^1 TM)$ show that the Oseen operator (5.1.20) is well defined.

Due to the technical details, our divergence free vector field ω has to satisfy the positivity condition[2]

$$\langle \nabla_{\omega} \mathbf{u}, \mathbf{u} \rangle_{\mathfrak{D}} \geq 0, \forall\ \mathbf{u} \in H^1(\mathfrak{D}, \Lambda^1 TM). \tag{5.1.23}$$

Note that (5.1.22) shows that $\nabla_{\omega} \mathbf{u} \in \widetilde{H}^{-1}(\mathfrak{D}, \Lambda^1 TM) = \big(H^1(\mathfrak{D}, \Lambda^1 TM)\big)'$, and hence the dual product in (5.1.23) is well defined, for any $u \in H^1(\mathfrak{D}, \Lambda^1 TM)$.

In the complete version, we show that the gradients of some harmonic functions can play the role of ω, since they divergence free. Because constants are the only harmonic functions on compact manifolds and our considerations require a non-zero vector field, we consider harmonic functions on the noncompact punctured manifold $M \setminus \{x_0\}$. Such a result has been obtained in our paper [42, Proposition 5.4.1].

5.1.5 The trace operator and the conormal derivative operator

Similary to the first part of this thesis, we introduce the trace operators and the generalized conormal derivative operator, which are needed in the sequel.

The trace operators on complementary Lipschitz domains. Let $\mathbf{x} \in \Gamma$ and let $\mathfrak{C}_{\pm}(\mathbf{x}) \subseteq \mathfrak{D}_{\pm}$ be non-tangential approach regions, i.e., some conical regions with vertex at $\mathbf{x}$ (see the definition given in (1.1.2) for the Euclidean setting). Then the non-tangential boundary limits of a function u on $\Gamma_{\pm}$ are defined as (see (1.3.2) for the Euclidean setting)

$$(\gamma^{\pm} u)(\mathbf{x}) := \lim_{\mathbf{y} \in \mathfrak{C}_{\pm}(\mathbf{x})} u(\mathbf{y}),\ \ \mathbf{x} \in \Gamma, \tag{5.1.24}$$

(see, e.g., [85, (3.23)]). These operators extend to Sobolev spaces similar to Lemma 1.3.1. The following variant of the trace lemma holds as well in the compact Riemannian setting (see, e.g., [86, Theorem 2.5.2], [85, 20]).

Lemma 5.1.4. *Let $s \in (0,1)$. Then there exist two continuous linear operators*

$$\gamma^{\pm} : H^{s+\frac{1}{2}}(\mathfrak{D}_{\pm}) \to H^s(\Gamma), \tag{5.1.25}$$

such that $\gamma^{\pm} u = u|_{\Gamma}$, $\forall\ u \in C^{\infty}(\overline{\mathfrak{D}}_{\pm})$, admitting (non-unique) linear, continuous right inverses

$$(\gamma^{\pm})^{-1} : H^s(\Gamma) \to H^{s+\frac{1}{2}}(\mathfrak{D}_{\pm}), \quad \gamma^{\pm}((\gamma^{\pm})^{-1}\phi) = \phi, \quad \forall\ \phi \in H^s(\Gamma). \tag{5.1.26}$$

The result in Lemma 5.1.4 holds also for trace operators acting on spaces of one forms $\gamma^{\pm} : H^{s+\frac{1}{2}}(\mathfrak{D}_{\pm}, \Lambda^1 TM) \to H^s(\Gamma, \Lambda^1 TM)$. Such operators are well defined, linear, bounded and onto (see [25, 47, 84]).

Let $\alpha \geq 0$ be given. In the sequel, we consider the special Sobolev space

$$\begin{aligned} \mathfrak{H}^1(\mathfrak{D}) := \big\{(\mathbf{u}, \pi, \mathbf{f}) \in & H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D}) \times \widetilde{H}^{-1}(\mathfrak{D}, \Lambda^1 TM)|_{\mathfrak{D}} : \\ & (L + \alpha \mathbb{I})\mathbf{u} + d\pi = \mathbf{f} \text{ and } \delta \mathbf{u} = 0 \text{ in } \mathfrak{D}\big\}. \end{aligned} \tag{5.1.27}$$

We denote by $d\sigma$ the surface measure on Γ and by $\boldsymbol{\nu}$ the outward unit conormal, which is defined almost everywhere on Γ, with respect to $d\sigma$.

The conormal derivative for the Stokes system on compact Riemannian manifolds is defined next (see, e.g., [81] [86, Theorem 10.4.1], [59, Lem 2.2]). Also, let us mention that a definition of the conormal derivative operator in the general case of Agmon-Douglis-Niremberg elliptic operators is given in Lemma 2.4 in [58].

[2] Recall that the notation $\langle \cdot, \cdot \rangle_X$ stands for the pairing between two dual Sobolev spaces defined on X.

Lemma 5.1.5. *Let $\alpha \geq 0$ be a given constant. Let M be a compact Riemannian manifold and $\mathfrak{D} \subset M$ be a Lipschitz domain. The conormal derivative operator*

$$
\begin{aligned}
&\mathbf{t}_\alpha^+ : \mathfrak{H}^1(\mathfrak{D}) \to H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM),\\
\langle \mathbf{t}_\alpha^+(\mathbf{u},\pi)_{\mathbf{f}}, \Phi\rangle_\Gamma &:= 2\langle \mathrm{Def}\mathbf{u}, \mathrm{Def}((\gamma^+)^{-1}\Phi)\rangle_{\mathfrak{D}} + \Big\langle \pi, \delta((\gamma^+)^{-1}\Phi)\Big\rangle_{\mathfrak{D}}\\
&- \langle \mathbf{f}, (\gamma^+)^{-1}\Phi\rangle_{\mathfrak{D}} + \alpha\langle \mathbf{u}, (\gamma^+)^{-1}\Phi\rangle_{\mathfrak{D}}, \quad \forall\ \Phi \in H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM) \qquad (5.1.28)
\end{aligned}
$$

is well-defined, linear and bounded, and is independent on the choice of the right inverse $(\gamma^+)^{-1} : H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM) \to H^1(\mathfrak{D}, \Lambda^1 TM)$ of the non-tangential boundary trace operator $\gamma^+ : H^1(\mathfrak{D}, \Lambda^1 TM) \to H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$. Also, for all $(\mathbf{u}, \pi, \mathbf{f}) \in \mathfrak{H}^1(\mathfrak{D})$ and any $\mathbf{w} \in H^1(\mathfrak{D}, \Lambda^1 TM)$ the following Green formula holds

$$
\langle \mathbf{t}_\alpha^+(\mathbf{u},\pi)_{\mathbf{f}}, \gamma^+\mathbf{w}\rangle_\Gamma = 2\langle \mathrm{Def}\mathbf{u}, \mathrm{Def}\mathbf{w}\rangle_{\mathfrak{D}} + \langle \pi, \delta\mathbf{w}\rangle_{\mathfrak{D}} - \langle \mathbf{f}, \mathbf{w}\rangle_{\mathfrak{D}} + \alpha\langle \mathbf{u}, \mathbf{w}\rangle_{\mathfrak{D}}. \qquad (5.1.29)
$$

Lemma 5.1.5 is a particular case with $g = 0$ in [59, Lemma 2.2].

5.2 The fundamental solution and the layer potential theory for the Stokes system

Mitrea and Taylor [85] and Dindos and Mitrea [25] have used the theory of pseudodifferential operators to show the existence of the fundamental solution for the Stokes system on compact Riemannian manifolds. One of the main assumptions needed in order to construct the fundamental solution for the Stokes system, is the assumption that the manifold lacks nontrivial Killing fields (see Definition 5.1.3, which guarantees that the deformation operator Def given in (5.1.14) is invertible.

The assumption that the Riemannian manifolds has no nontrivial Killing fields imposes no restrictions, since the manifold can be altered in order to satisfy this condition. A demonstration of this fact can be found at the beginning or Section 3 in [85].

An alternative technique to that of Mitrea and Taylor [85] has been developed by Kohr, Pintea and Wendland [57, Section 3] (see also [58]) in order to obtain the fundamental solution in the general case of Agmon-Douglis-Niremberg elliptic operators on compact Riemannian manifolds. In [58], Kohr, Pintea and Wendland provided the proof of the invertibility of a matrix of first- and second-order pseudodiferential operators, which special emphasis on a general Brinkman operator. The fundamental solution of such pseudodifferential operators is provided by the Schwartz kernels of two entries. Moreover, the authors derived layer potential theory for a pseudodifferential Brinkman operator on Lipschitz domains in Riemannian manifolds.

In their recent work, Kohr and Wendland [63] have developed the potential theory for the Stokes system with non-smooth coefficients of class L^∞ on compact Riemannian manifolds, starting from a variational method. In the particular case of the smooth coefficients, the authors found what Mitrea, Taylor [85] have obtained before.

This section is structured as follows. We begin by introducing the fundamental solution of the Stokes system, which enables us to define the corresponding single and double layer potentials, based on [85], [57], [58], [62]. Afterward, we give some invertibility and compactness properties for layer potential operators related to the mixed boundary problem for the Stokes system, following the ideas in our work [38]. Let us also mention, that this work is inspired by the papers [61], [76], [16].

5.2.1 The fundamental solution of the Stokes system on compact Riemannian manifolds

Using the pseudodifferential theory and the Hodge decomposition, Mitrea and Taylor in [85] have proved the existence of two operators $\Phi \in OPS^{-2}_{\mathrm{cl}}(\Lambda^1 TM, \Lambda^1 TM)$ and $\Psi \in$

$OPS^{-1}_{\rm cl}(\Lambda^1 TM, \mathbb{R})$ whose Schwartz kernels of their inverses $\mathcal{G}(\mathbf{x},\mathbf{y})$ and $\Pi(\mathbf{x},\mathbf{y})$ determine the fundamental solution of the Stokes operator B_0 on M. Thus

$$L_{\mathbf{x}}\mathcal{G}(\mathbf{x},\mathbf{y}) + d_{\mathbf{x}}\Pi(\mathbf{x},\mathbf{y}) = \mathrm{Dirac}_{\mathbf{y}}(\mathbf{x}), \quad \delta_{\mathbf{x}}\mathcal{G}(\mathbf{x},\mathbf{y}) = 0, \tag{5.2.1}$$

where $\mathrm{Dirac}_{\mathbf{y}}$ is the Dirac distribution centered at $\mathbf{y}$. The differential operators with subscript $\mathbf{x}$ added refers to the action of that operator on $\mathbf{x}$. Let us mention that Dindoš and Mitrea [25] have proved the existence of an operator $\Upsilon \in OPS^0_{\rm cl}(M,\mathbb{R})$ such that the following equivalence holds

$$\Psi L = \Upsilon\delta \Longleftrightarrow L\Psi^\top = d\Upsilon^\top. \tag{5.2.2}$$

Let us denote by Ξ the Schwartz kernel of of the classical pseudodifferential operator $\Upsilon^\top$, which is the transpose of Υ. By (5.2.2) it implies that

$$L_{\mathbf{x}}\Pi^\top(\mathbf{y},\mathbf{x}) = d_{\mathbf{x}}\Xi(\mathbf{x},\mathbf{y}) \tag{5.2.3}$$

(see [25, (3.22)]). Let us mentions that the fundamental solution for a pseudodifferential Brinkman operator has also been obtained in [57].

5.2.2 The Stokes Layer potential operators and related properties

Let $s \in (0,1)$. Then for $\mathbf{f} \in H^{s-1}(\Gamma, \Lambda^1 TM)$, $\mathbf{Vf}$ stands for the single-layer potential with density $\mathbf{f}$ for the Stokes system, and $Q^s\mathbf{f}$ denotes its corresponding pressure potential. Then,

$$(\mathbf{Vf})(\mathbf{x}) = \langle \mathcal{G}(\mathbf{x},\cdot), \mathbf{f}\rangle_\Gamma, \ (Q^s\mathbf{f})(\mathbf{x}) := \langle \Pi(\mathbf{x},\cdot), \mathbf{f}\rangle_\Gamma, \ \mathbf{x} \in M \setminus \Gamma. \tag{5.2.4}$$

The non-tangential boundary traces of $\mathbf{Vf}$ exist almost everywhere on Γ and are given by $\gamma^\pm(\mathbf{Vf})$ (cf, e.g., [85, Theorem 3.1]).

For $\mathbf{h} \in H^s_{\boldsymbol{\nu}}(\Gamma, \Lambda^1 TM)$, $\mathbf{Wh}$ denotes the double-layer potential with density $\mathbf{h}$ for the Stokes system, and $Q^d\mathbf{h}$ stands for be its corresponding pressure potential. Therefore,

$$(\mathbf{Wh})(\mathbf{x}) := \Big\langle -2\mathrm{Def}\ \mathcal{G}(\mathbf{x},\cdot)\boldsymbol{\nu} + \Pi^\top(\cdot,\mathbf{x})\boldsymbol{\nu}, \mathbf{h}\Big\rangle_\Gamma, \ \mathbf{x} \in M \setminus \Gamma, \tag{5.2.5}$$

$$(Q^d\mathbf{h})(\mathbf{x}) := \langle -2\mathrm{Def}\ \Pi(\mathbf{x},\cdot)\boldsymbol{\nu} - \Xi(\mathbf{x},\cdot)\boldsymbol{\nu}, \mathbf{h}\rangle_\Gamma, \ \mathbf{x} \in M \setminus \Gamma, \tag{5.2.6}$$

where $\Pi^\top(\cdot,\mathbf{x})$ is the transpose of $\Pi(\cdot,\mathbf{x})$, and $\Xi(\mathbf{x},\dot{)}$ is the Schwartz kernel of the pseudodifferential operator Υ (see (5.2.2) and (5.2.3)). The non-tangential boundary traces of $\mathbf{Wh}$ exist almost everywhere on Γ and are denoted by $\gamma^+(\mathbf{Wh})$ and $\gamma^-(\mathbf{Wh})$, respectively. The principal value of $\mathbf{Wh}$ is denoted by $\mathbf{Kh}$ and is defined at almost every point on $\mathbf{x} \in \Gamma$ by

$$\begin{aligned}(\mathbf{Kh})(\mathbf{x}) &:= \mathrm{p.v.}\int_\Gamma \langle -2\big[(\mathrm{Def}_y\ \mathcal{G}(\mathbf{x},\cdot))\boldsymbol{\nu}\big](\mathbf{y}) + \Pi^\top(\mathbf{y},\mathbf{x})\boldsymbol{\nu}(\mathbf{y}), \mathbf{h}(\mathbf{y})\rangle d\sigma(\mathbf{y}) \\ &= \lim_{\varepsilon\to 0}\int_{\{\mathbf{y}\in\Gamma:\ r(\mathbf{x},\mathbf{y})>\varepsilon\}} \langle -2\big[(\mathrm{Def}_{\mathbf{y}}\ \mathcal{G}(\mathbf{x},\cdot))\boldsymbol{\nu}\big](\mathbf{y}) + (\Pi)^\top(\mathbf{y},\mathbf{x})\boldsymbol{\nu}(\mathbf{y}), \mathbf{h}(\mathbf{y})\rangle d\sigma(\mathbf{y}),\end{aligned} \tag{5.2.7}$$

where $r(\mathbf{x},\mathbf{y})$ is the geodesic distance between $\mathbf{x}$ and $\mathbf{y} \in M$ (cf. [85, Lemma 3.2 and Proposition 3.3]).

In view of (5.2.1), we obtain that

$$\left.\begin{aligned} &L(\mathbf{Vf}) + d(Q^s\mathbf{f}) = 0, \ \delta\mathbf{Vf} = 0 \\ &L(\mathbf{Wh}) + d(Q^d\mathbf{h}) = 0, \ \delta\mathbf{Wh} = 0 \end{aligned}\right. \quad \text{in } M \setminus \Gamma. \tag{5.2.8}$$

Therefore, the pairs $(\mathbf{Vg}, Q^s\mathbf{g})$ and $(\mathbf{Wh}, Q^d\mathbf{h})$ satisfy the Stokes system in each of the domains $\mathfrak{D}_+$ and $\mathfrak{D}_-$, respectively.

The properties presented below, represent the Riemannian version of Theorem 2.4.2. For more details, we refer to [86, Propositions 4.2.5, 4.2.9, Corollary 4.3.2, Theorem 5.3.6, 5.4.1, 5.4.3, 10.5.3] for the Stokes system in the Euclidean setting, [25, Theorem 2.1, (3.5), Proposition 3.5], [85, Theorem 3.1, 6.1] for the Stokes system in compact Riemannian manifolds, and [58, Theorem 4.3, 4.9, 4.11, (131), (132), (137)] for a pseudodifferential Brinkman operator in compact Riemannian manifolds.

Theorem 5.2.1. *Let $\mathfrak{D} \subset M$ be a Lipschitz domain and let Γ be its boundary. Let $s \in (0,1)$. Assume that $\mathbf{f} \in H^{s-1}(\Gamma, \Lambda^1 TM)$ and $\mathbf{h} \in H^s_{\boldsymbol{\nu}}(\Gamma, \Lambda^1 TM)$. The following formulas hold almost everywhere on Γ:*

$$\gamma^+(\mathbf{Vf}) = \gamma^-(\mathbf{Vf}) := \mathcal{V}\mathbf{f}, \tag{5.2.9}$$

$$\gamma^+(\mathbf{Wh}) = \Big(\frac{1}{2}\mathbb{I} + \mathbf{K}\Big)\mathbf{h}, \ \gamma^-(\mathbf{Wh}) = \Big(-\frac{1}{2}\mathbb{I} + \mathbf{K}\Big)\mathbf{h}, \tag{5.2.10}$$

$$\mathbf{D}^+\mathbf{h} - \mathbf{D}^-\mathbf{h} \in \mathbb{R}\boldsymbol{\nu}, \tag{5.2.11}$$

$$\mathbf{t}^+(\mathbf{Vf}, \mathcal{Q}^s\mathbf{f}) = \Big(-\frac{1}{2}\mathbb{I} + \mathbf{K}^*\Big)\mathbf{f}, \quad \mathbf{t}^-(\mathbf{Vf}, \mathcal{Q}^s\mathbf{f}) = \Big(\frac{1}{2}\mathbb{I} + \mathbf{K}^*\Big)\mathbf{h}, \tag{5.2.12}$$

where $\mathbf{D}^\pm\mathbf{h} := \mathbf{t}^\pm(\mathbf{Wh}, \mathcal{Q}^d\mathbf{h})$, and $\mathbf{K}^*$ is the formal transpose of $\mathbf{K}$, i.e.,

$$(\mathbf{K}^*\mathbf{f})(\mathbf{x}) = \text{p.v.}\int_\Gamma \langle -2[(\text{Def}_{\mathbf{x}}\, \mathcal{G}(\cdot, \mathbf{y}))\boldsymbol{\nu}](\mathbf{x}) + \Pi(\mathbf{x}, \mathbf{y})\boldsymbol{\nu}(\mathbf{x}), \mathbf{f}(\mathbf{y})\rangle d\sigma(y\mathbf{y}). \tag{5.2.13}$$

In addition,

$$\mathbf{V}\boldsymbol{\nu} = 0, \ Q^s\boldsymbol{\nu} = c_\pm \in \mathbb{R} \text{ in } \mathfrak{D}_\pm, \tag{5.2.14}$$

$$\text{Ker}\Big(\mathcal{V} : H^{s-1}(\Gamma, \Lambda^1 TM) \to H^s(\Gamma, \Lambda^1 TM)\Big) = \mathbb{R}\boldsymbol{\nu}. \tag{5.2.15}$$

5.2.3 Invertibility results of the layer potential operators associated to mixed problems

Before we consider the mixed Dirichlet-Neumann boundary problem of the next section, let us analyse some properties of the layer potential operators needed in the sequel. These properties have been obtained in our work in Theorem 4.2 in [38]. They are the analogous versions of the results in the Euclidean setting.

To this end, let us recall the the operators $\mathcal{V}^D$, $\mathbf{K}^D$, $\mathbf{K}^{*N}$ and $\mathbf{D}^N$ have similar definitions to that given in (4.1.9), in the case of Euclidean setting.

Theorem 5.2.2. *Let $\mathfrak{D} \subset M$ be a bounded Lipschitz domain with connected boundary Γ as in Definition 5.1.2. Then the following operators are invertible:*

(*i*) *The single-layer integral operator*

$$\mathcal{V}^D : \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) \to H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \tag{5.2.16}$$

(*ii*) *The hypersingular integral operator*

$$\mathbf{D}^N : \widetilde{H}^{\frac{1}{2}}_{\boldsymbol{\nu}}(\Gamma_N, \Lambda^1 TM) \to H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)/\mathbb{R}\boldsymbol{\nu}. \tag{5.2.17}$$

5.2.4 Compactness of operators related to mixed boundary value problems

This subsection is devoted to the compactness property of some special double-layer integral operators, which play a major role in the analysis of the mixed Dirichlet-Neumann boundary problem for the Stokes system on compact Riemannian manifolds studied in the next chapter. There results are obtained in our work [38, Theorem 4.3].

Theorem 5.2.3. *Let $\mathfrak{D} \subset M$ be a Lipschitz domain with boundary decomposed as in Definition 5.1.2. Then the following operators are compact:*

$$\boldsymbol{K}^D : \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM) \to H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \text{ given by } \boldsymbol{K}^D\varphi = \boldsymbol{K}\varphi|_{\Gamma_D}, \tag{5.2.18}$$

$$\boldsymbol{K}^{*N} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) \to H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM), \text{ given by } \boldsymbol{K}^{*N}\psi = \boldsymbol{K}^*\psi|_{\Gamma_N}. \tag{5.2.19}$$

Chapter 6

Boundary value problems for the Stokes, Oseen and Navier-Stokes systems on compact Riemannian manifolds

This chapter is devoted to various boundary problems, related to the Stokes, Oseen and Navier-Stokes systems on compact Riemannian manifolds. The study of fluid flow on a compact, smooth Riemannian manifolds plays an important role in the analysis to the fundamental equations of meteorology and oceanography as pointed out in [109, 71], where a mathematical justification used to derive the primitive equations of the atmosphere and the ocean are derived as model of the Navier-Stokes equations in thin spherical shells (see also [108, 26]). Also, other types of flow equations, e.g., Stokes equations or Darcy-Forchheimer-Brinkman equations can be considered over compact surfaces (e.g., on the sphere S^2) which model the flow of water or other Newtonian fluids, passing through porous rocks or porous soil (see, e.g., [55]).

The complete version of the thesis starts with the study of a certain transmission problem for the Stokes system on compact Riemannian manifolds. Transmission problems have been intensively studied over the last decades, since they describe the flow within a stationary particle embedded into a fluid [86], [60]. Moreover, Dirichlet and Neumann boundary problems can be viewed as limiting cases to transmission problems as explained in [86, p. 1- 10], [58, Section 6].

The second part of this chapter is concerned with mixed boundary problems for the Stokes, Oseen and Navier-Stokes systems. Mixed boundary problems of Dirichlet-Neumann type on compact Riemannian manifolds could resemble a mathematical model for the fluid flow in a shallow ocean.

Kohr and Wendland [62, Theorem 7.9] have obtained well-posedness results on compact Riemannian manifolds for the mixed type boundary conditions and nonhomogeneous Poisson problem for the nonsmooth coefficient Brinkman system when the solution belongs to some L^p-based Sobolev spaces with p in a neighborhood of 2. Moreover, they proved in [63] the equivalence between some transmission problems for the Stokes system with nonsmooth coefficients in complementary Lipschitz domains, by using the Nečas-Babuška-Brezzi technique and by proving a well-posedness result of their mixed variational counterparts.

The main results of this chapter are based on the paper [42], which is obtained by joint work with M. Kohr, C. Pintea and W. L. Wendland and the paper [38]. The original results of this chapter are obtained in [42, Theorem 4.1 and 5.1]. Moreover, Theorem 6.1.4 is a generalization of the well-posedness results obtained in our work [38, Theorem 4.1].

6.1 Boundary value problems of mixed type for Lipschitz domains on compact Riemannian manifolds

This section analyses the mixed Dirichlet-Neumann boundary problems for the Stokes, Oseen and Navier Stokes systems. Let us mention that recently, Kohr and Wendland [62, Theorem 7.9] have obtained a well-posedness result for the nonhomogeneous Poisson problem with mixed boundary conditions for L^∞ coefficient Brinkman system with the solution belongs to the space $H_D^{1,p}(\mathfrak{D}, \Lambda^1 TM) \times L^p(\mathfrak{D})$, with p in some neighbourhood of 2, where $H_D^{1,p}(\mathfrak{D}, \Lambda^1 TM)$ is the subspace of the space $H^{1,p}(\mathfrak{D}, \Lambda^1 TM)$ (denoted by us $H_p^1(\mathfrak{D}, \Lambda^1 TM)$), whose elements have zero boundary traces on some part of the boundary Γ_D (see, also Theorem 7.4 in [63] and Remark 7.10 in [62]). In addition, let us mention the work of I. Mitrea and M. Mitrea [83, Theorem 8.2], which studies the mixed boundary problems for the Laplace operator on Besov spaces in the Euclidean setting.

Well-posedness results higher-order strongly elliptic system in an (ϵ, δ) domain in $\mathbb{R}^n$ with an Ahlfors-regular part of the boundary have been obtained in [10, Theorem 7.3]. Also, a general well-posedness result a higher order strongly elliptic operator in Lipschitz domains in $\mathbb{R}^n$ is developed by [76, Theorem 7.9] (see also [3] for the Euclidean setting). For the case mixed boundary problems on polyhedral domains we refer to [73, Theorem 5.2] and [74].

In this section we develop a different approach by those used in [62] and [83], in order to analyze mixed Dirichlet-Neumann problems for the Stokes system in Lipschitz domains on compact Riemannian manifolds.

6.1.1 The mixed Dirichlet-Neumann problem for the Stokes system

We start again with the assumption of the geometrical framework we are working with.

Assumption 6.1.1. Let $\mathfrak{D} \subset M$ be a bounded Lipschitz domain with connected boundary $\Gamma = \partial\mathfrak{D}$, which is decomposed into two adjacent, nonoverlapping parts Γ_D, Γ_N as in Definition 5.1.2.

Note that, the positive measure of both partitions is essential to our case as will be explained in the sequel. Then, we consider the mixed problem with Dirichlet and Neumann boundary conditions for the Brinkman system

$$\begin{cases} L\mathbf{u} + d\pi = 0 \text{ in } \mathfrak{D}, \\ \delta\mathbf{u} = 0 \text{ in } \mathfrak{D}, \\ \gamma^+\mathbf{u}|_{\Gamma_D} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \\ \mathbf{t}^+(\mathbf{u}, \pi)|_{\Gamma_N} = \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM), \end{cases} \tag{6.1.1}$$

where $(\cdot)|_{\Gamma_D}$, $(\cdot)|_{\Gamma_N}$ denote the restrictions of the spaces defined on the entire boundary Γ to the corresponding ones defined on Γ_D and Γ_N, respectively.

In order to match the system composed of (4.1.3) and (4.1.4) to the mixed boundary problem for the Stokes system (6.1.1), let us denote by $\mathbf{h}^* \in \widetilde{H}^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ and $\mathbf{g}^* \in \widetilde{H}^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)$, arbitrary extensions to the entire Γ of the corresponding boundary data $\mathbf{h}$, $\mathbf{g}$. Taking $\mathbf{h}^* \in \widetilde{H}^{\frac{1}{2}}_{\boldsymbol{\nu}}(\Gamma, \Lambda^1 TM)$ as described in (4.1.6), it follows that the boundary data is given by

$$\gamma^+\mathbf{u} = \varphi_N + \mathbf{h}_D^*, \quad \mathbf{t}^+(\mathbf{u}, \pi) = \psi_D + \mathbf{g}^*. \tag{6.1.2}$$

Obviously $\varphi_N \in \widetilde{H}^{\frac{1}{2}}_{\boldsymbol{\nu}}(\Gamma_N, \Lambda^1 TM)$ and $\psi_D \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)$, since $\varphi_N = 0$ on Γ_D.

In order to prove the well-posedness of the boundary problem (6.1.1), we will reformulate the problem as a system of boundary integral equations, inspired by the main ideas in [16] for the Laplace equation and [61] for the Stokes system. Let us mention also Theorem 7.9 in [76], which refers to a general strongly elliptic system.

Theorem 6.1.2. *Let $\mathfrak{D}, \Gamma_D$ and Γ_N be as in Assumption 6.1.1 and let $\mathbf{h} \in H^{\frac{1}{2}}(\Gamma, \Lambda^1 TM)$ and $\mathbf{g} \in H^{-\frac{1}{2}}(\Gamma, \Lambda^1 TM)$. Also, consider $\mathbf{g}^*$ and $\mathbf{h}_D^*$ given by (6.1.2). Then, the following assertions hold true:*

(*i*) *If $(\mathbf{u}, \pi) \in H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D})$ is a solution of (6.1.1), then (ψ_D, φ_N) given by*

$$\psi_D = \mathbf{t}^+(\mathbf{u}, \pi) - \mathbf{g}^*, \quad \varphi_N = \gamma^+ \mathbf{u} - \mathbf{h}_D^*, \tag{6.1.3}$$

is a solution of the integral equations

$$\begin{cases} -\mathcal{V}^D \psi_D + \mathbf{K}^D \varphi_N = f_1, & \mathbf{x} \in \Gamma_D \\ -\mathbf{K}^{*N} \psi_D + \mathbf{D}^N \varphi_N = f_2, & \mathbf{x} \in \Gamma_N \end{cases} \tag{6.1.4}$$

where (f_1, f_2) are given by

$$f_1 = \frac{1}{2}\mathbf{h} - \mathbf{K}^D \mathbf{h}_D^* + \mathcal{V}^D \mathbf{g}^*, \quad f_2 = -\mathbf{D}^N \mathbf{h}_D^* + \frac{1}{2}\mathbf{g} + \mathbf{K}^{*N} \mathbf{g}^*. \tag{6.1.5}$$

Moreover, the solution $(\mathbf{u}, \pi)$ can be represented by

$$\mathbf{u} = \mathbf{W}(\varphi_N + \mathbf{h}_D^*) - \mathbf{V}(\psi_D + \mathbf{g}^*), \quad \pi = Q^d(\varphi_N + \mathbf{h}_D^*) - Q^s(\psi_D + \mathbf{g}^*), \tag{6.1.6}$$

(*ii*) *If, $(\varphi_N, \psi_D) \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM) \times \widetilde{H}^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM)$ is a solution of the boundary integral equations (6.1.4), then the layer representations (6.1.6) define a solution of the Dirichlet-Neumann problem (6.1.1).*

The system of equations (6.1.4) can be written in matrix form as

$$\mathcal{A} \begin{bmatrix} \psi_D \\ \varphi_N \end{bmatrix} = \begin{bmatrix} -\mathcal{V}^D & \mathbf{K}^D \\ -\mathbf{K}^{*N} & \mathbf{D}^N \end{bmatrix} \begin{bmatrix} \psi_D \\ \varphi_N \end{bmatrix} = \mathbf{f}, \tag{6.1.7}$$

where $\mathbf{f} = [f_1, f_2]^T$. Let us decompose the matrix operator $\mathcal{A}$ as

$$\mathcal{A} = \begin{bmatrix} -\mathcal{V}^D & \mathbf{K}^D \\ -\mathbf{K}^{*N} & \mathbf{D}^N \end{bmatrix} = \begin{bmatrix} -\mathcal{V}^D & 0 \\ 0 & \mathbf{D}^N \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{K}^D \\ -\mathbf{K}^{N*} & 0 \end{bmatrix} = \mathfrak{B} + \mathfrak{P}. \tag{6.1.8}$$

where $\mathfrak{B}$ is some invertible matrix operator and $\mathfrak{P}$ is some compact matrix operator as we have already proved in [38], which will imply by the Fredholm Alternative that the operator $\mathcal{A}$ is a Fredholm operator of index zero. Now, we are ready to show the main theorem of this section.

Theorem 6.1.3. *Under Assumption 6.1.1, the system of integral equations (6.1.4) as a unique solution in the space $\widetilde{H}^{-\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) \times \widetilde{H}_{\nu}^{\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)$.*

Theorem 6.1.4. *Let $\mathfrak{D}, \Gamma_D$ and Γ_N be as in Assumption 6.1.1. Then the mixed problem with Dirichlet and Neumann conditions for the homogeneous Stokes system (6.1.1) has a unique solution $(\mathbf{u}, \pi) \in H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D})$. In addition, there is some constant $C = C(\Gamma_D, \Gamma_N)$ such that*

$$\|\boldsymbol{u}\|_{H^1(\mathfrak{D}, \Lambda^1 TM)} + \|\pi\|_{L^2(\mathfrak{D})} \leq C \|(\mathbf{h}, \mathbf{g})\|_{H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) \times H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM)}. \tag{6.1.9}$$

Having this result, we can extend the mixed boundary problem given in Theorem 6.1.4 to the Poisson problem for the system using Newtonian potentials. We state the result at this point due to its relevance in the sequel, but we omit the details of the proof for the sake of brevity. We refer to [55, Section 3.1 and 4.2], [42, Theorem 4.1] and [50, Section 8] for the Brinkman operator. Also, let us introduce for simplicity the notation for the given data

$$\mathcal{B} := \widetilde{H}^{-1}(\mathfrak{D}, \Lambda^1 TM) \times H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM) \times H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM). \tag{6.1.10}$$

Theorem 6.1.5. *Let $\mathfrak{D}, \Gamma_D$ and Γ_N be as in Assumption 6.1.1. Then the mixed Dirichlet-Neumann type boundary problem for the nonhomogeneous Stokes system*

$$\begin{cases} L\mathbf{u} + d\pi = \mathbf{f} \in \widetilde{H}^{-1}(\mathfrak{D}, \Lambda^1 TM), \\ \delta\mathbf{u} = 0 \ \textit{in}\ \mathfrak{D}, \\ \gamma^+\mathbf{u}|_{\Gamma_D} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \\ \mathbf{t}^+(\mathbf{u}, \pi)|_{\Gamma_N} = \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM), \end{cases} \tag{6.1.11}$$

has a unique solution $(\mathbf{u}, \pi) \in H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D})$. In addition, there is some constant $C = C(\Gamma_D, \Gamma_N)$ such that

$$\|\mathbf{u}\|_{\mathbf{H^1(\mathfrak{D},\Lambda^1TM)}} + \|\pi\|_{\mathbf{L^2(\mathfrak{D})}} \le \mathbf{C}\|(\mathbf{f}, \mathbf{h}, \mathbf{g})\|_{\mathcal{B}}. \tag{6.1.12}$$

Remark 6.1.6. Although, the positive measure condition from (5.1.4) is essential for our purpose, we can eliminate one of the partitions Γ_N or Γ_D, obtaining the Dirichlet respectively the Neumann problem. For more details we refer to [25, Theorem 6.1] for the Dirichlet problem for the Stokes system.

6.1.2 The mixed Dirichlet-Neumann problem for the Oseen system

Before we proceed with our analysis, let us mention that Russo and Simader have shown in [96] the well-posedness of the Oseen system in Euclidean Lipschitz domains (see, e.g., [97]). Furthermore, Tartaglione in [104] studied very weak solutions for boundary problems for the Stokes and Oseen in bounded and exterior domains in $\mathbb{R}^n, (n = 2, 3)$ of class $C^{k-1,1}$ for $k \ge 2$.

Theorem 6.1.7. *Under Assumption 6.1.1 there exists a pair $(\mathbf{u}, \pi) \in H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D})$, which satisfies the mixed Dirichlet-Neumann type boundary problem for the Oseen system*

$$\begin{cases} L\mathbf{u} + \nabla_\omega \mathbf{u} + d\pi = \mathbf{f} \ \textit{in}\ \mathfrak{D}, \\ \delta\mathbf{u} = 0 \ \textit{in}\ \mathfrak{D}, \\ \gamma^+\mathbf{u}|_{\Gamma_D} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \\ \mathbf{t}^+(\mathbf{u}, \pi)|_{\Gamma_N} = \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM), \end{cases} \tag{6.1.13}$$

and the following estimate

$$\|\mathbf{u}\|_{\mathbf{H^1(\mathfrak{D},\Lambda^1TM)}} + \|\pi\|_{\mathbf{L^2(\mathfrak{D})}} \le \mathbf{C}\|(\mathbf{f}, \mathbf{h}, \mathbf{g})\|_{\mathcal{B}}, \tag{6.1.14}$$

with some positive constant $C = C(\Gamma_D, \Gamma_N)$.

6.2 Mixed type boundary value problems for nonlinear systems on compact Riemannian manifolds

In this section, we show a solvability result for the mixed Dirichlet-Neumann boundary problems for the Navier-Stokes system on compact Riemannian manifolds. We give an alternative proof of the mixed Dirichlet-Neumann boundary problems for the Navier-Stokes system in Lipschitz domains on compact Riemannian manifolds, by using the well-posedness result for the Oseen system obtained in the previous section in Theorem 6.1.7. This result emphasizes the close relation between the Oseen and Navier-Stokes systems [39, Theorems 3.6 and 4.1].

6.2.1 The mixed Dirichlet-Neumann problem for the Navier-Stokes system

This section is concerned with the mixed Dirichlet-Neumann boundary problem for the Navier-Stokes system. We obtain a well-posedness result for the nonlinear Navier-Stokes system proving that the solution operator for the Oseen system has a fixed point [39, Theorem 4.1].

Theorem 6.2.1. *Under Assumption 6.1.1 and for* $\beta > 0$*, there exist two constants* $C_j \equiv C_j(\Gamma_D, \Gamma_N, \beta) > 0$*,* $j = 1, 2$*, with the property that for all data* $(\mathbf{f}, \mathbf{h}, \mathbf{g}) \in \mathcal{B}$ *satisfying*

$$\|\mathbf{f}\|_{\widetilde{H}^{-1}(\mathfrak{D},\Lambda^1 TM)} + \|\mathbf{h}\|_{H^{\frac{1}{2}}(\Gamma_D,\Lambda^1 TM)} + \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\Gamma_N,\Lambda^1 TM)} \le C_1, \tag{6.2.1}$$

the mixed Dirichlet-Neumann type problem for the Navier-Stokes system

$$\begin{cases} L\mathbf{u} + \beta\nabla_{\mathbf{u}}\mathbf{u} + d\pi = \mathbf{f}, & in\ \mathfrak{D} \\ \delta\mathbf{u} = 0, & in\ \mathfrak{D}, \\ \gamma^+\mathbf{u}|_{\Gamma_D} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma_D, \Lambda^1 TM), \\ \mathbf{t}^+(\mathbf{u}, \pi)|_{\Gamma_N} = \mathbf{g} \in H^{-\frac{1}{2}}(\Gamma_N, \Lambda^1 TM), \end{cases} \tag{6.2.2}$$

has a unique solution $(\mathbf{u}, \pi) \in H^1(\mathfrak{D}, \Lambda^1 TM) \times L^2(\mathfrak{D})$*, such that*

$$\|\mathbf{u}\|_{H^1(\mathfrak{D},\Lambda^1 TM)} \le C_2.$$

Part III

Numerical methods and applications related to mixed boundary value problems

Chapter 7

Numerical methods results for the lid-driven cavity flow problem filled with a porous medium

The solvability result for the Dirichlet problem for the nonlinear Darcy-Forchheimer-Brinkman system given in (4.3.2) has been provided in Theorem 4.3.1. Next we are concerning with a computational study of such a problem in a special Lipschitz domain. Thus in this chapter we describe some numerical methods concerning the well-known lid driven cavity flow problem with Dirichlet and Robin boundary conditions, denoted by short in the sequel as lid problem. The lid problem considers a square cavity consisting of three rigid walls on which non-slip boundary conditions are imposed and a tangentially moving lid with unit velocity. We consider that the motion of the fluid inside the square cavity is governed by the Darcy-Forchheimer-Brinkman system. In fact, we consider only the nonlinear term $\beta \mathbf{u} \cdot \nabla \mathbf{u}$ in our simulations, which means that we consider $\kappa = 0$, $\alpha > 0$ and $\beta > 0$ (see the desciption in the introduction (0.0.3) but also, e.g., [87]).

The lid problem has been the subject to many physical, theoretical and numerical studies, because it connects in its simple geometry all the relevant physical aspects to mathematical models and numerical methods. Therefore, the lid problem has become a benchmark problem for many authors who attempted to validate numerical methods (see, e.g., [1], [31]).

This chapter presents two important numerical methods used in order to study special lid problems in fluid mechanics that are mathematically described by mixed boundary problems. We start with a brief description of the non-dimensional equations related to such fluid flow problems (see, e.g., [44]), as well as the stream function-vorticity formulation (see, e.g., [60]) for the nonlinear Darcy-Forchheimer-Brinkman system. Such a formulation simplifies the numerical treatment of the equations in the two dimensional case. The central difference (CD) method and the Dual Reciprocity Boundary Element Method (DRBEM) considered in this chapter are briefly analyzed in order to evaluate the stability in the particular case of the two dimensional lid problem. In addition, we give a brief comparison of both methods for the related systems based on ([40, Section 4], [37, Section 3]), with classical results found in literature associated to the Navier-Stokes system.

Afterwards, we discuss some numerical results for the lid driven square cavity flow problem for the Darcy-Forchheimer-Brinkman system in two dimensions. Dirichlet and mixed Robin-Dirichlet boundary conditions are considered. The Robin condition is described by the physical meaning of a sliding parameter (see, e.g., [44], [43]). We describe the relation between the geometry of the stream lines and the following parameters: the Reynolds number, the Darcy number and the sliding parameter [87]. The results of this chapter are based on the paper [40, Section 4] written by joint work with T. Groşan and [37, Section 3].

7.1 The lid driven cavity flow problem. Statement and remarks

We next refer to the flow of a incompressible Newtonian fluid in a saturated porous medium located in a square cavity of length L (a special Lipschitz domain), where some specific Dirichlet and Robin boundary conditions are imposed. Three of the domain walls are fixed and the last one moves with a given velocity, which is tangential to the upper boundary. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be the fixed walls and Γ_4 be the moving wall (see Figure 7.1).

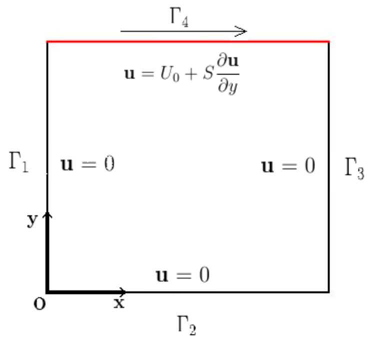


Fig. 7.1: The geometry of the fluid domain

Having become a benchmark problem to validate and test the numerical methods proposed by many authors throughout literature, the laminar incompressible flow in a square cavity whose top wall moves tangential with a uniform velocity has been known as the lid problem.

The zero-slip condition at the nonporous walls yields that the velocity field $\mathbf{u}$ as well as it's normal derivatives vanish on the entire boundary. As it is well known, no direct boundary conditions are provided for the vorticity Ω at the walls (see, e.g., [87]). However, most often the boundary conditions for the vorticity Ω are derived from the definition given in equation (7.2.10) below.

7.2 Non-dimensional analysis for the Darcy-Forchheimer-Brinkman system

For the purpose of solving numerically our physical problem, it is often convenient to derive a system of non-dimensional equations such that the length, height and velocity of the moving wall are scaled to unity. The next subsection gives the derivation of the abstract formulation of the non-dimensional form of the Darcy-Forchheimer-Brinkman system based on [44, pp. 307-309].

7.2.1 Non-dimensional form for the nonlinear Darcy-Forchheimer-Brinkman system

The nonlinear Darcy-Forchheimer-Brinkman system, i.e., the nonlinear PDE system (4.3.2) describes the flow of a incompressible, viscous fluid located in a square cavity filled with a saturated porous medium. Under the tangential motion of the upper wall, the fluid inside the cavity rotates until it arrives at an equilibrium state that is described by the equations in (4.3.2). Different boundary conditions are imposed on the moving wall, which correspond to either the Dirichlet conditions or to the mixed Dirichlet-Robin (4.3.2) boundary conditions.

Let $\mathbf{u}(x, y) = (u(x, y), v(x, y))$ be the velocity field of the flow and $\pi = \pi(x, y)$ the corresponding pressure field. Then the nonlinear Darcy-Forchheimer-Brinkman system (4.3.2) consists of the following scalar partial differential equations

$$\begin{cases} Ox: & \left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)-\alpha u-\dfrac{\partial \pi}{\partial x}=\beta\left(u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}\right), \\ Oy: & \left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)-\alpha v-\dfrac{\partial \pi}{\partial y}=\beta\left(u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}\right), \end{cases} \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{7.2.1}$$

Note that the constants α and β are related the physical properties of the Newtonian fluid and the porous medium. We describe them in subsection 7.2.3.

A main purpose of our analysis is the discretization of the fluid domain. To do so, we consider the following dimensionless variables, which represent a scaling of the length of the square cavity and of the velocity of the moving wall to unity, as follows (see, e.g., [40, Section 4.1])

$$X=\frac{x}{L},\ Y=\frac{y}{L},\ P=\frac{\pi L}{U_0},\ U=\frac{u}{U_0},\ V=\frac{v}{U_0}. \tag{7.2.2}$$

Then the first two equations (7.2.1) reduce to the non-dimensional equations

$$\begin{cases} \left(\dfrac{\partial^2 U}{\partial X^2}+\dfrac{\partial^2 U}{\partial Y^2}\right)-\alpha U-\dfrac{\partial P}{\partial X}=\beta\left(U\dfrac{\partial U}{\partial X}+V\dfrac{\partial U}{\partial Y}\right), \\ \left(\dfrac{\partial^2 V}{\partial X^2}+\dfrac{\partial^2 V}{\partial Y^2}\right)-\alpha V-\dfrac{\partial P}{\partial Y}=\beta\left(U\dfrac{\partial V}{\partial X}+V\dfrac{\partial V}{\partial Y}\right). \end{cases} \tag{7.2.3}$$

To these equations we add the non-dimensional continuity equation

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0. \tag{7.2.4}$$

Note that the scaling parameters are included also in the constants α, β from equation (7.2.3), but we use the same notations as in (7.2.1) for the sake of brevity.

Non-dimensional form of the boundary conditions

In view of the above definition of the non-dimensional variables (7.2.2), one obtains that the associated Dirichlet (non-slip) boundary conditions are of the analyzed lid problem have the form (see Figure 7.1)

$$\gamma^+\mathbf{u}=0 \text{ on } \Gamma_i,\ i=1,2,3, \tag{7.2.5}$$

$$\gamma^+\mathbf{u}=U_0 \text{ on } \Gamma_4, \tag{7.2.6}$$

where U_0 is the velocity on the moving wall. Furthermore, we consider Robin boundary condition for the moving wall Γ_4, which are expressed in terms of a sliding parameter S. Therefore, the Robin boundary condition considered in the sequel is

$$\gamma\mathbf{u}=U_0+S\frac{d\mathbf{u}}{d\boldsymbol{\nu}} \quad \text{on } \Gamma_4, \tag{7.2.7}$$

which lead to the corresponding non-dimensional mixed Dirichlet-Robin boundary conditions in terms of the stream function, given by

$$\frac{\partial\Psi}{\partial X}=0 \quad \text{on } \Gamma_1,\Gamma_3, \qquad \frac{\partial\Psi}{\partial Y}=0 \quad \text{on } \Gamma_2, \qquad \frac{\partial\Psi}{\partial Y}=1+S\frac{\partial^2\Psi}{\partial Y^2} \quad \text{on } \Gamma_4, \tag{7.2.8}$$

where Ψ is the stream function of the flow (we refer the reader to equations (7.2.9) below).

7.2.2 Stream function-vorticity formulation for the nonlinear Darcy-Forchheimer-Brinkman system

Since we are dealing only with the two dimensional case in this chapter, we rewrite the non-dimensional equations by using the stream function-vorticity formulation. Note that the stream function-vorticity formulation gives a simple way to analyze boundary problems in fluid mechanics, since the stream function represents the contours of fluid flow with the same velocity (see, e.g., [44]). This approach is not limited to the two dimensional case, but since the trajectories of of the fluid particles with the same velocity are not restricted to one plane, the mathematical formulation becomes more complex (see, e.g., [60]).

Now, taking into account the continuity equation (7.2.4), we introduce the stream function Ψ, defined by the following equations

$$\frac{\partial \Psi}{\partial Y} = U, \ \frac{\partial \Psi}{\partial X} = -V. \tag{7.2.9}$$

Let Ω be the vorticity field, given by

$$\Omega := \frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y}. \tag{7.2.10}$$

The stream function Ψ and the vorticity Ω are related through the relation

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = -\Omega. \tag{7.2.11}$$

By rearranging the equation (7.2.3) and by using the expressions (7.2.9), the vorticity Ω given by (7.2.10), and the continuity equation (7.2.4), as well as equation (7.2.11), we obtain the main equations of our numerical approach

$$\begin{cases} \dfrac{\partial^2 \Omega}{\partial X^2} + \dfrac{\partial^2 \Omega}{\partial Y^2} + \alpha\Omega = \beta\left(\dfrac{\partial \Psi}{\partial Y}\dfrac{\partial \Omega}{\partial X} - \dfrac{\partial \Psi}{\partial X}\dfrac{\partial \Omega}{\partial Y}\right), \\[2ex] \dfrac{\partial^2 \Psi}{\partial X^2} + \dfrac{\partial^2 \Psi}{\partial Y^2} = -\Omega. \end{cases} \tag{7.2.12}$$

7.2.3 Physical properties related to the fluid flow

The physical properties of a viscous fluid and of a porous medium are described by the following parameters (see, e.g., [87, Section 1.2, 1.5.2 and 1.5.3]):

- *The Reynolds number Re*, which is defined as the ratio of inertial forces to viscous forces and, consequently, quantifies the relative importance of these two forces for the flow conditions.
- *The Darcy number Da*, which represents the relative effect of the permeability of the medium versus its cross-sectional area, and is defined as the ratio between the permeability of the medium and the square of the diameter of the particles.
- *The porosity φ*, which is a fraction of the volume of voids over the total volume and represents a measure of the empty spaces in the fluid.
- *The viscosity coefficient μ* is related to the porosity of the medium.

The constants α and β, which appear in the mathematical model (4.3.2) are related by these physical parameters through the following formulas

$$\alpha = \frac{\varphi}{\mu Da}, \tag{7.2.13}$$

$$\beta = \frac{Re}{\varphi\mu}, \tag{7.2.14}$$

where α is called *the Darcy coefficient*, and β is *the convection coefficient.*

7.3 Central difference method (CD) and Dual Reciprocity Boundary Element Method (DRBEM)

In the complete version of this thesis, two numerical methods are described, the central difference (CD) method and the Dual Reciprocity Boundary Element Method (DRBEM).

A detailed description of the domain discretization and the stability of the central difference method combined with Gauss-Seidel iteration for the partial differential equations is presented.

The second numerical method considered in this thesis is the DRBEM, which can be applied to the integral form of the Darcy-Forchheimer-Brinkman system. The DRBEM method considers a series expansion of the Poisson term for the Laplacian with respect to the radial basis function (see, e.g., [9], [89], [35]).

7.4 Comparison of the numerical results with classical results in literature

Next we refer to the lid problem and present the numerical results obtained in [40] by using the central difference method, and those in [37] provided by the DRBEM. In order to validate these numerical results, we begin the analysis with the comparison of the maximal, absolute value of the stream function with classical results from literature, in the case when the fluid inside the square cavity is described by the Navier-Stokes equations.

The Navier-Stokes system is first assessed in order to verify the correctness of the numerical solution since ample results are available in literature, especially in the case $Re = 100$. Table 7.1 presents the results of our numerical simulations for different Reynolds numbers, and the results obtained in [31], [95], [72] and [27].

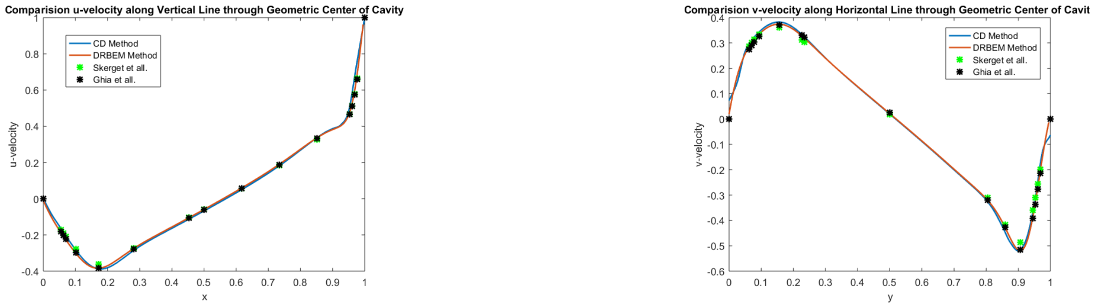
$\lvert\psi_{max}\rvert$ (center)	$Re = 10$	$Re = 100$	$Re = 1000$
Gutt [37]	0.1001 (0.5175, 0.7658)	0.1036 (0.6136, 0.7367)	0.1187 (0.5275, 0.5715)
Gutt and Groşan [40]	0.1000 (0.51, 0.77)	0.1034 (0.615, 0.74)	- -
Ghia et al. [31]	- -	0.1034 (0.6172, 0.7344)	0.1179 (0.5313, 0.5625)
Rek and Škerget [95]	- -	- -	0.113 (0524,0565)
Marchi et al. [72]	0.1001 (0.516, 0.7646)	0.1035 (0.616, 0.737)	0.1189 (0.531, 0.565)
Erturk et al. [27]	- -	0.1035 (0.6152, 0.7363)	0.1187 (0.5313, 0.5645)

Table 7.1: Comparison of our numerical result with classical results for the Navier-Stokes system.

The second approach in the validation of the numerical methods employed here requires the comparison of the velocity profiles u along the vertical line passing through the center of the cavity and v along the horizontal line, respectively, with the profiles given in [31] and [95].

Let us mention that the work of U. Ghia, K. Ghia and Shin [31] studies the vorticity-stream function formulation of the two-dimensional Navier-Stokes equations by using a coupled strongly implicit multigrid method. Their work studies the lid problem for the very high values of the Reynolds number up to 10.000 and has become the benchmark paper for this problem.

We also compare our numerical results with those of the paper of Rek and Škerget [95]. The authors have used BEM in their study. This comparison is important since in spite of the singularities at two of the cavity corners which lead inevitably to errors in the computation on the boundary integrals, the results presented here are in good agreement with the ones obtained by Rek and Škerget.



The profile of the component u of the velocity field along vertical line

The profile of the component v of the velocity field along horizontal line

Fig. 7.2: Comparison of the velocity profiles for u and v along the vertical line and horizontal line, respectively, passing through the geometric center of the cavity $Re = 1000$ with the results obtained by [31] and [95].

$Re = 10$

$Re = 100$

Vorticity profiles Ω for $Re = 1000$

Vorticity profiles Ω for $Re = 1000$

Fig. 7.3: Comparison of the profiles of the vorticity Ω along the moving boundary of the cavity $Re = 100$ and $Re = 1000$ with the results obtained by [31] and [95].

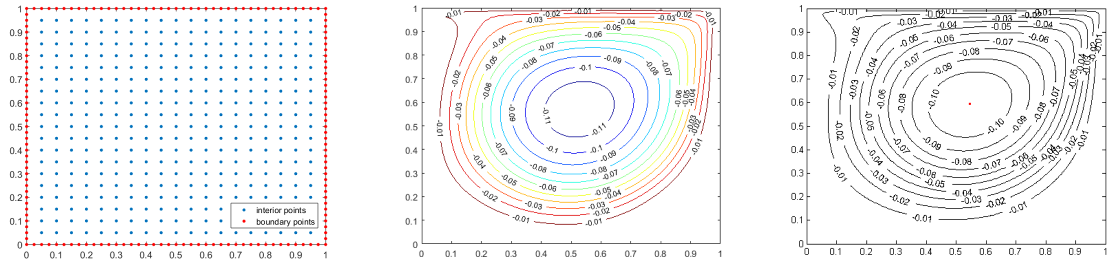
Figures 7.2 show the velocity profiles for u along vertical line and v along horizontal line for $Re = 1000$ passing through the geometric center of the cavity. We can observe that as the Reynolds number increases, the primary vortex moves up in the cavity, leading to the thinning of the boundary flow layer. For these values of the Reynolds number, the thinning is very slow, but increases more rapid for $Re > 5000$ (see, e.g., [31]).

Moreover, let us remark some phenomena that appear in the case of high Reynolds number, for which the numerical methods employed in the thesis are not adequate. We mention here this behavior, since it will occur for lower values of the Reynolds number in the case when a porous medium is involved, and accordingly, when the Darcy-Forchheimer-Brinkman system is considered.

As the velocity profiles in Figure 7.2 are beginning to suggest, at large Reynolds numbers, the near-linearity of the velocities throughout the cavity indicates the uniform vorticity region (see, e.g., [31]). Therefore, the component u of the velocity field bends rapidly near $y = 1$, whereas the velocity component v bends rapidly near $x = 1$.

For both values of $Re = 100$ and $Re = 1000$, the results obtained by the central difference method and those provided by BEM are in good agreement with the results reported by U. Ghia, K. Ghia and Shin [31] and [95].

a) BEM mesh b) DRBEM c) Gauss-Seidel iteration



● interior nodes L, ● boundary nodes N | $|\psi_{max}| = 0.1132$, $C = (0.5405, 0.5603)$ | $|\psi_{max}| = 0.1139$, $C = (0.5450, 0.5950)$

Fig. 7.4: (a) Domain discretization, (b)-(c) Comparison of the fluid flow in a porous medium in the case of the lid problem for the parameters $Re = 100$, $Da = 0.25$, $\phi = 0.2$.

Thirdly, we compare the vorticity profiles along the moving boundary wall, with the results given in [31] and [95]. The results obtained for $Re = 100$ and $Re = 1000$, are presented in Figure 7.3. As mentioned, the singularities at the two corners appear clearly and the results are in good agreement. Notice that for $Re = 1000$, the vorticity profile obtained by the BEM (for both the simulations presented in this thesis and that obtained in [95]) is above the vorticity profile obtained by central difference methods.

The last validation method we use requires a comparison between the numerical results obtained by using central difference methods and the Gauss-Seidel iterative scheme for the nonlinear Darcy-Forchheimer-Brinkman system (see [44] for more details), and the simulations obtained by using the DRBEM approach shown in Figure 7.4. The fluid and the porous medium parameters for the nonlinear Darcy-Forchheimer-Brinkman system are chosen as follows: $Re = 100$, $Da = 0.25$, $\phi = 0.2$ and $\mu = 1$.

The results obtained in Figures 7.4 (b)-(c) for the nonlinear Darcy-Forchheimer-Brinkman system by the CD method and DRBEM are in good agreement. The difference of the maximal absolute value of the stream function has an absolute error less than 10^{-3}, whereas the offset of the vortex center is explained by the fact that for the BEM we have used an interpolation procedure over a mesh of 1000×1000 point, whereas the discretization for the CD method has only 201×201 mesh points.

7.5 Numerical results and discussion related to the lid driven cavity flow problem filled with a porous domain

Since we have established the validation of the two methods, we now describe some numerical results regarding the nonlinear Darcy-Forchheimer-Brinkman system for the lid problem. The analysis begins with the case of Dirichlet boundary conditions, i.e., in the absence of the sliding parameter. For this case, we discuss the change of the streamlines with the variation of the Reynolds number Re between the values of 10 and 1000, for two cases when the porosity parameter is set to $\phi = 0.2$, as in [40], and $\phi = 0.5$ as in [37], respectively. Afterwards, we focus on the influence of the Darcy number Da on the streamline geometry, having $Re = 100$.

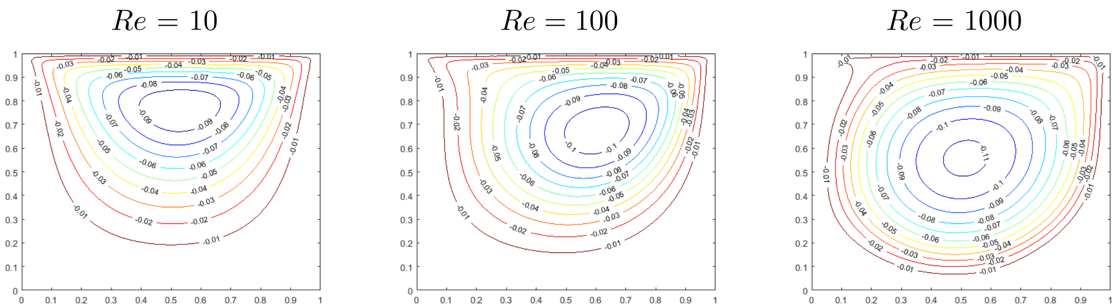
Next, we impose a sliding parameter on the upper moving wall which leads to a mixed Dirichlet-Robin boundary problem for the lid problem. The physical meaning of the sliding parameter is that not all the fluid in the neighborhood of the upper wall is engaged in the fluid motion. Therefore, as the sliding parameter increases, we expect that the strength of the primary vortex diminishes, a behavior that is in good agreement with the numerical results. However, this effect is only visible in the case when the DRBEM method is used, since the CD method is not stable for large enough values if the sliding parameter.

7.5.1 Variation of the physical parameters for the lid driven cavity flow problem in the absence of the sliding parameter

In this subsection, we analyze the structure of the streamlines for some representative fluid parameters (Re and Da) in the absence of the sliding parameter, i.e., $S = 0$ corresponding to Dirichlet boundary conditions since we have no sliding of the fluid in the neighborhood of the boundary.

Physical description on the variation of the Reynolds number

First, we analyze the dependency of the streamlines for $Re = 10, 100, 1000$. The Darcy parameter is assumed to be constant and equal to $Da = 0.25$, as well as the viscosity coefficient $\mu = 1$.



$Re = 10$: $|\psi_{max}| = 0.0984$, $C = (0.5315, 0.7667)$

$Re = 100$: $|\psi_{max}| = 0.1066$, $C = (0.6036, 0.6736)$

$Re = 1000$: $|\psi_{max}| = 0.1124$, $C = (0.5055, 0.5675)$

Fig. 7.5: Streamlines of the fluid flow in a porous medium with porosity $\phi = 0.5$, in the case of the lid problem for $Re = 10, 100, 1000$, obtained by using DRBEM.

As Re increases, the shortcoming of coarse meshes gradually becomes apparent, and the first order central difference method employed is no longer convergent. Therefore, the central difference method presented is only convergent for Reynolds numbers up to 200. However, since the porosity parameter ϕ occurs in the calculation of the convention parameter β, the value $\phi = 0.2$ corresponds to the value of $\beta = 1000$, similarly to the case of the Navier-Stokes system with $Re = 1000$. Therefore, the behavior of the streamlines change in the case of a porous medium is similar to that for higher Reynolds numbers. The streamline contours computed for the porosity parameter $\phi = 0.5$ by using DRBEM for the lid problem with Re increasing from 10 to 1000 are shown in Figure 7.5. For large Reynolds numbers, the maximum value of the streamfunction is larger than that in the case of the Navier-Stokes system, as the convection term has a significant role.

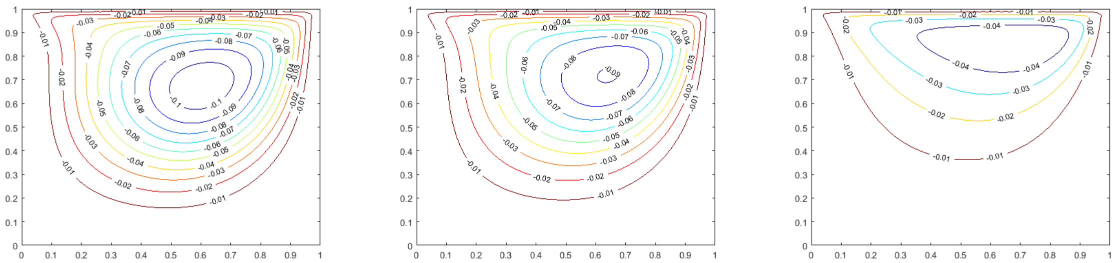
Physical description on the variation of the Darcy number

Now, we choose $Re = 100$ and $\mu = 1$, and analyze the dependency of the streamline geometry on the variation of Da for the values of $0.25, 0.025, 0.0025$, which are considered to be relevant values as mentiond by Nield and Bejan [87].

Notice that for both cases $\phi = 0.2$ in [40] and $\phi = 0.5$ in [37], the vortex is moving to the upper right corner of the cavity as the Darcy number decreases and that the strength of the vortex is directly proportional to the Darcy number. This dependency of the fluid flow is consistent with the physical behavior of the fluid [87].

For $\phi = 0.5$, the effect of the Darcy number is more visible beginning with larger values. From Figures 7.6, we observe that the primary vortex is located at a higher level in the cavity when the porosity is $\phi = 0.5$ then in the case $\phi = 0.2$ in [40], and that the strength of the vortex is weaker for the same value of the Darcy number. Therefore, in the case $\phi = 0.5$ we consider values of the Darcy number only near the value 0.0025.

$Da = 0.25$ $Da = 0.025$ $Da = 0.0025$



$|\psi_{max}| = 0.1066$, $C = (0.6036, 0.6736)$ $|\psi_{max}| = 0.0906$, $C = (0.6426, 0.7207)$ $|\psi_{max}| = 0.0489$, $C = (0.692, 0.8548)$

Fig. 7.6: Streamlines of the fluid flow in a porous medium with porosity $\phi = 0.5$ in the case of the lid problem for $Da = 0.25, 0.025, 0.0025$, computed by DRBEM.

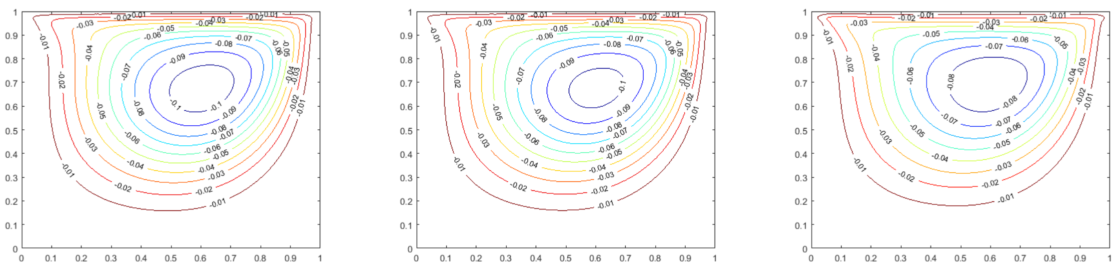
7.5.2 The lid driven cavity flow problem with a non-vanishing sliding parameter considered on the moving wall

This subsection is concerned with the mixed Dirichlet-Robin boundary problem associated to the nonlinear Darcy-Forchheimer-Brinkman system. Thus, we consider an additional sliding parameter imposed on the upper moving wall. A numerical study of the Navier slip condition can be found in [43]. This condition implies that not the entire fluid located in the neighborhood of the boundary is driven by the moving wall.

Let us mention that this kind of condition could be a particular case of a more general interface condition that takes into account of friction, adhesion and contact with memory (see [102] and [103]).

However, the DRBEM has provided a better convergence then in the case of central differences employed in [40], when the porosity is considered to be $\phi = 0.5$, such that higher values of the sliding parameter can be considered up to values of $S = 0.1$. For such values of the sliding parameter, a change of the streamlines is well visible.

$S = 0$ $S = 0.01$ $S = 0.1$



$|\psi_{max}| = 0.1066$, $C = (0.6036, 0.6736)$ $|\psi_{max}| = 0.1047$, $C = (0.6066, 0.6766)$ $|\psi_{max}| = 0.0882$, $C = (0.596, 0.695)$

Fig. 7.7: Streamlines of the fluid flow in a porous medium with porosity $\phi = 0.5$ computed by DRBEM for the sliding parameter $S = 0, 0.001, 0.01$.

Figures 7.7 shows that the structure of the streamlines of the fluid flow changes slightly with the increase of the sliding parameter and also the strength of the maximal absolute value of the stream function decreases as the sliding parameter increases. This behavior is in good agreement with the physical meaning of the parameter.

Conclusions

This thesis studies boundary value problems of mixed type (Dirichlet-Neumann, Dirichlet-Robin) for various elliptic systems in fluid mechanics and the theory of porous media, beginning with the analysis in the Euclidean settings, continuing the study on Riemannian manifolds and finally considering some numerical results regarding a special boundary problems with mixed boundary conditions.

We start with some introductory remarks regarding the geometrical and functional settings we are concerned in this thesis. As one of the first original results, we consider the connection (Theorem 1.3.3) between the nontangential trace and the Gagliardo trace operators 1.3.1 and Lemma 1.3.2). Afterward, we present the connection (Theorem 1.5.5) between the nontangential derivative operator (Eq. 1.5.4), the generalized derivative operator (Definition 1.5.5) and the canonical derivative operators (Definition 1.5.3).

In the next chapter we present the definitions of the potential operators associated to the Brinkman system and obtain some mapping properties of the Newtonian potential operator (Lemma 2.2.1), of the single-layer potential operator (Lemma 2.3.2 and Theorem 2.3.3), of the double-layer potential operator (Lemma 2.3.4 and Theorem 2.3.5) and the jump relations between them (Theorem 2.4.2).

Having the above mentioned results, the next chaper begins with the analysis of certain boundary problems in the Euclidean setting $\mathbb{R}^n$, with $n \geq 3$. In order to extend the mixed Dirichlet-Neumann boundary problem to L^p-based Sobolev spaces, we consider first that the boundary data belongs to L^2-based Sobolev spaces (Theorem 3.1.2) and introduce a Dirichlet-to-Neumann operator (Lemma 3.1.3) which provides the desired extension in Theorem 3.1.4. Finally, we are able to obtain the well-posedness result for the mixed boundary problem for the semilinear Darcy-Forchheimer-Brinkman system (Theorem 3.2.1).

The following chapter considers a similar outline as the previous one, but studies boundary problems in Euclidean settings of dimension $n = 2$, based mainly on a combination of a potential approach with a variational approach. The coeciveness of the associated layer potentials lead to the well-posedness of a variational problem related to the mixed Dirichlet-Neumann problem for the Brinkman system. Going further to the nonlinear Darcy-Forchheimer-Brinkman system, we give a proof for the mixed Dirichlet-Robin boundary conditions (Theorem 4.3.1).

In the second part of this thesis, we consider boundary problems for the Stokes, Oseen and Navier-Stokes systems on compact Riemannian manifolds. We present some original invertibility and compactness results regarding layer potential operators in Theorem 5.2.2 and Theorem 5.2.3. In order to obtain a well-posedness result for the Navier-Stokes system on compact Riemannian manifolds (Theorem 6.2.1), we consider the mixed Dirichlet-Neumann problem for the Stokes system (Theorem 6.1.4), moving on to the Oseen system (Theorem 6.1.7) and finally applying a fixed point theorem we obtain the final result.

The last part of this thesis considers numerical methods and results, which correspond to well-posedness results obtained throughout this work. The results provided by methods employed here are compared with the existing results in the literature. Finally we discus some numerical results for the lid driven cavity problem for the Darcy-Forchheimer-Brinkman system in two dimensions with Dirichlet and mixed Robin-Dirichlet boundary conditions.

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