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Representations and Evaluations of the Remainder Term of the Formulas of Interpolation and Numerical Integration with Applications

PhD Thesis Summary

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Contents

Keywords: interpolation, remainder term, optimal quadrature formula, corrected quadrature formula, Peano's theorem, Ostrowski inequality

Introduction

We have chosen this topic of great complexity and difficulty from the following objectives: study of the remainder term in the interpolation formulas and numerical integration and assessments of the remainder term in numerical integration formulas using the Ostrowski type inequalities.

The methodological, theoretical-scientific approach in the discussed field was highly important in the approach and analysis of the doctoral thesis topic, being based on a thorough theoretical research, which offered a clear image on the scientific theories, grounded in the field of numerical analysis.

We mention that the school of Numerical Analysis from Cluj has great results, well known around the international field of study of quadrature formulas. A large number of mathematicians have developed a number of valuable works relating to quadratures and numerical cubature: academic D. D. Stancu, P. Blaga, Gh. Coman, A. Coţiu, I. Gânscă, H. Roşcău, D. Acu, Gh. Micula, I. Gavrea, A. Lupaş, T. Vladislav and others. So currently, many mathematicians, formed around Numerical Analysis School of Cluj-Napoca, continue successful the scientific research in the field of numerical integration of functions of one and more variables.

The doctoral thesis has been divided into 3 chapters and a bibliography containing 130 titles, out of which 8 also signed by the author.

Chapter 1 is divided into 4 sections. The first section presents the main spaces of functions used. In section 1.2 is presented interpolation of functions on a rectangular field. In section 1.3 is presented interpolation of function on a standard triangle. Starting with the work of R. E. Barnhill, G. Birkhoff and W. J. Gordon [15], the triangle interpolation operators are studied extensively ([16], [18], [19], [29], [40], [58], [88], [89], [99]). In section 1.4 is presented interpolation of functions on a triangle with a curved side. Operator of type Lagrange, Hermite and Birkhoff on this triangle were also obtained by Gh. Coman and T. Cătinaş in [59]. I got these new operators of interpolations and representations of the rest term for them.

Chapter 2 is divided into 3 sections. In Section 2.1 are presented optimal quadrature formulas in the sense of Sard and Nikolski. I got here estimates of the remainder term of an optimal quadrature formula in the sense of Nikolski, in 2 points open type. Interesting results concerning optimal quadrature formulas in sense of Nikolski were also obtained by Gh. Coman, Gh. Micula, in papers [50], [48], [51], [49], [57]. The problem of constructing optimal quadrature formulas in various classes of functions has been studied in many articles. The first results were obtained by A. Sard, L. S. Meyers and S. M. Nikolski. In section 2.2 are presented the corrected quadrature formula of open type. I got here estimates of the remainder term for optimal quadrature formula in the sense of Nikolski, in 2 points open type, and then we derived formulas perturbed (corrected) thereof, giving also estimates for the remainder term. In section 2.3 are presented corrected quadrature formulas of closed type. I got here estimates of the remainder term for optimal quadrature formula in the sense of Nikolski, a closed 3-point type, but also some general optimal quadrature formulas and then corrected their derived formulas, giving also estimates for the remainder term.

Chapter 3 is divided into 3 sections. In sect Section 3.1 are presented the main results on Ostrowski type inequalities. In the last years, the Ostrowski's inequality occupied the attention of many authors ([10], [68], [67], [72], [73], [74], [101], [105], [122], [123],

[124]). In section 3.2 are presented mean value theorems used to obtain the Ostrowski type inequalities. Mean value theorems have been applied to demonstrate this kind of inequality. D. Pompeiu, S. S. Dragomir, E. C. Popa, J. Pecaric, S. Ungar, B. G. Pachpatte have achieved important results of applying the mean value theorems for obtaining Ostrowski type inequalities as seen in the papers [106], [69], [108], [105], [103]. I got this new inequalities of Ostrowski type using the mean value theorems. In section 3.2, are presented applications of Ostrowski's inequality in numerical integration. I gave here new estimates of the remainder term for a quadrature formula.

The original results are found within Section 1.4 (Theorem 1.4.3, Theorem 1.4.4, Theorem 1.4.5, Theorem 1.4.6, Theorem 1.4.7, Theorem 1.4.8, Theorem 1.4.9), 2.1 (Theorem 2.1.1, Theorem 2.1.2, Example 2.1.1, Example 2.1.2, Example 2.1.3), 2.2 (Theorem 2.2.1, Observation 2.2.1, Observation 2.2.2, Observation 2.2.2, Theorem 2.2.3, Observation 2.2.3, Observation 2.2.4, Observation 2.2.5, Observation 2.2.6, Observation 2.2.7, Theorem 2.2.4, Theorem 2.2.5, Theorem 2.2.6), 2.3 (Theorem 2.3.2, Observation 2.3.1, Observation 2.3.2, Theorem 2.3.3, Observation 2.3.3, Theorem 2.3.4, Observation 2.3.4, Observation 2.3.5, Theorem 2.3.5, Theorem 2.3.6, Observation 2.3.6, Observation 2.3.7, Theorem 2.3.7, Theorem 2.3.8, Theorem 2.3.9, Theorem 2.3.10, Observation 2.3.8, Theorem 2.3.11, Theorem 2.3.12, Theorem 2.3.13, Theorem 2.3.14, Observation 2.3.9), 3.2 (Theorem 3.2.5, Observation 3.2.2, Theorem 3.2.6, Theorem 3.2.7, Theorem 3.2.8, Theorem 3.2.9, Theorem 3.2.10), 3.3 (Theorem 3.3.2, Theorem 3.3.3).

A crucial reason in addressing the thesis is that, as a teacher, one must always be updated, possessing pertinent, relevant and scientifically sound information, and one must always be involved in the development of the scientifically field in which you work.

On this occasion I wish to thank professor PhD Petru Blaga, for the support I received during the realization of this work. I also bring sincere thanks to the Applied Mathematics department from Babes-Bolyai University from Cluj -Napoca.

Chapter 1

Interpolation functions of two variables

In this chapter is presented the notion of interpolation defined functions: a rectangular area on a standard triangle and a triangle with a curved side. The general concepts are presented, and the main results of interpolation in the areas mentioned above.

Starting from the operators defined on a triangle with a curved side in paper [59], we introduced in the last section a Lagrange-type operator that interpolates a function catheters, curved side, and on a line inside the triangle, when considering the interior is a median line. By using Peano's theorem for two-dimensional case we gave the evaluation of the remainder term for appropriate interpolation formula for this operator. We used this operator and an Lagrange-type operator defined in [59] and has built their product operators and sum Boolean, studying and the rest terms for their corresponding formulas. Afterwards by using the built Lagrange operator and Hermite-type operator defined in [59] we built new interpolation operators Boolean product and sum. We determined the properties of interpolation and degree of exactness for these operators. Also studied interpolation formulas generated, giving estimates for the remainder term. These results are contained in the papers [11] and [12].

1.1 Spaces of functions

1.2 Interpolation functions defined on a rectangular domain

Let $D \subset \mathbb{R}^n$ be a rectangular domain; $D = \prod_{i=1}^{n}[a_i, b_i]$ and $\mathcal{F}_n$ a set of functions defined on D. Also let $\bigwedge^{x_i}$ be a set of information about the function f in relation to variable $x_i, i = 1, ..., n$.

We call a projector a linear transformation P from a vector space to itself so $P^2 = P$. Let $P_i : \mathcal{F}_n \to \mathcal{G}_i$, that interpolates the function $f \in \mathcal{F}_n$ in relation to information $\bigwedge^{x_i}$. So, $\mathcal{G}_i$ are sets of functions of $n-1$ independent variables $(x_i, ..., x_{i-1}, x_{i+1}, ..., x_n)$. Assuming that the projectors $P_1, ..., P_n$ commute, is denoted by $\mathcal{P}_n$ lattice generated by these in relation to order relation " $\leq$ ".

Let be P *product*, and S *boolean sum* of all generators projectors $P_1, ..., P_n$ namely $P = P_1, ..., P_n, S = P_1 \oplus ... \oplus P_n$, i.e. $S = P_1 + ... + P_n - P_1P_2 - ... - P_{n-1}P_n + ... + (-1)^{n-1}P_1...P_n$.

In book [121] D. D. Stancu, Gh. Coman and P. Blaga show the following two theorems:

Theorem 1.2.1. *$P \leq Q \leq S$ for any $Q \in \mathcal{P}_n$.*

The problem relates to the error of approximation. In other words, Q generates the approximation formula f:

$$f = Qf + R_Q f.$$

The problem considered above is to study the remainder term $R_Q f$ or the remainder operator R_Q. For this purpose we note with $R_i = I - P_i$, (I the identic operator) the remainder operators corresponding generators operators.

For P and S, the product operator respectively boolean sum, the corresponding remainder operators are

$$R_P = R_1 \oplus ... \oplus R_n \tag{1.1}$$

and

$$R_S = R_1 ... R_n. \tag{1.2}$$

So take place the following decomposition of the identic operator:

$$I = P + R_P \tag{1.3}$$

and

$$I = S + R_S \tag{1.4}$$

In set of interpolation formulas generated by elements of $\mathcal{P}_n$ distinguish formulas given by elements P and S.

Definition 1.2.1. *Interpolation formula*

$$f = Pf + R_P f \tag{1.5}$$

is called algebraic minimal, and formula

$$f = Sf + R_S f \tag{1.6}$$

algebraic maximal

Remark 1.2.1. *Formula (1.5) is called tensor product interpolation formula, and (1.6) boolean sum interpolation formula.*

In terms of quality of approximation interpolation operator is characterized by *order approximation* ("ord"). We say that the operator $P_i : \mathcal{F}_n \to \mathcal{G}_i$ has the order approximation m if $Ker(P_i) = \mathcal{P}_m^n$, where $\mathcal{P}_m^n$ is the set of polynomials in n variable and global degree at most m.

The remainder term R_p of the minimal algebraic formula (1.5) is given by the boolean sum of the operators $R_1, ..., R_n$, while the remainder operator R_S of the maximal algebraic formula (1.6) is the product of operators $R_1, ..., R_n$. It follows that

$$ord(P) = min\{ord(P_1), ..., ord(P_n)\},$$

and

$$ord(S) = ord(P_1) + ... + ord(P_n).$$

Note that

$$ord(P) \leq ord(Q) \leq ord(S), \ Q \in \mathcal{P}_n.$$

Thus we have the remarkable property of maximal algebraic approximation formula :

$$ord(S) = \max_{Q \in \mathcal{P}_n} ord(Q).$$

1.3 Interpolation functions defined on a triangle T_h

In order to study the remainder of such interpolation formulas we need a Peano type theorem for the case of multidimensional, in particular two-dimensional, for functional defined to Sard type spaces. We first present two such spaces.

1. Sard space $B_{pq}(a,c), (p,q \in \mathbb{N}, p+q = m)$ of functions $f : D \to \mathbb{R}, D = [a,b] \times [c,d]$, with properties:

1) $f^{(p,q)} \in C(D)$
2) $f^{(m-j,j)} \in C[a,b], j = 0,1,...,q-1$
3) $f^{(i,m-i)} \in C[a,b], i = 0,1,...,p-1.$

Theorem 1.3.1. *If $f \in B_{pq}(a,c)$ then*

$$f(x,y) = \sum_{i+j<m} \frac{(x-a)^i}{i!} \frac{(y-c)^j}{j!} f^{(i,j)}(a,c) + (R_m f)(x,y) \tag{1.7}$$

where

$$\begin{aligned}(R_m f)(x,y) &= \sum_{j<q} \frac{(y-c)^j}{j!} \int_a^b \frac{(x-s)_+^{m-j-1}}{(m-j-1)!} f^{(m-j,j)}(s,c)ds + \\ &+ \sum_{i<p} \frac{(x-a)^i}{i!} \int_c^d \frac{(y-t)_+^{m-i-1}}{(m-i-1)!} f^{(i,m-i)}(a,t)dt + \\ &+ \iint_D \frac{(x-s)_+^{p-1}}{(p-1)!} \frac{(y-t)_+^{q-1}}{(q-1)!} f^{(p,q)}(s,t)dsdt,\end{aligned}$$

and $z_+ = z$ for $z \geq 0$ and $z_+ = 0$ for $z < 0$.

Remark 1.3.1. *Taylor's formula 1.7 holds for any plan domain Ω with the property that there is a point $(a,c) \in \Omega$ (which is near the Taylor development) so rectangle $[a,x] \times [c,y] \subseteq \Omega$ whatever $(x,y) \in \Omega$.*

In this case the formula (1.7)can be written as

$$f(x,y) = \sum_{i+j<m} \frac{(x-a)^i}{i!} \frac{(y-c)^j}{j!} f^{(i,j)}(a,c) + (R_m f)(x,y)$$

with

$$
\begin{aligned}
(R_m f)(x,y) &= \sum_{j<q} \frac{(y-c)^j}{j!} \int_{I_1} \frac{(x-s)_+^{m-j-1}}{(m-j-1)!} f^{(m-j,j)}(s,c)ds + \\
&+ \sum_{i<p} \frac{(x-a)^i}{i!} \int_{I_2} \frac{(y-t)_+^{m-i-1}}{(m-i-1)!} f^{(i,m-i)}(a,t)dt + \\
&+ \iint_{\Omega} \frac{(x-s)_+^{p-1}}{(p-1)!} \frac{(y-t)_+^{q-1}}{(q-1)!} f^{(p,q)}(s,t)dsdt,
\end{aligned}
$$

where

$$I_1 = \{(x,y) \in \mathbb{R}^2 | y = c\} \cap \Omega$$

$$I_2 = \{(x,y) \in \mathbb{R}^2 | x = a\} \cap \Omega.$$

Examples of such domains are triangle $T_h = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq h\}$ with $(a,c) = (0,0)$ or any circle with center (a,c).

2. Space $B_{pq}^r(a,c), (p,q \in \mathbb{N}, p+q = m, r \in \mathbb{R}, r \geq 1)$ of functions $f : D \to \mathbb{R}, D = [a,b] \times [c,d]$, with properties:

1) $f^{(i,j)} \in C(D), i < p, j < q$
2) $f^{(m-j-1,j)}$ is absolutely continuous on $[a,b]$ and $f^{(m-j,j)} \in L^r[a,b], j < q$
3) $f^{(i,m-i-1)}$ is absolutely continuous on $[c,d]$ and $f^{(i,m-i)} \in L^r[c,d], i < p$
4) $f^{(p,q)} \in L^r(D)$.

Peano's theorem

Theorem 1.3.2. *Let $L : H^m[a,b] \to R$ be a linear functional with form*

$$L(f) = \sum_{i=0}^{m-1} \int_a^b f^{(i)}(x) d\mu_i(x),$$

where μ_i are functions of bounded variation on $[a,b]$.
If $Ker(L) = P_{m-1}$, then

$$L(f) = \int_a^b K_m(t) f^{(m)}(t) dt,$$

where

$$K_m(t) = L^x \left[\frac{(x-t)_+^{m-1}}{(m-1)!} \right].$$

Function K_m is called *Peano kernel.* For two-dimensional case we have:

Theorem 1.3.3. *Fie $L : B^r_{pq}(a,c) \to \mathbb{R}$ a linear functional and $f \in B_{pq}(a,c)$. If $Ker(L) = P^2_{m-1}$, then*

$$\begin{aligned} L(f) &= \sum_{j<q} \int_a^b K_{m-j,j}(s) f^{(m-j,j)}(s,c)ds + \\ &+ \sum_{i<p} \int_a^b K_{i,m-i}(t) f^{(i,m-i)}(a,t)dt + \\ &+ \iint_D K_{pq}(s,t) f^{(p,q)}(s,t)dsdt, \end{aligned}$$

where

$$\begin{aligned} K_{m-j,j}(s) &= L^{(x,y)}\left[\frac{(x-s)_+^{m-j-1}}{(m-j-1)!}\frac{(y-c)^j}{j!}\right], j<q \\ K_{i,m-i}(t) &= L^{(x,y)}\left[\frac{(x-a)^i}{i!}\frac{(y-t)_+^{m-i-1}}{(m-i-1)!}\right], i<p \\ K_{pq}(s,t) &= L^{(x,y)}\left[\frac{(x-s)_+^{p-1}}{(p-1)!}\frac{(y-t)_+^{q-1}}{(q-1)!}\right], \end{aligned}$$

are Peano kernel

Let

$$\mathbf{T}_h = \{(x,y) \in R^2 | x \geq 0,\ y \geq 0,\ x+y \leq h\}$$

be the standard triangle, as for any triangle $\mathbf{T}$ of plan there is an affine transformation of $\mathbf{T}$ in $\mathbf{T}_h$.

Be so $f : \mathbf{T}_h \to \mathbb{R}$.

Let (x,y) be a point inside the triangle $\mathbf{T}_h$ (see Figure 1.1). Each parallel to one side of the triangle and passing through the point (x,y) intersects the other two sides by two points. For example, cathetus parallel to Ox axis intersects the other cathetus and hypotenuse respectively in points $(0,y)$ and $(h-y,y)$. Noting by P_1 the Lagrange interpolation operator relative to the nodes 0 and $h-y$ and dropping the y, we obtain

$$(P_1 f)(x,y) = \frac{h-x-y}{h-y} f(0,y) + \frac{x}{h-y} f(h-y,y),$$

where f is a function defined on $\mathbf{T}_h$.

Immediately verify that

$$(P_1 f)(x,y) = f(0,y)$$

and

$$(P_1 f)(h-y,y) = f(h-y,y),\ y \in [0,h],$$

i.e. $P_1 f$ interpolates the function f on two sides of the triangle $\mathbf{T}_h$ (on hypotenuse and catheters located on the axis Oy).

In a similar manner are constructed the functions

$$(P_2 f)(x,y) = \frac{h-x-y}{h-x} f(x,0) + \frac{y}{h-x} f(x,h-x)$$

and

$$(P_3f)(x,y) = \frac{x}{x+y}f(x+y,0) + \frac{y}{x+y}f(0,x+y),$$

which interpolates the function f on all the pairs of sides of $\mathbf{T}_h$.

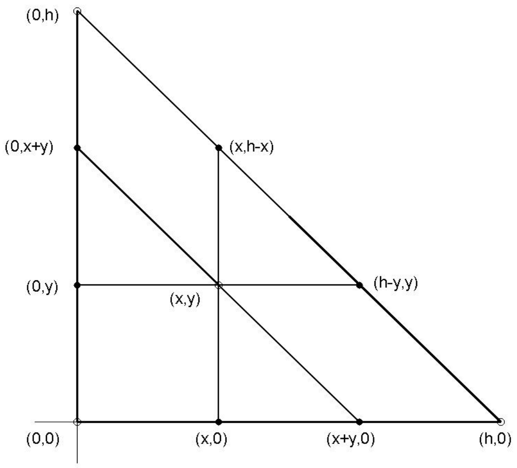

Figure 1.1: Standard triangle $\mathbf{T}_h$

Therefore, each of the operators P_1, P_2, P_3 generates a function that interpolates the given function f on two sides of the triangle $\mathbf{T}_h$.

Starting with the paper of R. E. Barnhill, G. Birkhoff and W. J. Gordon [15], triangle interpolation operators are studied extensively ([16], [18], [19], [29], [40], [58], [88], [89], [99]).

1.4 Interpolation functions defined on a triangle with one curved side

In [59] Gh. Coman and T. Cătinaş build some type operators Lagrange, Hermite and Birkhoff, which interpolates a given function and some of its derivatives on the boundary of a triangle with a curved side, and also build product and boolean sum operators of some of them. They also study the properties of interpolation and degree of exactness of construct operators, respectively the remainder term of corresponding interpolation formulas.

They consider a standard triangle, $\tilde{T}_h$, having the vertices $V_1 = (h,0), V_2 = (0,h)$ and $V_3 = (0,0)$, two straight sides Γ_1, Γ_2, along the coordinate axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e. $y = f(x)$ and $x = g(y)$ with $f(0) = g(0) = h$.(Figure 1.2)

Let F be a real-valued function defined on $\tilde{T}_h$.

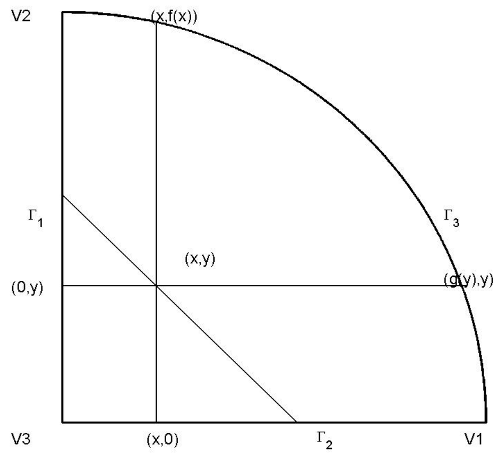

Figure 1.2: Standard triangle $\tilde{T}_h$

Let L_1, L_2 and L_3 be Lagrange operators defined by

$$
\begin{aligned}
(L_1F) &= \frac{g(y)-x}{g(y)}F(0,y) + \frac{x}{g(y)}F(g(y),y), \\
(L_2F) &= \frac{f(x)-y}{f(x)}F(x,0) + \frac{y}{f(x)}F(f(x),x), \\
(L_3F) &= \frac{x}{x+y}F(x+y,0) + \frac{y}{x+y}F(0,x+y),
\end{aligned}
\tag{1.8}
$$

(1) Each of the operators L_1, L_2 and L_3 interpolates the function F along the two sides of the triangle $\tilde{T}_h$, i.e.,

$$
\begin{aligned}
(L_1F)(0,y) &= F(0,y),\ y \in [0,h], \\
(L_1F)(g(y),y) &= F(g(y),y),\ y \in [0,h], \\
(L_2F)(x,0) &= F(x,0),\ x \in [0,h], \\
(L_2F)(x,f(x)) &= F(x,f(x)),\ x \in [0,h], \\
(L_3F)(x+y,0) &= F(x+y,0),\ x,y \in [0,h], \\
(L_3F)(0,x+y) &= F(0,x+y),\ x,y \in [0,h],
\end{aligned}
$$

(2) The degree of exactness: $gex(L_i) = 1,\ i = 1,2,3.$

(3) Regarding the remaining term, $R_i^L F, i = 1,2,3$, if interpolation formulas

$$F = L_iF + R_i^L F \ i = 1,2,3$$

we have

Theorem 1.4.1. *If* $F \in B_{11}(0,0)$ *then*

$$\begin{aligned}(R_1^L F)(x,y) &= \frac{x[x-g(y)]}{2} F^{(2,0)}(\xi,0)\\ &+ \frac{xy[g(y)-x]}{g(y)}\big[F^{(1,1)}(\xi_1,\eta_1) - F^{(1,1)}(\xi_2,\eta_2)\big],\end{aligned}$$

with $\xi \in [0,h], (\xi_1,\eta_1) \in [0,x] \times [0,y]$ *and* $(\xi_2,\eta_2) \in [x, g(y)] \times [0,y]$, *respectively*

$$|(R_1^L F)(x,y)| \le \frac{h^2}{8}\Big[\|F^{(2,0)}(\cdot,0)\|_\infty + \|F^{(1,1)}\|_\infty\Big], \tag{1.9}$$

where $\|\cdot\|_\infty$ *denotes the Chebysev norm.*

Then consider Hermite operators H_1 and H_2 defined by

$$\begin{aligned}(H_1F)(x,y) &= \frac{[x-g(y)]^2}{g^2(y)}F(0,y) + \frac{x[2g(y)-x]}{g^2(y)}F(g(y),y)\\ &+ \frac{x[x-g(y)]}{g(y)}F^{(1,0)}(g(y),y),\\ (H_2F)(x,y) &= \frac{[y-f(x)]^2}{f^2(x)}F(x,0) + \frac{y[2f(x)-y]}{f^2(x)}F(x,f(x))\\ &+ \frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x)).\end{aligned} \tag{1.10}$$

Corresponding interpolation formula is

$$F = H_iF + R_i^H F,\ i = 1,2,$$

where $R_i^H F, i = 1,2$ is the remainder term, for we have:

Theorem 1.4.2. *If* $F \in B_{12}(0,0)$ *then we have the following inequalities*

$$\begin{aligned}|(R_1^H F)(x,y)| &\le \frac{x[g(y)-x]^2}{6}\|F^{(3,0)}(\cdot,0)\|_\infty + \frac{xy[g(y)-x]^2}{2g(y)-x}\|F^{(2,1)}(\cdot,0)\|_\infty\\ &+ \frac{xy^2[g(y)-x][3g(y)-2x]}{2g^2(y)}\|F^{(1,2)}(\cdot,\cdot)\|_\infty,\end{aligned} \tag{1.11}$$

şi

$$\begin{aligned}|(R_1^H F)(x,y)| &\le \frac{2h^3}{81}\|F^{(3,0)}(\cdot,0)\|_\infty + \frac{xy[g(y)-x]^2}{2g(y)-x}\|F^{(2,1)}(\cdot,0)\|_\infty\\ &+ \frac{xy^2[g(y)-x][3g(y)-2x]}{2g^2(y)}\|F^{(1,2)}(\cdot,\cdot)\|_\infty.\end{aligned} \tag{1.12}$$

In [11] we built a Lagrange-type operator which interpolates the function F on a cathetus, on curved side , but the interior line of the triangle $\tilde{T}_h$. We consider the case when the interior line is a median. Thus we have introduced the operator L_2^x which interpolates the function F in relation to x in the points $(0,y)$, $\left(\frac{h-y}{2},y\right)$ and $(g(y),y)$:

$$
\begin{aligned}
(L_2^x F)(x,y) &= \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}F(0,y)+\frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]}F\left(\frac{h-y}{2},y\right)\\
&+\frac{x(2x-h-y)}{g(y)[2g(y)-h+y]}F(g(y),y). \qquad (1.13)
\end{aligned}
$$

So the operator L_2^x interpolates the function F on cathetus V_2V_3, on curved side and on median V_2M(Figure 1.3)

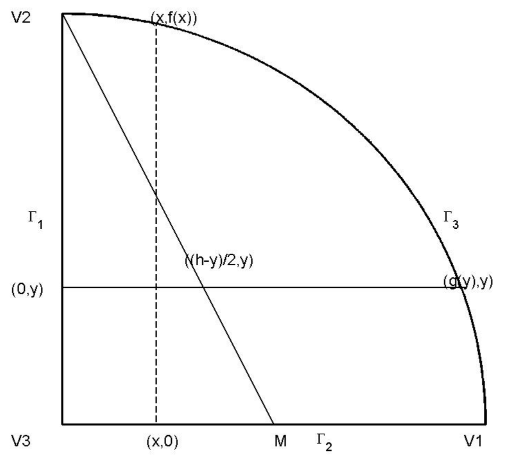

Figure 1.3: Standard triangle $\tilde{T}_h$

Regarding the remainder term $R_2^L F$, of the interpolation formula $F = L_2F + R_2^L F$, we have the following theorem:

Theorem 1.4.3. *If $F \in B_{12}(0,0)$, then the following inequality holds*

$$
\begin{aligned}
|R_2^L F| &\le \frac{x(h-y-2x)[4g(y)(h-y)(h-y-x)+(h-y)[-(h-y)^2+3(2g(y)-h+y)]]}{48g(y)(h-y)}\\
&\left\|F^{(3,0)}(\cdot,0)\right\|_\infty + \frac{xy[(h-y)^2-4x^2][g(y)-x]}{(h-y)[h-y-2x+2g(y)]}\left\|F^{(2,1)}(\cdot,0)\right\|_\infty \qquad (1.14)\\
&+\frac{xy^2(h-y-2x)[2(g(y)-x)+h-y]}{2g(y)(h-y)}\left\|F^{(1,2)}(\cdot,\cdot)\right\|_\infty .
\end{aligned}
$$

Next I built the product operator. We denote by L_1^y the operator L_2(1.8).
The product of the operators L_1^y and L_2^x is given by:

$$
\begin{aligned}
(P_{21}^L F)(x,y) &= \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}\left[\frac{h-y}{h}F(0,0)+\frac{y}{h}F(0,h)\right]+\frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]} \\
&\left[\frac{f\left(\frac{h-y}{2}\right)-y}{f\left(\frac{h-y}{2}\right)}F\left(\frac{h-y}{2},0\right)+\frac{y}{f\left(\frac{h-y}{2}\right)}F\left(\frac{h-y}{2},f\left(\frac{h-y}{2}\right)\right)\right] \quad (1.15)\\
&+ \frac{x(2x-h+y)}{g(y)[2g(y)-h+y]}F(g(y),y).
\end{aligned}
$$

Regarding the remainder term of the corresponding interpolation formula $F = P_{21}^L F + R_{21}^P F$, we have :

Theorem 1.4.4. *If $F \in B_{11}(0,0)$, then the following inequality holds*

$$
\begin{aligned}
|(R_{21}^L F)(x,y)| &\leq \left\|F^{(2,0)}(\cdot,0)\right\|_\infty \int_0^h |K_{20}(x,y,s)|ds + \left\|F^{(0,2)}(0,\cdot)\right\|_\infty \int_0^h |K_{02}(x,y,t)|dt \\
&+ \left\|F^{(1,1)}(\cdot,\cdot)\right\|_\infty \iint_{\tilde{T}_h} |K_{11}(x,y,s,t)|dsdt. \quad (1.16)
\end{aligned}
$$

The boolean sum of the operators L_1^y and L_2^x is given by

$$
\begin{aligned}
&(S_{21}^L F)(x,y) \\
&= \frac{x-g(y)}{h-y}\left[\frac{2x-h+y}{g(y)}F(0,y)+\frac{4x}{h-y-2g(y)}F\left(\frac{h-y}{2},y\right)\right] \\
&+ \frac{1}{f(x)}[(f(x)-y)F(x,0)+yF(x,f(x))] \\
&- \frac{(2x-h+y)(x-g(y))}{(h-y)g(y)}\left[\frac{h-y}{h}F(0,0)+\frac{y}{h}F(0,h)\right] \\
&- \frac{4x(x-g(y))}{(h-y)[h-y-2g(y)]}\left[\frac{f\left(\frac{h-y}{2}\right)-y}{f\left(\frac{h-y}{2}\right)}F\left(\frac{h-y}{2},0\right)\right. \\
&+ \left.\frac{y}{f\left(\frac{h-y}{2}\right)}F\left(\frac{h-y}{2},f\left(\frac{h-y}{2}\right)\right)\right].
\end{aligned}
$$

For the remainder of the corresonding interpolation formula, $F = S_{21}^L F + R_{21}^S F$, we have

Theorem 1.4.5. *If $F \in B_{11}(0,0)$, then the following inequality holds*

$$
\begin{aligned}
|(R_{21}^S F)(x,y)| &\leq \left\|F^{(0,2)}(0,\cdot)\right\|_\infty \int_0^h |K_{02}(x,y,t)|dt \\
&+ \left\|F^{(1,1)}(\cdot,\cdot)\right\|_\infty \iint_{\tilde{T}_h} |K_{11}(x,y,s,t)|dsdt. \quad (1.17)
\end{aligned}
$$

In [12] we built new interpolation operators on triangle $\tilde{T}_h$ using the product and boolean sum and we determined their interpolation properties and the degree of exactness. We consider the Hermite operator H_2^y given in (1.10):

$$\begin{aligned}(H_2^yF)(x,y) &= \frac{[y-f(x)]^2}{f^2(x)}F(x,0)+\frac{y[2f(x)-y]}{f^2(x)}F(x,f(x))\\ &+ \frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x)),\end{aligned}$$

respectively the Lagrange operaor L_2^x given in (1.13):

$$\begin{aligned}(L_2^xF)(x,y) &= \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}F(0,y)\\ &+ \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]}F\Big(\frac{h-y}{2},y\Big)\\ &+ \frac{x(2x-h+y)}{g(y)[2g(y)-h+y]}F(g(y),y). \qquad (1.18)\end{aligned}$$

Let P be

$$P := H_2^yL_2^x$$

and

$$F = PF + R_1 \qquad (1.19)$$

Theorem 1.4.6. *Let consider $F:\tilde{T}_h\to R$. If there exist $F^{(0,1)}$ on the side Γ_3 then P verifies the interpolation properties:*

$$\begin{aligned}PF &= F,\ on\,\Gamma_2\cup\Gamma_3\\ (PF)^{(0,1)} &= F^{(1,0)},\ on\,\Gamma_3\end{aligned}$$

and $gex(P)=2$.

Theorem 1.4.7. *If $F\in B_{1,2}(0,0)$ then the following inequality holds*

$$\begin{aligned}|(R_1F)(x,y)| &\le \frac{x[y-f(x)]^2(h-2x)(h-x)}{12f^2(x)}\|F^{(3,0)}(\cdot,0)\|_\infty\\ &+ \frac{xy[y-f(x)]^2(2x-h)(x-h)}{f^2(x)(3h-2x)}\|F^{(2,1)}(\cdot,0)\|_\infty\\ &+ \frac{y[y-f(x)]^2}{6}\|F^{(0,3)}(0,\cdot)\|_\infty\\ &+ \frac{xy[f(x)-y]^2}{2f(x)-y}\|F^{(1,2)}(\cdot,\cdot)\|_\infty. \qquad (1.20)\end{aligned}$$

Let S be

$$S := H_2^y\oplus L_2^x$$

and

$$F = SF + R_2F \qquad (1.21)$$

approximation formula generated by S.

Theorem 1.4.8. *Let consider $F : \tilde{T}_h \to \mathbb{R}$ then:*

1. $SF = F$, *on* $\partial \tilde{T}_h$.
2. $gex(S) = 2$.

Theorem 1.4.9. *If $F \in B_{1,2}(0,0)$ then*

$$
\begin{aligned}
(R_2F)(x,y) &= \int_0^h K_{30}(x,y,s)F^{(3,0)}(s,0)ds + \\
&+ \int_0^h K_{21}(x,y,s)F^{(2,1)}(s,0)ds + \\
&+ \int_0^h K_{03}(x,y,t)F^{(0,3)}(0,t)dt + \\
&+ \iint_{\tilde{T}_h} K_{12}(x,y,s,t)F^{(1,2)}(s,t)dsdt,
\end{aligned} \tag{1.22}
$$

with the Peano's kernels

$$
\begin{aligned}
K_{30}(x,y,s) &= \frac{(x-s)_+^2}{2} - \frac{[y-f(x)]^2}{f^2(x)} \cdot \frac{(x-s)_+^2}{2} \\
&- \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]} \cdot \frac{\left(\frac{h-y}{2}-s\right)_+^2}{2} - \frac{x(2x-h-y)}{g(y)[2g(y)-h+y]} \cdot \frac{[g(y)-s]_+^2}{2} \\
&+ \frac{[y-f(x)]^2}{f^2(x)} \cdot \left[-\frac{4x(x-h)}{h^2} \cdot \frac{\left(\frac{h}{2}-s\right)_+^2}{2} + \frac{x(2x-h)}{h^2} \cdot \frac{(h-s)^2}{2} \right] \\
K_{21}(x,y,s) &= y(x-s)_+ - \frac{y[y-f(x)]^2}{f^2(x)}(x-s)_+ \\
&- \frac{4xy[x-g(y)]}{(h-y)[h-y-2g(y)]}\left(\frac{h-y}{2}-s\right)_+ - \frac{xy(2x-h-y)}{g(y)[2g(y)-h+y]}[g(y)-s]_+ \\
&+ \frac{[y-f(x)]^2 y}{f^2(x)}\left[-\frac{4x(x-h)}{h^2}\left(\frac{h}{2}-s\right)_+ + \frac{x(2x-h)}{h^2}(h-s) \right] \\
K_{03}(x,y,t) &= 0 \\
K_{12}(x,y,s,t) &= (y-t)_+ \left[(x-s)_+^0 - \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]}\left(\frac{h-y}{2}-s\right)_+^0 \right. \\
&\left. - \frac{x(2x-h-y)}{g(y)[2g(y)-h-y]}[g(y)-s]_+^0 \right]
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|(R_2F)(x,y)| &\leq \|F^{(3,0)}(\cdot,0)\|_\infty \int_0^h |K_{30}(x,y,s)|ds \\
&+ \|F^{(2,1)}(\cdot,0)\|_\infty \int_0^h |K_{21}(x,y,s)|ds \\
&+ \|F^{(1,2)}(\cdot,\cdot)\|_\infty \iint_{\tilde{T}_h} |K_{12}(x,y,s,t)|dsdt,
\end{aligned} \tag{1.23}
$$

Chapter 2

Optimal quadrature formulas

This chapter is dedicated to the optimal quadrature formulas, making optimal quadrature formulas presented in sense Sard and Nikolski and the corrected formulas corresponding for open and closed type.

In the first section we obtained an optimal quadrature formula of 2 nodes open type and we have shown that there are situations when this formula is a better representation of the remainder term than the well-known formula of Gauss to 2 knots. In the second section we obtained optimal quadrature formula in the Nikolski sense, 2 nodes open type, giving estimates of the remainder term for a variety of rules involving the second derivative. We built then the corrected formulas of these optimal formulas. In the third section we obtained optimal quadrature formula in the Nikolski sense, closed type with 3 nodes, i.e. optimal formulas in the Nikolski general sense, giving also estimates of the remainder term for a variety of rules involving the second derivative, afterwards building the corrected formulas of these optimal formulas. These formulas have higher degree of exactness than the original. We have shown that the error estimates are better than in the corrected formulas then the original ones. These results are contained in the papers [3], [4], [5], [14] and [13].

2.1 Optimal quadrature formulas in the sense Sard and Nikolski

Definition 2.1.1. *It is called a quadrature formula or formula of numerical integration, the following formula*

$$I[f]=\int_{\mathbb{R}} f(x)d\lambda(x)=\sum_{i=0}^{m} A_i\lambda_i[f]+\mathcal{R}_m[f], \tag{2.1}$$

where $\lambda_i[f]$, $i=\overline{0,m}$ *are the punctual (local) information relating of function* f*, which to integrate with respect to the measure* $d\lambda$*,* A_i*,* $i=\overline{0,m}$ *are called the coefficients of quadrature formula, and* $\mathcal{R}_m[f]$ *is the remainder term.*

Definition 2.1.2. *A quadrature formula has degree of exactness equal n, if*

$$\mathcal{R}_m[e_0]=0\,,\ \mathcal{R}_m[e_1]=0\,,\ \cdots\,,\ \mathcal{R}_m[e_n]=0,$$

where $e_j(t) = t^j$. *Moreover, if*

$$\mathcal{R}_m[e_{n+1}] \neq 0$$

then the quadrature formula has degree of exactness effectively equal n.

Lemma 2.1.1. *If* $-\infty < \alpha < \beta < +\infty$ *and* w *is a weight on* (α, β) *and*

$$\int_\alpha^\beta f(t)w(t)dt = \sum_{i=0}^m A_i f(x_i) + r_m[f], \; f \in L_w^1(\alpha, \beta),$$

then

$$W(x) = w\left(\alpha + (\beta - \alpha)\frac{x-a}{b-a}\right), \; x \in (a,b), \; -\infty < a < b < +\infty,$$

is a weight on (a, b) *and*

$$\int_a^b F(x)W(x)dx = \frac{b-a}{\beta-\alpha}\sum_{i=0}^m A_i F\left(a + (b-a)\frac{x_i - \alpha}{\beta - \alpha}\right) + \mathcal{R}_m[F],$$

where $F \in L_w^1(a, b)$ *and*

$$\mathcal{R}_m[F] = \frac{b-a}{\beta-\alpha} r_m[\tilde{F}], \; \tilde{F}(t) = F\left(a + (b-a)\frac{t-\alpha}{\beta-\alpha}\right).$$

Let be the quadrature formula

$$\int_a^b f(x)dx = \sum_{k=0}^m A_{m,k} f(a_k) + R_m[f], \tag{2.2}$$

with the degree of exacness $n-1$, where nodes satisfies the inequality $a \leq a_0 < a_1 < \cdots < a_m \leq b$.

If $f \in H^n[a,b]$, i.e. $f \in C^{n-1}[a,b]$ and $f^{(n-1)}$ is absolutely continuous on the interval $[a,b]$, using Peano's theorem, we have the following integral representation of the remainder term

$$R_m[f] = \int_a^b K_{m,n}(t) f^{(n)}(t)dt, \tag{2.3}$$

where

$$\begin{aligned} K_{m,n}(t) &= R_m\left[\frac{(x-t)_+^{n-1}}{(n-1)!}\right] \\ &= \frac{1}{(n-1)!}\left[\int_a^b (x-t)_+^{n-1}dx - \sum_{k=0}^m A_{m,k}(a_k - t)_+^{n-1}\right], \end{aligned} \tag{2.4}$$

is called *Peano kernel.*

Note

$$H^{n,p}[a,b] := \left\{ f \in C^{n-1}[a,b], f^{(n-1)} \text{ absolutely continuous }, \; \left\| f^{(n)} \right\|_p < \infty \right\}$$

with

$$\|f\|_p := \left\{ \int_a^b |f(x)|^p \, dx \right\}^{\frac{1}{p}}, \quad \text{for} \quad 1 \le p < \infty,$$
$$\|f\|_\infty := \sup_{x \in [a,b]} |f(x)| .$$

We can get the following estimates of the remainder term

$$|R_m[f]| \le M_n^{[\infty]}[f] \int_a^b |K_{m,n}(t)| dt, \quad M_n^{[\infty]}[f] = \sup_{t\in[a,b]} |f^{(n)}(t)|, \tag{2.5}$$

$$[R_m[f]]^2 \le M_n^{[2]}[f] \int_a^b |K_{m,n}(t)|^2 dt, \quad M_n^{[2]}[f] = \int_a^b [f^{(n)}(t)]^2 dt, \tag{2.6}$$

$$|R_m[f]| \le M_n^{[1]}[f] \sup_{t\in[a,b]} |K_{m,n}(t)| dt, \quad M_n^{[1]}[f] = \int_a^b |f^{(n)}(t)| dt, \tag{2.7}$$

respectively when $f \in H^{n,\infty}[a,b], f \in H^{n,2}[a,b], f \in H^{n,1}[a,b]$. Optimality quadrature formula returns to determine that quadrature formula, ie the coefficients, possibly the nodes, placing condition as factor of the remainder term evaluation depending on the Peano kernel to be minimal. When nodes are fixed will reach optimality in the sense of Sard, and otherwise to optimality in the sense of Nikolski.

Consider that in formula (2.2) integrated function f is continuous, with continuous derivatives up to order $n-1$, and the derivative of order n is square integrable, ie $f \in H^{n,2}[a,b]$. The integral representation of remainder term given by Peano's theorem occurs, as well as evaluating (2.6). Suppose also that the nodes of quadrature formula are known.

Definition 2.1.3. *Quadrature formula (2.2) is optimal in the sense of Sard if*

$$\int_a^b [K_{m,n}(t)]^2 dt \to minimum.$$

Consider that in formula (2.2) ntegrated function f is continuous, with continuous derivatives up to order $n-1$, and the derivative of order n is integrable to power $p \ge 1$, ie $f \in H^{n,p}[a,b]$. If the degree of exactness of quadrature formula is $n-1$, the integral representation of the remainder term is given by (2.3) and (2.4) of Peano's theorem. Moreover, for the remainder term we have the evaluation

$$|R_m[f]| \le \left[M_n^{[p]}[f] \right]^{\frac{1}{p}} \left[\int_a^b |K_{m,n}(t)|^q dt \right]^{\frac{1}{q}}, \tag{2.8}$$

where

$$M_n^{[p]}[f] = \int_a^b |f^{(n)}(t)|^p dt, \frac{1}{p} + \frac{1}{q} = 1,$$

with observation that in cases $p = 1$ and $p = \infty$ this evaluation becomes respectively

$$|R_m[f]| \le M_n^{[1]}[f] \sup_{t\in[a,b]} |K_{m,n}(t)|, \tag{2.9}$$

$$|R_m[f]| \le M_n^{[\infty]}[f] \int_a^b |K_{m,n}(t)| dt, \tag{2.10}$$

where

$$M_n^{[1]}[f] = \int_a^b |f^n(t)|dt, M_n^{[\infty]}[f] = \sup_{t\in[a,b]} |f^{(n)}(t)|$$

We assume that nodes of the quadrature formula, as the coefficients are unknown parameters.

Definition 2.1.4. *Quadrature formula (2.2) is optimal in the sense of Nikolski in $H^{n,p}[a,b]$ if*

$$\int_a^b [K_{m,n}(t)]^q dt \to minimum, \ \frac{1}{p} + \frac{1}{q} = 1.$$

Problem of constructing optimal quadrature formulas has been studied by many authors. The first result was obtained by A. Sard, L. S. Meyers and S. M. Nikolski. In recent years a number of authors have obtained optimal quadrature formulas in many different ways ([41], [50], [51], [91], [126], [127]).

In [125], N. Ujević obtained the following quadrature formula

$$\int_{-1}^1 f(t)dt = f(x) + f(y) + R[f], \ f \in H^{2,2}[-1,1] \tag{2.11}$$

where

$$R[f] = \int_{-1}^1 K(x,y,t) f''(t)dt, \tag{2.12}$$

$$K(x,y,t) = \begin{cases} \frac{1}{2}(t+1)^2, \ t\in[-1,x], \\ \frac{1}{2}t^2 + x + \frac{1}{2}, \ t \in (x,y) \\ \frac{1}{2}(t-1)^2, \ t\in[x,1]. \end{cases} \tag{2.13}$$

Using the relation $|R[f]| \leq ||K(x,y,.)||_2 \, ||f''||_2$ and putting the condition that $||K(x,y,.)||_2^2 \to min$, to obtain $x = \sqrt{6} - 3$, respectively the following quadrature formula:

$$\int_{-1}^1 f(t)dt = f(\sqrt{6}-3) + f(-\sqrt{6}+3) + R[f], \tag{2.14}$$

$$|R[f]| \leq \sqrt{\frac{98}{5} - 8\sqrt{6}} \, ||f''||_2 \, . \tag{2.15}$$

N. Ujević compare this result with the 2-point Gauss formula, namely

$$\int_{-1}^1 f(t)dt = f\Big(-\frac{\sqrt{3}}{3}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big) + R_1[f], \tag{2.16}$$

$$|R_1[f]| \leq \sqrt{-\frac{34}{135} + \frac{4}{27}\sqrt{3}} \, ||f''||_2 \, , \tag{2.17}$$

and show that the estimate (2.15) is better than the estimate (2.17).

In [3] we obtained a optimal 2-point quadrature formula of open type.

Theorem 2.1.1. *If $f \in H^{3,2}[-1,1]$, then we have*

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) + R_2[f], \tag{2.18}$$

$$|R_2[f]| \leq \sqrt{-\frac{4}{405}\sqrt{3} + \frac{148}{8505}}\,||f^{(3)}||_2\,. \tag{2.19}$$

Using Lemma 2.1.1 we obtain the quadrature formula on the interval $[a, b]$:

Theorem 2.1.2. *If $f \in H^{3,2}[a,b]$, then*

$$\int_{a}^{b} f(t)dt = \frac{b-a}{2}[f(x_1) + f(x_2)] + \tilde{R}_2[f], \tag{2.20}$$

where

$$x_1 = \frac{a+b}{2} - \frac{\sqrt{3}}{3}.\frac{b-a}{2},\; x_2 = \frac{a+b}{2} + \frac{\sqrt{3}}{3}.\frac{b-a}{2}$$

$$\left|\tilde{R}_2[f]\right| \leq \frac{(b-a)^{7/2}}{4\sqrt{2}}\sqrt{-\frac{1}{405}\sqrt{3} + \frac{148}{8505}}\,||f^{(3)}||_2\,. \tag{2.21}$$

Now, we shoe that there are the situations when the estimates (2.19) is better then the estimates (2.15) and (2.17).

Example 2.1.1. *If $f(x) = e^x + x$, $x \in [-1,1]$, then*

$$|R[f]| \leq \sqrt{\frac{98}{5} - 8\sqrt{6}}.\frac{\sqrt{2}}{2}.\sqrt{\frac{e^4-1}{e^2}} \cong 0,1208 \tag{2.22}$$

$$|R_1[f]| \leq \sqrt{-\frac{34}{135} + \frac{4}{27}\sqrt{3}}.\frac{\sqrt{2}}{2}.\sqrt{\frac{e^4-1}{e^2}} \cong 0,1303 \tag{2.23}$$

$$|R_2[f]| \leq \sqrt{-\frac{4}{405}\sqrt{3} + \frac{148}{8505}}.\frac{\sqrt{2}}{2}.\sqrt{\frac{e^4-1}{e^2}} \cong 0,0325 \tag{2.24}$$

Example 2.1.2. *If $f(x) = \dfrac{1}{x^2+4}$, $x \in [-1,1]$, then*

$$|R[f]| \leq \sqrt{\frac{98}{5} - 8\sqrt{6}}.\frac{1}{2000}.\sqrt{39246 + 46875arctg\left(\frac{1}{2}\right)} \cong 0,0079 \tag{2.25}$$

$$|R_1[f]| \leq \sqrt{-\frac{34}{135} + \frac{4}{27}\sqrt{3}}.\frac{1}{2000}.\sqrt{39246 + 46875arctg\left(\frac{1}{2}\right)} \cong 0,0085 \tag{2.26}$$

(2.27)

$$|R_2[f]| \leq \sqrt{-\frac{4}{405}\sqrt{3} + \frac{148}{8505}}.\frac{1}{280000}.\sqrt{1329179460 + 1722656250arctg\left(\frac{1}{2}\right)} \cong 0,0028$$

Example 2.1.3. *If* $f(x) = \frac{x^3}{e^x}$, $x \in [-1, 1]$, *then*

$$|R[f]| \leq \sqrt{\frac{98}{5} - 8\sqrt{6}}.\frac{1}{4}.\sqrt{-62e^{-2} + 470e^2} \cong 0,9338 \tag{2.28}$$

$$|R_1[f]| \leq \sqrt{-\frac{34}{135} + \frac{4}{27}\sqrt{3}}.\frac{1}{4}.\sqrt{-62e^{-2} + 470e^2} \cong 1,0071 \tag{2.29}$$

$$|R_2[f]| \leq \sqrt{-\frac{4}{405} + \frac{148}{8505}}.\frac{1}{4}.\sqrt{-134e^{-2} + 3998e^2} \cong 0,7326. \tag{2.30}$$

2.2 Corrected quadrature formulas of open type

In recent years some authors have considered so called perturbed (corrected) quadrature rules (see [43], [44], [45], [71], [80], [130]). By corrected quadrature rule we mean the formula which involves the values of the first derivative in end points of the interval not only the values of the function in certain points. These formulae have a higher degree of exactness than the original rule. The estimates of the error in corrected rule are better then in the original rule, in generally.

In [14] we derived a 2-points quadrature formula which is optimal in sense Nikolski.

Let

$$\int_0^1 f(x)dx = A_1 f(a_1) + A_2 f(a_2) + \mathcal{R}_2^{[p]}[f], \tag{2.31}$$

be a quadrature formula with degree of exactness equal 1. We will calculate the coefficients and the nodes such that the quadrature formula to be optimal, considering that the remainder term is evaluate in sense of (2.8) in the cases $p = 1, p = 2$ and $p = \infty$.

Since the quadrature formula has degree of exactness 1, the remainder term verifies the conditions $\mathcal{R}_2^{[p]}[e_i] = 0,\ e_i(x) = x^i,\ i = 0, 1$, namely

$$\begin{cases} A_1 + A_2 = 1, \\ A_1 a_1 + A_2 a_2 = \frac{1}{2}, \end{cases} \tag{2.32}$$

and using Peano's theorem the remainder term has the following integral representation

$$\mathcal{R}_2^{[p]}[f] = \int_0^1 K_2(t) f''(t) dt, \text{ where} \tag{2.33}$$

$$K_2(t) = \mathcal{R}_2^{[p]}[(x - t)_+] = \begin{cases} \frac{1}{2}t^2, & 0 \leq t < a_1, \\ \frac{(1-t)^2}{2} + A_2 t - a_2 A_2, & a_1 \leq t \leq a_2, \\ \frac{1}{2}(1-t)^2, & a_2 < t \leq 1. \end{cases} \tag{2.34}$$

Theorem 2.2.1. *For $f \in H^{2,\infty}[0,1]$, the quadrature formula of the form (2.31), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{1}{2}\left[f\left(\frac{2\sqrt{3}-3}{2}\right) + f\left(\frac{5-2\sqrt{3}}{2}\right)\right] + \mathcal{R}_2^{[\infty]}[f], \tag{2.35}$$

with

$$\mathcal{R}_2^{[\infty]}[f] = \int_0^1 K_2(t)f''(t)dt, \tag{2.36}$$

where

$$K_2(t) = \begin{cases} \frac{1}{2}t^2, & 0 \le t < \frac{2\sqrt{3}-3}{2}, \\ \frac{(1-t)^2}{2} + \frac{1}{2}t - \frac{5-2\sqrt{3}}{4}, & \frac{2\sqrt{3}-3}{2} \le t \le \frac{5-2\sqrt{3}}{2}, \\ \frac{1}{2}(1-t)^2, & \frac{5-2\sqrt{3}}{2} < t \le 1. \end{cases}$$

Remark 2.2.1. *For the remainder term of quadrature formula (2.35) can be established the following estimations*

$$\left|\mathcal{R}_2^{[\infty]}[f]\right| \le \|f''\|_\infty \int_0^1 |K_2(t)|dt = \frac{7-4\sqrt{3}}{8}\|f''\|_\infty \approx 0.0089\|f''\|_\infty,\ f \in H^{2,\infty}[0,1],$$

$$\left|\mathcal{R}_2^{[\infty]}[f]\right| \le \left[\int_0^1 (K_2(t))^2\, dt\right]^{\frac{1}{2}} \|f''\|_2 = \frac{\sqrt{2400\sqrt{3}-4155}}{120}\|f''\|_2 \approx 0.0156\|f''\|_2,\ f \in H^{2,2}[0,1],$$

$$\left|\mathcal{R}_2^{[\infty]}[f]\right| \le \sup_{t\in[0,1]} |K_2(t)| \cdot \|f''\|_1 = \frac{3(7-4\sqrt{3})}{8}\|f''\|_1 \approx 0.0269\|f''\|_1,\ f \in H^{2,1}[0,1].$$

Theorem 2.2.2. *For $f \in H^{2,2}[0,1]$, the quadrature formula of the form (2.31), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{1}{2}\left[f\left(\frac{\sqrt{6}-2}{2}\right) + f\left(\frac{4-\sqrt{6}}{2}\right)\right] + \mathcal{R}_2^{[2]}[f], \tag{2.37}$$

with

$$\mathcal{R}_2^{[2]}[f] = \int_0^1 K_2(t)f''(t)dt, \tag{2.38}$$

where

$$K_2(t) = \begin{cases} \frac{1}{2}t^2, & 0 \le t < \frac{\sqrt{6}-2}{2}, \\ \frac{(1-t)^2}{2} + \frac{1}{2}t - \frac{4-\sqrt{6}}{4}, & \frac{\sqrt{6}-2}{2} \le t \le \frac{4-\sqrt{6}}{2}, \\ \frac{1}{2}(1-t)^2, & \frac{4-\sqrt{6}}{2} < t \le 1. \end{cases}$$

Remark 2.2.2. *For the remainder term of quadrature formula (2.37) can be established the following estimations*

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \left[\int_0^1 (K_2(t))^2\, dt\right]^{\frac{1}{2}} \|f''\|_2 = \frac{(5-2\sqrt{6})\sqrt{5}}{20}\|f''\|_2 \approx 0.0113\|f''\|_2,\ f \in H^{2,2}[0,1],$$

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \int_0^1 |K_2(t)|dt\|f''\|_\infty = \frac{(1+\sqrt{2})(9\sqrt{6}-22)}{12}\|f''\|_\infty \approx 0.0091\|f''\|_\infty,\ f \in H^{2,\infty}[0,1],$$

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \sup_{t\in[0,1]} |K_2(t)| \cdot \|f''\|_1 = \frac{5-2\sqrt{6}}{4}\|f''\|_1 \approx 0.0252\|f''\|_1,\ f \in H^{2,1}[0,1].$$

Theorem 2.2.3. *For $f \in H^{2,1}[0,1]$, the quadrature formula of the form (2.31), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{1}{2}\left[f\left(\frac{\sqrt{2}-1}{2}\right) + f\left(\frac{3-\sqrt{2}}{2}\right)\right] + \mathcal{R}_2^{[1]}[f], \tag{2.39}$$

with

$$\mathcal{R}_2^{[1]}[f] = \int_0^1 K_2(t) f''(t)dt, \tag{2.40}$$

where

$$K_2(t) = \begin{cases} \frac{1}{2}t^2,\ 0 \leq t < \frac{\sqrt{2}-1}{2}, \\ \frac{(1-t)^2}{2} + \frac{1}{2}t - \frac{3-\sqrt{2}}{4},\ \frac{\sqrt{2}-1}{2} \leq t \leq \frac{3-\sqrt{2}}{2}, \\ \frac{1}{2}(1-t)^2,\ \frac{3-\sqrt{2}}{2} < t \leq 1. \end{cases}$$

Remark 2.2.3. *For the remainder term of quadrature formula (2.39) can be established the following estimations*

$$\left|\mathcal{R}_2^{[1]}[f]\right| \leq \sup_{t\in[0,1]} |K_2(t)| \cdot \|f''\|_1 = \frac{3-2\sqrt{2}}{8}\|f''\|_1 \approx 0.0214\|f''\|_1,\ f \in H^{2,1}[0,1].$$

$$\left|\mathcal{R}_2^{[1]}[f]\right| \leq \int_0^1 |K_2(t)|dt \cdot \|f''\|_\infty = \frac{32\sqrt{2}-45}{24}\|f''\|_\infty \approx 0.0106\|f''\|_\infty,\ f \in H^{2,\infty}[0,1],$$

$$\left|\mathcal{R}_2^{[1]}[f]\right| \leq \left[\int_0^1 (K_2(t))^2 dt\right]^{\frac{1}{2}} \|f''\|_2 = \frac{\sqrt{4245-3000\sqrt{2}}}{120}\|f''\|_2 \approx 0.0128\|f''\|_2,\ f \in H^{2,2}[0,1].$$

Then we constructed the corrected quadrature formulas of the optimal quadrature formulas in sense Nikolski and we will show that the estimations of the error in terms of variety on norms are better in the corrected formula than in the original formula. Let

$$\int_0^1 f(x)dx = A_1 f(a_1) + A_2 f(a_2) + A\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2^{[p]}[f], \tag{2.41}$$

where

$$\tilde{\mathcal{R}}_2^{[p]}[e_i]=0,\ i=0,1,\ \text{şi}\ A=\int_0^1 K_2(t)dt$$

be the corrected quadrature formula of the rule (2.31).

Since the remainder term has degree of exactness 1 we can write

$$\tilde{\mathcal{R}}_2^{[p]}[f]=\int_0^1 \tilde{K}_2(t)f''(t)dt,\ \text{where} \tag{2.42}$$

$$\tilde{K}_2(t)=\tilde{\mathcal{R}}_2^{[p]}\left[(x-t)_+\right]=K_2(t)-A. \tag{2.43}$$

From the relation (2.43) we remark that $\int_0^1 \tilde{K}_2(t)dt=0$. If we consider $f(x)=\frac{x^2}{2}$ in the optimal quadrature (2.31), where $A_1=A_2=\frac{1}{2}$, we find

$$A=\frac{1}{2}a_1a_2-\frac{1}{12}. \tag{2.44}$$

Using relations (2.43) and (2.44) we construct the following corrected quadrature formula of (2.35), (2.37), respectively (2.39):

$$\int_0^1 f(x)dx=\frac{1}{2}\left[f\left(\frac{2\sqrt{3}-3}{2}\right)+f\left(\frac{5-2\sqrt{3}}{2}\right)\right]+\frac{48\sqrt{3}-83}{24}\left[f'(1)-f'(0)\right]+\tilde{\mathcal{R}}_2^{[\infty]}[f], \tag{2.45}$$

where

$$\tilde{\mathcal{R}}_2^{[\infty]}[f]=\int_0^1 \tilde{K}_2(t)f''(t)dt, \tag{2.46}$$

$$\tilde{K}_2(t)=\begin{cases}\frac{1}{2}t^2-\frac{48\sqrt{3}-83}{24}, & 0\le t<\frac{2\sqrt{3}-3}{2},\\ \frac{1}{2}t^2-\frac{1}{2}t+\frac{65-36\sqrt{3}}{24}, & \frac{2\sqrt{3}-3}{2}\le t\le\frac{5-2\sqrt{3}}{2},\\ \frac{1}{2}(1-t)^2-\frac{48\sqrt{3}-83}{24}, & \frac{5-2\sqrt{3}}{2}<t\le 1.\end{cases}$$

$$\int_0^1 f(x)dx=\frac{1}{2}\Big[f\Big(\frac{\sqrt{6}-2}{2}\Big)+f\Big(\frac{4-\sqrt{6}}{2}\Big)\Big]+\frac{9\sqrt{6}-22}{12}\left[f'(1)-f'(0)\right]+\tilde{\mathcal{R}}_2^{[2]}[f], \tag{2.47}$$

where

$$\tilde{\mathcal{R}}_2^{[2]}[f]=\int_0^1 \tilde{K}_2(t)f''(t)dt, \tag{2.48}$$

$$\tilde{K}_2(t)=\begin{cases}\frac{1}{2}t^2-\frac{9\sqrt{6}-22}{12}, & 0\le t<\frac{\sqrt{6}-2}{2},\\ \frac{1}{2}t^2-\frac{1}{2}t+\frac{8-3\sqrt{3}}{6}, & \frac{\sqrt{6}-2}{2}\le t\le\frac{4-\sqrt{6}}{2},\\ \frac{1}{2}(1-t)^2-\frac{9\sqrt{6}-22}{12}, & \frac{4-\sqrt{6}}{2}<t\le 1.\end{cases}$$

respectively

$$(2.49) \qquad \int_0^1 f(x)dx=\frac{1}{2}\Big[f\Big(\frac{\sqrt{2}-1}{2}\Big)+f\Big(\frac{3-\sqrt{2}}{2}\Big)\Big]+\frac{12\sqrt{2}-17}{24}\left[f'(1)-f'(0)\right]+\tilde{\mathcal{R}}_2^{[1]}[f],$$

where

$$(2.50) \qquad \tilde{\mathcal{R}}_2^{[1]}[f]=\int_0^1 \tilde{K}_2(t)f''(t)dt,$$

$$\tilde{K}_2(t)=\begin{cases} \frac{1}{2}t^2-\frac{12\sqrt{2}-17}{24}, & 0\le t<\frac{\sqrt{2}-1}{2},\\ \frac{1}{2}t^2-\frac{1}{2}t+\frac{11-6\sqrt{2}}{24}, & \frac{\sqrt{2}-1}{2}\le t\le\frac{3-\sqrt{2}}{2},\\ \frac{1}{2}(1-t)^2-\frac{12\sqrt{2}-17}{24}, & \frac{3-\sqrt{2}}{2}<t\le 1.\end{cases}$$

Remark 2.2.4. *For the remainder term of quadrature formula (2.45) can be established the following estimations*

$$\left|\tilde{\mathcal{R}}_2^{[\infty]}[f]\right|\le\|f''\|_\infty\int_0^1|K_2(t)|dt=\frac{1}{54}\Big[-62\sqrt{108\sqrt{3}-186}+108\sqrt{36\sqrt{3}-62}$$

$$+144\sqrt{48\sqrt{3}-83}-83\sqrt{144\sqrt{3}-249}\Big]\|f''\|_\infty\approx 0.0088\|f''\|_\infty,\ f\in H^{2,\infty}[0,1],$$

$$\left|\tilde{\mathcal{R}}_2^{[\infty]}[f]\right|\le\left[\int_0^1(K_2(t))^2\,dt\right]^{\frac{1}{2}}\|f''\|_2=\frac{\sqrt{50400\sqrt{3}-87295}}{60}\|f''\|_2\approx 0.006\|f''\|_2,\ f\in H^{2,2}[0,1],$$

$$\left|\tilde{\mathcal{R}}_2^{[\infty]}[f]\right|\le\sup_{t\in[0,1]}|K_2(t)|\cdot\|f''\|_1=\frac{73-42\sqrt{3}}{12}\|f''\|_1\approx 0.0211\|f''\|_1,\ f\in H^{2,1}[0,1].$$

Remark 2.2.5. *For the remainder term of quadrature formula (2.47) can be established the following estimations*

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right|\le\left[\int_0^1(K_2(t))^2\,dt\right]^{\frac{1}{2}}\|f''\|_2=\frac{\sqrt{9000\sqrt{6}-22045}}{60}\|f''\|_2\approx 0.0106\|f''\|_2,\ f\in H^{2,2}[0,1],$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right|\le\int_0^1|K_2(t)|dt\|f''\|_\infty=\frac{1}{54}\Big[-29\sqrt{36\sqrt{6}-87}+36\sqrt{24\sqrt{6}-58}$$

$$+108\sqrt{9\sqrt{6}-22}-44\sqrt{54\sqrt{6}-132}\Big]\|f''\|_\infty\approx 0.0084\|f''\|_\infty,\ f\in H^{2,\infty}[0,1],$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right|\le\sup_{t\in[0,1]}|K_2(t)|\cdot\|f''\|_1=\frac{37-15\sqrt{6}}{12}\|f''\|_1\approx 0.0214\|f''\|_1,\ f\in H^{2,1}[0,1].$$

Remark 2.2.6. *For the remainder term of quadrature formula (2.49) can be established the following estimations*

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \leq \sup_{t\in[0,1]} |K_2(t)| \cdot \|f''\|_1 = \frac{13-9\sqrt{2}}{12}\|f''\|_1 \approx 0.0226\|f''\|_1,\ f \in H^{2,1}[0,1].$$

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \leq \int_0^1 |K_2(t)|dt \cdot \|f''\|_\infty = \frac{2(3-2\sqrt{2})\sqrt{9\sqrt{2}-12}}{27}\|f''\|_\infty \approx 0.0108\|f''\|_\infty,\ f \in H^{2,\infty}[0,1],$$

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \leq \left[\int_0^1 (K_2(t))^2 dt\right]^{\frac{1}{2}} \|f''\|_2 = \frac{\sqrt{1800\sqrt{2}-2545}}{60}\|f''\|_2 \approx 0.0097\|f''\|_2,\ f \in H^{2,2}[0,1].$$

Remark 2.2.7. *The estimates of the error in the corrected rules (2.45), respectively (2.47) are better then in the original rules (2.35), respectively (2.37).*

The corrected quadrature formulas (2.45), (2.47) and (2.49), respectively have degree of exactness 3, which is higher than the original rule, namely for $p \in \{\infty, 2, 1\}$, $\tilde{R}_2^{[p]}[e_i] = 0$ and $\tilde{R}_2^{[p]}[e_4] \neq 0$, where $e_i(x) = x^i,\ i = \overline{0,4}$. Using Peano's Theorem, the remainder term can be written

$$\tilde{\mathcal{R}}_2^{[p]}[f] = \int_0^1 \overline{K}_2(t) f^{(4)}(t),\ \ \overline{K}_2(t) = \tilde{\mathcal{R}}_2^{[p]}\left[\frac{(x-t)_+^3}{3!}\right]. \tag{2.51}$$

Next using relation (2.51), we will give new estimations of the remainder term in quadrature formulas (2.45), (2.47), and (2.49), respectively.

Theorem 2.2.4. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (2.45) has the integral representation*

$$\tilde{\mathcal{R}}_2^{[\infty]}[f] = \int_0^1 \overline{K}_2(t) f^{(4)}(t)dt,\ \ \textit{where}$$

$$\overline{K}_2^{[\infty]}(t) = \begin{cases} \frac{1}{24}t^2\left(t^2 - \frac{48\sqrt{3}-83}{2}\right), 0 \leq t \leq \frac{2\sqrt{3}-3}{2}, \\ \frac{1}{24}(1-t)^4 - \frac{1}{12}\left(\frac{5-2\sqrt{3}}{2} - t\right)^3 - \frac{48\sqrt{3}-83}{48}(1-t)^2,\ \ \frac{2\sqrt{3}-3}{2} \leq t \leq \frac{5-2\sqrt{3}}{2}, \\ \frac{1}{24}(1-t)^2\left[(1-t)^2 - \frac{48\sqrt{3}-83}{2}\right],\ \ \frac{5-2\sqrt{3}}{2} < t \leq 1. \end{cases}$$

and the following estimations holds

$$
\begin{aligned}
\left|\tilde{\mathcal{R}}_2^{[\infty]}[f]\right| &\leq \sqrt{\int_0^1 \left(\overline{K}_2(t)\right)^2 dt}\sqrt{\int_0^1 [f^{(4)}(t)]^2\, dt} \\
&= \frac{\sqrt{2166615360\sqrt{3}-3752687855}}{40320}\|f^{(4)}\|_2 \approx 1.335\times 10^{-4}\|f^{(4)}\|_2, \\
\left|\tilde{\mathcal{R}}_2^{[\infty]}[f]\right| &\leq \int_0^1 \left|\overline{K}_2(t)\right| dt\cdot \sup_{t\in[0,1]} |f^{(4)}(t)| \\
&= -\frac{9}{4}\sqrt{3}+\frac{22447}{5760}+\sqrt{-62-\sqrt{8397-4848\sqrt{3}}+36\sqrt{3}} \\
&\times \left(-\frac{9727}{720}+\frac{39}{5}\sqrt{3}+\sqrt{8397-4848\sqrt{3}}\left(\frac{\sqrt{3}}{20}-\frac{31}{360}\right)\right)\cdot\|f^{(4)}\|_\infty \\
&\approx 0.938\times 10^{-4}\cdot\|f^{(4)}\|_\infty, \\
\left|\tilde{\mathcal{R}}_2^{[\infty]}[f]\right| &\leq \sup_{t\in[0,1]} |\overline{K}_2(t)|\cdot\int_0^1 |f^{(4)}(t)|dt \\
&= \frac{384\sqrt{3}-665}{384}\cdot\|f^{(4)}\|_1 \approx 2.7997\times 10^{-4}\cdot\|f^{(4)}\|_1.
\end{aligned}
$$

Theorem 2.2.5. *If $f\in C^4[0,1]$, then the remainder term of quadrature formula (2.47) has the integral representation*

$$
\begin{aligned}
\tilde{\mathcal{R}}_2^{[2]}[f] &= \int_0^1 \overline{K}_2(t) f^{(4)}(t)dt, \ \textit{where} \\
\overline{K}_2(t) &= \begin{cases}
\frac{1}{24}t^2[t^2-(9\sqrt{6}-22)],\ 0\leq t\leq\frac{\sqrt{6}-2}{2}, \\
\frac{1}{24}(1-t)^4-\frac{1}{12}\left(\frac{4-\sqrt{6}}{2}-t\right)^3-\frac{9\sqrt{6}-22}{24}(1-t)^2,\ \ \frac{\sqrt{6}-2}{2}\leq t\leq\frac{4-\sqrt{6}}{2}, \\
\frac{1}{24}(1-t)^2[(1-t)^2-(9\sqrt{6}-22)],\ \frac{4-\sqrt{6}}{2}<t\leq 1.
\end{cases}
\end{aligned}
$$

and the following estimations holds

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| &\le \sqrt{\int_0^1 \left(\overline{K}_2(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\ &= \frac{\sqrt{27305005-11147220\sqrt{6}}}{10080}\|f^{(4)}\|_2 \approx 1.972\times 10^{-4}\|f^{(4)}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| &\le \int_0^1 \left|\overline{K}_2(t)\right| dt\cdot \sup_{t\in[0,1]}|f^{(4)}(t)| \\ &= \frac{64(485-198\sqrt{6})\sqrt{9\sqrt{6}-22}+630\sqrt{6}-1543}{1440}\cdot\|f^{(4)}\|_\infty \approx 1.337\times 10^{-4}\cdot\|f^{(4)}\|_\infty,\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| &\le \sup_{t\in[0,1]}|\overline{K}_2^{[2]}(t)|\cdot\int_0^1 |f^{(4)}(t)|dt \\ &= \frac{96\sqrt{6}-235}{384}\cdot\|f^{(4)}\|_1 \approx 3.993\times 10^{-4}\cdot\|f^{(4)}\|_1.\end{aligned}$$

Theorem 2.2.6. *If $f\in C^4[0,1]$, then the remainder term of quadrature formula (2.49) has the integral representation*

$$\tilde{\mathcal{R}}_2^{[1]}[f] = \int_0^1 \overline{K}_2(t) f^{(4)}(t)dt, \ \textit{where}$$

$$\overline{K}_2(t)=\begin{cases}\frac{1}{24}t^2\Big(t^2-\frac{12\sqrt{2}-17}{2}\Big), 0\le t\le\frac{\sqrt{2}-1}{2}, \\ \frac{1}{24}(1-t)^4-\frac{1}{12}\Big(\frac{3-\sqrt{2}}{2}-t\Big)^3-\frac{12\sqrt{2}-17}{48}(1-t)^2, \ \frac{\sqrt{2}-1}{2}\le t\le\frac{3-\sqrt{2}}{2}, \\ \frac{1}{24}(1-t)^2\Big[(1-t)^2-\frac{12\sqrt{2}-17}{2}\Big], \ \frac{3-\sqrt{2}}{2}<t\le 1,\end{cases}$$

and the following estimations holds

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| &\le \sqrt{\int_0^1 \left(\overline{K}_2(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\ &= \frac{\sqrt{30974545-21902160\sqrt{2}}}{40320}\|f^{(4)}\|_2 \approx 3.622\times 10^{-4}\|f^{(4)}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| &\le \int_0^1 \left|\overline{K}_2(t)\right| dt\cdot \sup_{t\in[0,1]}|f^{(4)}(t)| \\ &= \frac{600\sqrt{2}-847}{5760}\cdot\|f^{(4)}\|_\infty \approx 2.653\times 10^{-4}\cdot\|f^{(4)}\|_\infty,\end{aligned}$$

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| &\le \sup_{t\in[0,1]}|\overline{K}_2(t)|\cdot\int_0^1 |f^{(4)}(t)|dt \\ &= \left(-\frac{15}{128}+\frac{\sqrt{2}}{12}\right)\cdot\|f^{(4)}\|_1 \approx 6.636\times 10^{-4}\cdot\|f^{(4)}\|_1.\end{aligned}$$

2.3 Corrected quadrature formulas of closed type

In [129], N. Ujević and L. Mijić constructed a class of quadrature formulas of close type with 3 nodes. Let

$$K_2(\alpha,\beta,\gamma,\delta;t)=\begin{cases}\frac{1}{2}(t-\alpha)(t-\beta), & t\in\left[a,\frac{a+b}{2}\right],\\ \\ \frac{1}{2}(t-\gamma)(t-\delta), & t\in\left(\frac{a+b}{2},b\right],\end{cases}$$

be a function which depends on the parameters $\alpha,\beta,\gamma,\delta\in\mathbb{R}$.

Integrating by parts the integral $\int_a^b K_2(\alpha,\beta,\gamma,\delta;t)f''(t)dt$, and putting conditions that the coefficients of the first derivatives to be zero, N. Ujević and L. Mijić where constructed the following class of quadrature formulas of close type

$$\int_a^b f(t)dt=A_0(\alpha,\beta,\gamma,\delta)f(a)+A_1(\alpha,\beta,\gamma,\delta)f\left(\frac{a+b}{2}\right)+A_2(\alpha,\beta,\gamma,\delta)f(b)+\mathcal{R}[f],$$

where

$$\mathcal{R}[f]=\int_a^b K_2(\alpha,\beta,\gamma,\delta)f''(t)dt.$$

The parameters $\alpha,\beta,\gamma,\delta$ are obtained putting conditions that the remainder term which is evaluate in sense of (2.10) to be minimal, namely $\int_a^b |K_2(\alpha,\beta,\gamma,\delta)|\,dt$ to attains the minimum value.

The main result obtained by N. Ujević and L. Mijić, in the above described procedure is formulated bellow.

Theorem 2.3.1. *Let $I\subset\mathbb{R}$ be an open interval such that $[0,1]\subset I$ and let $f:I\to\mathbb{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then we have*

$$\left|\int_0^1 f(t)dt-\frac{\sqrt{2}}{8}f(0)-\left(1-\frac{\sqrt{2}}{4}\right)f\left(\frac{1}{2}\right)-\frac{\sqrt{2}}{8}f(1)\right|\le\frac{2-\sqrt{2}}{48}\|f''\|_\infty. \tag{2.52}$$

Motivated by this result, in paper [4] we derived a quadrature formula of close type with 3-points which is optimal in sense Nikolski, namely we calculate the coefficients A_i, $i=\overline{0,2}$ and the node $a_1\in(a,b)$ such that the quadrature formula

$$\int_a^b f(t)dt=A_0f(a)+A_1f(a_1)+A_2f(b)+\mathcal{R}_2[f],$$

to be optimal, considering that the remainder term is evaluate in sense of (2.8) in the cases $p=1$, $p=2$ şi $p=\infty$.

For the simplicity we choose $[a,b]=[0,1]$. Switching to range $[a,b]$ can be done using Lemma 2.1.1.

Let

$$\int_0^1 f(x)dx=A_0f(0)+A_1f(a_1)+A_2f(1)+\mathcal{R}_2[f] \tag{2.53}$$

be a quadrature formula with degree of exactness equal 1.

Since the quadrature formula has degree of exactness 1, the remainder term verifies the conditions $\mathcal{R}_2[e_i] = 0,\ e_i(x) = x^i,\ i = 0, 1$, namely

$$\begin{cases} A_0 + A_1 + A_2 = 1 \\ A_1 a_1 + A_2 = \dfrac{1}{2} \end{cases} \tag{2.54}$$

and using Peano's theorem the remainder term has the following integral representation

$$\mathcal{R}_2[f] = \int_0^1 K_2(t) f''(t)dt, \text{ where} \tag{2.55}$$

$$K_2(t) = \mathcal{R}_2\left[(x-t)_+\right] = \begin{cases} \dfrac{1}{2}t^2 - A_0 t, & 0 \le t \le a_1, \\ \dfrac{1}{2}(1-t)^2 - A_2(1-t), & a_1 < t \le 1. \end{cases} \tag{2.56}$$

Theorem 2.3.2. *For $f \in H^{2,\infty}[0,1]$, the quadrature formula of the form (2.53), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{\sqrt{2}}{8} f(0) + \frac{4-\sqrt{2}}{4} f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8} f(1) + \mathcal{R}_2^{[1]}[f], \tag{2.57}$$

with

$$\mathcal{R}_2^{[1]}[f] = \int_0^1 K_2^{[1]}(t) f''(t)dt,\ K_2^{[1]}(t) = \begin{cases} \dfrac{1}{2}t^2 - \dfrac{\sqrt{2}}{8}t, & 0 \le t \le \dfrac{1}{2}, \\ \dfrac{1}{2}(1-t)^2 - \dfrac{\sqrt{2}}{8}(1-t), & \dfrac{1}{2} < t \le 1, \end{cases} \tag{2.58}$$

$$\left|\mathcal{R}_2^{[1]}[f]\right| \le \frac{2-\sqrt{2}}{48}\|f''\|_\infty \approx 0.0122\|f''\|_\infty.$$

Remark 2.3.1. *The optimal quadrature (2.57) coincides with the quadrature formula (2.52) obtained by N. Ujević and L. Mijić in [129], but this quadrature formula was obtained in different way than in [129]. This result motivate us to seek the quadrature formulas of type (2.53) such that the estimation of its error to be best possible in p-norm for $p = 2$ and $p = 1$.*

Remark 2.3.2. *For the remainder term of quadrature formula (2.57) can be established the following two estimations*

$$\left|\mathcal{R}_2^{[1]}[f]\right| \le \left[\int_0^1 \left(K_2^{[1]}(t)\right)^2 dt\right]^{\frac{1}{2}} \|f''\|_2 = \frac{1}{16}\sqrt{\frac{22-15\sqrt{2}}{15}}\|f''\|_2 \approx 0.0143\|f''\|_2,\ f \in H^{2,2}[0,1],$$

$$\left|\mathcal{R}_2^{[1]}[f]\right| \le \sup_{t\in[0,1]} |K_2^{[1]}(t)| \cdot \|f''\|_1 = \frac{2-\sqrt{2}}{16}\|f''\|_1 \approx 0.0366\|f''\|_1,\ f \in H^{2,1}[0,1].$$

Theorem 2.3.3. *For $f \in H^{2,2}[0,1]$, the quadrature formula of the form (2.53), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{3}{16}f(0) + \frac{5}{8}f\left(\frac{1}{2}\right) + \frac{3}{16}f(1) + \mathcal{R}_2^{[2]}[f], \tag{2.59}$$

with

$$\mathcal{R}_2^{[2]}[f]=\int_0^1 K_2^{[2]}(t)f''(t)dt, \quad K_2^{[2]}(t) = \begin{cases} \frac{1}{2}t^2-\frac{3}{16}t, & 0\le t\le\frac{1}{2}, \\ \frac{1}{2}(1-t)^2-\frac{3}{16}(1-t), & \frac{1}{2}<t\le 1, \end{cases} \tag{2.60}$$

and

$$\left|\mathcal{R}_2^{[2]}[f]\right| \le \frac{\sqrt{5}}{160}\|f''\|_2 \approx 0.0140\|f''\|_2.$$

Remark 2.3.3. *For the remainder term of quadrature formula (2.59) can be established the following two estimations*

$$\left|\mathcal{R}_2^{[2]}[f]\right| \le \int_0^1 |K_2^{[2]}(t)|dt \cdot \|f''\|_\infty = \frac{19}{1536}\|f''\|_\infty \approx 0.0124\|f''\|_\infty, \ f \in H^{2,\infty}[0,1],$$

$$\left|\mathcal{R}_2^{[2]}[f]\right| \le \sup_{t\in[0,1]} |K_2^{[2]}(t)| \cdot \|f''\|_1 = \frac{1}{32}\|f''\|_1 \approx 0.0313\|f''\|_1, \ f \in H^{2,1}[0,1].$$

Theorem 2.3.4. *For $f \in H^{2,1}[0,1]$, the quadrature formula of the form (2.53), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{\sqrt{2}-1}{2}f(0) + (2-\sqrt{2})f\left(\frac{1}{2}\right) + \frac{\sqrt{2}-1}{2}f(1) + \mathcal{R}_2^{[3]}[f], \tag{2.61}$$

with

$$\mathcal{R}_2^{[3]}[f]=\int_0^1 K_2^{[3]}(t)f''(t)dt, \ K_2^{[3]}(t)=\begin{cases} \frac{1}{2}t^2-\frac{\sqrt{2}-1}{2}t, & 0\le t\le\frac{1}{2}, \\ \frac{1}{2}(1-t)^2-\frac{\sqrt{2}-1}{2}(1-t), & \frac{1}{2}<t\le 1, \end{cases} \tag{2.62}$$

and

$$\left|\mathcal{R}_2^{[3]}[f]\right| \le \frac{3-2\sqrt{2}}{8}\|f''\|_1 \approx 0.0214\|f''\|_1.$$

Remark 2.3.4. *For the remainder term of quadrature formula (2.61) can be established the following two estimations*

$$\left|\mathcal{R}_2^{[3]}[f]\right|\le\int_0^1 |K_2^{[3]}(t)|dt \cdot \|f''\|_\infty=\frac{37\sqrt{2}-52}{24}\|f''\|_\infty\approx 0.0136\|f''\|_\infty, \ f \in H^{2,\infty}[0,1],$$

$$\left|\mathcal{R}_2^{[3]}[f]\right|\le\left[\int_0^1\left(K_2^{[3]}(t)\right)^2 dt\right]^{\frac{1}{2}}\|f''\|_2=\frac{1}{8}\sqrt{\frac{78-55\sqrt{2}}{15}}\|f''\|_2 \approx 0.0151\|f''\|_2, \ f \in H^{2,2}[0,1].$$

Remark 2.3.5. *If we denote by* $C_p^{[i]}$ *the constants which appear in estimations of the following type*

$$\left|\mathcal{R}_2^{[i]}[f]\right| \le C_p^{[i]} \left\|f''\right\|_p,$$

where $i = 1,2,3$, $p = \infty, 2$, *respectively 1, and* $f \in H^{2,p}[0,1]$, *from the above results the inequalities* $C_\infty^{[1]} \le C_\infty^{[2]} \le C_\infty^{[3]}$, $C_2^{[2]} \le C_2^{[1]} \le C_2^{[3]}$ *and* $C_1^{[3]} \le C_1^{[2]} \le C_1^{[1]}$ *are true. Therefore, we can assert that our results are better than Ujević and Mijić's result, if we consider* 2*-norm, respectively* 1*-norm.*

Similarly to the procedure described in the previous paragraph, we derived corrected rule of the optimal quadrature formulae obtained above. We also showed that the corrected formula improves the original formula. We mention that the corrected formula of (2.57) was considered by N. Ujević and L. Mijić in [129].

Let

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f\left(\frac{1}{2}\right) + A_2 f(1) + A\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2[f], \tag{2.63}$$

where

$$\tilde{\mathcal{R}}_2[e_i] = 0,\ i = 0,1, \text{ and } A = \int_0^1 K_2(t)dt$$

be the corrected quadrature formula of the rule (2.53).

Since the remainder term has degree of exactness 1 we can write

$$\tilde{\mathcal{R}}_2[f] = \int_0^1 \tilde{K}_2(t) f''(t)dt, \text{ where} \tag{2.64}$$

$$\tilde{K}_2(t) = \tilde{\mathcal{R}}_2\left[(x-t)_+\right] = K_2(t) - A. \tag{2.65}$$

From the relation (2.65) we remark that $\int_0^1 \tilde{K}_2(t)dt = 0$. If we consider $f(x) = \frac{x^2}{2}$ in (2.63) we find

$$A = \frac{1}{6} - \frac{1}{2}A_1 - \frac{1}{2}A_2. \tag{2.66}$$

Using relations (2.65) and (2.66)we construct the following corrected quadrature formula of (2.57), (2.59), respectively (2.61):

$$\int_0^1 f(x)dx = \frac{\sqrt{2}}{8} f(0) + \frac{4-\sqrt{2}}{4} f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8} f(1) + \frac{4-3\sqrt{2}}{96}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2^{[1]}[f], \tag{2.67}$$

where

$$\tilde{\mathcal{R}}_2^{[1]}[f] = \int_0^1 \tilde{K}_2^{[1]}(t) f''(t)dt, \tag{2.68}$$

$$\tilde{K}_2^{[1]}(t) = \begin{cases} \frac{1}{2}t^2 - \frac{\sqrt{2}}{8}t - \frac{4-3\sqrt{2}}{96}, & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}(1-t)^2 - \frac{\sqrt{2}}{8}(1-t) - \frac{4-3\sqrt{2}}{96}, & \frac{1}{2} < t \le 1, \end{cases}$$

$$(2.69)\qquad \int_0^1 f(x)dx=\frac{3}{16}f(0)+\frac{5}{8}f\left(\frac{1}{2}\right)+\frac{3}{16}f(1)-\frac{1}{192}\left[f'(1)-f'(0)\right]+\tilde{\mathcal{R}}_2^{[2]}[f],$$

where

$$(2.70)\qquad \tilde{\mathcal{R}}_2^{[2]}[f]=\int_0^1 \tilde{K}_2^{[2]}(t)f''(t)dt,$$

$$\tilde{K}_2^{[2]}(t)=\begin{cases}\frac{1}{2}t^2-\frac{3}{16}t+\frac{1}{192}, & 0\le t\le \frac{1}{2}\\ \frac{1}{2}(1-t)^2-\frac{3}{16}(1-t)+\frac{1}{192}, & \frac{1}{2}<t\le 1,\end{cases}$$

respectively

$$(2.71)\qquad \int_0^1 f(x)dx=\frac{\sqrt{2}-1}{2}f(0)+(2-\sqrt{2})f\left(\frac{1}{2}\right)+\frac{\sqrt{2}-1}{2}f(1)$$

$$+\frac{4-3\sqrt{2}}{24}\left[f'(1)-f'(0)\right]+\tilde{\mathcal{R}}_2^{[3]}[f],$$

where

$$(2.72)\qquad \tilde{\mathcal{R}}_2^{[3]}[f]=\int_0^1 \tilde{K}_2^{[3]}(t)f''(t)dt,$$

$$\tilde{K}_2^{[3]}(t)=\begin{cases}\frac{1}{2}t^2-\frac{\sqrt{2}-1}{2}t-\frac{4-3\sqrt{2}}{24}, & 0\le t\le \frac{1}{2}\\ \frac{1}{2}(1-t)^2-\frac{\sqrt{2}-1}{2}(1-t)-\frac{4-3\sqrt{2}}{24}, & \frac{1}{2}<t\le 1,\end{cases}$$

Denote by $\tilde{C}_p^{[i]}$ the constant which appear in estimations of the remainder term of corrected quadrature formulas, namely

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right|\le \tilde{C}_p^{[i]}\left\|f''\right\|_p,$$

where $i=1,2,3$, $p=\infty,2$, respectively 1, and $f\in H^{2,p}[0,1]$. The constants $\tilde{C}_p^{[i]}$ can be calculated in a similar way with the constants $C_p^{[i]}$ defined in Remark 2.3.5. From the bellow table follows that for $p=\infty$ and $p=2$ the corrected formula improves the original formula.

$\diagdown i$	1	2	3
$C_\infty^{[i]}$	$\dfrac{2-\sqrt{2}}{48} \approx 0.0122$	$\dfrac{19}{1536} \approx 0.0124$	$\dfrac{37\sqrt{2}-52}{24} \approx 0.0136$
$\tilde{C}_\infty^{[i]}$	$\dfrac{5}{96}\sqrt{6} - \dfrac{29}{432}\sqrt{3} \approx 0.0113$	$\dfrac{19}{13824}\sqrt{57} \approx 0.0104$	$\dfrac{\sqrt{3(13-9\sqrt{2})^3}}{27} \approx 0.0091$
$C_2^{[i]}$	$\dfrac{1}{16}\sqrt{\dfrac{22-15\sqrt{2}}{15}} \approx 0.0143$	$\dfrac{\sqrt{5}}{160} \approx 0.0140$	$\dfrac{1}{8}\sqrt{\dfrac{78-55\sqrt{2}}{15}} \approx 0.0151$
$\tilde{C}_2^{[i]}$	$\dfrac{1}{480}\sqrt{470-300\sqrt{2}} \approx 0.0141$	$\dfrac{\sqrt{155}}{960} \approx 0.0130$	$\left(\dfrac{1}{90} - \dfrac{1}{128}\sqrt{2}\right) \approx 0.0112$
$C_1^{[i]}$	$\dfrac{2-\sqrt{2}}{16} \approx 0.0366$	$\dfrac{1}{32} \approx 0.0313$	$\dfrac{3-2\sqrt{2}}{8} \approx 0.0214$
$\tilde{C}_1^{[i]}$	$\left(\dfrac{1}{12} - \dfrac{1}{32}\sqrt{2}\right) \approx 0.0391$	$\dfrac{7}{192} \approx 0.0365$	$\left(\dfrac{5}{24} - \dfrac{1}{8}\sqrt{2}\right) \approx 0.0316$

Theorem 2.3.5. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$ and there exist real number $m[f], M[f]$ such that $m[f] \le f''(t) \le M[f]$, $t \in [0,1]$. Then*

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \le \frac{M[f]-m[f]}{2}\left(\frac{5\sqrt{6}}{96} - \frac{29\sqrt{3}}{432}\right) \approx 11306 \times 10^{-6} \cdot \frac{M[f]-m[f]}{2}, \tag{2.73}$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \le \frac{M[f]-m[f]}{2} \cdot \frac{19\sqrt{57}}{13824} \approx 10377 \times 10^{-6} \cdot \frac{M[f]-m[f]}{2}, \tag{2.74}$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \le \frac{M[f]-m[f]}{2} \cdot \frac{\sqrt{3(13-9\sqrt{2})^3}}{27} \approx 9104 \times 10^{-6} \cdot \frac{M[f]-m[f]}{2}. \tag{2.75}$$

If there exist a real number $m[f]$ such that $m[f] \le f''(t)$, $t \in [0,1]$, then

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \le \frac{1}{4}\left(\frac{1}{3} - \frac{\sqrt{2}}{8}\right)(f'(1)-f'(0)-m[f]) \approx 39139 \times 10^{-6}\,(f'(1)-f'(0)-m[f]), \tag{2.76}$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \le \frac{7}{192}\,(f'(1)-f'(0)-m[f]) \approx 36458 \times 10^{-6}\,(f'(1)-f'(0)-m[f]), \tag{2.77}$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \le \frac{5-3\sqrt{2}}{24} \cdot (f'(1)-f'(0)-m[f]) \approx 31557 \times 10^{-6}\,(f'(1)-f'(0)-m[f]). \tag{2.78}$$

If there exist a real number $M[f]$ such that $f''(t) \le M[f]$, $t \in [0,1]$, then

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \le \frac{1}{4}\left(\frac{1}{3} - \frac{\sqrt{2}}{8}\right)[M[f]-(f'(1)-f'(0))] \approx 39139 \times 10^{-6}[M[f]-(f'(1)-f'(0))],$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \le \frac{7}{192}\,[M[f]-(f'(1)-f'(0))] \approx 36458 \times 10^{-6}\,[M[f]-(f'(1)-f'(0))],$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \le \frac{5-3\sqrt{2}}{24}\,[M[f]-(f'(1)-f'(0))] \approx 31557 \times 10^{-6}\,[M[f]-(f'(1)-f'(0))].$$

Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$. The functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt, \tag{2.79}$$

is well known in the literature as the Čebyšev functional. It was proved that $T(f,f) \geq 0$ and $|T(f,g)| \leq \sqrt{T(f,f)} \cdot \sqrt{T(g,g)}$. Denote by $\sigma(f,a,b) = \sqrt{T(f,f)}$.

Theorem 2.3.6. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L_2[0,1]$. Then*

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \leq \sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}} \cdot \sigma(f'';0,1) \approx 14089 \times 10^{-6} \sigma(f'';0,1), \tag{2.80}$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \leq \frac{\sqrt{155}}{960} \cdot \sigma(f'';0,1) \approx 12969 \times 10^{-6} \sigma(f'';0,1), \tag{2.81}$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \leq \frac{1}{120}\sqrt{320 - 225\sqrt{2}} \cdot \sigma(f'';0,1) \approx 11186 \times 10^{-6} \sigma(f'';0,1). \tag{2.82}$$

Remark 2.3.6. *The inequalities (2.80), (2.81),and (2.82, respectively, are sharp in the sense that the constants $\sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}}$, $\frac{\sqrt{155}}{960}$ and $\frac{1}{120}\sqrt{320 - 225\sqrt{2}}$ respectively, cannot be replaced by a smaller ones. To prove that we define the functions*

$$F^{[i]}(x) = \int_0^x \left(\int_0^t K_2^{[i]}(u)du \right) dt, \;\; i = 1,2,3. \tag{2.83}$$

For the function (2.83) the right-hand side of the inequalities (2.80), (2.81) and (2.82) respectively, are equal with $T(K_2^{[i]}, K_2^{[i]})$, $i = 1,2,3$, respectively, and the left-hand side becomes

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| = \left| \int_0^1 \tilde{K}_2^{[i]}(t) \cdot K_2^{[i]}(t) dt \right| =$$

$$\left| \int_0^1 \left[K_2^{[i]}(t) - \int_0^1 K_2^{[i]}(t)dt \right] K_2^{[i]}(t)dt \right| = T(K_2^{[i]}, K_2^{[i]}), \;\; i = 1,2,3.$$

Remark 2.3.7. *Denote by Z_i, $i = 1,2,3$, the constants which appears in one of the following types of estimations obtained in Theorem 2.3.5 and Theorem 2.3.6, namely $\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot \frac{M[f] - m[f]}{2}$, $\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot (f'(1) - f'(0) - m[f])$, $\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot (M[f] - [f'(1) - f'(0)])$ or $\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot \sigma(f'';0,1)$. Since for every $i = 1,2,3$ we have $Z_3 \leq Z_2 \leq Z_1$, for the corrected quadrature formulas, our results are better than Ujević and Mijić's results obtained in în [129].*

The corrected quadrature formulas (2.67), (2.69), and (2.71), respectively have degree of exactness 3, which is higher than the original rule, namely for $j = \overline{1,3}$, $\tilde{R}_2^{[j]}[e_i] = 0$ and $\tilde{R}_2^{[j]}[e_4] \neq 0$, unde $e_i(x) = x^i$, $i = \overline{0,4}$. Using Peano's Theorem, the remainder term can be written

$$\mathcal{R}_4[f] = \int_0^1 K_4(t) f^{(4)}(t), \;\; K_4(t) = \mathcal{R}_4 \left[\frac{(x-t)_+^3}{3!} \right], \tag{2.84}$$

where by $\mathcal{R}_4$ we denote the new integral representation of the remainder term of these quadrature formulas.

Next, using relation (2.84), we will give new estimations of the remainder term in quadrature formulas (2.67), (2.69), and (2.71), respectively.

Theorem 2.3.7. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (2.67) has the integral representation*

$$\mathcal{R}_4^{[1]}[f] = \int_0^1 K_4^{[1]}(t) f^{(4)}(t) dt, \text{ where}$$

$$K_4^{[1]}(t) = \begin{cases} \dfrac{1}{24} t^2 \left(t^2 - \dfrac{\sqrt{2}}{2} t - \dfrac{4-3\sqrt{2}}{8} \right), & 0 \le t \le \dfrac{1}{2}, \\ \dfrac{1}{24}(1-t)^2 \left((1-t)^2 - \dfrac{\sqrt{2}}{2}(1-t) - \dfrac{4-3\sqrt{2}}{8} \right), & \dfrac{1}{2} < t \le 1, \end{cases}$$

and the following estimations holds

$$\begin{aligned} \left| \mathcal{R}_4^{[1]}[f] \right| &\le \sqrt{\int_0^1 \left(K_4^{[1]}(t) \right)^2 dt} \sqrt{\int_0^1 \left(f^{(4)}(t) \right)^2 dt} \\ &= \frac{\sqrt{23170 - 15645\sqrt{2}}}{80640} \| f^{(4)} \|_2 \approx 4.008 \times 10^{-4} \| f^{(4)} \|_2, \end{aligned}$$

$$\begin{aligned} \left| \mathcal{R}_4^{[1]}[f] \right| &\le \int_0^1 \left| K_4^{[1]}(t) \right| dt \cdot \sup_{t \in [0,1]} |f^{(4)}(t)| \\ &= \frac{200 - 171\sqrt{2} - (90 - 68\sqrt{2})\sqrt{5 - 3\sqrt{2}} + 2(15 - 8\sqrt{2})\sqrt{43 - 30\sqrt{2}}}{11520} \cdot \| f^{(4)} \|_\infty \\ &\approx 2.946 \times 10^{-4} \cdot \| f^{(4)} \|_\infty, \end{aligned}$$

$$\begin{aligned} \left| \mathcal{R}_4^{[1]}[f] \right| &\le \sup_{t \in [0,1]} |K_4^{[1]}(t)| \cdot \int_0^1 |f^{(4)}(t)| dt \\ &= \frac{2 - \sqrt{2}}{768} \cdot \| f^{(4)} \|_1 \approx 7.627 \times 10^{-4} \cdot \| f^{(4)} \|_1. \end{aligned}$$

Theorem 2.3.8. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (2.69)has the integral representation*

$$\mathcal{R}_4^{[2]}[f] = \int_0^1 K_4^{[2]}(t) f^{(4)}(t) dt, \text{ where}$$

$$K_4^{[2]}(t) = \begin{cases} \dfrac{1}{24} t^2 \left(t^2 - \dfrac{3}{4} t + \dfrac{1}{16} \right), & 0 \le t \le \dfrac{1}{2}, \\ \dfrac{1}{24}(1-t)^2 \left((1-t)^2 - \dfrac{3}{4}(1-t) + \dfrac{1}{16} \right), & \dfrac{1}{2} < t \le 1, \end{cases}$$

and the following estimations holds

$$\begin{aligned}\left|\mathcal{R}_4^{[2]}[f]\right| &\le \sqrt{\int_0^1 \left(K_4^{[2]}(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt}\\ &= \frac{\sqrt{2905}}{161280}\|f^{(4)}\|_2 \approx 3.342\times 10^{-4}\|f^{(4)}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[2]}[f]\right| &\le \int_0^1 \left|K_4^{[2]}(t)\right| dt\cdot \sup_{t\in[0,1]} |f^{(4)}(t)|\\ &= \frac{125\sqrt{5}-103}{737280}\cdot\|f^{(4)}\|_\infty \approx 2.394\times 10^{-4}\cdot\|f^{(4)}\|_\infty,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[2]}[f]\right| &\le \sup_{t\in[0,1]} |K_4^{[2]}(t)|\cdot \int_0^1 |f^{(4)}(t)|dt\\ &= \frac{1}{1536}\cdot\|f^{(4)}\|_1 \approx 6.51\times 10^{-4}\cdot\|f^{(4)}\|_1.\end{aligned}$$

Theorem 2.3.9. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (2.71)has the integral representation*

$$\mathcal{R}_4^{[3]}[f] = \int_0^1 K_4^{[3]}(t) f^{(4)}(t)dt, \ \textit{where}$$

$$K_4^{[3]}(t) = \begin{cases} \dfrac{1}{24}t^2\left(t^2 - 2(\sqrt{2}-1)t - \dfrac{4-3\sqrt{2}}{2}\right), \ 0 \le t \le \dfrac{1}{2},\\[2ex] \dfrac{1}{24}(1-t)^2\left((1-t)^2 - 2(\sqrt{2}-1)(1-t) - \dfrac{4-3\sqrt{2}}{2}\right), \ \dfrac{1}{2} < t \le 1,\end{cases}$$

and the following estimations holds

$$\begin{aligned}\left|\mathcal{R}_4^{[3]}[f]\right| &\le \sqrt{\int_0^1 \left(K_4^{[3]}(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt}\\ &= \frac{\sqrt{68530-48405\sqrt{2}}}{40320}\|f^{(4)}\|_2 \approx 2.148\times 10^{-4}\|f^{(4)}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[3]}[f]\right| &\le \int_0^1 \left|K_4^{[3]}(t)\right| dt\cdot \sup_{t\in[0,1]} |f^{(4)}(t)|\\ &= \frac{78470-55487\sqrt{2}-32(550-389\sqrt{2})\sqrt{10-7\sqrt{2}}}{5760}\cdot\|f^{(4)}\|_\infty\\ &\approx 1.461\times 10^{-4}\cdot\|f^{(4)}\|_\infty,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[3]}[f]\right| &\le \sup_{t\in[0,1]} |K_4^{[3]}(t)|\cdot \int_0^1 |f^{(4)}(t)|dt\\ &= \frac{3-2\sqrt{2}}{384}\cdot\|f^{(4)}\|_1 \approx 4.468\times 10^{-4}\cdot\|f^{(4)}\|_1.\end{aligned}$$

Problem of constructing quadrature formulas can be generalized for more than 3 points. For example in the paper [121] were constructed the quadrature formulas of open type, optimal in sense Nikolski for m ($m \in N$) points. In [5] we treated the case of quadrature formula of closed type, optimal in sense Nikolski , building their corrected quadrature formulas.

We will calculate the coefficients A_i, $i = \overline{0, m}$ and the nodes a_i, $i = \overline{1, m-1}$ such that the following quadrature formula which degree of exactness 1

$$\int_0^1 f(x)dx = \sum_{i=0}^{m} A_i f(a_i) + \mathcal{R}[f], \text{ where } 0 = a_0 < a_1 < \cdots < a_m = 1, \tag{2.85}$$

to be optimal, considering that the remainder term is evaluate in sense of (2.10).

Theorem 2.3.10. *For $f \in H^{2,\infty}[0,1]$ the quadrature formula of the form (2.85), optimal to the error has the following nodes and coefficients*

$$a_1 = \frac{\sqrt{3(2+\sqrt{2})}}{2}h, \; a_{m-1} = 1 - \frac{\sqrt{3(2+\sqrt{2})}}{2}h, \tag{2.86}$$
$$a_k = 1 - \frac{4(m-k-1) + \sqrt{3(2+\sqrt{2})}}{2}h, \; k = \overline{2, m-2},$$

$$A_0 = A_m = \frac{\sqrt{6(2+\sqrt{2})}}{8}h, \; A_k = 2h, \; k = \overline{2, m-2} \tag{2.87}$$
$$A_1 = A_{m-1} = \frac{(4-\sqrt{2})\sqrt{3(2+\sqrt{2})}+8}{8}h, \; \textit{where}\, h = \frac{1}{2(m-2) + \sqrt{3(2+\sqrt{2})}}.$$

The remainder term has the following evaluation $|\mathcal{R}[f]| \leq \dfrac{h^2}{8}\|f''\|_\infty$.

Similar to the algorithm described in the previous sections we obtain the following corrected quadrature formula of rule (2.85)

$$\int_0^1 f(x)dx = \sum_{i=0}^{m} A_i f(a_i) + A\left(f'(1) - f'(0)\right) + \tilde{\mathcal{R}}[f], \tag{2.88}$$

where the nodes a_i, respectively the coefficients A_i, $i = \overline{0, m}$ are given in relations (2.86), respectively (2.87) and

$$A = \int_0^1 K(t)dt = \int_0^1 \tilde{K}(u)du = \frac{h^3}{48}\left[4(m-2) + 3(1-\sqrt{2})\sqrt{3(2+\sqrt{2})}\right].$$

Remark 2.3.8. *If we consider in Theorem 2.3.10 the particular case m=2 we obtained the following optimal quadrature formula of close type with 3-points*

$$\int_0^1 f(x)dx = \frac{\sqrt{2}}{8}f(0) + \frac{4-\sqrt{2}}{4}f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8}f(1) + \mathcal{R}[f], \tag{2.89}$$

and the corrected rule of this quadrature formulas is given by

$$\int_0^1 f(x)dx = \frac{\sqrt{2}}{8}f(0) + \frac{4-\sqrt{2}}{4}f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8}f(1) + \frac{4-3\sqrt{2}}{96}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}[f]. \tag{2.90}$$

The optimal quadrature (2.89)and the corrected rule (2.90) were obtained by N. Ujević and L. Mijić in [129].

Next we consider a corrected version of the optimal quadrature with 4-points and we show that this rule provides a better approximation than the original rule.

Considering $m = 3$ in Theorem 2.3.10 we have the following optimal quadrature formula

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(a_1) + A_2 f(a_2) + A_3 f(1) + \mathcal{R}[f], \text{ where} \tag{2.91}$$

$$A_0 = A_3 = \frac{\sqrt{6(2+\sqrt{2})}}{8}h, \ A_1 = A_2 = \frac{(4-\sqrt{2})\sqrt{3(2+\sqrt{2})}+8}{8}h,$$

$$a_1 = \frac{\sqrt{3(2+\sqrt{2})}}{2}h, \ a_2 = 1 - a_1, \ h = \frac{1}{2+\sqrt{3(2+\sqrt{2})}}.$$

The remainder term has the following representation $\mathcal{R}[f] = \int_0^1 K(t)f''(t)dt$, where

$$K(t) = \begin{cases} \frac{1}{2}t^2 - A_0 t, \ 0 \le t \le a_1, \\ \frac{1}{2}t^2 - (A_0 + A_1)t + A_1 a_1, \ a_1 \le t \le a_2, \\ \frac{1}{2}(1-t)^2 - A_3(1-t), \ a_2 \le t \le 1. \end{cases}$$

Using (2.88) we obtain the following corrected quadrature formula of (2.91)

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(a_1) + A_2 f(a_2) + A_3 f(1) + A\left(f'(1) - f'(0)\right) + \tilde{\mathcal{R}}[f], \tag{2.92}$$

where $A = \frac{h^3}{48}\left[4 + 3(1-\sqrt{2})\sqrt{3(2+\sqrt{2})}\right]$ and

$$\tilde{\mathcal{R}}[f] = \int_0^1 \tilde{K}(t)f''(t)dt, \text{ with } \tilde{K}(t) = K(t) - A.$$

Theorem 2.3.11. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$ and there exist real number $m[f], M[f]$ such that $m[f] \le f''(t) \le M[f]$, $t \in [0,1]$. Then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le 0.00462291793614 \cdot \frac{M[f] - m[f]}{2}.$$

Theorem 2.3.12. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$. If there exist a real number $m[f]$ such that $m[f] \le f''(t)$, $t \in [0,1]$, then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le \frac{1}{48}\frac{32 + (15+3\sqrt{2})\sqrt{6+3\sqrt{2}}}{(2+\sqrt{6+3\sqrt{2}})^3} \cdot (f'(1) - f'(0) - m[f]).$$

Theorem 2.3.13. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$. If there exist a real number $M[f]$ such that $f''(t) \le M[f]$, $t \in [0,1]$,then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le \frac{1}{48}\frac{32 + (15+3\sqrt{2})\sqrt{6+3\sqrt{2}}}{(2+\sqrt{6+3\sqrt{2}})^3} \cdot (M[f] - f'(1) + f'(0)).$$

Theorem 2.3.14. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L_2[0,1]$. Then*

$$\left|\tilde{\mathcal{R}}[f]\right| \le C \cdot \sigma(f''; 0, 1) \text{ where} \tag{2.93}$$

$$C = \left(\frac{1}{11520} \cdot \frac{2374 + (12\sqrt{2}+1080)\sqrt{6+3\sqrt{2}} + 783\sqrt{2}}{(2+\sqrt{6+3\sqrt{2}})^6}\right)^{1/2}.$$

The inequality (2.93) is sharp in the sense that the constant C cannot be replaced by a smaller ones.

Remark 2.3.9. *Considering that the remainder term of original, respectively corrected quadrature formula is evaluate in sense of (2.8) with $p \in \{1,2\}$ we obtain the following inequalities*

$$|\mathcal{R}[f]| \le \|K\|_\infty \cdot \|f''\|_1 = \frac{3}{8} \cdot \frac{1}{(2+\sqrt{6+3\sqrt{2}})^2} \cdot \|f''\|_1 \approx 0.01386614276036 \cdot \|f''\|_1,$$

$$\left|\tilde{\mathcal{R}}[f]\right| \le \|\tilde{K}\|_\infty \cdot \|f''\|_1 = \frac{1}{48} \cdot \frac{32 + (15+3\sqrt{2})\sqrt{6+3\sqrt{2}}}{(2+\sqrt{6+3\sqrt{2}})^3} \cdot \|f''\|_1$$
$$\approx 0.01386273025465 \cdot \|f''\|_1,$$

$$|\mathcal{R}[f]| \le \|K\|_2 \cdot \|f''\|_2 = \left(-\frac{1}{1920} \cdot \frac{-92 + (9\sqrt{2}-54)\sqrt{6+3\sqrt{2}}}{(2+\sqrt{6+3\sqrt{2}})^5}\right)^{1/2} \cdot \|f''\|_2$$
$$\approx 0.00553941461773 \cdot \|f''\|_2,$$

$$\left|\tilde{\mathcal{R}}[f]\right| \le \|\tilde{K}\|_2 \cdot \|f''\|_2 = \left(\frac{1}{11520} \cdot \frac{2374 + (12\sqrt{2}+1080)\sqrt{6+3\sqrt{2}} + 783\sqrt{2}}{(2+\sqrt{6+3\sqrt{2}})^6}\right)^{1/2} \cdot \|f''\|_2$$
$$\approx 0.00553941356661 \cdot \|f''\|_2.$$

We can remark that the estimations of the remainder term in corrected rule are better than in original quadrature formula.

Chapter 3

Evaluations of the remainder term in numerical integration formulas using Ostrowski type inequalities

In this chapter are presented Ostrowski type inequalities, mean value theorems used to obtain these inequalities and applications in numerical integration, more accurate assessments of the remainder term in numerical integration formulas using Ostrowski type inequalities.

In the second section we gave a generalization of a mean value theorem D. Pompeiu ([106]). Using the mean value theorem we obtained inequalities of Ostrowski type in p norm for $p = 2, \infty$ and 1. In the last part of this section we considered the case of weighted Ostrowski type inequality. In the third section, the same as the reasoning used by S. S. Dragomir and E. C. Popa in [69] and [108], we gave new estimates of the remainder term in quadrature formula obtained by E. C. Popa. The results are contained in [6].

3.1 Ostrowski type inequalities

The following result is known in the literature as Ostrowski's inequality ([101]).

$$\left|\frac{1}{b-a} \cdot \int_a^b f(y)dy - f(x)\right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right) \cdot (b-a) \cdot ||f'||_{\infty}, \tag{3.1}$$

where $f \in C^1[a,b], x \in [a,b]$. The constant $\frac{1}{4}$ is best possible. One can easily notice that

$$\left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right) \cdot (b-a) = \frac{(x-a)^2 + (b-x)^2}{2 \cdot (b-a)} \tag{3.2}$$

In the last years the inequalities of Ostrowski type have occupied the attention of many authors.

3.2 The mean value theorems to obtain Ostrowski type inequalities

In this section we present some inequalities of Ostrowski type obtained using the mean value theorems. In 1946, D. Pompeiu ([106]) derive a variant of Lagrange's mean value theorem, later used by S. S. Dragomir ([69]) în obtaining an Ostrowski type inequalities.

Theorem 3.2.1. *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ in (x_1, x_2) such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{3.3}$$

Theorem 3.2.2. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $[a, b]$ not containing 0. Then for any $x \in [a, b]$, we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \cdot \|f - lf'\|_\infty, \tag{3.4}$$

where $l(t) = t$, $t \in [a, b]$.

In [108], E. C. Popa using a mean value theorem obtained a generalization of Dragomir's result.

Theorem 3.2.3. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then for any $x \in [a, b]$ we have the inequality*

$$\left| \left[\frac{a+b}{2} - \alpha \right] f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \cdot (b-a) \cdot \|f - lf'\|_\infty, \tag{3.5}$$

where $\alpha \notin [a, b]$ şi $l(t) = t - \alpha$, $t \in [a, b]$.

Also, in [105] J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1 \leq p \leq \infty$ which for $p = \infty$ give the Dragomir's result.

Theorem 3.2.4. *Let the function $f : [a, b] \to \mathbb{R}$be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leq p, q \leq \infty$, and all $x \in [a, b]$, the following inequality holds:*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq PU(x, p) \cdot \|f - lf'\|_p, \tag{3.6}$$

where $l(t) = t$, $t \in [a, b]$, and

$$\begin{aligned} PU(x,p) &= (b-a)^{\frac{1}{p}-1} \cdot \left[\left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \right. \\ &+ \left. \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{3.7}$$

Note that in cases $(p,q)=(1,\infty)$, $(\infty,1)$ and $(2,2)$ the constant $PU(x,p)$ has to be taken as the limit as $p\to 1,\infty$ and 2, respectively.

In [6] we obtained new inequalities of Ostrowski type using mean value theorems, generalizing some results of S. S. Dragomir, J. Pečarić, S. Ungar şi E. C. Popa (vezi [69], [105], [108]). The inequalities for p-norm are also given and the weighted case is considered.

The following result is a generalization of Pompeiu's mean value theorem ([6]).

Theorem 3.2.5. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1,x_2\in[a,b]$, $x_1\neq x_2$, there exists a point ξ in (x_1,x_2) such that*

$$\frac{(x_1-\alpha)f(x_2)-(x_2-\alpha)f(x_1)}{x_1-x_2}=f(\xi)-(\xi-\alpha)f'(\xi),\tag{3.8}$$

where $\alpha\notin[a,b]$.

Remark 3.2.1. *If we choose $\alpha=0$ to obtain the Pompeiu's mean value theorem.*

Remark 3.2.2. *From the relation (3.8) we obtained*

$$|(x_1-\alpha)f(x_2)-(x_2-\alpha)f(x_1)|\le\sup_{\xi\in[a,b]}|f(\xi)-(\xi-\alpha)f'(\xi)|\cdot|x_1-x_2|.\tag{3.9}$$

Integrating (3.9) over $x_1\in[a,b]$ we find the the Ostrowski inequality (3.5) obtained by E.C. Popa in [106].

In paper [6] we obtained inequalities of Ostrowski type in p-norm. First of all we will consider the particular cases $p=2,\infty$, respectively 1.

Theorem 3.2.6. *Let the function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for all $x\in[a,b]$ the following inequality holds*

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\le\frac{(b-a)^{\frac12}}{3}\|f-lf'\|_2\cdot\left[\Phi(a,\alpha,x)^{\frac12}+\Phi(b,\alpha,x)^{\frac12}\right],$$

where $\alpha\notin[a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ and

$$\Phi(s,\alpha,x)=\ln\left(\frac{x-\alpha}{s-\alpha}\right)^3+\left(\frac{s-\alpha}{x-\alpha}\right)^3-1,\ s\in[a,b].$$

Theorem 3.2.7. *Let the function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for all $x\in[a,b]$ the following inequality holds*

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\le\begin{cases}\|f-lf'\|_\infty\cdot\Psi(a,b,\alpha,x), & for\ \alpha<a,\\ -\|f-lf'\|_\infty\cdot\Psi(a,b,\alpha,x), & for\alpha>b,\end{cases}\tag{3.10}$$

where $\alpha\notin[a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ and

$$\Psi(a,b,\alpha,x)=\frac{1}{2(x-\alpha)}\left[(b-x)^2+(x-a)^2\right].$$

Remark 3.2.3. *The inequality (3.10) coincides with the Ostrowski inequality (3.5) obtained by E.C. Popa in [108]. The proof of the inequality (3.5) was done in a different manner than in [108].*

Theorem 3.2.8. *Let the function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for all $x\in[a,b]$ the following inequality holds*

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\leq(b-a)\|f-lf'\|_1\Omega(a,b,\alpha,x), \tag{3.11}$$

where $\alpha\notin[a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ and

$$\Omega(a,b,\alpha,x)=\begin{cases}\dfrac{1}{a-\alpha}+\dfrac{b-\alpha}{(x-\alpha)^2}, & \textit{for } \alpha<a,\\[2ex] \dfrac{\alpha-a}{(\alpha-x)^2}+\dfrac{1}{\alpha-b}, & \textit{for } \alpha>b.\end{cases}$$

Theorem 3.2.9. *Let the function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for $\frac{1}{p}+\frac{1}{q}=1$, with $1<p,q<\infty$, $p,q\neq 2$ and all $x\in[a,b]$ the following inequality holds*

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\leq \tag{3.12}$$

$$\begin{cases}(b-a)^{\frac{1}{p}}\|f-lf'\|_p\left[\Theta(a,x,\alpha)^{\frac{1}{q}}+\Theta(b,x,\alpha)^{\frac{1}{q}}\right], & \textit{for } \alpha<a,\\[2ex] -(b-a)^{\frac{1}{p}}\|f-lf'\|_p\left[\Theta(a,x,\alpha)^{\frac{1}{q}}+\Theta(b,x,\alpha)^{\frac{1}{q}}\right], & \textit{for } \alpha>b,\end{cases}$$

where $\alpha\notin[a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ and

$$\Theta(s,x,\alpha)=\frac{1}{1-2q}\left\{\frac{(x-\alpha)^{2-q}}{q+1}+\frac{(x-\alpha)^{2-q}}{q-2}-\frac{(x-\alpha)^{1-2q}(s-\alpha)^{q+1}}{q+1}-\frac{(s-\alpha)^{2-q}}{q-2}\right\},\ s\in[a,b].$$

Theorem 3.2.10. *Let the function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$, and let $w:[a,b]\to\mathbb{R}$ be a nonnegative integrable function. Then $\frac{1}{p}+\frac{1}{q}=1$, with $1\leq p,q\leq\infty$, and for all $x\in[a,b]$the following inequality holds*

$$\left|\frac{f(x)}{x-\alpha}\int_a^b(t-\alpha)w(t)dt-\int_a^b f(t)w(t)dt\right|\leq(b-a)^{\frac{1}{p}}\|f-lf'\|_p\Lambda(a,b,\alpha,x), \tag{3.13}$$

where $\alpha\notin[a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ and

$$\Lambda(a,b,\alpha,x)=\left[\int_a^x\left(\int_t^x\frac{|t-\alpha|^q w(t)^q}{(u-\alpha)^{2q}}du\right)dt\right]^{\frac{1}{q}}+\left[\int_x^b\left(\int_x^t\frac{|t-\alpha|^q w(t)^q}{(u-\alpha)^{2q}}du\right)dt\right]^{\frac{1}{q}}.$$

3.3 Applications in numerical integration

In this section we present some applications of Ostrowski type inequalities in numerical integration. Using the idea of S. S. Dragomir in [69], E. C. Popa consider in [108] the division of the interval $[a, b]$ given by

$$\Delta : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

and (ξ_i) sequence of intermediate points $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$ and $h_i = x_{i+1} - x_i$. Defines the following quadrature formula

$$\int_a^b f(t)dt = S_\Delta(f, \xi_i) + R_\Delta(f, \xi_i), \text{ where}$$

$$S_\Delta(f, \xi_i) = \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i - \alpha} \left(\frac{x_{i+1} + x_i}{2} - \alpha \right) h_i,$$

and obtain the following estimation for remainder term $\mathcal{R}_\Delta(f, \xi_i)$ ([108]):

Theorem 3.3.1. *Assume that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then we have*

$$|R_\Delta(f, \xi_i)| \le \frac{1}{2h} \|f + lf'\|_\infty \cdot \sum_{i=0}^{n-1} h_i^2,$$

where $h = \min\{|a - \alpha|, |b - \alpha|\}$ and $l(t) = t - \alpha$, $t \in [a, b]$.

In [6], in the same way with the reasoning used in [69] and [108] we gave new estimations of the remainder term $R_\Delta(f, \xi_i)$ in the quadrature formula constructed by E. C. Popa in [108]. :

Theorem 3.3.2. *Assume that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then we have*

$$|\mathcal{R}_\Delta(f, \xi_i)| \le \frac{2}{3} \mathbb{K}(a, b, \alpha) \cdot \|f - lf'\|_2 \sum_{i=0}^{n-1} h_i^{\frac{1}{2}}, \tag{3.14}$$

where $\mathbb{K}(a, b, \alpha) = \left[\ln \left(\frac{b - \alpha}{a - \alpha} \right)^3 + \left(\frac{b - \alpha}{a - \alpha} \right)^3 - 1 \right]^{\frac{1}{2}}$ *and $l(t) = t - \alpha$, $t \in [a, b]$.*

Theorem 3.3.3. *Assume that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then we have*

$$|\mathcal{R}_\Delta(f, \xi_i)| \le \tilde{\Omega}(a, b, \alpha) \|f - lf'\|_1 \cdot \sum_{i=0}^{n-1} h_i, \tag{3.15}$$

where $l(t) = t - \alpha$, $t \in [a, b]$, and $\tilde{\Omega}(a, b, \alpha) = \begin{cases} \dfrac{a + b - 2\alpha}{(a - \alpha)^2}, & \text{for } \alpha < a, \\ \dfrac{2\alpha - a - b}{(\alpha - b)^2}, & \text{for } \alpha > b. \end{cases}$

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