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CONTRIBUTIONS TO KOROVKIN-TYPE APPROXIMATION

PhD thesis abstract

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Introduction

1. *About Korovkin-type Approximation Theory*

We present the notion of *approximation scheme.* Let (X, d) be a metric space. An approximation method requires a set of approximating functions, $\mathcal{F}$ say, which is a subset of X. Given any $f \in X$, the method picks the element e_f, say, from $\mathcal{F}$, which is regarded as an approximation to f. To find how good is the chosen method, e_f should be compared with the best approximation to $f \in X$ from $\mathcal{F}$; this is an element $e^* \in \mathcal{F}$ such that

$$d(f, e^*) = \inf\{d(f, e) : \; e \in \mathcal{F}\} := \mathrm{dist}(f, \mathcal{F}).$$

Conditions for the existence, uniqueness and for the characterization of the best approximation in the case when $\mathcal{F}$ is a Hilbert space can be found, e.g., in [27, Section 1.4].

So, an important issue is to determine what type of approximating functions we use. One of the directions of Approximation Theory is given by positive linear approximation processes. It is relatively new trend that came to light in the fifties due to the research of T. Popoviciu, H. Bohman and P.P. Korovkin. Their famous theorem characterizing sequences of positive linear operators that approximate the identity operator, based on easily checked, simple criteria.

The following three aspects are most vital in this direction: the construction of these processes, the study of the degree of approximation, their ability to mimic qualitative properties of the approximated function such as monotonicity, convexity, shape preservation. After the pioneer work carried out by the mentioned mathematicians, a new theory was born on which we may now call Korovkin-type approximation theory, in short KAT. The development of KAT in $C(X)$-spaces was pursued and enriched by Wulbert [103], Berens and Lorentz [19], [20], Bauer and Donner [17]. KAT has been developed also in the framework of Banach lattices. For documentation in this area, we used, as primary source, the monograph of Altomare and Campiti [11]. All basic information are concentrated in Chapter 1 of this thesis.

Among the many approaches to the field called KAT, recently studied topic is the analysis of linear processes by using statistical convergence and the matrix summability method. The first steps were made in 2002 by Gadjiev and Orhan [45]. This vein of research was proved to be extremely fertile, consequently many mathematicians have developed this subject.

This thesis aims at this direction of investigation. Our goal is to construct different classes of linear positive operators of discrete or integral type and to study their statistical approximation properties. In terms of statistical convergence and A-statistical convergence we study both classical operators and new introduced operators which may depend on parameters.

The work combined classical results, new results appeared in the last decade and personal research aspects. We tried to make everything as simple as possible, but not simpler.

2. *The architecture of the thesis*

The thesis is structured in three chapters.

Chapter 1 gives a collection of some significant developments in the area of KAT. Here we meet definitions, examples, the classical Korovkin theorems, results on the rate of convergence of a sequence of positive linear operators. All the involved mathematical entities are fully described and specific. In a distinct section we detailed the concept of statistical convergence and its use in KAT. Also, elements of q-Calculus are delivered. Having established some formulas in q-Calculus, we go on to harvest the array of applications in the construction and in the study of q-approximation linear processes.

Chapter 2 treats classes of modified operators. We refer to operators which fix the monomial of the second degree. After a stopover on genuine King operators introduced in 2003, we deal with a family of discrete operators which preserves certain polynomials. The last section is devoted to approximation properties of a new class of q-Szász-Mirakjan operators. The exposed results usually involve various moduli of smoothness.

Chapter 3 begins by presenting some recent Korovkin-type theorems created to study the A-statistical convergence of sequences of positive linear operators. Author's original results are presented in three different sections and they address the following: a bivariate extension in q-Calculus of Stancu operators, new results on statistical approximation of Lupaş operators and of a class of Kantorovich-type operators, an investigation of mixed summation-integral operators based on discrete Jain operators. We mention that values of the errors for the appearing convergences (uniform, in L_p-norm, A-statistical) are found in explicit form.

In the construction of this work we tried first to include our personal results. Although the temptation was great, we did not want to present numerous existing results in this field. It would be transformed into a broad synthesis, which is not the primary purpose of a PhD thesis. Following this line, we inserted only the results we actually used in the papers published.

The exposed results come from single or joint papers of the author and the following: Octavian Agratini, Tudor Andrica, Cristina Radu, Andreea Veţeleanu. So far we have published 6 articles.

3. *Original results*

Our results are disseminated in Chapter 2 and Chapter 3 as follows.

Section 2.2: Theorem 2.2.1, Lemma 2.2.2, Theorem 2.2.6, Lemma 2.2.7, Theorem 2.2.8, Lemma 2.2.10, Theorem 2.2.11 published in [9].

Section 2.3: Lemma 2.3.2, Theorem 2.3.3, Theorem 2.3.4, Theorem 2.3.5, Corollary 2.3.6 published in [86].

Section 2.4.: Theorem 2.4.7, Corollary 2.4.8 published in [10].

Section 3.2: Theorem 3.2.1, Theorem 3.2.3, Theorem 3.2.5, Theorem 3.2.7 published in [10].

Section 3.3: Theorem 3.3.1, Theorem 3.3.2 published in [96].

Section 3.4: Lemma 3.4.1, Lemma 3.4.2, Theorem 3.4.3, Theorem 3.4.4, Theorem

3.4.5, Theorem 3.4.6 published in [95].

Also, the results presented in Lemma 2.4.10, Theorem 2.4.11, Theorem 3.1.9, Lemma 3.2.8 and Theorem 3.4.8 are so far unpublished.

We emphasize that in the above list was not included any personal remark or didactic example which were presented in this thesis.

Remark. We mention that in this abstract the numbering of theorems, lemmas and of all relationships was preserved as in the original thesis.

Chapter 1. Preliminaries

In three sections we collect notations, formulas and outstanding results which will be used in the presentation of our achievements.

1.1. Positive approximation processes. Classical approach

In the present section we shall deal with a basic topic of Korovkin-type approximation theory.

Since our goal is to study approximation properties of positive linear operators, a question arises. If $(L_n)_{n\geq 1}$ is such a sequence, what are the sufficient conditions to guarantee that $(L_n f)_{n\geq 1}$ converges uniformly to f for each continuous function f? H. Bohman [23] and P.P. Korovkin [65] have found the answer by giving a very simple criterion in order to decide the convergence of a given sequence of positive linear operators to the identity operator. Also, we point that the result was independently earlier established by Tiberiu Popoviciu [82] whose contribution remained unknown for a long time.

Set e_j, $j \in \mathbb{N}_0$, the monomial of degree j, where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Theorem 1.1.6 (Popoviciu-Bohman-Korovkin). *Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators of $C([a,b])$ into itself. Suppose that $(L_n e_j)_{n\geq 1}$ converges uniformly to e_j for $j \in \{0,1,2\}$.*

Then $(L_n f)_{n\geq 1}$ converges uniformly to f on $[a,b]$ for all functions $f \in C([a,b])$.

Usually, e_0, e_1, e_2 are called *test-functions* of the space $C([a,b])$ with respect to Popoviciu-Bohman-Korovkin theorem.

1.2. The concept of statistical convergence

Sixty years ago, the notion of statistical convergence was introduced by H. Fast [40]. In this paper the author recognizes the merits of H. Steinhaus who, at February 18^{th} 1949 in the frame of Polish Mathematics Society (Wroclaw), presented the first proof of the statement: for measurable sequences of functions, the statistical convergence and the asymptotic-statistical convergence are equivalent. The concept of statistical convergence is based on the notion of the asymptotic density of subsets of $\mathbb{N}$.

Even though the notion of statistical convergence was introduced long time ago, its application to the study of positive linear operators was attempted only in 2002. A.D. Gadjiev and C. Orhan [45] obtained Korovkin-type theorems via statistical convergence. The main result will be read as follows.

Theorem 1.2.14 [45, Theorem 1] *If the sequence of positive linear operators* $L_n : C([a,b]) \to B([a,b])$ *satisfies the conditions*

$$st - \lim_{n} \|L_n e_j - e_j\| = 0,\ j = 0, 1, 2, \tag{1.2.7}$$

then, for any function $f \in C([a,b])$, *one has*

$$st - \lim_{n} \|L_n f - f\| = 0. \tag{1.2.8}$$

1.3. Elements of q-Calculus and related formulas

Quantum Calculus is equivalent to traditional infinitesimal calculus without the notion of limits. It defined *q-Calculus* and *h-Calculus*.

The h-Calculus is just the calculus of finite differences which had been firstly studied by George Boole (1815-1864).

The q-Calculus, while dating in a sense back to Leonhard Euler (1707-1783) and Carl Gustav Jacobi (1804-1851), is only recently beginning to see more usefulness in quantum mechanics. Besides this implication, it has a lot of applications in different mathematical areas, such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions.

The aim of this Section is to present definitions, notations and basic results regarding q-Calculus. For this brief introduction we documented in the book of Kac and Cheung [59, pp. 7-13].

Chapter 2. Classes of modified operators

This chapter is focused on linear positive operators having the degree of exactness null and fixing the monomial of the second degree. The starting point is presented by J.P. King's paper [61] appeared in 2003. In the first paragraph we sum up results obtained in the past five years on these operators. Further on, in other paragraphs we introduce and study various new classes of operators following the construction created by King.

2.1. On genuine King operators

It is well known, if a linear positive operator reproduces all three test functions of Popoviciu-Bohman-Korovkin criterion, then it is the identity operator of the space.

A question arises: What is known about operators which fix the monomials e_0 and e_2?

J.P. King [61] was the first to present an example of operators enjoying this property.

2.2. On a King-type family of operators preserving certain polynomials

This section contains results published by Saddika Tarabie as co-author in [9].

Our aim is to introduce a general class of discrete type operators, to reproduce e_0 and $e_2+\alpha e_1$. This family is defined on certain subspaces of $C(J)$, $J \subset \mathbb{R}$. We take into account two types of intervals: $J=[0,1]$ and $J=\mathbb{R}_+ := [0,\infty)$, respectively. For the first case, the local and global rate of convergence is established by using the classical modulus ω_f associated to any function $f \in C([0,1])$. As usual, this space is endowed with the sup-norm $\|\cdot\|$. For the second case, the approximation property of our class is given in the frame of spaces of functions with polynomial growth. The involved spaces are defined via certain weights. More precisely, for a given $p \geq 2$, we consider the weight w_p, $w_p(x) = (1+x^p)^{-1}$, $x \geq 0$, and the corresponding space

$$C_p(\mathbb{R}_+) = \{f \in C(\mathbb{R}_+) : \ w_p(x)f(x) \text{ is convergent as } x \text{ tends to infinity}\} \quad (2.2.1)$$

endowed with the norm $\|\cdot\|_{C_p}$, $\|f\|_{C_p} = \sup\limits_{x\geq 0} w_p(x)|f(x)|$.

We notice, since $p \geq 2$, the test functions e_j, $j \in \{0,1,2\}$, belong to $C_p(\mathbb{R}_+)$. Further on, we detail the construction of the announced family of operators as it was indicated in [9].

For each $n \geq 2$, let $\Delta_n = (x_{n,k})_{k\in I_n}$ be a net on the interval J, where $I_n \subset \mathbb{N}$ is a set of indices consistent with J, this meaning $\{x_{n,k} : \ k \in I_n\} \subset J$. We consider the operators L_n having the form

$$(L_n f)(x) = \sum_{k\in I_n} u_{n,k}(x) f(x_{n,k}), \quad x \in J, \quad (2.2.2)$$

where $u_{n,k} \in C(J)$, $u_{n,k} \geq 0$, for every $(n,k) \in \{2,3,\ldots\} \times I_n$ and

$$f \in \mathcal{F}(J) = \{g \in C(J) : \text{ the series in (2.2.2) is convergent}\}.$$

Further on, we assume that the following identities

$$(L_n e_0)(x) = 1, \quad (L_n e_1)(x) = x, \quad (L_n e_2)(x) = a_n x^2 + b_n x, \quad x \in J, \quad (2.2.3)$$

are fulfilled for each $n \geq 2$. Moreover, we assume

$$a_n > 0, \quad b_n > 0, \quad \lim_n a_n = 1, \quad \lim_n b_n = 0. \quad (2.2.4)$$

Let $\alpha \geq 0$ be fixed. For each $n = 2, 3, \ldots$ setting

$$c_{n,\alpha} = \frac{b_n + \alpha}{2a_n},$$

we define the functions $v_{n,\alpha} : J \to \mathbb{R}_+$,

$$v_{n,\alpha}(x) = -c_{n,\alpha} + \sqrt{c_{n,\alpha}^2 + \frac{x^2 + \alpha x}{a_n}}, \quad x \in J. \tag{2.2.5}$$

Clearly, $v_{n,\alpha} \in C(J)$. Taking into account (2.2.2), we consider the linear and positive operators

$$(L_{n,\alpha}^* f)(x) = \sum_{k \in I_n} u_{n,k}(v_{n,\alpha}(x)) f(x_{n,k}), \quad x \in J, \tag{2.2.6}$$

where $f \in \mathcal{F}(J)$.

Theorem 2.2.1 *Let $L_{n,\alpha}^*$, $n = 2, 3, \ldots$, be defined by (2.2.6). The following relations hold.*

(i) $L_{n,\alpha}^* e_0 = e_0$, $L_{n,\alpha}^* e_1 = v_{n,\alpha}$, $L_{n,\alpha}^*(e_2 + \alpha e_1) = e_2 + \alpha e_1$.

(ii) $(L_{n,\alpha}^* \varphi_x^2)(x) = (2x + \alpha)(x - v_{n,\alpha}(x))$, $x \in J$, *where* $\varphi_x : J \to \mathbb{R}_+$ *is defined by* $\varphi_x(t) = |t - x|$.

Lemma 2.2.2 *Let $v_{n,\alpha}$, $n = 2, 3, \ldots$ be defined by (2.2.5). For each $x \in J$ one has*

(i) $0 \leq v_{n,\alpha}(x) \leq x$,

(ii) $\lim_n v_{n,\alpha}(x) = x$.

Theorem 2.2.6 *Let $L_{n,\alpha}^*$, $n = 2, 3, \ldots$, be defined by (2.2.6), where $J = [0, 1]$. For any $f \in C([0, 1])$ one has*

$$\lim_n \|L_{n,\alpha}^* f - f\| = 0.$$

For exploring the rate of convergence of $L_{n,\alpha}^*$ ($n \geq 2$, $\alpha \in \mathbb{R}_+$) operators, we need the following technical result.

Lemma 2.2.7 *Let $v_{n,\alpha}$, $n = 2, 3, \ldots$, be given by (2.2.5).*

(i) For $\alpha > 0$, one has $x - v_{n,\alpha}(x) \leq \dfrac{(a_n - 1)x^2 + b_n x}{b_n + \alpha}$.

(ii) For $\alpha = 0$, one has $x - v_{n,\alpha}(x) \leq \dfrac{|a_n - 1|}{\sqrt{a_n}} x + \dfrac{b_n}{2a_n}$.

Theorem 2.2.8 *Let $L_{n,\alpha}^*$, $n = 2, 3, \ldots$, be defined by (2.2.6), where $J = [0, 1]$. We assume that the sequence $((a_n - 1)/b_n)_{n \geq 2}$ is bounded. Let f belong to $C([0, 1])$.*

(i) For $\alpha > 0$, one has

$$|(L_{n,\alpha}^* f)(x) - f(x)| \leq \left(1 + (2x + \alpha)x \frac{(a_n - 1)x + b_n}{b_n(b_n + \alpha)}\right) \omega_1(f; \sqrt{b_n}). \tag{2.2.10}$$

(ii) For $\alpha = 0$, one has

$$|(L_{n,0}^* f)(x) - f(x)| \leq \left(1 + \frac{x}{a_n}\left(1 + 2\sqrt{a_n}\frac{|a_n - 1|}{b_n} x\right)\right) \omega_1(f; \sqrt{b_n}). \tag{2.2.11}$$

Lemma 2.2.10 *Let $L^*_{n,\alpha}$, $n = 2, 3, \ldots$, be defined by (2.2.6), where $J = \mathbb{R}_+$.*

(i) For any $p \geq 2$ one has

$$\frac{|(L^*_{n,\alpha}e_1)(x) - x|}{1 + x^p} \leq \left|1 - \frac{1}{\sqrt{a_n}}\right| + \frac{|b_n(\sqrt{a_n} + 1) - \alpha(\sqrt{a_n} - 1)|}{\sqrt{a_n}(b_n + \alpha)}, \quad x \geq 0; \quad (2.2.12)$$

(ii) $\lim\limits_{n} \|L^*_{n,\alpha}e_1 - e_1\|_{C_p} = 0$.

At this moment we show that the sequence $(L^*_{n,\alpha})_{n\geq 2}$ furnishes a new strong approximation process on the weighted space $C_p(\mathbb{R}_+)$, $p \geq 2$.

Theorem 2.2.11 *Let $L^*_{n,\alpha}$, $n = 2, 3, \ldots$, be defined by (2.2.6), where $J = \mathbb{R}_+$. For every $f \in \mathcal{F}(\mathbb{R}_+) \cap C_p(\mathbb{R}_+)$, $p \geq 2$, the following identity*

$$\lim_{n\to\infty} \|L^*_{n,\alpha}f - f\|_{C_p} = 0 \quad (2.2.13)$$

holds.

2.3. Approximation properties of a new class of q-Szász-Mirakjan operators

Our aim is to present a q-generalization of Szász-Mirakjan operators and to investigate their rate of convergence. The main tool is a certain weighted modulus of smoothness.

The established results relating to this sequence of operators represent the fruit of joint activities carried out during the common doctoral program in 2008-2009 between the PhD students Cristina Radu, Saddika Tarabie and Andreea Veţeleanu. Later, these results were structured and, in spring 2011, were published in *Studia Universitatis Babeş-Bolyai* journal, see [86].

Throughout this paragraph we consider $q \in (0, 1)$.

In [15] A. Aral introduced the first q-analogue of the classical Szász-Mirakjan operators.

Motivated by this work, for $q \in (0, 1)$ we give another q-analogue of the same class of operators as follows

$$S_{n,q}(f; x) = \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!} q^{k(k-1)} E_q(-[n]_q q^k x) f\left(\frac{[k]_q}{[n]_q q^{k-1}}\right), \quad (2.3.1)$$

$x \geq 0$ and $f \in \mathcal{F}(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{R},$ the series in (2.3.1) is convergent$\}$.

For $q \to 1^-$ the above operators reduce to the classical Szász-Mirakjan operators. In this case the approximation function $S_{n,q}f$ is defined on $\mathbb{R}_+$ for each $n \in \mathbb{N}$.

Easier to handle this construction, we will use the following q-difference operator

$$\Delta_q^0 f_{k,s} = f_{k,s}, \quad (2.3.2)$$

$$\Delta_q^{r+1} f_{k,s} = q^r \Delta_q^r f_{k+1,s} - \Delta_q^r f_{k,s-1}, \quad r \in \mathbb{N}_0, \quad (2.3.3)$$

where $f_{k,s} = f(x_{k,s})$ and $x_{k,s} = \dfrac{[k]_q}{q^s[n]_q}$, $k \in \mathbb{N}_0$, $s \in \mathbb{Z}$.

As usual, $[t_0, t_1, \dots, t_n; f]$ denotes the divided difference of the function f with respect to the distinct points $t_0, t_1, \dots, t_n$. We recall, it is defined recursively

$$[t_0; f] = f(t_0) \text{ and } [t_0, t_1, \dots, t_n; f] = \frac{[t_1, \dots, t_n; f] - [t_0, \dots, t_{n-1}; f]}{t_n - t_0}.$$

Following Ivan [56, p. 20], the term *divided difference* was introduced in mathematics by Augustus de Morgan (1842).

Lemma 2.3.2 *For all $k, r \in \mathbb{N}_0$, $s \in \mathbb{Z}$, one has*

$$[x_{k,s-1}, \dots, x_{k+r,s+r-1}; f] = \frac{q^{r(r+2s-1)/2}[n]_q^r}{[r]_q!}\Delta_q^r f_{k,r+s-1}, \qquad (2.3.5)$$

where the nodes were indicated in (2.3.3).

Theorem 2.3.3 *Let $q \in (0,1)$ and $S_{n,q}$, $n \in \mathbb{N}$, be defined by (2.3.1). For any $f \in \mathcal{F}(\mathbb{R}_+)$ we have*

$$S_{n,q}(f; x) = \sum_{r=0}^{\infty} \frac{([n]_q x)^r}{[r]_q!} q^{r(r-1)/2}\Delta_q^r f_{0,r-1}, \quad x \geq 0. \qquad (2.3.6)$$

In what follows we consider a sequence $(q_n)_n$, $0 < q_n < 1$, such that

$$\lim_n [n]_{q_n} = \infty. \qquad (2.3.13)$$

Theorem 2.3.4 *Let $(q_n)_n$ be a sequence satisfying (2.3.13) and let the operators S_{n,q_n}, $n \in \mathbb{N}$, be defined by (2.3.1). For any compact $J \subset \mathbb{R}_+$ and for each $f \in C(\mathbb{R}_+)$ we have*

$$\lim_{n\to\infty} S_{n,q_n}(f; x) = f(x), \text{ uniformly in } x \in J.$$

Theorem 2.3.5 *Let $(q_n)_n$ be a sequence satisfying (2.3.13) and let the operators S_{n,q_n}, $n \in \mathbb{N}$, be defined by (2.3.1). Let $q_0 = \inf_{n\in\mathbb{N}} q_n$ and $\alpha \geq 2$. For each $n \in \mathbb{N}$ and every $f \in B_\alpha(\mathbb{R}_+)$ one has*

$$|S_{n,q_n}(f; x) - f(x)| \leq C_{\alpha,q_0}(1 + x^{\alpha+1})\Omega_\alpha(f; \sqrt{1/[n]_{q_n}}), \quad x \geq 0, \qquad (2.3.16)$$

where C_{α,q_0} is a positive constant independent of f and n.

On the basis of this theorem we are able to give the following global estimate.

Corollary 2.3.6 *Under the hypothesis of Theorem 2.3.5 one has*

$$\|S_{n,q_n} f - f\|_{B_{\alpha+1}} \leq C_{\alpha,q_0}\Omega_\alpha(f; \sqrt{1/[n]_{q_n}}),$$

where C_{α,q_0} is a positive constant independent of f and n.

2.4. A sequence of summation integral type operators

This section is devoted to study a mixed summation-integral type of linear positive operators that approximate certain functions defined on $\mathbb{R}_+$. We obtain the pointwise rate of convergence. Within this general class of operators we highlight specific cases already studied in the literature. The material presented is based on [13], an article written jointly by Andrica Tudor and Tarabie Saddika.

Here are some examples of mixed operators which use different basis functions. Setting

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \quad v_{n,k}(x) = \frac{1}{(1+x)^n}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k, \tag{2.4.3}$$

$k \in \mathbb{N}_0$, $x \in \mathbb{R}_+$, and $C_\gamma(\mathbb{R}_+) = \{g \in C(\mathbb{R}_+) : \ |g(t)| \le Me^{\gamma t} \text{ for some } M > 0\}$, $\gamma > 0$ fixed, some examples are outlined.

Example 2.4.1 *Szász-Durrmeyer type operators with Baskakov basis*

$$(L_n f)(x) = n\sum_{\nu=0}^{\infty} v_{n,\nu}(x)\int_0^\infty s_{n,\nu}(t)f(t)dt,$$

$n \in \mathbb{N}$, $x \ge 0$, for $f \in L_p(\mathbb{R}_+)$, $p \ge 1$, see Gupta and Srivastava [51].

Example 2.4.2 *Baskakov-Durrmeyer type operators with Szász basis*

$$(L_n f)(x) = (n-1)\sum_{\nu=1}^{\infty} s_{n,\nu}(x)\int_0^\infty v_{n,\nu-1}f(t)dt + e^{-nx}f(0), \tag{2.4.4}$$

$n \ge 2$, $x \in \mathbb{R}_+$, where $f \in C_\gamma(\mathbb{R}_+)$, see [52, Eq. (1.1)].

Example 2.4.3 *Szász-Durrmeyer type operators with Beta basis*

$$(L_n f)(x) = \sum_{\nu=1}^{\infty} \beta_{n,\nu}(x)\int_0^\infty s_{n,\nu-1}(t)f(t)dt + (1+x)^{-n-1}f(0), \tag{2.4.5}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}_+$, where $f \in C_\gamma(\mathbb{R}_+)$, see [53, Section 4] and [54]. Here the weights $\beta_{n,\nu}$ are given by Beta function as follows

$$\beta_{n,\nu}(x) = \frac{1}{B(n,\nu+1)}\,\frac{x^\nu}{(1+x)^{n+\nu+1}}, \quad \nu \ge 1. \tag{2.4.6}$$

Inspired by the above constructions, in what follows we introduce a general class of hybrid integral type operators.

Let $(a_{n,k})_{k\ge 0}$, $(b_{n,k})_{k\ge 0}$ be two sequences of continuous and positive functions defined on $\mathbb{R}_+$ such that the following relations hold

$$\sum_{k=0}^{\infty} a_{n,k} = \mathbf{1}, \quad \sum_{k=0}^{\infty} b_{n,k} = \mathbf{1}, \quad \int_0^\infty b_{n,k}(t)dt := c_{n,k} < \infty, \tag{2.4.7}$$

where $\mathbf{1}$ denotes the constant function on $\mathbb{R}_+$ of constant value 1. For each $n \in \mathbb{N}$ we define the operator

$$(V_n f)(x) = f(0)a_{n,0}(x) + \sum_{k=1}^{\infty} \frac{a_{n,k}(x)}{c_{n,k}} \int_0^{\infty} b_{n,k}(t)f(t)dt, \quad x \in \mathbb{R}_+, \tag{2.4.8}$$

where $f \in \mathcal{F}(\mathbb{R}_+)$, this space consisting of all real valued functions f defined on $\mathbb{R}_+$ with the following two properties: $b_{n,k}f$ belongs to the Lebesgue space $L_1(\mathbb{R}_+)$ for each $k \in \mathbb{N}$ and the series from the right hand side of relation (2.4.8) is convergent.

As regards our operators V_n, $n \in \mathbb{N}$, we impose that the polynomials of first and respectively second degree to be transformed into polynomials of first respectively second degree which vanish at the origin. This means

$$(V_n e_1)(x) = (1+\alpha_n)x \quad \text{and} \quad (V_n e_2)(x) = (1+\beta_n)x^2 + \gamma_n x, \quad x \in \mathbb{R}_+. \tag{2.4.10}$$

Theorem 2.4.7 *Let $\tau > 0$ be fixed. Let V_n, $n \in \mathbb{N}$, be the operators defined by (2.4.8) such that (2.4.10) takes place. For each $f \in C_2(\mathbb{R}_+)$ the following relation*

$$|(V_n f)(x) - f(x)| \le M_{f,\tau}\delta_n^2(x) + 2\omega(f;\delta_n(x))_{[0,\tau+1]} \tag{2.4.17}$$

holds, where

$$\delta_n(x) = \sqrt{(\beta_n - 2\alpha_n)x^2 + \gamma_n x}, \quad x \in [0,\tau], \tag{2.4.18}$$

and $M_{f,\tau}$ is a constant depending only on f and τ.

Corollary 2.4.8 *Under the assumptions of Theorem 2.4.7 one has*

$$\|V_n f - f\|_{[0,\tau]} \le M_{f,\tau}\|\delta_n^2\|_{[0,\tau]} + 2\omega(f;\|\delta_n\|_{[0,\tau]})_{[0,\tau+1]}, \quad n \in \mathbb{N}, \tag{2.4.19}$$

where δ_n is defined at (2.4.18).

In the final part we will study the statistical convergence of $(V_n)_{n\ge1}$.

Lemma 2.4.10 *Let $(\alpha_n)_{n\ge1}$, $(\beta_n)_{n\ge1}$, $(\gamma_n)_{n\ge1}$ be real sequences. If*

$$st - \lim_n(\alpha_n - 2\beta_n) = 0, \quad st - \lim_n \gamma_n = 0, \tag{2.4.20}$$

then

$$st - \lim_n \|\delta_n^2\|_{[0,\tau]} = 0 \quad \textit{and} \quad st - \lim_n \omega(f;\|\delta_n\|_{[0,\tau]}) = 0, \tag{2.4.21}$$

where δ_n, $n \in \mathbb{N}$, are given by (2.4.18).

Theorem 2.4.11 *Let $\tau > 0$ be fixed. Let V_n, $n \in \mathbb{N}$, be the operators defined by (2.4.8) such that (2.4.10) takes place.*

If (2.4.20) holds, then one has

$$st - \lim_n \|V_n f - f\|_{[0,\tau]} = 0, \quad f \in C_2(\mathbb{R}_+).$$

Chapter 3. Statistical convergence of some classes of linear operators

First we present results subsequent to Theorem 1.2.14, revealing what has been achieved notably in Approximation Theory by using statistical convergence. Next we continue to investigate other classes of positive linear approximation operators. The original results obtained are included in separate sections of this chapter and they refer to the statistical convergence of Stancu, Lupaş and Jain type operators. These results have been published in papers [10], [96], [95], respectively.

3.1. Korovkin-type theorems via statistical convergence

We give another application that aims Lupaş operators introduced by using elements of q-Calculus [69]. For $q \in (0,1]$, these operators are defined as follows.

$$B_n^q : C([0,1]) \to C([0,1]),$$

$$(B_n^q f)(x) = \frac{1}{v_n(x;q)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k (1-x)^{n-k} f\left(\frac{[k]_q}{[n]_q}\right), \tag{3.1.14}$$

where $v_n(x;q) = \prod_{k=1}^{n} (1 - x + xq^{k-1})$, $x \in [0,1]$.

Theorem 3.1.9 *Let $0 < q_n < 1$, $n \in \mathbb{N}$, and let A be a non-negative regular summability matrix. Let $(B_n^{q_n})_{n\geq 1}$ be defined as in (3.1.14).*

If $st_A - \lim_n [n]_{q_n} = \infty$, then, for all functions $f \in C([0,1])$, we have

$$st_A - \lim_n \|B_n^{q_n} f - f\| = 0. \tag{3.1.17}$$

3.2. A bivariate extension of Stancu operators

Starting from Markov-Pólya urn scheme, D.D. Stancu [92] has introduced and investigated a linear operator $P_n^{\langle\alpha\rangle}$ which maps the space $C([0,1])$ into itself and is defined by

$$(P_n^{\langle\alpha\rangle} f)(x) = \sum_{k=0}^{n} w_{n,k}(x;\alpha) f\left(\frac{k}{n}\right), \tag{3.2.1}$$

where

$$w_{n,k}(x;\alpha) = \binom{n}{k} \frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha) \prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+\overline{n-1}\alpha)}, \tag{3.2.2}$$

α being a parameter which may depend only on the natural number n. Note, an empty product is taken to be equal to 1. If α is non-negative, then these operators preserve the positivity of the function f.

Recently, G. Nowak [77] introduced a q-analogue of Stancu operators. We also introduced [10] an extension of this class acting on the space of real valued functions defined on a rectangular domain. For $f \in C([0,1])$, $\alpha \geq 0$ and each $n \in \mathbb{N}$, in [77] have been defined the operators

$$(B_n^{q,\alpha} f)(x) = \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0,1], \tag{3.2.3}$$

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(x+\alpha[i]_q) \prod_{s=0}^{n-1-k}(1-q^s x+\alpha[s]_q)}{\prod_{i=0}^{n-1}(1+\alpha[i]_q)}, \quad 0 \leq k \leq n. \tag{3.2.4}$$

This class contains as special cases the following three well-known sequences.

i) For $\alpha = 0$, $B_n^{q,0} \equiv B_n^q$ represents q-Bernstein operator introduced by Phillips [80].

ii) For $\alpha = 0$ and $q = 1$, $B_n^{1,0} \equiv B_n$ is the classical Bernstein operator.

iii) For $q = 1$, $p_{n,k}^{1,\alpha}$ become fundamental Stancu polynomials $w_{n,k}(\cdot;\alpha)$, $k = \overline{0,n}$, see (3.2.2), and $B_n^{1,\alpha} \equiv P_n^{\langle\alpha\rangle}$ turns into Stancu operator defined by (3.2.1).

Theorem 3.2.1 *Let the sequences* $(q_n)_n$, $(\alpha_n)_n$ *be given such that* $0 < q_n < 1$ *and* $\alpha_n \geq 0$, $n \in \mathbb{N}$. *Let the operators* $B_n^{q_n,\alpha_n}$, $n \in \mathbb{N}$, *be defined as in (3.2.3). If*

$$st - \lim_n [n]_{q_n} = \infty \quad and \quad st - \lim_n \alpha_n = 0, \tag{3.2.7}$$

then, for each $f \in C([0,1])$, *one has*

$$st - \lim_n \|B_n^{q_n,\alpha_n} f - f\| = 0. \tag{3.2.8}$$

Theorem 3.2.3 *Let the sequences* $(q_n)_n$, $(\alpha_n)_n$ $(0 < q_n < 1,\ \alpha_n > 0,\ n \in \mathbb{N})$ *be given such that the relation (3.2.7) takes place and they satisfy the following conditions*

$$k\Delta q_{k+1} \geq -c_1, \quad k\Delta\alpha_{k+1} \geq -c_2, \tag{3.2.9}$$

for some $c_1 > 0$, $c_2 > 0$ *and for any* $k \in \mathbb{N}$.

If the operators $B_n^{q_n,\alpha_n}$, $n \in \mathbb{N}$, *are defined as in (3.2.3), then the sequence* $(B_n^{q_n,\alpha_n})_n$ *converges uniformly on* $C([0,1])$ *to the identity operator.*

Following [10], set $K = [0,1] \times [0,1]$ the unit square and let the vector $q(q_1, q_2)$ belong to the interior of K. We consider the parameter $\alpha(\alpha_1, \alpha_2) \in \mathbb{R}_+ \times \mathbb{R}_+$. For each $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$ we define the operator involving a cartesian product grid and acting on $C(K)$ as follows

$$(B_{n_1,n_2}^{(q,\alpha)} f)(x_1, x_2) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} f(\lambda_{n_1,k_1,q_1}, \lambda_{n_2,k_2,q_2}) p_{n_1,k_1}^{q_1,\alpha_1}(x_1) p_{n_2,k_2}^{q_2,\alpha_2}(x_2), \qquad (3.2.10)$$

$(x_1, x_2) \in K$, where $\lambda_{n_j,k_j,q_j} = [k_j]_{q_j}/[n_j]_{q_j}$ and $p_{n_j,k_j}^{q_j,\alpha_j}$, $0 \le k_j \le n_j$, are defined by (3.2.4), $j = 0, 1$.

Theorem 3.2.5 *Let $q(q_1, q_2) \in (0,1) \times (0,1)$ and $\alpha(\alpha_1, \alpha_2) \in \mathbb{R}_+ \times \mathbb{R}_+$. The operators $B_{n_1,n_2}^{(q,\alpha)}$, $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$, defined by (3.2.10) verify the following identities*

$$\begin{aligned} B_{n_1,n_2}^{(q,\alpha)} e_{i,j} &= e_{i,j}, \ (i,j) \in \{(0,0), (0,1), (1,0)\}, \\ (B_{n_1,n_2}^{(q,\alpha)} e_{2,0})(x_1, x_2) &= (B_{n_1}^{q_1,\alpha_1} e_2)(x_1), \\ (B_{n_1,n_2}^{(q,\alpha)} e_{0,2})(x_1, x_2) &= (B_{n_2}^{q_2,\alpha_2} e_2)(x_2), \end{aligned}$$

for each $(x_1, x_2) \in K$.

Theorem 3.2.7 *For each $n(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$, in (3.2.10) we substitute α by $\alpha_n(\alpha_{1,n_1}, \alpha_{2,n_2})$, $\alpha_{j,n_j} \ge 0$, and q by $q_n(q_{1,n_1}, q_{2,n_2})$, $0 < q_{j,n_j} < 1$, where $j = 0, 1$. If*

$$st - \lim_{n_j} (1/[n_j]_{q_j,n_j}) = st - \lim_{n_j} \alpha_{j,n_j} = 0, \quad j = 0, 1, \qquad (3.2.11)$$

then, for each $f \in C(K)$, one has

$$st - \lim_{n_1,n_2} \| B_{n_1,n_2}^{(q_n,\alpha_n)} f - f \| = 0.$$

If in (3.2.11) we replace the statistical limit by ordinary limit, then we obtain the uniform convergence of the sequence $(B_{n_1,n_2}^{(q_n,\alpha_n)})_n$ to the identity operator.

Returning to the one-dimensional case and examining the relation (3.2.7) a question can raise: what sufficient conditions can be imposed to the sequence $(q_n)_{n \ge 1}$ such that $st - \lim_n [n]_{q_n} = \infty$ to take place? A variant of answer is given in the following.

Lemma 3.2.8 *Let $(q_n)_{n \ge 1}$ be a real sequence such that $0 < q_n < 1$, $n \in \mathbb{N}$. If*

$$st - \lim_n q_n = 1 \text{ and } st - \lim_n [n]_{q_n} \text{ exists},$$

then $st - \lim_n [n]_{q_n} = \infty$.

3.3. On A-statistical approximation of Lupaş and Kantorovich type operators

In this section we are concerned with A-statistical convergence of two sequences of linear positive operators. The first is of discrete type and the second is of integral type. The results were published in [96].

In [70] Lupaş proposed studying the following sequence of linear and positive operators

$$(\Lambda_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \ x \geq 0, \ f : \mathbb{R}_+ \to \mathbb{R}, \tag{3.3.1}$$

where $(nx)_0 = 1$ and $(nx)_k = nx(nx+1)\dots(nx+k-1)$, $k \geq 1$, indicate the *rising factorial* or *upper factorial.* In [3] was presented an integral extension in Kantorovich sense of these operators defined as follows

$$(K_n f)(x) = n2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{q^k k!} \int_{k/n}^{(k+1)/n} f(t)dt, \quad x \geq 0, \tag{3.3.2}$$

and f belongs to the class of local integrable functions defined on $\mathbb{R}_+$.

Our aim is to study the A-statistical convergence of the above two sequences, $(\Lambda_n)_{n\geq 1}$ and $(K_n)_{n\geq 1}$, respectively. We work in the weighed spaces $B_\rho(I)$ and $C_\rho(I)$.

To establish our results we use the following weight functions

$$\rho_1(x) = 1 + x^2, \quad \rho_2(x) = 1 + x^{2\lambda}, \quad \lambda > 1, \ x \in \mathbb{R}_+. \tag{3.3.5}$$

Theorem 3.3.1 *Let $A = (a_{j,n})$ be a non-negative regular summability matrix and let ρ_1, ρ_2 be weight functions introduced by (3.3.5). The operators Λ_n, $n \in \mathbb{N}$, defined by (3.3.1) satisfy the following identity*

$$st_A - \lim_n \|\Lambda_n f - f\|_{\rho_2} = 0 \textit{ for any } f \in C_{\rho_1}(\mathbb{R}_+). \tag{3.3.6}$$

Theorem 3.3.2 *Let $A = (a_{j,n})$ be a non-negative regular summability matrix and let ρ_1, ρ_2 be weight functions introduced by (3.3.5). The operators K_n, $n \in \mathbb{N}$, defined by (3.3.2) satisfy the following identity*

$$st_A - \lim_n \|K_n f - f\|_{\rho_2} = 0 \textit{ for any } f \in C_{\rho_1}(\mathbb{R}_+). \tag{3.3.8}$$

3.4. On Jain-Beta linear operators

Starting from a sequence of linear positive operators introduced by G.C. Jain [58], we present an integral version of it. Approximation properties and the rate of convergence are investigated in our paper [95]. Also, an extension for smooth functions is given.

We present the construction of our mixed summation-integral type operators and their approximation properties. We are working in the space $C_{\rho_\lambda}(\mathbb{R}_+)$, where the weight $\rho_\lambda : \mathbb{R}_+ \to \mathbb{R}$ is given by $\rho_\lambda(x) = 1 + x^{2+\lambda}$, $\lambda \geq 0$. We introduce a sequence of operators calling it Jain-Beta, as follows

$$(J_n^{[\beta]} f)(x) = \sum_{k=1}^{\infty} \frac{w_\beta(k; nx)}{B(n+1,k)} \int_0^{\infty} f(t) \frac{t^{k-1}}{(1+t)^{n+k+1}} dt + e^{-nx} f(0), \quad x \geq 0, \tag{3.4.4}$$

where $n \geq 2$, $f \in C_{\rho_0}(\mathbb{R}_+)$ and $w_\beta(k; nx)$ is the Poisson-type distribution given by

$$w_\beta(k;\alpha) = \frac{\alpha}{k!}(\alpha + k\beta)^{k-1}e^{-(\alpha+k\beta)}, \quad k \in \mathbb{N}_0.$$

Lemma 3.4.1 *The operators $J_n^{[\beta]}$, $n \geq 2$, defined by (3.4.4) satisfy the following relations*

$$\begin{cases} J_n^{[\beta]}e_0 = e_0, \quad J_n^{[\beta]}e_1 = \dfrac{e_1}{1-\beta}, \\ \\ J_n^{[\beta]}e_2 = \dfrac{n}{(n-1)(1-\beta)^2}\left(e_2 + \dfrac{1+(1-\beta)^2}{n(1-\beta)}e_1\right). \end{cases} \tag{3.4.6}$$

Lemma 3.4.2 *The first and the second central moment of $J_n^{[\beta]}$, $n \geq 2$, operators, are given by*

$$\begin{aligned} (J_n^{[\beta]}\varphi_x)(x) &= \frac{\beta}{1-\beta}x, \\ (J_n^{[\beta]}\varphi_x^2)(x) &= \left(\frac{n}{(n-1)(1-\beta)^2} - \frac{1+\beta}{1-\beta}\right)x^2 + \frac{1+(1-\beta)^2}{(n-1)(1-\beta)^3}x, \end{aligned} \tag{3.4.7}$$

respectively.

Theorem 3.4.3 *Let $J_n^{[\beta]}$, $n \geq 2$, be defined by (3.4.4). For any function f belonging to $C_{\rho_0}(\mathbb{R}_+)$ one has*

$$|(J_n^{[\beta]}f)(x) - f(x)| \leq (1 + \sqrt{x(x+1)})\omega(f; \sqrt{\delta_{n,\beta}})_{[0,a]}, \quad x \in [0,a],$$

where $\delta_{n,\beta} = \dfrac{n+2}{(n-1)(1-\beta)^3} - \dfrac{1}{1-\beta}$.

Examining the relation (3.4.6) and based on famous Popoviciu-Bohman-Korovkin criterion, it is clear that $(J_n^{[\beta]})_{n\geq 2}$ does not form an approximation process. The next step is to transform it for enjoying of this property. For each $n \geq 2$, the constant β will be replaced by a number $\beta_n \in [0,1)$. If

$$\lim_n \beta_n = 0, \tag{3.4.10}$$

then Lemma 3.4.1 ensures $\lim_n (J_n^{[\beta_n]}e_j)(x) = x^j$, $j = 0,1,2$, uniformly on any interval compact $K \subset \mathbb{R}_+$. Consequently, based on the mentioned criterion, we can state

Theorem 3.4.4 *Let $J_n^{[\beta_n]}$, $n \geq 2$, be defined as in (3.4.4), where $(\beta_n)_{n\geq 2}$ satisfies (3.4.10). For any compact $K \subset \mathbb{R}_+$ and for each $f \in C_{\rho_0}(\mathbb{R}_+)$ one has*

$$\lim_n (J_n^{[\beta_n]}f)(x) = f(x), \ \textit{uniformly in } x \in K.$$

Our next concern is the study of statistical convergence of Jain-Beta sequence of operators.

Theorem 3.4.5 *Let $A = (a_{n,k})$ be a non-negative regular summability matrix and let $\lambda > 0$ be fixed. Let $J_n^{[\beta_n]}$, $n \geq 2$, be defined as in (3.4.4), where $(\beta_n)_{n\geq 2}$, $0 \leq \beta_n < 1$, satisfies*

$$st_A - \lim_n \beta_n = 0. \tag{3.4.11}$$

One has

$$st_A - \lim_n \|J_n^{[\beta_n]} f - f\|_{\rho_\lambda} = 0, \quad f \in C_{\rho_0}(\mathbb{R}_+). \tag{3.4.12}$$

To increase the rate of convergence we can replace $J_n^{[\beta]}$ by its generalization of the r-th order, see [6]. We point out some details of the idea used. The disadvantage of the positive linear approximating sequences is definitely determined by the fact that they don't react to the improvement of the smoothness of functions they are generated from. To overcome this fact, G. Kirov and L. Popova [63] proposed a generalization of the r-th order, $r \in \mathbb{N}$. For a given positive linear operator, this generalization is obtained by the action of the operator not directly on the signal f, but on its Taylor polynomial of r-th degree. The new operator keeps the linearity property but loose the positivity.

In what follows, we apply the technique of Kirov and Popova to Jain-Beta operators. Let $f \in C^r(\mathbb{R}_+)$ such that $e_s f^{(s)} \in C_{\rho_0}(\mathbb{R}_+)$ for $s = 0, 1, \ldots, r$, and let $T_r f(x; \cdot)$ be the r-th degree Taylor polynomial associated to the function f at the point $x \in \mathbb{R}_+$. For $n \geq 2$ and any $x \geq 0$ we define the linear operators

$$\begin{aligned}(J_{n,r}^{[\beta_n]} f)(x) &= J_n^{[\beta_n]}(T_r f; x) \\ &= \sum_{k=1}^{\infty} \frac{w_{\beta_n}(k; nx)}{B(n+1, k)} \sum_{s=0}^{r} \frac{1}{s!} \int_0^\infty f^{(s)}(t) \frac{(x-t)^s t^{k-1}}{(1+t)^{n+k+1}} dt \\ &+ e^{-nx} f(0).\end{aligned} \tag{3.4.13}$$

Theorem 3.4.6 *Let A be a non-negative regular summability matrix. Let $r \in \mathbb{N}$ be fixed, $\alpha \in (0, 1]$ and $M > 0$. Let the operators $J_n^{[\beta_n]}$ and $J_{n,r}^{[\beta_n]}$, $n \geq 2$, be defined by (3.4.4) and (3.4.13), respectively. Suppose $st_A - \lim_n \beta_n = 0$.*

If $x \geq 0$ and $\varphi_x^{r+\alpha} \in C_{\rho_0}(\mathbb{R}_+)$ such that

$$st_A - \lim_n (J_n^{[\beta_n]} \varphi_x^{r+\alpha})(x) = 0, \tag{3.4.14}$$

then

$$st_A - \lim_n |(J_{n,r}^{[\beta_n]} f)(x) - f(x)| = 0 \tag{3.4.15}$$

holds for any function $f \in C^r(\mathbb{R}_+) \cap C_{\rho_0}(\mathbb{R}_+)$ with the properties $e_s f^{(s)} \in C_{\rho_0}(\mathbb{R}_+)$, $s = 0, 1, \ldots, r$ and $f^{(r)} \in Lip_M \alpha$.

Here φ_x is given by $\varphi_x(t) = t - x$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$.

We mention that a generalization of Kantorovich type for Jain operators was obtained in [99]. These integral operators have the following construction

$$(K_n^{[\beta]} f)(x) = n \sum_{k=0}^{\infty} w_\beta(k; nx) \int_{k/n}^{(k+1)/n} f(t) dt, \tag{3.4.16}$$

where f belongs to the Lebesgue space $L_1(\mathbb{R}_+)$.
Theorem 3.4.8 *Let A be a non-negative regular summability matrix and let $\lambda > 0$ be fixed. Let $K_n^{[\beta_n]}$, $n \geq 1$, be defined as in (3.4.16), where $(\beta_n)_{n\geq 1}$, $0 \leq \beta_n < 1$, satisfies relation (3.4.11). One has*

$$st_A - \lim_n \|K_n^{[\beta_n]} f - f\|_{\rho_\lambda} = 0, \quad f \in C_{\rho_0}(\mathbb{R}_+).$$

Comparing this result with Theorem 3.4.5 we see that both integral generalizations of Jain operators enjoy the same A-statistical approximation property.

As has been seen in this chapter, Jain operators are introduced by using the Poisson-type distribution.

Approximation linear positive operators can be obtained starting from other types of distributions. We refer here to compound distributions, see, e.g., V. Preda [83] or using the density of a linear combination of some given random variables, as was established in [84].

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