Compression-Expansion Theorems of Krasnoselskii Type and Applications
(Ph. D. Thesis Summary)

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Contents

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Introduction

The compression-expansion theorems of Krasnoselskii represent a main tool in studying nonlinear problems for integral, ordinary differential and partial differential equations and systems. They are used to prove not only the existence of solutions, but also to localize solutions in a conical annulus or other domains of this type. This, in particular, allows us to obtain multiple solutions to nonlinear problems when nonlinearities are oscillating. In the last three decades a rich literature has been produced on this subject both in theory and applications.

From a theoretical point of view, two major approaches of the subject are known: the first one is in the framework of the fixed point theory and uses essentially Schauder's fixed point theorem and its generalizations, while the second approach uses topological degree theory.

The original contributions of Krasnoselskii were given using the first approach and the present thesis follows the same way. Therefore all our theoretical results are based on fixed point theory.

Here are the original Krasnoselskii's compression-expansion fixed point theorems.

Theorem 0.1 *(M. A. Krasnoselskii [36]) Let X be a real Banach space and let $K \subset X$ be a cone. Let N be a positive completely continuous operator with $N\theta = \theta$. If numbers $r, R > 0$ can be found such that*

$$\begin{cases} Nx \not\leq x \text{ for all } x \in K \text{ with } \mid x \mid \leq r, x \neq \theta \\ \text{and for all } \varepsilon > 0 \\ Nx \not\geq (1+\varepsilon)x \text{ for all } x \in K \text{ with } \mid x \mid \geq R, \end{cases} \tag{1}$$

then the operator N has at least one non-zero fixed point in K.

Theorem 0.2 *(M. A. Krasnoselskii [36]) Let X be a real Banach space and let $K \subset X$ be a cone. Let N be a positive completely continuous operator with $N\theta = \theta$. If numbers $r, R > 0$ can be found such that for all $\varepsilon > 0$*

$$\begin{cases} Nx \not\geq (1+\varepsilon)x \text{ for all } x \in K \text{ with } \mid x \mid \leq r, x \neq \theta \\ \text{and} \\ Nx \not\leq x \text{ for all } x \in K \text{ with } \mid x \mid \geq R, \end{cases} \tag{2}$$

then the operator N has at least one non-zero fixed point in K.

In literature, for $r < R$, the compression conditions from Theorem 0.1 is in most cases replaced by the conditions:

$$\begin{cases} \| Nu \| \geq \| u \|, \text{ if } \| u \| = r, \\ \| Nu \| \leq \| u \|, \text{ if } \| u \| = R. \end{cases} \tag{3}$$

Similarly, the expansion conditions from Theorem 0.2 is replaced by:

$$\begin{cases} \| Nu \| \leq \| u \|, \text{ if } \| u \| = r, \\ \| Nu \| \geq \| u \|, \text{ if } \| u \| = R. \end{cases} \tag{4}$$

In this thesis we obtain a number of generalizations of Krasnoselskii's fixed point theorems in the cone K, where the fixed point is localized in a conical generalized annulus of one of the forms:

$$K_{r,R} := \{u \in K : r \leq \varphi(u) \leq \psi(u) \leq R\}, \tag{5}$$

$$K_{r,R} := \{u \in K : r \leq \psi(u), \varphi(u) \leq R\}, \tag{6}$$

where $\varphi \leq \psi$ on K,

$$K_{r,R} := \{x \in K : r \leq \delta(x) \leq R\} \tag{7}$$

The compression conditions (1) and (3) and the expansion conditions (2) and (4) will be generalized using the functionals φ, ψ and δ as follows:

$$\begin{cases} \varphi(u) \leq \varphi(Nu) \text{ if } \varphi(u) = r, \\ \psi(u) \geq \psi(Nu) \text{ if } \psi(u) = R, \end{cases} \quad \text{(compression conditions)}$$

$$\begin{cases} \varphi(u) \geq \varphi(Nu) \text{ if } \varphi(u) = r, \\ \psi(u) \leq \psi(Nu) \text{ if } \psi(u) = R, \end{cases} \quad \text{(expansion conditions).}$$

for the forms (5), (6) of the annulus, and

$$\begin{cases} \varphi(u) \leq \varphi(Nu) \text{ if } \delta(u) = r \\ \psi(u) \geq \psi(Nu) \text{ if } \delta(u) = R, \end{cases} \quad \text{(compression conditions)},$$

$$\begin{cases} \varphi(u) \geq \varphi(Nu) \text{ if } \delta(u) = r \\ \psi(u) \leq \psi(Nu) \text{ if } \delta(u) = R, \end{cases} \quad \text{(expansion conditions)},$$

for the form (7) of the annulus.

Compared to the literature, our results complement, extend and generalize in several directions the results obtained by R. Legget, L. Williams [38], R. Avery, J. Henderson, D. O'Regan [4], [5], R. Precup [58], [59] and others. Such extensions using functionals φ, ψ and δ have been motivated by concrete applications where this kind of functionals appeared in a natural way. For instance, we may consider the functionals(see Section 3.5):

$$\begin{aligned} \varphi(u) &: = \min_{t \in I} u(t), \\ \psi(u) &: = \max_{t \in [0,1]} u(t), \\ \delta(u) &: = \begin{cases} \frac{(\min_{t \in I} u(t))^2}{\max_{t \in [0,1]} u((t)}, & \text{if } u \text{ is not identically zero}, \\ 0, & \text{if } u \equiv 0. \end{cases} \end{aligned}$$

We note that functional δ is used in this thesis for the first time.

A consistent part of the thesis is devoted to applications of the compression-expansion type theorems to several classes of problems: two point boundary value problems for second order ordinary differential equations, functional-differential equations, systems of ordinary differential equations and p-Laplace equations.

Our results concerning the applications complement and extend those given by L.H. Erbe, H. Wang [14], D. O'Regan, R. Precup [54], R. Avery, J. Henderson, D. O'Regan [4] and others.

This thesis is based on our papers S. Budisan [8], [9], [11], [12] and S. Budisan, R. Precup [10].

The thesis is divided into three chapters, an **Introduction** and a list of **References**.

1

Preliminaries

In this section we present the basic notions of "cone", "compression of the cone", "expansion of the cone" in the way that they appear at Krasnoselskii. Also, we present Krasnoselskii's fixed point theorems using the original formulation of M. A. Krasnoselskii [36] and other theorems related to the subject.

Definition 1.1 *(M. A. Krasnoselskii [36]) Let X be a real Banach space. A set $K \subset X$ is called a cone if the following conditions are satisfied:*
(a) the set K is closed;
(b) if $x, y \in K$ then $\alpha x + \beta y \in K$ for all $\alpha, \beta \geq 0$;
(c) $K \cap (-K) = \{\theta\}$, where θ is the zero of the space X.

From property (b), it follows, in particular, that a cone K is a convex set. The cone K induces in the Banach space X an order relation defined in the following manner:

$$x \leq y \text{ if and only if } y - x \in K.$$

Let us denote by $|\ .\ |$ the norm of the Banach space X.

Definition 1.2 *(M. A. Krasnoselskii [36]) We say that the operator $N(N\theta = \theta)$ is a compression of the cone K if numbers $r, R > 0$ can be found such that*

$$Nx \nleq x \text{ for all } x \in K \text{ with } |\ x\ | \leq r, x \neq \theta \tag{1.1}$$

and for all $\varepsilon > 0$

$$Nx \ngeq (1+\varepsilon)x \text{ for all } x \in K \text{ with } |\ x\ | \geq R. \tag{1.2}$$

The operator N is *positive* if it transforms the cone K into itself.

Theorem 1.3 *(M. A. Krasnoselskii [36], Theorem 4.12) Let the positive completely continuous operator N be a compression of the cone K.*

Then the operator N has at least one non-zero fixed point in K.

We denote $K_{r,R} := \{ x \in K : r \leq\mid x \mid\leq R\}$.

Lemma 1.4 *(M. A. Krasnoselskii [36], Lemma 4.4)Let K be a cone in a finite-dimensional space X. Let N be a continuous operator from $K_{r,R}$ to K such that the following conditions hold:*

$$Nx = v_0 \text{ if } x \in K, \mid x \mid= r$$

and

$$Nx = u_0 \text{ if } x \in K, \mid x \mid= R,$$

where

$$\mid v_0 \mid< r < R <\mid u_0 \mid .$$

Then the operator N has at least one fixed point in $K_{r,R}$.

Lemma 1.5 *(M. A. Krasnoselskii [36], Lemma 4.5) Let $N : K_{r,R} \to K$ be a completely continuous operator. Assume that*

$$Nx = \begin{cases} v_0, & \text{if } x \in K, \mid x \mid= r, \\ u_0, & \text{if } x \in K, \mid x \mid= R, \end{cases}$$

where $\mid v_0 \mid< r < R <\mid u_0 \mid$.

Then the operator N has at least one fixed point in $K_{r,R}$.

Definition 1.6 *(M. A. Krasnoselskii [36])We say that the positive operator $N(N\theta = \theta)$ is an expansion of the cone K if numbers $r, R > 0$ can be found such that for all $\varepsilon > 0$*

$$Nx \not\geq (1 + \varepsilon)x \text{ for all } x \in K \text{ with } \mid x \mid\leq r, x \neq \theta \tag{1.3}$$

and

$$Nx \not\leq x \text{ for all } x \in K \text{ with } \mid x \mid\geq R. \tag{1.4}$$

Theorem 1.7 *(M. A. Krasnoselskii [36], Theorem 4.14)Let the positive completely continuous operator N be an expansion of the cone K. Then the operator N has at least one non-zero fixed point in K.*

We present now some results from R. Legget, L. Williams [38]. To present some results of these authors we need the following definition.

Definition 1.8 *(see R. Legget, L. Williams [38]) Let K be a cone of the Banach space $(E, \| \cdot \|)$. α is a* ***concave positive functional*** *on K if α :*

$K \to [0, \infty)$ *is continuous and satisfies*

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y), \ 0 \leq \lambda \leq 1.$$

We observe that if α is a concave positive functional on a cone K, a set of the form

$$S(\alpha, a, b) = \{x \in K : a \leq \alpha(x) \text{ and } \parallel x \parallel \leq b\}$$

is closed, bounded and convex in K.Let $K_c := \{x \in K : \parallel x \parallel \leq c\}$, where $0 < c < \infty$.

Most of the results involve fixed point index, the basic properties which are in the following lemma.

Next we present a fixed point result from R. Legget, L. Williams [38].

Theorem 1.9 *(R. Legget, L. Williams [38]) Suppose $N : K_c \to K$ is completely continuous and suppose there exist a concave positive functional α with $\alpha(x) \leq \parallel x \parallel$ $(x \in K)$ and numbers $b > a > 0$ $(b \leq c)$ satisfying the following conditions:*

(1) $\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \phi$ and $\alpha(Nx) > a$ if $x \in S(\alpha, a, b)$;

(2) $Nx \in K_c$ if $x \in S(\alpha, a, c)$;

(3) $\alpha(Nx) > a$ for all $x \in S(\alpha, a, c)$ with $\parallel Nx \parallel > b$.

Then N has a fixed point x in $S(\alpha, a, c)$.

In R. Avery, J. Henderson, D. O'Regan [4] the authors give an existence result based on the following properties:

Property A1 (R. Avery, J. Henderson, D. O'Regan [4]) Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A1 if one of the following conditions hold:

(i) β is convex, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$, and $\inf_{x \in P \cap \partial\Omega} \beta(x) > 0$,

(ii) β is sublinear, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$, and $\inf_{x \in P \cap \partial\Omega} \beta(x) > 0$,

(iii) β is concave and unbounded.

Property A2. (R. Avery, J. Henderson, D. O'Regan [4]) Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A2 if one of the following conditions hold:

(i) β is convex, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$,

(ii) β is sublinear, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$,

(iii) $\beta(x+y) \geq \beta(x) + \beta(y)$ for all $x, y \in P$, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$.

We present now the existence result.

Theorem 1.10 *(R. Avery, J. Henderson, D. O'Regan [4]) Let $\Omega_1$and Ω_2 be two bounded open sets in a Banach Space E such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subseteq \Omega_2$ and P is a cone in E. Suppose $N : P \cap (\bar{\Omega}_2 - \Omega_1) \to P$ is completely continuous, α and ψ are nonnegative continuous functionals on P, and one of the two conditions:*

(K1) α satisfies Property A1 with $\alpha(Nx) \geq \alpha(x)$, for all $x \in P \cap \partial\Omega_1$, and ψ satisfies Property A2 with $\psi(Nx) \leq \psi(x)$, for all $x \in P \cap \partial\Omega_2$; or

(K2) α satisfies Property A2 with $\alpha(Nx) \leq \alpha(x)$, for all $x \in P \cap \partial\Omega_1$, and ψ satisfies Property A1 with $\psi(Nx) \geq \psi(x)$, for all $x \in P \cap \partial\Omega_2$,

is satisfied. Then N has at least one fixed point in $P \cap (\bar{\Omega}_2 - \Omega_1)$.

Theorem 1.10 provides a generalization of some compression-expansion arguments that have utilized the norm /or functionals in obtaining the existence of at least one fixed point. Avery, Henderson and O'Regan proved the result above using the fixed point index.

Other authors give generalizations of Krasnoselskii's fixed point theorem in cones. For example, in R. Precup [57], is given the following result:

Theorem 1.11 *(R. Precup [57]) Let $(X, |\ .\ |)$ be a normed linear space, $K_1, K_2 \subset X$ two cones; $K := K_1 \times\ K_2$; $r, R \in \mathbf{R}_+^2$ with $0 < r < R (r = (r_1, r_2)$, $R = (R_1, R_2)$ and $r < R$ if and only if $r_i < R_i$ for $i \in \{1, 2\})$, and $N : K_{r,R} \to K$, $N = (N_1, N_2)$ a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $K_{r,R}$,where $u = (u_1, u_2)$:*

(a) $N_i(u) \not< u_i$ if $| u_i |= r_i$, and $N_i(u) \not> u_i$ if $| u_i |= R_i$;

(b) $N_i(u) \not> u_i$ if $| u_i |= r_i$, and $N_i(u) \not< u_i$ if $| u_i |= R_i$.

Then N has a fixed point u in K with $r_i \leq| u_i |\leq R_i$ for $i \in \{1, 2\}$.

Some other versions and extensions of Krasnoselskii's compression-expansion fixed point theorems can be found in R. P. Agarwal, D. O'Regan [1], M. K. Kwong [37], R. Precup [58], [59], M. Zima [91].

A powerful tool that is used in our thesis and in many other papers is the following version of Krasoselskii's fixed point theorem in cones. We present it in two similar forms. The first form of the theorem is the following one:

Theorem 1.12 *(M.A. Krasnoselskii [36]) Suppose that N is a completely continuous operator on a Banach space X, and let K be a cone in X. If there exist bounded open sets Ω_1, Ω_2 containing the zero element of X such that $\overline{\Omega_1} \subset \Omega_2$, $N : K \cap (\overline{\Omega_2} - \Omega_1) \to K$, and either:*

(i) $u \nleq Nu$ *for* $u \in K \cap \partial\Omega_1$ *and* $Nu \nleq u$ *for* $u \in K \cap \partial\Omega_2$*; or*
(ii) $Nu \nleq u$ *for* $u \in K \cap \partial\Omega_1$ *and* $u \nleq Nu$ *for* $u \in K \cap \partial\Omega_2$,
then there is an element $u_0 \in \Omega_2 - \bar{\Omega}_1$ *satisfying* $Nu_0 = u_0$.

The second form of the theorem is the following one:

Theorem 1.13 *(M. A. Krasnoselskii [36]) Let* $(X, |\ .\ |)$ *be a normed linear space,* $C \subset X$ *a cone,* $\leq$ *the partial order relation induced by* C, $0 < r < R$ *and* $C_{r,R} = \{x \in C : r \leq\mid x \mid\leq R\}$*. Assume that* $N : C_{r,R} \to C$ *is a compact map and one of the following conditions is satisfied:*
(a) $x \nleq Nx$ *for* $\mid x \mid= r$ *and* $Nx \nleq x$ *for* $\mid x \mid= R$*;*
(b) $x \nleq Nx$ *for* $\mid x \mid= R$ *and* $Nx \nleq x$ *for* $\mid x \mid= r$*.*
Then N *has a fixed point* x *with* $r <\mid x \mid< R$*.*

Other forms of Krasnoselskii's fixed point theorem are the following two theorems:

Theorem 1.14 *(M. A. Krasnoselskii [36]) Let* $(X, |\ .\ |)$ *be a normed linear space,* $K \subset X$ *a cone,* $\leq$ *the partial order relation induced by* K, $0 < r < R$ *and* $K_{r,R} = \{x \in K : r \leq\mid x \mid\leq R\}$*. Assume that* $N : K_{r,R} \to K$ *is a compact map and one of the following conditions is satisfied:*
(a) $x \nless Nx$ *for* $\mid x \mid= r$ *and* $Nx \nless x$ *for* $\mid x \mid= R$*;*
(b) $x \nless Nx$ *for* $\mid x \mid= R$ *and* $Nx \nless x$ *for* $\mid x \mid= r$*.*
Then N *has a fixed point* x *with* $r \leq\mid x \mid\leq R$*.*

Theorem 1.15 *(M. A. Krasnoselskii, [36]) Let* X *be a Banach space and* $K \subset X$ *be a cone. Assume that* Ω_1 *and* Ω_2 *are open subsets of* X *with* $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$*, and let*

$$N : K \cap (\overline{\Omega_2} - \Omega_1) \to K$$

Using more or less the same technique, many authors gave existence and localization results for different classes of equations, such as [2], [6], [13], [16]-[35], [39]-[52], [55], [56], [60]-[69], [71]-[87], [89], [90].

2

Generalizations of Krasnoselskii's fixed point theorem in cones

This chapter contains the main theoretical contributions of this thesis. They are concerning with compression and expansion conditions expressed in terms of different functionals which totally or only partially replace the norm of the space. This chapter relies on the papers S. Budisan [11], [12].

2.1 Fixed point theorems in a usual conical annulus.

The presentation from this section is based on the paper S. Budisan [11]. In this section we give a number of generalizations of Krasnoselskii's fixed point theorems in the cone K, where the fixed point is localized in the conical annulus

$$K_{r,R} := \{u \in K : r \leq\mid u \mid\leq R\}.$$

The compression-expansion conditions are expressed with two functionals, namely:

$$\begin{cases} \varphi(u) \leq \varphi(Nu) & \text{if } \mid u \mid= r \\ \psi(u) \geq \psi(Nu) & \text{if } \mid u \mid= R, \end{cases} \quad \text{(compression conditions)}$$

respectively

$$\begin{cases} \varphi(u) \geq \varphi(Nu) & \text{if } \mid u \mid= r \\ \psi(u) \leq \psi(Nu) & \text{if } \mid u \mid= R. \end{cases} \quad \text{(expansion conditions)}$$

The first theorem from this section deals with the compression of the conical annulus, while the following two theorems deal with the expansion of the conical annulus.

Throughout this section we consider $(X, |\ .\ |)$ be a normed linear space, $K \subset X$ a positive cone, "$\preceq$" the order relation induced by K, "$\prec$" the strict order relation induced by K and $\mathbf{R}_+ = [0, \infty)$.

Theorem 2.1 *(S. Budisan [11]) Let $r, R \in \mathbf{R}_+$ be such that $0 < r < R$. Assume that $N : K_{r,R} \to K$ is a completely continuous operator and let $\varphi : K \to \mathbf{R}_+, \psi : K \to \mathbf{R}$. Also, assume that the following conditions are satisfied:*

$$(i1) \begin{cases} \varphi(0) = 0 \text{ and there exists } h \in K - \{0\} \text{ such that} \\ \qquad \varphi(\lambda h) > 0, \text{ for all } \lambda \in (0,1], \\ \varphi(x+y) \geq \varphi(x) + \varphi(y) \text{ for all } x, y \in K, \end{cases}$$

$(i2)$ *$\psi(\alpha x) > \psi(x)$ for all $\alpha > 1$ and for all $x \in K$ with $| x |= R$,*

$$(i3) \begin{cases} \varphi(u) \leq \varphi(Nu) \text{ if } | u |= r \\ \psi(u) \geq \psi(Nu) \text{ if } | u |= R. \end{cases}$$

Then N has a fixed point in $K_{r,R}$.

Remark 2.2 *(S. Budisan [11]) (1) If $X := C[0,1], \eta > 0, I \subset [0,1]$, $I \neq [0,1]$, $\| x \| := \max_{t \in [0,1]} x(t)$ and $K := \{x \in C[0,1] : x \geq 0$ on $[0,1], x(t) \geq \eta \| x \|$ for all $t \in I\}$, a functional that satisfies (i1) is*

$$\varphi(x) := \min_{t \in I} x(t).$$

Indeed, $\varphi(0) = 0$, there exists $h \in K - \{0\}$ such that $\varphi(\lambda h) > 0$, for all $\lambda \in (0,1]$ and

$$\varphi(x+y) = \min_{t \in I} [x(t) + y(t)] \geq \min_{t \in I} x(t) + \min_{t \in I} y(t) = \varphi(x) + \varphi(y).$$

(2) The norm is an example of functional that satisfies (i2).

Theorem 2.3 *(S. Budisan [11]) Let $r, R \in \mathbf{R}_+$ be such that $0 < r < R$. Assume that $N : K_{r,R} \to K$ is a completely continuous operator and let $\varphi, \psi : K \to \mathbf{R}$. Suppose that the following conditions are satisfied:*

$(ii1)$ *φ is strictly decreasing,*

$(ii2)$

$$\psi(\alpha x) < \psi(x) \text{ for all } \alpha > 1 \text{ and for all } x \in K \text{ with } | x |= R,$$

$(ii3)$

$$\begin{cases} \varphi(u) \geq \varphi(Nu) \text{ if } | u |= r, \\ \psi(u) \leq \psi(Nu) \text{ if } | u |= R. \end{cases}$$

Then N has a fixed point in $K_{r,R}$.

Remark 2.4 *(S. Budisan [11]) The functional $\psi(x) := \frac{1}{|x|+1}$ is an example of mapping that satisfies (ii2). Indeed, for $\alpha > 1$ and $\mid x \mid= R$, we have that $\psi(\alpha x) = \frac{1}{\alpha|x|+1} < \frac{1}{|x|+1} = \psi(x)$. Also, if $\mid . \mid$ is strictly increasing, i.e., $x < y$ implies $\mid x \mid<\mid y \mid$, then $\varphi(x) := \frac{1}{|x|+1}$ is strictly decreasing, so it satisfies (ii1).*

Theorem 2.5 *(S. Budisan [11]) Let $r, R \in \mathbf{R}_+$ be such that $0 < r < R$. Assume that $N : K_{r,R} \to K$ is a completely continuous operator and $\varphi, \psi : K \to \mathbf{R}_+$. Suppose that the following conditions are satisfied:*

$$(iii1)\begin{cases} \varphi(\alpha x) = \alpha\varphi(x), \text{ for all } \alpha > 0 \text{ and } x \in K, \\ \varphi(\alpha x) > \varphi(x), \text{ for all } \alpha > 1 \text{ and for all } x \in K \text{ with } \mid x \mid= R, \end{cases}$$

$$(iii2)\begin{cases} \psi(0) = 0 \text{ and there exists } h \in K \setminus \{0\} \text{ such that} \\ \qquad \psi(\lambda h) > 0 \text{ for all } \lambda \in (0, 1], \\ \psi(\alpha x) = \alpha\psi(x) \text{ for all } \alpha > 0 \text{ and } x \in K, \\ \psi(x + y) \geq \psi(x) + \psi(y) \text{ for all } x, y \in K, \end{cases}$$

(iii3)

$$\begin{cases} \varphi(u) \geq \varphi(Nu) \text{ if } \mid u \mid= r, \\ \psi(u) \leq \psi(Nu) \text{ if } \mid u \mid= R. \end{cases}$$

Then N has a fixed point in $K_{r,R}$.

2.2 Fixed point theorems in a conical annulus defined by one or two functionals

The presentation in this section is based on the paper S. Budisan [12]. In this section we give a number of generalizations of Krasnoselskii's fixed point theorems in the cone K, where the fixed point is localized in a conical generalized annulus of one of the forms:

$$K_{r,R} := \{u \in K : r \leq \varphi(u) \leq R\},$$

$$K_{r,R} := \{u \in K : r \leq \varphi(u) \leq \psi(u) \leq R\}, \text{ where } \varphi \leq \psi \text{ on } K,$$

or

$$K_{r,R} := \{u \in K : r \leq \psi(u), \varphi(u) \leq R\}, \text{ where } \varphi \leq \psi \text{ on } K.$$

The both compression-expansion conditions and boundary conditions are expressed using the same functionals φ and ψ, namely:

$$\begin{cases} \varphi(u) \leq \varphi(Nu) \text{ if } \varphi(u) = r \\ \psi(u) \geq \psi(Nu) \text{ if } \psi(u) = R, \end{cases} \quad \text{(compression conditions)}$$

or

$$\begin{cases} \varphi(u) \geq \varphi(Nu) \text{ if } \varphi(u) = r \\ \psi(u) \leq \psi(Nu) \text{ if } \psi(u) = R, \end{cases} \quad \text{(expansion conditions).}$$

The first three theorems from this section refer to the compression of the annulus. The fourth theorem refers to its expansion.

In this section we will assume throughout that $(X, |\,.\,|)$ is a normed linear space, $K \subset X$ is a positive cone, "$\preceq$" is the order relation induced by K, "$\prec$" the strict order relation induced by K, $\mathbf{R}_+ := [0, \infty), \mathbf{R}_+^* := (0, \infty)$, and for $r \in \mathbf{R}$ we have $\varphi^{-1}(r) := \{x \in K : \varphi(x) = r\}$, where $\varphi : K \to \mathbf{R}$ is a functional. From our results, we list here the following six theorems and remarks.

Theorem 2.6 *(S. Budisan [12]) Let $K_{r,R} = \{x \in K : r \leq \varphi(x) \leq R\}$ be a nonempty set, where $r, R \in \mathbf{R}_+^*$, $r < R$ and let $\varphi, \psi : K \to \mathbf{R}_+$ be continuous functionals. Assume that $N : K_{r,R} \to K$ is a continuous operator with $N(K_{r,R})$ relatively compact, $\varphi(0) = 0$ and φ is strictly increasing with respect to order relation induced by K (in the sense that $x, y \in K$ with $x \prec y$ implies $\varphi(x) < \varphi(y)$). Suppose that the following conditions are satisfied:*

(i) $\psi \geq \varphi$ on K,

(ii) $\varphi(\alpha x) = \alpha\varphi(x)$, for all $x \in K$ and for all $\alpha \in (0, \infty)$,

(iii) $\psi(\alpha x) = \alpha\psi(x)$ for all $x \in K$ and for all $\alpha \in (0, 1)$,

(iv) $\psi(\alpha x) \geq \alpha\psi(x)$ for all $x \in K$ with $\psi(x) \geq R$

and for all $\alpha \in (1, \infty)$,

(v) $N(\varphi^{-1}(r))$ is bounded,

(vi)

$$\varphi(x) \leq \varphi(Nx) \ \textit{if} \ \varphi(x) = r,$$
$$\psi(x) \geq \psi(Nx) \ \textit{if} \ \psi(x) \geq R.$$

Then N has a fixed point u^* in $K_{r,R}$.

Theorem 2.7 *(S. Budisan [12]) Let $K_{r,R} = \{x \in K : r \leq \varphi(x) \leq R\}$ be a nonempty set, where $r, R \in \mathbf{R}_+^*$, $r < R$ and let $\varphi, \psi : K \to \mathbf{R}_+$ be continuous functionals. Assume that $N : K_{r,R} \to K$ is a continuous operator with $N(K_{r,R})$*

relatively compact, $\varphi(0) = 0$ *and* $h \in K \setminus \varphi^{-1}(0)$ *such that* $\varphi(\lambda h) > 0$ *for all* $\lambda > 0$*. Also, assume that the following conditions are satisfied:*

(i) $\psi \geq \varphi$ *on* K*,*

(ii) $\varphi(\alpha x) = \alpha\varphi(x)$ *for all* $x \in K$ *and for all* $\alpha \in (0, \infty)$*,*

(iii) $\varphi(x + y) \geq \varphi(x) + \varphi(y)$*, for all* $x, y \in K$*,*

(iv) $\psi(\alpha x) = \alpha\psi(x)$ *for all* $x \in K$ *and for all* $\alpha > 0$*,*

(v) $N(\varphi^{-1}(r))$ *is bounded,*

(vi)

$$\begin{aligned} \varphi(x) &\leq \varphi(Nx) \ \text{ if } \varphi(x) = r, \\ \psi(x) &\geq \psi(Nx) \ \text{ if } \psi(x) \geq R. \end{aligned}$$

Then N *has a fixed point* $u^* \in K_{r,R}$*.*

Remark 2.8 *(S. Budisan [12]) If* $X := C([0,1], \mathbf{R}_+), I \subset [0,1], I \neq [0,1], \eta > 0, \| x \| := \max_{t\in[0,1]} x(t)$*,* $K := \{x \in X : x(t) \geq \eta \| x \|$ *for all* $t \in I\}$ *then* $\varphi(x) := \min_{t\in I} x(t)$ *and* $\psi(x) := \max_{t\in[0,1]} x(t)$ *are functionals that satisfy the hypothesis of Theorem 2.7.*

Theorem 2.9 *(S. Budisan [12]) Let* $\varphi, \psi : K \to \mathbf{R}_+$ *be continuous functionals,* $\varphi(0) = 0$ *and* $h \in K \setminus \varphi^{-1}(0)$ *such that* $\varphi(\lambda h) > 0$ *for all* $\lambda > 0$*. Assume that* $\psi \geq \varphi$ *on* K *and let* $K_{r,R} := \{x \in K : r \leq \varphi(x) \leq \psi(x) \leq R\}$ *be a nonempty set, where* $r, R \in \mathbf{R}_+^*$*,* $r < R$*. Also, assume that* $N : K_{r,R} \to K$ *is a continuous operator with* $N(K_{r,R})$ *relatively compact . Suppose that the following conditions are satisfied:*

(i) $\varphi(\alpha x) = \alpha\varphi(x)$ *for all* $x \in K$ *and for all* $\alpha \in (1, \infty)$*,*

(ii) $\varphi(x + y) \geq \varphi(x) + \varphi(y)$*, for all* $x, y \in K$*,*

(iii) $\psi(\alpha x) = \alpha\psi(x)$ *for all* $x \in K$ *and for all* $\alpha \geq 0$*,*

(iv) $R\varphi(x) \geq r\psi(x)$ *for all* $x \in K$*,*

(v) $N(\varphi^{-1}(r))$ *is bounded,*

(vi)

$$\begin{aligned} \varphi(x) &\leq \varphi(Nx) \ \text{ if } \varphi(x) = r, \\ \psi(x) &\geq \psi(Nx) \ \text{ if } \psi(x) = R. \end{aligned}$$

Then N *has a fixed point* $u^* \in K_{r,R}$*.*

Theorem 2.10 *(S. Budisan [12]) Let* r, R *be positive numbers with* $0 < r < R$ *and consider* $\varphi, \psi : K \to \mathbf{R}_+$ *continuous functionals ,* $\psi(0) = 0$ *and* $h \in K \setminus \psi^{-1}(0)$ *such that* $\psi(\lambda h) > 0$ *for all* $\lambda > 0$*.Assume that* $\psi \geq \varphi$*,* $R\varphi \geq r\psi$ *on* K *and denote* $c_1 := \frac{r}{R}, c_2 := \frac{R}{r}$*. Let* $K_{r,R} := \{x \in K : r \leq \psi(x), \varphi(x) \leq R\}$

be a nonempty set . Assume that $N : K_{r,R} \to K$ is a continuous operator with $N(K_{r,R})$ relatively compact. Suppose that the following conditions are satisfied:

(i) $\psi(\alpha x) = \alpha\psi(x)$ for all $x \in K$ and for all $\alpha \geq 0$,

(ii) $\psi(x+y) \geq \psi(x) + \psi(y)$ for all $x, y \in K$,

(iii) $\varphi(\alpha x) = \alpha\varphi(x)$ for all $x \in K$ and for all $\alpha \geq 0$,

(iv) $N(\psi^{-1}(r))$ is bounded,

(v) φ and ψ are increasing on K,

(vi) $c_1 N(x) \leq N(c_1 x)$ for all $x \in K_{r,R}$ with $\psi(x) = R$ and $c_2 N(x) \geq N(c_2 x)$ for all $x \in K_{r,R}$ with $\varphi(x) = r$,

(vii)

$$\begin{aligned} \varphi(x) &\geq \varphi(Nx) \quad \text{if } \varphi(x) = r, \\ \psi(x) &\leq \psi(Nx) \quad \text{if } \psi(x) = R. \end{aligned}$$

Then N has a fixed point $u_0 \in K_{r,R}$.

Remark 2.11 *(S. Budisan [12]) Notice that function ψ from Theorem 2.9 does not satisfies neither Property A1, nor Property A2. Also, φ from Theorem 2.10 does not satisfies neither Property A1, nor Property A2. Thus our theorems clearly extend Theorem 1.10.*

2.3 Krasnoselskii type fixed point theorems with respect to three functionals

The presentation in this section is based on the paper S. Budisan [12]. In this section we give two generalizations of Krasnoselskii's fixed point theorems in the cone K, where the "conical annulus" is defined as

$$K_{r,R} := \{x \in K : r \leq \delta(x) \leq R\},$$

by means of a functional δ, and the compression-expansion conditions are given in the terms of two functionals φ and ψ, while the boundary conditions are given using the same functional like in the conical annulus definition, namely:

$$\begin{cases} \varphi(u) \leq \varphi(Nu) \ \text{ if } \delta(u) = r \\ \psi(u) \geq \psi(Nu) \ \text{ if } \delta(u) = R, \end{cases} \quad \text{(compression conditions)},$$

respectively

$$\begin{cases} \varphi(u) \geq \varphi(Nu) \ \text{ if } \delta(u) = r \\ \psi(u) \leq \psi(Nu) \ \text{ if } \delta(u) = R, \end{cases} \quad \text{(expansion conditions)}.$$

In this section we will assume throughout that $(X, |\ .\ |)$ is a normed linear space, $K \subset X$ is a positive cone, "$\preceq$" is the order relation induced by K , "$\prec$" the strict order relation induced by K, $\mathbf{R}_+ := [0, \infty), \mathbf{R}_+^* := (0, \infty)$, and for $r \in \mathbf{R}$ we have $\varphi^{-1}(r) := \{x \in K : \varphi(x) = r\}$, where $\varphi : K \to \mathbf{R}$ is a functional.

Theorem 2.12 *(S. Budisan [12]) Let $\varphi, \psi, \delta : K \to \mathbf{R}_+$ be continuous functionals, $\delta\,(0) = 0$ and $h \in K \setminus \delta^{-1}(0)$ such that $\varphi(\lambda h) > 0$ for all $\lambda > 0$. Let $K_{r,R} := \{x \in K : r \leq \delta(x) \leq R\}$ be a nonempty set, where $r, R \in \mathbf{R}_+^*$, $r < R$. Assume that $N : K_{r,R} \to K$ is a continuous operator with $N(K_{r,R})$ relatively compact . Suppose that the following conditions are satisfied:*

(i) $\delta\,(\alpha x) = \alpha\delta\,(x)$ for all $x \in K$ and for all $\alpha \geq 0$,

(ii) $\varphi(x + y) \geq \varphi(x) + \varphi(y)$, for all $x, y \in K$,

(iii) $\psi\,(\alpha x) \geq \alpha\psi\,(x)$ for all $x \in K$ with $\delta(x) = R$ and for all $\alpha \in (1, \infty)$,

(iv) $\psi\,(x) > 0$ for all $x \in K$ with $\delta(x) = R$,

(v) $N(\delta^{-1}(r))$ is bounded,

(vi)

$$\varphi\,(x) \leq \varphi\,(Nx) \ \ \textit{if}\ \delta\,(x) = r,$$
$$\psi\,(x) \geq \psi\,(Nx) \ \ \textit{if}\ \delta\,(x) = R.$$

Then N has a fixed point $u^* \in K_{r,R}$.

Theorem 2.13 *(S. Budisan [12]) Let $\varphi, \psi, \delta : K \to \mathbf{R}_+$ be continuous functionals, $\delta\,(0) = 0$ and $h \in K\setminus\ \delta^{-1}(0)$ such that $\psi(\lambda h) > 0$ for all $\lambda > 0$. Let $K_{r,R} := \{x \in K : r \leq \delta(x) \leq R\}$ be a nonempty set, where $r, R \in \mathbf{R}_+^*$, $r < R$. Assume that $N : K_{r,R} \to K$ is a continuous operator with $N(K_{r,R})$ relatively compact . Suppose that the following conditions are satisfied:*

(i) $\delta\,(\alpha x) = \alpha\delta\,(x)$ for all $x \in K$ and for all $\alpha \geq 0$,

(ii)$\psi(x + y) \geq \psi(x) + \psi(y)$, for all $x, y \in K$,

(iii) $\psi\left(\frac{R}{r}x\right) = \frac{R}{r}\psi\,(x)$ for all $x \in K$ with $\delta(x) = r$,

(iv) $\varphi\,(x) > 0$ for all $x \in K$ with $\delta(x) = R$,

(v) $\begin{cases} \varphi(\frac{r}{R}x) = \frac{r}{R}\varphi(x) \text{ for all } x \in K \text{ with } \delta(x) = R, \\ \varphi(\alpha x) \geq \alpha\varphi(x) \text{ for all } x \in K \text{ with } \delta(x) = R \text{ and for all } \alpha \in (1, \infty), \end{cases}$

(vi) $N(\delta^{-1}(R))$ is bounded,

(vii)

$$\varphi\,(x) \geq \varphi\,(Nx) \ \ \textit{if}\ \delta\,(x) = r,$$
$$\psi\,(x) \leq \psi\,(Nx) \ \ \textit{if}\ \delta\,(x) = R.$$

Then N has a fixed point $u_0 \in K_{r,R}$.

2.4 Krasnoselskii type theorem for a coincidence equation

We give an abstract Krasnoselskii type theorem for the coincidence equation

$$Lx = Nx$$

For our result we need Theorem 1.14.

Theorem 2.14 *Let $(X, |\,.\,|)$ be a normed linear space,$K \subset X$ a cone, "$\prec$" the order relation induced by K. Let $r, R \in \mathbf{R}_+$, $0 < r < R$, $K_{r,R} := \{u \in K : r \leq | u | \leq R\}$ and $L, N : K_{r,R} \to K$. Suppose that there exists $P : K \to K$ so that $L + P$ is a bijection, $(L + P)^{-1}$ is completely continuous, $N + P$ is continuous and $(L + P)^{-1}$ is nondecreasing. Also assume that one of the following conditions (a) or (b) is satisfied:*

(a) there exists $u_1 \in K$ with $| u_1 | = r$ so that

$$Lu_1 \succeq Nu_1$$

and

there exists $u_2 \in K$ with $| u_2 | = R$ so that

$$Lu_2 \preceq Nu_2$$

(b) there exists $u_1 \in K$ with $| u_1 | = r$ so that

$$Lu_1 \preceq Nu_1$$

and

there exists $u_2 \in K$ with $| u_2 | = R$ so that

$$Lu_2 \succeq Nu_2$$

*Then (**??**) has a solution $u^* \in K_{r,R}$.*

3

Applications

This chapter contains the main applications of this thesis. They are concerning with existence and localization results for the solutions of several classes of problems involving ordinary differential equations and systems, and partial differential equations with p-Laplacian. The chapter relies on the papers S. Budisan [8], [9], [12] and S. Budisan, R. Precup [10].

3.1 Applications of Krasnoselskii's fixed point theorem in cones to a delay ordinary differential equations

The results of this section were established in the paper S. Budisan [8]. Consider the following bilocal problem:

$$\begin{cases} u''(t) + a(t)\ f(u(g(t))) = 0, \quad 0 < t < 1 \\ \alpha\, u(0) - \beta\, u'(0) = 0, \\ \gamma\, u(1) + \delta u'(1) = 0, \\ u(t) = k,\ -\theta \le t < 0. \end{cases} \tag{3.1}$$

Here $g : [0,1] \to [-\theta, 1]\,,\ \ \theta \ge 0$, and $g(t) \le t$ for all $t \in [0,1]$ and $k \ge 0$.

The following conditions will be assumed throughout:

(A1) $f \in C([0,\infty), [0,\infty))$ and $g \in C([0,1], [-\theta,1])$;

(A2) $a \in C([0,1], [0,\infty))$ and $a(t)$ is not identically zero on any proper subinterval of $[0,1]$;

(A3) $\alpha, \beta, \gamma, \delta \ge 0$ and $\rho := \gamma\,\beta + \alpha\,\gamma + \alpha\,\delta > 0$.

The purpose here is to give an existence result for positive solutions to (3.1), assuming that f is either superlinear or sublinear. We seek solutions of (3.1) which are positive in the sense that $u(t) > 0$ for $0 < t < 1$. We introduce the notations

$$f_0 := \lim_{u\to 0}\frac{f(u)}{u}, \qquad f_\infty := \lim_{u\to\infty}\frac{f(u)}{u}.$$

The situation $f_0 = 0$ and $f_\infty = \infty$ corresponds to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ to the sublinear case.

The proof of our main result, Theorem 3.1, is based on Theorem 1.15 due to Krasnoselskii.

Theorem 3.1 *(S. Budisan [8]) Assume that conditions (A1)-(A3) hold. If* $g \in C^1[0,1]$, $g' > 0$, $g(1) > 0$, *then problem (3.1) has at least one positive solution in each of the cases:*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear);*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Remark 3.2 *(S. Budisan [8]) If we choose* $g(t) = t - h, h \in [0,1)$, *we note that* $g(1) > 0 \geq g(0), g' \equiv 1$, *so g satisfies the hypothesis of Theorem 3.1.*

Remark 3.3 *(S. Budisan [8]) If we choose* $g(t) = \frac{t}{\xi}$, $\xi > 1$, *we note that* $g(1) > 0 \geq g(0), g' \equiv \frac{1}{\xi}$ *so g satisfies the hypothesis of Theorem 3.1.*

3.2 Applications to p-Laplacian equations

The content of this section is based on the paper S. Budisan [9], and we present below some of the results and reasonings. In this section we give some applications of Krasnoselskii's fixed point theorem in cones to the following boundary value problem for a system of equations with p-Laplacian($p \geq 2$):

$$\begin{cases} -\operatorname{div}(\mid \nabla u \mid_s^{p-2} \nabla u) = f(u) & \text{for } \mid x \mid < T, \\ u > 0 & \text{for } 0 < \mid x \mid < T, \\ u = 0 & \text{for } x = 0, \\ \nabla u = 0 & \text{for } \mid x \mid = T. \end{cases} \tag{3.2}$$

Here $x = (x_1, x_2, ..., x_N) \in \mathbf{R}^N$, $u = (u_1(x), u_2(x), ..., u_n(x))$ and

$$\mid x \mid_e = \sqrt{\sum_{j=1}^{N} x_j^2} = r$$

is the Euclidian norm. Also $\nabla u = (\nabla u_1, \nabla u_2, ..., \nabla u_n)$ is the gradient of u in the following sense:

$$\nabla u_i = \left(\frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, ..., \frac{\partial u_i}{\partial x_N}\right) \text{ and } \mid \nabla u \mid_s = \sum_{i=1}^{n} \mid \nabla u_i \mid_e,$$

and

$$\begin{aligned} \mathrm{div}(\quad | \quad & \nabla u \,|_s^{p-2} \nabla u) \\ = \quad & (\mathrm{div}(| \nabla u \,|_s^{p-2} \nabla u_1), \mathrm{div}(| \nabla u \,|_s^{p-2} \nabla u_2), ..., \mathrm{div}(| \nabla u \,|_s^{p-2} \nabla u_n)). \end{aligned}$$

We seek a radial solution of (3.2), that is, a function $u(x) = v(| x |_e)$.

This means that v satisfies the following conditions:

$$\begin{cases} [(\sum\limits_{i=1}^{n} | v_i^{'}(r) |)^{p-2} v^{'}(r)]^{'} + \frac{N-1}{r}[(\sum\limits_{i=1}^{n} | v_i^{'}(r) |)^{p-2} v^{'}(r)] \\ \qquad = -f(v(r)),\ 0 < r < T, \\ \qquad v^{'}(T) = v(0) = 0, \\ \qquad v > 0 \text{ on } (0, T). \end{cases}$$

Theorem 3.4 *(S. Budisan [9]) Let* $f \in C(\mathbf{R}_+^n; \mathbf{R}_+^n)$*. Assume that there are numbers* $\alpha, \beta > 0$*,* $\alpha \neq \beta$ *and functions* $\varphi, \psi \in L^1([0, T]; \mathbf{R}_+)$ *such that*

$$\frac{S_1}{\alpha^{p-1}} < [\int_0^T (\int_s^T \tau^{-N} \varphi(\tau) d\tau)^{\frac{1}{p-1}} ds]^{1-p}, \tag{3.3}$$

and

$$\frac{I_1}{\beta^{p-1}} > [\int_0^T (\int_s^T \psi(\tau) d\tau)^{\frac{1}{p-1}} ds]^{1-p}, \tag{3.4}$$

where

$$S_1 := \sup_{x \in \mathbf{R}_+^n,\ |x| \le \alpha,\ r \in (0,T)} \frac{| f(x(r)) + (N-1) r^{-N} \int_r^T t^{N-1} f(x(t)) dt |}{r^{-N} \varphi(r)},$$

$$I_1 := \inf_{x \in \mathbf{R}_+^n,\ |x| \in [\frac{r}{T}\beta, \beta],\ r \in (0,T)} \frac{| f(x(r)) + (N-1) r^{-N} \int_r^T t^{N-1} f(x(t)) dt |}{\psi(r)}.$$

Here the norm $| \,.\, |$ *is* $| \,.\, |_s$*.*

Then (3.2) has at least one solution $u \in K$ *with* $| u |_s$ *increasing, concave and*

$$\min\{\alpha, \beta\} <| u |_\infty < \max\{\alpha, \beta\}.$$

Remark 3.5 *(S. Budisan [9]) 1) If* f *is nondecreasing on* $[0, \max\{\alpha, \beta\}]$*, condition (3.3) can be replaced with*

$$\frac{T^N f(\alpha)}{\alpha^{p-1}} \frac{1}{\inf\limits_{r \in (0,T)} \varphi(r)} < [\int_0^T (\int_s^T \tau^{-N} \varphi(\tau) d\tau)^{\frac{1}{p-1}} ds]^{1-p}, \tag{3.5}$$

for $\inf\limits_{r \in (0,T)} \varphi(r) > 0$*.*

2) If in addition we suppose $\varphi \equiv 1$, then condition (3.5) becomes

$$\frac{f(\alpha)T^N}{\alpha^{p-1}} < [\int_0^T (\int_s^T \tau^{-N} d\tau)^{\frac{1}{p-1}} ds]^{1-p} \tag{3.6}$$

and for $p > N$, (3.6) can be replaced by the sufficient condition

$$\frac{f(\alpha)}{\alpha^{p-1}} < (\frac{p-N}{p-1})^{p-1} \frac{N-1}{T^p}. \tag{3.7}$$

3) If $\psi(t) = 0$ for $0 \le t \le a < T$ and $\psi(t) = 1$ for $a < t \le T$, then condition (3.4) becomes

$$I_2 :=$$

$$= \inf_{x \in \mathbf{R}^n_+,\ |x| \in [\frac{r}{T}\beta, \beta],\ r \in (a,T)} \frac{| f(x(r)) + (N-1) r^{-N} \int_r^T t^{N-1} f(x(t)) dt |}{\beta^{p-1}}$$

$$> (\frac{p}{p-1})^{p-1} \frac{1}{(T-a)^p}.$$

If, in addition, f is nondecreasing on $[0, \max\{\alpha, \beta\}]$, condition (3.4) becomes

$$\frac{f(\frac{a}{T}\beta)}{\beta^{p-1}} > (\frac{p}{p-1})^{p-1} \frac{1}{(T-a)^p}. \tag{3.8}$$

For a given compact interval $[c, d]$, let λ_1 and ϕ_1 be the first eigenvalue and a corresponding positive eigenfunction of the problem

$$\begin{cases} -\operatorname{div}(| \nabla u |_s^{p-2} \nabla u) = \lambda \mid u \mid^{p-2} u, \\ u(c) = u'(d) = 0. \end{cases} \tag{3.9}$$

Notice that

$$\lambda_1 = \inf\{\frac{\int_c^d | \nabla u |^p dt}{\int_c^d | u |^p dt} : u \in C^1[c, d] \setminus \{0\}, u(c) = u'(d) = 0\},\ \lambda_1 > 0,$$

and there exists a function $\phi_1 \in C^1[c, d]$ with $\phi_1(c) = \phi_1'(d) = 0$ and $\phi_1(t) > 0$ on (c, d), for which the above inf is reached. In the sequel we shall always assume that $| \phi_1 |_\infty = \max_{t \in [c,d]} \phi_1(t) = 1$.

Theorem 3.6 *(S. Budisan [9]) Let $f \in C(\mathbf{R}^n_+; \mathbf{R}^n_+)$. Assume that there exist intervals $[a, b]$ and $[A, B]$ with $[a, b] \subseteq [0, T] \subseteq [A, B]$, such that if λ, ϕ and Λ, Φ denote the first eigenvalue and the first positive eigenfunction for the interval $[a, b]$ and respectively $[A, B]$, then* *(i) there are constants $c, C > 0$ with*

$$c\phi(t)^{p-1} \leq 1 \text{ a.e. } t \in (a,b), \tag{3.10}$$

$$1 \leq C\Phi(t)^{p-1} \text{ a.e. } t \in (0,T); \tag{3.11}$$

(ii) there are numbers $\alpha, \beta > 0$, $\alpha \neq \beta$ with

$$C\frac{\max\limits_{x\in\mathbf{R}_+^n,\ |x|\leq\alpha} \mid f(x) \mid}{\alpha^{p-1}}\left[1+(N-1)\sup_{r\in(0,T)}\frac{\displaystyle\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\right] < \Lambda, \tag{3.12}$$

$$c\frac{\min_{x\in\mathbf{R}_+^n,\ |x|\in[\frac{r}{T}\beta,\beta],\ r\in[a,b]}|f(x(r))|}{\beta^{p-1}}\cdot \tag{3.13}$$
$$\left[1+(N-1)\inf_{r\in(0,T)}\frac{\displaystyle\int_r^T t^{N-1}\phi(t)^{p-1}dt}{r^N\phi(r)^{p-1}}\right] > \lambda M,$$

where $M > 1$ is such that

$$\phi(r)^{p-1} + (N-1)r^{-N}\int_r^T t^{N-1}\phi(t)^{p-1}dt \leq M\phi(r)^{p-1} \text{ for } r \in [0,T]. \tag{3.14}$$

Then (3.2) has at least one solution $u \in K$ with $\mid u \mid_s$ increasing, concave and

$$\min\{\alpha,\beta\} <\mid u \mid_\infty < \max\{\alpha,\beta\}.$$

Theorem 3.7 *(S. Budisan [9]) Let $f \in C(\mathbf{R}_+^n;\mathbf{R}_+^n)$ and suppose that there exist the intervals $[a,b]$ and $[A,B]$ with $[a,b] \subset [0,T] \subseteq [A,B]$ and $a > 0$ so that the condition (i) from Theorem 3.6 is fulfiled (λ, ϕ, Λ, Φ are from Theorem 3.6). Also, suppose that condition (3.14) is satisfied. In addition, assume that one of the following two conditions holds:*

(a)

$$Ch_0\left[1+(N-1)\sup_{r\in(0,T)}\frac{\displaystyle\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\right] < \Lambda$$

and

$$(\frac{a}{T})^{p-1}ch_\infty\left[1+(N-1)\inf_{r\in(0,T)}\frac{r^{-N}\displaystyle\int_r^T t^{N-1}\phi(t)^{p-1}dt}{\phi(r)^{p-1}}\right] > \lambda M$$

or

(b)

$$(\frac{a}{T})^{p-1}ch_0\left[1+(N-1)\inf_{r\in(0,T)}\frac{r^{-N}\displaystyle\int_r^T t^{N-1}\phi(t)^{p-1}dt}{\phi(r)^{p-1}}\right]>\lambda M$$

and

$$Ch_\infty\left[1+(N-1)\sup_{r\in(0,T)}\frac{\displaystyle\int_r^T t^{N-1}\Phi(t)^{p-1}dt}{r^N\Phi(r)^{p-1}}\right]<\Lambda.$$

Then (3.2) has a solution.

Remark 3.8 *(S. Budisan [9]) 1) The relation (3.8) from Remark 3.5, 3) is also found in D. O'Regan,R. Precup [54](Remark 2.3, 3^0).*

2) For $p>N$, the relation (3.7), namely

$$\frac{f(\alpha)}{\alpha^{p-1}}<(\frac{p-N}{p-1})^{p-1}\frac{N-1}{T^p},$$

from Remark 3.5, 2) is similar to the relation

$$\frac{f(\alpha)}{\alpha^{p-1}}<(\frac{p}{p-1})^{p-1}\frac{N-1}{T^p}$$

from D. O'Regan,R. Precup [54], Remark 2.3, 3^0.

3.3 Applications of the vector version of Krasnoselskii's fixed point theorem

This section relies on the paper S. Budisan, R. Precup [10]**.** We give an application of Krasnoselskii's fixed point theorem in cones to the second-order functional-differential system

$$\begin{cases} u_1''(t)+a_1(t)f_1(u_1(g(t)),u_2\left(g\left(t\right)\right))=0, \\ u_2''(t)+a_2(t)f_2(u_1(g(t)),u_2\left(g\left(t\right)\right))=0 \end{cases} \tag{3.15}$$

$(0<t<1)$ under the boundary conditions

$$\begin{cases} \alpha_i u_i(0)-\beta_i u_i'(0)=0, \\ \gamma_i u_i(1)+\delta_i u_i'(1)=0, \\ u_i(t)=k_i \quad \text{for } -\theta\le t<0 \quad (i=1,2). \end{cases} \tag{3.16}$$

Here $\theta > 0$, $g : [0,1] \to [-\theta, 1]$ and $g(t) \le t$ for all $t \in [0,1]$. We seek positive solutions to (3.15)-(3.16), that is a couple $u = (u_1, u_2)$ with $u_i(t) > 0$ for $0 < t < 1$ and $i = 1, 2$.

We shall assume that the following conditions are satisfied for $i \in \{1, 2\}$:

(A1) $f_i \in C([0,\infty)^2, [0,\infty))$ and $g \in C([0,1], [-\theta, 1])$;

(A2) $a_i \in C([0,1], [0,\infty))$ and $a_i(t)$ is not identically zero on any proper subinterval of $[0,1]$;

(A3) $\alpha_i, \beta_i, \gamma_i, \delta_i, k_i \ge 0$, $\rho_i := \gamma_i\beta_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$.

Our existence result is based on the vector version of Krasnoselskii's fixed point theorem in cones, Theorem 1.11, due to R. Precup [57] and extends to systems the main result due to S. Budisan [8]. We need to introduce some notations and notions. Let $(X, \| \, . \, \|)$ be a normed linear space, let K_1, K_2 be two cones of X and let $K := K_1 \times K_2$. We shall use the same symbol $\preceq$ to denote the partial order relation induced by K in X^2 and by K_1, K_2 in X. Similarly, the same symbol $\prec$ will be used to denote the strict order relation induced by K_1 and K_2 in X. Also, in X^2, the symbol $\prec$ will have the following meaning: $u \prec v$ $(u, v \in X^2)$ if $u_i \prec v_i$ for $i = 1, 2$. For $r, R \in \mathbf{R}_+^2$, $r = (r_1, r_2), R = (R_1, R_2)$, we write $0 < r < R$ if $0 < r_1 < R_1$ and $0 < r_2 < R_2$ and we use the notations:

$$\begin{aligned} (K_i)_{r_i,R_i} \quad &: \quad = \{v \in K_i : r_i \le \| v \| \le R_i\} \quad (i = 1, 2) \\ K_{r,R} \quad &: \quad = \{u = (u_1, u_2) \in K : r_i \le \| u_i \| \le R_i \text{ for } i = 1, 2\}. \end{aligned}$$

Clearly, $K_{r,R} = (K_1)_{r_1,R_1} \times (K_2)_{r_2,R_2}$.

Remark 3.9 *(S. Budisan, R. Precup [10])* In Theorem 1.11 four cases are posible for $u \in K_{r,R}$:

(c1) $N_1(u) \nprec u_1$ if $\| u_1 \| = r_1$, and $N_1(u) \nsucc u_1$ if $\| u_1 \| = R_1$,
$N_2(u) \nprec u_2$ if $\| u_2 \| = r_2$, and $N_2(u) \nsucc u_2$ if $\| u_2 \| = R_2$;

(c2) $N_1(u) \nprec u_1$ if $\| u_1 \| = r_1$, and $N_1(u) \nsucc u_1$ if $\| u_1 \| = R_1$,
$N_2(u) \nsucc u_2$ if $\| u_2 \| = r_2$ and $N_2(u) \nprec u_2$ if $\| u_2 \| = R_2$;

(c3) $N_1(u) \nsucc u_1$ if $\| u_1 \| = r_1$ and $N_1(u) \nprec u_1$ if $\| u_1 \| = R_1$,
$N_2(u) \nprec u_2$ if $\| u_2 \| = r_2$, and $N_2(u) \nsucc u_2$ if $\| u_2 \| = R_2$;

(c4) $N_1(u) \nsucc u_1$ if $\| u_1 \| = r_1$ and $N_1(u) \nprec u_1$ if $\| u_1 \| = R_1$,
$N_2(u) \nsucc u_2$ if $\| u_2 \| = r_2$ and $N_2(u) \nprec u_2$ if $\| u_2 \| = R_2$.

The main result of this section is the following existence theorem.

Theorem 3.10 *(S. Budisan, R. Precup [10]) Assume that conditions* (A1)-(A3) *hold. In addition assume that $g \in C^1[0,1]$, $g' > 0$, $g(1) > 0$. Then problem* (3.15)-(3.16) *has at least one positive solution $u := (u_1, u_2)$ in each* *of the following four cases:*

$$\begin{cases} (f_1)_0(x_2) = 0 & \textit{uniformly for all } x_2 \geq 0, \\ (f_1)_\infty(x_2) = \infty & \textit{uniformly for all } x_2 \geq 0, \\ (f_2)_0(x_1) = 0 & \textit{uniformly for all } x_1 \geq 0, \\ (f_2)_\infty(x_1) = \infty & \textit{uniformly for all } x_1 \geq 0; \end{cases} \tag{3.17}$$

$$\begin{cases} (f_1)_0(x_2) = \infty & \textit{uniformly for all } x_2 \geq 0, \\ (f_1)_\infty(x_2) = 0 & \begin{array}{l} \textit{uniformly for all } x_2 \geq 0, \ \textit{if } \ f_1 \ \textit{is unbounded} \\ \textit{(and in this case suppose that at least } f_1(.,x_2) \\ \textit{is unbounded),} \end{array} \\ (f_2)_0(x_1) = \infty & \textit{uniformly for all } x_1 \geq 0, \\ (f_2)_\infty(x_1) = 0 & \begin{array}{l} \textit{uniformly for all } x_1 \geq 0, \ \textit{if } \ f_2 \ \textit{is unbounded} \\ \textit{(and in this case suppose that at least } f_2(x_1,.) \\ \textit{is unbounded);} \end{array} \end{cases} \tag{3.18}$$

$$\begin{cases} (f_1)_0(x_2) = \infty & \textit{uniformly for all } x_2 \geq 0, \\ (f_1)_\infty(x_2) = 0 & \begin{array}{l} \textit{uniformly for all } x_2 \geq 0, \ \textit{if } \ f_1 \ \textit{is unbounded} \\ \textit{(and in this case suppose that at least } f_1(.,x_2) \\ \textit{is unbounded),} \end{array} \\ (f_2)_0(x_1) = 0 & \textit{uniformly for all } x_1 \geq 0, \\ (f_2)_\infty(x_1) = \infty & \textit{uniformly for all } x_1 \geq 0; \end{cases} \tag{3.19}$$

$$\begin{cases} (f_1)_0(x_2) = 0 & \textit{uniformly for all } x_2 \geq 0, \\ (f_1)_\infty(x_2) = \infty & \textit{uniformly for all } x_2 \geq 0, \\ (f_2)_0(x_1) = \infty & \textit{uniformly for all } x_1 \geq 0, \\ (f_2)_\infty(x_1) = 0 & \begin{array}{l} \textit{uniformly for all } x_1 \geq 0, \ \textit{if } f_2 \ \textit{is unbounded} \\ \textit{(and in this case suppose that at least } f_2(x_1,.) \\ \textit{is unbounded).} \end{array} \end{cases} \tag{3.20}$$

Remark 3.11 *(S. Budisan, R. Precup [10])* 1) An example of function $f = (f_1, f_2)$ like in (3.17) is the following:

$$\begin{aligned} f_1(x_1,x_2) &= \frac{x_1^p(x_2+1)}{x_1+x_2+1}, \\ f_2(x_1,x_2) &= x_2^p[\frac{1}{(x_1+1)^q}+1] + x_2^r[\frac{1}{(x_1+1)^s}+1], \end{aligned}$$

where $r > 1$; $p > 2$; $q, s > 0$.

2) An example of function $f = (f_1, f_2)$ like in (3.18) is the following:

$$\begin{aligned} f_1(x_1, x_2) &= x_1^p[\frac{1}{(x_2+1)^q} + 1] + x_1^r[\frac{1}{(x_2+1)^s} + 1], \\ f_2(x_1, x_2) &= x_2^p[\frac{qx_1}{x_1+1} + 1] + x_2^r[\frac{sx_1}{x_1+1} + 1], \end{aligned}$$

where $0 < r, p < 1$; $q, s > 0$.

3.4 Existence of positive periodic solutions of differential systems

In this section we complete the results from R. Precup [57] by providing sufficient conditions that assure the hypothesis of Theorem 3.12(Theorem 3.1, R. Precup [57]).

The author R. Precup [57] gives sufficient conditions in only three different cases and it is our goal in this section to obtain sufficient conditions in other two cases.

These conditions guarantee the existence and localization of positive periodic solutions of the nonlinear differential system

$$\begin{cases} u_1'(t) = -a_1(t)u_1(t) + \varepsilon_1 f_1(t, u_1(t), u_2(t)) \\ u_2'(t) = -a_2(t)u_2(t) + \varepsilon_2 f_2(t, u_1(t), u_2(t)), \end{cases} \tag{3.21}$$

where for $i \in \{1, 2\} : a_i \in C(\mathbf{R}, \mathbf{R})$, $\int_0^\omega a_i(t)dt \neq 0$, $\varepsilon_i = sign \int_0^\omega a_i(t)dt$, $f_i \in C(\mathbf{R} \times \mathbf{R}_+^2, \mathbf{R}_+)$, and a_i, $f_i(., u_1, u_2)$ are ω-periodic functions for some $\omega > 0$. We seek positive ω-periodic solutions to system (3.21), i.e., pairs $u := (u_1, u_2)$ of ω-periodic functions from $C(\mathbf{R}, (0, \infty))$ which satisfy (3.21). Let $X = \{v : v \in C(\mathbf{R}, \mathbf{R}),\ v(t) = v(t + \omega)\}$ be endowed with norm $\mid v \mid_\infty = \max_{t \in [0,\omega]} \mid v(t) \mid$, and let P be the cone of all nonnegative functions from X. Let

$$H_i(t, s) = \varepsilon_i \frac{\exp(\int_t^s a_i(\tau)d\tau)}{\exp(\int_0^\omega a_i(\tau)d\tau) - 1} = \frac{\exp(\int_t^s a_i(\tau)d\tau)}{\mid \exp(\int_0^\omega a_i(\tau)d\tau) - 1 \mid},$$

$i \in \{1, 2\}$. It is easily seen that the problem of finding nonnegative ω-periodic solutions for system (3.21) is equivalent to the integral system in P^2,

$$\begin{cases} u_1(t) = \int\limits_t^{t+\omega} H_1(t,s) f_1(s, u_1(s), u_2(s)) ds \\ u_2(t) = \int\limits_t^{t+\omega} H_2(t,s) f_2(s, u_1(s), u_2(s)) ds. \end{cases}$$

Let

$$\begin{aligned} A_i &: = \min\{H_i(t,s) : 0 \le t \le \omega, t \le s \le t+\omega\}, \\ B_i &: = \max\{H_i(t,s) : 0 \le t \le \omega, t \le s \le t+\omega\}, \\ M_i &: = \frac{A_i}{B_i} \ (i = 1, 2). \end{aligned}$$

Notice that $A_i = H_i(t,t) > 0$ and $B_i = H_i(t, t+\omega) > 0$ if $\varepsilon_i = +1$, and $A_i = H_i(t, t+\omega) > 0$ and $B_i = H_i(t,t) > 0$ when $\varepsilon_i = -1$. Also note that $0 < M_i < 1$. Before stating Theorem 3.12, we introduce the following notations. For α_i, $\beta_i > 0$ with $\alpha_i \neq \beta_i$ we let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ $(i = 1, 2)$, and

$$\begin{aligned} \gamma_1 &= \min\{f_1(t, u_1, u_2) : 0 \le t \le \omega, M_1\beta_1 \le u_1 \le \beta_1, M_2 r_2 \le u_2 \le R_2\}, \\ \gamma_2 &= \min\{f_2(t, u_1, u_2) : 0 \le t \le \omega, M_1 r_1 \le u_1 \le R_1, M_2\beta_2 \le u_2 \le \beta_2\}, \\ \Gamma_1 &= \max\{f_1(t, u_1, u_2) : 0 \le t \le \omega, M_1\alpha_1 \le u_1 \le \alpha_1, M_2 r_2 \le u_2 \le R_2\}, \\ \Gamma_2 &= \max\{f_2(t, u_1, u_2) : 0 \le t \le \omega, M_1 r_1 \le u_1 \le R_1, M_2\alpha_2 \le u_2 \le \alpha_2\}. \end{aligned}$$

Theorem 3.12 *(Theorem 3.1,R. Precup [57]) Assume that there exist α_i, $\beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, such that*

$$\begin{aligned} B_1\omega\Gamma_1 &\le \alpha_1, \quad A_1\omega\gamma_1 \ge \beta_1, \\ B_2\omega\Gamma_2 &\le \alpha_2, \quad A_2\omega\gamma_2 \ge \beta_2. \end{aligned} \tag{3.22}$$

Then (3.21) has a positive ω-periodic solution $u = (u_1, u_2)$ with $r_i \le | u_i |_\infty \le R_i$, $i = 1, 2$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$. Moreover, the corresponding orbit of u is included in the rectangle $[M_1 r_1, R_1] \times [M_2 r_2, R_2]$.

Now we give sufficient conditions for (3.22), if $f_1 = f_1(u_1, u_2)$ and $f_2 = f_2(u_1, u_2)$ do not depend on t, where we shall assume for simplicity that $M_1 = M_2 =: M$:

Theorem 3.13 *Assume that f_1 is nondecreasing in u_1 and u_2, f_2 is nonincreasing in u_1 and u_2, and suppose that*

$$\lim_{x\to\infty} \frac{f_1(Mx,0)}{x} = \infty, \tag{3.23}$$

$$\lim_{x\to 0} \frac{f_1(x,y)}{x} = 0 \text{ for all } y > 0, \tag{3.24}$$

$$\lim_{y\to\infty} \frac{f_2(x,My)}{y} = 0, \text{ uniformly for all } x > 0. \tag{3.25}$$

$$\lim_{y\to 0} \frac{f_2(x,y)}{y} = \infty, \text{ for all } x > 0. \tag{3.26}$$

Then conditions (3.22) are satisfied for some suitable $\alpha_1, \alpha_2, \beta_1, \beta_2$.

Remark 3.14 *The functions $f_1(x,y) = x^2(y^2+1)$ and $f_2(x,y) = \frac{1}{(x+1)(y+1)}$ satisfy the conditions from the previous theorem.*

Theorem 3.15 *Assume that f_1, f_2 are nondecreasing in u_1 and nonincreasing in u_2 and suppose that*

$$\lim_{x\to\infty} \frac{f_1(x,0)}{x} = 0, \tag{3.27}$$

$$\lim_{x\to 0} \frac{f_1(Mx,y)}{x} = \infty, \text{ for all } y > 0, \tag{3.28}$$

$$\lim_{y\to\infty} \frac{f_2(x,My)}{y} = 0, \text{ for all } x > 0, \tag{3.29}$$

$$\lim_{y\to 0} \frac{f_2(x,y)}{y} = \infty, \text{ for all } x > 0. \tag{3.30}$$

Then conditions (3.22) are satisfied for some suitable $\alpha_1, \alpha_2, \beta_1, \beta_2$.

Remark 3.16 *The functions $f_1(u_1,u_2) = \frac{u_1^{\frac{1}{4}}}{u_2+1}$ and $f_2(u_1,u_2) = \frac{u_1}{u_2+1}$ satisfy the conditions from previous theorem.*

3.5 Applications of the generalizations of Krasnoselskii's fixed point theorems

In this section we give applications of our abstract theorems from Chapter 2. The presentation from this section relies on the paper S. Budisan [12]. We are concerned with the existence of at least one positive solution for the second order boundary value problem,

$$u''(t) + f(t,u(t)) = 0, \; 0 \le t \le 1, \tag{3.31}$$

$$u(0) = u(1) = 0, \tag{3.32}$$

where $f : [0,1] \times \mathbf{R}_+ \to \mathbf{R}_+$ is continuous. We look for solutions $u \in C^2[0,1]$ of (3.31)-(3.32), which are both nonnegative and concave on $[0,1]$. We will apply the theorems from the previous chapter to a completely continuous operator whose kernel is the Green's function $G(t,s)$ of the equation

$$u'' = 0, \tag{3.33}$$

under the boundary conditions (3.32). We have that

$$G(t,s) = \begin{cases} t(1-s),\ 0 \le t \le s \le 1, \\ s(1-t),\ 0 \le s \le t \le 1. \end{cases} \tag{3.34}$$

$G(t,s)$ has the following properties, which are essential for our results:

$$\begin{cases} G(t,s) \le G(s,s), & 0 \le t, s \le 1, \\ \frac{1}{4}G(s,s) \le G(t,s), & 0 \le s \le 1,\ \frac{1}{4} \le t \le \frac{3}{4}, \\ \int\limits_0^1 G(s,s)ds = \frac{1}{6}, & \int\limits_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds = \frac{11}{96}. \end{cases} \tag{3.35}$$

Let

$$\begin{cases} I := [\frac{1}{4}, \frac{3}{4}], \\ K := \{u \in C[0,1] : u \ge 0 \text{ on } [0,1],\ u(t) \ge \frac{1}{4} \parallel u \parallel \text{ for all } t \in I\}, \\ \varphi, \psi : K \to \mathbf{R}_+,\ \varphi(u) := \min\limits_{t\in I} u(t),\ \psi(u) := \max\limits_{t\in[0,1]} u(t) = \parallel u \parallel . \end{cases} \tag{3.36}$$

This technique of using min and max functionals is also used in R. Avery, J. Henderson, D. O'Regan [4] and in many other papers.

It is obviously that K is a cone. Also, $u \in K$ is a solution of (3.31)-(3.32) if and only if

$$u(t) = \int\limits_0^1 G(t,s) f(s, u(s))ds. \tag{3.37}$$

Firstly, we will impose conditions on f which ensure the existence of at least one positive solution of (3.31)-(3.32) by applying Theorem 2.9.

Theorem 3.17 *(S. Budisan [12]) Let M, r, R be positive numbers such that $0 < r < R$, $r \le \frac{M}{35}, \frac{64}{11}r \le R$ and let $f_1 : [0,1] \times K \to [0, \infty)$ be defined by $f_1(s,x) = f(s, x(s))$. Assume that f_1 satisfies the following conditions:*

(i) $f_1(s,x) \le M$ for all $s \in [0,1]$, for all $x \in K$ with $\varphi(x) = r$,

(ii) $f_1(s,x) \ge \frac{384}{11} r$ for all $s \in I$, for all $x \in K$ with $\varphi(x) = r$,

(iii) $f_1(s,x) \le 6R$ for all $s \in [0,1]$, for all $x \in K$ with $\psi(x) = R$.

Then (3.31)-(3.32) has a solution u^* such that

$$r \leq \min_{t\in[\frac{1}{4},\frac{3}{4}]} u^*(t) \;\leq\; \max_{t\in[0,1]} u^*(t) \leq R.$$

Remark 3.18 *(S. Budisan [12])The function $f(s,x)= \frac{s+1}{s+2}\cdot\frac{x+2}{x+3}$ is an example of mapping that satisfies Theorem 3.17.*

Theorem 3.19 *(S. Budisan [12]) Let M, r, R be positive numbers such that $0 < r < R$, $0 < M$, $r \leq \frac{M}{35}$ and let $f_1 : [0,1] \times K \to [0,\infty)$ be defined by $f_1(s,x) = f(s,x(s))$. Assume that f_1 satisfies the following conditions:*

(i) $f_1(s,x) \leq M$ for all $s \in [0,1]$, for all $x \in K$ with $\delta(x) = r$,

(ii) $f_1(s,x) \geq \frac{384}{11} r$ for all $s \in I$, for all $x \in K$ with $\delta(x) = r$,

(iii) $f_1(s,x) \leq 6R$ for all $s \in [0,1]$, for all $x \in K$ with $\delta(x) = R$,

where $\delta(x) := \frac{\varphi(x)+\psi(x)}{2}$.

Then (3.31)-(3.32) has a solution u^* such that

$$r \leq \frac{\min_{t\in[\frac{1}{4},\frac{3}{4}]} u^*(t) + \max_{t\in[0,1]} u^*(t)}{2} \leq R \tag{3.38}$$

and

$$r \leq \max_{t\in[0,1]} u^*(t) \leq 4R \text{ and } \frac{r}{4} \leq \min_{t\in[\frac{1}{4},\frac{3}{4}]} u^*(t) \;\leq R. \tag{3.39}$$

Remark 3.20 *(S. Budisan [12]) The function $f(s,x) := \frac{s+1}{x+1}$ is an example of a mapping that satisfies the hypothesis of Theorem 3.19.*

Let

$$\varphi_1, \psi_1 : K \to \mathbf{R}_+,\ \psi_1(u) := \min_{t\in I} u(t),\ \varphi_1(u) := \max_{t\in[0,1]} u(t) = \| u \| .$$

We make these new notations to make easier the application of Theorem 2.13.

Now we give an application of Theorem 2.13.

Theorem 3.21 *(S. Budisan [12]) Let r and R be positive numbers such that $0 < r < R$ and let $f_1 : [0,1]\times K \to [0,\infty)$ be defined by $f_1(s,x) = f(s,x(s))$.Suppose f_1 satisfies the following conditions:*

(i) $f_1(s,x) \leq 6r$ for all $s \in [0,1]$, for all $x \in K$ with $\delta(x) = r$,

(ii) $f_1(s,x) \geq \frac{384}{11} R$ for all $s \in I$, for all $x \in K$ with $\delta(x) = R$,

where $\delta(x) := \frac{\varphi_1(x)+\psi_1(x)}{2}$.

Then (3.31)-(3.32) has a solution u^* such that

$$r \leq \frac{\min_{t\in[\frac{1}{4},\frac{3}{4}]} u^*(t) + \max_{t\in[0,1]} u^*(t)}{2} \leq R \tag{3.40}$$

and

$$r \leq \max_{t\in[0,1]} u^*(t) \leq 4R \text{ and } \frac{r}{4} \leq \min_{t\in[\frac{1}{4},\frac{3}{4}]} u^*(t) \ \leq R. \tag{3.41}$$

Remark 3.22 *(S. Budisan [12]) In Theorem 3.19 and Theorem 3.21 (the relations (3.39) and (3.41)) we obtain similar localization results with the results from R. Avery, J. Henderson, D. O'Regan [4]. In this paper the authors obtain the following localization result(see Theorem 4.1):*

$$r \leq \max_{t\in[0,1]} u^*(t) \text{ and } \min_{t\in[\frac{1}{2},\frac{3}{4}]} u^*(t) \ \leq R. \tag{3.42}$$

Now we give an other application of Theorem 2.13, where $f(.,.)$ may be unbounded. In our proof we use a functional δ, that gives a similar result like in (3.42). We note that functional δ is used in S. Budisan [12] for the first time.

Theorem 3.23 *(S. Budisan [12]) Let r and R be positive numbers such that $0 < 16r < R$ and suppose f satisfies the following conditions:*
(i) $f(s,x) \leq 6r$ for all $s \in [0,1]$, for all $x \in [0,16r]$,
(ii) $f(s,x) \geq \frac{6144}{11}R$ for all $s \in I$, for all $x \in [R,16R]$.
Then (3.31)-(3.32) has a solution u_0 with

$$r \leq \min_{t\in[\frac{1}{4},\frac{3}{4}]} u_0 \leq 4R$$

and

$$r \leq \max_{t\in[0,1]} u_0 \leq 16R.$$

Remark 3.24 *(S. Budisan [12]) An example of unbounded function that satisfies the conditons of Theorem 3.23 is $f(s,x) = \frac{s+1}{s+2}x^2$.*

As an application of Theorem 3.23 we give the following multiplicity result.

Corollary 3.25 *(S. Budisan [12]) Suppose that there exist positive numbers r_1, r_2, R_1 and R_2 such that $0 < 16r_1 < R_1 \leq \frac{11r_2}{1024}$, $16r_2 < R_2$ and f satisfies the following conditions:*
(i) $f(s,x) \leq 6r_1$ for all $s \in [0,1]$, for all $x \in [0,16r_1]$,
(ii) $f(s,x) \geq \frac{6144}{11}R_1$ for all $s \in I$, for all $x \in [R_1,16R_1]$,
(iii) $f(s,x) \leq 6r_2$ for all $s \in [0,1]$, for all $x \in [0,16r_2]$,
(iv) $f(s,x) \geq \frac{6144}{11}R_2$ for all $s \in I$, for all $x \in [R_2,16R_2]$,

Then (3.31)-(3.32) has at least two solutions u_1 and u_2 with

$$r_1 \leq \min_{t\in[\frac{1}{4},\frac{3}{4}]} u_1 \leq 4R_1,$$

$$r_1 \leq \max_{t\in[0,1]} u_1 \leq 16R_1,$$

and

$$r_2 \leq \min_{t\in[\frac{1}{4},\frac{3}{4}]} u_2 \leq 4R_2,$$

$$r_2 \leq \max_{t\in[0,1]} u_2 \leq 16R_2.$$

Remark 3.26 *(S. Budisan [12]) An example of function f that satisfies the conditions from Corollary 3.25 is*

$$f(s,x) = \begin{cases} 6r_1 \exp(x-16r_1), \text{ if } x \in [0,16r_1], \\ a(x-R_1)\exp(x-16r_1) + \frac{6144}{11}R_1\exp(x-R_1), \text{ if } x \in (16r_1, R_1), \\ \frac{6144}{11}R_1\exp(x-R_1), \text{ if } x \in [R_1, 16R_1], \\ 6cr_2\exp(x-16r_2) + d(x-16r_2), \text{ if } x \in (16R_1, 16r_2], \\ \frac{6144}{11}R_2\exp(x-R_2) + p(x-R_2)\exp(x-16r_2), \text{ if } x \in (16r_2, R_2), \\ \frac{6144}{11}R_2\exp(x-R_2), \text{ if } x \in [R_2, \infty), \end{cases}$$

where, beside the conditions from Corollary 3.25, $r_1$and R_1 satisfy the conditions

$$\begin{cases} R_1 - 16r_1 \leq 1, \\ \frac{1024}{11}R_1\exp(16r_2 - R_1) < r_2 \textit{(it is posible for } R_1 \textit{ small enough),} \end{cases}$$

with r_2 and R_2 satisfying the conditions from Corollary 3.25. Also,

$$\begin{cases} a := \frac{6r_1 - \frac{6144}{11}R_1\exp(16r_1 - R_1)}{16r_1 - R_1}, \\ 1 \geq c > \frac{1024}{11}\frac{R_1}{r_2}\exp(16r_2 - R_1), \\ p := \frac{6cr_2 - \frac{6144}{11}R_2\exp(16r_2 - R_2)}{16r_2 - R_2}, \\ d := \frac{\frac{6144}{11}R_1\exp(15R_1) - 6cr_2\exp(16R_1 - 16r_2)}{16R_1 - 16r_2} \end{cases}$$

The values of the constants a, c, p, d assure us that f is continuous.

Finally we give an application of Theorem 2.10.

Theorem 3.27 *(S. Budisan [12]) Let r, R be positive numbers with $R > 4r > 0$ and denote $c_1 := \frac{r}{R}, c_2 := \frac{R}{r}$. Let us define $\varphi, \psi : K \to [0,\infty)$, $\varphi(x) := \frac{1}{4}\max_{t\in[0,1]} x(t) := \frac{1}{4} \parallel x \parallel$, $\psi(x) := \min_{t\in I} x(t)$ and $K_{r,R} := \{x \in K : r \leq \psi(x), \varphi(x) \leq R\}$, $f(s,x) = g(s)h(x)$ where $g : [0,1] \to [0,\infty)$, $h : [0,\infty) \to$*

$[0, \infty)$ *are continuous. Also, we define* $h_1 : [0,1] \times K \to [0, \infty), h_1(s, x) = h(x(s))$. *Assume that the following conditions are satisfied:*

(i) $\begin{cases} h_1(s, c_1 x) \geq c_1 h_1(s, x) \text{ for all } s \in [0,1] \text{ and } x \in K \text{ with } \psi(x) = R, \\ h_1(s, c_2 x) \leq c_2 h_1(s, x) \text{ for all } s \in [0,1] \text{ and } x \in K \text{ with } \varphi(x) = r, \end{cases}$

(ii) $f(s, x) \leq 24r$ *for all* $s \in [0, 1]$, *for all* $x \in [0, 4r]$,

(iii) $f(s, x) \geq \frac{384}{11} R$ *for all* $s \in I$, *for all* $x \geq R$.

Then (3.31)-(3.32) has a solution u_0 *with*

$$r \leq \min_{t \in I} u_0(t) \leq \| u_0 \| \leq 4R.$$

Remark 3.28 *(S. Budisan [12]) Comparing the condition (ii) from Theorem 3.27 with the following condition*

$$f(x) \leq 8r \text{ for all } x \in [0, r]$$

(the condition (ii) from Theorem 4.1 from R. Avery, J. Henderson, D. O'Regan [4]), we note that Theorem 3.27 extends Theorem 4.1 from R. Avery, J. Henderson, D. O'Regan [4].

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