Babeş-Bolyai University, Cluj-Napoca, Romania

Doctoral School of Mathematics and Computer Science

PhD Thesis - Summary

CONTRIBUTIONS TO THE FIXED POINT THEORY OF SINGLE-VALUED AND MULTI-VALUED OPERATORS DEFINED ON GENERALIZED METRIC SPACES

Scientific Advisor

Prof. Dr. Adrian-Olimpiu Petruşel

PhD Student

Cristian-Daniel Alecsa

Cluj-Napoca, Romania
2019

Contents

Keywords: fixed point results, convex metric spaces, Picard iteration, Mann iteration, Ćirić operators, Presić operators, convex contractions, admissible perturbation, data dependence, multi-valued operators.

Introduction

Fixed Point Theory can be considered as one of the most dynamical research domains of Mathematics, that has many applications in different scientific domains, such as physics, economics, engineering, computer science and so on. The theory of single-valued and multi-valued operators can be considered as a part of the mathematical field of nonlinear analysis, the latter being a fruitful domain of activity, with a very dynamic development. We recall first a brief description of some of the main ideas of the Fixed Point Theory, which are related to our PhD Thesis :
Stefan Banach, in [30], proved that any contraction on a complete metric space has a unique fixed point, i.e. there exists and is unique $x^* \in X$, such that $f(x^*) = x^*$. The well known Banach contraction principle was first extended in [126] by S.B. Presić to operators defined on a cartesian product of sets. In [103], M. Păcurar obtained a convergence result regarding Kannan-Presić operators that is an interesting generalization of Presić contractions, in the sense that they are discontinuous. At the same time, I.A. Rus [140] and M. Păcurar [104] proved some fixed points and common fixed points results through a multi-step type iterative method. In this way, the field of Fixed Point Theory was briefly connected to the subfields of Numerical Analysis.
Quite interestingly, in [38], A. Branciari introduced a new metric-type functional in which the well-known triangle inequality is replaced by an inequality which involves four different elements. This is called a rectangular metric space or a generalized metric space. At the same time (see [67]) was proved that convergent sequences and Cauchy sequences can be introduced in a similar manner as in metric spaces. This type of topological spaces are crucial due to the fact that they generalize the well-known metric spaces. Eventually, this led to an increasing research regarding fixed point results in the setting of different type of generalized metric-type spaces.
Now, we turn our attention to the work of F.F. Bonsall [35] and S.B. Nadler Jr. [97], who have studied some stability results regarding sequences of contractions, that are defined on a metric space (X, d). An interesting extension of the previous results was made by M. Păcurar [105], who has developed pointwise and uniform convergence results, for the sequence of fixed points of almost contractions. On the other hand, L. Barbet and K. Nachi [31] considered some convergence results for the fixed points of a contraction mapping in the usual setting of a metric space (X, d), but for operators defined on subsets and not on the entire metric space (X, d).
Last but not least, we end this introduction with the work of W. Takahashi (see [157]), who has introduced a new concept of convexity in metric spaces and proved that all normed convex spaces and their convex subsets are convex metric spaces. Further, he gave some examples of convex metric spaces. At the same time, generalized type of contractype mappings were studied in the framework of convex metric spaces.

The present PhD Thesis is divided into four chapters, where each chapter contains several sections and subsections.

Chapter 1 : **Preliminaries**.
The aim of this chapter is to recall some important concepts that will be used through the entire thesis. The used terminology concerns the basic notions of Presić single-valued operators, convex metric spaces, b-rectangular metric spaces, cone metric spaces over Banach algebras and different qualitative notions regarding single-valued and also multi-valued operators. This chapter contains the following sections :
• *Qualitative notions for single-valued and multi-valued mappings* :
In this section we recall the most important concepts regarding fixed points and strict fixed points of single-valued ande multi-valued mappings, Ulam-Hyers stability, well-posedness and the convergence of fixed point

iterations. The basic articles concerning these concepts are the following : [32], [53], [79], [80], [96], [112], [115], [116], [117], [120], [138], [139].
• *Presić operators* :
In this section we present a survey concerning generalized Presić-type contractions, introduced by a metrical type approach. The theory behind it is based upon the convergence of a k-step fixed point iteration and the data dependence of the fixed points of Presić single-valued operators. The most important references are the following : [76], [37], [148], [140], [104], [47].
• *b-Rectangular metric spaces* :
In this section we describe generalized contractions in the framework of b-rectangular metric spaces, along with some topological and metric properties of these spaces. Also, we briefly recall the idea of b-metric spaces and rectangular metric spaces. The most important papers regarding these generalzied type metric spaces are the following : [52], [67], [136].
• *Cone metric spaces over Banach algebras* : This section concerns the pointwise and uniform convergence for a family of mappings defined on subsets of cone metric spaces over a given Banach algebra. Also, we present the idea of equicontinuity of a family of mappings under these generalized metric-type spaces. The most important references that are used are the following : [35], [97], [105].
• *Convex metric spaces* : This section contains several common fixed point results in the setting of convex metric spaces for generalized single-valued contractions. Some articles that were used through this entire section are the following : [34], [51], [55],[91], [92], [93], [159].

Chapter 2 : **Fixed point results for single-valued generalized contractions in generalized metric spaces**.
The aim of this chapter is to present some existence and uniqueness fixed point theorems and data dependence results for Presić-type generalized contractions in the setting of complete metric spaces. Also, we give some extended fixed point results for a generalized type contraction mappings in b-rectangular metric spaces. Last but not least, we present the idea of the convergence of the sequence of fixed points of contractions defined on cone metric spaces over Banach algebras. Finally, we give some applications to systems of functional and differential equations. The sections of this chapter are :
• *Some results for Istrăţescu-Presić and Presić-type operators via simulation functions for single-valued mappings* :
In this section, we present some fixed point results for Presić-type single-valued operators. We introduce new type of operators that generalizez the well-known Presić contractions and Istrăţescu convex contractions of second order. Our original contributions of this section are the following :
- [Definition 2.1.1] and [Definition 2.1.2] in which we introduce new types of Presić operators, namely Presić convex contractions of the first and second kind, respectively.
- [Theorem 2.1.4] which is an existence and uniqueness result for Presić convex contractions of the first and second kind, respectively.
- [Corollary 2.1.5] which is a consequence of an existence and uniqueness result for Presić convex contractions of second order.
- [Theorem 2.1.7] which is a data dependence result of the fixed point between Presić convex contractions of the first and second kind and a mapping on a cartesian product of a complete metric space with at least a fixed point.
- [Example 2.1.9], [Example 2.1.10], [Example 2.1.11] in which we present some non-trivial examples of Presić convex contractions of the first and second kind, respectively.
- [Definition 2.1.12] in which we introduce the idea of a k-simulation mapping.
- [Example 2.1.13], [Example 2.1.14], [Example 2.1.15], [Example 2.1.16], [Example 2.1.17], [Example 2.1.18] in which we give some examples of k-simulation mappings that will be used in the rest of this section.
- [Definition 2.1.19], [Definition 2.1.20], [Definition 2.1.21], [Definition 2.1.22] in which we introduce some Presić-type single-valued operators that are based upon the idea of a k-simulation function.
- [Theorem 2.1.23] which is the main result of the second part of this section in which we prove an existence and uniqueness result for the so-called $P-(\mathcal{Z},g)$-ordered contractions.
- [Corollary 2.1.25], [Corollary 2.1.26], [Corollary 2.1.27], [Corollary 2.1.28], [Corollary 2.1.29], [Corollary 2.1.30] that are some consequences of the main result.
- [Example 2.1.32], [Example 2.1.33], [Example 2.1.34] that validates the already mentioned corollaries.
• *Generalized contractions, fixed points and b-rectangular metric-type spaces* :

In this section we prove some existence and uniqueness results for generalized contraction-type mappings under the framework of a complete b-rectangular metric space, the latter being a natural generalization of the well-known b-metric spaces and rectangular metric spaces, respectively. Our original contributions from this section are the following :

- [Theorem 2.2.1] which is an existence result for the fixed points of some generalized contractions.
- [Example 2.2.3] which is an example of a generalized contraction mapping under a suitable complete b-rectangular metric space.
- [Example 2.2.4] in which we construct an example of a b-rectangular metric space that will be used throughout the entire section.
- [Example 2.2.5] which is also an example of a generalized contraction mapping in the setting of a complete b-rectangular metric space.
- [Lemma 2.2.7] which is an important result that will be used in the analysis of expansive mappings.
- [Theorem 2.2.9] which is the second important result of this section, in which we study the existence of fixed points for expansive mappings.
- [Example 2.2.10] which is an example that validates the already mentioned theorem.
- [Theorem 2.2.13] which is an existence of fixed points for expansive type mappings, under suitable assumptions on the generalized contraction.
- [Example 2.2.15] which is an example of an expansive mapping under the setting of a given b-rectangular metric space.

• *Contraction sequences in cone metric spaces over Banach algebras. Applications to nonlinear systems of equations and systems of differential equations* :

In this section we extend the concepts of G-convergence, H-convergence and continuity type properties of mappings that are defined on a cone metric space over a given Banach algebra. Here, we validate our theoretical results through some applications for the solutions of systems of functional and differential equations. Our main contributions are the following results :

- [Definition 2.3.1] in which we extend the idea of pointwise convergence of a sequence of operators to the framework of cone metric spaces over Banach algebras.
- [Definition 2.3.2] in which we extend the idea of uniform convergence of a sequence of operators to the framework of cone metric spaces over Banach algebras.
- [Definition 2.3.3] and [Definition 2.3.4] in which we extend the idea of pointwise and uniform equicontinuity for a family of mappings defined on cone metric spaces over Banach algebras.
- [Example 2.3.5] which is an example of a complete metric space over a Banach algebra.
- [Example 2.3.6] in which we present an example of a sequence of mappings that converge uniformly in the setting of cone metric spaces over a Banach algebra.
- [Example 2.3.8] in which we present an example of a sequence of mappings that are endowed with pointwise convergence in the setting of cone metric spaces over a Banach algebra.
- [Theorem 2.3.9] which is the main result of this section in which we present the convergence of a sequence of fixed points in the framework of cone metric spaces, under the assumption that these elements are fixed points for some operators that are endowed with the uniform convergence.
- [Theorem 2.3.10] in which we present the convergence of a sequence of fixed points in the framework of cone metric spaces, under the hypothesis that these elements are fixed points for some operators that are endowed with the pointwise convergence.
- [Definition 2.3.11] and [Definition 2.3.12] in which we extend the concepts of G and H convergence to mappings defined on cone metric spaces over Banach algebras.
- [Proposition 2.3.14] and [Theorem 2.3.15] that are related to the existence and uniqueness of the (G)-limit of a family of operators.
- [Theorem 2.3.16] and [Corollary 2.3.18] in which we are concerned with the convergence of a sequence of fixed points that belong to a family of mappings that is endowed with property (G).
- [Theorem 2.3.19] in which we are concerned with the relationship between the pointwise convergence of a sequence of mappings in the setting of a cone metric space and the equicontinuity of the family of mappings.
- [Theorem 2.3.20], [Theorem 2.3.21], [Theorem 2.3.22] in which we present some theoretical results regarding the (H) and (G) limit of a family of operators.
- [Theorem 2.3.24] in which we present an application to nonlinear systems of equations.
- [Theorem 2.3.25] and [Theorem 2.3.26] in which we present an application to systems of differential

equations.

Chapter 3 : **Fixed point results for multi-valued generalized contractions**.
The aim of this chapter is to to present some fixed point results for multi-valued operators using the idea of altering distances. Furthermore, we present some fixed point and strict fixed point extended principles for Ćirić multi-valued operators. This chapter contains the following sections :
• *Fixed point results for multi-valued operators. The altering distance technique* :
In this section we consider some existence and uniqueness results on fixed points of multi-valued operators that are introduced by technique of altering distances. Our original contributions are the following :
- [Theorem 3.1.1] in which we consider some fixed point results for multi-valued operators that are endowed with a metrical condition, the latter being based on a comparison-type mapping.
- [Remark 3.1.3] and [Remark 3.1.4] in which we have some observations regarding the comparison function that is used in the main result of this section.
- [Theorem 3.1.5] which is the second theoretical result of this section, that differs from the first theorem in the assumptions that we impose.
• *Extended fixed point principles for Ćirić multi-valued contractions* :
In this section we present some extended fixed point principles and strict fixed point principles for multi-valued Ćirić contractions. Our original contributions are :
- [Theorem 3.2.1] in which we consider a fixed point principle for Ćirić multi-valued operators. We discuss some fixed point concepts such as : Ulam-Hyers stability, well-posedness, selection results, data dependence, the compactness of the set of fixed points.
- [Theorem 3.2.2] which is the second result of the present section, in which we consider metrical and topological results for the strict fixed points of the Ćirić multi-valued contractions.

Chapter 4 : **Iterative schemes for generalized contractions in complete metric spaces**.
In this chapter we present some theoretical results for fixed points of single-valued generalized contractions in the setting of convex metric spaces. Also, in the particular framework of hyperbolic spaces, we consider the fixed point analysis of the Mann iteration for multi-valued mappings. Last but not least, we present some results in which we compare the rate of convergence of different iterative schemes. Our original contributions from this chapter are :
• *Fixed point analysis of some generalized contractions through Ishikawa's iteration* :
In this section we present some fixed point results for some generalized contractive-type mappings in the setting of convex metric spaces. Our original results are the following :
- [Theorem 4.1.1] which is the first important result regarding the convergence of the Picard successive approximation sequence for mappings satisfying a generalized contractive metrical condition.
- [Corollary 4.1.3] which is a consequence of the previous result, for the iterates of the self-mapping.
- [Example 4.1.4] which is an example that validates our theoretical result, in the setting of a generalized contraction mapping.
• *Qualitative properties and stability results for Mann's algorithm for multi-valued mappings* :
The present section consists in some theoretical results regarding the convergence of Mann iterative scheme in the context of multivalued operators introduced by admissible perturbations. The original contributions from this section are the following :
- [Definition 4.2.3] in which we consider the concept of the convergence of Mann algorithm given by the idea of admissible perturbations for multi-valued type operators.
- [Theorem 4.2.5] which is a data dependence result for Mann iterative algorithm, in the setting of multi-valued operators.
- [Theorem 4.2.7] which represents a useful lemma that is used in proving the T-stability of the Mann iterative sequence.
- [Theorem 4.2.10] in which we deal with the data dependence of the Mann iteration in the framework of complete hyperbolic metric spaces.
• *Convergence results for iterative schemes in the setting of convex metric spaces* :
The last section of this chapter is based upon the rate of convergence of some new iterative schemes introduced by us. These results rely on the idea of a convex metric spaces. Also, our original contributions

are :
- [Theorem 4.2.12] deals with the property of Ulam stability with respect to Mann iteration.
- [Theorem 4.2.13] in which we prove the convergence of a sequence of contractions with respect to Mann iteration.
- [Theorem 4.2.14] which deals with the well-posedness with respect to Mann's iterative scheme.
- [Theorem 4.2.15] in which we consider the property of limit shadowing with respect to Mann iteration for multi-valued contractions.
- [Theorem 4.3.1], [Theorem 4.3.3], [Theorem 4.3.4], in which we consider the convergence of newly introduced iterative sequences to the fixed point of a single-valued contraction in the setting of a complete convex metric space.

The authors contributions included in this PhD thesis are part of the following scientific papers :

- C.D. Alecsa, *Common fixed points of Presić operators via simulation functions*, J. Nonlinear Convex Anal. **20** (2019), no. 3, 1-15.
- C.D. Alecsa, *Approximating fixed points for nonlinear generalized mappings using Ishikawa iteration*, Rend. Circ. Mat. Palermo, II. Ser. **68** (2019), no. 1, 163-191.
- C.D. Alecsa, *Fixed point theorems for generalized contraction mappings on b-rectangular metric spaces*, Studia Universitatis Babeş-Bolyai Mathematica **62** (2017), no. 4, 495-520.
- C.D. Alecsa, *Some fixed point results linked to $\alpha - \beta$ rational contractions and modified multivalued Hardy-Rogers operators*, J. Fixed Point Theory, 2018:3 (2018), Article ID 3, 1-22.
- C.D. Alecsa, *Sequences of contractions on cone metric spaces over Banach algebras and applications to nonlinear systems of equations and systems of differential equations*, arXiv:1906.06261, 26 pages, (*submitted*).
- C.D. Alecsa, A. Petruşel, *Some variants of Ćirić's multi-valued contraction principle*, Anal. Univ. de Vest Timisoara Ser. Mat. Inf., vol. 1, 2019, (*accepted*).
- C.D. Alecsa, *Stability results and qualitative properties for Mann's algorithm via admissible perturbations technique*, Applied Anal. Optim. **1** (2017), no. 2, 327-344.
- C. Alecsa, *On new faster fixed point iterative schemes for contraction operators and comparison of their rate of convergence in convex metric spaces*, Int. J. Nonlin. Anal. Appl. **8** (2017), no. 1, 353-388, doi:10.22075/IJNAA.2017.11144.1543.
- C.D. Alecsa, A. Petruşel, *On some fixed point theorems for multi-valued operators by altering distance technique*, J. Nonlin. Var. Anal. **1** (2017), 237-251.
- C.D. Alecsa, *Some fixed point results regarding convex contractions of Presić type*, J. Fixed Point Theory and Appl. 20:**20** (2018), no. 1, doi:10.1007/s11784-018-0488-7.

An important part of the original results proved by us were presented at the following scientific conferences :

- Interdisciplinary Conference for Phd Students, Baru Mare, Hunedoara, 3-5 June 2016
- Student Scientific Session, Phd Students, Cluj-Napoca, 31 May 2016
- Student Scientific Session, Phd Students, Cluj-Napoca, 6 June 2017
- Numerical Analysis, Approximation and Modeling (Symposium), Cluj-Napoca, 14 June 2017
- National Session of Mathematics Scientific Communications, Iaşi, 6-9 July 2017
- Numerical Analysis, Approximation and Modeling (Symposium), Cluj-Napoca, 16 April 2019

Acknowledgments :

In the memory of my father,

I will like to express my sincere gratitude to my PhD Scientific Advisor, Prof. **Adrian Olimpiu Petruşel** for his guidance and for his valuable suggestions given throughout my PhD Thesis. Also, the present thesis would not have been made without the profound scientific support of the members of the **Nonlinear Operators and Differential Equations** research group from **Babeş-Bolyai University**.
My sincere thanks are addressed to Associate Prof. **Szilard Csaba László** from **Technical University of Cluj-Napoca**, Lecturer **Adrian Viorel** from **Babeş-Bolyai University of Cluj-Napoca** and Prof. **Radu Boţ** from **University of Vienna** for their important contributions to my scientific thinking during the participation as a researcher on the national grant PN-III-P1-1.1-TE-2016- 0266.
Further, I will like to thank the mathematicians from **Tiberiu Popoviciu Institute of Numerical Analysis**, especially CS I **Nicolae Suciu**, who have guided my attention to the connection between the field of Fixed Point Theory and the applications from Numerical Analysis and Scientific Computing.
Last but not least, I will like to express my gratitude to all the members of the **Applied Mathematics Research Group** from **Babeş-Bolyai University**, especially to Associate Prof. **Teodor Groşan**, Prof. **Mirela Kohr** and Prof. **Gabriela Kohr** from **Mechanics and Astronomy Research Group** who have encouraged me through the entire time I was in the PhD program and who outlined the idea behind the purpose of Applied Mathematics.
Above all, I don't have enough words to thank my entire family that supported me unconditionally through the entire time I was in the PhD program.

Chapter 1

Preliminaries

1.1 Qualitative notions for single-valued and multi-valued mappings

In this section, we recall some general notions in the framework of single-valued and multi-valued analysis theory. Also, for the following preliminary notions and lemmas (such as : multi-valued weakly Picard operators, data dependence of the fixed point set, Haussdorf metric properties) we refer the reader to [112], [141] and [142]. Also, through the present thesis, we will use the concepts and notions from [116]. Let (X, d) be a metric space. Firstly, we remind some important functionals used in multi-valued fixed point theory, which depend on the metric d of the space X. We simplify the notations avoiding the subscript d, for example we use H instead of H_d if it is no confusion.
The generalized gap functional :

$$D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad D(A, B) := inf\{d(a, b)/a \in A, b \in B\}.$$

The generalized diameter functional :

$$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad \delta(A, B) := sup\{d(a, b)/a \in A, b \in B\}.$$

The generalized excess functional :

$$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad \rho(A, B) := sup\{D(a, B)/a \in A\}.$$

The generalized Pompeiu-Hausdorff functional :

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad H(A, B) := max\{\rho(A, B), \rho(B, A)\}.$$

We denote the following class of sets:

$$\mathcal{P}(X) := \{Y/Y \subset X\}, \quad P(X) := \{Y \subset \mathcal{P}(X)/Y \neq \emptyset\}.$$

Let $T : X \to P(X)$ be a multi-valued operator. Throughout this section, we will denote

$$F_T := \{x \in X/x \in Tx\} \text{ and } (SF)_T := \{x \in X/\{x\} = Tx\}$$

the fixed point set of T and the strict fixed point set of T, respectively. Additionally,

$$Graph(T) := \{(x, y) \in X \times X/y \in Tx\}$$

will denote the graph of the multi-valued operator T.
We will introduce some useful notations such as :

$$P_b(X) := \{Y \in P(X)/Y \text{ bounded}\}, \quad P_{cl}(X) := \{Y \in P(X)/Y \text{ closed}\},$$
$$P_{cp}(X) := \{Y \in P(X)/Y \text{ compact}\}, \quad P_{b,cl}(X) := P_b(X) \cap P_{cl}(X).$$

Also, if $T : X \to P(X)$, then by

$$T^0 := 1_X, \ T^1 := T, \ ..., \ T^{n+1} := T \circ T^n,$$

for $n \in \mathbb{N}$ we denote the iterates of the operator T, where

$$T(A) := \bigcup_{a \in A} Ta,$$

for $A \subset X$. Last but not least, $\overline{B}(x_0, r)$ means the closure in (X, d) of the ball $B(x_0, r)$, where

$$B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$$

is the open ball with radius $r > 0$ and the center $x_0 \in X$. By

$$\widetilde{B}(x_0; r) := \{x \in X | d(x_0, x) \leq r\}$$

we denote the closed ball centered in x_0 with radius r.
From [143], we recall that the set

$$V^0(Y; \varepsilon) := \{x \in X \ / \ D(x, Y) < \varepsilon\}$$

is called the (open) ε-neighborhood of $Y \in P(X)$.
Now, we recall some useful results concerning the Pompeiu-Hausdorff generalized functional H(see [116]).

Lemma 1.1.1. *1) Let (X, d) be a metric space, $A, B \in P(X)$ and $q > 1$. Then, for any $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq qH(A, B).$$

2) Let (X, d) be a metric space, $A, B \in P(X)$ and $\varepsilon > 0$. Then, for any $a \in A$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Lemma 1.1.2. *Let (X, d) be a metric space and $A, B \in P(X)$.*
Suppose that there exists $\eta > 0$ such that :

$$\begin{aligned} &(i) \ \ \text{for each } a \in A, \ \text{there exists } b \in B \text{ such that } d(a, b) \leq \eta, \\ &(ii) \ \ \text{for each } b \in B, \ \text{there exists } a \in A \text{ such that } d(a, b) \leq \eta. \end{aligned}$$

Then, it follows that

$$H(A, B) \leq \eta.$$

Lemma 1.1.3. *Let $A \in P(X)$ and $x \in X$. Then*

$$x \in \overline{A} \text{ if and only if } D(x, A) = 0.$$

Definition 1.1.4. *Let (X, d) be a metric space. An operator $T : X \to P(X)$ is called a multi-valued weakly Picard operator, briefly a MWP, if for all $x \in X$ and each $y \in Tx$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of X, such that :*

$$\begin{aligned} &(i) \ x_0 = x \text{ and } x_1 = y, \\ &(ii) \ x_{n+1} \in Tx_n, \ \forall n \in \mathbb{N}, \\ &(iii) \text{ the sequence } (x_n) \text{ is convergent to an element } x^*(x, y) \in F_T. \end{aligned}$$

Remark 1.1.5. We consider the following observations that will be useful in this sequel :

(1) The sequence $(x_n)_{n \in \mathbb{N}}$ defined above by (i) and (ii) is called the successive approximation sequence for T starting from $(x, y) \in Graph(T)$.
(2) If $T : X \to P(X)$ is a MWP-operator, then we can define the multi-valued operator $T^\infty : Graph(T) \to P(F_T)$, by $T^\infty(x, y) = \{z \in F_T$ / there exists a sequence of successive approximations of T starting from (x, y) which converges to $z\}$, for each $(x, y) \in Graph(T)$.

Definition 1.1.6. *Let (X, d) be a metric space and $T : X \to P(X)$ a MWP. Then T is called ψ-MWP if and only if it satisfies :*

$$(i)\ \psi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ increasing, continuous in 0, with } \psi(0) = 0,$$
$$(ii) \text{ there exists } t^\infty \text{ a selection for } T^\infty,$$

such that $d\left(x, t^\infty(x, y)\right) \leq \psi\left(d(x, y)\right)$, for each $(x, y) \in Graph(T)$.

A useful particular case is when ψ has a linear representation, i.e. there exists $c > 0$ such that $\psi(t) = ct$, for all $t \in \mathbb{R}_+$. In this case, T is called a c-MWP.

Following [116] we recall the notions of multi-valued contractions and multi-valued Lipschitz operators.

Definition 1.1.7. *Let (X, d) and (Y, d') be two metric spaces and $T : X \to P(Y)$. Then T is called :*

(a) *$\alpha -$ Lipschitz, if $\alpha \geq 0$ and $H_{d'}(Tx_1, Tx_2) \leq \alpha d(x_1, x_2)$, for each $x_1, x_2 \in X$;*

(b) *$\alpha -$ contraction, if T is $\alpha -$ Lipschitz with $\alpha < 1$.*

Remark 1.1.8. Let (X, d) be a complete metric space. If $T : X \to P_{cl}(X)$ is an α-contraction, then T is $1/(1-\alpha)$-MWP.

For some other examples and results in the MWP operator theory, see [112].

We will use the concepts from [32], [53] and [115]. Moreover, for the sake of completeness, we present some of them here. On the other hand, we consider (X, d) a metric space, $T : X \to P(X)$ a multi-valued operator and we refer to the fixed point inclusion $x \in Tx$. Also, we recall the basic concepts for the qualitative properties of the fixed point inclusion and of the fixed point iteration. The first two definitions are related to the well-posedness of the fixed point problem. For the concept of well-posedness, we let the reader follow [80] and [120].

Definition 1.1.9 (T-stability)**.** *Let (X, d) be a metric space and $T : X \to P_{cl}(X)$ a multi-valued operator. For $x_0 \in X$, let*

$$x_{n+1} \in f(T, x_n), \text{ for each } n \in \mathbb{N} \tag{1.1.0.1}$$

denote an iteration algorithm. Let the sequence (x_n) be convergent to a fixed point p of T. Also let (y_n) be a sequence in X and set

$$\varepsilon_n = H\left(y_{n+1}, f\left(T, y_n\right)\right).$$

If $\lim\limits_{n\to\infty} \varepsilon_n = 0$ implies that $\lim\limits_{n\to\infty} y_n = p$, then the iteration defined in (1.1.0.1) is said to be T-stable or stable with respect to the operator T.

Remark 1.1.10. In the above definition, it is easy to see that the iteration procedure $f\left(T, x_n\right)$ can be represented as an admissible perturbation $T_{G_n}(x_n)$.

Definition 1.1.11 (Ulam-Hyers stability of the inclusion $x \in Tx$)**.** *Let (X, d) be a metric space and $T : X \to P(X)$. By definition, the fixed point inclusion*

$$x \in Tx \tag{1.1.0.2}$$

is called generalized Ulam-Hyers stable if and only if there exists an increasing, continuous in 0 function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi(0) = 0$, such that : for every $\varepsilon > 0$ and for each $y^* \in X$ for which $D(y^*, Ty^*) \leq \varepsilon$, there exists a solution x^* a solution of the fixed point inclusion 1.1.0.2, such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

Definition 1.1.12. *Let (X, d) be a metric space and $T : Y \to P(X)$. By definition, the strict fixed point inclusion*

$$\{x\} = Tx \tag{1.1.0.3}$$

is called generalized Ulam-Hyers stable if and only if there exists an increasing, continuous in 0 function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi(0) = 0$, such that :
for every $\varepsilon > 0$ and for each $y^* \in X$ for which $H(y^*, Ty^*) \leq \varepsilon$, there exists a solution x^* a solution of the strict fixed point inclusion 1.1.0.3, such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

Finally, following [56], [61] and [125], we recall the last important concepts.

Definition 1.1.13. *Let $X \neq \emptyset$ and $T : X \to P(X)$ be a multi-valued operator. By definition, T has the approximate endpoint property if*

$$\inf_{x \in X} \sup_{y \in Tx} d(x, y) = 0.$$

For the study of the generalized Ulam-Hyers stability we refer to [110].

Remark 1.1.14. In the above assumptions, if T is a multivalued contraction, then the fixed point inclusion $x \in Tx$ is Ulam-Hyers stable, with $\psi(\varepsilon) = \dfrac{\varepsilon}{1-\alpha}$.

Definition 1.1.15 (Well posedness of the fixed point problem wrt to D)**.** *Let (X, d) be a metric space and $T : X \to P(X)$ a multi-valued operator. We consider the fixed point inclusion $x \in Tx$.*
The fixed point problem is well-posed with respect to D if the following implication hold :
If $(x_n)_{n\in\mathbb{N}} \in X$ is a sequence such that

$$\lim_{n\to\infty} D(x_n, Tx_n) = 0,$$

then $\lim_{n\to\infty} x_n = x^*$.

Definition 1.1.16. *Let (X, d) be a metric space, $Y \in P(X)$ and let a multi-valued operator $T : Y \to P_{cl}(X)$. Then the fixed point problem is well posed in the generalized sense (respectively well-posed) for T with respect to the gap functional D if and only if :*

(*i*) $F_T \neq \emptyset$*(respectively* $F_T = \{x^*\}$*),*
(*ii*) *if (x_n) is a sequence in Y such that $D(x_n, Tx_n) \to 0$, then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that $x_{n_k} \to x^* \in F_T$(respectively $x_n \to x^* \in F_T$).*

Definition 1.1.17 (Limit shadowing property of the multi-valued operator)**.** *Let (X, d) be a metric space and $T : X \to P(X)$ a multi-valued operator. We consider the fixed point inclusion $x \in Tx$. The fixed point problem has the limit shadowing property if the following implication holds :*
If $\{y_n\} \in X$ is a sequence such that

$$\lim_{n\to\infty} D(y_{n+1}, Ty_n) = 0,$$

then there exists $\{x_n\}$ a sequence of successive approximations such that

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Moreover, we recall two important lemmas which are very useful for T-stability and limit shadowing property. For details, see [58] and [119] :

Lemma 1.1.18 (Harder & Hicks)**.** *Let $c \in \mathbb{R}$, with $0 < |c| < 1$.*
Let $(b_n)_{n\in\mathbb{N}}$ be a sequence, such that $\lim_{k\to\infty} b_k = 0$. Then, we have

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} c^{n-k} b_k \right) = 0$$

More generally we have :

Lemma 1.1.19 (Cauchy's lemma)**.** *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be two sequences such that* $\lim\limits_{n\to\infty} b_n = 0$ *and* $\sum\limits_{k=0}^{+\infty} a_k < +\infty$. *Then, we have*

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n} a_{n-k}b_k\right) = 0$$

Further, we shall present some new concepts and definitions regarding Ulam-Hyers stability, limit shadowing and well-posedness of the multi-valued operator using a general iterative function of the form $f(T, x_n)$. For this, let X be a nonempty set. For a multi-valued operator $T : X \to P(X)$, in the following sequel we consider an associated operator $f_T : X \to P(X)$, with respect to the operator T. For conciseness, we denote $f_T(x)$ as $f(T, x)$ for every $x \in X$. Moreover, $f(T, \cdot)$ will be used as a notation for $f_T(\cdot)$.

Definition 1.1.20 (Ulam Hyers stability of the inclusion $x \in f(T, x)$)**.** *Let* (X, d) *be a metric space,* $T : X \to P(X)$ *and* $f(T, \cdot) \in P(X)$.
Let $\varepsilon > 0$ *and* $x \in X$*, such that*

$$D(x, f(T, x)) \le \varepsilon.$$

If there exists $x^* \in F_T$*, such that*

$$d(x, x^*) \le \psi(\varepsilon),$$

and a ψ *function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *increasing, continuous in 0 and with* $\psi(0) = 0$*, then the fixed point inclusion* $x \in T(x)$ *is called generalized Ulam-Hyers stable with respect to the inclusion* $x \in f(T, x)$.

Definition 1.1.21 (Well posedness of the fixed point problem wrt to D and $f(T, x_n)$ iteration)**.** *Let* (X, d) *be a metric space,* $T : X \to P(X)$ *and* $f(T, \cdot) \in P(X)$.
The fixed point problem $x \in f(T, x)$ *is called well-posed if the following implication holds :*
If $\{x_n\} \in X$ *is a sequence such that*

$$\lim_{n\to\infty} D(x_n, f(T, x_n)) = 0,$$

then $\lim\limits_{n\to\infty} x_n = x^*$.

Definition 1.1.22 (Limit shadowing property wrt to $f(T, x_n)$ iteration)**.** *Let* (X, d) *be a metric space,* $T : X \to P(X)$. *We say that the fixed point problem* $x \in f(T, x)$ *has the limit shadowing property if the following implication holds :*
If $(y_n)_{n\in\mathbb{N}} \in X$ *is a sequence such that*

$$\lim_{n\to\infty} D(y_{n+1}, f(T, y_n)) = 0,$$

then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ *satisfying* $x_{n+1} \in f(T, x_n)$ *such that*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Definition 1.1.23. *Let* (X, d) *be a metric space,* $Y \in P(X)$ *and let* $T : Y \to P_{cl}(X)$. *Then the fixed point problem is well posed in the generalized sense (respectively well-posed) for* T *with respect to the Pompeiu-Hausdorff functional* H *if and only if :*

- *(i)* $(SF)_T \neq \emptyset$*(respectively* $(SF)_T = \{x^*\}$*),*
- *(ii) if* (x_n) *is a sequence in* Y *such that* $H(x_n, Tx_n) \to 0$*, then there exists a subsequence* $(x_{n_k})_{k\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$*, such that* $x_{n_k} \to x^* \in (SF)_T$*(respectively* $x_n \to x^* \in (SF)_T$*).*

Now, the second important concept related to the fixed point problem is the Ostrowski property, which can be found in [80] and [79].

Definition 1.1.24 (Ostrowski property)**.** *Let* (X, d) *be a metric space and* $T : X \to P(X)$ *be a multi-valued operator. By definition, the multi-valued operator* T *has the Ostrowski property, if*

$$F_T = \{x^*\}$$

and for any sequence $(y_n)_{n\in\mathbb{N}} \subset X$*, such that*

$$D(y_{n+1}, Ty_n) \to 0,$$

we have that

$$(y_n)_{n\in\mathbb{N}} \to x^*, \ \textit{as } n \to \infty.$$

1.2 Presić operators

The purpose behind this section is to investigate the existence and uniqueness of coincidence and common fixed points for some new Presić operators in the framework of metric spaces endowed with a partial order. Since our aim is to introduce the idea of simulation function for Presić-type mappings, we shall recall first the concept of simulation functions. Also, we shall remind some generalizations of this notion that led to some interesting fixed point results in this relatively new subfield of fixed point theory.
In [76], Khojasteh et.al., introduced the idea of simulation function.

Definition 1.2.1. *Let* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *be a mapping, then* ζ *is called a simulation function if it satisfies the following conditions :*
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$, *for all* $t, s > 0$;
(ζ_3) *if* $(t_n), (s_n)$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$*, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Furthermore, the set of all simulation functions is denoted by $\mathcal{Z}$ or $\mathcal{Z}_d$ if the context is a metric-type space endowed with a distance d. Also, the authors in [76] studied the existence and uniqueness of fixed points for self-mappings entitled with the name $\mathcal{Z}-$contractions, which are defined below.

Definition 1.2.2. *Let* (X, d) *be a metric space,* $T : X \to X$ *a mapping and* $\zeta \in \mathcal{Z}$*. Then* T *is called a* $\mathcal{Z}-$*contraction with respect to* ζ *if the following condition is satisfied*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0, \ \textit{for all } x, y \in X.$$

Also, an interesting article was that of Roldán-López-de-Hierro et. al., [135], in which the authors studied the existence and uniqueness of coincidence points for mappings modified by simulation functions. In [Definition 3.2] of the same article, the authors relaxed the condition (ζ_3) and redefined the condition of a simulation function.

Definition 1.2.3. *Let* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *be a mapping, then* ζ *is called a simulation function if it satisfies the following conditions :*
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$, *for all* $t, s > 0$
(ζ_3) *if* $(t_n), (s_n)$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ *and* $t_n < s_n$ *for all* $n \in \mathbb{N}$*, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Furthermore, using the framework of a metric space (X, d), they have considered some fixed point results for $(\mathcal{Z}_d, g)$-contractions. Also, we shall recall here this type of contraction.

Definition 1.2.4. *Let* (X, d) *be a metric space and let* $T, g : X \to X$ *be self-mappings. We say that* T *is a* $(\mathcal{Z}_d, g)$*-contraction if there exists* $\zeta \in \mathcal{Z}$ *such that*

$$\zeta(d(Tx, Ty), d(gx, gy)) \geq 0, \ \textit{for all } x, y \in X, \ \textit{such that } gx \neq gy.$$

In [26], Ansari et. al., generalized the notion of a simulation function developed by Khojasteh, Shukla and Radenovic in [76]. For the sake of completeness, we recall it here.

Definition 1.2.5. *A mapping $G : [0,\infty)^2 \to \mathbb{R}$ is called a $\mathcal{C}-$class function if it is continuous and satisfied the following conditions :*
(i_1) *$G(s,t) \leq s$;*
(i_2) *$G(s,t) = s$ implies that $s = 0$ or $t = 0$, for all $t, s \geq 0$.*

Definition 1.2.6. *A mapping $G : [0,\infty)^2 \to \mathbb{R}$ has the property C_G, if there exists an $C_G \geq 0$, such that*
(i_3) *$G(s,t) > C_G$ implies that $s > t$;*
(i_4) *$G(t,t) \leq C_G$, for all $t \geq 0$.*

Definition 1.2.7. *A C_G-simulation function is a mapping $\zeta : [0,\infty)^2 \to \mathbb{R}$ satisfying the following :*
(i_5) *$\zeta(t,s) < G(s,t)$, for all $t, s > 0$, where G is a $\mathcal{C}-$class function;*
(i_6) *if $(t_n), (s_n)$ are sequences in $(0,\infty)$, such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < C_G.$$

Furthermore, Radenović and Chandok in [127] considered a pair of operators (f,g) defined on a metric space, such that they satisfy

$$\zeta(d(fx,fy), d(gx,gy)) \geq C_G, \text{ for all } x, y \in X \text{ with } gx \neq gy.$$

The authors have called the mapping f a $(\mathcal{Z}_G, g)$-contraction and studied the existence and uniqueness of coincidence and common fixed points for the pair (f,g). Also, for the case when $C_G = 0$ and $G(s,t) = s - t$, Radenović et. al. in [128] presented an easy approach to the study of common fixed points for mappings altered by simulation functions in complete metric spaces.
Moreover in [84] and [108], the authors of these articles generalized the main results of [127], namely one can find some interesting fixed point results regarding Ćirić and Ćirić-Suzuki type self-mappings through a C_G-simulation function.
On the other hand, for the study of fixed points for self-mapping endowed with the property of α-admissiblity in the framework of simulation functions, we refer to [21] and [71].
Finally, for the study of fixed points in other types of spaces endowed with some generalized distances, such as b-metric spaces, quasi-metric spaces and $0-\sigma$ metric spaces, we let the reader follow [21], [23], [50] and [162].

In [126], S.B. Presić extended the well known Banach contraction principle to operators that are mappings that have more than one variable. For the sake of completeness, we recall the main result of [126].

Theorem 1.2.8. *Let (X,d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a mapping satisfying the following contractive type condition :*

$$d(f(x_1,...,x_k), f(x_2,...,x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}),$$

for every $x_1,...,x_{k+1} \in X$, where $q_1,...,q_k$ are nonegative constants such that $q_1 + ... + q_k < 1$. Then there exists a unique point $x \in X$ such that $f(x,...,x) = x$. Moreover, if $x_1,...,x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = f(x_n,...,x_{n+k-1})$, then the sequence (x_n) is convergent and $\lim x_n = f(\lim x_n,...,\lim x_n)$.

From now on, for a given mapping $f : X^k \to X$, with k a fixed positive integer, we consider the 'one step' sequence (x_n), defined by

$$x_{n+1} = f(x_n,...,x_n), \tag{1.2.0.1}$$

and the 'k-step' sequence (x_n), defined as

$$x_{n+k} = f(x_n,...,x_{n+k-1}),\ n = 1,2,\ldots \tag{1.2.0.2}$$

Now, we remind that in [103], M. Păcurar obtained a convergence result regarding Kannan-Presić operators.

Theorem 1.2.9. *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a given mapping. Suppose that there exists $a \in \mathbb{R}$ satisfying $0 < ak(k+1) < 1$ such that*

$$d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \le a \sum_{i=1}^{k+1} d(x_i, f(x_i, ..., x_i)),$$

for all $(x_1, ..., x_{k+1}) \in X^{k+1}$. Then,
(i) f has a unique fixed point $x^* \in X$;
(ii) for any arbitrary points $x_1, ..., x_k \in X$, the sequence (x_n) defined by (1.2.0.2) converges to x^*.

On the other hand, starting from the article [37] of Boyd and Wong, Shukla and Radenović in [148] proved some convergence results for Presić-Boyd-Wong type operators for the case of complete metric spaces endowed with a partial order, denoted by $(X, \preceq, d)$. In their article, some coincidence and common fixed points were studied for a pair (f, g), where $f : X^k \to X$ and $g : X \to X$ satisfy some conditions regarding the partial order "$\preceq$" and also the following metrical type inequality

$$d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \le \psi(d(gx_1, gx_2), ..., d(gx_k, gx_{k+1})), \tag{1.2.0.3}$$

where the function $\psi : \mathbb{R}_+^k \to \mathbb{R}_+$ satisfies the following conditions :

$$\begin{aligned} &(\psi_1) \text{ for } t_n \in \mathbb{R}_+ \text{ and } t_n \downarrow t \ge 0 \text{ implies that } \limsup_{n\to\infty} \psi(t_n, ..., t_n) \le \psi(t, ..., t); \\ &(\psi_2)\ \psi(t, ..., t) < t, \text{ for each } t > 0; \\ &(\psi_3)\ \psi(t, 0, ..., 0) + \ldots + \psi(0, ..., 0, t) \le \psi(t, ..., t), \text{ for each } t \ge 0. \end{aligned} \tag{1.2.0.4}$$

Moreover, regarding [Corollary 6] from the same article [148], we observe that when $g = I_X$ is the identity mapping, then the sequence defined by (1.2.0.1) converges to the fixed point of the operator f.
Now, turning our attention to the work of I.A. Rus [140] and M. Păcurar [104], some fixed points and common fixed points respectively, were studied through the multi-step iterative method (1.2.0.2) (or a generalized form of that if we are referring to the study of coincidence and common fixed points for Presić-type pair of mappings). There, the operators satisfying

$$d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \le \varphi(d(x_1, x_2), ..., d(x_k, x_{k+1})), \tag{1.2.0.5}$$

or

$$d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \le \varphi(d(gx_1, gx_2), ..., d(gx_k, gx_{k+1})), \tag{1.2.0.6}$$

had a unique fixed point (coincidence point for the second case). The mapping $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ satisfied the following conditions :

$$\begin{aligned} &(\varphi_1)\ \varphi(r) \le \varphi(s), \text{ for } r, s \in \mathbb{R}_+^k, \text{ with } r \le s; \\ &(\varphi_2)\ \psi(r, ..., r) < r, \text{ for each } r \in \mathbb{R}_+, r > 0; \\ &(\varphi_3)\ \psi(r, 0, ..., 0) + \ldots + \psi(0, ..., 0, r) \le \psi(r, ..., r), \text{ for each } r \in \mathbb{R}_+; \\ &(\varphi_4)\ \varphi \text{ is continuous ;} \\ &(\varphi_5)\ \sum_{i=0}^{\infty} \varphi^i(r) < \infty. \end{aligned} \tag{1.2.0.7}$$

Now, we observe that the conditions $(\varphi_1) - (\varphi_5)$ are stronger than the conditions $(\psi_1) - (\psi_3)$ and the former conditions were necessary for the convergence of the sequence (1.2.0.2). At the same time, conditions $(\psi_1) - (\psi_3)$ were more relaxed, since in [148] only a converge result regarding the sequence (1.2.0.1) was given.
Finally, for other important results on Presić-type single-valued operators, we let the reader follow [102] and for an exhaustive study concerning fixed point results for mappings defined on the cartesian product of a complete metric space, we kindly refer to [145].

On the other hand, it is worth mentioning that the k-step iterative sequence given by equation 1.2.0.2 can be regarded as a nonlinear difference equation. Furthermore, if the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent, then

the limit of the sequence is a fixed point for the mapping f.
Now, we mention that very many authors have generalized the contractive condition given by Presić. One of the earliest generalization was made by Presić and Ćirić in [44]. We recall the theorem given by the aforementioned authors.

Theorem 1.2.10. *Let (X,d) be a complete metric space. Consider a mapping $f : X^k \to X$ satisfying the following contractive-type condition*

$$d(f(x_0,...,x_{k-1}), f(x_1,...,x_k)) \leq \lambda \max\{d(x_i,x_{i+1}) : 0 \leq i \leq k-1\}, \tag{1.2.0.8}$$

where $\lambda \in (0,1)$ and $x_0,...,x_k$ are arbitrary given elements of X.
Then, there exists a point $x^* \in X$ for which $f(x^*,...,x^*) = x^*$.
Furthermore, if $x_0,...,x_{k-1}$ are arbitrary elements from X, then the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_{n+k} = f(x_n,...,x_{n+k-1}) \tag{1.2.0.9}$$

is convergent and satisfies $\lim\limits_{n\to\infty} x_n = f\left(\lim\limits_{n\to\infty} x_n,...,\lim\limits_{n\to\infty} x_n\right)$.
Additionally, if on the diagonal $\Delta \subset X^k$, we have that

$$d(f(u,...,u), f(v,...,v)) < d(u,v), \tag{1.2.0.10}$$

for each $u,v \in X$, with $u \neq v$, then x^* is the unique point in X.

Also, in 1981, I.A. Rus [140] generalized [Theorem 1.2.8] and proved the existence and uniqueness of a fixed point in X, i.e. a point x^* that satisfy $x^* = f(x^*,...,x^*)$, for a mapping $f : X^k \to X$, satisfying the following condition

$$d(f(x_0,...,x_{k-1}), f(x_1,..,x_k)) \leq \varphi\left(d(x_0,x_1),...,d(x_{k-1},x_k)\right), \tag{1.2.0.11}$$

where the function φ satisfies the following

$$\begin{cases} \varphi(r) \leq \varphi(s), \text{ where } r \leq s, \text{ with } r,s \in \mathbb{R}^k_+, \\ \varphi \text{ is continuous}, \\ \varphi(t,...,t) < t, \text{ for each } t \in \mathbb{R}_+, \\ \sum\limits_{i=0}^{\infty} \varphi^i(r) < +\infty, \text{ for each } r \in \mathbb{R}^k_+, \\ \varphi(t,0,...,0) + \varphi(0,t,0,...,0) + ... + \varphi(0,...,0,t) \leq \varphi(t,...,t), \forall t \in \mathbb{R}_+. \end{cases}$$

Furthermore, we mention that in [104], M. Păcurar gave a generalization for the theorem developed by I.A. Rus and studied the existence of coincidence and common fixed points for a pair of mappings (f,g), where $f : X^k \to X$ and $g : X \to X$, such that

$$d(f(x_0,...,x_{k-1}), f(x_1,...,x_k)) \leq \varphi\left(d(g(x_0),g(x_1)),...,d(g(x_{k-1}),g(x_k))\right), \tag{1.2.0.12}$$

where $\varphi : \mathbb{R}^k_+ \to \mathbb{R}^k_+$ is endowed with the properties presented above.
Some generalizations of Presić-type mappings were made by Shukla, Radenovic et.al. They have studied existence and uniqueness of fixed points of various mappings in complete metric spaces and in complete ordered metric spaces. For these, we let the reader follow [149], [150] and [151]. The most general one, i.e. Presić-Hardy-Rogers, has the following form

$$d(f(x_0,...,x_{k-1}), f(x_1,...,x_k)) \leq \sum_{i=0}^{k-1} d(x_i,x_{i+1}) + \sum_{i=0}^{k}\sum_{j=0}^{k} \beta_{i,j} d(x_i, f(x_j,...,x_j)). \tag{1.2.0.13}$$

Also, existence and uniqueness of fixed points for various Presić type mappings were studied. We let the reader follow [3], [75] and [100]. Furthermore, we remind that the mentioned authors gave examples in metric spaces and b-metric spaces. Also, in [100] applications to matrix equations were given.
Finally, since our aim is to study the existence and uniqueness of fixed points for some new types of mappings $f : X^k \to X$, for more research papers regarding Presić-type mappings, we let the reader follow [33], [102], [107] and [126].

Since we shall extend the concept of convex contractions to Presić operators, we remind the definition of convex contraction of second order, given by Istrăţescu in [63].

Definition 1.2.11. *Let* (X, d) *be a metric space. Consider a continuous mapping* $f : X \to X$*.* f *is said to be a convex contraction of order 2 if there exists* $a, b \in (0, 1)$*, such that for each* $x, y \in X$

$$d(f^2x, f^2y) \leq ad(fx, fy) + bd(x, y), \tag{1.2.0.14}$$

where $a + b < 1$*.*

Furthermore, in the same paper, Istrăţescu introduced convex contractions of order n, like follows.

Definition 1.2.12. *Let* (X, d) *be a metric space. Consider a continuous mapping* $f : X \to X$*.* f *is said to be a convex contraction of order* n *if there exists* $a_0, ..., a_{n-1} \in (0, 1)$*, such that for each* $x, y \in X$

$$\begin{aligned} d(f^n x, f^n y) \leq a_0 d(x, y) + a_1 d(fx, fy) + ... \\ + a_{n-1} d(f^{n-1}x, f^{n-1}y), \end{aligned} \tag{1.2.0.15}$$

where $a_0 + ... + a_{n-1} < 1$*.*

Even though in the definitions given by Istrăţescu, in the case of convex contractions of order 2, the coefficients a, b are in (0,1) and in the case of convex contractions of order $n \geq 2$, the coefficients $a_0, ..., a_{n-1}$ lie in the interval (0,1), we shall employ the fact that the coefficients can be in $[0, 1)$ as in [144]. This change will be very useful for our examples.
Also, Istrăţescu studied other types of continuous operators in [62] and [64] and Satry, Rao et.al. in [144] studied the existence and uniqueness principles for convex contractions of order $m \geq 2$. Moreover, we remind that V. Mureşan and A.S. Mureşan in [95] gave theorems regarding data dependence and qualitative properties for convex contractions of order 2.
Finally, other authors have studied qualitative properties and developed existence and uniqueness theorems for convex contractions of order 2 and for other type of operators, such as convex contractions with diminishing diameters. We let the reader follow [20], [22] and [88].

1.3 b-Rectangular metric spaces

In this section we shall present some useful lemmas and definitions regarding rectangular and b-rectangular metric spaces. Also, we shall present some recent results in the field of fixed point theory regarding expansive and some generalized contraction mappings.
In [38], A. Branciari introduced a new metric-type space, where triangle inequality is replaced by an inequality which involves four different elements. This is called a rectangular metric space or a generalized metric space (g.m.s.)

Definition 1.3.1 (Rectangular metric spaces)**.** *Let* $X \neq \emptyset$*,* $d : X \times X \to [0, \infty)$*, such that for each* $x, y \in X$ *and* $u, v \in X$ *(each distinct from* x *and* y*), we have that*

$$\begin{aligned} &(1)\ d(x, y) = 0 \Longleftrightarrow x = y, \\ &(2)\ d(x, y) = d(y, x), \\ &(3)\ d(x, y) \leq d(x, u) + d(u, v) + d(u, y). \end{aligned}$$

Furthermore, from [67] we mention that convergent sequences and Cauchy sequences can be introduced in a similar manner as in metric spaces.
Also, from the same paper, we know that if (X, d) is a rectangular metric space and if (x_n) is a b-rectangular Cauchy sequence with the property that $x_n \neq x_m$, for each $n \neq m$, then (x_n) converge to at most one point, i.e. the property that (X, d) is Hausdorff becomes superfluous.
Moreover, from [47], [52], [136], we recall the definition of b-rectangular metric spaces (or b-generalized metric spaces), or briefly b-g.m.s.

Definition 1.3.2 (b-g.m.s.)**.** *Let* $X \neq \emptyset$*,* $s \geq 1$ *a given real number and* $d : X \times X \to [0, \infty)$*, such that for each* $x, y \in X$ *and* $u, v \in X$ *(each distinct from* x *and* y*), we have that*

$$\begin{aligned} &(1)\ d(x, y) = 0 \Longleftrightarrow x = y, \\ &(2)\ d(x, y) = d(y, x), \\ &(3)\ d(x, y) \leq s\left[d(x, u) + d(u, v) + d(u, y)\right]. \end{aligned}$$

As in metric spaces, we recall the basic notions regarding sequences in b-g.m.s :

Definition 1.3.3. *Let (X,d) be a b-g.m.s, $x \in X$ and $(x_n) \subset X$ a given sequence. Then*
(a) (x_n) is convergent in (X,d) to an element $x \in X$, if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $d(x_n, x) < \varepsilon$, for each $n > n_0$. We denote this by $\lim_{n\to\infty} x_n = x$.
(b) (x_n) is Cauchy in (X,d) (b-rectangular Cauchy, briefly b-g.m.s.), if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $d(x_n, x_{n+p}) < \varepsilon$, for each $n > n_0$ and for each $p > 0$. We denote this by $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$, for each $p > 0$.
(c) (X,d) is said to be complete b-g.m.s, if every Cauchy sequence in X converges to some $x \in X$.

We recall the following important remark from [47] :

Remark 1.3.4. (1) Every metric space and every rectangular metric space (g.m.s) is b-g.m.s.
(2) The limit of a sequence in a b-rectangular metric space is not unique.
(3) Every convergent sequence in a b-g.m.s is not necessarily a b-g.m.s Cauchy.

For this, we recall a crucial lemma from [47], i.e. [Lemma 1.3.5], that specify when a b-rectangular Cauchy sequence can't have two limits in a b-g.m.s.

Lemma 1.3.5. *Let (X,d) be a b-rectangular metric space, with the coefficient $s \geq 1$. Let (x_n) be a b-rectangular Cauchy sequence in X, such that $x_n \neq x_m$, for each $n \neq m$. Then (x_n) can converge to at most one point.*

Also, we recall from [69] and [47] the following lemma.

Lemma 1.3.6. *Let (X,d) be a b-rectangular metric space, with the coefficient $s \geq 1$. Also, let (x_n) be a sequence such that $x_n \neq x_m$, for each $n \neq m$, with $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.*
If (x_n) is not a b-rectangular Cauchy sequence, then there exists $\varepsilon > 0$ and there exists $(m(k))_{k\in\mathbb{N}}$ and $(n(k))_{k\in\mathbb{N}}$ two sequences of positive integers, that satisfies

$$\begin{aligned}
&d(x_{m(k)}, x_{n(k)}) \geq \varepsilon,\\
&\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)-2}) \leq \varepsilon \text{ and}\\
&\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)-1}).
\end{aligned}$$

In [136], another crucial lemma regarding sequences in b-rectangular metric spaces was presented. For the convenience, we remind it

Lemma 1.3.7. *Let (X,d) be a b-g.m.s., with coefficient $s \geq 1$.*
(a) Consider two sequences (x_n) and (y_n), such that x_n converges to $x \in X$ and y_n converges to $y \in X$, with $x \neq y$. Also, suppose that for each $n \in \mathbb{N}$, $x_n \neq x$ and $y_n \neq y$. Then

$$\frac{1}{s} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s d(x,y).$$

(b) Consider an element $y \in X$ and a b-rectangular Cauchy sequence (x_n), such that $x_n \neq x_m$, for each $n \neq m$. Moreover, suppose that the sequence (x_n) converges to an element $x \neq y$. Then

$$\frac{1}{s} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y) \leq \limsup_{n\to\infty} d(x_n, y) \leq s d(x,y).$$

1.4 Cone metric spaces over Banach algebras

In this section we present some notions that are linked to the convergence of sequences of contractions defined on cone metric spaces over Banach algebras. First of all we need to recall that F.F. Bonsall [35] and S.B. Nadler Jr. [97] studied some stability results regarding sequences of contractions defined on a whole metric space (X,d). Furthermore, an interesting extension of the previous results was made by M.

Păcurar [105], who developed some fixed point results for the convergence of the sequence of fixed points of almost contractions. M. Păcurar presented two interesting theorems, the first one regarding the pointwise convergence and the second one concerning uniform convergence of a sequence of almost contractions defined by the same coefficients. Now, our first aim of the present section is to remind some mathematical notions that are well established in the field of nonlinear analysis. For more information regarding these concepts, we kindly refer to [35] and [105]. We first present the idea of pointwise convergence.

Definition 1.4.1. *Let (X, d) be a metric space. Also, let $T : X \to X$ and $T_n : X \to X$ be some given mappings for each $n \in \mathbb{N}$. By definition, the sequence $(T_n)_{n\in\mathbb{N}}$ converges pointwise to T on X, briefly $T_n \xrightarrow{p} T$, if for each $\varepsilon > 0$ and for every $x \in X$, there exists $N = N(\varepsilon, x) > 0$, such that for each $n \geq N$, we have that $d(T_n x, T x) < \varepsilon$.*

We easily observe that in [Definition 1.4.1], one can replace the strict inequality $d(T_n x, T x) < \varepsilon$ by the non-strict inequality without changing the idea behind the concept of pointwise convergence.
Similarly, the particular notion of uniform convergence of a sequence of mappings is given as follows.

Definition 1.4.2. *Let (X, d) be a metric space. Also, let $T : X \to X$ and $T_n : X \to X$ be some given mappings for each $n \in \mathbb{N}$. By definition, the sequence $(T_n)_{n\in\mathbb{N}}$ converges uniformly to T on X, briefly $T_n \xrightarrow{u} T$, if for each $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$, such that for each $n \geq N$ and for every $x \in X$, one has the following : $d(T_n x, T x) < \varepsilon$.*

Also, for a family of mappings we can briefly recall the fundamental notions of equicontinuity and uniform equicontinuity, respectively.

Definition 1.4.3. *Let (X, d) be a metric space and $T_n : X \to X$ be some given mappings, for every $n \in \mathbb{N}$. The family $(T_n)_{n\in\mathbb{N}}$ is called equicontinuous if and only if for every $\varepsilon > 0$ and for each $x \in X$, there exists $\delta = \delta(\varepsilon, x) > 0$, such that for every $y \in X$ satisfying $d(x, y) < \delta$, one has that $d(T_n x, T_n y) < \varepsilon$.*

Now, regarding uniform equicontinuity of a family of operators, we employ the following definition.

Definition 1.4.4. *Let (X, d) be a metric space and $T_n : X \to X$ be some given mappings, for every $n \in \mathbb{N}$. The family $(T_n)_{n\in\mathbb{N}}$ is called uniformly equicontinuous if and only if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for every x and y in X, satisfying $d(x, y) < \delta$, one has that $d(T_n x, T_n y) < \varepsilon$.*

As before, one can easily replace the strict inequality with the non-strict one, such that the two definitions are equivalent to each other.
Now, it is time to remind that the starting point of the present section is the paper of L. Barbet and K. Nachi. According to [31], the authors considered some fixed point results regarding the convergence of fixed points of contraction mappings in the regular setting of a metric space (X, d). The novelty of the already mentioned paper consists on redefining pointwise and uniform convergence, respectively, but for operators defined on subsets of the whole space and not on the entire metric space (X, d). Pointwise convergence was generalized by G-convergence and uniform convergence was extended as H-convergence. For the sake of completeness, we recall these two notions here.

Definition 1.4.5. *Let (X, d) be a metric space and X_n be nonempty subsets of X, for each $n \in \mathbb{N}$. Let $T_n : X_n \to X$ for every $n \in \mathbb{N}$ and $T_\infty : X_\infty \to X$ be some given mappings. By definition T_∞ is the G-limit mapping of the sequence $(T_n)_{n\in\mathbb{N}}$, whenever $(T_n)_{n\in\mathbb{N}}$ satisfies property (G), i.e.*

$$(G) : \forall x \in X_\infty,\ \exists (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n,\ \text{s.t. } x_n \to x \text{ and } T_n x_n \to T_\infty x.$$

Regarding the generalization of uniform convergence for mappings that are not defined on the whole metric space, we remind the following concept from [31].

Definition 1.4.6. *Let (X, d) be a metric space and X_n be nonempty subsets of X, for each $n \in \mathbb{N}$. Let $T_n : X_n \to X$ for every $n \in \mathbb{N}$ and $T_\infty : X_\infty \to X$ be some given mappings. By definition T_∞ is the H-limit mapping of the sequence $(T_n)_{n\in\mathbb{N}}$, whenever $(T_n)_{n\in\mathbb{N}}$ satisfies property (H), i.e.*

$$(H) : \forall (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n, \exists (y_n)_{n\in\mathbb{N}} \subset X_\infty,\ \text{s.t. } d(x_n, y_n) \to 0 \text{ and } d(T_n x_n, T_\infty y_n) \to 0.$$

Now, considering $\mathcal{A}$ to be a Banach algebra with zero element $\theta \in \mathcal{A}$ and unit element $e \in \mathcal{A}$,we recall the notion of a cone from [83].

Definition 1.4.7. *A nonempty closed subset P of $\mathcal{A}$ is called a cone if the following conditions hold :*

$$\begin{aligned}
&(P1) \quad \theta \text{ and } e \text{ are in } P,\\
&(P2) \quad \alpha P + \beta P \subset P, \text{ for every } \alpha, \beta \geq 0,\\
&(P3) \quad P^2 \subseteq P,\\
&(P4) \quad P \cap (-P) = \{\theta\}.
\end{aligned}$$

Furthermore, we recall that P is called a solid cone if $int(P) \neq \emptyset$, where $int(P)$ represent the topological interior of the set P. Now, as in [59], one can define a partial ordering $\preceq$ with respect to the cone P, such as if x and y are in $\mathcal{A}$, then $x \preceq y$ if and only if $y - x \in P$. Also, we shall write $x \prec y$ in order to specify that $x \neq y$ and $x \preceq y$. At the same time, for $x, y \in \mathcal{A}$, we denote by $x \ll y$ the fact that $y - x \in int(P)$, based on the assumption that we will always suppose that the cone P is solid.
From [Definition 1.6] of [83] and [Definition 1.1] of [85], we introduce the well-known cone metric distances over the Banach algebra $\mathcal{A}$ and present some useful terminologies.

Definition 1.4.8. *Let X be a nonempty set and $d : X \times X \to \mathcal{A}$ be a mapping that satisfies the following conditions :*

$$\begin{aligned}
&(D1) \quad \theta \preceq d(x,y), \text{ for each } x, y \in X, \text{ and } d(x,y) = \theta \text{ if and only if } x = y,\\
&(D2) \quad d(x,y) = d(y,x), \text{ for each } x, y \in X,\\
&(D3) \quad d(x,y) \preceq d(x,z) + d(z,y), \text{ for every } x, y, z \in X.
\end{aligned}$$

Then (X,d) is called a cone metric space over the Banach algebra $\mathcal{A}$.

Furthermore, from [160], we recall the following concepts.

Definition 1.4.9. *Let (X,d) be a complete cone metric space over the Banach algebra $\mathcal{A}$. Also, let x be an element of X and $(x_n)_{n\in\mathbb{N}} \subset X$ be given. Then, we have the following :*

- (i) *$(x_n)_{n\in\mathbb{N}}$ converges to x, briefly $\lim_{n\to\infty} x_n = x$, if for every $c \gg \theta, \exists N = N(c) > 0$, such that $d(x_n, x) \ll c$, $\forall n \geq N$.*
- (ii) *$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence , if for every $c \gg \theta, \exists N = N(c) > 0$, such that $d(x_n, x_m) \ll c$, $\forall n, m \geq N$.*
- (iii) *(X,d) is complete if each Cauchy sequence is convergent.*

In Definition 1.4.9, $c \gg \theta$ represent an useful notation for $\theta \ll c$, so it lies no confusion in the rest of this section. Now, following the well-known Rudin's book of Functional Analysis [137], for the sake of completeness, we recall the idea of the spectral radius of an element of the Banach algebra $\mathcal{A}$.

Lemma 1.4.10. *Let $k \in \mathcal{A}$ be a given element. Then, by definition we consider the spectral radius of k, by*

$$\rho(k) = \lim_{n\to\infty} \|k^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|k^n\|^{\frac{1}{n}}.$$

If $\lambda \in \mathbb{C}$ and $\rho(k) < |\lambda|$, then the element $\lambda e - k$ is invertible. Also, one has that :

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}.$$

Now, from [83], we present some important properties regarding the spectral radius of an element of a Banach algebra $\mathcal{A}$ and some notions concerning the idea of a c-sequence, respectively.

Definition 1.4.11. *A sequence $(d_n)_{n\in\mathbb{N}}$ from a Banach algebra $\mathcal{A}$ endowed with a solid cone P is called a c-sequence if and only if for every $c \gg \theta$, there exists $N = N(c) \in \mathbb{N}$, for which one has $d_n \ll c$, for each $n > N$.*

Alternatively, it is easy to see that it is of no loss if we take $n \geq N$ in the above definition. Moreover, one can use, as in the case of an usual metric space, alternative definitions such as the [Proposition 3.2] from [160] when the sequence $(d_n)_{n\in\mathbb{N}}$ is from P. Also, we remind the fact that one can rewrite the definition of convergent sequences and Cauchy sequences respectively, using the [Definition 1.4.11] and [Definition 1.8] from [83].
Furthermore, we have the following properties that can be put together in a single lemma. Regarding these properties, one can follow [59], [66], [83] and [160].

Lemma 1.4.12. *Consider $\mathcal{A}$ be a Banach algebra. Then, we have the following :*

(1) *if $u \preceq v \ll w$ or $u \ll v \preceq w$, then $u \ll w$,*
(2) *if $\theta \preceq u \ll c$, for every $c \gg \theta$, then $u = \theta$,*
(3) *if P is a cone, $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ are two c-sequences in $\mathcal{A}$ and α, β are in P, then $(\alpha u_n + \beta v_n)_{n\in\mathbb{N}}$ is also a c-sequence,*
(4) *if P is a cone and $k \in P$ with $\rho(k) < 1$, then $((k)^n)_{n\in\mathbb{N}}$ is a c-sequence,*
(5) *if $k \in P, k \succeq \theta$, wih $\rho(k) < 1$, then $(e-k)^{-1} \succeq \theta$.*

On the other hand, we end this section by reminding the readers that for interesting examples of complete cone metric spaces over Banach algebras and for useful applications to functional and integral equations, we refer to [59], [60], [83], and [163]. From now on, if T is an operator, then by F_T we denote the set of fixed points of the mapping T.
Finally, since our aim is to use the fixed point techniques in order to develop applications that have a meaningful connection with nonlinear systems of functional and differential equations, we kindly refer to [59] and [83] for some important applications to nonlinear differential problems through fixed point results.

1.5 Convex metric spaces

In [157], W. Takahashi introduced a new concept of convexity in metric spaces and proved that all normed spaces and their convex subsets are convex metric spaces. Moreover, he gave some examples of convex metric spaces. We recall the basic definitions and properties of convex metric spaces. For details, we let the reader follow [5], [51], [147] and [157]. Also, for basic notions and results in the theory of Takahashi's convex metric spaces, we refer to [77], [157]. In this section we will recall some of them.

Definition 1.5.1. *Let (X, d) be a metric space. If there exists a function $W : X \times X \times [0,1] \to X$ such that*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y), \text{ for each } u, x, y \in X,$$

then W is a convex structure and (X, d) is called a $W-$convex metric space or a Takahashi convex metric space, briefly a TCS.

Let X be a convex metric space. Takahashi showed that the open balls and the closed balls are convex subsets of X. If $\{K_\alpha\}_{\alpha\in J}$ is a family of nonempty, convex subsets of X, then the intersection $\bigcap_{\alpha\in J} K_\alpha$ is a convex subset of X. In the case of convex metric spaces, we often use the following property (see [51] and [122]).

Remark 1.5.2. For each $x, y, z \in X$ and $\lambda \in [0,1]$, we have that

$$d(z, W(x, y, \lambda)) \geq (1-\lambda) d(z, y) - \lambda d(z, x).$$

Also, from [91], [92], [93] and [159], we recall some important lemmas regarding convex metric spaces :

Lemma 1.5.3. *Let (X, d, W) be a convex metric space. Then the following statements hold :*

(i) $d(x, y) = d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))$, *for each* $(x, y) \in X \times X$ *and* $\lambda \in [0,1]$.
(ii) $d(x, W(x, y, \lambda)) = (1-\lambda) d(x, y)$, *for each* $x, y \in X$.
(iii) $d(y, W(x, y, \lambda)) = \lambda d(x, y)$, *for each* $x, y \in X$.

Lemma 1.5.4. *Let (X, d, W) be a convex metric space. Then :*

$$d\left(x, W\left(x, y, \frac{1}{2}\right)\right) = d\left(y, W\left(x, y, \frac{1}{2}\right)\right) = \frac{1}{2}d(x, y),$$

for each $x, y \in X$.

We notice now that, if (X, d) is a TCS, then for each $x, y \in X$ and $\lambda \in [0, 1]$, we have some important properties :

Lemma 1.5.5.

$$\begin{aligned} &(1)\ W(x, y, 1) = x \text{ and } W(x, y, 0) = y; \\ &(2)\ W(x, x, \lambda) = x; \end{aligned}$$

Now, we give the definition of hyperbolic spaces that are particular cases of convex metric spaces. The concept of hyperbolic metric spaces will be used for the study of Mann iteration concerning multi-valued operators.

Definition 1.5.6. *A hyperbolic space (X, d, W) is a metric space (X, d) together with a convex structure $W : X \times X \times [0, 1] \to X$, satisfying:*

$$\begin{aligned} &(W1)\ d(z, W(x, y, \lambda)) \le (1 - \lambda)d(z, x) + \lambda d(z, y); \\ &(W2)\ d(W(x, y, \lambda_1), W(x, y, \lambda_2)) \le |\lambda_1 - \lambda_2| d(x, y); \\ &(W3)\ W(x, y, \lambda) = W(y, x, 1 - \lambda); \\ &(W4)\ d(W(x, z, \lambda), W(y, w, \lambda)) \le (1 - \lambda)d(x, y) + \lambda d(z, w); \end{aligned}$$

for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$.

Remark 1.5.7. From the property (W1) from [Definition 1.5.6] , taking $x = z = y$, we get that

$$d(x, W(x, x, \lambda)) \le \lambda d(x, x) = 0,$$

so it follows that

$$x = W(x, x, \lambda),$$

for each $\lambda \in [0, 1]$. Property (2) from [Lemma 1.5.5] of a TCS remains valid also in hyperbolic spaces.

Evidently, every hyperbolic space is a Takahashi convex metric space, but, in general, the converse is not true.

Further, we present a useful lemma regarding hyperbolic metric spaces. Since the definition of Takahashi convex metric spaces and the property (W1) in hyperbolic spaces differ by the fact that $\lambda \in [0, 1]$ is replaced by $1 - \lambda \in [0, 1]$ in hyperbolic metric spaces, our next lemma will differ from [Lemma 1.5.3].

Lemma 1.5.8. *Let (X, d, W) be a hyperbolic space. Then*

$$d(x, W(x, y, \lambda)) = \lambda d(x, y) \text{ and } d(y, W(x, y, \lambda)) = (1 - \lambda)d(x, y),$$

for each $x, y \in X$ and $\lambda \in [0, 1]$.

In the entire fixed point literature, there are a lot of classical iteration schemes defined on normed linear spaces and on metric spaces endowed with a convexity structure. Following [51] and [41], we shall remind some of them, but with the remark that, in the research article [41], the authors use a modified version of convex metric space, that is the hyperbolic space in the sense of Goebel and Kirk. So, in the last section of our thesis, we shall use the iterative schemes defined with the inverse order of the two sequence terms appearing in the convexity structure W. Furthermore, let C be a convex subset of the convex metric space (X, d, W) and $T : C \to C$ be a contraction mapping. Moreover, let α_n, b_n a_n be sequences in $(0, 1)$. Moreover, for the sake of completeness we remind the classical iterative processes , such as Krasnoselskii, Mann and Ishikawa in convex metric spaces, etc. In [28], M. Asadi used Krasnoselskii iteration for a contractive-type nonlinear mapping, i.e.

$$x_n = W\left(x_{n-1}, Tx_{n-1}, \lambda\right),$$

with $\lambda \in [0, 1]$. Also, the Mann iteration is defined as :

$$x_n = W\left(x_{n-1}, Tx_{n-1}, \alpha_n\right),$$

with $\alpha_n \in [0, 1]$, for each $n \in \mathbb{N}$.
Moreover, the classical Ishikawa iteration is as follows

$$\begin{cases} x_{n+1} = W\left(x_n, Ty_n, \alpha_n\right) \\ y_n = W\left(x_n, Tx_n, \beta_n\right) \end{cases}, \tag{1.5.0.1}$$

with $\alpha_n, \beta_n \in [0, 1]$, for each $n \in \mathbb{N}$.

Now, we make the following crucial remark : although Berinde, Assadi and Moosaei used the same definition of convex metric spaces, in [51], Berinde defined Mann iteration as

$$x_n = W\left(Tx_{n-1}, x_{n-1}, \alpha_n\right),$$

with $\alpha_n \in [0, 1]$, for each $n \in \mathbb{N}$.
Since many authors use the property that $W(x, y, \lambda) = W(y, x, 1 - \lambda)$, the two forms of Mann iteration can be transformed one to another, so it lies no confusion. Thus, in some parts of the last chapter of this thesis, we shall employ the classical iterative schemes based on the already mentioned observation. As we said before, this does not lead to any confusion, since an iterative scheme can be transformed to a similar form through the convexity structure of the underlying space. This means that, in several sections of our thesis, we employ a condition from hyperbolic spaces, which is satisfied in linear normed spaces, i.e. : $W(x, y, \lambda) = W(y, x, 1 - \lambda)$, for each $x, y \in X$ and $\lambda \in [0, 1]$. This conditions is not at all restrictive and it has the advantage that the iteration terms in the convexity structure W can be swapped one with another and this does not affect convergence of the fixed point iteration.
On the other hand, we recall that the classical Noor iteration can be written as

$$\begin{cases} x_{n+1} = W\left(Ty_n, x_n, \alpha_n\right) \\ y_n = W\left(Tz_n, x_n, b_n\right) \\ z_n = W\left(Tx_n, x_n, a_n\right). \end{cases} \tag{1.5.0.2}$$

Putting $a_n = 0$ we have that $z_n = x_n$, for each $n \in \mathbb{N}$, we get the well know Ishikawa iteration in convex metric spaces:

$$\begin{cases} x_{n+1} = W\left(Ty_n, x_n, \alpha_n\right) \\ y_n = W\left(Tx_n, x_n, b_n\right). \end{cases} \tag{1.5.0.3}$$

Ishikawa iteration 1.5.0.3 differs from the alternative variant 1.5.0.1 by the remark made above.
Putting $a_n = b_n = 0$, then $y_n = z_n = x_n$, for each $n \in \mathbb{N}$, we get the well know Mann iteration in convex metric spaces:

$$x_{n+1} = W\left(Tx_n, x_n, \alpha_n\right), \tag{1.5.0.4}$$

which differs from the alternative form presented above, as in the case with Ishikawa's iteration.
Moreover, we recall the basic fixed point iteration which appears in Banach contraction principle, that is Picard iteration :

$$x_{n+1} = Tx_n, \text{for each n } \in \mathbb{N} \tag{1.5.0.5}$$

Other interesting iteration algorithms are the implicit iterations. Following [41], we recall :
The implicit Noor iteration

$$\begin{cases} x_{n+1} = W\left(Tx_{n+1}, y_n, \alpha_n\right) \\ y_n = W\left(Ty_n, z_n, b_n\right) \\ z_n = W\left(Tz_n, x_n, a_n\right). \end{cases} \tag{1.5.0.6}$$

Putting $a_n = 0$, then $z_n = x_n$, for each $n \in \mathbb{N}$, we get the implicit Ishikawa iteration in convex metric spaces:

$$\begin{cases} x_{n+1} = W\left(Tx_{n+1}, y_n, \alpha_n\right) \\ y_n = W\left(Ty_n, x_n, b_n\right). \end{cases} \tag{1.5.0.7}$$

Additionally, putting $a_n = b_n = 0$, it follows that $y_n = z_n = x_n$, for each $n \in \mathbb{N}$; we get the implicit Mann iteration:

$$x_{n+1} = W\left(Tx_{n+1}, x_n, \alpha_n\right). \tag{1.5.0.8}$$

Now, we recall sufficient conditions for the convergence to the fixed point of a contraction mapping of Noor iteration, respectively implicit Noor iteration.

Remark 1.5.9. Since Noor iteration is more general than Ishikawa and Mann iterations, we shall remind that, the classical Noor iteration 1.5.0.2 is convergent to the fixed point p of the contraction mapping T, if $\sum_{k=0}^{\infty} \alpha_k = \infty$. In a similar way, since implicit Noor iteration is more general than implicit Mann and implicit Ishikawa iterations, we remind that implicit Noor algorithm 1.5.0.6 is convergent when $\sum_{k=0}^{\infty}(1-\alpha_k) = \infty$.

Chapter 2

Fixed point results for single-valued generalized contractions in generalized metric spaces

2.1 Some results for Istrățescu-Presić and Presić-type operators via simulation functions for single-valued mappings

In this section we present new types of Presić single-valued operators that are generalizations of the well-known second order convex contractions. Our aim is to present some fixed point results for this kind of operators. Also, we give a data dependence theorem. Further. we emphasize that our results contain the particular case when $k = 1$, where k is the dimension of the cartesian product of a given complete metric space. First of all, we introduce new types of Presić that will be frequently used in the sequel.

Definition 2.1.1. *Let (X, d) be a metric space. Let $x_0, x_1, ..., x_{k-1}, x_k$ be some arbitrary elements from X. Let $\alpha_i \in [0, 1)$, with $i = \overline{0, k-1}$ and $\beta_{ij} \in [0, 1)$, with $i, j = \overline{0, k}$.*
A mapping $f : X^k \to X$ satisfying the following inequality :

$$
\begin{aligned}
&d\left(f\left(f\left(x_0, ..., x_0\right), ..., f\left(x_{k-1}, ..., x_{k-1}\right)\right), f\left(f\left(x_1, ..., x_1\right), ..., f\left(x_k, ..., x_k\right)\right)\right)\\
&\leq \sum_{i=0}^{k-1} \alpha_i d(x_i, x_{i+1}) + \sum_{i,j=0}^{k} \beta_{ij} d\left(f\left(x_i, ..., x_i\right), f\left(x_j, ..., x_j\right)\right)
\end{aligned}
$$

is called a ***Presić convex contraction of the first kind****.*

The second type of Presić operators that we consider are the following.

Definition 2.1.2. *Let (X, d) be a metric space. Let $x_0, x_1, ..., x_{k-1}, x_k$ be arbitrary elements from X. Let $\alpha_i \in [0, 1)$, with $i = \overline{0, k-1}$ and $\beta_{ij} \in [0, 1)$, with $i, j = \overline{0, k}$.*
A mapping $f : X^k \to X$ satisfying the following inequality :

$$
\begin{aligned}
&d\left(f\left(f\left(x_0, ..., x_{k-1}\right), ..., f\left(x_0, ..., x_{k-1}\right)\right), f\left(f\left(x_1, ..., x_k\right), ..., f\left(x_1, ..., x_k\right)\right)\right)\\
&\leq \sum_{i=0}^{k-1} \alpha_i d(x_i, x_{i+1}) + \sum_{i,j=0}^{k} \beta_{ij} d\left(f\left(x_i, ..., x_i\right), f\left(x_j, ..., x_j\right)\right)
\end{aligned}
$$

is called a ***Presić convex contraction of the second kind****.*

Definition 2.1.3. *Let (X, d) be a metric space and $f : X^k \to X$ a mapping. Then, the operator $F : X \to X$, defined as $F(x) = f(x, ..., x)$, for each $x \in X$ is called the associated operator of f.*

The main result of the present section is concerned with the existence and uniqueness of the fixed point of f, i.e. the element $x^* \in X$ that satisfies $x^* = f(x^*, ..., x^*)$. More precisely, we shall develop a theorem involving the fixed point of the associated operator F, i.e. $x^* \in X$ that satisfies $x^* = F(x^*)$ for the Presić convex contractions of the first kind and for the fixed point of the associated operator of Presić convex contractions of the second kind, respectively.

Theorem 2.1.4. *Let (X, d) be a complete metric space. Consider the following :*
(i) Let $f : X^k \to X$ a continuous Presić convex contraction of the first kind. Suppose that the coefficients of the mapping f from [Definition 2.1.1] satisfy

$$\sum_{i=0}^{k-1} \alpha_i + 2\sum_{p=1}^{k}\left(\sum_{i=0}^{k-p}\sum_{j=k-p+1}^{k}\beta_{ij}\right) \in (0,1). \tag{2.1.0.1}$$

Then f has a unique fixed point x^* and the sequence $(x_n)_{n\in\mathbb{N}}$ defined by 1.2.0.1, is convergent to x^*.
(ii) Consider $f : X^k \to X$ a continuous Presić convex contraction of the second kind. Suppose that the coefficients from the [Definition 2.1.2] satisfy the same condition as before, namely 2.1.0.1.
Then, as in the previous case, f has a unique fixed point x^* and the sequence defined by 1.2.0.1, is convergent to x^*.

Based on our previous results, we consider the particular case when $k = 1$. This means that the fixed point conclusions holds also for the continuous convex contractions of second order.

Corollary 2.1.5. *For $k = 1$, the Presić convex contractions of the first kind and Presić convex contractions of the second kind satisfy the metric-type inequality*

$$d(f^2x_0, f^2x_1) \leq \alpha_0 d(x_0, x_1) + [\beta_{01} + \beta_{10}]\, d(fx_0, fx_1),$$

with $\alpha_0 + [\beta_{10} + \beta_{01}] \in [0, 1)$ for each x_0 and x_1 arbitrary elements of X. This means that our new types of Presić mappings are valid generalizations of the well-known convex contractions of second order.

On the other hand, since we have used the associate operator, we present two forms in which the operator can be computed :

Remark 2.1.6. For the associated self-mapping F of f, the second iterate can be computed in two different manners :
a) $F^2(x) = F(Fx) = f(Fx, ..., Fx) = f\left(f(x, ..., x), ..., f(x, ..., x)\right)$,
b) $F^2(x) = F\left(f(x, ..., x)\right) = F(y)$, with $y = f(x, ..., x)$, where $F(y) = f(y, ..., y)$. So $F^2(x) = f(y, ..., y) = f\left(f(x, ..., x), ..., f(x, ..., x)\right)$.

In [Corollary 2.1.5] we have shown that our newly introduced operators can be considered as generalized operators of the second order convex contractions in higher dimension metric spaces. Furthermore, it is easy to observe that the Presić convex contractions of the first kind are also generalizations for the well-known Presić operators. So far we have presented some remarks and a fixed point result for Presić operators of the first and second kind, respectively. These metric-type operators are important due to the fact that they generalize Istrăţescu convex contractions of the second order and also the basic Presić contractions. Now, another important property regarding fixed points of generalized type contractions is the idea of data dependence of the fixed points (see [95]), that will be presented below.

Theorem 2.1.7. *Let (X, d) be a complete metric space. Also, consider $g : X^k \to X$ be an arbitrary mapping with at least a fixed point $x_g^* \in X$. Further, consider the following assumptions : (i) Let $f : X^k \to X$ be a continuous Presić convex contraction of the first kind, satisfying the conditions from (i) of [Theorem 2.1.4]. Furthermore, suppose that :*
(a) there exists $\eta_1 > 0$, such that $d(f(x, ..., x), g(x, ..., x)) \leq \eta_1$, for each $x \in X$,
(b) there exists $\eta_2 > 0$, such that for each $x \in X$, we have that

$$d(f(f(x, ..., x), ..., f(x, ..., x)), g(g(x, ..., x), ..., g(x, ..., x))) \leq \eta_2\ .$$

If x_f^* denotes the unique fixed point of f, one has that

$$d(x_f^*, x_g^*) \leq \frac{\eta_2 + 2\sum\limits_{p=1}^{k}\sum\limits_{i=0}^{k-p}\left(\sum\limits_{j=k-p+1}^{k}\beta_{ij}\right)\cdot\eta_1}{1 - \left[\left(\sum\limits_{i=0}^{k-1}\alpha_i\right) + 2\sum\limits_{p=1}^{k}\sum\limits_{i=0}^{k-p}\left(\sum\limits_{j=k-p+1}^{k}\beta_{ij}\right)\right]}\ ,$$

where x_f^* is the unique fixed point of f.
(ii) Consider $f : X^k \to X$ a continuous Presić convex contraction of the second kind, satisfying the conditions from (ii) of [Theorem 2.1.4].
In a similar manner, suppose that f and g satisfy the previous conditions (a) and (b). Then, $d(x_f^*, x_g^*)$ has the same major bound as before.

Below we will present the particular case when the dimension of the cartesian product of the metric space $X$is 1. So, the data dependence of the fixed points between two self-mappings, where one of them is a Presić operator, is considered as follows.

Corollary 2.1.8. *Taking $k = 1$ in the previous theorem, we obtain that $2\sum_{p=1}^{k}\left(\sum_{i=0}^{k-p}\sum_{j=k-p+1}^{k}\beta_{ij}\right) = 2\beta_{1,0} = \beta_{1,0} + \beta_{0,1}$. It follows that the data dependence results is valid for the case of $k = 1$ as in [95], where $d(x_f^*, x_g^*) \leq \dfrac{\eta_2 + \delta \cdot \eta_1}{1 - (\alpha + \delta)}$, with $\delta := 2\beta_{10}$ and $\alpha := \alpha_0$.*

Regarding operators endowed with Presić-type regularity, in the fixed point literature, very few non-trivial examples are given. In the framework of a complete metric space (X, d), in general, some classical examples of mappings $f : X^2 \to X$ are given, where f is either linear or a piecewise mapping. In the following sequel, we construct some non-trivial examples defined on a cartesian product of an interval denoted by $[0, r]$, with $r > 0$. Interestingly enough, we also present some useful situations where some particular examples of operators are not of Presić type. Furthermore, in our constructions, we shall modify some coefficients such that the Presić-type operators given in [Definition 2.1.1] and [Definition 2.1.2] are also Presić contractions.

Example 2.1.9. Let $f : [0, r]^2 \to [0, r]$, with $r > 0$, defined as $f(x, y) = \dfrac{(xy)^2}{a}$, with $a > 0$.
For $r = 2$ and $a = 12$, f is a Presić convex contraction of the first kind, but it is not a Presić mappings. Also, for $r = 2$ and $a = 34$, f is a Presić convex contraction of the first kind that is also a Presić operator.

Our second example concerns the construction of another type of mapping, satisfying the conditions from (ii) of [Theorem 2.1.4].

Example 2.1.10. Consider the function $f : [0, r]^2 \to [0, r]$, defined as $f(x, y) = \alpha x^2 + \beta y^2$, where $r, \alpha, \beta > 0$. For $r = 2$, $\beta = 0.27$ and $\alpha = 0.03$, f is a Presić convex contraction of the second kind, that is not a Presić mapping. On the other hand, for $r = 3$, $\beta = \dfrac{1}{7}$ and $\alpha = \dfrac{1}{47}$, f is a Presić convex contraction of the second kind, that it is also a Presić operator.

Now, we present a case of a convex contraction of Presić-type of the first kind, following the conditions from (i) of [Theorem 2.1.4].

Example 2.1.11. Let the mapping f, defined as $f(x, y) = (\tau_1 x - \tau_2 y)^2$, where $f : X^2 \to X$, with $X = [0, r]$, where $r > 0$ and $\tau_1, \tau_2 > 0, \tau_1 \neq \tau_2$. For $r = 2$, $\tau_2 = 0.8$ and $\tau_1 = 0.6$, f is a Presić convex contraction of the first kind that it is not a Presić mapping. On the other hand, for $r = 2$, $\tau_2 = -\dfrac{1}{4} + \dfrac{\sqrt{2}}{4}$ and $\tau_1 - 0.45$, f is a Presić convex contraction of the first kind, that is also a Presić operator.

So far we have presented some new types of Presić operators that extend the well-known convex contractions of second order and Presić contractions, respectively. Quite surprisingly, Banach contraction mappings were generalized through the idea of simulation functions. The second purpose of the present section is to introduce an appropriate concept of simulation function in order to generalize fixed point results for Presić-type single-valued operators. All of the present analysis is based upon the research paper of the Boyd-Wong-Presić self-mappings that were considered in [148]. We shall give an intuitive definition of a simulation function that is necessary for operators defined on a cartesian product of nonempty sets, endowed with a metric function and a partial order relation. This new concept will allow us to assert that the Boyd-Wong-Presić operators can be regarded as particular cases of our new operators, given through the so-called k-simulation functions.

Definition 2.1.12. *Let $\zeta : [0,\infty) \times [0,\infty)^k \to \mathbb{R}$ be a given mapping. We call ζ a k-simulation function if the following conditions are satisfied :*
(I.) If $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = L > 0$, where $s_n \downarrow L$ and $t_n < s_n$, for every $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n, \ldots, s_n) < 0;$$

(II.) If $t, s > 0$, then $\zeta(t, s, \ldots, s) < s - t$;
(III.) If $s \geq 0$ and $t_1, \ldots, t_k > 0$, then the mapping ζ is endowed with the property that

$$\zeta(t_1, s, 0, \ldots, 0) + \zeta(t_2, 0, s, \ldots, 0) + \ldots \zeta(t_k, 0, \ldots, 0, s) \leq \zeta(t_1 + t_2 + \ldots t_k, s, \ldots, s).$$

Now, following the examples from [71], [76], [127], [135] and [162] of the classical case of a simulation function, our first aim is to extend and make those examples appropriate for k-simulation functions, where k is a given positive integer. So, from now on, we shall consider a fixed nonnegative integer k.
The first example consists on choosing the simulation function as a necessary tool for the case when our Presić operator is of Boyd-Wong type.

Example 2.1.13. Let $\zeta : [0,\infty) \times [0,\infty)^k \to \mathbb{R}$ be a mapping defined as

$$\zeta(t, s_1, \ldots, s_k) = \psi(s_1, \ldots, s_k) - t, \text{ where } t \geq 0 \text{ and } s_1, \ldots, s_k \geq 0,$$

where the function $\psi : \mathbb{R}_+^k \to \mathbb{R}$ satisfies the following conditions :

$$\psi(t, \ldots, t) < t, \text{ for each } t > 0;$$
$$\psi(t, 0, \ldots, 0) + \psi(0, t, 0, \ldots, 0) + \ldots + \psi(0, 0, \ldots, 0, t) \leq \psi(t, \ldots, t), \text{ for each } t \geq 0;$$
$$\limsup_{n\to\infty} \psi(s_n, \ldots, s_n) \leq \psi(L, \ldots, L), \text{ with } s_n \downarrow L > 0, \text{ i.e. } \psi \text{ is usc at right.}$$

Example 2.1.14. Let $\zeta : [0,\infty) \times [0,\infty)^k \to \mathbb{R}$ be a mapping defined as

$$\zeta(t, s_1, \ldots, s_k) = \phi_1(s_1, \ldots, s_k) - \phi_2(t), \text{ where } t \geq 0 \text{ and } s_1, \ldots, s_k \geq 0,$$

where the functions $\phi_1 : \mathbb{R}_+^k \to \mathbb{R}$ and $\phi_2 : \mathbb{R}_+ \to \mathbb{R}$ satisfy the following conditions :

$$\phi_1(s, \ldots, s) < s, \text{ for each } s > 0;$$
$$\phi_2(t) \geq t, \text{ for each } t > 0;$$
$$\phi_1(s, 0, \ldots, 0) + \phi_1(0, s, 0, \ldots, 0) + \ldots + \phi_1(0, 0, \ldots, 0, s) \leq \phi_1(s, \ldots, s), \text{ for each } s \geq 0;$$
$$\phi_2(t_1 + \ldots + t_k) \leq \phi_2(t_1) + \ldots \phi_2(t_k), \text{ for each } t_1, \ldots, t_k > 0;$$
$$\phi_2 \text{ is lsc}$$
$$\limsup_{n\to\infty} \phi_1(s_n, \ldots, s_n) \leq \phi_1(L, \ldots, L), \text{ where } s_n \downarrow L > 0.$$

Example 2.1.15. Let $\zeta : [0,\infty) \times [0,\infty)^k \to \mathbb{R}$ be a mapping defined as

$$\zeta(t, s_1, \ldots, s_k) = \left(\frac{s_1 + \ldots + s_k}{k}\right) \phi(s_1, \ldots, s_k) - t, \text{ where } t \geq 0 \text{ and } s_1, \ldots, s_k \geq 0,$$

where the function $\phi : \mathbb{R}_+^k \to [0, 1)$ satisfies the following conditions :

$$\phi(s, 0, \ldots, 0) + \phi(0, s, 0, \ldots, 0) + \ldots + \phi(0, 0, \ldots, 0, s) \leq k\phi(s, \ldots, s), \text{ for each } s \geq 0;$$
$$\limsup_{s\to r^+} \phi(s, \ldots, s) < 1, \text{ for each } r > 0.$$

Example 2.1.16. Let $\zeta : [0,\infty) \times [0,\infty)^k \to \mathbb{R}$ be a mapping defined as

$$\zeta(t, s_1, \ldots, s_k) = s_1\phi_1(s_1) + \ldots + s_k\phi_k(s_k) - t, \text{ where } t \geq 0 \text{ and } s_1, \ldots, s_k \geq 0,$$

where the functions $\phi_1, \ldots, \phi_k : \mathbb{R}_+ \to \mathbb{R}$ satisfy the following conditions :

$$\phi_1(s) + \phi_2(s) + \ldots + \phi_k(s) < 1, \text{ for each } s > 0;$$
$$\phi_1, \ldots, \phi_k \text{ are usc at right.}$$

Now, the last two examples that we consider are linked to more concrete types of mappings that appear in the choice of the k-simulation functions. Since they are crucial to some particular examples of Presić-type operators, we present them in here.

Example 2.1.17. Let $\zeta : [0,\infty) \times [0,\infty)^k \to \mathbb{R}$ be a mapping defined as

$$\zeta(t, s_1, \ldots, s_k) = \frac{s_1 + \ldots + s_k}{k} - \frac{t+2}{t+1}t, \text{ where } t \geq 0 \text{ and } s_1, \ldots, s_k \geq 0,$$

Now, taking $\phi_1(t) = \ldots = \phi_k(t) = \dfrac{1}{k(t+1)}$, for each $t \geq 0$ in [Example 2.1.16], we get the following example as a particular case.

Example 2.1.18. Let $\zeta : [0,\infty) \times [0,\infty)^k \to \mathbb{R}$ be a mapping defined as

$$\zeta(t, s_1, \ldots, s_k) = \frac{1}{k}\left[\frac{s_1}{s_1+1} + \ldots + \frac{s_k}{s_k+1}\right] - t, \text{ where } t \geq 0 \text{ and } s_1, \ldots, s_k \geq 0,$$

Now, using [Definition 2.1.12] from the previous section, we introduce some operators that satisfy a generalized Presić-contractive type inequality through the already given concept of a $k-$simulation function.

Definition 2.1.19. *Let (X, d) be a metric space and $T : X^k \to X$ be a given mapping. The operator T is called a Presić-$\mathcal{Z}$-contraction (briefly $P - \mathcal{Z}-$contraction), if it satisfies the following inequality :*

$$\zeta\left(d\left(T(z_1, \ldots, z_k), T(z_2, \ldots, z_{k+1})\right), d(z_1, z_2), \ldots, d(z_k, z_{k+1})\right) \geq 0,$$

for each $z_1, \ldots, z_{k+1}$ arbitrary points in X.

Now, using an additional self-mapping $g : X \to X$, we can extend the previous definition.

Definition 2.1.20. *Let (X, d) be a metric space. Also, let $T : X^k \to X$ and $g : X \to X$ be two given mappings. The operator T is called a Presić-$(\mathcal{Z}, g)$-contraction (briefly $P-(\mathcal{Z}, g)-$contraction), if it satisfies the following inequality :*

$$\zeta\left(d\left(T(z_1, \ldots, z_k), T(z_2, \ldots, z_{k+1})\right), d(gz_1, gz_2), \ldots, d(gz_k, gz_{k+1})\right) \geq 0,$$

for each $z_1, \ldots, z_{k+1}$ arbitrary points in X.

Definition 2.1.21. *Let $(X, \preceq, d)$ be an ordered metric space, where " $\preceq$ " is a partial order on X. Let k be a nonnegative integer and $T : X^k \to X$ be a given mapping. The operator T is called a Presić$-\mathcal{Z}$-ordered contraction (briefly a $P - \mathcal{Z}$-ordered contraction), it it satisfies the following condition :*

$$\zeta\left(d(T(z_1, \ldots, z_k), T(z_2, \ldots, z_{k+1})), d(z_1, z_2), \ldots, d(z_k, z_{k+1})\right) \geq 0,$$

for every arbitrary points $z_1, \ldots, z_{k+1} \in X$, such that $z_1 \preceq z_2 \preceq \ldots \preceq z_k \preceq z_{k+1}$.

Definition 2.1.22. *Let $(X, \preceq, d)$ be an ordered metric space, where " $\preceq$ " is a partial order on X. Let k be a nonnegative integer and $T : X^k \to X$, $g : X \to X$ be two given mappings. The operator T is called a Presić-$(\mathcal{Z}, g)$-ordered contraction (briefly a $P - (\mathcal{Z}, g)$-ordered contraction), it it satisfies the following condition :*

$$\zeta\left(d(T(z_1, \ldots, z_k), T(z_2, \ldots, z_{k+1})), d(gz_1, gz_2), \ldots, d(gz_k, gz_{k+1})\right) \geq 0,$$

for every arbitrary points $z_1, \ldots, z_{k+1} \in X$, such that $gz_1 \preceq gz_2 \preceq \ldots \preceq gz_k \preceq gz_{k+1}$.

Armed with Definitions 2.1.19, 2.1.20, 2.1.21 and 2.1.22, we are ready to present our main result regarding Presić-type operators given by k-simulation functions. In the setting of a complete metric space with a partial order, we present a fixed point result for single-valued mappings. For more details of metric spaces with a partial order, we let the reader follow [148]. Moreover, for the terminology related to coincidence and common fixed points, well-ordered sets and weakly compatible mappings, we emphasize the fact that, in the next theorem, we follow the same concepts as in [104] and [148].

Theorem 2.1.23. *Let $(X, \preceq, d)$ be a complete ordered metric space and $k \geq 1$ a fixed integer. Let $T : X^k \to X$ and $g : X \to X$ be two mappings, such that $T(X^k) \subset g(X)$ and $g(X)$ is a closed subspace of X. Furthermore, suppose that the following conditions are satisfied :*

(H1) The operator T is a $P - (\mathcal{Z}, g)$-ordered contraction,
(H2) There exists $x_1 \in X$, such that $gx_1 \preceq T(x_1, \ldots, x_1)$,
(H3) The operator T is g-increasing,
(H4) If $(gx_n)_{n\in\mathbb{N}}$ is a increasing sequence that converges to $gu \in X$, then $gx_n \preceq gu$, for each $n \in \mathbb{N}$, and $gu \preceq ggu$.

Under these conditions, we have the following conclusion :
1.) T and g have at least a coincidence point,
2.) If T and g are weakly compatible, then these mappings are endowed with a common fixed point,
3.) The set of all common fixed points for the pair (T, g) is $g-$well ordered is equivalent to the fact that the operators T and g have a unique common fixed point.

In the following remark, we present a direct connection between $P - \mathcal{Z}-$contractions and Presić-type operators, which are defined by a strict inequality. The idea behind it is to choose an arbitrary pair $(x_1, \ldots, x_{k+1})$ of elements of the nonempty set X from another pair $(z_1, \ldots, z_{k+1})$, under the condition that

$$d(x_1, x_2) = \ldots = d(x_k, x_{k+1}) > 0 \ .$$

This remark will be based upon [Definition 2.1.19] and [Remark 1.2] of E. Karapinar's article [71].

Remark 2.1.24. If $T : X^k \to X$ is a $P - \mathcal{Z}$-contraction, then

$$d(T(x_1, \ldots, x_k), T(x_2, \ldots, x_{k+1})) < d(x_i, x_{i+1}), \text{ with } i = \overline{1, k}, \tag{2.1.0.2}$$

where $(x_1, \ldots, x_{k+1})$ is the pair reminded before the present remark.
Now, we prove this assertion under the assumptions that $d(x_i, x_{i+1}) > 0$, for each $i = \overline{1, k}$. Then, if $T(x_1, \ldots, x_k) = T(x_2, \ldots, x_{k+1})$, we have a valid answer. On the other hand, we can suppose that $d(T(x_1, \ldots, x_k), T(x_2, \ldots, x_{k+1})) > 0$. Applying property $(II.)$ of the simulation function on the pair $(x_1, \ldots, x_{k+1})$, we find that

$$\begin{aligned} 0 &\leq \zeta(d(T(x_1, \ldots, x_k), T(x_2, \ldots, x_{k+1})), d(x_i, x_{i+1}), \ldots, d(x_i, x_{i+1})) \\ &< d(x_i, x_{i+1}) - d(T(x_1, \ldots, x_k), T(x_2, \ldots, x_{k+1})), \ i = \overline{1, k}, \end{aligned}$$

so 2.1.0.2 holds properly.

Now, as applications, we present some direct corollaries linked to [Theorem 2.1.23]. We also infer some remarks on the applications in complete metric spaces with the usual partial order. Finally, some examples are given concerning the already mentioned corollaries in the setting of common and coincidence fixed points of Presić-type operators, introduced by us through the concept of a k-simulation function.
Following [Example 2.1.13] and [Theorem 2.1.23], we have the following consequence, i.e. the main result in [148] is a particular case of [Theorem 2.1.23], where we have a fixed point theorem for Boyd-Wong-Presić operators.

Corollary 2.1.25. *Let $(X, \preceq, d)$ be a complete ordered metric space and $k \geq 1$ a fixed integer. Let $T : X^k \to X$ and $g : X \to X$ be two mappings, such that $T(X^k) \subset g(X)$ and $g(X)$ is a closed subspace of X. Furthermore, suppose that the following condition is satisfied :*

$$d\left(T(z_1, \ldots, z_k), T(z_2, \ldots, z_{k+1})\right) \leq \psi(d(gz_1, gz_2), \ldots, d(gz_k, gz_{k+1})),$$

for every arbitrary points $z_1, \ldots, z_{k+1} \in X$, such that $gz_1 \preceq gz_2 \preceq \ldots \preceq gz_k \preceq gz_{k+1}$, where the function ψ is the mapping from the [Example 2.1.13]. Furthermore, suppose that the operator T is endowed with properties $(H2)$,$(H3)$ and $(H4)$ from [Theorem 2.1.23]. Under these conditions, the conclusions 1.), 2.) and 3.) from [Theorem 2.1.23] are valid.

Now, using the other examples from the second section, we have the following consequences.

Corollary 2.1.26. *Let $(X,\preceq,d)$ be a complete ordered metric space and $k\geq 1$ a fixed integer. Let $T : X^k\to X$ and $g:X\to X$ be two mappings, such that $T(X^k)\subset g(X)$ and $g(X)$ is a closed subspace of X. Furthermore, suppose that the following condition is satisfied :*

$$\phi_2(d\left(T(z_1,\ldots,z_k),T(z_2,\ldots,z_{k+1})\right))\leq \phi_1(d(gz_1,gz_2),\ldots,d(gz_k,gz_{k+1})),$$

for every arbitrary points $z_1,\ldots,z_{k+1}\in X$, such that $gz_1\preceq gz_2\preceq\ldots\preceq gz_k\preceq gz_{k+1}$, where the functions ϕ_1 and ϕ_2 are the ones from [Example 2.1.14]. Furthermore, suppose that the operator T is endowed with properties $(H2)$,$(H3)$ and $(H4)$ from [Theorem 2.1.23]. Under these conditions, the conclusions 1.), 2.) and 3.) from [Theorem 2.1.23] are valid.

Corollary 2.1.27. *Let $(X,\preceq,d)$ be a complete ordered metric space and $k\geq 1$ a fixed integer. Let $T : X^k\to X$ and $g:X\to X$ be two mappings, such that $T(X^k)\subset g(X)$ and $g(X)$ is a closed subspace of X. Furthermore, suppose that the following condition is satisfied :*

$$d\left(T(z_1,\ldots,z_k),T(z_2,\ldots,z_{k+1})\right)\leq \frac{d(gz_1,gz_2)+\ldots+d(gz_k,gz_{k+1})}{k}\cdot\phi(d(gz_1,gz_2),\ldots,d(gz_k,gz_{k+1})),$$

for every arbitrary points $z_1,\ldots,z_{k+1}\in X$, such that $gz_1\preceq gz_2\preceq\ldots\preceq gz_k\preceq gz_{k+1}$, where the function ϕ is the one from [Example 2.1.15]. Furthermore, suppose that the operator T is endowed with properties $(H2)$,$(H3)$ and $(H4)$ from [Theorem 2.1.23]. Under these conditions, the conclusions 1.), 2.) and 3.) from [Theorem 2.1.23] are valid.

Corollary 2.1.28. *Let $(X,\preceq,d)$ be a complete ordered metric space and $k\geq 1$ a fixed integer. Let $T : X^k\to X$ and $g:X\to X$ be two mappings, such that $T(X^k)\subset g(X)$ and $g(X)$ is a closed subspace of X. Furthermore, suppose that the following condition is satisfied :*

$$d\left(T(z_1,\ldots,z_k),T(z_2,\ldots,z_{k+1})\right)\leq d(gz_1,gz_2)\phi_1(d(gz_1,gz_2))+\ldots+d(gz_k,gz_{k+1})\phi_k(d(gz_k,gz_{k+1})),$$

for every arbitrary points $z_1,\ldots,z_{k+1}\in X$, such that $gz_1\preceq gz_2\preceq\ldots\preceq gz_k\preceq gz_{k+1}$, where the functions $\phi_1,\ldots,\phi_k$ are taken from [Example 2.1.16]. Furthermore, suppose that the operator T is endowed with properties $(H2)$,$(H3)$ and $(H4)$ from [Theorem 2.1.23]. Under these conditions, the conclusions 1.), 2.) and 3.) from [Theorem 2.1.23] are valid.

Following the choice of the k-simulation function in [Example 2.1.17], we have the following consequence of our main result.

Corollary 2.1.29. *Let $(X,\preceq,d)$ be a complete ordered metric space and $k\geq 1$ a fixed integer. Let $T : X^k\to X$ and $g:X\to X$ be two mappings, such that $T(X^k)\subset g(X)$ and $g(X)$ is a closed subspace of X. Furthermore, suppose that the following condition is satisfied :*

$$\frac{d\left(T(z_1,\ldots,z_k),T(z_2,\ldots,z_{k+1})\right)+2}{d\left(T(z_1,\ldots,z_k),T(z_2,\ldots,z_{k+1})\right)+1}d\left(T(z_1,\ldots,z_k),T(z_2,\ldots,z_{k+1})\right)\leq\frac{d(gz_1,gz_2)+\ldots+d(gz_k,gz_{k+1})}{k},$$

for every arbitrary points $z_1,\ldots,z_{k+1}\in X$, such that $gz_1\preceq gz_2\preceq\ldots\preceq gz_k\preceq gz_{k+1}$. Furthermore, suppose that the operator T is endowed with properties $(H2)$,$(H3)$ and $(H4)$ from [Theorem 2.1.23]. Under these conditions, the conclusions 1.), 2.) and 3.) from [Theorem 2.1.23] are valid.

Now, the last corollary consists on the choice of the k-simulation function, following [Example 2.1.18].

Corollary 2.1.30. *Let $(X,\preceq,d)$ be a complete ordered metric space and $k\geq 1$ a fixed integer. Let $T : X^k\to X$ and $g:X\to X$ be two mappings, such that $T(X^k)\subset g(X)$ and $g(X)$ is a closed subspace of X. Furthermore, suppose that the following condition is satisfied :*

$$d\left(T(z_1,\ldots,z_k),T(z_2,\ldots,z_{k+1})\right)\leq\frac{1}{k}\left[\frac{d(gz_1,gz_2)}{d(gz_1,gz_2)+1}+\ldots+\frac{d(gz_k,gz_{k+1})}{d(gz_k,gz_{k+1})+1}\right],$$

for every arbitrary points $z_1,\ldots,z_{k+1}\in X$, such that $gz_1\preceq gz_2\preceq\ldots\preceq gz_k\preceq gz_{k+1}$. Furthermore, suppose that the operator T is endowed with properties $(H2)$,$(H3)$ and $(H4)$ from [Theorem 2.1.23]. Under these conditions, the conclusions 1.), 2.) and 3.) from [Theorem 2.1.23] are valid.

As in [Theorem 7] and [Corollary 6] of [148] and following our main result and from the above corollaries, we end this section by recalling that the same fixed point results are valid even for the case of complete metric spaces.

Remark 2.1.31. For the case when X is a nonempty set endowed only with a complete metric d, the conclusion from [Theorem 2.1.23] and from all the corollaries of this section remain valid, where the metric-type inequalities take place for any arbitrary points $z_1, \ldots, z_{k+1} \in X$. On the other hand, we can take the mapping g to be the identity mapping 1_X in order to obtain some fixed point results just for the operator $T : X^k \to X$.

Now, our purpose is to validate [Corollary 2.1.25], [Corollary 2.1.29] and [Corollary 2.1.30], based on [Remark 2.1.31], in the framework of complete metric spaces endowed with the usual partial order. Furthermore, based on our main result, namely [Theorem 2.1.23], we shall naturally extend and generalize some of the examples of Boyd-Wong-type and some examples when $k = 1$, that can be found in [76],[127] and [148], respectively. Based upon [Example 3] in [148], we consider an example of a mapping defined on X^2 with values in X, such that is not a Presić contraction, but is of Boyd-Wong type. Moreover, this examples justifies the coincidence and common fixed point result obtained by us in [Corollary 2.1.25].

Example 2.1.32. Let $X = [0, \infty)$,$T : X^2 \to X$ and $g : X \to X$, be some given mappings, defined as

$$T(x_1, x_2) = \frac{p(x_1 + x_2) + 2x_1x_2}{(x_1 + x_2)(q + 1) + x_1x_2(q + 2) + qp^2} \text{ and } g(x) = \frac{x}{p},$$

where $p \geq 1$ and $q \geq 3$. Furthermore, consider the mapping $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, defined as

$$\psi(t_1, t_2) = \frac{t_1 + t_2 + 2t_1t_2}{3t_1t_2 + 3t_1 + 3t_2 + 3 + |t_1 - t_2|}.$$

The operators T and g commute at their unique common fixed point and so they are weakly compatible and the mapping ψ is continuous, with $\psi(t, t) = \frac{2}{3}\frac{t}{t+1} < t$ and $\psi(0, t) + \psi(t, 0) = \frac{2t}{3(t+1)+t} \leq \psi(t, t)$, so this mapping has the properties from [Corollary 2.1.25].

Now, inspired by [76] and [127], we give now two examples for mappings satisfying [Corollary 2.1.29] and [Corollary 2.1.30], respectively, in a general context, i.e. for a mapping $T : X^k \to X$, where k is a fixed arbitrary non-negative integer. The second to last example refers to [Corollary 2.1.29] and the last example is linked to [Corollary 2.1.30].

Example 2.1.33. Let $X = [1, \infty)$, $T : X^k \to X$ and $g : X \to X$ be some mappings, defined as :

$$T(x_1, \ldots, x_k) = \frac{1 + \left(\dfrac{x_1^2 + \ldots + x_k^2}{k}\right)}{2} = \frac{k + (x_1^2 + \ldots + x_k^2)}{2k},$$
$$g(x) = x^2.$$

The pair (T, g) is weakly compatible and the only common fixed point is $x = 1$.
Furthermore, one can see that the operator T is not a Presić contraction. On the other hand, one can show that the inequality from [Corollary 2.1.29] is valid.

Finally, the last example concerns [Corollary 2.1.30]. We give it below.

Example 2.1.34. Let's consider $X = [0, 1]$, $k \geq 1$ and $p \geq 2$ a fixed integer. We define the mappings $T : X^k \to X$ and $g : X \to X$, by :

$$T(x_1, \ldots, x_k) = \frac{1}{k}\left[\frac{x_1}{x_1 + p} + \ldots + \frac{x_k}{x_k + p}\right],$$
$$g(x) = \frac{x}{p}.$$

The operators T and g commute at their unique common fixed point, $x = 0$. Also, the contraction-type condition from [Corollary 2.1.30] holds true.

2.2 Generalized contractions, fixed points and b-rectangular metric-type spaces

For the convenience of the reader, we remind some important results in b-rectangular metric spaces that will be used throughout this section. In [52], George et.al.studied basic contraction-type mappings in b-rectangular metric spaces, like Kannan operators, i.e.

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx) + d(y, Ty) \right], \text{ with } \lambda \in \left[0, \frac{1}{s+1} \right].$$

In [47], Radenovic et.al. extended the results to mappings satisfying

$$d(fx, gy) \leq ad(gx, gy) + b \left[d(gx, fx) + d(gy, fy) \right],$$

for each $x, y \in X$ and studied unique coincidence and common fixed points for the pair of operators (f, g), when they satisfy certain conditions.
Also, for more results in b-rectangular metric spaces and for a consistent survey on different generalized metric-type spaces, we recommend [68] and [69].

Regarding generalized contraction mappings we recall some recent advances in this subfield of fixed point theory.
In [72], Karapinar studied unique fixed points for some generalized contractions on cone Banach spaces satisfying the following contractive-type conditions

$$d(x, Tx) + d(y, Ty) \leq pd(x, y),$$

where $p \in [0, 2)$ and

$$ad(Tx, Ty) + b \left[d(x, Tx) + d(y, Ty) \right] \leq sd(x, y),$$

with $0 \leq s + |a| - 2b < 2(a + b)$.
Moreover, in 2009, Kumar [78] presented some theorems for two maps satisfying the following

$$d(fx, fy) \geq qd(gx, gy),$$

with $q > 1$, where f is onto and g is one-to-one.
Moosaei, Azizi, Asadi and Wang generalized the results of Karapinar as follows :
In [91], Moosaei used Krasnoselskii's iteration defined in convex metric spaces, for the following mappings, that satisfy

$$d(Tx, Ty) + d(x, Tx) + d(y, Ty) \leq rd(x, y),$$

where $r \in [2, 5)$, respectively

$$ad(x, fx) + bd(y, fy) + cd(fx, fy) \leq kd(x, y),$$

with $2b - |c| \leq k < 2(a + b + c) - |c|$.
In [93], Moosaei and Azizi extended the results to generalized contraction-type operators, studying coincidence points for various mappings, such as

$$ad(Sx, Tx) + bd(Sy, Ty) + cd(Tx, Ty) \leq ed(x, y),$$

where $T(K) \subset S(K)$, K and $S(K)$ are closed and convex subsets of a convex metric space and the coefficients satisfy $2b - |c| \leq e < 2(a + b + c) - |c|$.
Nevertheless, in 2014, Moosaei [92] studied a more generalized pair of contractions (S, T), where

$$\alpha d(Tx, Ty) + \beta \left[d(Sx, Tx) + d(Sy, Ty) \right] + \gamma \left[d(Sx, Ty) + d(Sy, Tx) \right] \leq \eta d(Sx, Sy),$$

with some assumptions on contractive-coefficients, i.e.

$$2\beta + \gamma - |\gamma| - \alpha \leq \eta < \alpha + 2\beta + 3\gamma - |\gamma| \text{ and } \beta + \gamma \leq 0.$$

Asadi in [28], using the same iteration (Krasnoselskii) on convex metric spaces, studied fixed points for generalized Hardy-Rogers type-mappings, as follows

$$ad(x,Tx)+bd(y,Ty)+cd(Tx,Ty)+ed(Ty,x)+fd(y,Tx)\leq kd(x,y),$$

where

$$\frac{b+e-|f|(1-\lambda)-|c|\lambda}{1-\lambda}\leq k<\frac{a+b+c+e+f-|c|\lambda-|f|(1-\lambda)}{1-\lambda},$$

with $\lambda\in[0,1]$ is the coefficient of Krasnoselskii's iteration.
Furthermore, Wang and Zhang, in [159] extended the above results for pairs of generalized Hardy-Rogers type contractions.
Now, expansive and expansive-type mappings can be considered a particular case of generalized contractions. Regarding the former ones, we recall some recent development into the study of type of operators.
In 2011, Aage [1] considered expansive mappings in cone metric spaces. The more general form of these mappings, with some underlying assumptions, are

$$d(Tx,Ty)\geq kd(x,y)+ld(x,Tx)+pd(y,Ty),$$

where the above coefficients satisfy

$$K\geq -1,\ p<1,\ l>1 \text{ and } k+l+p>1.$$

Aydi et.al. studied in [29] some fixed point theorems for pairs of expansive mappings for spaces endowed with c-distances. We recall them using the standard notations for metric spaces, i.e.

$$d(Tx,Ty)\geq ad(fx,fy)+bd(Tx,fx)+cd(Ty,fy),$$

with $b<1$, $a\neq 0$, $f(X)\subseteq T(X)$ and $(T(X),d)\subset(X,d)$ complete.
Also, in cone rectangular metric spaces, some fixed point theorems were studied. For example, in [101], pair of mappings satisfying

$$d(fx,fy)\geq \alpha d(gx,gy)+\beta d(fx,gx)+\gamma d(fy,gy)$$

were studied, with some assumptions on the coefficients α,β and γ and on the range of g and f.
These pairs of generalized mappings were extended by Olaoluwa and Olaleru in [98], but in the framework of b-metric spaces and for a pair of four mappings, as follows

$$d(fx,gy)\geq a_1d(Sx,Ty)+a_2d(fx,Sx)+a_3d(gy,Ty)+a_4d(fx,Ty)+a_5d(gy,Sx).$$

Also, for the sake of convenience, we recall other studies in metric-type spaces and for expansive-type mappings, as follows : in [164] generalized mappings were studied on cone rectangular metric spaces using scalarizing, in [129] mappings that satisfy

$$d(Tx,Ty)\leq\varphi(d(x,y))$$

were studied on cone rectangular metric spaces and in [99], fixed point theorems for a general type of expansive mappings were developed, satisfying

$$\phi(d(S^2x,TSy))\geq\frac{1}{3}\left[d(Sx,S^2x)+d(TSy,Sy)+d(Sx,Sy)\right],$$

where ϕ is a functional that satisfies some assumptions, were studied.
Also, in the context of dislocated metric spaces, Daheriya et.al. [45] discussed rational-type expansive mappings, and in [19] Alghamdi studied fixed points for generalized expansive mappings in b-metric like spaces.
The purpose of this section is to present some extensions regarding fixed results for a hybrid class of generalized contractive-type mappings and for some expansive-type operators in the context of b-rectangular metric spaces. Moreover, at the end of the second section, we shall let and open problem.
In [91], Moosaei developed some fixed point theorems for generalized contractions using Krasnoselskii's iterative scheme in the setting of complete convex metric spaces. Nevertheless, this results hold even for Picard iterative scheme. Our purpose of the present section is to extend these theoretical results for Picard

successive approximation scheme in the framework of complete b-rectangular metric spaces. Also, we extend the results of Aage [1] from cone metric spaces to b-rectangular metric spaces. Moreover, we aim to generalize the fixed point results of Patil from [101]. In order to validate our theorems, we use examples similar to thse presented in [1], [69] and [101], respectively. The generalized contractive-type operators that we shall consider in this section are mappings $f : X \to X$ defined on a b-rectangular metric space X, satisfying the following condition :

$$ad(x, fx) + bd(y, fy) + cd(fx, fy) \leq kd(x, y).$$

We will analyze two separate cases : when $c > 0$ and $c < 0$. Also, for expansive-type mappings, i.e. when $c < 0$, we consider two types of sequence, namely the classical Picard iteration $x_{n+1} = fx_n$, for each $n \in \mathbb{N}$ and the 'inverse' Picard iteration, i.e. $x_n = fx_{n+1}$, for each $n \in \mathbb{N}$, where we require that the operator f is onto. Our first result is a theorem for the existence and uniqueness of the fixed point of a mapping satisfying the contractive condition from above. The technique we will used is based on [Lemma 1.3.6].

Theorem 2.2.1. *Let (X, d) be a complete b-rectangular metric space (b-gms), with coefficient $s > 1$. Consider a mapping $f : X \to X$, satisfying the following contractive condition*

$$ad(x, fx) + bd(y, fy) + cd(fx, fy) \leq kd(x, y),$$

where $0 \leq k - b < \dfrac{a + c}{s}$. Also, suppose the following assumptions are satisfied :

$$(A)\ \textit{If } c > 0 \textit{ and } k \geq 0, \textit{ then } \frac{k}{c} < \frac{1}{s},$$

$$(B)\ \textit{If } c > 0 \textit{ and } k \leq 0, \textit{ then we have no additional conditions },$$

$$(C)\ \textit{If } c < 0 \textit{ and } k < 0, \textit{ then } \frac{k}{c} > s^2.$$

Then, the Picard sequence (x_n), defined as $x_{n+1} = fx_n$, for each $n \in \mathbb{N}$ converges to a fixed point of the mapping f.

Relative to [Theorem 2.2.1], we give two examples that validate cases (A) and(C):
From [69], we recall an example of a complete b-rectangular metric space.

Example 2.2.2. Let $X = A \cup B$, where $A = \left\{1/n \middle| n = \overline{2, 5}\right\}$ and $B = [1, 2]$. We define $d : X \times X \to [0, \infty)$, such that $d(x, y) = d(y, x)$ and :

$$d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{3}{100},$$
$$d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{2}{100},$$
$$d\left(\frac{1}{4}, \frac{1}{3}\right) = d\left(\frac{1}{5}, \frac{1}{3}\right) = \frac{6}{100},$$
$$d(x, y) = (x - y)^2, \text{otherwise}.$$

Then (X, d) is a complete b-rectangular metric space, with coefficient $s = 3$. Furthermore, (X, d) is neither a metric space nor a rectangular metric space.

We give the following example of a mapping f that satisfies the assumptions from case (A) of [Theorem 2.2.1].

Example 2.2.3. Let (X, d) be the b-rectangular metric space defined above, with $s = 3$.
Also, define $f : X \to X$, such as

$$f(x) = \begin{cases} \dfrac{1}{3}, & x \in A \\ \\ \dfrac{1}{5}, & x \in B \end{cases}$$

It is easy to observe that f has a unique fixed point $1/3$, and that f satisfies :

$$1 \cdot d(fx, fy) \leq \frac{1}{52}d(x, y) + \frac{1}{4}d(x, fx) + \frac{23}{100}d(y, fy),$$

for each $x, y \in X$.

Now, we present an example of a complete b-rectangular metric space, which will be used further.

Example 2.2.4. Let $X = \{1, 2, 3, 4\}$ and define $d : X \times X \to [0, \infty)$, such as

$$\begin{aligned} d(1,2) &= d(2,1) = \frac{6}{10} \\ d(1,3) &= d(3,1) = \frac{1}{10} \\ d(2,3) &= d(3,2) = \frac{1}{10} \\ d(1,4) &= d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = \frac{2}{10} \end{aligned}$$

Then (X, d) is a b-rectangular metric space with coefficient $s = 3/2$, which is not a rectangular metric space.

We further give an example of a self-mapping f that satisfies the assumptions from Case (C) of [Theorem 2.2.1].

Example 2.2.5. Consider $X = \{1, 2, 3, 4\}$ to be the b-rectangular metric space defined above, with coefficient $s = \frac{3}{2}$.

Let $f(x) = \begin{cases} 3, & x \neq 4 \\ 1, & x = 4 \end{cases}$ a self-mapping defined on X.

Then f satifies

$$d(fx, fy) \geq (-3)d(x, y) - 5d(x, fx) + 3d(y, fy)$$

and also the conditions from *case (C)* of [Theorem 2.2.1].

Remark 2.2.6. We observe that the contractive condition when $c > 0$, can be written as :

$$d(fx, fy) \leq \frac{k}{c} d(x, y) - \frac{a}{c} d(x, fx) - \frac{b}{c} d(y, fy),$$

for each $x, y \in X$.
Taking $k > 0, a < 0$ and $b < 0$, it follows that the operator f is Reich-type, so the above theorem (when $k > 0$) is similar with the results of [47].

Now, we present an useful lemma for expansive-type mappings in b-rectangular metric spaces, following the technique in [98].

Lemma 2.2.7. *Let (X, d) a b-rectangular metric space. Also, consider $\lambda \in \mathbb{R}$ and x, y, z, w arbitrary elements of X, each distinct from each other. Then*

$$\begin{aligned} \lambda d(x,z) \geq & \left[\frac{1+s^2}{2s}\lambda + \frac{1-s^2}{2s}|\lambda|\right] d(x,y) + \left[\frac{s-1}{2}\lambda - \frac{s+1}{2}|\lambda|\right] d(z,w) \\ & + \left[\frac{s-1}{2}\lambda - \frac{s+1}{2}|\lambda|\right] d(w,y). \end{aligned}$$

For expansive-type mappings, i.e. when $c < 0$, we make the following important remark.

Remark 2.2.8. We studied contraction-type mappings, that satisfy

$$\begin{aligned} ad(x, fx) + bd(y, fy) + cd(fx, fy) &\leq kd(x, y) \\ cd(fx, fy) &\leq kd(x, y) - ad(x, fx) - bd(y, fy) \\ d(fx, fy) &\geq \frac{k}{c} d(x, y) - \frac{a}{c} d(x, fx) - \frac{b}{c} d(y, fy) \end{aligned}$$

By some substitutions we can make the mapping f satisfies

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy)$$

where

$$\begin{cases} \alpha = k/c \\ \beta = -a/c \\ \gamma = -b/c \end{cases}$$

We will analyze the cases when $\alpha \leq 0$ and $\alpha \geq 0$, so, when $k \geq 0, c < 0$, respectively $k \leq 0, c < 0$.

The second important result of the present section involves the study of the convergence for the Picard iterative scheme of expansive-type mappings in the framework of b-gms.

Theorem 2.2.9. *Let (X, d) a complete b-rectangular metric space, with coefficient $s \geq 1$. Also, consider $f : X \to X$ a mapping satisfying*

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy),$$

for each $x, y \in X$. Moreover, suppose the following conditions are satisfied :

$$(i) \beta < 1 - s,\ \gamma > s,\ \alpha + \gamma < \frac{1-\beta}{s},$$

$$(ii)\ \textit{If } \alpha > \gamma,\ \textit{then we have the additional assumptions } \alpha + 1 < \gamma\left(1 + \frac{1}{s}\right).$$

If $\alpha < \gamma$, then we have the additional assumptions :

$$\alpha > 1 \textit{ and } 1 - \alpha < \left(\frac{1}{s} - 1\right).$$

Then, the mapping f has a fixed point.

Concerning [Theorem 2.2.9], we are ready to present an example that justifies our theoretical result.

Example 2.2.10. Let (X, d), with $X = \{1, 2, 3, 4\}$ be the b-rectangular metric space, endowed with the b-rectangular metric from [Example 2.2.2]. Consider f, defined as : $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(4) = 4$. It is obviously that f has as a unique fixed point the element $4 \in X$. Then, f satisfies

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy)\ ,$$

with $\alpha = \frac{9}{50}$, $\beta = -\frac{101}{5}$ and $\gamma = \frac{17}{100}$.

Now, we recall Lemma 2 from [33], that is crucial for inequalities involving difference inequalities.

Lemma 2.2.11. *Let (a_n) and (b_n) be two sequences of nonnegative real numbers, such that*

$$a_{n+1} \leq \alpha_1 a_n + \alpha_2 a_{n-1} + ... + \alpha_k a_{n-k+1} + b_n,$$

where $n \geq k - 1$. If $\alpha_1, ..., \alpha_k \in [0, 1)$, $\sum\limits_{i=1}^{k} \alpha_i < 1$ and $\lim\limits_{n\to\infty} b_n = 0$, then it follows that $\lim\limits_{n\to\infty} a_n = 0$.

Remark 2.2.12. In the proof of the previous result, we have that the following estimation is valid

$$d_n^* = d(x_{n+2}, x_n) \leq a_2^n d_0^* + \frac{\delta^n - a_2^n}{\delta - a_2} a_1 D_0.$$

So, based on this lemma, we give a nonconstructive approach for evaluating (x_n) as a Cauchy sequence
In the above lemma, let's take $k = 1$. Then, we get that $a_{n+1} \leq \alpha_1 a_n + b_n$, with $\alpha_1 \in [0, 1)$ and $\lim\limits_{n\to\infty} b_n = 0$. Then $\lim\limits_{n\to\infty} a_n = 0$. Now, we have proved that

$$d_n^* \leq a_2 d_{n-1}^* + a_1 \delta^{n-1} D_0.$$

Let's define the following : $\alpha_1 := a_2$ and $b_n := a_1 D_0 \delta^{n-1}$. Since $\delta < \frac{1}{s} < 1$ and $a_2 \in [0, 1)$, then apply Lemma 2 from [33] with the particular case when $k = 1$, we get that $\lim\limits_{n\to\infty} d_n^* = 0$.

Now, we give a theoretical result for expansive-type mappings under the new assumption such that the mapping f is onto and we shall use the 'inverse' Picard iterative process.

Theorem 2.2.13. *Let (X, d) be a complete b-rectangular metric space and $f : X \to X$ a mapping satisfying*

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy)\ .$$

Let f continuous and onto. Suppose that

$$(i)\beta < 1,\ \alpha + \gamma > 0 \text{ and } 1 - \beta < \frac{\alpha + \gamma}{s}.$$

Also, suppose the following additional assumptions :
Case (E1), i.e. $\alpha > 0$: Suppose that the following assumptions are satisfied:

$$(ii)\alpha > 1$$

Case (E2), i.e. $\alpha < 0$: Suppose the following assumptions are satisfied :

$$\begin{cases} (ii)\alpha < -1,\ \gamma > 0 \\ (iii)s\left(1 - \dfrac{\alpha}{\gamma}\right) < 1 + \dfrac{1}{\alpha} \end{cases}$$

Then, the mapping f has a fixed point in X.

The following example that we shall consider is that of a complete b-gms, that is also a b-rectangular metric space. In this manner, we validate [Theorem 2.2.13] through another example, showing that the assumptions from our theoretical result are easily fulfilled.

Example 2.2.14. Let $X = [0, \infty)$, endowed with $d : X \times X \to \mathbb{R}_+$, such that $d(x, y) = (x - y)^2$, for each $x, y \in X$. Then (X, d) is a complete b-metric space, with coefficient $s = 2$. Then, it is also a complete b-rectangular metric space, with coefficient $s = 4$.

Example 2.2.15. Let $X = [0, \infty)$, where d is the above b-rectangular metric, with $s = 4$.
Define $f : X \to X$ as $f(x) = \dfrac{x + \delta_1}{\delta_2}$, with $\delta_1, \delta_2 \geq 0$. It is easy to see that f is continuous. Also, for each $y \in X$, there exists $x = y\delta_2 - \delta_1 \geq 0$,, since δ_1 and δ_2 are positive, so f is onto. Moreover :

$$d(fx, fy) = (fx - fy)^2 = \Big|\frac{x + \delta_1}{\delta_2} - \frac{y + \delta_1}{\delta_2}\Big| = \frac{1}{\delta_2}|x - y|^2 = \frac{1}{\delta_2}d(x, y).$$

Let's take $\beta = 0$, $\gamma = 0$ and $\alpha = 10$. Also, let $\delta < \dfrac{1}{s}$, i.e. $\delta_2 < \dfrac{1}{4}$. For example : $\delta_2 = \dfrac{1}{10}$ and $\delta_1 = 1$. Then f satisfies $d(fx, fy) \geq 10d(x, y)$, for each $x, y \in X$.

2.3 Contraction sequences in cone metric spaces over Banach algebras. Applications to nonlinear systems of equations and systems of differential equations

In this section we deal with sequences of contractions with respect to some fixed point results in the setting of cone metric spaces over Banach algebras. First of all, we extend some concepts regarding continuity properties, equicontinuity of a family of mappings, (H)-convergence and (G)-convergence to the framework of cone metric spaces over Banach algebras. Finally, we validate the theoretical significance of these new concepts through interesting applications to nonlinear systems of equations and systems of differential equations, that are in deep connection with the idea of f sequences of contractions.
From now on, by $\mathcal{A}$ we will denote an ordinary Banach algebra. Also, P will represent the underlying solid cone over $\mathcal{A}$. Moreover, the first step is to extended the definition of equicontinuity of a family of operators and the pointwise and uniform convergence, respectively. First we introduce the concept of pointwise convergence in the framework of a cone metric space over $\mathcal{A}$.
Now, since we have reminded the basic concepts crucially important in our fixed point analysis, we make

the following remark that in [Theorem 2] from [31] and in [Theorem 1] from [97], the authors considered the contractions to be defined on a metric space and on subset of a metric space, respectively. Moreover, they have supposed that the contractions have at least a fixed point. On the other hand, M. Păcurar in [105] considered that the almost contractions were defined on a complete metric space and, in this case, each of them have a unique fixed point. For this, see This means that in our case it is of no importance if we consider or not the completeness of the cone metric space over the given Banach algebra. Similarly, in [Theorem 2] of Nadler's article, that author considered the pointwise convergence of a sequence of fixed points under the assumption that the contractions are defined on a locally compact metric space (X, d). Additionally, in [105], M. Păcurar extended this result for the case of almost contractions that are defined on a complete metric space, because these mappings are not continuous so it is not properly to talk about the equicontinuity of a family of almost contractions. For this, see the observation made by M. Păcurar before Theorem 2.6 in [105]. So, in our framework of a cone metric space over a Banach algebra, it is of no loss to employ the analysis of M. Păcurar when dealing with the completeness of such a space. Finally, for other interesting results concerning the stability of fixed points in 2-metric spaces, stability of fixed points for sequences of (ψ, ϕ)-weakly contractive mappings and mappings defined on an usual metric space, we let the reader follow [89], [90] and [154], respectively.
Now, it is time to shift our focus to some articles regarding fixed point results in the setting of cone metric spaces over Banach algebras. H. Liu and S. Xu [85] introduced the concept of cone metric spaces with Banach algebras in order to study fixed point results, replacing Banach spaces by Banach algebras. Furthermore, S. Xu and S. Radenović [160] considered mappings defined on cone metric spaces over Banach algebras but one solid cones, without the usual assumption of normality. An interesting generalization was made by H. Huang and S. Radenović [59], considering cone b-metric spaces over Banach algebras. They have studied common fixed points of generalized Lipschitz mappings. Also, P. Yan et. al. [163] developed coupled fixed point theorems for mappings in the setting of cone metric spaces. Finally, the idea of replacing the Banach space by a Banach algebra was motivated by [66] and [73] in which some remarks about the connection between fixed point theorems for different mappings and in the case of usual normal cones of Banach spaces and usual metric spaces was given.
Now, we are ready to review some necessary concepts and theorems regarding cone metric spaces over Banach algebras.

Definition 2.3.1. *Let (X, d) be a cone metric space over the Banach algebra $\mathcal{A}$. Also, let $T : X \to X$ and $T_n : X \to X$ be some given mappings for each $n \in \mathbb{N}$. By definition, the sequence $(T_n)_{n\in\mathbb{N}}$ converges pointwise to T on X, briefly $T_n \xrightarrow{p} T$, if for each $c \gg \theta$, $c \in \mathcal{A}$ and for every $x \in (X, d)$, there exists $N > 0$ that depends on c and x, such that for each $n \geq N$, we have that $d(T_n x, Tx) \ll c$.*

In a similar way, the particular notion of uniform convergence of a sequence of mappings can be constructed as follows.

Definition 2.3.2. *Let (X, d) be a cone metric space over the Banach algebra $\mathcal{A}$. Also, let $T : X \to X$ and $T_n : X \to X$ be some given mappings for each $n \in \mathbb{N}$. By definition, the sequence $(T_n)_{n\in\mathbb{N}}$ converges uniformly to T on X, briefly $T_n \xrightarrow{u} T$, if for each $c \gg \theta$, $c \in \mathcal{A}$, there exists $N > 0$ that depedens only on c, such that for each $n \geq N$ and for every $x \in (X, d)$, one has the following : $d(T_n x, Tx) \ll c$.*

At the same time, for a given family of operators we define the concepts of equicontinuity and uniformly equicontinuity on a cone metric space over a given Banach algebra $\mathcal{A}$.

Definition 2.3.3. *Let (X, d) be a cone metric space over the Banach algebra $\mathcal{A}$ and $T_n : X \to X$ be some given mappings, for every $n \in \mathbb{N}$. The family $(T_n)_{n\in\mathbb{N}}$ is called equicontinuous if and only if for every $c_1 \gg \theta$, $c_1 \in \mathcal{A}$ and for each $x \in (X, d)$, there exists $c_2 \gg \theta$, $c_2 \in \mathcal{A}$ that depends on c_1 and x, such that for every $y \in (X, d)$ satisfying $d(x, y) \ll c_2$, one has that $d(T_n x, T_n y) \ll c_1$, for every $n \in \mathbb{N}$.*

Definition 2.3.4. *Let (X, d) be a cone metric space over the Banach algebra $\mathcal{A}$ and $T_n : X \to X$ be some given mappings, for every $n \in \mathbb{N}$. The family $(T_n)_{n\in\mathbb{N}}$ is called uniformly equicontinuous if and only if for every $c_1 \gg \theta$, $c_1 \in \mathcal{A}$, there exists $c_2 \gg \theta$, $c_2 \in \mathcal{A}$ that depends only on c_1, such that for every x and y in (X, d) with $d(x, y) \ll c_2$, one has that $d(T_n x, T_n y) \ll c_1$, for every $n \in \mathbb{N}$.*

Inspired by [Example 2.17] of [59] in which the authors presented a complete cone b-metric space over a Banach algebra with coefficient $s = 2$, we are ready to present a modified version in which we have a complete metric space over a Banach algebra.

Example 2.3.5. Let's consider $\mathcal{A}$ to be set of all the matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$, where α and β are from $\mathbb{R}$. On $\mathcal{A}$, we define a norm $\|\cdot\|$, such as for every matrix from $\mathcal{A}$, one has that $\left\| \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \right\| = |\alpha| + |\beta|$. Also, on $\mathcal{A}$ we have the usual matrix multiplication. Moreover, one can see that $P = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} / \alpha, \beta \geq 0 \right\}$ is a nonempty solid cone on $\mathcal{A}$. Finally, one can easily show that $\mathcal{A}$ is a Banach algebra.

Now, based on the [Example 2.3.5], we shall present also an example, in which we have the uniform convergence of a sequence of mappings defined on the previous cone metric space (X, d) over the Banach algebra $\mathcal{A}$ given above.
Moreover, from now on we specify that the notation $\lim\limits_{\substack{n\to\infty \\ (\mathcal{A})}} x_n = \theta$ means the convergence under the Banach algebra $\mathcal{A}$, i.e. $(x_n)_{n\in\mathbb{N}}$ is a given sequence that satisfies the fact that is a c-sequence. Furthermore, for a real given sequence $(y_n)_{n\in\mathbb{N}}$ that converges to a real number y, we denote $\lim\limits_{\substack{n\to\infty \\ (\mathbb{R})}} y_n = y$. Finally, we make the observation that if we work with sequences of mappings, the latter covergence can be understood pointwise or uniformly, depending on the given context.

Example 2.3.6. For every $n \in \mathbb{N}$, let $f_n : [0,1] \to [0,1]$, such as $f_n(x) = \dfrac{x}{n}$, for each $x \in [0,1]$. Then, $f_n \xrightarrow{u} f$ with respect to the cone metric d from [Example 2.3.5].

From [83], we recall an example of a cone metric space over a Banach algebra, which will be used further in this paper.

Example 2.3.7. Let $\mathcal{A} = \mathbb{R}^2$. Then $\mathcal{A}$ is a Banach algebra, with the norm given by $\|(u_1, u_2)\| = |u_1| + |u_2|$, for any arbitrary element (u_1, u_2) of $\mathcal{A}$. Moreover, we have the multiplication $u \cdot v = (u_1v_1, u_1v_2 + u_2v_1)$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are given elements. Also $P = \{u = (u_1, u_2) \,/\, u_1, u_2 \geq 0\}$ is a solid cone over $\mathbb{R}^2$. Taking $\tilde{X} = \mathbb{R}^2$, we can define the operator $d : \tilde{X} \times \tilde{X} \to \mathcal{A}$, by $d(x, y) = (|x_1 - y_1|, |x_2 - y_2|)$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then $(\tilde{X}, d)$ is a cone metric space over $\mathbb{R}^2$.

We mention that if we take $X = [0, 1) \times [0, 1) \subset \tilde{X}$, then it is easy to see that (X, d) is also a cone metric space over $\mathbb{R}^2$, where d is defined in [Example 2.3.7]. Also, based on the previous example, we shall present a sequence of mappings that converges pointwise and does not converge uniformly toward the null mapping, with respect to the cone metric d.

Example 2.3.8. For every $n \in \mathbb{N}$, let $T_n : [0, 1) \times [0, 1) \to [0, 1) \times [0, 1)$, defined as $T_n(x) = (x_1^{n^2}, x_2^n)$, where $x = (x_1, x_2) \in [0, 1)^2$. Also, we consider the null operator T, i.e. $T(x) = (0, 0)$, where $x \in [0, 1)^2$ and $T : [0, 1) \times [0, 1) \to \{0\} \times \{0\} \subset [0, 1) \times [0, 1)$. Then, $T_n \xrightarrow{p} T$, but $T_n \not\xrightarrow{u} T$.

We present our main result of the present section that consists of the convergence of a sequence of operators in the setting of a cone metric space of a given Banach algebra $\mathcal{A}$. The theoretical result that we will present concerns the pointwise and uniform convergence, respectively.

Theorem 2.3.9. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$. Also, consider $T_n, T : X \to X$, for each $n \in \mathbb{N}$ such that they satisfy the following assumptions :*

(i) for every $n \in \mathbb{N}, T_n$ has at least a fixed point, i.e. there exists $x_n \in T_n x_n$,

(ii) the operator T is an $\alpha -$ contraction with respect to the cone metric d, i.e. there exists $\alpha \in P$, with $\rho(\alpha) < 1$, such that $d(Tx, Ty) \preceq \alpha d(x, y)$, for all $x, y \in X$,

$(iii) T_n \xrightarrow{u} T$ as $n \to \infty$, with respect to the cone metric,

$(iv)(X, d)$ is a complete cone over the Banach algebra $\mathcal{A}$.

Then, following the fact that x^* is the unique fixed point of the operator T, we have that $(d(x_n, x^*))_{n\in\mathbb{N}}$ is a c-sequence.

Now we are ready to present our second result concerning the pointwise convergence of a sequence of operators with respect to a given Banach algebra $\mathcal{A}$.

Theorem 2.3.10. *Let (X,d) be a cone metric space over a Banach algebra $\mathcal{A}$. Also, consider $T_n, T : X \to X$, for each $n \in \mathbb{N}$ such that they satisfy the following assumptions :*

(i) *the operator T_n is an $\alpha -$ contraction with respect to the cone metric d, i.e. there exists $\alpha \in P$, with $\rho(\alpha) < 1$, such that $d(T_n x, T_n y) \preceq \alpha d(x,y)$, for all $x, y \in (X,d)$ and $n \in \mathbb{N}$,*

(ii) *the operator T is an $\alpha_0 -$ contraction with respect to the cone metric d, i.e. there exists $\alpha_0 \in P$ with $\rho(\alpha_0) < 1$, such that $d(Tx, Ty) \preceq \alpha_0 d(x,y)$, for all $x, y \in X$,*

$(iii) T_n \xrightarrow{p} T$ *as* $n \to \infty$,

$(iv)(X,d)$ *is a complete cone over the Banach algebra $\mathcal{A}$.*

Then, following the fact that x_n^* are the unique fixed points of the operators T_n, we have that $(d(x_n^*, x^*))_{n\in\mathbb{N}}$ is a c-sequence.

Now, as in [Theorem 2.3.9], one can observe that we can use an equivalent definition of pointwise convergence using non-strict inequalities, and this does not influence the obtained results.

Our further purpose of this section is to extend the concepts of (G)-convergence and (H)-convergence from [31] to the setting of cone metric spaces over Banach algebras. Also, we extend this notions for the case of sequences of operators that admit different domains. First and foremost, we consider the idea of pointwise convergence for operators that do not have the same domain of definition.

Definition 2.3.11. *Let X_n, X_∞ be subsets of X, where (X,d) is a cone metric space (not necessarily complete) over a given Banach algebra $\mathcal{A}$. Also, let's consider for each $n \in \mathbb{N}$ some operators $T_n : X_n \to X$ and $T_\infty : X_\infty \to X$. By definition, T_∞ is a (G)-limit of the sequence $(T_n)_{n\in\mathbb{N}}$, when the family of mappings $(T_n)_{n\in\mathbb{N}}$ satisfies the following property :*

$$(G): \text{ for each } x \in X_\infty, \text{ there exists a sequence } (x_n)_{n\in\mathbb{N}}, \text{ with } x_n \in X_n \ (n \in \mathbb{N}), \text{ such that :}$$
$$(d(x_n, x))_{n\in\mathbb{N}} \text{ is a c-sequence and } (d(T_n x_n, T_\infty x))_{n\in\mathbb{N}} \text{ is also a c-sequence.}$$

Now, the next definition of the present section concerns a generalization of the uniform convergence, but for mappings that do not have the same domain.

Definition 2.3.12. *Let X_n, X_∞ be subsets of X, where (X,d) is a cone metric space (not necessarily complete) over a given Banach algebra $\mathcal{A}$. Also, let's consider for each $n \in \mathbb{N}$ some operators $T_n : X_n \to X$ and $T_\infty : X_\infty \to X$. By definition, T_∞ is a (H)-limit of the sequence $(T_n)_{n\in\mathbb{N}}$, when the family of mappings $(T_n)_{n\in\mathbb{N}}$ satisfies the following property :*

$$(H): \text{ for each sequence } (x_n)_{n\in\mathbb{N}}, \text{ with } x_n \in X_n, \text{ for every } n \in \mathbb{N},$$
$$\text{there exists a sequence } (y_n)_{n\in\mathbb{N}} \subset X_\infty, \text{ such that :}$$
$$(d(x_n, y_n))_{n\in\mathbb{N}} \text{ is a c-sequence and } (d(T_n x_n, T_\infty y_n))_{n\in\mathbb{N}} \text{ is also a c-sequence.}$$

Further, we show that the (H)-limit of a sequence of operators is also a (G)-limit, under suitable circumstances. Moreover, since we need the idea of continuity of an operator, we can employ two definitions : an extension of the definition of continuity from the case of metric spaces to the case of cone metric spaces over Banach algebras and the second one the idea of sequential continuity (for this see (iii) of Definition 2.1 from [83]). These are given as follows.

Remark 2.3.13. If (X,d) is a cone metric space over a Banach algebra $\mathcal{A}$, then :
a) An operator T is continuous in $x_0 \in (X,d)$ if and only if for each $c \gg \theta$, $c \in \mathcal{A}$, there exists $\bar{c} \in P$ that depends on c, such that for every $x \in (X,d)$, satisfying $d(x,x_0) \ll \bar{c}$, one has that $d(T(x), T(x_0)) \ll c$. Moreover, the operator T is continuous if it is continuous at every point of it's domain.
b) An operator T is sequential continuous if for every sequence $(y_n)_{n\in\mathbb{N}}$ convergent to $x \in X$, i.e. satisfying $(d(y_n, x))_{n\in\mathbb{N}}$ is a c-sequence, then $(d(Ty_n, Tx))_{n\in\mathbb{N}}$ is also a c-sequence.

Proposition 2.3.14. *Let (X,d) be a cone metric space over a given Banach algebra $\mathcal{A}$. Also, for each $n \in \mathbb{N}$, let X_n be some nonempty subsets of X. Also, consider another nonempty subset of X, namely X_∞.*

Furthermore, suppose that the following conditions are satisfied :

(i) *if* $x \in X_\infty$, *then there exists* $(x_n)_{n\in\mathbb{N}}$, *with* $x_n \in X_n$ *for every* $n \in \mathbb{N}$, *such that* $(d(x_n, x))_{n\in\mathbb{N}}$ *is a c-sequence,*
(ii) $T_\infty : X_\infty \to X$ *is sequential continuous,*
(iii) T_∞ *is a (H)-limit for the family* (T_n).

Then, T_∞ *is a (G)-limit for the family* (T_n).

The following result is based upon the question : under what conditions the (G)-limit of a sequence of operators is unique? We give an answer in the theorem from below.

Theorem 2.3.15. *Let* (X, d) *be a cone metric space over a given Banach algebra* $\mathcal{A}$. *Also, consider* X_n *(for every* $n \in \mathbb{N}$*) and* X_∞ *be some nonempty subsets of* X. *Suppose that the following assumptions are satisfied:*

(i) *for all* $n \in \mathbb{N}$, *let* T_n *to be a k-Lipschitz with respect to the Banach algebra* $\mathcal{A}$, *i.e. there exists* $k \in P$, *such that* $d(T_n(x), T_n(y)) \preceq k \cdot d(x, y)$, *for each* $x, y \in X_n$
(ii) $T_\infty : X_\infty \to X$ *is a (G)-limit for the family* (T_n).

Then, T_∞ *is the unique (G)-limit on* X_∞.

Our third result from this section concerns the convergence of a sequence of fixed points of a family of mappings that has property (G), with respect to a given Banach algebra.

Theorem 2.3.16. *Let* (X, d) *be a cone metric space over a given Banach algebra* $\mathcal{A}$. *Also, consider* X_n *(for* $n \in \mathbb{N}$*) and* X_∞ *to be some nonempty subsets of* X. *Also, consider some mappings* $T_n : X_n \to X$ *and* $T_\infty : X_\infty \to X$ *that satisfy the following assumptions :*

(i) *for each* $n \in \mathbb{N}$, T_n *is a k-contraction, i.e. there exists* $k \in P$ *with* $\rho(k) < 1$, *such that* $d(T_n(x), T_n(y)) \preceq k \cdot d(x, y)$, *for each* $x, y \in X_n$,
(ii) *the family* (T_n) *has property (G),*
(iii) *there exists* $x_\infty \in F_{T_\infty}$, *i.e.* x_∞ *is a fixed point of* T_∞.

Then, $(d(x_n, x_\infty))_{n\in\mathbb{N}}$ *is a c-sequence.*

Remark 2.3.17. In [Theorem 2.3.16] we have supposed that indeed there exists $x_n \in F_{T_n}$. An alternative way is to suppose that (X, d) is a complete cone metric space over $\mathcal{A}$ and after that one can establish a local variant of existence and uniqueness of fixed points for the mappings T_n, since they are contractions with respect to the cone metric, but not on the whole metric space.

Now, it is time to present a consequence of [Theorem 2.3.16] in which we refer to the connection between the pointwise convergence of a sequence of self-mappings and the (G)-property of the same sequence.

Corollary 2.3.18. *Let* (X, d) *be a cone metric space over a Banach algebra* $\mathcal{A}$. *Also, consider* $T_n, T_\infty : X \to X$ *some given mappings. Suppose the following assumptions are satisfied :*

(i) $T_n \xrightarrow{p} T_\infty$ *as* $n \to \infty$,
(ii) T_n *is a k-contraction with respect to the cone metric, for eac* $n \in \mathbb{N}$,
(iii) *there exists* $x_n \in F_{T_n}$ *and* $x_\infty \in F_{T_\infty}$.

Then, $(T_n)_{n\in\mathbb{N}}$ *has the property* (G), *with* T_∞ *as the* (G)*-limit.*

Now, we shall present a theorem in which we are concerned with the relationship between the pointwise convergence of a sequence of mappings in the setting of cone metric spaces and the equicontinuity of the family of mappings.

Theorem 2.3.19. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and M be a nonempty subset of X. Furthermore, let $T_n : M \to X$ be a given operator such that the family (T_n) has the (G) property with the (G)-limit T_∞. Also, let's suppose that the following conditions are satisfied :*

$$(i)\ \ \text{the family } (T_n) \text{ is equicontinuous on } M,$$
$$(ii)\ \text{there exists } x_n \in F_{T_n},\ \text{for each } n \in \mathbb{N} \text{ and } x_\infty \in F_{T_\infty}.$$

Then $T_n \xrightarrow{p} T_\infty$.

Now, the next theorem of this section is an existence result for the fixed points of the (G)-limit mapping of a sequence of contractions with respect to the cone metric space over a given Banach algebra.

Theorem 2.3.20. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$. Also, consider X_n and X_∞ some given nonempty subsets of X. Let $T_n : X_n \to X$ and $T_\infty : X_\infty \to X$ be some mappings that satisfy :*

$$(i)\ \ \text{the family } (T_n) \text{ has property (G) with the (G)-limit } T_\infty,$$
$$(ii)\ T_n \text{ are } k\text{-contractions in the sense of the given cone metric},$$
$$(iii)\ \ \text{there exists } x_n \in F_{T_n}.$$

Then, there exists $x_\infty \in F_{T_\infty}$ if and only if the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in X_∞ in the sense of the Banach algebra (i.e. there exists $y \in X_\infty$ such that $(d(x_n, y))_{n\in\mathbb{N}}$ is a c-sequence).

Now, we are ready to present our last two theoretical results from the present section regarding the link between the uniform convergence with respect to the cone metric and the (H)-property of a given sequence of operators.

Theorem 2.3.21. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$. Also, consider $M \subset X$ a nonempty set. Let $T_n, T_\infty : M \to X$ some given mappings.*
a) If $T_n \xrightarrow{u} T_\infty$, then T_∞ is the (H)-limit of the family (T_n).
b) If T_∞ is the (H)-limit of (T_n) and if T_∞ is uniformly continuous on M, then $T_n \xrightarrow{u} T_\infty$.

Now, our last theorem of this section concerns the convergence of a sequence of fixed points of a family of mappings to the fixed point of the (H)-limit of the same family of operators.

Theorem 2.3.22. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$. Consider X_n for each $n \in \mathbb{N}$ and X_∞ to be some nonempty subset of X. Also, let $T_n : X_n \to X$ and $T_\infty : X_\infty \to X$ be some mappings that satisfy the following assumptions :*

$$(i)\ x_n \in F_{T_n},$$
$$(ii)\ (T_n) \text{ has the property (H) with the (H)-limit } T_\infty,$$
$$(iii)\ T_\infty \text{ is a } k_\infty - \text{contraction with respect to the cone metric } d,$$
$$(iv)\ \ \text{there exists and is unique } x_\infty \in F_{T_\infty}.$$

Then $(d(x_n, x_\infty))_{n\in\mathbb{N}}$ is a c-sequence.

At last, we are ready to give a crucial remark regarding the assumption (iv) of [Theorem 2.3.22].

Remark 2.3.23. One can omit condition (iv) from the previous theorem if we suppose that (X, d) is a complete cone over the Banach algebra $\mathcal{A}$. With this assumption, since T_∞ is a contraction on a subset of the space in the sense of the cone metric d, then one can prove a local variant principle in which the operator has a unique fixed point.

In the following sequel we justify all of our theoretical results through some applications with respect to functional coupled equations and systems of differential equations, respectively. Also, we show that some of our fixed point results represent a viable tool in order to study the convergence of the unique solution of different types of sequences concerning families of operators for nonlinear equations and differential equations. By [Theorem 3.1] of [59], we present the convergence of solutions of some coupled equations, based on the idea of a cone metric space over an appropriately chosen Banach algebra.

Theorem 2.3.24. *Let $F_n, G_n, \tilde{F}, \tilde{G} : \mathbb{R}^2 \to \mathbb{R}^2$ be some given mappings (for $n \in \mathbb{N}$). Also, consider the following systems of coupled functional equations :*

$$\begin{cases} F_n(x,y) = 0 \\ G_n(x,y) = 0 \end{cases} , \quad \text{with } (x,y) \in \mathbb{R}^2 , \tag{2.3.0.1}$$

and

$$\begin{cases} \tilde{F}(x,y) = 0 \\ \tilde{G}(x,y) = 0 \end{cases} , \quad \text{with } (x,y) \in \mathbb{R}^2. \tag{2.3.0.2}$$

Suppose that the mappings F_n, G_n, $\tilde{F}$ and $\tilde{G}$ satisfy the following assumptions :
(1) There exists $M > 0$, such as for $n \in \mathbb{N}$, there exists $L_n > 0$ satisfying $\max\limits_{n\geq 1} L_n \leq M < 1$, such that

$$\begin{cases} |F_n(x_1,y_1) - F_n(x_2,y_2) + x_1 - x_2| \leq L_n|x_1 - x_2| \\ |G_n(x_1,y_1) - G_n(x_2,y_2) + y_1 - y_2| \leq L_n|y_1 - y_2| \end{cases}$$

where (x_1,x_2) and (y_1,y_2) are from $\mathbb{R}^2$.
(2) There exists $\tilde{L} \in (0,1)$, such that

$$\begin{cases} |\tilde{F}(x_1,y_1) - \tilde{F}(x_2,y_2) + x_1 - x_2| \leq \tilde{L}|x_1 - x_2| \\ |\tilde{G}(x_1,y_1) - \tilde{G}(x_2,y_2) + y_1 - y_2| \leq \tilde{L}|y_1 - y_2| \end{cases}$$

where (x_1,x_2) and (y_1,y_2) are from $\mathbb{R}^2$.
(3) The sequence $(F_n)_{n\in\mathbb{N}}$ converges pointwise to $\tilde{F}$ and $(G_n)_{n\in\mathbb{N}}$ also converges pointwise to $\tilde{G}$ in the classical sense, i.e. :

$$\begin{cases} F_n \xrightarrow{p} \tilde{F} \\ G_n \xrightarrow{p} \tilde{G} \end{cases} , \quad \text{i.e.} \quad \begin{cases} \lim\limits_{n\to\infty} F_n(x) = \tilde{F}(x) \\ \lim\limits_{n\to\infty} G_n(x) = \tilde{G}(x) \end{cases} , \quad \text{for each } x \in \mathbb{R}^2.$$

Then x_n converges to $\tilde{x}$ and y_n converges to $\tilde{y}$, where (x_n, y_n) is the unique solution of 2.3.0.1 and $(\tilde{x}, \tilde{y})$ is the unique solution of 2.3.0.2.

Now, our next result is linked to an application to systems of differential equations. In fact, using the results obtained by us and the idea of a Banach algebra, we recall the well-known existence and uniqueness theorem that is based upon an alternative proof, concerning the solution of a nonlinear system of differential equations.

Theorem 2.3.25. *Let $D \subset \mathbb{R}^3$ and $(\alpha, \beta, \gamma) \in D$. Also, consider $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ sufficiently small such that the compact set $\Delta := \{(x,y,z) \ / \ |x - a| \leq \bar{a},\ |y - \beta| \leq \bar{\beta},\ |z - \gamma| \leq \bar{\gamma}\}$ is a subset of D.*
Consider the following nonlinear system of differential equations :

$$\begin{cases} y'(x) = f(x, y(x), z(x)) \\ z'(x) = g(x, y(x), z(x)) \\ y(a) = \beta \\ z(a) = \gamma \end{cases} , \quad \text{where } x \in I := [a - \bar{a}, a + \bar{a}]. \tag{2.3.0.3}$$

Also, suppose the following assumptions are satisfied :

(1) *the mappings f and g are continuous on D,*
(2) *there exists $L_1, L_2 > 0$, such that*

$$\begin{cases} |f(x,y,z) - f(x,\bar{y},\bar{z})| \leq L_1|y - \bar{y}| \\ |g(x,y,z) - g(x,\bar{y},\bar{z})| \leq L_2|z - \bar{z}| \end{cases} , \quad \text{for every } (x,y,z) \text{ and } (x,\bar{y},\bar{z}) \in D.$$

Then, there exists a unique solution for the nonlinear differential system 2.3.0.3 on $I = [a - \bar{a}, a + \bar{a}]$.

Finally, we present our last result of this section regarding the convergence of a sequence of solutions for a family of nonlinear differential systems. Furthermore, the following theorem is crucial, in the sense that we extend the application of S.B. Nadler Jr. from [97].

Theorem 2.3.26. *Let D, Δ and I as in [Theorem 2.3.25]. Consider the following nonlinear systems of differential equations :*

$$\begin{cases} y'(x) = f_n(x, y(x), z(x)) \\ z'(x) = g_n(x, y(x), z(x)) \\ y(a) = \beta \\ z(a) = \gamma \end{cases}, \quad \textit{for each } n \geq 1 \textit{ and } x \in I. \tag{2.3.0.4}$$

Furthermore, consider another system of differential equations :

$$\begin{cases} y'(x) = f(x, y(x), z(x)) \\ z'(x) = g(x, y(x), z(x)) \\ y(a) = \beta \\ z(a) = \gamma \end{cases}, \tag{2.3.0.5}$$

where the functions f_n, g_n, f and g are continuous on D. Moreover, suppose the following assumptions are satisfied :

(1) *there exists $M \in (0,1)$, such that for all $n \in \mathbb{N}$ there exists $k_n, h_n, k, h \geq 0$, for which one has :*

$$\begin{cases} |f_n(x,y,z) - f_n(x,\bar{y},\bar{z})| \leq k_n|y - \bar{y}| \\ |g_n(x,y,z) - g_n(x,\bar{y},\bar{z})| \leq h_n|z - \bar{z}| \end{cases}, \textit{ for every } (x,y,z) \textit{ and } (x,\bar{y},\bar{z}) \textit{ from } D.$$

(2) $$\begin{cases} |f(x,y,z) - f(x,\bar{y},\bar{z})| \leq k|y - \bar{y}| \\ |g(x,y,z) - g(x,\bar{y},\bar{z})| \leq h|z - \bar{z}| \end{cases}, \textit{ for every } (x,y,z) \textit{ and } (x,\bar{y},\bar{z}) \textit{ from } D,$$

with $k_n, h_n, k, h > 0$, for each $n \in \mathbb{N}$, satisfying $\max\{k_n, h_n, k, h\} \leq M < 1$.

(3) *The pointwise convergence of the families (f_n) and (g_n), i.e.*

$$\begin{cases} f_n \xrightarrow{p} f \\ g_n \xrightarrow{p} g \end{cases}, \textit{ i.e. } \begin{cases} \lim\limits_{n\to\infty} f_n(x,y,z) = f(x,y,z) \\ \lim\limits_{n\to\infty} g_n(x,y,z) = g(x,y,z) \end{cases}, \textit{ for every } (x,y,z) \in D.$$

(4) *If the mappings f_n and g_n are bounded, for each $n \in \mathbb{N}$ by M_n and $\tilde{M}_n$ respectively, then there exist $M_f, M_g \geq 0$, such that $M_n \leq M_f$ and $\tilde{M}_n \leq M_g$, for each $n \in \mathbb{N}$.*

If (y_n, z_n) is the unique solution of 2.3.0.4 and (y,z) is the unique solution of 2.3.0.5, then

$$\begin{cases} y_n \xrightarrow{u} y \\ z_n \xrightarrow{u} z \end{cases}, \textit{ i.e. } \begin{cases} \lim\limits_{n\to\infty} \|y_n - y\|_{B,\tau_1} = 0 \\ \lim\limits_{n\to\infty} \|z_n - z\|_{B,\tau_2} = 0 \end{cases}.$$

Chapter 3

Fixed point results for multi-valued generalized contractions

3.1 Fixed point results for multi-valued operators. The altering distance technique

In this section, we will present some preliminary notions and fixed point results for single-valued self-mappings satisfying some altering distance type conditions in a complete metric space.
In [74], Khan gave sufficient conditions such that an operator has a unique fixed point, under the following contractive-type condition

$$\psi(d(Tx, Ty)) \leq k\psi(d(x, y)) \ , \forall x, y \in X \ ,$$

where (X, d) is a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ is endowed with the following properties

$$\psi(t) = 0 \text{ if and only if } t = 0,$$
$$\psi \text{ is continuous and increasing.}$$

Furthermore, Alber and Guerre-Delabriere in [8] gave a different generalization, for mappings satisfying the assumption

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \text{ where}$$
$$\varphi : [0, \infty) \to [0, \infty) \text{ is also increasing and continuous}$$
$$\varphi(t) = 0 \text{ if and only if } t = 0,$$
$$\lim_{t\to\infty} \varphi(t) = \infty.$$

Then in [133], Rhoades showed that the last assumption is not necessary for the existence and uniqueness of the fixed points of the above self-mappings. Generalizations for this type of mappings were done by Dutta et. al. in [49] for self-mappings f defined on a complete metric space (X, d), satisfying

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \text{ where}$$
$$\psi, \varphi : [0, \infty) \to [0, \infty) \text{ are both increasing and continuous functions,}$$
$$\psi(t) = \varphi(t) = 0 \text{ if and only if } t = 0.$$

A very interesting approach was done by Amini-Harandi and Petruşel in [24], where the authors studied sufficient conditions for the existence and uniqueness of the fixed points for an operator T satisfing the following assumption

$$u(d(Tx, Ty)) \leq v(d(x, y)),$$

where the self-mappings u and v defined on $[0, \infty)$ satisfy some relaxed conditions. The authors also gave some interesting corollaries showing that their theorem is a real generalization of the already presented type of mappings.
Moreover, regarding the weakly contractive condition for a single-valued operator, Rhoades et. al. [134]

presented other types of generalizations in the framework of partially ordered metric spaces. As an example for this type of mappings we suppose that

$$d(fx, fy) \leq \varphi(d(x, y)) - \psi(d(x, y))$$

where the operators ψ, φ have the following properties:

$i)$ φ, ψ are both positive on $(0, \infty)$ with $\psi(0) = \varphi(0)$,
$ii)$ $\varphi(t) - \psi(t) < t$,
$iii)$ φ upper semicontinuous and increasing,
$iv)$ ψ lower semicontinuous and nonincreasing.

Last, but not least, for fixed point results involving other generalizations of the weakly contractive condition for single-valued mappings we refer to [94], [123] and [161].

Concerning the case of multi-valued operators, T. Lazăr et. al. [80] presented an exhaustive study of some qualitative properties concerning Reich type multi-valued operators. Moreover, V. Lazăr [79] extended the results concerning the case of multi-valued φ-type contractions.
T.P. Petru and M. Boriceanu [109] gave some fixed point results for φ-contractions in a set endowed with two metrics.
In all the articles [80], [79] and [109], the comparison function φ used for the case of φ-contractions satisfy the following properties:

$i)$ $\varphi(0) = 0$,
$ii)$ $\varphi(t) > 0$ for $t > 0$,
$iii) \varphi^k(t) \to 0$, for each $t > 0$ for $k \to \infty$.

Notice that φ is not necessarily continuous on $[0, \infty)$, but in [109] the continuity of the mapping was additionally assumed. Furthermore, an important property of comparison functions is the fact that $\varphi(t) < t$, for each $t > 0$.
As a conclusion, there are two distinct classes of mappings involved in the generalizations of contractive-type operators. There are comparison functions, on one hand, in many cases denoted by φ. On the other, there is the case of altering distance functions for which the most important conditions are continuity or semicontinuity properties and a certain monotonicity. We also notice that the weakly contractive mappings used in [134] are combination of these types of self-mappings.

Regarding the case of multi-valued operators, in 2011, Kamran and Kiran [70] presented some rezults involving altering distance type functionals. In this article, more precisely, in [Theorem 4.2] in [70], a special type of altering distance function denoted by θ was used. This mapping satisfies the following conditions on an interval $[0, A)$, where A is real number strictly greater than 0, i.e.,

(i) θ is increasing on $[0, A)$,
(ii) $\theta(t) > 0$, for each $t \in (0, A)$,
(iii) θ subadditive on $(0, A)$ and
(iv) $\theta(at) \leq a\theta(t)$, for each $a > 0$ and $t \in [0, A)$.

Also, in 2012, Liu et. al. [86] gave a similar theorem, namely [Theorem 2.3], were the functional φ is similar to the functional α from [70]. From the same article we observe that the conditions put upon the altering distance mapping θ are somewhat different. For the sake of completeness, we recall them here

(a) θ is increasing on $\mathbb{R}^+$,
(b) $\theta(t) > 0$, when $t \in (0, \infty)$,
(c) θ is subadditive on $(0, \infty)$,
(d) $\theta(\mathbb{R}^+) = \mathbb{R}^+$ and
(e) θ is strictly inverse on $\mathbb{R}^+$.

Finally, concerning weakly contractive $(\varphi - \psi)$ contractive type multi-valued operators, G. Petruşel et.al. in [118] presented a fixed point result for this kind of operators in the context of complete ordered b-metric spaces with coefficient $s \geq 1$, along with some theorems involving coupled fixed points. From [Theorem 2.2] of [118], a self multi-valued operator $T : X \to P_{cl}(X)$ was defined by a contractive-type inequality, i.e.

$$\varphi(H(Tx, Ty)) \leq \varphi(d(x, y)) - \psi(d(x, y))$$

and sufficient conditions for the existence of fixed points for this kind of operators were studied. Here, the altering distance function $\varphi : [0, \infty) \to [0, \infty)$ satisfy the following

$$\begin{aligned} &(i_\varphi)\ \varphi \text{ continuous and strictly increasing,}\\ &(ii_\varphi)\ \varphi(t) < t,\ \text{for each } t > 0,\\ &(iii_\varphi)\ \varphi(a + b) \leq \varphi(a) + b,\ \text{for } a, b \in [0, \infty),\\ &(iv_\varphi)\ \varphi(st) \leq s\varphi(t),\ \text{for each } t \in [0, \infty). \end{aligned}$$

Also, the other altering distance function $\psi : [0, \infty) \to [0, \infty)$ satisfy

$$\begin{aligned} &(i_\psi)\ \limsup_{t \to r} \psi(t) > 0,\ \text{for all } r > 0 \text{ and}\\ &(ii_\psi)\ \lim_{t \to 0^+} \psi(t) = 0. \end{aligned}$$

Moreover, we recognize that in contrast to the original usage of altering functions as in [49], the conditions from [118] on ψ were relaxed and the conditions on the mapping φ were made more restrictive, since the condition (ii_φ) is a comparison type condition. So, in this sense, these weakly contractive-type self-mappings are a combination of altering distances and comparison functions as the operators defined in [134].
Finally, in [46] the authors presented some fixed point theorems for multi-valued operators and in [124] Popescu et.al. extended these types of comparison based multi-valued operators, i.e. $T : X \to P_{b,cl}(X)$, such that $H(Tx, Ty) \leq \varphi(d(x, y))$, for each x, y from the complete metric space (X, d). Here, the mapping φ satisfies the following assumptions

$$\begin{aligned} &(1)\ \varphi(x) \leq x,\ \text{for each } x \in [0, \infty),\\ &(2)\ \varphi(x + y) \leq \varphi(x) + \varphi(y),\ \text{for all } x, y \in [0, \infty),\\ &(3)\ \varphi(x) = 0 \text{ if and only if } x = 0 \text{ and}\\ &(4)\ \text{for any } \varepsilon > 0,\ \text{there exists } \delta > \varepsilon \text{ such that } \varepsilon < t < \delta \text{ implies } \varphi(t) \leq \varepsilon. \end{aligned}$$

Furthermore, condition (4) defined for the above mapping φ can be considered as a local type comparison function. Also, the authors in [124] extended the results of the authors of [46] from the case of hyperconvex metric spaces to the usual metric spaces. Furthermore, since they worked in a less restrictive framework, they have put an important assumption regarding the well known diameter functional and so they used the condition that the multi-valued operator has bounded values.

In the present section we consider some fixed point results on $(\varphi - \psi)$ multi-valued operators, by using the altering distance technique. The starting point is the article of G. Petruşel et. al. [118], where the authors have considered some theoretical results for an abstract type of multi-valued operators in complete ordered b-metric spaces. The main idea behind this article was the construction of a sequence of successive approximations that is showed to converge to a fixed point of the multi-valued operator. On the other hand, the proof of [Theorem 2.2] from [118] contains a certain gap. The authors consider an $\tilde{\varepsilon}$ at each step and then, using the reductio ad absurdum argument, chose $\tilde{\varepsilon}$, such that $\tilde{\varepsilon} < \lim_{n \to \infty} \psi(\delta_n)$. Since $\tilde{\varepsilon}$ was already constructed at each step (as $\tilde{\varepsilon_n}$) and the sequence δ_n was not given there, this technique is not valid. Furthermore, trying to show that the sequence (x_n) is Cauchy, the authors used the fact that $d(x_{m(k)+1}, x_{n(k)+1}) \leq H(T(x_{m(k)}), T(x_{n(k)})) + \tilde{\varepsilon}$, which, by the well known Nadler's lemma, is not necessarily true. So, our aim is to present a correction to these arguments by imposing an additional assumption on the iterates of the multi-valued operator. This assumption is inspired by the articles [46] and [124]. Finally, the second purpose is to impose more relaxed continuity conditions on the altering distance function and to get rid off the property that is frequently used for comparison-type functions.

Theorem 3.1.1. *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ a multi-valued operator, that satisfies the following*

$$\varphi(H(Tx, Ty)) \leq \varphi(d(x, y)) - \psi(d(x, y)), \text{ for each } x, y \in X,$$

where the mappings φ and ψ satisfy

$$\begin{cases} (H1)\ \varphi, \psi : [0, \infty) \to [0, \infty) \\ (H2)\ \varphi \text{ is usc and } \psi \text{ is lsc} \\ (H3)\ \varphi(0) = \psi(0) = 0 \text{ and } \varphi(t), \psi(t) > 0, \text{ for each } t > 0 \\ (H4)\ \varphi \text{ is (strictly) increasing} \\ (H5)\ \varphi(a + b) \leq \varphi(a) + b, \text{ for each } a > 0 \text{ and } b \geq 0 \end{cases}$$

Also, suppose that

$$\delta(T^n x) \to 0 \text{ as } n \to \infty, \text{ for each } x \in X.$$

Then, the multi-valued operator T has at least one fixed point $x^* \in F_T$.
Moreover, if $(SF)_T \neq \emptyset$, then $F_T = (SF)_T = \{x^*\}$.

Remark 3.1.2. In [70], the authors have used the fact that $\varphi : [0, \infty) \to [0, \infty)$ is increasing, so it implies that the functional φ is strictly inverse isotone on $[0, \infty)$, i.e.

$$\varphi(t_1) < \varphi(t_2) \Rightarrow t_1 < t_2, \text{ where } t_1, t_2 \in [0, \infty).$$

In our case, the function is (strictly) increasing. This means that φ is indeed strictly increasing, so $t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$. By logical transposition, it follows that

$$\varphi(t_2) \leq \varphi(t_1) \Rightarrow t_2 \leq t_1$$

Furthermore, we have used in the proof of the previous result that for $\varphi(t_1) < \varphi(t_2) \Rightarrow t_1 \leq t_2$. This is equivalent to the following fact : if $t_1 > t_2 \Rightarrow \varphi(t_1) \geq \varphi(t_2)$. By the fact that φ is strictly increasing, we get that the last inequality is a particular case, i.e. a strict inequality.
Also, some parts from the proof of the above result, we have used that φ is only increasing. Now, since φ is strictly increasing, then this functional is one-to-one, so $\varphi(t_1) = \varphi(t_2) \Rightarrow t_1 = t_2$. Combining this with the fact that φ is strictly increasing, we have simplified the proof by taking

$$\varphi(t_1) \leq \varphi(t_2) \Rightarrow t_1 \leq t_2.$$

Also, since φ is strictly increasing we get that $t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$. Moreover, for $t_1 = t_2$, it follows that $\varphi(t_1) = \varphi(t_2)$ from the definition of a regular function. Combining both of these, it follows that for $t_1 \leq t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$.

Remark 3.1.3. An important observation is that in the assumption (H5) of the previous theorem we have that $\varphi(a + b) \leq \varphi(a) + b$, for each $b \geq 0$ and $a > 0$. If the assumption was defined for each $a \geq 0$, then taking $a = 0$, it follows that $\varphi(a) \leq a$. So, we get the property of comparison functions. In this case, if (H5) seems restrictive, one can use the previous remark.

The second theoretical result of this present section regards a fixed point theorem for $(\varphi - \psi)$ multi-valued operators. The next theorem is an improvement of the previous one, where an assumption involving the diameter functional δ is imposed. Following [124], we shall relax the assumptions of the first result. Under certain circumstances, we obtain also the uniqueness of the fixed point for the multi-valued operator. Firstly, we consider a crucial observation that we shall use throughout the remaining part of the section.

Remark 3.1.4. Let φ be a functional endowed with the properties (H1),(H3),(H4) and (H5) from the previous theorem. Also, consider a sequence $(d_n)_{n\in\mathbb{N}}$, such that $d_n \geq d$ and with $\lim_{n\to\infty} d_n = d$. Then, $d_n - d \geq 0$. It follows that

$$\varphi(d_n) = \varphi(d + d_n - d) \leq \varphi(d) + [d_n - d],$$

which is valid since $d > 0$ and $d_n - d \geq 0$. It follows that

$$\varphi(d_n) \leq \varphi(d) + (d_n - d).$$

Taking the upper limit, we have that

$$\limsup_{n\to\infty} \varphi(d_n) \leq \varphi(d).$$

We remark that if φ was usc, then

$$\limsup_{n\to\infty} \varphi(d_n) \leq \varphi(d).$$

So, because of the assumptions on φ, it follows that the above property is more relaxed, in the sense that the functional φ must satisfy the usc condition only for the case when $d_n \geq d$.

The second fixed point theorem is the following.

Theorem 3.1.5. *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ be a multi-valued operator satisfying the following assumption*

$$\varphi(H(Tx, Ty)) \leq \varphi(d(x, y)) - \psi(d(x, y)), \text{ for each } x, y \in X,$$

where the mappings φ and ψ satisfy

$$\begin{cases} (H1)\ \varphi, \psi : [0, \infty) \to [0, \infty) \\ (H2)\ \varphi \text{ is usc in } 0 \text{ and} \psi \text{ is lsc in } 0 \\ (H3)\ \varphi(0) = \psi(0) = 0 \text{ and } \varphi(t), \psi(t) > 0, \text{ for each } t > 0 \\ (H4)\ \varphi \text{ is (strictly) increasing} \\ (H5)\ \varphi(a + b) \leq \varphi(a) + b, \text{ for each } a > 0 \text{ and } b \geq 0\ . \end{cases}$$

We also suppose that

$$\delta(T^n x) \to 0 \text{ as } n \to \infty, \text{ for each } x \in X.$$

Then, the multi-valued operator T has a unique fixed point.

Based on [118] and [124], in this article, we presented two theorems regarding fixed point results for $(\varphi - \psi)$ multi-valued operators. The additional assumptions based on the diameter functional δ was an important hypothesis such that the multi-valued operator has a fixed point. Also, the second theorem is an improvement of the first one. The two main results were constructed to correct some arguments from [118] for this type of operators. Furthermore, the restrictive condition on δ lead to the case when the multi-valued operators had unique contractive fixed points.

3.2 Extended fixed point principles for Ćirić multi-valued contractions

The aim of this section is to present a study on Ćirić type multi-valued operators. Moreover, the starting point is the article of A. Petruşel [113], in which the author studied some saturated fixed point results, respectively strict fixed point results for multi-valued contractions in the sense of Nadler [96]. In [131] Reich developed some fixed point theorems for multi-valued operators and Hardy and Rogers [57] generalized those fixed point theorems. Moreover, C. Chifu and G. Petruşel in [40] studied qualitative properties concerning Hardy-Rogers multi-valued operators in the framework of b-metric spaces. A fully comprehensive study on Reich operators was made in [80] by T. Lazăr et. al. Also, qualitative properties, namely data dependence, Ulam-Hyers stability and so on, were studied for the case of multi-valued φ-contractions by V.L. Lazăr in [79] and extended by T.P. Petru and M. Boriceanu in [109] to the case of multi-valued operators defined on a set endowed with two metrics. Also, in [36], M. Boriceanu studied existence and uniqueness of the fixed point and data dependence for multi-valued Ćirić type operators in the context of b-metric spaces.
At the same time, Ćirić multi-valued operators were studied extensively in [36], [42], [82], [111] and [120]. Now, regarding the terminology and the basic concepts related to fixed point problems for multi-valued operators, especially for multi-valued Nadler contractions, we will follow [25],[65], [119] and [130]. Furthermore, for the approximation of strict fixed points (also called end-points) of multi-valued mappings, we refer to [56], [61] and [125].
Finally, regarding data dependence, multi-valued fractal operators, selections and qualitative properties for the fixed point inclusion and for multi-valued fractals, we will refer to [48], [81] and [121].

Last but not least, we are concerned with some theoretical results in which some fixed point extended principles of the multi-valued Ćirić operators. The starting point is the article of A. Petruşel [113], in which the author studied some extended principles for fixed point problems for Nadler multi-valued contractions. Our main results consists of extended principles for the fixed points, respectively extended principles for the strict fixed points of the Ćirić multi-valued operators.

Theorem 3.2.1 (An extended fixed point principle for multi-valued Ćirić operators)**.** *Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multi-valued α-Ćirić type operator, i.e., there exists $\alpha \in (0,1)$, such that*

$$H(Tx,Ty) \leq \alpha \cdot M(x,y), \text{ for each } x,y \in X,$$

where

$$M(x,y) := \max\{d(x,y), D(x,Tx), D(y,Ty), \frac{1}{2}[D(x,Ty)+D(y,Tx)]\}.$$

Then, the following conclusions hold :
(a) there exists $x^* \in F_T$;
(b) for each $(x,y) \in Graph(T)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from (x,y), that converges to a fixed point of T;
(c) there exists a selection $t^\infty : Graph(T) \to F_T$ of T^∞, such that

$$d(x,t^\infty(x,y)) \leq \frac{1}{1-\alpha}d(x,y), \forall (x,y) \in Graph(T);$$

(d) F_T is closed in (X,d);
(e) if $(x_n)_{n\in\mathbb{N}}$ is a sequence of successive approximations for T, starting from a pair $(x,y) \in Graph(T)$, which converges to a fixed point $x^*(x,y)$ of T, then

$$d(x_n,x^*) \leq \frac{\alpha^n}{1-\alpha}d(x,y), \forall n \in \mathbb{N}^*;$$

(f) if $G : X \to P_{cl}(X)$ is a Ćirić-type multi-valued operator with coefficient β, and there exists $\eta > 0$, such that

$$H(Tx,Gx) \leq \eta, \forall x \in X,$$

then $H(F_T,F_G) \leq \eta \cdot \max\left\{\frac{1}{1-\alpha}, \frac{1}{1-\beta}\right\}$;
(g) if $T_n : X \to P_{cl}(X)$ is a sequence of multi-valued α-Ćirić-type operators, with $T_n x \overset{H}{\to} Tx$ as $n \to \infty$, uniformly with respect to $x \in X$, then

$$\lim_{n\to\infty} H(F_{T_n},F_T) = 0;$$

(h) if there exists $x_0 \in X$ and $r > 0$, such that $D(x_0,Tx_0) < (1-\alpha)r$, then there exists $x^* \in F_T \cap B(x_0,r)$;
(i) if there exists $x_0 \in X$ and $r > 0$ such that $\delta(x_0,Tx_0) < (1-\alpha)r$, then

$$T : \tilde{B}(x_0,r) \to P\left(\tilde{B}\left(x_0, \frac{1}{1-\alpha}r\right)\right)$$

and there exists $x^* \in F_T \cap B(x_0,r)$;
(j) if X is also a Banach space, U an open subset of X and $T : U \to P_{cl}(X)$ is a Ćirić multi-valued operator, then the associated multi-valued field $G : U \to P(X)$, defined by $Gx := x - Tx$, is open;
(k) there exists a Caristi selection of T;
(m) if, additionally, $T : X \to P_{cp}(X)$, then the fixed point inclusion $x \in Tx$ associated to the multi-valued Ćirić-type operator T is generalized Ulam-Hyers stable;
(n) the multi-valued operator T has the approximate fixed point property, i.e., $\inf_{x\in X} D(x,Tx) = 0$;
(o) if the multi-valued operator T is lsc, then it has the approximate endpoint property if and only if it has a unique strict fixed point;
(p) if $T : X \to P_{cp}(X)$ is α-multi-valued Ćirić type operator, with coefficient $\alpha < \frac{1}{2}$, then the fixed point set F_T is compact.
(q) if $T : X \to P_{cl,bd}(X)$ is a Ćirić-type multi-valued operator, then for each $p > 0$, one has

$$H(F_p^*,F_T) \leq \frac{p}{1-\alpha}, \text{ where } F_p^* := \{x \in X \mid D(x,Tx) < p\}.$$

The second theoretical result concerning this section is a extended strict fixed point principle for multi-valued Ćirić operators. We use the conclusions from [Theorem 3.2.1] that hold true for the particular case when $(SF)_T$ is nonempty. So, in our next theorem we present only the metrical conclusions that are new.

Theorem 3.2.2 (An extended strict fixed point principle for multi-valued Ćirić operators)**.** *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be an α-Ćirić type operator, i.e. there exists $\alpha \in (0, 1)$, such that*

$$H(Tx, Ty) \leq \alpha \cdot M(x, y), \text{ for each } x, y \in X,$$

where

$$M(x, y) := \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Tx)]\}.$$

Furthermore, suppose that $(SF)_T \neq \emptyset$. Then, the following conclusions hold :
(a) $(SF)_T = F_T = \{x^*\}$;
(b) if $\alpha < \frac{1}{2}$, then the multi-valued operator T has the Ostrowski property;
(c) the fixed point inclusion $x \in Tx$ associated to the multi-valued Ćirić-type operator T is generalized Ulam-Hyers stable;
(d) the strict fixed point inclusion $\{x\} = Tx$ associated to the multi-valued Ćirić-type operator T is generalized Ulam-Hyers stable;
(e) the fixed point problem is well-posed for T, with respect to the gap functional D, respectively to H;
(f) $H(Tx, x^*) \leq \alpha d(x, x^*)$, for each $x \in X$;
(g) $d(x, x^*) \leq \frac{1}{1-\alpha} H(x, Tx)$, for each $x \in X$;
(h) if $G : X \to P(X)$ is a multi-valued operator with $F_G \neq \emptyset$, and there exists $\eta > 0$, such that

$$H(Tx, Gx) \leq \eta, \forall x \in X,$$

then $H(F_T, F_G) \leq \eta \cdot \frac{1}{1-\alpha}$.

Chapter 4

Iterative schemes for generalized contractions in complete metric spaces

4.1 Fixed points analysis of some generalized contractions through Ishikawa's iteration

In the present section, our aim is to study some results concerning existence and uniqueness of fixed points for generalized contraction mappings. The setting is that of contraction-type operators, in the following sense, i.e.

$$cd(Tx,Ty)+a\left[d(x,Tx)+d(y,Ty)\right]\leq kd(x,y)\ .$$

Further, we make the observation that our techniques are valid also for more complex type of generalized contractions. Karapinar, Moosaei and other authors have used Krasnoselskii's iteration in the particular case when the auxiliary parameter λ is equal to 1/2.
Now, since we work with generalized type of contractype mappings, we recall some generalization of Banach contraction principle in convex metric spaces.
In [72] Karapinar developed existence and uniqueness theorems for mappings satisfying certain contractive conditions in cone Banach spaces, such as

$$\begin{aligned}&d(Tx,Ty)\geq ad(x,y),\\&d(x,Tx)+d(y,Ty)\leq pd(x,y),\\&ad(Tx,Ty)+b\left[d(x,Tx)+d(y,Ty)\right]\leq sd(x,y).\end{aligned}$$

Karapinar used the following conditions on this contractive type-operators, such as :
$a>1$, $p\in[0,2)$, respectively $0\leq s+|a|-2b<2(a+b)$.
In [28], Asadi generalized the above conditions for mappings endowed with more coefficients, i.e.

$$ad(x,Tx)+bd(y,Ty)+cd(Tx,Ty)+ed(x,Ty)+fd(y,Tx)\leq kd(x,y).$$

Asadi, developed existence and uniqueness for above contractions, using the condition that:

$$\frac{b+e-|f|(1-\lambda)-|c|\lambda}{1-\lambda}\leq k<\frac{a+b+c+e+f-|c|\lambda-|f|(1-\lambda)}{1-\lambda}$$

Independently, in [159] Wang and Zhang developed a theorem for the existence of fixed points of a pair of contractive mappings, such as

$$kd(Tx,Sy)\leq ad(x,y)+b\left[d(x,Tx)+d(y,Sy)\right]+c\left[d(x,Sy)+d(y,Tx)\right],$$

using a Jungk-Krasnoselskii iteration, as follows

$$\begin{cases}x_{2n+1}=W\left(x_{2n},Tx_{2n},\lambda\right)\\x_{2n+2}=W\left(x_{2n+1},Tx_{2n+1},\lambda\right)\end{cases},\tag{4.1.0.1}$$

and give an exemple of a mapping satisfying the above condition.
Also, Wang and Zhang imposed the conditions $a+2c<k$ and $(k,a,b,c)\in\Gamma_\rho$, with $\rho\in\{1,2,3,4\}$, where

Γ_ρ were defined in terms of the coefficients a, b, c and k.
In [91] Moosaei developed a theorem for a Banach operator pair (f, g) satysfing

$$ad(gx, fx) + bd(gy, fy) + cd(fx, fy) \leq kd(gx, gy),$$

with $2b - |c| \leq k < 2(a + b + c) - |c|$ and in [93] the same author presented a theorem for existence and uniqueness of a common fixed point of a pair of mappings (S, T), where (S, T) weakly compatible pair, i.e. :

$$ad(Sx, Tx) + bd(Sy, Ty) + cd(Tx, Ty) \leq ed(Sx, Sy).$$

Likewise, in [92], Moosaei presented sufficient conditions for the existence of a coincidence point of a generalized contraction pair (S, T), as follows

$$\alpha d(Tx, Ty) + \beta\left[d(Sx, Ty) + d(Sy, Tx)\right] + \gamma\left[d(Sx, Ty) + d(Sy, Tx)\right] \leq \eta d(Sx, Sy),$$

using as Wang and Zhang a Jungk-Krasnoselskii iteration.
In [2], Abbas et. al. discussed the simplicity of the modified Krasnoselskii iteration

$$x_{n+1} = (1 - \lambda)x_n + \lambda T^n x_n,$$

where $\lambda \in (0, 1)$ and the operator T is defined on a nonempty, closed convex subset of a uniformly convex Banach space. Furthermore, one can easily extend this iteration to the case of convex metric spaces in the sense of Takahashi. Abbas, Khan and Rhoades suggested that a number of fixed point iteration problems can be solved using the above iteration, which is much simpler than Ishikawa iteration. However, the simplification principle does not directly apply to the setting of our research since we are dealing with a generalized type of nonlinear contractions in contrast with the case when the operator T is defined by it's iterates, for example when T is an asymptotically nonexpansive mapping. In this case it satisfies the following condition : there exists a sequence $(k_n)_{n\in\mathbb{N}} \subset [1, \infty)$, with $\sum\limits_{n\in\mathbb{N}} (k_n - 1) < \infty$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

So, in the above case, the condition would act on the iterates of the operator T, contrary to the case when T is a nonlinear generalized contraction-type mapping. In our case, as we do not assume an asymptotic condition for the nonlinear operator of interest, especially from a computational point of view, the modified Krasnoselskii iteration process would be less cost-efficient than the well-known Ishikawa iteration. The cause of this relies on the fact that the modified Krasnoselskii iteration requires the computation of T^n in each step.
Our purpose is to give a new method of proof for the more complex Ishikawa iterative scheme. Finally, this will be validated through an example.

Theorem 4.1.1. *Let (X, d) be a complete convex metric space.*
Let K a nonempty, closed and convex subset of X and $T : K \to K$ be a map satisfying the following contractive condition

$$cd(Tx, Ty) + a\left[d(x, Tx) + d(y, Ty)\right] \leq kd(x, y),$$

for each $x, y \in K$.
Let (α_n) and (β_n) be two sequences in $(0, 1)$ and let $a_1, a_2, b_1, b_2 \in [0, 1]$, such that $\alpha_n \in [a_1, a_2] \subset [0, 1)$ and $\beta_n \in [b_1, b_2] \subset [0, 1)$, for each $n \in \mathbb{N}$.
Let :

$$\varepsilon \in \left(0, \frac{1}{2}\right),\ w_1 := \begin{cases} \dfrac{a}{1 - a_2}, & a \geq 0 \\ \\ \dfrac{2a}{3(1 - a_1)}, & a < 0 \end{cases},\ w_2 := \begin{cases} \dfrac{a(1 + b_2)}{1 - b_2}, & a \geq 0 \\ \\ \dfrac{a(1 + b_1)}{1 - b_1}, & a < 0 \end{cases}$$

$$w_3 := \begin{cases} \dfrac{\frac{1}{2}(a + c) - |a|(1 + b_2)}{1 - b_1}, & k > 0 \\ \\ \dfrac{\frac{1}{2}(a + c) - |a|(1 + b_2)}{1 - b_2}, & k < 0 \end{cases},\ \tau := \begin{cases} \frac{1}{3}, & a_2 < \frac{1}{2} \\ \frac{1}{3} \cdot \frac{a_2}{1 - a_2}, & a_1 > \frac{1}{2} \end{cases},$$

$$s := \begin{cases} \dfrac{3(1-a_1)}{2(a+c)}, & a \geq 0 \ and \ c \geq 0, \\ \\ \dfrac{a}{2(1-a_1)} + \dfrac{c}{\varepsilon}\dfrac{(1+b_2)}{(1-a_2)(1-b_2)}, & a \geq 0 \ and \ c < 0, \\ \\ \dfrac{a}{\varepsilon}\dfrac{(1+b_2)}{(1-a_2)(1-b_2)} + \dfrac{c}{2(1-a_1)}, & a < 0 \ and \ c \geq 0 \end{cases}$$

Suppose the following conditions hold :

$$\begin{cases} (1) & c \neq 0, \\ (2) & a + c > 0, \\ (3) & k \geq w_1, \\ (4) & k \geq w_2, \\ (5) & k < w_3, \\ (6) & s > 0, \\ (7) & |c|\tau < s - k - \dfrac{|a|}{\varepsilon}\dfrac{(1+b_2)}{(1-a_2)(1-b_2)}. \end{cases}$$

Then, the operator T admits a fixed point.
Moreover, if $k < c$, then the fixed point is unique.

Remark 4.1.2. For the sake of completeness involving the proof from [Theorem 4.1.1], we make the following crucial remark : if $a \geq 0$, then $k \geq 0$.

As a consequence of the above result, we can obtain a fixed point theorem for a certain iterate of a self-operator defined on a complete convex metric space.

Corollary 4.1.3. *Let (X, d) be a complete convex metric space. Let K a nonempty, closed and convex subset of X and $T : K \to K$ be a self-mapping. Suppose, there exists $p > 1$, such that T satisfies :*

$$cd(T^p x, T^p y) + a\left[d(x, T^p x) + d(y, T^p y)\right] \leq kd(x, y),$$

for each $x, y \in K$. Also, suppose the operator $T^p : K \to K$ satisfies conditions (1)-(8) from [Theorem 4.1.1]. Then, T has a fixed point. Moreover, if $k < c$ then the fixed point is unique.

Finally, we give a result that validates our main result of this section.

Example 4.1.4. Let $X := [0, \infty)$ and $T : X \to X$, such that :

$$T(x) = \begin{cases} \frac{x}{194}, & x \in [0, x_0) \\ \frac{x}{250}, & x \in [x_0, \infty) \end{cases},$$

where x_0 can be any point in $[0, \infty)$. We remark that because x_0 is a discontinuity point for T, then T can not be a contraction type mapping. Then T satisfies the inequality :

$$1800d(Tx, Ty) + \frac{1}{2}d(x, y) \leq 10\left[d(x, Tx) + d(y, Ty)\right].$$

Also, T has a unique fixed point, i.e. $0 \in X$.
Furthemore, the sequences (α_n) and (β_n) from the Ishikawa iteration process are :

$$\begin{cases} \alpha_n = \dfrac{1}{n+7} + \dfrac{1}{4}, \\ \\ \beta_n = \dfrac{1}{n+3} + \dfrac{1}{2}. \end{cases}$$

4.2 Qualitative properties and stability results for Mann's algorithm for multi-valued mappings

The purpose of this section is to investigate the results on limit shadowing, Ulam-Hyers stability, T-stability and well-posedness involving multi-valued contractions operators. The original definitions consist in working with Picard iteration, that is $x_{n+1} \in Tx_n$, with $x_0 \in X$ an arbitrary element. Here we will modify the original definitions in order to be suitable for the context of Mann's iterative scheme. Firstly, we will give some theoretical results regarding the data dependence of the fixed point set for Mann's algorithms given through admissible perturbations.

Definition 4.2.1. *Let X be a nonempty set, $T : X \to P(X)$ be a multi-valued operator and $G : X \times X \to X$ an operator satisfying the following conditions :*

$$(A1)\ G(x,x) = x,\ \textit{for each } x \in X,$$
$$(A2)\ \textit{for each } x, y \in X \textit{ with } G(x,y) = x,\ \textit{we have that } x = y.$$

Let $T_G : X \to P(X)$ a multi-valued operator, defined by

$$T_G x := G\left(x, Tx\right) := \{G(x,u)/u \in Tx\}.$$

Then, by definition, T_G is called the admissible perturbation of T corresponding to G.

Lemma 4.2.2. $F_{T_G} = F_T$ *and* $(SF)_{T_G} = (SF)_T$.

For some examples, see [Example 5.3] - [Example 5.6] from [114].

Now we present Mann's iterative algorithm through admissible perturbations, which will be used further in this paper.

Definition 4.2.3 (Mann algorithm)**.** *Let (X,d) be a metric space, $T : X \to P(X)$ and $G_n : X \times X \to X$ be such that $T_{G_n} : X \to P(X)$ is the admissible perturbation of T corresponding to G_n, for $n \in \mathbb{N}$.*
Mann's algorithm, or simply GM-algorithm, corresponding to the family of operators $G := (G_n)_{n\in\mathbb{N}}$ is defined as

$$x_0 \in X \textit{ arbitrary and } x_{n+1} \in G_n\left(x_n, Tx_n\right) = T_{G_n}x_n,\ \textit{for each } n \in \mathbb{N}.$$

By definition, the GM-algorithm is convergent if $\forall x \in X$ and $y \in G_0\left(x, Tx\right) = T_{G_0}x$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ from X, such that

$$(i)\ x_0 = x \textit{ and } x_1 = y,$$
$$(ii)\ x_{n+1} \in G_n\left(x_n, Tx_n\right) = T_{G_n}x_n,$$
$$(iii)\ \textit{the sequence } (x_n) \textit{ is convergent to some } x^*(x,y) \in F_T,$$
$$(iv)\ x^*(x,x) = x,\ \textit{for each } x \in F_T.$$

In terms of multi-valued weakly Picard operators, we have that if GM algorithm is convergent, then the operator T_{G_n} is MWP, for each $n \in \mathbb{N}$. In case of convergence, we can define the single-valued operator $t^\infty : Graph(T_{G_0}) \to X$, by

$$t^\infty(x,y) := x^*(x,y),$$

for each $(x,y) \in Graph(T_{G_0})$.

Definition 4.2.4 (Mann algorithm)**.** *Let (X,d) be a metric space and let the operators $G_n : X \times X \to X$ such that T_{G_n} is the admissible perturbation of T corresponding to G_n, for each $n \in \mathbb{N}$. The GM algorithm satisfies condition (ψ) with respect to the multi-valued operator $T : X \to P(X)$ if the following assumptions are true :*

$$(i)\ \psi : \mathbb{R}_+ \to \mathbb{R}_+ \textit{ is increasing, continuous in 0 and } \psi(0) = 0,$$
$$(ii)\ \textit{the GM algorithm is convergent,}$$
$$(iii)\ d\left(x, t^\infty(x,y)\right) \le \psi\left(d(x,y)\right),\ \textit{for each } (x,y) \in Graph(T_{G_0}).$$

Our first result concerns the data dependence problem for *GM* algorithm. Our theorem is related to Theorem 7.2 in [114] with an additional assumption which is necessary for the proof.

Theorem 4.2.5 (GM-algorithm)**.** *Let (X, d) be a metric space, $T, S : X \to P(X)$ two multi-valued operators and $G := (G_n)$, where $G_n : X \times X \to X$ operators satisfying assumptions $(A1)$ and $(A2)$, for each $n \in \mathbb{N}$. We suppose that :*

(i) *the GM algorithm satisfies condition (ψ) with respect to T and S,*

(ii) *there exists $l_{G_0} > 0$ such that $d(G_0(x, y), G_0(x, z)) \leq l_{G_0} d(y, z)$, for each $x, y, z \in X$,*

(iii) *there exists $\eta > 0$ such that $H(Tx, Sx) \leq \eta$, for each $x \in X$,*

(iv) *if $q > 1$, then $\psi(qt) \leq q\psi(t)$, for each $t \in \mathbb{R}_+$.*

Then,

$$H(F_T, F_S) \leq \psi(l_{G_0}\eta).$$

Remark 4.2.6. If in [Theorem 4.2.5], we have that if GM algorithm satisfies condition (ψ_1) with respect to T and condition (ψ_2) with respect to S, it follows that :
For $x \in F_S$, we have that

$$d(x, t^{\infty}(x, G_0(x, Tx))) \leq q\psi_1(l_{G_0}\eta).$$

Similar, for $z \in F_T$, we have that

$$d(z, s^{\infty}(z, G_0(z, Sz))) \leq q\psi_2(l_{G_0}\eta).$$

Using the definition of H and letting $q \searrow 1$, the conclusion is that

$$H(F_T, F_S) \leq max\{\psi_1(l_{G_0}\eta), \psi_2(l_{G_0}\eta)\}.$$

Our further purpose is to study the data dependence property for Mann's iteration. We shall employ the same technique as in [143]. Also, for the convergence of these algorithms in different types of spaces, such as complete metric spaces, hyperbolic spaces and Banach spaces, we let the reader follow [41],[43] and [146].
Moreover, for further use of hyperbolic spaces, we notice that if $G(x, y) := W(x, y, \lambda)$, for $\lambda \in [0, 1]$ fixed and $x, y \in X$, we have that $W(x, x, \lambda) = x$, for each $x \in X$, by [Remark 1.5.7], so the assumption $(A1)$ is satisfied and if, for each $x, y \in X$, $W(x, y, \lambda) = x$, then by [Lemma 1.5.8] in the same section, we get that

$$0 = d(W(x, y, \lambda), x) = \lambda d(x, y) \ ,$$

so $x = y$ for $\lambda \in (0, 1)$. Hence, the assumption $(A2)$ is also satisfied.
Furthermore, our aim is to exemplify the data dependence for Mann's iterative algorithm for the case of multi-valued Nadler contractions.

From now on, we denote by (X, d, W) a hyperbolic space X, endowed with a metric d. Also, we shall work only in the context of hyperbolic metric spaces.

Furthermore, we shall use the following notation :

$$W(A, B, \lambda) := \{W(a, b, \lambda) / a \in A \text{ and } b \in B\}.$$

In the proof of our next theorem, we will use the property $(W4)$ of the hyperbolic spaces. Moreover, our next theorem will be very useful in this section for [Theorem 4.2.10] , [Remark 4.2.11] and for qualitative properties of the fixed point problem, actually in [Theorem 4.2.15], regarding the T-stability.

Theorem 4.2.7. *Let (X, d, W) be a hyperbolic space and Y_1, Y_2, Z_1 and Z_2 from $P(X)$. Then, we have that:*

$$H(W(Y_1, Y_2, \lambda), W(Z_1, Z_2, \lambda)) \leq (1 - \lambda)H(Y_1, Z_1) + \lambda H(Y_2, Z_2).$$

Remark 4.2.8. Using the definition of H, if we have linear structure, it is easy to show that

$$H(\lambda A, \lambda B) \leq \lambda H(A, B),$$

for each $\lambda \geq 0$ and $A, B \in P(X)$.

Corollary 4.2.9. *One can easily prove that*

$$H\left(W(A,B,\lambda),W(A,C,\lambda)\right)\leq\lambda H(B,C),$$

for each $\lambda\geq0$ *and for each* $A,B,C\in P(X)$.

Our second result is the data dependence of the fixed point set for multi-valued contractions, using Mann's iterative scheme.

Theorem 4.2.10 (Data dependence)**.** *Let* (X,d,W) *be a complete hyperbolic metric space and* $T,S:X\to P_{cl}(X)$ *two multi-valued contractions, i.e.*
- *there exists* $\alpha_T\in(0,1)$ *such that* $H(Tx,Ty)\leq\alpha_T d(x,y)$, *for each* $x\in X$ *and*
- *there exists* $\alpha_S\in(0,1)$ *such that* $H(Sx,Sy)\leq\alpha_S d(x,y)$, *for each* $x\in X$.

For $\alpha_n\in[a,1-a]\subset(0,1)$ *a sequence, we consider the admissible perturbations of* T, *respectively* S *corresponding to* W *:*

$$T_{\alpha_n}x:=W\left(x,Tx,\alpha_n\right)=\{W(x,u,\alpha_n)/u\in Tx\},$$
$$S_{\alpha_n}x:=W\left(x,Sx,\alpha_n\right)=\{W(x,u,\alpha_n)/u\in Sx\}.$$

Additionally, let's suppose the following assumptions are satisfied :
(1) there exists $\eta>0$ *, such that* $H(Tx,Sx)\leq\eta$, *for each* $x\in X$,
(2) The lower bound a *of the sequence* (α_n) *satisfy :*

$$a\in\left(\frac{\alpha_S-2+\sqrt{8+\alpha_S^2}}{2(1+\alpha_S)},\frac{1}{2}\right)\ \textit{and}\ a\in\left(\frac{\alpha_T-2+\sqrt{8+\alpha_T^2}}{2(1+\alpha_T)},\frac{1}{2}\right).$$

Then,

$$H(F_T,F_S)\leq\alpha_0\eta\cdot\max\left\{\frac{a}{(3a-1)-a(1-a)(1+\alpha_S)},\frac{a}{(3a-1)-a(1-a)(1+\alpha_T)}\right\}.$$

Remark 4.2.11. a) Using property $(W4)$ of hyperbolic spaces and since $G_0(x,y)$ is $W(x,y,\alpha_0)$ for $\alpha_0\in(0,1]$ fixed, we get that

$$d\left(G_0(x,y),G_0(x,z)\right)=d\left(W(x,y,\alpha_0),W(x,z,\alpha_0)\right)\leq\alpha_0 d(y,z).$$

Then l_{G_0} can be taken as $\alpha_0>0$. Next, we will apply [Theorem 4.2.5] to the case of multi-valued contractions of the previous theorem:
We have shown that

$$d(x_0,x^*)\leq q\alpha_0\eta\cdot\frac{a}{(3a-1)-a(1-a)(1+q\alpha_T)},$$

for $x_0\in F_S$ and, by [Theorem 4.2.5] and by [Remark 4.2.6] and taking $q\searrow1$ and $d(x_0,x^*)\leq\psi_1(\alpha_0\eta)$, this implies that

$$\psi_1(t)=c_1t,$$

with $t=\alpha_0\eta$ and $c_1=\dfrac{a}{(3a-1)-a(1-a)(1+\alpha_T)}>0$.
For $y_0\in F_T$ and ψ_2 we have an analogous remark, i.e.

$$d(y_0,y^*)\leq q\alpha_0\eta\cdot\frac{a}{(3a-1)-a(1-a)(1+q\alpha_S)},$$

for $y_0\in F_T$, by [Theorem 4.2.5] and by [Remark 4.2.6], one has that

$$d(y_0,y^*)\leq\psi_2(\alpha_0\eta).$$

This implies that $\psi_2(t)=c_2t$, with $t=\alpha_0\eta$ and

$$c_2=\frac{a}{(3a-1)-a(1-a)(1+\alpha_S)}>0.$$

b) If (X,d) is complete and $T,S:X\to P_{cl}(X)$, if GM algorithm is convergent and since we have showed that T_{α_n} and S_{α_n} are multi-valued contractions with coefficients $(1-a)(1+\alpha_S)$, respectively $(1-a)(1+\alpha_T)$, we reason that T_{α_n} and S_{α_n} are indeed ψ_1-MWP and ψ_2-MWP, by [Remark 1.1.8], where $\psi_1(t)=\dfrac{t}{\alpha_n(1-\alpha_T)}$ and $\psi_2(t)=\dfrac{t}{\alpha_n(1-\alpha_S)}$, for each $t\in\mathbb{R}_+$.

Now, our next result concerns the Ulam-Hyers stability of the fixed point inclusion related to Mann's iteration.

Theorem 4.2.12 (Ulam Stability with respect to Mann iteration)**.** *Let (X, d, W) be a complete hyperbolic space and $T : X \to P(X)$ a multi-valued operator, such that $T_{\alpha_0} : X \to P_{cp}(X)$.*
Let $\alpha_n \in [a, 1-a] \subset (0,1)$.
Let's suppose following hypotheses hold :

(a) *there exists $\alpha_T \in (0,1)$ such that $H(Tx, Ty) \leq \alpha_T d(x,y)$, for each $x, y \in X$,*

(b) $a \in \left(\dfrac{\alpha_T - 2 + \sqrt{8 + \alpha_T^2}}{2(1+\alpha_T)}, \dfrac{1}{2} \right)$.

Let $\varepsilon > 0$ and $x \in X$, such that

$$D(x, T_{\alpha_0} x) \leq \varepsilon.$$

Then, there exists $x^* \in F_T$, such that

$$d(x, x^*) \leq \varepsilon \cdot \frac{a}{(3a-1) - a(1-a)(1+\alpha_T)}.$$

Theorem 4.2.13 (Sequences of contractions with respect to Mann's iteration)**.** *Let (X, d, W) be a complete hyperbolic space and $T : X \to P_{cl}(X)$ an α_T contraction with respect to the functional H. Let $T_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ a sequence of multi-valued α_T-contractions with respect to the Pompeiu-Hausdorff metric H, such that $T_n x \overset{H}{\to} Tx$ as $n \to +\infty$, uniformly with respect with $x \in X$. Then,*

$$F_{T_n} \overset{H}{\to} F_T \text{ as } n \to +\infty.$$

Theorem 4.2.14 (Well-posedness with respect to D and Mann's iteration)**.** *Let (X, d, W) be a complete hyperbolic space and $T : X \to P_{cl}(X)$, with $(SF)_T \neq \emptyset$ and $x^* \in (SF)_T$.*
Let $\alpha_n \subset (0,1)$, for each $n \in \mathbb{N}$ and T_{α_n} be the admissible perturbations with respect to the multi-valued α_T-contraction T.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence from X, such that

$$D(x_n, T_{\alpha_n} x_n) \to 0 \text{ as } n \to +\infty.$$

Additionally, let's suppose there exists $a > 0$, such that $\alpha_n \geq a$, for each $n \in \mathbb{N}$. Then,

$$x_n \overset{d}{\to} x^* \text{ as } n \to +\infty.$$

Theorem 4.2.15 (T-stability with respect to Mann iteration)**.** *Let (X, d, W) be a complete hyperbolic space and $T : X \to P_{cl}(X)$ an α_T-contraction. Let $x_0 \in X$ and $x_{n+1} \in T_{\alpha_n} x_n$, with $\{\alpha_n\} \in [a, 1-a] \subset (0,1)$, for each $n \in \mathbb{N}$. Suppose that the lower bound of the sequence $(\alpha_n)_{n\in\mathbb{N}}$, i.e. a satisfies the condition from [Theorem 4.2.10]. Furthermore, let $\{y_n\}$ a sequence and set*

$$\varepsilon_n := H(y_{n+1}, T_{\alpha_n} y_n),$$

for each $n \in \mathbb{N}$. Moreover, let $x^* \in F_T$. If Tx^* is a singleton, we get that

$$y_n \to x^* \text{ as } n \to +\infty \text{ if and only if } \varepsilon_n \to 0 \text{ as } n \to +\infty.$$

Theorem 4.2.16 (Limit shadowing property with respect to Mann's iteration)**.** *Let (X, d, W) be a complete hyperbolic space and $T : X \to P_{cl}(X)$ be an α_T-contraction. Let $x_0 \in X$ and $x_{n+1} \in T_{\alpha_n} x_n$, with $(\alpha_n) \in [a, 1-a] \subset (0,1)$, for each $n \in \mathbb{N}$. Suppose that the lower bound of the sequence $(\alpha_n)_{n\in\mathbb{N}}$, i.e. a satisfies the condition from [Theorem 4.2.10].*
Let (y_n) be a sequence and set

$$D(y_{n+1}, T_{\alpha_n} y_n) \to 0, \text{ as } n \to \infty.$$

Consider $x^* \in F_T$. Then if Tx^* is a singleton, we have that

$$d(y_n, x_n) \to 0 \text{ as } n \to \infty$$

4.3 Convergence results for iterative schemes in the setting of convex metric spaces.

Most of the real world problems of applied sciences are, in general, functional equations. Such equations can be written as fixed point equations. Then, it is necessary to develop an iterative process which approximate the solution of these equations that has a good rate of convergence. Many studies in the field of fixed point theory concerning the existence and uniqueness of fixed points of single-valued contractions have been developed using basic iterative algorithms, such as : Picard iteration, Krasnoselksii, Mann and Ishikawa iterative processes. Over the years the interest regarding the speed of convergence of such iterations grew very fast. For example, many authors considered numerous iteration processes and studied their rate of convergence. For this see [4], [34], [51], [55], [41] and [156]. Some iterations were introduced to study the fixed points of contractions. Also, others [132] were introduced for the context of nonexpansive mappings. Furthermore, some authors [51] compared the rate of convergence for some iterative algorithms for the class of quasi-contractions. Finally, since the class of convex metric spaces is larger than the well-known class of linear normed spaces, we shall work in the context of convex metric spaces introduced by W. Takahashi.
Our aim is to introduce new iteration processes and prove that these are convergent under suitable circumstances. In the present section, we work on a nonlinear domain, more explicitly on a convex metric space. Also, we remind two important basic example of convex metric spaces : $CAT(0)$ spaces and linear normed spaces. For details, we let the reader follow [39] and [157]. Other important examples are : hyperbolic spaces introduced by Goebel and Kirk and hyperbolic spaces in the sense of Reich and Safrir. For details one can follow : [54] and [132].
In order to simplify some existing iteration processes from the literature, we recall the definition of Machado from [87] of general convex combinations defined on convex metric spaces :
For $a_1, ..., a_n \in X$ and $\varphi_1, ..., \varphi_n \in [0,1]$ with $\sum_{i=1}^{n} \varphi_i = 1$, we define the multiple convex combination of $a_1, ..., a_n$

$$W(a_1, ..., a_n; \varphi_1, ..., \varphi_n) = W\left(W\left(a_1, ..., a_{n-1}; \frac{\varphi_1}{1-\varphi_n}, ..., \frac{\varphi_{n-1}}{1-\varphi_n}\right), a_n; 1-\varphi_n\right),$$

if

$$\varphi_n \neq 1$$

and

$$W(a_1, ..., a_n; 0, ..., 1) = a_n,$$

if $\varphi_n = 1$.
We will work in the cases when $n = 2$ and $n = 3$. For simplicity of this remark, we consider that $\varphi_n \neq 1$. The other case is obvious and follows from the above definition.
We make the convention that, for $n = 2$, we have :

$$\begin{aligned} W(a_1, a_2; \varphi_1, \varphi_2) &= W\left(W\left(a_1, a_1; \frac{\varphi_1}{1-\varphi_2}\right), a_2; 1-\varphi_2\right) \\ &= W(a_1, a_2; 1-\varphi_2) \\ &= W(a_1, a_2, \varphi_1), \end{aligned}$$

where $\varphi_1 + \varphi_2 = 1$.
Furthermore, we remind that we have used the following property of convex metric spaces, i.e. that $W(x, x, \lambda) = x$, $\forall x \in X$ and $\lambda \in [0,1]$.
For $n = 3$, we have that

$$\begin{aligned} W(a_1, a_2, a_3; \varphi_1, \varphi_2, \varphi_3) &= W\left(W\left(a_1, a_2; \frac{\varphi_1}{1-\varphi_3}, \frac{\varphi_2}{1-\varphi_3}\right), a_3; 1-\varphi_3\right) \\ &= W(b_3, a_3; 1-\varphi_3), \end{aligned}$$

where

$$\begin{aligned} b_3 &= W\left(a_1, a_2; \frac{\varphi_1}{1-\varphi_3}, \frac{\varphi_2}{1-\varphi_3}\right) \\ &= W\left(a_1, a_2; 1 - \frac{\varphi_2}{1-\varphi_3}\right), \end{aligned}$$

as in the case when $n = 2$. Also, we have that $\varphi_1 + \varphi_2 + \varphi_3 = 1$. Furthermore, let C be a nonempty convex subset of a normed space E and $T : C \to C$ a δ-contraction mapping. In 2005 Suantai [156] introduced a modified Noor iterative method with the sequences
$(\alpha_n), (\beta_n), (a_n), (b_n), (c_n) \subseteq [0, 1]$,$x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n T y_n + \beta_n T z_n + (1 - \alpha_n - \beta_n) x_n, \ n \geq 1 \\ y_n = b_n T z_n + c_n T x_n + (1 - b_n - c_n) x_n \\ z_n = a_n T x_n + (1 - a_n) x_n. \end{cases} \tag{4.3.0.1}$$

In the case when C is a nonempty convex subset of a convex metric space E, Berinde modified the above iteration with the use of the convexity structure W and defined the iteration as follows

$$\begin{cases} x_{n+1} = W\left(T y_n, W\left(T z_n, x_n, \dfrac{\beta_n}{1 - \alpha_n}\right), \alpha_n\right) \\ y_n = W\left(T z_n, W\left(T x_n, x_n, \dfrac{c_n}{1 - b_n}\right), b_n\right) \\ z_n = W\left(T x_n, x_n, a_n\right). \end{cases} \tag{4.3.0.2}$$

Moreover, in [6], Agarwal et al. presented a new iteration defined on nonempty convex subset C on normed spaces, that can be adapted easily on convex metric spaces. This iteration is defined by $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n T y_n + (1 - \alpha_n) T x_n \\ y_n = b_n T x_n + (1 - b_n) x_n. \end{cases} \tag{4.3.0.3}$$

In the context of nonempty convex subset C of a convex metric space, the above iteration is defined by $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = W\left(T y_n, T x_n, \alpha_n\right) \\ y_n = W\left(T x_n, x_n, b_n\right). \end{cases} \tag{4.3.0.4}$$

In [4] Abbas and Nazir improved the above iteration and they presented a three-step iteration. We will present it in the context of convex metric space, as follows

$$\begin{cases} x_{n+1} = W\left(T z_n, T y_n, \alpha_n\right) \\ y_n = W\left(T z_n, T x_n, b_n\right) \\ z_n = W\left(T x_n, x_n, a_n\right). \end{cases} \tag{4.3.0.5}$$

In the fixed point literature we can find other classical iterations. From [55], we will recall them in the context of convex metric spaces
SP iteration, with $x_0 = x \in C$ and

$$\begin{cases} x_{n+1} = W\left(T y_n, y_n, \alpha_n\right) \\ y_n = W\left(T z_n, z_n, b_n\right) \\ z_n = W\left(T x_n, x_n, a_n\right). \end{cases} \tag{4.3.0.6}$$

S iteration, with $x_0 = x \in C$ and

$$\begin{cases} x_{n+1} = W\left(T y_n, T x_n, \alpha_n\right) \\ y_n = W\left(T x_n, x_n, b_n\right). \end{cases} \tag{4.3.0.7}$$

CR iteration, with $x_0 = x \in C$ and

$$\begin{cases} x_{n+1} = W\left(T y_n, y_n, \alpha_n\right) \\ y_n = W\left(T z_n, T x_n, b_n\right) \\ z_n = W\left(T x_n, x_n, a_n\right). \end{cases} \tag{4.3.0.8}$$

Additionally, in [55] Gursoy and Karakaya presented a modified Picard-S hybrid iteration.

$$\begin{cases} x_{n+1} = T y_n \\ y_n = W\left(T z_n, T x_n, b_n\right) \\ z_n = W\left(T x_n, x_n, a_n\right). \end{cases} \tag{4.3.0.9}$$

Also, regarding the question of Agarwal, Sintunavarat and Pitea in [155] introduced a new iteration that is qualitatively better than that of Agarwal's and of Picard. That is
S_n *iteration*, with $x_0 = x \in C$ and

$$\begin{cases} x_{n+1} = W\left(Ty_n, Tz_n, \alpha_n\right) \\ y_n = W\left(Tx_n, x_n, b_n\right) \\ z_n = W\left(y_n, x_n, a_n\right). \end{cases} \tag{4.3.0.10}$$

Even though we are working in the setting of a convex metric space, we shall employ an additional property of hyperbolic spaces in the sense of Goebel and Kirk, that is $W(x, y, \lambda) = W(y, x, 1-\lambda)$, for each $x, y \in X$ and $\lambda \in [0, 1]$. This property is easily satisfied in a linear normed space.
The first main result of this section improve Suantai's iteration 4.3.0.2 on convex metric spaces. The next iteration is an implicit algorithm made by multiple convex combinations. Let's call it *Suantai implicit*

$$\begin{cases} x_{n+1} = W(y_n, Ty_n, Tx_{n+1}; 1-\alpha_n-\beta_n, \beta_n, \alpha_n) \\ y_n = W(z_n, Tz_n, Ty_n; 1-b_n-c_n, c_n, b_n) \\ z_n = W(Tz_n, x_n, a_n). \end{cases}$$

In terms of simple convex combinations, this iteration is

$$\begin{cases} x_{n+1} = W\left(Tx_{n+1}, W\left(Ty_n, y_n, \dfrac{\beta_n}{1-\alpha_n}\right); \alpha_n\right) \\ y_n = W\left(Ty_n, W\left(Tz_n, z_n, \dfrac{c_n}{1-b_n}\right); b_n\right) \\ z_n = W(Tz_n, x_n, a_n). \end{cases} \tag{4.3.0.11}$$

Our first result of this section concerns under what condition iteration 4.3.0.11 is convergent to the unique fixed point of a δ-contraction.

Theorem 4.3.1. *Let C be a nonempty, closed and convex subset of a complete convex metric space X. Let $T : C \to C$ be a δ-contraction. Let (a_n), (b_n), (c_n), (α_n), (β_n), $(b_n + c_n)$ and $(\alpha_n + \beta_n)$ sequences in $[0, 1]$ such that $\sum_{k=0}^{\infty}(\alpha_k + \beta_k) = \infty$. Then (x_n) in 4.3.0.11 is convergent to the unique fixed point p of T.*

As particular cases of iteration 4.3.0.11 we get classical iterations, such as implicit Noor, respectively implicit Ishikawa iterative processes.

Remark 4.3.2. In 4.3.0.11, taking $\beta_n = c_n = 0$, we get *Implicit Noor iteration* 1.5.0.6 and taking $\beta_n = c_n = a_n = 0$, we get *Implicit Ishikawa iteration* 1.5.0.7.

Finally, we will study two iterations which are modified implicit Noor II-type iterations through multiple convex combinations.
Let's call the first one *Implicit Noor II with multiple convex combinations*, or simply, *IN II m.c.c.* This is defined, through multiple convex combinations, as

$$\begin{cases} x_{n+1} = W(Ty_n, Tz_n, Tx_{n+1}; 1-\alpha_n-\beta_n, \beta_n, \alpha_n) \\ y_n = W(Tz_n, Tx_n, Ty_n; 1-b_n-c_n, c_n, b_n) \\ z_n = W(Tz_n, x_n, a_n). \end{cases}$$

With simple convex combinations, the iteration becomes

$$\begin{cases} x_{n+1} = W\left(Tx_{n+1}, W\left(Tz_n, Ty_n; \dfrac{\beta_n}{1-\alpha_n}\right); \alpha_n\right) \\ y_n = W\left(Ty_n, W\left(Tx_n, Tz_n; \dfrac{c_n}{1-b_n}\right); b_n\right) \\ z_n = W(Tz_n, x_n, a_n). \end{cases} \tag{4.3.0.12}$$

Concerning the convergence of the iteration 4.3.0.12, we have the following theorem.

Theorem 4.3.3. *Let C be a nonempty, closed and convex subset of a complete convex metric space X. Let $T : C \to C$ be a δ-contraction. Let (a_n), (b_n), (α_n) ,$(b_n + c_n)$, $(\alpha_n + \beta_n)$ sequences in $(0, 1)$. Then (x_n) in 4.3.0.12 is convergent to the unique fixed point p of T.*

We present the last iteration. Let's call it *Double Implicit Noor II with multiple convex combinations*, or simply, *DIN II m.c.c.* This is defined, through multiple convex combinations, as

$$\begin{cases} x_{n+1} = W(Ty_n, Tx_{n+1}, Tx_{n+1}; 1-\alpha_n-\beta_n, \beta_n, \alpha_n) \\ y_n = W(Tz_n, Ty_n, Ty_n; 1-b_n-c_n, c_n, b_n) \\ z_n = W(Tz_n, x_n, a_n). \end{cases}$$

With simple convex combinations, the iteration becomes

$$\begin{cases} x_{n+1} = W\left(Tx_{n+1}, W\left(Tx_{n+1}, Ty_n; \dfrac{\beta_n}{1-\alpha_n}\right); \alpha_n\right) \\ y_n = W\left(Ty_n, W\left(Ty_n, Tz_n; \dfrac{c_n}{1-b_n}\right); b_n\right) \\ z_n = W(Tz_n, x_n, a_n). \end{cases} \tag{4.3.0.13}$$

In our last theorem of this section, sufficient conditions for the convergence of the iterative process 4.3.0.13 are presented.

Theorem 4.3.4. *Let C be a nonempty, closed and convex subset of a complete convex metric space X. Let $T : C \to C$ be a δ-contraction. Let (a_n), (b_n), (α_n) ,$(b_n + c_n)$, $(\alpha_n + \beta_n)$ sequences in $(0, 1)$. Then (x_n) in 4.3.0.13 is convergent to the unique fixed point p of T.*

Bibliography

[1] C.T. Aage, J.N. Salunke, *Some fixed point theorems for expansion onto mappings on cone metric spaces*, Acta Mathematica Sinica, English Series **27** (2011), no. 6, 1101-1106, doi:10.1007/s10114-011-9606-9.

[2] M. Abbas, S.H. Khan, B.E. Rhoades, *Simpler is also better in approximating fixed points*, Appl. Math. Comput. **205** (2008), no. 1, 428-431.

[3] M. Abbas, D. Ilić and T. Nazir, *Iterative approximation of fixed points of generalized weak Presic type k-step iterative method for a class of operators*, Filomat **29** (2015), no. 4, 713-724.

[4] M. Abbas and T. Nazir,*A new faster iteration process applied to constrained minimization and feasibility problems*, Matematicki Vesnik **66:2** (2014), 223-234.

[5] R.P. Agarwal, D. O'Regan, D.R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, Series : Topological Fixed Point Theory and Its Applications, **6**, Springe, New York (2009).

[6] R. Agarwal, D. O'Regan, D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, Journal of Nonlinear and Convex Analysis **8** (2007), no. 1, 61-79.

[7] S. Akbulut and M. Ozdemir, *Picard iteration converges faster than Noor iteration for a class of quasi-contractive operators*, Chiang Mai J.Sci. **39** (2012), no. 4, 688-692.

[8] Y. I. Alber, S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces*, New results in operator theory and its applications: the Israel M. Glazman memorial volume, ser. Oper. Theory, Adv. Appl., I. Gohberg, Ed. Basel: Birkhauser, 1997, vol. 98, 7-22.

[9] **C.D. Alecsa**, *Common fixed points of Prešić operators via simulation functions*, J. Nonlinear Convex Anal. **20** (2019), no. 3, 1-15.

[10] **C.D. Alecsa**, *Approximating fixed points for nonlinear generalized mappings using Ishikawa iteration*, Rend. Circ. Mat. Palermo, II. Ser. **68** (2019), no. 1, 163-191.

[11] **C.D. Alecsa**, *Fixed point theorems for generalized contraction mappings on b-rectangular metric spaces*, Studia Universitatis Babeş-Bolyai Mathematica **62** (2017), no. 4, 495-520.

[12] **C.D. Alecsa**, *Some fixed point results linked To $\alpha-\beta$ rational contractions and modified multivalued Hardy-Rogers operators*, J. Fixed Point Theory, 2018:3 (2018), Article ID 3, 1-22.

[13] **C.D. Alecsa**, *Sequences of contractions on cone metric spaces over Banach algebras and applications to nonlinear systems of equations and systems of differential equations*, arXiv:1906.06261, 26 pages, *(submitted)*.

[14] **C.D. Alecsa, A. Petruşel**, *Some variants of Ćirić's multi-valued contraction principle*, Anal. Univ. de Vest Timişoara Ser. Mat. Inf. , vol. 1, 2019, (*accepted*).

[15] **C.D. Alecsa**, *Stability results and qualitative properties for Mann's algorithm via admissible perturbations technique*, Applied Anal. Optim. **1** (2017), no. 2, 327-344.

[16] **C. Alecsa**, *On new faster fixed point iterative schemes for contraction operators and comparison of their rate of convergence in convex metric spaces*, Int. Journal of Nonlinear Anal. and Appl. **8** (2017), no. 1, 353-388, doi:10.22075/IJNAA.2017.11144.1543.

[17] **C.D. Alecsa, A. Petruşel**, *On some fixed point theorems for multi-valued operators by altering distance technique*, J. Nonlinear Var. Anal. **1** (2017), 237-251.

[18] **C.D. Alecsa**, *Some fixed point results regarding convex contractions of Presić type*, J. Fixed Point Theory and Appl. 20:**20** (2018), no. 1, doi:10.1007/s11784-018-0488-7.

[19] M.A. Alghamdi, N. Hussain, P. Salimi, *Fixed point and coupled fixed point theorems on b-metric-like spaces*, J. Ineq. Appl. (2013), 2013:**402**, doi:10.1186/1029-242X-2013-402.

[20] M.A. Alghamdi, S.H. Alnafei, S. Radenovic and N. Shahzad, *Fixed point theorem for convex contraction mappings on cone metric spaces*, Math. Comput. Model. **54** (2011), 2020-2026.

[21] A.S.S. Alharbi, H.H. Alsulami, E. Karapinar, *On the power of simulation and admissible functions in metric fixed point theory*, J. Funct. Spaces, https://www.hindawi.com/journals/jfs/aip/2068163.

[22] S.H. Alnafei, S. Radenovic and N. Shahzad, *Fixed point theorems for mappings with convex diminishing diameters on cone metric spaces*, Appl. Math. Lett. **24** (2011), 2162-2166.

[23] H.H. Alsulami, E. Karapinar, F. Khojasteh, A.F. Roldán-López-de-Hierro, *A proposal to the study of contractions in quasi-metric spaces*, Discrete Dyn. Nat. Soc. (2014), Article ID 269286, 10 pages, doi:10.1155/2014/269286.

[24] A. Amini-Harandi, A. Petruşel, *A fixed point theorem by altering distance technique in complete metric spaces*, Miskolc Math. Notes, **14** (2013), no. 1, 11-17.

[25] J. Andres, L. Górniewicz, *On The Banach Contraction Principle For Multivalued Mappings*, Approximation, Optimization and Mathematical Economics (M. Lassonde ed.), 23 pages.

[26] A.H. Ansari, H. Işik, S. Radenović, *Coupled fixed point theorems for contractive mappings involving new function classes and applications*, Filomat **31** (2017), no. 7, 1893-1907.

[27] H. Argoubi, B. Samet, C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order*, J. Nonlinear Sci. Appl. **8** (2015), 1082-1094.

[28] M. Asadi, *Some results of fixed point theorems in convex metric spaces*, Nonlinear Func. Anal. Appl. **19** (2014), no. 2, 171-175.

[29] H. Aydi, E. Karapinar, S. Moradi, *Coincidence points for expansive mappings under c-distance in cone metric spaces*, Int. Journal of Math. and Math. Sci., vol. 2012, Article ID 308921, doi:10.1155/2012/308921.

[30] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math. **3** (1922), 133-181.

[31] L. Barbet, K. Nachi, *Sequences of contractions and convergence of fixed points*, Monografias del Seminario Matemático García de Galdeano **33** (2006), 51-58.

[32] V. Berinde, *Iterative Approximation of Fixed Points*, Springer-Verlag Berlin Heidelberg, 2007.

[33] V. Berinde, M. Păcurar, *Stability of k-step fixed point iterative methods for some Presic type contractive mappings*, J. Inequal. Appl. 2014:**149** (2014), doi:10.1186/1029-242X-2014-149.

[34] V. Berinde, *Picard iteration converges faster than Man iteration for a class of quasi-contractive operators*, Fixed Point Theory Appl. **1** (2004), 1-9.

[35] F.F. Bonsall, *Lectures on Some Fixed Point Theorems of Functional Analysis*, Tata Institute of Fundamental Research, Bombay, 1962.

[36] M. Boriceanu, *Fixed point theory for multivalued generalized contraction on a set with two b-metrics*, Stud. Univ.Babeş-Bolyai **54** (2009), no. 3.

[37] D.W. Boyd, J.S. Wong, *On nonlinear contractions*, Proc. Am. Math. Soc. **20** (1969), 458C464.

[38] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen **57** (2000), 31-37.

[39] M.R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[40] C. Chifu, G. Petruşel, *Fixed point results for multivalued Hardy-Rogers Contractions in b-metric spaces*, Filomat **31** (2017), no. 8, 2499-2507.

[41] R. Chugh, P. Malik, V. Kumar, *On a new faster implicit fixed point iterative scheme in convex metric spaces*, J. Function Spaces, Article ID 905834 (2015), 149-154.

[42] L. Ćirić, *Fixed points for generalized multi-valued contractions*, Mat. Vesnik, **9** (1972), 265-272.

[43] L.B. Ciric, N.T. Nikolic, *Convergence of the Ishikawa iterates for multi-valued mappings in metric spaces of hyperbolic type*, Matematicki Vesnik, **60** (2008), 149-154.

[44] L.B. Ćirić and L.B. Prešić, *On Presic type generalisation of Banach contraction mapping principle*, Acta Math. Univ. Comenian **76** (2007), no. 2, 143-147.

[45] R.D. Daheriya, R. Jain, M. Ughade, *Some fixed point theorem for expansive type mapping in dislocated metric spaces*, Internat. Sch. Research Network, ISRN Math. Analysis, vol. 2012, Article ID 376832, 5 pages, doi:10.5402/2012/376832.

[46] S. Dhompongsa, H. Yingtaweesittikul, *Diametrically contractive multivalued mappings*, Fixed Point Theory Appl. (2007), Article ID 19745.

[47] H.S. Ding, V. Ozturk, S. Radenovic, *On some new fixed point results in b-rectangular metric spaces*, J. Nonlinear Sci. Appl. **8** (2015), 378-386.

[48] N.V. Dung, A. Petruşel, *On iterated functions systems consisting of Kannan maps, Reich maps, Chatterjea type maps, and related results*, J. Fixed Point Theory Appl. (2017), doi:10.1007/s11784-017-0419-z.

[49] P. N. Dutta, B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory Appl., vol. 2008, (2008), 8 pages.

[50] Z.M. Fadail, A.G.B. Ahmad, A.H. Ansari, S. Radenović, M. Rajović, *Some common fixed point results of mappings in* $0-\sigma-$*complete metric-like spaces via new function*, Appl. Math. Sci. **9** (2015), no. 83, 4109-4127.

[51] H. Fukhar-ud-din and V. Berinde, *Iterative methods for the class of quasi-contractive type operators and comparsion of their rate of convergence in convex metric spaces*, Filomat **30** (2016), no. 1, 223-230.

[52] R. George, S. Radenovic, K.P. Reshma, S. Shukla, *Rectangular b-metric space and contraction principles*, J. Nonlinear Sci. Appl. **8** (2015), 1005-1013.

[53] V. Glăvan, V. Guţu, *On the dynamics of contracting relations*, Analysis and Optimization of Differential Systems (Barbu V., Lasiecka I., Tiba D., Varsan C. eds.), Kluwer Acad. Publ. (2003), 179-188.

[54] K. Goebel and W. Kirk, *Iteration processes for nonexpansive mappings*, Topological Methods in Nonlinear Functional Analysis, Contemporary Mathematics **21** (1983), Amer. Math. Soc. Providence, 115-123.

[55] F. Gursoy and V.Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument*, http://arxiv.org/abs/1403.2546.

[56] A.A. Harandi, *Endpoints of set-valued contractions in metric spaces*, Nonlinear Anal. **72** (2010), 132-134.

[57] G.E. Hardy, A.D. Rogers, *A generalisation of fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), 201-208.

[58] A.M. Harder, T.L. Hicks, *Stability results for fixed point iteration procedures*, Math. Japon. **33** (1988), 693-706.

[59] H. Huang, S. Radenović, *Common fixed point theorems of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras and applications*, J. Nonlinear Sci. Appl. **8** (2015), 787-799.

[60] H. Huang, S. Hu, B.Z. Popović, S. Radenović, *Common fixed point theorems for four mappings on cone b-metric spaces over Banach algebras*, J. Nonlinear Sci. Appl. **9** (2016), 3655-3671.

[61] N. Hussain, A.A. Harandi, Y.J. Cho, *Approximate endpoints for set-valued contractions in metric spaces*, Fixed Point Theory Appl., vol. 2010, 13 pages, doi:10.1155/2010/614867.

[62] V. Istrăţescu, *Some fixed point theorems for convex contraction mappings and convex nonexpansive mappings (I)*, Lib. Math. **1** (1981), 151-164.

[63] V. Istrăţescu, *Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters - I*, Ann. Mat. Pura Appl. **130** (1982), 89-104.

[64] V. Istrăţescu, *Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters, II*, Ann. Mat. Pura Appl. **134** (1983), 327-362.

[65] J.R. Jachymski, *Caristi's fixed point theorem and selections of set-valued contractions*, J. Math. Anal. Appl. **227** (1998), 55-67.

[66] S. Janković, Z. Kadelburg, S. Radenović, *On cone metric spaces : A survey*, Nonlinear Anal. **74** (2011), no. 7, 2591-2601.

[67] Z. Kadelburg, S. Radenovic, *Fixed point results in generalized metric spaces without Hausdorff property*, Math Sci. **125** (2014), no. 8, doi:10.1007/s40096-014-0125-6.

[68] Z. Kadelburg, S. Radenovic, *On generalized metric spaces : A survey*, TWMS J. Pure Appl. Math. **5** (2014), no. 1, 3-13.

[69] Z. Kadelburg, S. Radenovic, *Pata-type common fixed point results in b-metric and b-rectangular metric spaces*, J. Nonlinear Sci. Appl. **8** (2015), 944-954.

[70] T. Kamran, Q. Kiran, *Fixed point theorems for multi-valued mappings obtained by altering distances*, Math. and Comp. Modell., **54** (2011), 2772-2777.

[71] E. Karapinar, *Fixed points results via simulation functions*, Filomat **30** (2016), no. 8, 2343-2350.

[72] E. Karapinar, *Fixed point theorems in cone Banach spaces*, Fixed Point Theory Appl. (2009), Article ID 609281, doi:10.1155/2009/609281.

[73] A. Khamsi, *Remarks on cone metric spaces and fixed point theorems of contractive mappings*, Fixed Point Theory Appl. (2010), Article ID 2010:315398, 7 pages.

[74] M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984), 1-9.

[75] M.S. Khan, M. Berzig and B. Samet, *Some convergence results for iterative sequences of Prešić type and applications*, Adv. Difference. Equ. vol. 2012 (2012), Article 38.

[76] F. Khojasteh, S. Shukla, S. Radenović, *A new approach to the study of fixed point theory for simulation functions*, Filomat **29** (2015), no. 6, 1189-1194.

[77] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Amer. Math. Soc. **357** (2005), no. 1, 89-128.

[78] S. Kumar, S.K. Garg, *Expansion mapping theorems in metric spaces*, Int. J. Contemp. Math. Sciences **4** (2009), no. 36, 1749 - 1758.

[79] V. L. Lazăr, *Fixed point theory for multivalued φ-contractions*, Fixed Point Theory and Appl. 2011:**50** (2011).

[80] T. Lazăr, G. Moţ, G. Petruşel, S. Szentesi, *The theory of Reich's fixed point theorem for multivalued operators*, Fixed Point Theory and Appl., vol. 2010, Article ID 178421, 10 pages, doi:10.1155/2010/178421.

[81] T. A. Lazăr, A. Petruşel, N. Shahzad, *Fixed points for non-self operators and domain invariance theorems*, Nonlinear Anal. **70** (2009), 117-125.

[82] T. Lazăr, D. O'Regan, A. Petruşel, *Fixed points and homotopy results for Ćirić-type multivalued operators on a set with two metrics*, Bull. Korean Math. Soc. **45** (2008), no. 1, 67-73.

[83] B. Li, H. Huang, *Fixed point results for weak φ-contractions in cone metric spaces over Banach algebras and applications*, Journal of Function Spaces (2017), Article ID 5054603, 6 pages.

[84] X.L. Liu, A.H. Ansari, S. Chandok, S. Radenović, *On some results in metric spaces using auxiliary simulation functions via new functions*, J. Comput. Anal. Appl. **24** (2018), 1103-1114.

[85] H. Liu, S. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*, Fixed Point Theory and Appl. (2013), Article ID 2013:320, 1-10.

[86] Z. Liu, Z. Wu, S.M. Kang, S. Lee, *Some fixed point theorems for nonlinear set-valued contractive mappings*, J. of Appl. Math., volume 2012, Article ID 786061, 13 pages, doi:10.1155/2012/786061.

[87] H.V. Machado, *A characterization of convex subsets of normed spaces*, Kodai Math. Sem. Rep. **25** (1973), 307-320.

[88] R. Miculescu and A. Mihail, *A generalization of $Matkowski's$ fixed point theorem and $Istr\check{a}\c{t}escu's$ fixed point theorem concerning convex contractions*, J. Fixed Point Theory Appl. (2017).

[89] S.N. Mishra, R. Pant, R. Panicker, *Sequences of (ψ, ϕ)-weakly contractive mappings and stability of fixed points*, Int. Journal of Math. Analysis **7** (2013), no. 22, 1085-1096.

[90] S.N. Mishra, S.L. Singh, R. Pant, *Some new results on stability of fixed points*, Chaos, Solitons & Fractals **45** (2012), 1012-1016.

[91] M. Moosaei, *Fixed point theorems in convex metric spaces*, Fixed Point Theory Appl. 2012:**164** (2012), doi:10.1186/1687-1812-2012-164.

[92] M. Moosaei, *Common fixed points for some generalized contraction pairs in convex metric spaces*, Fixed Point Theory Appl. 2014:**98** (2014), doi:10.1186/1687-1812-2014-98.

[93] M. Moosaei, A. Azizi, *On coincidence points of generalized contractive pair mappings in convex metric spaces*, J. of Hyperstructures **2** (2015), no. 4, 136-141.

[94] S. Moradi, A. Farajzadeh, *On the fixed point of $(\psi - \varphi)$-weak and generalized $(\psi - \varphi)$-weak contraction mappings*, Appl. Math. Lett. **25** (2012), 1257-1262.

[95] V. Mureşan and A.S. Mureşan, *On the theory of fixed point theorems for convex contraction mappings*, Carpatian J. Math. **31** (2015), no. 3, 365-371.

[96] S.B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475-488.

[97] S.B. Nadler Jr., *Sequences of contractions and fixed points*, Pacific J. Math. **27** (1968), 579-585.

[98] H. Olaoluwa, J.O. Olaleru, *A hybrid class of expansive-contractive mappings in cone b-metric spaces*, Afr. Mat., doi:10.1007/s13370-015-0381-0.

[99] H.K. Pathak, *Some fixed points of expansion mappings*, Internat. J. Math & Math. Sci. **19** (1996), no. 1, 97-102.

[100] H.K. Pathak, R. George, H.A. Nabwey, M. S. El-Paoumy and K. P. Reshma, *Some generalized fixed point results in b-metric space and application to matrix equations*, Fixed Point Theory Appl. **101** (2015).

[101] S.R. Patil, J.N. Salunke, *Fixed point theorems for expansion mappings in cone rectangular metric spaces*, Gen. Math. Notes **29** (2015), no. 1, 30-39.

[102] M. Păcurar, *Approximating common fixed points of Presić-Kannan type operators by a multi-step iterative method*, An. Stiint. Univ. Ovidius Constanţa Ser. Mat. **17** (2009), no. 1, 153-168.

[103] M. Păcurar, *A multi-step iterative method for approximating fixed points of Prešić-Kannan operators*, Acta Math. Univ. Comenianae **79** (2010), no. 1, 77-88.

[104] M. Păcurar, *A multi-step iterative method for approximating common fixed points of Presić-Rus type operators on metric spaces*, Studia Univ. "Babeş-Bolyai", Math. **LV** (2010), no. 1, 149-162.

[105] M. Păcurar, *Sequences of almost contractions and fixed points*, Carpathian J. Math. **24** (2008), no. 2, 101-109.

[106] M. Păcurar, *A multi-step method for approximating common fixed points of Presić-Rus type operators on metric spaces.* Studia Univ. Babeş-Bolyai Math., vol. LV (2010), no. 1, 149-162.

[107] M. Păcurar, *Fixed points of almost Presić operators by a k-step iterative method*, An. Ştiinţ. Univ. AI.I. Cuza Iaşi Mat. **57** (2011).

[108] D.K. Patel, P.R. Patle, L. Budhia, D. Gopal, *Coincidence point results involving a generalized class of simulation functions*, arXiv:1708.05693.

[109] T.P. Petru, M. Boriceanu, *Fixed point results for generalized φ-contraction on a set with two metrics*, Top. Methods Nonlin. Anal. **33** (2009), 315-326.

[110] T.P. Petru, A. Petruşel, J.C. Yao, *Ulam-Hyers stability of operatorial equations and inclusions via nonself operators*, Taiwanese J. Math. **15** (2011), no. 5, 2195-2212.

[111] A. Petruşel, *Ćirić type fixed point theorems*, Stud. Univ. Babeş-Bolyai, **59** (2014), no. 2, 233-245.

[112] A. Petruşel, *Multivalued weakly Picard operators and applications*, Sci. Math. Jpn. **59** (2004), 169-202.

[113] A. Petruşel, *Some variants of the contraction principle for multi-valued operators, generalizations and applications*, preprint.

[114] A. Petruşel, I.A. Rus, *An abstract point of view on iterative approximation schemes of fixed points for multivalued operators*, J. Nonlinear Sci. Appl. **6** (2013), 97-107.

[115] A. Petruşel, I.A. Rus, *Well-posedness of the fixed point problem for multivalued operators*, Applied Analysis and Differential Equations (Cârjă, O., Vrabie I.I. eds.), World Scientific (2007), 295-306.

[116] A. Petruşel, *Operatorial Inclusions*, House of the Book of Science, Cluj-Napoca, Romania, 2002.

[117] A. Petruşel, I.A. Rus, *Multivalued Picard and weakly Picard operators*, Fixed Point Theory and Appl. (J. Garcia Falset, E. Llorens Fuster, B. Sims eds.), Yokohama Publisher (2004), 207-226.

[118] G. Petrusel, T. Lazar, V.L. Lazar, *Fixed points and coupled fixed points for multi-valued $(\psi - \varphi)$-contractions in b-metric spaces*, Applied Analysis and Optimization **1** (2017), no. 1, 99-112.

[119] A. Petruşel, I.A. Rus, *The theory of a metric fixed point theorem for multivalued operators*, Proc. Ninth International Conference on Fixed Point Theory and Applications, Changhua, Taiwan, July 16-22 (2009), (L.J. Lin, A. Petruşel, H.K. Xu - eds.), Yokohama Publ. 2010, 161-175.

[120] A. Petruşel, I.A. Rus, J.C. Yao, *Well-posedness in the generalized sense of the fixed point problems for multivalued operators*, Taiwanese J. Math. **11** (2007), no. 3, 903-914.

[121] A. Petruşel, G. Petruşel, *Selection theorems for multivalued generalized contractions*, Math. Morav. **9** (2005), 43-52.

[122] W. Phuengrattana and S. Suantai, *Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach Spaces*, Thai J. Math. **11** (2013), no. 11, 217-226.

[123] O. Popescu, *Fixed points for $(\psi - \varphi)$-weak contractions*, Appl. Math. Letters, **24** (2011), 1-4.

[124] O. Popescu, G. Stan, *A generalization of Nadler's fixed point theorem*, Results Math. (2017), doi:10.1007/s00025-017-0694-4.

[125] B. Prasad, R. Sahni, *Endpoints of multivalued contraction operators*, vol. 2013, 7 pages, doi:10.1155/2013/542302.

[126] S.B. Prešić, *Sur une classe d'inéquations aux differences finies et sur la convergence de certain suites*, Pub. de l'Inst. Math. Belgrade **5** (1965), no. 19, 75-78.

[127] S. Radenović, S. Chandok, *Simulation type functions and coincidence point results*, Filomat, *to appear*.

[128] S. Radenović, F. Vetro, J. Vujaković, *An alternative and easy approach to fixed point results via simulation functions*, Demonstr. Math. **50** (2017), 223-230, https://doi.org/10.1515/dema-2017-0022.

[129] R.A. Rashwan, *Some fixed point theorems in cone rectangular metric spaces*, Math. Aeterna, **2** (2012), no. 6, 573-587.

[130] J.S. Raymond, *Multivalued contractions*, Set-Valued Analysis **2** (1994), 559-571.

[131] S. Reich, *Fixed points of contractive functions*, Bolletino U.M.I. **4** (1972), no. 5, 26-42.

[132] S. Reich and I. Safrir, *Nonexpansive iteration in hyperbolic spaces*, Nonlinear. Anal. **15** (1990), 537-558.

[133] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., Theory Methods Appl. **47** (2001), no. 4, 2683-2693.

[134] B. E. Rhoades, H.K. Pathak, S.N. Mishra, *Some weakly contractive mappings theorems in partially ordered spaces and applications*, Demonstratio Mathematica, vol. XLV (2012), no. 3.

[135] A.F. Roldán-López-de-Hierro, E. Karapinar, C. Roldán-López-de-Hierro, J. Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. **275** (2015), 345-355.

[136] J.R. Roshan, N. Hussain, V. Parvaneh, Z. Kadelburg, *New fixed point results in rectangular b-metric spaces*, Nonlinear Analysis: Modelling and Control **21** (2016), no. 5, 614-634, doi:10.15388/NA.2016.5.4.

[137] W. Rudin, *Functional Analysis (second edition)*, McGraw-Hill, New York, 1991.

[138] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[139] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, Romania, 2008.

[140] I.A. Rus, *An iterative method for the solution of the equation $x = f(x, ..., x)$*, Rev. Anal. Numer. Theor. Approx. **10** (1981), no. 1, 95-100.

[141] I. A. Rus, *Picard operators and applications*, Sci. Math. Jpn. **58** (2003), 191-219.

[142] I.A. Rus, A. Petruşel, A. Sîntămărian, *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal. **52** (2003), no.8, 1947-1959 .

[143] I.A. Rus, A. Petruşel, A. Sîntămărian, *Data dependence of the fixed point set of multivalued weakly Picard operators*, Stud. Univ. Babeş-Bolyai Math. **46** (2001), no. 2, 111-121.

[144] K.P.R. Sastry, Ch. S. Rao, C. Sekhar and M. Balaiah, A *Fixed Point Theorem for Cone Convex Contractions of Order $m \geq 2$*, Int. J. Math. Sci. Eng. Appl. **6**, no. 1, 263-271.

[145] M.A. Şerban, *The Fixed Point Theory for the Operators on Cartesian Product*, (Romanian), Cluj University Press, Cluj-Napoca, 2002.

[146] N. Shahzad, H. Zegeye, *On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces*, Nonlinear Anal. **71** (2009), 838-844.

[147] T. Shimizu, W. Takahashi, *Fixed point theorems in certain convex metric spaces*, Math. Japon., **37** (1992), 855-859.

[148] S. Shukla, S. Radenović, *Prešić-Boyd-Wong type results in ordered metric spaces*, Int. J. Anal. Appl. **5** (2014), no. 2, 154-166.

[149] S. Shukla and S. Radenovic, *Presić-Maya type theorems in ordered metric spaces*, Gulf J. Math. **2** (2014), no. 2, 73-82.

[150] S. Shukla, S. Radenovic and S. Pantelić, *Some fixed point theorems for Prešić-Hardy-Rogers type contractions in metric spaces*, J.Math. (2013), doi:10.1155/2013/295093.

[151] S. Shukla and S. Radenovic, *Some generalizations of Prešić type mappings and applications.* An. Ştiinţ. Univ. AI.I. Cuza Iaşi Mat., doi:10.1515/aicu-2015-0026.

[152] S.L. Singh, V. Chada, *Round-off stability of iterations for multivalued operators*, C.R. Math. Rep. Acad. Sci. Canada. **17** (1995), no. 5, 187-192.

[153] S.L. Singh, S.N. Mishra, S. Jain, *Round-off stability for multivalued mappings*, Fixed Point Th. Appl. **12** (2012), doi:10.1186/1687-1812-2012-12.

[154] S.P. Singh, W. Russell, *A note on a sequence of contraction mappings*, Can. Math. Bull. **12** (1969), 513-516.

[155] W. Sintunavarat and A. Pitea, *On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis*, J. Nonlinear Science Appl. **9** (2016), 2553-2562.

[156] S. Suantai, *Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **311** (2005), 506-517.

[157] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings I*, Kodai. Math. Sem. Rep. **22** (1970), 142-149.

[158] L.A. Talman, *Fixed points for condensing multifunctions in metric spaces with convex structure*, Kodai Math. Sem. Rep. **29** (1977), 62-70.

[159] C. Wang, T. Zhang, *Approximating common fixed points for a pair of generalized nonlinear mappings in convex metric spaces*, J. Nonlinear. Sci. Appl. **9** (2016), 1-7.

[160] S. Xu, S. Radenović, *Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality*, Fixed Point Theory and Appl. (2014), Article ID 2014:102, 1-12.

[161] Z. Xue, *Fixed points theorems for generalized weakly contractive mappings*, Bull. Aust. Math. Soc. **93** (2016), 321-329, doi:10.1017/S0004972715001069.

[162] O. Yamaod, W. Sintunavarat, *An approach to the existence and uniqueness of fixed point results in $b-$metric spaces via $s-$simulation functions*, J. Fixed Point Theory Appl. (2017), doi:10.1007/s11784-017-0453-x.

[163] P. Yan, J. Yin, Q. Leng, *Some coupled fixed point results on cone metric spaces over Banach algebras and applications*, J. Nonlinear Sci. Appl. **9** (2016), 5661-5671.

[164] P. Zangenehmehr, A.P. Farajzadeh, R. Lashkaripour, Ar. Karamian, *On fixed point theory for generalized contractions in cone rectangular metric spaces via scalarizing*, Thai Journal of Math., vol. XLV (2012), no. 3, 717-724.