Ordinary Differential Equations and Contact Problems: Modeling, Analysis and Numerical Methods

Ph.D. Thesis Summary

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Contents

Introduction

Keywords: elastic constitutive law, quasistatic processes, viscoelastic constitutive law, numerical method, Cauchy problem, normal compliance, Runge-Kutta method, memory term, Steffensen method, unilateral constraint, convergence order, local truncation error, variational inequality, zero-stability

The purpose of this thesis is twofold. The first one is to introduce the reader to some representative notions of numerical methods for Cauchy problems like consistency, zero-stability, convergence, order of convergence, local truncation error, etc. The second one is to introduce the reader a mathematical theory of contact problems involving deformable bodies. This concern the mathematical modelling and the variational analysis of the models, including existence, uniqueness and convergence results. More precisely, quasistatic contact processes are treated in the infinitesimal strain theory and the material behavior is modeled with elastic and viscoelastic constitutive laws. The contact is frictionless and modeled with various conditions, including normal compliance and memory term.

The thesis is divided into two parts and eight chapters. **Part I**, containing Chapters 1–3 is devoted to the numerical methods for initial value problems for ordinary differential equations. In writing this part of the thesis the books [17], [23], [54], [36] were followed especially. **Part II** refers to the modelling and analysis of some frictionless contact problems for nonlinear elastic or viscoelastic materials. It contains Chapters 4–8. In writing this part the books [39], [89], [95] were followed especially.

The original contributions in the first part of the thesis consist in the analysis of some numerical methods for Cauchy problems for first order ordinary differential equations. Thus,

> in Chapter 2 the combination of a Steffensen type method with the trapezoidal rule for approximating solutions of scalar initial value problems for first order differential equations is studied (Section 2.4). Conditions under which this method provides bilateral approximations are provided (Theorem I.2.22– I.2.25). These results were published in [70].

> in Chapter 3 an interpolation formula is used to introduce a class of numerical methods for approximating the solutions of scalar initial value problems for first order differential equations. These methods can be identified as explicit Runge-Kutta methods. Bounds for the local truncation error are determined and the convergence order and the absolute stability region are compared with those for explicit Runge-Kutta methods, which have

convergence order equal with number of stages (Sections 3.2 and 3.3). The contents of this chapter represent the object of the papers [71] and [72].

The original contributions in the second part of the thesis consist in modeling and analyzing some new contact problems. Thus,

in Chapter 6 some ideas in [95] are used. There, the Signorini contact problem for nonlinear elastic materials and static process was considered. The originality of the results in this chapter arises in the fact that the Signorini condition is replaced with the normal compliance condition with unilateral constraint. It is shown that this new problem leads to an elliptic variational inequality for the displacement field (Section 6.1). An existence and uniqueness result for the weak solution of the model is proved (Theorem II.6.1) and this result is recovered by using a penalization method (Theorem II.6.2). To this end, a penalized problem with normal compliance and infinite penetration is considered (Section 6.3). The contents of this chapter will make the object of the forthcoming paper [8].

in Chapter 7 a new quasistatic contact problem for nonlinear elastic materials is studied. The novelty consists in the fact that the contact is moeled with normal compliance and memory term. A variational formulation of the problem in the form of a history-dependent variational equation for displacement field is derived (Section 7.1). Also, the unique weak solvability of the model is proved (Theorem II.7.1) and the continuous dependence of the solution with respect to the data is provided (Theorem II.7.2). The results of this chapter will be included in [74] and the numerical results were published in [11].

in Chapter 8 a new contact problem is considered. Here the novelty consists in the fact that the material 's behavior is described with a viscoelastic constitutive law with long memory and the contact is modeled with normal compliance, memory term and unilateral constraint. A variational formulation of the problem in the form of a history-dependent variational inequality for displacement field, which involves two Volterra integral terms is provided (Section 8.1). The unique weak solvability of the problem (Theorem II.8.1) and two convergent results are proved. The first one shows the continuous dependence of the solution with respect to the data (Theorem II.8.2), The second one proves that the weak solution of the problem represents the limit of the weak solution of a contact problem with normal compliance and memory term, as the stiffness coefficient of the foundation convergences to infinity (Theorem II.8.3). The material presented in this chapter has made the object of the paper [97].

Part I

Numerical Methods for Ordinary Differential Equations

Chapter 1

Preliminaries

1.1 Ordinary Differential Equations

In the first chapter of the thesis basic notions concerning first order differential equations (ODE) are introduced and related to them notions concerning autonomous ODE, high order ODE and initial value problems are presented.

Definition 1.1. *An* ordinary differential equation of first order *is an equation of the form*

$$y' = f(x, y) \quad x \in I, \tag{1.1}$$

where $I \subseteq \mathbb{R}$ is a bounded or unbounded interval and $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function. A function $y : \mathbb{R}^n \to \mathbb{R}^n$ is called solution *of this equation if it verifies* (1.1) *for all $x \in I$.*

Definition 1.2. *For $x_0 \in I$ and $y_0 \in \mathbb{R}^n$ an additional condition of the form*

$$y(x_0) = y_0 \tag{1.2}$$

is called initial condition *and the obtained problem is called* initial value problem *or* Cauchy problem*.*

Thus, an *initial value problem* has the form

$$\begin{cases} y' = f(x, y) \quad x \in I, \\ y(x_0) = y_0. \end{cases} \tag{1.3}$$

A briefly description of linear differential equations is also provided.

1.2 Existence and Uniqueness of Solutions

In the second part of the chapter the following existence and uniqueness theorems and the following results concerning the dependence of solution on the data are mentioned.

Definition 1.3. *The function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is said to satisfy a* Lipschitz condition in its second variable *if there exists a constant L, named the* Lipschitz constant*, such that*

$$\|f(x, y) - f(x, z)\| \leq L\|y - z\| \tag{1.4}$$

for any $x \in I$, y, $z \in \mathbb{R}^n$.

Theorem 1.4. *If f satisfies a Lipschitz condition with constant L and w and z are each solutions of the equation*

$$y' = f(x, y),$$

then

$$\|w(x) - z(x)\| \leq \|w(x_0) - z(x_0)\| \exp(L(x - x_0)) \quad \forall\, x \geq x_0. \tag{1.5}$$

Definition 1.5. *The function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies a* one-sided Lipschitz condition *if there exists a constant l, named the* one-sided Lipschitz constant*, such that*

$$\langle f(x, y) - f(x, z),\, y - z\rangle \leq l\|y - z\|^2 \tag{1.6}$$

for any $x \in I$, y, $z \in \mathbb{R}^n$.

Theorem 1.6. *If f satisfies a one-sided Lipschitz condition with constant l and w and z are solutions of the equation*

$$y' = f(x, y),$$

then

$$\|w(x) - z(x)\| \leq \|w(x_0) - z(x_0)\| \exp(l(x - x_0)) \quad \forall\, x \geq x_0. \tag{1.7}$$

Definition 1.7. *The function $y : I_0 \to \mathbb{R}^n$ is called* local solution *of initial value problem* (1.3) *if $I_0 \subset I$ is a neighborhood of x_0 and y verifies* (1.3)*. If $I = I_0$ then y is called* global solution.

Theorem 1.8. (The Cauchy-Peano Theorem) *If f is continuous on $I \times \mathbb{R}^n$ then for all $y_0 \in \mathbb{R}^n$ there exists a unique local solution of the initial value problem* (1.3)*.*

Theorem 1.9. (The Cauchy-Lipschitz Theorem) *If f is continuous and satisfies a Lipschitz condition in its second variable then for all $y_0 \in \mathbb{R}^n$ there exists a unique global solution of the initial value problem* (1.3)*.*

The material mentioned in this chapter is standard and can be found in many books and surveys, e.g. [3], [13], [38], [40], [62], [81], [101]. For a comprehensive treatment of basic aspects concerning the existence and uniqueness of solution, the dependence of solution on the data or the regularity of solution the reader is referred to [12], [23], [38], [62], [81], [101].

Chapter 2

Numerical Methods for Cauchy Problems

2.1 Background

In this section the general form of a numerical method for Cauchy problems, classifications concerning the number of steps or implicitness and properties like convergence, consistency, zero-stability are introduced. Thus, the following results are provided.

A solution of Cauchy problem (1.3) is sought on the interval $I = [x_0,\, x_0 + T]$, where $T > 0$ is finite. This continuous interval is replaced by the discrete point set

$$\{x_i |\, x_i = x_{i-1} + h_i,\, i = \overline{1,\, N}\},$$

where $N \in \mathbb{N}^*$ is the number of points and h_i represents the distance between x_{i-1} and x_i. The length of the steps h_i can be constant $h_i = h$, $i = \overline{1,\, N}$ or can be changed after a prescribed rule with the requirement $\max\limits_{i=\overline{1,\, N}} h_i \le h$.

The numerical methods of the form

$$\sum_{j=0}^{k} \alpha_j y_{i+j} = h\phi_f(y_{i+k}, y_{i+k-1}, \ldots, y_i, x_i; h), \tag{2.1}$$

is considered. Here $\alpha_j \in \mathbb{R}$, $j = 0, \ldots, k$, are constants and the subscript f on the right-hand side indicates that the dependence of ϕ on $y_{i+k},\ y_{i+k-1},\ \ldots y_i,\ x_i$ is through the function $f(x, y)$.

The following classifications of (2.1) concerning number of steps and concerning implicitness are mentioned.

Definition 2.1. *The numerical method* (2.1) *for the approximation of the solution of the initial value problem* (1.3) *is called* one-step *method if for all $i > 0$ the value y_{i+1} depends only on y_i. Otherwise the method is called* multistep *method.*

Definition 2.2. *A numerical method is called* explicit *if y_{i+1} can be computed directly in terms of the previous values y_k, $k \leq i$. A method is said to be* implicit *if y_{i+1} depends implicitly on itself through f.*

The concept of *convergent scheme* is presented.

Definition 2.3. *The method defined by* (2.1) *is said to be* convergent *if for all initial values problems satisfying the hypotheses of Theorem* 1.9 *we have that*

$$\max_{0 \leq i \leq N} \|y(x_i) - y_i\| \to 0 \quad \text{as} \quad h \to 0. \tag{2.2}$$

To give necessary and sufficient conditions for convergence of the numerical method (2.1) notions like: *local truncation error, consistency, zero-stability, root-condition* and their properties are provided.

Definition 2.4. The local truncation error *of* (2.1) *at x_{i+k} is given by*

$$T_{i+k} = \sum_{j=0}^{k} \alpha_j y(x_{i+j}) - h\phi_f(y(x_{i+k}), y(x_{i+k-1}), \ldots, y(x_i), x_i; h). \tag{2.3}$$

Definition 2.5. *The method defined by* (2.1) *is said to be* consistent *if for all initial value problems satisfying the hypotheses of Theorem* 1.9 *the quantity T_{i+k} defined by* (2.3) *satisfies*

$$\lim_{h \to 0} \frac{1}{h} T_{i+k} = 0. \tag{2.4}$$

Theorem 2.6. *The method* (2.1) *is consistent if it satisfies the conditions*

$$\begin{cases} \displaystyle\sum_{j=0}^{k} \alpha_j = 0, \\ \phi_f(y(x_i), y(x_i), \ldots, y(x_i), x_i; 0) = f(x_i, y(x_i)) \cdot \big(\displaystyle\sum_{j=0}^{k} j\alpha_j\big). \end{cases} \tag{2.5}$$

Definition 2.7. *The method* (2.1) *is said to satisfy* root-condition *if all the roots of the polynomial*

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \quad \xi \in \mathbb{C}.$$

have modulus less than or equal to unity, and those with modulus equal to unity are simple.

Definition 2.8. *The method defined by* (2.1) *is said to be* zero-stable *if it verifies* root-condition.

Theorem 2.9. *The method* (2.1) *is convergent if and only if it is consistent and zero-stable.*

Also, some notions and properties of iterative methods for nonlinear algebraic equations of the form

$$y = \phi(y) \quad \phi : \mathbb{R}^n \to \mathbb{R}^n. \tag{2.6}$$

are mentioned.
The first method is the *fixed point iteration* method, which consists of generating a sequence of approximations $\{y^{(\nu)}\}$ given by

$$y^{(\nu+1)} = \phi(y^{(\nu)}) \quad \nu = 0, 1, \dots, \tag{2.7}$$

where initial guess $y^{(0)}$ is arbitrary.

The following theorem states conditions under which (2.6) has a unique solution such that the iteration generated by (2.7) converges.

Theorem 2.10. *Let ϕ satisfy the Lipschitz condition*

$$\|\phi(y) - \phi(z)\| \leq L\|y - z\| \tag{2.8}$$

for all y, $z \in \mathbb{R}^n$, where $0 \leq L < 1$. Then there exists a unique solution y^* of (2.6) *and the sequence generated by* (2.7) *converges to y^* as $\nu \to \infty$.*

The second iterative method is the *Newton iteration* method and, in general, it is applied when the hypotheses of Theorem 2.10 are not satisfied and the sequence $\{y^{(\nu)}\}$ given by (2.7) diverges. The Newton iteration applied to the system

$$F(y) = 0, \tag{2.9}$$

with $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is given by

$$y^{(\nu+1)} = y^{(\nu)} - J^{-1}(y^{(\nu)})F(y^{(\nu)}) \quad \nu = 0, 1, \dots , \tag{2.10}$$

where $J = \frac{\partial F}{\partial y}$ is the Jacobian matrix of F with respect to y.

When

$$F(y) = 0, \quad F : \mathbb{R} \to \mathbb{R} \tag{2.11}$$

the Newton iteration is given by

$$y^{(\nu+1)} = y^{(\nu)} - \frac{F(y^{(\nu)})}{F'(y^{(\nu)})} \quad \nu = 0, 1, \dots , \tag{2.12}$$

and, substituting $F(y) = y - \phi(y)$ in (2.11) the following formula

$$y^{(\nu+1)} = y^{(\nu)} - \left[1 - \phi'(y^{(\nu)})\right]^{-1}[y^{(\nu)} - \phi(y^{\nu})] \quad \nu = 0, 1, \dots \tag{2.13}$$

is obtained.

A method that does not require the calculation of the derivative of F is the *Steffensen method* given by

$$y^{(\nu+1)} = y^{(\nu)} - \frac{F^2(y^{(\nu)})}{F(y^{(\nu)} + F(y^{(\nu)})) - F(y^{(\nu)})} \quad \nu = 0, 1, \dots , \tag{2.14}$$

or, in equivalent form,

$$y^{(\nu+1)} = y^{(\nu)} - \frac{F(y^{(\nu)})}{[y^{(\nu)}, y^{(\nu)} + F(y^{(\nu)}); F]} \quad \nu = 0, 1, \dots , \tag{2.15}$$

where $[u, v; F]$ represents the *first order divided difference* of F at the points u, v defined by

$$[u, v; F] = \frac{F(u) - F(v)}{u - v}. \tag{2.16}$$

A generalized method for (2.14) and (2.15) is defined by

$$y^{(\nu+1)} = y^{(\nu)} - \frac{F(y^{(\nu)})}{[y^{(\nu)}, g(y^{(\nu)}); F]} \quad \nu = 0, 1, \dots , \tag{2.17}$$

where g is a decreasing function such that it has y^* as a fixed point, i.e. y^* verifies the equation

$$g(y) = y. \tag{2.18}$$

The following hypothesis for the functions F and g are considered.

(α) F is a continuous function given in such a way that equation (2.11) has a unique solution y^* in the bounded interval (c, d);

(β) the equations (2.11) and (2.18) are equivalent;

(γ) g is a continuous and decreasing function on $[c, d]$.

Results in the study of these kind of methods are provided in many papers, e.g., [19], [20], [25], [26], [76], [77] and [79].

The next theorem is proved in [75].

Theorem 2.11. *Assume that the functions F and g satisfy the conditions $(\alpha) - (\gamma)$ and moreover, the following conditions hold:*

(i) F is increasing and convex on $[c, d]$;

(ii) $F(y^{(0)}) < 0$;

(iii) $g(y^{(0)}) \leq d$.

Then the elements of the sequences $\{y^{(\nu)}\}$ and $\{g(y^{(\nu)})\}$ belong to the interval $[c, d]$ and, moreover, the following properties hold:

(j) the sequence $\{y^{(\nu)}\}$ defined by (2.17) is increasing and convergent;

(jj) the sequence $\{g(y^{(\nu)})\}$ is decreasing and convergent;

(jjj) $y^{(\nu)} \leq y^* \leq g(y^{(\nu)}) \quad \nu = 0, 1, \dots$;

(jv) $\lim_{\nu\to\infty} y^{(\nu)} = \lim_{\nu\to\infty} g(y^{(\nu)}) = y^*$;

(v) $|y^* - y^{(\nu)}| \leq |g(y^{(\nu)}) - y^{(\nu)}|, \quad \nu = 0, 1, \dots$.

2.2 Linear Multistep Methods

In this section a special sub-class of (2.1) is presented. This class is represented by *linear multistep methods* or a *linear k-steps methods.* The general form of these methods is given by

$$\sum_{j=0}^{k} \alpha_j y_{i+j} = h \sum_{j=0}^{k} \beta_j f_{i+j} \tag{2.19}$$

where $f_{i+j} = f(x_{i+j}, y_{i+j})$, $j = \overline{0,\, k}$, and α_j, β_j are constants which verify the conditions

$$\alpha_k = 1, \quad |\alpha_0| + |\beta_0| \neq 0. \tag{2.20}$$

The notions of convergence order, local truncation error, polynomial of stability, absolute stability and region of absolute stability are introduced.

Definition 2.12. *The* linear difference operator $\mathcal{L}$ *associated with the linear multistep method* (2.19) *is defined by*

$$\mathcal{L}[z(x); h] = \sum_{j=0}^{k} [\alpha_j z(x + jh) - h\beta_j z'(x + jh)], \tag{2.21}$$

where $z \in C^1(I)$ is an arbitrary function.

Expanding $z(x + jh)$ and $z'(x + jh)$ in Taylor series with respect to x is obtained

$$\mathcal{L}[z(x); h] = C_0 z(x) + C_1 h z'(x) + \ldots + C_q h^q z^{(q)}(x) + \ldots \tag{2.22}$$

where C_q are constants.

Definition 2.13. *The linear multistep method* (2.19) *and the associated linear difference operator $\mathcal{L}$ defined by* (2.21) *are said to be of* order p *if in* (2.22) *we have* $C_0 = C_1 = \ldots = C_p = 0, \quad C_{p+1} \neq 0$.
In this case C_{p+1} is called error constant.

Definition 2.14. *The* local truncation error (LTE) *of the method* (2.19) *at x_{i+k} denoted by T_{i+k} is defined by*

$$T_{i+k} = \mathcal{L}[y(x_i); h], \tag{2.23}$$

where $\mathcal{L}$ is the associated difference operator given by (2.21) *and y is the exact solution of initial value problem* (1.3).

Definition 2.15. *The* polynomial of stability *associated to the method* (2.19) *is defined by*

$$\pi(r, z) = \rho(r) - z\sigma(r), \tag{2.24}$$

where $z = h\lambda$ and ρ, σ are given by

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \quad \sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j, \quad \xi \in \mathbb{C} \tag{2.25}$$

Definition 2.16. *The method* (2.19) *is said to be* absolutely stable *for a given z if all the roots r_j of the stability polynomial* (2.24) *satisfy*

$$|r_j| < 1, \quad j = 1, \dots, k. \tag{2.26}$$

The set of all these points z is called region of absolute stability *for the method* (2.19) *and is denoted in general by $\mathcal{R}_A$.*

2.3 Runge-Kutta Methods

The second special sub-class of (2.1) analyzed in this chapter is represented by *Runge-Kutta methods*. The general form of these methods is given by

$$\begin{cases} y_{i+1} = y_i + h \displaystyle\sum_{t=1}^{s} b_t k_t, \\ k_t = f(x_i + c_t h,\, y_i + h \displaystyle\sum_{j=1}^{s} a_{tj} k_j), \quad t = 1, 2, \dots, s \end{cases} \tag{2.27}$$

where $b_t,\, c_t,\, a_{tj},\, t,\, j = \overline{1,\, s}$ are constants.

Using

$$k_t = f(x_i + c_t h, Y_t), \quad t = \overline{1,\, s} \tag{2.28}$$

the alternative form of (2.27) given by

$$\begin{cases} y_{i+1} = y_i + h \displaystyle\sum_{t=1}^{s} b_t f(x_i + c_t h, Y_t) \\ Y_t = y_i + h \displaystyle\sum_{j=1}^{s} a_{tj} f(x_i + c_t h, Y_j), \quad t = 1, 2, \dots, s \end{cases} \tag{2.29}$$

is provided.

The notions of stability function, absolute stability and region of absolute stability are introduced.

Definition 2.17. *The* stability function *of the method* (2.27) *is defined by*

$$R(z) = 1 + z b^T (I - zA)^{-1} e \tag{2.30}$$

or, in alternative form,

$$R(z) = \frac{\det[I - z(A - e b^T)]}{\det(I - zA)}, \tag{2.31}$$

where $e = [1, 1, \dots, 1]^T \in \mathbb{R}^s$.

The method (2.27) *is said to be* absolutely stable *for a given z if*

$$|R(z)| < 1. \tag{2.32}$$

The region of absolute stability for (2.27) *is defined by*

$$\mathcal{R}_A = \{z \in \mathbb{C} : |R(z)| < 1\}. \tag{2.33}$$

For Runge-Kutta methods a practicable way to determine the absolute stability region is *scanning technique* presented in [54]. Additional information concerning Runge-Kutta methods can be found in [17] or [36].
For various results, details and comments on the material in previous sections the following references [4], [22], [25], [44], [82], [99], [100] are provided. These papers abound in information concerning the topic of numerical analysis, including notions about numerical methods for initial value problems of ordinary differential equations and numerical methods for nonlinear algebraic equations, as well.

2.4 Application of Steffensen Method to the Trapezoidal Rule

In this section a combination of a Steffensen type method with the trapezoidal rule for approximating solutions of scalar initial value problems is studied. Thus, an implicit equation of the form

$$y = hA\phi(x, y) + \psi \tag{2.34}$$

is obtained after the application of the trapezoidal rule with constant step-size h. Here A is a constant determined by the numerical method and ψ is a known value.

Thus, the approximations y_i for the exact solution at the points x_i, $i = \overline{1, N}$ are the solutions of the equation

$$y = hA\phi(x_i, y) + \psi_i, \tag{2.35}$$

where $\psi_i = \psi_i(x_0, h, y_{i-1}, y_{i-2}, \dots, y_0)$ are known values.

For each $i = \overline{1, N}$ $F_i : [c, d] \to \mathbb{R}$ is defined by

$$F_i(y) = y - hA\phi(x_i, y) - \psi_i. \tag{2.36}$$

and equation (2.35) is rewritten in the form

$$F_i(y) = 0.$$

To approximate bilaterally the solution y_i^*, $i = \overline{1, N}$, of (2.35) the sequence $\{y_i^{(\nu)}\}_{\nu \geq 0}$ is generated by

$$y_i^{(\nu+1)} = y_i^{(\nu)} - \frac{F_i(y_i^{(\nu)})}{[y_i^{(\nu)}, g(y_i^{(\nu)}); F_i]}, \quad \nu = 0, 1, \dots \tag{2.37}$$

or, using (2.36),

$$y_i^{(\nu+1)} = \frac{hA(\phi(x_i, y_i^{(\nu)}) - y_i^{(\nu)}[y_i^{(\nu)}, g(y_i^{(\nu)}); \phi(x_i, \cdot)]) + \psi_i}{1 - hA[y_i^{(\nu)}, g(y_i^{(\nu)}); \phi(x_i, \cdot)]}. \tag{2.38}$$

The *trapezoidal rule* to integrate the initial value problem (1.3), for $n = 1$, and the Steffensen method (2.17) to solve the equation (2.35) are considered.

For any point x_i, $i = \overline{1, N}$, the bellow equation

$$y_i - \frac{h}{2}f(x_i, y_i) - \frac{h}{2}f(x_{i-1}, y_{i-1}) - y_{i-1} = 0 \tag{2.39}$$

is obtained and, in this case,

$$F_i(y) = y - \frac{h}{2} f(x_i, y) - \frac{h}{2} f(x_{i-1}, y_{i-1}) - y_{i-1}.$$

Thus, in (2.35) the values are given by $A = \frac{1}{2}$, $\phi(x_i, y) = f(x_i, y)$ and $\psi_i = \frac{h}{2} f(x_{i-1}, y_{i-1}) + y_{i-1}$, $i = \overline{1, N}$.

For simplicity only the autonomous case, i.e. $f = f(y)$, is considered and in this case equation (2.39) becomes

$$y_i - \frac{h}{2} f(y_i) - \frac{h}{2} f(y_{i-1}) - y_{i-1} = 0 \tag{2.40}$$

with

$$F_i(y) = y - \frac{h}{2} f(y) - \psi_i,$$

where $\psi_i = \frac{h}{2} f(y_{i-1}) + y_{i-1}$, $i = \overline{1, N}$.

Using the fact that

$$[u, v; F_i] = 1 - \frac{h}{2} [u, v; f] \quad \forall\, u,\, v \in [c, d], \tag{2.41}$$

and

$$[u, v, w; F_i] = -\frac{h}{2} [u, v, w; f] \quad \forall\, u,\, v,\, w \in [c, d], \tag{2.42}$$

for all $i = \overline{1, N}$, the auxiliary function g is given by

$$g(y) = \frac{\frac{h}{2}(f(y) - y[d - \varepsilon, d; f]) + \psi_i}{1 - \frac{h}{2}[d - \varepsilon, d; f]}$$

or

$$g(y) = \frac{\frac{h}{2}(f(y) - y[c, c + \varepsilon; f]) + \psi_i}{1 - \frac{h}{2}[c, c + \varepsilon; f]},$$

where ε is sufficiently small such that the exact solution y_i^* of the equation $F_i(y_i) = 0$, $i = 1, \ldots, N$, belongs to the interval $[c + \varepsilon, d - \varepsilon]$.

For each $i = \overline{1, N}$ the followings quantities

$$\psi_{\max}^i = \max\{y_k + \tfrac{h}{2} f(y_k) | k = 0, \ldots, i - 1\},$$

$$\psi_{\min}^i = \min\{y_k + \tfrac{h}{2} f(y_k) | k = 0, \ldots, i - 1\}$$

are introduced.

The following original results are proved.

Theorem 2.18. *Assume that the function f, the step-size h and the initial guesses $y_i^{(0)}$, $i = 1, \ldots, N$, satisfy the following conditions*

(*i*) $[u, v, w; f] \leq 0$ *for all* $u,\ v,\ w \in [c, d]$*;*

(ii) *($m \le [u,v;f] \le M \le 0$ for all u, $v \in [c,d]$) or ($0 \le m \le [u,v;f] \le M$ for all u, $v \in [c,d]$ and $h \le \frac{2}{M}$);*

(iii) $y_i^{(0)} - \frac{h}{2} f(y_i^{(0)}) < \psi^i_{\min}$*;*

(iv) $y_i^{(0)} M - f(y_i^{(0)}) \ge \frac{2}{h}[d(M\frac{h}{2} - 1) + \psi^i_{\max}]$.

Then the elements of the sequences $\{y_i^{(\nu)}\}_{\nu\ge0}$, $\{g(y_i^{(\nu)})\}_{\nu\ge0}$, $i = 1,\dots,N$, belong to the interval $[c,d]$ and, moreover, the following properties hold:

(j) $\{y_i^{(\nu)}\}_{\nu\ge0}$ *defined by* (2.17) *is increasing and convergent;*

(jj) $\{g(y_i^{(\nu)})\}_{\nu\ge0}$ *is decreasing and convergent;*

(jjj) $y_i^{(\nu)} \le y_i^* \le g(y_i^{(\nu)})$, $\nu = 0,1,\dots$*;*

(jv) $\lim\limits_{\nu\to\infty} y_i^{(\nu)} = \lim\limits_{\nu\to\infty} g(y_i^{(\nu)}) = y_i^*$*;*

(v) $|y_i^* - y_i^{(\nu)}| \le |g(y_i^{(\nu)}) - y_i^{(\nu)}|, \quad \nu = 0,1,\dots$.

Theorem 2.19. *Assume that the function f, the step-size h and the initial guesses $y_i^{(0)}$, $i = 1,\dots,N$, satisfy the following conditions*

(i) $[u,v,w;f] \le 0$ *for all* u, v, $w \in [c,d]$*;*

(ii) $0 \le m \le [u,v;f] \le M$, *for all* u, $v \in [c,d]$*;*

(iii) $y_i^{(0)} - \frac{h}{2} f(y_i^{(0)}) < \psi^i_{\min}$*;*

(iv) $y_i^{(0)} m - f(y_i^{(0)}) \ge \frac{2}{h}[c(m\frac{h}{2} - 1) + \psi^i_{\max}]$*;*

(v) $\frac{2}{m} \le h$.

Then the elements of the sequences $\{y_i^{(\nu)}\}_{\nu\ge0}$, $\{g(y_i^{(\nu)})\}_{\nu\ge0}$, $i = 1,\dots,N$, belong to the interval $[c,d]$ and, moreover, the following properties hold

(j) $\{y_i^{(\nu)}\}_{\nu\ge0}$ *defined by* (2.17) *is decreasing and convergent;*

(jj) $\{g(y_i^{(\nu)})\}_{\nu\ge0}$ *is increasing and convergent;*

(jjj) $g(y_i^{(\nu)}) \le y_i^* \le y_i^{(\nu)}$ $\nu = 0,1,\dots$*;*

(jv) $\lim\limits_{\nu\to\infty} y_i^{(\nu)} = \lim\limits_{\nu\to\infty} g(y_i^{(\nu)}) = y_i^*$*;*

(v) $|y_i^* - y_i^{(\nu)}| \le |g(y_i^{(\nu)}) - y_i^{(\nu)}|, \quad \nu = 0,1,\dots$.

Theorem 2.20. *Assume that the function f, the step-size h and the initial guesses $y_i^{(0)}$, $i = 1,\dots,N$, satisfy the following conditions*

(i) $[u,v,w;f] \geq 0$ *for all* u, v, $w \in [c,d]$;

(ii) *(*$m \leq [u,v;f] \leq M \leq 0$ *for all* u, $v \in [c,d]$*) or (*$0 \leq m \leq [u,v;f] \leq M$ *for all* u, $v \in [c,d]$, *and* $h \leq \frac{2}{M}$*);*

(iii) $y_i^{(0)} - \frac{h}{2} f(y_i^{(0)}) > \psi_{\max}^i$;

(iv) $y_i^{(0)} M - f(y_i^{(0)}) \leq \frac{2}{h}[c(M\frac{h}{2} - 1) + \psi_{\min}^i]$.

Then the elements of the sequences $\{y_i^{(\nu)}\}_{\nu \geq 0}$, $\{g(y_i^{(\nu)})\}_{\nu \geq 0}$, $i = 1, \ldots, N$, *belong to the interval* $[c,d]$ *and, moreover, the following properties hold*

(j) $\{y_i^{(\nu)}\}_{\nu \geq 0}$ *defined by* (2.17) *is decreasing and convergent;*

(jj) $\{g(y_i^{(\nu)})\}_{\nu \geq 0}$ *is increasing and convergent;*

(jjj) $g(y_i^{(\nu)}) \leq y_i^* \leq y_i^{(\nu)}$, $\nu = 0, 1, \ldots$;

(jv) $\lim\limits_{\nu \to \infty} y_i^{(\nu)} = \lim\limits_{\nu \to \infty} g(y_i^{(\nu)}) = y_i^*$;

(v) $|y_i^* - y_i^{(\nu)}| \leq |g(y_i^{(\nu)}) - y_i^{(\nu)}|, \quad \nu = 0, 1, \ldots$.

Theorem 2.21. *Assume that the function* f, *the step-size* h *and the initial guesses* $y_i^{(0)}$, $i = 1, \ldots, N$, *satisfy the following conditions:*

(i) $[u,v,w;f] \geq 0$ *for all* u, v, $w \in [c,d]$;

(ii) $0 \leq m \leq [u,v;f] \leq M$ *for all* u, $v \in [c,d]$;

(iii) $y_i^{(0)} - \frac{h}{2} f(y_i^{(0)}) > \psi_{\max}^i$;

(iv) $y_i^{(0)} m - f(y_i^{(0)}) \leq \frac{2}{h}[d(m\frac{h}{2} - 1) + \psi_{\min}^i]$;

(v) $\frac{2}{m} \leq h$.

Then the elements of the sequences $\{y_i^{(\nu)}\}_{\nu \geq 0}$, $\{g(y_i^{(\nu)})\}_{\nu \geq 0}$, $i = 1, \ldots, N$, *belong to the interval* $[c,d]$ *and, moreover, the following properties hold*

(j) $\{y_i^{(\nu)}\}_{\nu \geq 0}$ *defined by* (2.17) *is increasing and convergent;*

(jj) $\{g(y_i^{(\nu)})\}_{\nu \geq 0}$ *is decreasing and convergent;*

(jjj) $y_i^{(\nu)} \leq y_i^* \leq g(y_i^{(\nu)})$ $\nu = 0, 1, \ldots$;

(jv) $\lim\limits_{\nu \to \infty} y_i^{(\nu)} = \lim\limits_{\nu \to \infty} g(y_i^{(\nu)}) = y_i^*$;

(v) $|y_i^* - y_i^{(\nu)}| \leq |g(y_i^{(\nu)}) - y_i^{(\nu)}|, \quad \nu = 0, 1, \ldots$.

The results presented in this section are original and were published in [70].

2.5 Numerical example

The chapter ends with a numerical example to exemplify the results presented in the previous section.

Chapter 3

A Special Class of Runge-Kutta Methods

3.1 An Approximation Formula for Functions

In this section some remarks concerning an approximation formula for functions given in [78] are provided. This represents a generalization of some interpolation formulae defined in [103].

First of all, a $(2q+1)$-times derivable function $h : I \to \mathbb{R}$, $q \in \mathbb{N}$ is considered, where the interval I is finite and has the form $I = [x_0,\, x_0 + T]$, $T > 0$. Also, the class G of functions

$$\begin{aligned} G = \{g : g(x) \quad &= \quad h(x_0) + (x - x_0)\sum_{i=1}^{q} \alpha_i h'(x_0 + \beta_i(x - x_0)), \\ & \alpha_i,\, \beta_i \in \mathbb{R}, \quad i = \overline{1,\, q}, \quad x \in I\} \end{aligned} \tag{3.1}$$

is given, and the following problem is considered.
Find a function $\overline{g} \in G$ *such that*

$$h^{(i)}(x_0) = \overline{g}^{(i)}(x_0), \quad i = \overline{1,\, m}, \tag{3.2}$$

where $m \in \mathbb{N}^*$.

The coefficients β_i are taken as the roots of the Legendre polynomial w_q of degree q, i.e. the roots of the equation

$$w_q(x) := \frac{q!}{(2q)!}\frac{d^q}{dx^q}[x^q(x-1)^q] = 0, \tag{3.3}$$

and the coefficients α_i are given by the following formula

$$\alpha_i = \frac{(q!)^4}{[(2q)!]^2\beta_i(1-\beta_i)[w_q'(\beta_i)]^2}, \quad i = \overline{1,\, q}. \tag{3.4}$$

Next, is presented a theorem, given in [78], which shows that the above problem has a unique solution for $m = 2q$ and establishes the forms of the coefficients α_i, β_i.

Theorem 3.1. *Assume that $h : I \to \mathbb{R}$ is a $(2q+1)$-times derivable function on I. Then there exists a unique function $\overline{g} \in G$ which verifies conditions (3.2) for $m = 2q$. Moreover, the coefficients $\{\alpha_i\}_{i=1}^{q}$ are given by the formula (3.4) and $\{\beta_i\}_{i=1}^{q}$ are the roots of the equation (3.3).*

This section ends with the estimation

$$|r[h]| \leq \frac{M_{2q+1}}{(2q+1)!} \frac{2[(2q)!]^2 - [q!]^4}{[(2q)!]^2} |x - x_0|^{2q+1} \tag{3.5}$$

for the remainder given by

$$h(x) = \overline{g}(x) + r[h], \tag{3.6}$$

where

$$M_{2q+1} = \sup_{x \in I} |h^{(2q+1)}(x)|. \tag{3.7}$$

The interpolation formula is used in the rest of the chapter to introduce a class of numerical methods for approximating the solutions of scalar initial value problems for first order differential equations.

3.2 The Particular Case $q = 1$

In this section the approximation formula for the particular case $q = 1$ is used. Thus, α_1, β_1 have the values $\beta_1 = \frac{1}{2}$, $\alpha_1 = 1$, the function $\overline{g} \in G$ in Theorem 3.1 has the form

$$\overline{g}(x) = h(x_0) + (x - x_0)h'(x_0 + \tfrac{1}{2}(x - x_0)) \tag{3.8}$$

and satisfies

$$h^{(i)}(x_0) = \overline{g}^{(i)}(x_0), \quad i = \overline{0,\, 2} \tag{3.9}$$

and

$$|h(x) - \overline{g}(x)| \leq \tfrac{7}{24} M_3 |x - x_0|^3, \tag{3.10}$$

where $M_3 = \sup\limits_{x \in I} |h'''(x)|$.

Using (3.8) a class of numerical methods for approximating the solutions of an autonomous scalar initial value problem (1.3), i.e. when $f = f(y)$ and $n = 1$, is introduced. The following numerical method is obtained

$$y_{i+1} = y_i + h_i f(y_i + \tfrac{1}{2} h_i f(y_i + \tfrac{1}{2^2} h_i f(y_i + \ldots + \tfrac{1}{2^{p_0 - 1}} h_i f(y_i)) \ldots), \tag{3.11}$$

where $h_i = x_{i+1} - x_i$, $i = 0, \ldots, N-1$ represents the length of the step.
For this method the following equivalence result is proved.

Theorem 3.2. *The method* (3.11) *is equivalent with a* p_0*-stages explicit Runge-Kutta method with the Butcher array given by*

$$
\begin{array}{c|ccccccc}
 & 0 & & & & & & \\
\frac{1}{2^{p_0-1}} & \frac{1}{2^{p_0-1}} & 0 & & & & & \\
\frac{1}{2^{p_0-2}} & 0 & \frac{1}{2^{p_0-2}} & 0 & & & & \\
\vdots & & \dots & & & & & \\
\frac{1}{2^2} & 0 & 0 & 0 & \dots & \frac{1}{2^2} & 0 & \\
\frac{1}{2} & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 \\
\hline
 & 0 & 0 & 0 & \dots & 0 & 0 & 1
\end{array}
$$

For the method (3.11) and for particular cases the local truncation error, absolute–stability region, consistency, convergence order are studied. First of all, as in [83], it is supposed that

$$\|f\| < M \text{ and } \|f^{(j)}\| < \frac{L^j}{M^{j-1}} \text{ on } I, \tag{3.12}$$

where $\|f\| = \sup\{|\mathrm{f}(t)| : t \in I\}$ and M, L are positive real numbers.

The convergence order of the method (3.11) is provided in the following result.

Theorem 3.3. *The method* (3.11) *has convergence order* 2 *and the coefficient of principal local truncation error* C_3 *has the following bound*

$$\|C_3\| \le \tfrac{1}{12} M L^2. \tag{3.13}$$

The method (3.11) is a *zero-stable* method because it verifies *root-condition.* Also, since the convergence order is 2 we conclude that it satisfies the *consistency condition.* It follows that the method (3.11) represents a convergent method.

Also, the stability function and the absolute–stability region for the method (3.11) are defined.

Theorem 3.4. *The method* (3.11) *has the stability function given by*

$$R(z) = 1 + \sum_{k=1}^{p_0} \frac{1}{2^{\frac{k(k-1)}{2}}} z^k, \quad z \in \mathbb{C}. \tag{3.14}$$

The *absolute-stability region* is given by

$$\mathcal{R} = \{z \in \mathbb{C} : |1 + \sum_{k=1}^{p_0} \frac{1}{2^{\frac{k(k-1)}{2}}} z^k| < 1\}. \tag{3.15}$$

In the rest of the section two particular cases are analyzed and are compared with explicit Runge-Kutta methods, which have convergence order equal with number of stages. This part was written following original paper [71].

3.3 The Particular Case $q = 2$

In this section the approximation formula for the particular case $q = 2$ is used. Thus, the coefficients α_i, β_i have the values

$$\beta_1 = \tfrac{3-\sqrt{3}}{6},\ \beta_2 = \tfrac{3+\sqrt{3}}{6},\ \alpha_1 = \alpha_2 = \tfrac{1}{2}, \tag{3.16}$$

the function $\overline{g} \in G$ in Theorem 3.1 has the form

$$\overline{g}(x) = h(x_0) + \tfrac{1}{2}\left[h'(x_0 + \tfrac{3-\sqrt{3}}{6}(x - x_0)) + h'(x_0 + \tfrac{3+\sqrt{3}}{6}(x - x_0))\right] \tag{3.17}$$

and satisfies

$$h^{(i)}(x_0) = \overline{g}^{(i)}(x_0), \quad i = \overline{0,\ 4} \tag{3.18}$$

and

$$|h(x) - \overline{g}(x)| \leq \tfrac{71}{4320} M_5 |x - x_0|^5, \tag{3.19}$$

where $M_5 = \sup\limits_{x \in I} |h^{(5)}(x)|$.

Using for each x_i, $i = \overline{1,\ N}$ the notation

$$u^i_{qr} = y_i + \tfrac{1}{2}\beta_1^q \beta_2^r h_i [f(u^i_{q+1r}) + f(u^i_{qr+1})],$$

the following numerical method is obtained

$$\begin{aligned} y_{i+1} &= y_i + \tfrac{1}{2} h_i [f(u^i_{10}) + f(u^i_{01})], \\ u^i_{k-jj} &= y_i + \tfrac{1}{2}\beta_1^{k-j}\beta_2^j h_i [f(u^i_{k-j+1j}) + f(u^i_{k-jj+1})], \\ &\qquad j = \overline{0,\ k}, \quad k = \overline{1, p_0 - 1}, \\ u^i_{p_0-jj} &= y_i, \quad j = \overline{0,\ p_0}, \end{aligned} \tag{3.20}$$

where $h_i = x_{i+1} - x_i$, $i = \overline{0, N-1}$ represents the length of the step.

In the rest of the section the particular cases of this method for $p_0 = 2$, 3, 4 are analyzed.

For $p_0 = 2$ the method

$$\begin{aligned} y_{i+1} = y_i + \tfrac{1}{2} h_i [&f(y_i + \beta_1 h_i f(y_i)) \\ &+ f(y_i + \beta_2 h_i f(y_i))], \quad i = \overline{1,\ N-1} \end{aligned} \tag{3.21}$$

is obtained.

The convergence order of the method (3.21) is provided in the following result.

Theorem 3.5. *The method* (3.21) *has convergence order* 2 *and the coefficient of principal local truncation error* C_3 *has the following bound*

$$\|C_3\| \leq \tfrac{1}{6} M L^2. \tag{3.22}$$

The *stability function* and *absolute stability region* of the method (3.21) are provided in the following result.

Theorem 3.6. *The method* (3.21) *has the stability function given by*

$$R(z) = 1 + z + \frac{z^2}{2}, \quad z \in \mathbb{C}. \tag{3.23}$$

For $p_0 = 3$ the method

$$\begin{aligned} y_{i+1} &= y_i + \tfrac{1}{2} h_i [f(u_{10}^i) + f(u_{01}^i)], \\ u_{10}^i &= y_i + \tfrac{1}{2} \beta_1 h_i [f(u_{20}^i) + f(u_{11}^i)], \\ u_{01}^i &= y_i + \tfrac{1}{2} \beta_2 h_i [f(u_{11}^i) + f(u_{02}^i)], \\ u_{2-jj}^i &= y_i + \beta_1^{2-j} \beta_2^j h_i f(y_i), \quad j = \overline{0,\, 2}, \end{aligned} \tag{3.24}$$

is obtained.

The convergence order of the method (3.24) is provided in the following result.

Theorem 3.7. *The method* (3.24) *has the convergence order* 3 *and the coefficient of the principal local truncation error* C_4 *has the bound*

$$\|C_4\| \leq \tfrac{1}{24} M L^3.$$

The *stability function* and *absolute stability region* of the method (3.24) are provided in the following result.

Theorem 3.8. *The method* (3.24) *has the stability function given by*

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6}, \quad z \in \mathbb{C}. \tag{3.25}$$

For $p_0 = 4$ the method

$$\begin{aligned} y_{i+1} &= y_i + \tfrac{1}{2} h_i [f(u_{10}^i) + f(u_{01}^i)], \\ u_{k-jj}^i &= y_i + \tfrac{1}{2} \beta_1^{k-j} \beta_2^j h_i [f(u_{k-j+1j}^i) + f(u_{k-jj+1}^i)], \\ & \qquad j = \overline{0,\, k}, \quad k = \overline{1,\, 2}, \\ u_{3-jj}^i &= y_i + \beta_1^{3-j} \beta_2^j h_i f(y_i), \quad j = \overline{0,\, 3}, \end{aligned} \tag{3.26}$$

is obtained. This method has convergence order 4 and the stability function is given by

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}. \tag{3.27}$$

This part was written following original paper [72].

3.4 Numerical Examples

The chapter ends with a numerical examples to exemplify the results presented in the previous sections.

Part II

Modeling and Analysis of Contact Problems

Chapter 4

Preliminaries of Functional Analysis

This chapter contains some preliminaries and basic results on functional analysis which will be used in the next chapters.

4.1 Banach Spaces and Hilbert Spaces

In this section are presented some basic definitions and properties of the normed spaces, including results on Banach and Hilbert spaces. It is started with the *Banach fixed point theorem* for Banach spaces.

Theorem 4.1. (The Banach Fixed Point Theorem) *Let K be a nonempty closed subset of a Banach space $(X, \|\cdot\|_X)$. Assume that $\Lambda : K \to K$ is a* contraction*, i.e. there exists a constant $\alpha \in [0,1)$ such that*

$$\|\Lambda u - \Lambda v\|_X \leq \alpha \, \|u - v\|_X \quad \forall \, u, \, v \in K. \tag{4.1}$$

Then there exists a unique $u \in K$ such that $\Lambda u = u$.

To introduce *projection operators* the following existence and uniqueness result, named *The Projection Lemma* is provided.

Theorem 4.2. (The Projection Lemma) *Let K be a nonempty, closed, convex subset of a Hilbert space X. Then, for each $f \in X$ there exists a unique element $u \in K$ such that*

$$\|u - f\|_X = \min_{v \in K} \|v - f\|_X. \tag{4.2}$$

Using this theorem the following definition is presented.

Definition 4.3. *Let K be a nonempty, closed, convex subset of a Hilbert space X. Then, for each $f \in X$ the element u which satisfies* (4.2) *is called the* projection *of f on K and is usually denoted $\mathcal{P}_K f$. Moreover, the operator $\mathcal{P}_K : X \to K$ is called the* projection operator *onto K.*

Also, the following characterizations of the projection and *Riesz Representation Theorem* are provided.

Proposition 4.4. *Let K be a nonempty, closed, convex subset of a Hilbert space X and let $f \in X$. Then $u = \mathcal{P}_K f$ if and only if*

$$u \in K, \quad (u, v-u)_X \geq (f, v-u)_X \quad \forall\, v \in K. \tag{4.3}$$

Proposition 4.5. *Let K be a nonempty, closed, convex subset of a Hilbert space X. Then the projection operator $\mathcal{P}_K$ satisfies the following inequalities:*

$$(\mathcal{P}_K u - \mathcal{P}_K v, u - v)_X \geq 0 \quad \forall\, u,\, v \in X, \tag{4.4}$$

$$\|\mathcal{P}_K u - \mathcal{P}_K v\|_X \leq \|u - v\|_X \quad \forall\, u,\, v \in X. \tag{4.5}$$

Theorem 4.6. (The Riesz Representation Theorem) *Let $(X, (\cdot,\cdot)_X)$ be a Hilbert space and let $\ell \in X'$. Then there exists a unique $u \in X$ such that*

$$\ell(v) = (u, v)_X \quad \forall\, v \in X. \tag{4.6}$$

Moreover,

$$\|\ell\|_{X'} = \|u\|_X. \tag{4.7}$$

This introductory section ends with a particular case of the well-known *Eberlein-Smulyan Theorem.*

Theorem 4.7. *If X is a Hilbert space then any bounded sequence in X has a weakly convergent subsequence.*

Theorem 4.8. *Let X be a Hilbert space and let $\{u_n\}$ be a bounded sequence of elements in X such that each weakly convergent subsequence of $\{u_n\}$ converges weakly to the same limit $u \in X$. Then $u_n \rightharpoonup u$ in X.*

4.2 Monotone Operators

In this section are presented several results on *monotone operators* which will be useful in the study of variational inequalities.

Definition 4.9. *Let X be a space with inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$ and let $A : X \to X$ be an operator. The operator A is said to be* monotone *if*

$$(Au - Av, u - v)_X \geq 0 \quad \forall\, u,\, v \in X.$$

The operator A is strictly monotone *if*

$$(Au - Av, u - v)_X > 0 \quad \forall\, u,\, v \in X,\ u \neq v,$$

and strongly monotone *if there exists a constant $m > 0$ such that*

$$(Au - Av, u - v)_X \geq m\, \|u - v\|_X^2 \quad \forall\, u,\, v \in X. \tag{4.8}$$

The operator A is nonexpansive *if*

$$\|Au - Av\|_X \leq \|u - v\|_X \quad \forall\, u,\, v \in X$$

and Lipschitz continuous *if there exists $M > 0$ such that*

$$\|Au - Av\|_X \leq M\|u - v\|_X \quad \forall\, u,\, v \in X. \tag{4.9}$$

Finally, the operator A is hemicontinuous *if the real valued mapping*

$$\theta \mapsto (A(u + \theta v), w)_X \text{ is continuous on } \mathbb{R}, \quad \forall\, u,\, v,\, w \in X,$$

and continuous *if*

$$u_n \to u \text{ in } X \implies Au_n \to Au \text{ in } X.$$

Proposition 4.10. *Let $(X, (\cdot,\cdot)_X)$ be an inner product space and let $A : X \to X$ be a monotone hemicontinuous operator. Assume that $\{u_n\}$ is a sequence of elements in X which converges weakly to the element $u \in X$, i.e.*

$$u_n \rightharpoonup u \quad \text{in} \quad X \quad \text{as} \quad n \to \infty. \tag{4.10}$$

Moreover, assume that

$$\limsup_{n\to\infty}\, (Au_n, u_n - u)_X \leq 0. \tag{4.11}$$

Then, for all $v \in X$, the following inequality holds

$$\liminf_{n\to\infty}\, (Au_n, u_n - v)_X \geq (Au, u - v)_X. \tag{4.12}$$

Theorem 4.11. *Let X be a Hilbert space and let $A : X \to X$ be a strongly monotone Lipschitz continuous operator. Then, for each $f \in X$ there exists a unique element $u \in X$ such that $Au = f$.*

4.3 Elliptic Variational Inequalities

In this section is provided an extension of the existence and uniqueness result in Theorem 4.11. First of all, it is considered the problem of finding an element u such that

$$u \in K, \quad (Au, v - u)_X \geq (f, v - u)_X \quad \forall\, v \in K, \tag{4.13}$$

where X is a given Hilbert space, $A : X \to X$ is an operator, $K \subset X$ and $f \in X$.

Next is presented the following theorem which guarantees the existence and uniqueness of solution of *elliptic variational inequality of the first kind* (4.13).

Theorem 4.12. *Let X be a Hilbert space and let $K \subset X$ be a nonempty, closed, convex subset. Assume that $A : K \to X$ is a strongly monotone Lipschitz continuous operator, i.e. it satisfies conditions* (4.8) *and* (4.9)*. Then, for each $f \in X$ the variational inequality* (4.13) *has a unique solution.*

4.4 History-dependent Variational Inequalities

In the last section of this chapter is extended the existence and uniqueness result in Theorem 4.12 to a special class of time-dependent variational inequalities. To this end it is introduced some background of spaces of functions defined on a time interval with values in an abstract Hilbert space. Also, it is provided a fixed point result which is useful to prove the solvability of nonlinear equations and variational inequalities with history-dependent operators and *the Gronwall inequality.*

Proposition 4.13. *Let $\Lambda : C([0,T];X) \to C([0,T];X)$ be an operator which satisfies the following property: there exist $k \in [0,1)$ and $c \geq 0$ such that*

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X \leq k\,\|\eta_1(t) - \eta_2(t)\|_X + c\int_0^t \|\eta_1(s) - \eta_2(s)\|_X\,ds$$

$$\forall\,\eta_1,\,\eta_2 \in C([0,T];X), \quad t \in [0,T]. \tag{4.14}$$

Then, there exists a unique element $\eta^* \in C([0,T];X)$ such that $\Lambda\eta^* = \eta^*$.

Lemma 4.14. (The Gronwall Inequality) *Let $f,\,g \in C([0,T])$ and assume that there exists $c > 0$ such that*

$$f(t) \leq g(t) + c\int_0^t f(s)\,ds \quad \forall\,t \in [0,T]. \tag{4.15}$$

Then

$$f(t) \leq g(t) + c\int_0^t g(s)\,e^{c\,(t-s)}\,ds \quad \forall\,t \in [0,T]. \tag{4.16}$$

Moreover, if g is nondecreasing, then

$$f(t) \leq g(t)\,e^{c\,t} \quad \forall\,t \in [0,T].$$

In the rest of this section it is introduced the concept of *history-dependent quasivariational inequalities* for which it is provided an existence and uniqueness result. As in the previous section it is considered the problem of finding a function $u \in C([0,T];X)$ such that, for all $t \in [0,T]$, the inequality below holds

$$u(t) \in K, \quad (Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X$$

$$\geq (f(t), v - u(t))_X \qquad \forall\,v \in K, \tag{4.17}$$

where

$$K \text{ is a nonempty closed convex subset of } X \tag{4.18}$$

$A : X \to X$ is a strongly monotone Lipschitz continuous operator, i.e.

$$\left\{\begin{array}{l} \text{(a) There exists } m > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \geq m\,\|u_1 - u_2\|_X^2 \\ \qquad\qquad \forall\,u_1,\,u_2 \in X. \\ \text{(b) There exists } M > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \leq M\,\|u_1 - u_2\|_X \quad \forall\,u_1,\,u_2 \in X. \end{array}\right. \tag{4.19}$$

$\mathcal{S} : C([0,T];X) \to C([0,T];X)$ satisfies

$$\left\{ \begin{array}{l} \text{There exists } L_S > 0 \text{ such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \leq L_S \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \qquad\quad \forall\, u_1,\, u_2 \in C([0,T];X),\ \forall\, t \in [0,T] \end{array} \right. \tag{4.20}$$

and

$$f \in C([0,T];X). \tag{4.21}$$

Finally, it is provided the following results which guarantees the existence and uniqueness of solution of history-dependent quasivariational inequalities (4.17).

Theorem 4.15. *Let X be an Hilbert space and assume that* (4.18)–(4.21) *hold. Then, the variational inequality* (4.17) *has a unique solution $u \in C([0,T];K)$.*

Corollary 4.16. *Let X be a Hilbert space and assume that* (4.19)-(4.21) *hold. Then there exists a unique function $u \in C([0,T];X)$ such that*

$$(Au(t), v)_X + (\mathcal{S}u(t), v)_X = (f(t), v)_X \quad \forall\, v \in X, \quad \forall\, t \in [0,T]. \tag{4.22}$$

The material presented in this chapter is standard and can be found in many books of functional analysis. For more information in the field we refer the reader to the books [5], [16]. Existence, uniqueness and regularity results for nonlinear equations with monotone operators in Hilbert spaces can be found in [14], [15]. The literature in the study of elliptic variational inequalities is extensive see for instance the surveys [58], [64], [95] and the references therein. Interest in variational inequalities originates in mechanical problems, as shown in [29], where many problems in mechanics and physics are formulated in framework of variational inequalities. More recent references in the mathematical analysis of variational inequalities include [6], [48], [50], [68], [95].

Chapter 5

Modelling of Contact Problems

This chapter contains the preliminary material which is needed to introduce the mathematical models analyzed in Chapters 6-8 of the manuscript.

5.1 Function Spaces in Contact Mechanics

The aim of the first section is to provide the functional spaces with their basic properties in which the data and the unknowns belong. Thus, it is denoted by $\mathbb{R}^d$ the d-dimensional real linear space, and the symbol $\mathbb{S}^d$ is used for the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{v} &= u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \forall\, \boldsymbol{u}=(u_i),\, \boldsymbol{v}=(v_i)\in\mathbb{R}^d,\\ \boldsymbol{\sigma}\cdot\boldsymbol{\tau} &= \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \forall\, \boldsymbol{\sigma}=(\sigma_{ij}),\, \boldsymbol{\tau}=(\tau_{ij})\in\mathbb{S}^d, \end{aligned}$$

respectively. Notation $\boldsymbol{I}_d$ will represent the identity operator on $\mathbb{R}^d$ or, equivalently, the unit matrix of order d. And, as usual, the zero elements of $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $\mathbf{0}_{\mathbb{R}^d}$ and $\mathbf{0}_{\mathbb{S}^d}$, respectively.

The following spaces used in the rest of this manuscript are provided.

$C^m(\overline{\Omega})$ – the space of functions whose derivatives up to and including order m are continuous up to the boundary Γ;

$C^\infty(\overline{\Omega})$ – the space of infinitely differentiable functions up to the boundary Γ;

$C_0^\infty(\Omega)$ – the space of infinitely differentiable functions with compact support in Ω;

$L^p(\Omega)$ – the Lebesgue space of p-integrable functions on Ω, with the usual modification if $p=\infty$;

$L^p(\Gamma)$ – the Lebesgue space of p-integrable functions on Γ, with the usual modification if $p=\infty$;

$L^p(\Gamma_2)$ – the Lebesgue space of p-integrable functions on Γ_2, with the usual modification if $p=\infty$;

$L^p(\Gamma_3)$ – the Lebesgue space of p-integrable functions on Γ_3, with the usual modification if $p = \infty$;

$L^1_{loc}(\Omega)$ – the space of locally integrable functions on Ω;

$W^{m,p}(\Omega)$ – the Sobolev space of functions whose weak derivatives of orders m or less are p-integrable on Ω;

$H^m(\Omega) \equiv W^{m,2}(\Omega)$, for positive integer m.

Also, if X represents one of the above spaces the notations X^d and $X_s^{d\times d}$ are used for the spaces

$$X^d = \{\ \boldsymbol{x} = (x_i)\ :\ x_i \in X,\ 1 \le i \le d\ \},$$
$$X_s^{d\times d} = \{\ \boldsymbol{x} = (x_{ij})\ :\ x_{ij} = x_{ji} \in X,\ 1 \le i,\, j \le d\ \}$$

and, in particular, are used the spaces

$$L^2(\Omega)^d = \big\{\,\boldsymbol{v} = (v_i)\ :\ v_i \in L^2(\Omega),\ 1 \le i \le d\,\big\}, \tag{5.1}$$
$$Q = L^2(\Omega)_s^{d\times d} = \big\{\,\boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ 1 \le i,\, j \le d\,\big\}. \tag{5.2}$$

These are Hilbert spaces with the canonical inner products

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_\Omega u_i\, v_i\, dx = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\, \tau_{ij}\, dx = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms denoted by $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_Q$, respectively.

The *deformation* operator $\boldsymbol{\varepsilon} : H^1(\Omega)^d \to Q$ is defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \tfrac{1}{2}\,(u_{i,j} + u_{j,i}).$$

The quantity $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the *linearized* (or *small*) *strain tensor* associated with the displacement $\boldsymbol{u}$.

Besides the function spaces introduced above specific function spaces for the displacement and the stress field are provided. Displacements are sought in the space

$$H^1(\Omega)^d = \big\{\,\boldsymbol{v} = (v_i) : v_i \in H^1(\Omega),\ 1 \le i \le d\,\big\}$$

or its subspaces or subsets, depending on prescribed boundary conditions. The space $H^1(\Omega)^d$ is a Hilbert space with the canonical inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{H^1(\Omega)^d} = \int_\Omega (u_i v_i + u_{i,j} v_{i,j})\ dx,$$

and the corresponding norm

$$\|\boldsymbol{v}\|_{H^1(\Omega)^d} = \Big(\int_\Omega (v_i\, v_i + v_{i,j}\, v_{i,j})\ dx\Big)^{1/2}. \tag{5.3}$$

For an element $\boldsymbol{v} \in H^1(\Omega)^d$ its *trace* $\gamma : H^1(\Omega)^d \to L^2(\Gamma)^d$ and its *normal* component and *tangential* part on the boundary are defined. Thus,

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}. \tag{5.4}$$

In the study of contact problems, the subspace V of $H^1(\Omega)^d$ given by

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1 \,\} \tag{5.5}$$

is frequently used. Here, the condition " $\boldsymbol{v} = \boldsymbol{0}$ a.e. on Γ_1" is understood in the sense of trace, i.e. $\gamma \boldsymbol{v} = \boldsymbol{0}$ a.e. on Γ_1. On the space V the inner product $(\cdot\,,\cdot)_V$

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \tag{5.6}$$

and its induced norm

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \tag{5.7}$$

are defined. It is showed that there exists a positive constant c_0, depending on Ω, Γ_1, and Γ_3, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0 \, \|\boldsymbol{v}\|_V \quad \forall \, \boldsymbol{v} \in V. \tag{5.8}$$

To define a function space for stress fields which is useful in the study of contact problems the definition of the *divergence* of a regular tensor field is extended. Thus, the concept of the *weak divergence* is introduced directly.

Definition 5.1. *Let $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\boldsymbol{w} = (w_i)$ be such that $\sigma_{ij} = \sigma_{ji} \in L^1_{\mathrm{loc}}(\Omega)$, $w_i \in L^1_{\mathrm{loc}}(\Omega)$, for all $1 \le i,\ j \le d$. Then $\boldsymbol{w}$ is called a* weak divergence *of $\boldsymbol{\sigma}$ if*

$$\int_\Omega \sigma_{ij}\varphi_{i,j}\, dx = - \int_\Omega w_i\, \varphi_i \, dx \quad \forall \, \boldsymbol{\varphi} = (\varphi_i) \in C_0^\infty(\Omega)^d.$$

Using the above definition is defined the space

$$Q_1 = \{\, \boldsymbol{\tau} \in Q : \operatorname{Div} \boldsymbol{\tau} \in L^2(\Omega)^d \,\}, \tag{5.9}$$

which is a Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_{L^2(\Omega)^d}$$

and the associated norm $\|\cdot\|_{Q_1}$.

This section ends with the definitions of *normal* component and the *tangential* part of the stress field $\boldsymbol{\sigma}$ on the boundary

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu} \tag{5.10}$$

and Green's formula

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \forall \, \boldsymbol{v} \in H^1(\Omega)^d. \tag{5.11}$$

5.2 Physical Setting and Constitutive Laws

In this section the physical setting of contact process is provided. Also, some constitutive laws are presented.

First of all, it is assumed that a deformable body occupies, in the reference configuration, an open bounded connected set $\Omega \subset \mathbb{R}^d$ with boundary Γ, composed of three sets $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$, with the mutually disjoint relatively open sets Γ_1, Γ_2 and Γ_3. The body is clamped on Γ_1. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and volume forces of density $\boldsymbol{f}_0$ act in Ω. In the reference configuration the body is in contact on Γ_3 with an obstacle, the so-called *foundation*.

The general elastic constitutive law

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{5.12}$$

is considered, where $\mathcal{F}$ is the *elasticity operator*, assumed to be nonlinear. Also, $\mathcal{F}$ depends on the location of the point and the short-hand notation $\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ for $\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}))$ is used.

It is assumed that the operator $\mathcal{F}$ satisfies the following conditions

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\
\qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1,\, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\
\qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} \, \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1,\, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^d}) \text{ belongs to } Q.
\end{array}\right. \tag{5.13}$$

Some particular cases for (5.12), when $d = 3$, are provided, e.g.,

$$\mathcal{F}(\boldsymbol{\varepsilon}) = 2\,\mu\,\boldsymbol{\varepsilon} + \lambda\, tr(\boldsymbol{\varepsilon})\, \boldsymbol{I}_3,$$

where $\lambda > 0$, $\mu > 0$ are the *Lamé coefficients* and $tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ denotes the trace of the tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$,

$$tr(\boldsymbol{\varepsilon}(\boldsymbol{u})) = \varepsilon_{ii}(\boldsymbol{u}).$$

Also, the general *viscoelastic* constitutive law with *long memory*

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{R}(t - s,\, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds, \tag{5.14}$$

is considered, where the *elasticity operator* $\mathcal{F}$ and the *relaxation operator* $\mathcal{R}$ depend on the location of the point. Also, the operator $\mathcal{F}$ satisfies condition (5.13) and the relaxation operator $\mathcal{R}$ is such that

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{R}\colon \Omega \times [0,T] \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) } \mathcal{R}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}) = (r_{ijkl}(\boldsymbol{x}, t)\varepsilon_{kl}) \text{ for all } \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^d, \\
\qquad t \in [0,T], \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) } r_{ijkl} = r_{jikl} = r_{klij} \in C([0,T]; L^\infty(\Omega)), \\
\qquad 1 \le i, j, k, l \le d.
\end{array}\right. \tag{5.15}$$

Using (5.15) the constitutive law (5.14) can be written in the form

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds. \tag{5.16}$$

At the end of the section the space of fourth order tensor fields $\mathbf{Q}_\infty$ is defined by

$$\mathbf{Q}_\infty = \{\, \mathcal{E} = (e_{ijkl}) \ : \ e_{ijkl} = e_{jikl} = e_{klij} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d \,\}.$$

More information concerning the elastic and viscoelastic constitutive laws can be found in [21, 27, 28, 34, 39, 59, 80, 89, 93, 102]. Experimental background and elements of surface physics which justify some of the contact and frictional boundary conditions can be found in [49, 67].

5.3 Contact Conditions

In the last section of this chapter the balance equation and the boundary conditions are presented. It is assumed that the volume forces and surface tractions do not depend on time and the inertia of the mechanical system is negligible. In other words, the acceleration term is negligible in the equation of motion and, therefore, the balance equation for the stress field is

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \mathbf{0}. \tag{5.17}$$

Equation (5.17) is called the *equation of equilibrium.* It is satisfied in $\Omega \times (0,T)$ if the process is time dependent and in Ω if it is time independent. Here Div is the divergence operator, that is $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$, $\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$. Equation (5.17) shows that the applied external force $\boldsymbol{f}_0$ is fully balanced by the internal forces that are represented by $-\operatorname{Div}\boldsymbol{\sigma}$ and is derived from the principle of momentum conservation.

It is assumed that the body is held fixed on Γ_1 and, therefore,

$$\boldsymbol{u} = \mathbf{0}, \tag{5.18}$$

which represents the *displacement boundary condition.* It is satisfied in $\Gamma_1 \times (0,T)$ if the process is time dependent and in Γ_1 if it is time independent. Also, it is supposed that known tractions of density $\boldsymbol{f}_2$ act on the portion Γ_2 thus,

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2. \tag{5.19}$$

This condition is called the *traction boundary condition.* It is satisfied in $\Gamma_2 \times (0,T)$ if the process is time dependent and in Γ_2 if it is time independent.

Finally, the various boundary conditions on the contact surface Γ_3 are described. These are divided naturally into the conditions in the normal direction, called *contact conditions* or *contact laws*, and those in the tangential direction, called also *frictional conditions* or *friction laws.* First of all, the *normal compliance contact condition* is mentioned. This condition describes a deformable foundation. It assigns a reactive normal pressure that depends on the interpenetration of the asperities on the body's surface and those of the foundation. A general expression for the normal reactive pressure is

$$-\sigma_\nu = p(u_\nu), \tag{5.20}$$

where p is a nonnegative prescribed function which vanishes for negative argument. Condition (5.20) shows that the pressure exerts by the foundation on the body depends on the penetration.

The following examples of the normal compliance function p are provided:

$$p(r) = c_\nu r^+, \tag{5.21}$$

where $c_\nu > 0$ is the surface *stiffness coefficient*, and $r^+ = \max\{r, 0\}$ denotes the positive part of r;

$$p(r) = \begin{cases} c_\nu r^+ & \text{if } r \leq \alpha, \\ c_\nu \alpha & \text{if } r > \alpha, \end{cases} \tag{5.22}$$

where α is a positive coefficient related to the wear and hardness of the surface. In this case the contact condition (5.20) means that when the penetration is too large, i.e., when it exceeds α, the obstacle offers no additional resistance to penetration.

The *Signorini contact condition* is mentioned. This condition represents an idealization of the normal compliance in which the foundation is assumed to be perfectly rigid. Thus, the complementarity form of Signorini contact condition

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu u_\nu = 0 \tag{5.23}$$

is mentioned.

Also, the *normal compliance condition with unilateral constraint* is given by

$$\begin{cases} u_\nu \leq g, \quad \sigma_\nu + p(u_\nu) \leq 0, \\ (u_\nu - g)(\sigma_\nu + p(u_\nu)) = 0, \end{cases} \tag{5.24}$$

where $g > 0$. This condition is satisfied in $\Gamma_3 \times (0, T)$ if the process is time dependent and in Γ_3 if it is time independent. It can be interpreted physically in the following way: the foundation is assumed to be made of a hard material covered by a thin layer of a soft material with thickness g; the soft material is deformable and allows penetration, which is modeled with normal compliance; the hard material is rigid and, therefore, it does not allow penetration. It follows from above that the foundation has an elastic-rigid behavior; the elastic behavior is given by the layer of the soft material, and the rigid behavior is given by the hard material.

Taking into account that in various situations the reaction of the foundation at the moment t depends on the history of the penetration and, therefore, it cannot be determinate as a function of the current value $u_\nu(t)$, it is assumed that the normal stress satisfies a condition of the form

$$-\sigma_\nu(t) = \int_0^t b(t-s)\, u_\nu^+(s)\, ds, \tag{5.25}$$

in which b represents a given function, the so-called *surface memory function.*

In the tangential direction only *frictionless* condition is considered, i.e.,

$$\boldsymbol{\sigma}_\tau = \mathbf{0}. \tag{5.26}$$

The normal compliance contact condition was first introduced in [67] and since then used in many publications, see, e.g., [49, 51, 52, 60] and references therein. The term *normal compliance* was first introduced in [51, 52]. The Signorini condition was first introduced in [90] and then used in many papers, e.g., [89] and references. Condition (5.24) was first introduced in [45] and conditions of the form (5.25) were considered in [63] in the study of a lumped model with contact and friction. More details and information on the friction laws can be found in the books [39], [89], [93], [95].

Chapter 6

Analysis of a Static Contact Problem

In this chapter is studied a frictionless contact problem for nonlinear elastic materials. The process is static and the contact is described with normal compliance and unilateral constraint.

6.1 Problem Statement

In this section is presented the classical formulation of the problem and it is listed the assumptions on the data. Thus, it is considered the following problem.
Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega, \tag{6.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \mathbf{0} \quad \text{in} \quad \Omega, \tag{6.2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma_1, \tag{6.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{6.4}$$

$$\left.\begin{array}{l} u_\nu \le g, \quad \sigma_\nu + p(u_\nu) \le 0 \\ (\sigma_\nu + p(u_\nu))(u_\nu - g) = 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{6.5}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on} \quad \Gamma_3, \tag{6.6}$$

where the elasticity operator $\mathcal{F}$ satisfies condition (5.13), the body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d \tag{6.7}$$

and the normal compliance function p is such that

$$\begin{cases} \text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \leq L_p \, |r_1 - r_2| \\ \qquad\quad \forall \, r_1, \, r_2 \in \mathbb{R}, \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))(r_1 - r_2) \geq 0 \\ \qquad\quad \forall \, r_1, \, r_2 \in \mathbb{R}, \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \\ \qquad \text{for any } r \in \mathbb{R}. \\ \text{(e) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \leq 0, \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{6.8}$$

To derive the variational formulation of this problem it is introduced the set of admissible displacements U by

$$U = \{ \, \boldsymbol{v} \in V : \, v_\nu \leq g \ \text{ a.e. on } \Gamma_3 \, \} \tag{6.9}$$

and using the Riesz representation Theorem (4.6) it is defined the element $\boldsymbol{f} \in V$ by equality

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v} \, da \quad \forall \, \boldsymbol{v} \in V \tag{6.10}$$

and the operator $P : V \to V$ by equality

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu da \quad \forall \, \boldsymbol{v} \in V. \tag{6.11}$$

Using the material presented above it is provided the variational formulation of the problem $\mathcal{P}$,

Problem $\mathcal{P}^V$. *Find a displacement field* $\boldsymbol{u} \in U$ *such that*

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + (P\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \, \boldsymbol{v} \in U \tag{6.12}$$

which represents an elliptic variational inequality of the first kind for the displacement field.

6.2 Existence and Uniqueness

In this section it is proved an existence and uniqueness result for the weak solution, using arguments of elliptic variational inequalities.

Theorem 6.1. *Assume* (5.13)*,* (6.7) *and* (6.8) *hold. Then there exists a unique solution* $\boldsymbol{u} \in U$ *to Problem* $\mathcal{P}^V$*.*

6.3 Penalization

In this section the result (6.1) is recovered by using a penalization method. For simplicity, it is assumed that the function p does not depend on $\boldsymbol{x} \in \Gamma_3$, i.e. it is considered the homogeneous case. In this case assumption (6.8) is written as follows:

$$\begin{cases} \text{(a) } p : \mathbb{R} \to \mathbb{R}_+, \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2|, \quad \forall\, r_1,\, r_2 \in \mathbb{R}, \\ \text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \ge 0, \quad \forall\, r_1,\, r_2 \in \mathbb{R}, \\ \text{(d) } p(r) = 0 \text{ for all } r < 0. \end{cases} \tag{6.13}$$

It is introduced the function q which satisfies

$$\begin{cases} \text{(a) } q :\ [g, +\infty] \to \mathbb{R}_+, \\ \text{(b) There exists } L_q > 0 \text{ such that} \\ \qquad |q(r_1) - q(r_2)| \le L_q |r_1 - r_2|, \quad \forall\, r_1,\, r_2 \ge g, \\ \text{(c) } (q(r_1) - q(r_2))(r_1 - r_2) > 0, \quad \forall\, r_1,\, r_2 \ge g, \quad r_1 \ne r_2, \\ \text{(d) } q(g) = 0 \end{cases} \tag{6.14}$$

and for $\mu > 0$ is defined the function $p_\mu : \mathbb{R} \to \mathbb{R}$ by

$$p_\mu(r) = \begin{cases} p(r) & \text{if } \ r \le g, \\ \frac{1}{\mu}\, q(r) + p(g) & \text{if } \ r > g. \end{cases} \tag{6.15}$$

The function p_μ satisfies condition (6.13), i.e.

$$\begin{cases} \text{(a) } p_\mu : \mathbb{R} \to \mathbb{R}_+, \\ \text{(b) There exists } L_\mu > 0 \text{ such that} \\ \qquad |p_\mu(r_1) - p_\mu(r_2)| \le L_\mu |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}, \\ \text{(c) } (p_\mu(r_1) - p_\mu(r_2))(r_1 - r_2) \ge 0 \quad \forall\, r_1,\, r_2 \in \mathbb{R}, \\ \text{(d) } p_\mu(r) = 0 \text{ for all } r < 0. \end{cases} \tag{6.16}$$

With these notation, it is considered the following contact problem.

Problem $\mathcal{P}_\mu$. *Find a displacement field* $\boldsymbol{u}_\mu : \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}_\mu : \Omega \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}_\mu = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu) \qquad \text{in } \Omega, \tag{6.17}$$

$$\mathrm{Div}\, \boldsymbol{\sigma}_\mu + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{6.18}$$

$$\boldsymbol{u}_\mu = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{6.19}$$

$$\boldsymbol{\sigma}_\mu \boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{6.20}$$

$$-\sigma_{\mu\nu} = p_\mu(u_{\mu\nu}) \qquad \text{on } \Gamma_3, \tag{6.21}$$

$$\boldsymbol{\sigma}_{\mu\tau} = \boldsymbol{0} \qquad \text{on } \Gamma_3. \tag{6.22}$$

The difference between problems $\mathcal{P}$ and $\mathcal{P}_\mu$ arises in the fact that here the contact condition with normal compliance and unilateral constraint (6.5) is replaced with the contact condition with normal compliance (6.21). In this condition μ represents a penalization parameter which may be interpreted as a deformability of the foundation, and then $\frac{1}{\mu}$ is the surface stiffness coefficient. Indeed, when μ is smaller the reaction force of the foundation to penetration is larger and so the same force will result in a smaller penetration, which means that the foundation is less deformable. When μ is larger the reaction force of the foundation to penetration is smaller, and so the foundation is less stiff and more deformable.

Using arguments similar to those used in the study of Problem $\mathcal{P}$ the following variational formulation of Problem $\mathcal{P}_\mu$ is obtained.

Problem $\mathcal{P}_\mu^V$. *Find a displacement field $\boldsymbol{u}_\mu \in V$ such that*

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P_\mu \boldsymbol{u}_\mu, \boldsymbol{v})_V = (\boldsymbol{f}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V. \tag{6.23}$$

Here the operator $P_\mu : V \to V$ is defined by

$$(P_\mu \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\mu(u_\nu) v_\nu \, da \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{6.24}$$

In the rest of the section the following existence, uniqueness and convergence result is proved.

Theorem 6.2. *Assume* (5.13), (6.7) *and* (6.16). *Then:*

1) *For each $\mu > 0$ there exists a unique solution $\boldsymbol{u}_\mu \in V$ to Problem $\mathcal{P}_\mu^V$.*

2) *There exists a unique solution $\boldsymbol{u} \in U$ to Problem $\mathcal{P}^V$.*

3) *The solution $\boldsymbol{u}_\mu$ of Problem $\mathcal{P}_\mu^V$ converges strongly to the solution $\boldsymbol{u}$ of Problem $\mathcal{P}^V$, i.e.*

$$\boldsymbol{u}_\mu \to \boldsymbol{u} \quad in \quad V \quad as \quad \mu \to 0. \tag{6.25}$$

The proof of Theorem 6.2 is carried out in several steps and it is based on arguments similar to those used in [95] in the study of the Signorini contact problem. Original contributions in this proof consist to handle the nonlinear term involving the operator P. This leads to some new mathematical difficulties.

The convergence result in Theorem 6.2 is extended to the weak solution of the corresponding contact problems $\mathcal{P}$ and $\mathcal{P}_\mu$. Thus, it is showed that

$$\|\boldsymbol{\sigma}_\mu - \boldsymbol{\sigma}\|_{Q_1} \to 0 \qquad \text{as} \quad \mu \to 0. \tag{6.26}$$

In addition to the mathematical interest in the convergence result (6.25), (6.26), it is important from the mechanical point of view, since it shows that the weak solution of the elastic contact problem with normal compliance and unilateral constraint may be approached as closely as one wishes by the solution of the elastic contact problem with normal compliance with a sufficiently small deformability coefficient.

6.4 Numerical Solution

In this section the numerical solution of the contact problem $\mathcal{P}$ is provided.

6.5 Numerical Example

The chapter ends with numerical simulations which validate the convergence result described in the penalization method.

In writing this chapter some ideas in [95] were used. There, the Signorini contact problem for nonlinear elastic materials was considered and its unique solvability was proved by using a penalization method. The originality in this chapter arises in the fact that the results in [95] are extended to the case when the Signorini condition is replaced with the normal compliance condition with unilateral constraint. The contents of this chapter will make the object of the forthcoming paper [8].

Chapter 7

Analysis of a Quasistatic Contact Problem

In this chapter is studied a frictionless contact problem for nonlinear elastic materials. In contrast with the problem in Chapter 6, the process is quasistatic and the contact is modeled with normal compliance and memory term.

7.1 Problem Statement

In this section is presented the classical formulation of the problem and the assumptions on the data are listed . Thus, the following problem is considered.

Problem $\mathcal{Q}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that, for each* $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \qquad \text{in} \quad \Omega, \tag{7.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{7.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{7.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on} \quad \Gamma_2, \tag{7.4}$$

$$\sigma_\nu(t) + p(u_\nu(t)) + \int_0^t b(t-s)u_\nu^+(s)ds = 0 \qquad \text{on} \quad \Gamma_3, \tag{7.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3, \tag{7.6}$$

where the elasticity operator $\mathcal{F}$ satisfies condition (5.13), the body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in C([0,T];L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C([0,T];L^2(\Gamma_2)^d), \tag{7.7}$$

the normal compliance function p verifies (6.8) and the surface memory function satisfies

$$b \in C([0,T];L^\infty(\Gamma_3)). \tag{7.8}$$

The novelty consists in condition (7.5) which shows that the contact follows a normal compliance condition with memory term. At the moment t, the reaction of the foundation depends both on the current value of the penetration (represented by the term $p(u_\nu(t))$) as well as on the history of the penetration (represented by the integral term).

As in Chapter 6 it is provided the variational formulation of the problem $\mathcal{Q}$,

Problem $\mathcal{Q}^V$. *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$ *such that, for all* $t \in [0,T]$, *the equality below holds*

$$\begin{aligned}
&\boldsymbol{u}(t) \in V, \quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (p(u_\nu(t)), v_\nu)_{L^2(\Gamma_3)} \\
&+\Big(\int_0^t b(t-s)u_\nu^+(s)\,ds, v_\nu\Big)_{L^2(\Gamma_3)} \\
&= (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2)^d} \quad \forall \boldsymbol{v} \in V
\end{aligned} \tag{7.9}$$

which represents a nonlinear variational equation for the displacement field involving a Volterra type integral term.

7.2 Existence and Uniqueness

In this section the following existence and uniqueness result is presented.

Theorem 7.1. *Assume that* (5.13), (6.8), (7.7) *and* (7.8) *hold. Then, Problem* $\mathcal{Q}^V$ *has a unique solution which satisfies* $\boldsymbol{u} \in C([0,T];V)$.

Theorem 7.1 provides the unique weak solvability of Problem $\mathcal{Q}$. Moreover, the regularity of the weak solution is $\boldsymbol{u} \in C([0,T];V)$, $\boldsymbol{\sigma} \in C([0,T];Q_1)$.

7.3 A Continuous Dependence Result

In this section the dependence of the solution with respect to the data is studied and a convergence result is proved. To this end, for each $\rho > 0$, p_ρ, b_ρ, $\boldsymbol{f}_{0\rho}$ and $\boldsymbol{f}_{2\rho}$ are considered perturbations of p, b, $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ which satisfy conditions (6.8), (7.8), (7.7), and the following variational problem is considered.

Problem $\mathcal{Q}_\rho^V$. *Find a displacement field* $\boldsymbol{u}_\rho : [0,T] \to V$ *such that, for all* $t \in [0,T]$, *the equality below holds*:

$$\begin{aligned}
&\boldsymbol{u}_\rho(t) \in V, \quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (p_\rho(u_{\rho\nu}(t)), v_\nu)_{L^2(\Gamma_3)} \\
&+\Big(\int_0^t b_\rho(t-s)u_{\rho\nu}^+(s)\,ds, v_\nu\Big)_{L^2(\Gamma_3)} \\
&= (\boldsymbol{f}_{0\rho}(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_{2\rho}(t), \boldsymbol{v})_{L^2(\Gamma_2)^d} \quad \forall \boldsymbol{v} \in V,
\end{aligned} \tag{7.10}$$

where $u_{\rho\nu}$ *represents the normal component of the function* $\boldsymbol{u}_\rho$.

Using the following assumptions on the data

$$b_\rho \to b \quad \text{in} \quad C([0,T]; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0, \tag{7.11}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \quad C([0,T]; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0, \tag{7.12}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \quad C([0,T]; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0. \tag{7.13}$$

$$\left\{\begin{array}{l} \text{There exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \beta \in \mathbb{R}_+ \text{ such that} \\ \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \leq G(\rho)(|r| + \beta) \\ \qquad \forall\, r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for each } \rho > 0, \\ \text{(b) } \lim\limits_{\rho \to 0} G(\rho) = 0. \end{array}\right. \tag{7.14}$$

the following convergence result is proved.

Theorem 7.2. *Under assumptions* (7.11)–(7.14) *the solution* $\boldsymbol{u}_\rho$ *of Problem* $\mathcal{Q}_\rho^V$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{Q}^V$*, i.e.*

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad \textit{in} \quad C([0,T]; V) \quad \textit{as} \quad \rho \to 0. \tag{7.15}$$

The convergence result in Theorem 7.2 is extended to the corresponding stress function, i.e.

$$\boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad \text{in} \quad C([0,T]; Q_1) \quad \text{as } \rho \to 0. \tag{7.16}$$

In addition to the mathematical interest in the convergence result (7.15), (7.16), it is of importance from mechanical point of view, since it states that the weak solution of problem (7.1)–(7.6) depends continuously on the normal compliance function, the surface memory function and the densities of body forces and surface tractions.

7.4 Numerical Examples

The chapter ends with numerical simulations, for one-dimensional and bi-dimensional examples, which validate the convergence result. The results of this chapter are original and will be included in [74]. The numerical results were published in [11].

Chapter 8

Analysis of a Viscoelastic Contact Problem

In this chapter a frictionless contact problem for nonlinear viscoelastic materials is studied. In contrast with the problems in Chapter 6 and Chapter 7, the process is quasistatic, the material behavior is described with a viscoelastic constitutive law with long memory and the contact is modeled with normal compliance, memory term and unilateral constraint.

8.1 Problem Statement

The chapter begins with the classical formulation of the problem and the assumptions on the data. Thus, the following problem is considered.

Problem $\mathcal{M}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that, for each* $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \qquad \text{in} \quad \Omega, \tag{8.1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{8.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{8.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on} \quad \Gamma_2, \tag{8.4}$$

$$\left.\begin{array}{c} u_\nu(t) \le g, \\ \sigma_\nu(t) + p(u_\nu(t)) + \displaystyle\int_0^t b(t-s)u_\nu^+(s)ds \le 0, \\ (u_\nu(t) - g)\Big(\sigma_\nu(t) + p(u_\nu(t)) \\ + \displaystyle\int_0^t b(t-s)u_\nu^+(s)ds\Big) = 0 \end{array}\right\} \qquad \text{on} \quad \Gamma_3, \tag{8.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3, \tag{8.6}$$

where the elasticity operator $\mathcal{F}$ verifies (5.13), the relaxation operator $\mathcal{R}$ satisfies (5.15), $g > 0$

is a given bound for the normal displacement, the surface memory function b has the regularity (7.8), the normal compliance function p verifies

$$\left\{\begin{array}{l}\text{(a) } p:\mathbb{R}\to\mathbb{R}_+,\\ \text{(b) There exists } L_p>0 \text{ such that}\\ \qquad |p(r_1)-p(r_2)|\le L_p|r_1-r_2| \quad \forall\, r_1,\, r_2\in\mathbb{R},\\ \text{(c) } (p(r_1)-p(r_2))(r_1-r_2)\ge 0 \quad \forall\, r_1,\, r_2\in\mathbb{R},\\ \text{(d) } p(r)=0 \ \text{ for all } r<0.\end{array}\right. \tag{8.7}$$

and the body forces and the traction forces have the regularity (7.7).

Using similar arguments as in Chapter 6 and Chapter 7 the following variational formulation of Problem $\mathcal{M}$ is derived.

Problem $\mathcal{M}^V$. *Find a displacement field $\boldsymbol{u}:[0,T]\to V$ such that the inequality below holds, for all $t\in[0,T]$:*

$$\begin{aligned}&\boldsymbol{u}(t)\in U,\quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q\\ &\quad+\Big(\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)_Q\\ &\quad+(p(u_\nu(t)),v_\nu-u_\nu(t))_{L^2(\Gamma_3)}\\ &\quad+\Big(\int_0^t b(t-s)u_\nu^+(s)\,ds,v_\nu-u_\nu(t)\Big)_{L^2(\Gamma_3)}\\ &\quad\ge(\boldsymbol{f}_0(t),\boldsymbol{v}-\boldsymbol{u}(t))_{L^2(\Omega)^d}\\ &\qquad+(\boldsymbol{f}_2(t),\boldsymbol{v}-\boldsymbol{u}(t))_{L^2(\Gamma_2)^d}\quad\forall\,\boldsymbol{v}\in U,\end{aligned} \tag{8.8}$$

which represents an evolutionary variational inequality for the displacement field which involves two Volterra integral terms.

8.2 Existence and Uniqueness

In the study of the problem $\mathcal{M}^V$ the following existence and uniqueness result is presented.

Theorem 8.1. *Assume that* (5.13), (5.15), (7.7), (7.8) *and* (8.7) *hold. Then, Problem $\mathcal{M}^V$ has a unique solution which satisfies $\boldsymbol{u}\in C([0,T];V)$.*

Theorem 8.1 provides the unique weak solvability of Problem $\mathcal{M}$. Moreover, the regularity of the weak solution is $\boldsymbol{u}\in C([0,T];V)$, $\boldsymbol{\sigma}\in C([0,T];Q_1)$.

8.3 A First Convergence Result

In this section the dependence of the solution of Problem $\mathcal{M}^V$ with respect to perturbations of the data is studied. To this end, the following variational problem is considered.

Problem $\mathcal{M}_\rho^V$. *Find a displacement field* $\boldsymbol{u}_\rho : [0,T] \to V$ *such that, for all* $t \in [0,T]$*, the inequality below holds :*

$$
\begin{aligned}
&\boldsymbol{u}_\rho(t) \in U, \quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\
&\quad + \Big(\int_0^t \mathcal{R}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) \Big)_Q \\
&\quad\quad + (p_\rho(u_{\rho\nu}(t)), v_\nu - u_{\rho\nu}(t))_{L^2(\Gamma_3)} \\
&\quad\quad\quad + \Big(\int_0^t b_\rho(t-s) u_{\rho\nu}^+(s)\, ds, v_\nu - u_{\rho\nu}(t) \Big)_{L^2(\Gamma_3)} \\
&\quad\quad\quad \geq (\boldsymbol{f}_{0\rho}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_{L^2(\Omega)^d} \\
&\quad\quad\quad\quad + (\boldsymbol{f}_{2\rho}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_{L^2(\Gamma_2)^d} \quad \forall \boldsymbol{v} \in U,
\end{aligned}
\tag{8.9}
$$

where, for each $\rho > 0$, $\mathcal{R}_\rho$, p_ρ, b_ρ, $\boldsymbol{f}_{0\rho}$ and $\boldsymbol{f}_{2\rho}$ are perturbations of $\mathcal{R}$, p, b, $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ which satisfy conditions (5.15), (8.7), (7.8) and (7.7), respectively.

Using the following assumptions:

$$\mathcal{R}_\rho \to \mathcal{R} \quad \text{in} \quad C([0,T]; \mathbf{Q}_\infty) \quad \text{as} \quad \rho \to 0, \tag{8.10}$$
$$b_\rho \to b \quad \text{in} \quad C([0,T]; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0, \tag{8.11}$$
$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \quad C([0,T]; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0, \tag{8.12}$$
$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \quad C([0,T]; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0, \tag{8.13}$$

and the functions p_ρ and p verify (7.14), the following convergence result is provided.

Theorem 8.2. *Under assumptions* (7.14), (8.10)–(8.13)*, the solution* $\boldsymbol{u}_\rho$ *of Problem* $\mathcal{M}_\rho^V$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{M}^V$*, i.e.*

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad \textit{in} \quad C([0,T]; V) \quad \textit{as} \quad \rho \to 0. \tag{8.14}$$

The convergence result in Theorem 8.2 is extended to the corresponding stress functions, i.e.

$$\boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \text{ in } C([0,T]; Q_1) \text{ as } \rho \to 0. \tag{8.15}$$

In addition to the mathematical interest in the convergence result (8.14), (8.15), it is of importance from mechanical point of view, since it states that the weak solution of problem (8.1)–(8.6) depends continuously on the relaxation operator, the normal compliance function, the surface memory function and the densities of body forces and surface tractions.

8.4 A Second Convergence Result

A convergence result in the study of Problem $\mathcal{M}$, based on the penalization of the unilateral constraint is provided in this section. To this end, the following contact problem is considered.

Problem $\mathcal{M}_\mu$. *Find a displacement field $\boldsymbol{u}_\mu : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}_\mu : \Omega \times [0,T] \to \mathbb{S}^d$ such that, for all $t \in [0,T]$,*

$$\boldsymbol{\sigma}_\mu(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\, ds \qquad \text{in} \quad \Omega, \tag{8.16}$$

$$\operatorname{Div} \boldsymbol{\sigma}_\mu(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{8.17}$$

$$\boldsymbol{u}_\mu(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{8.18}$$

$$\boldsymbol{\sigma}_\mu(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on} \quad \Gamma_2, \tag{8.19}$$

$$-\sigma_{\mu\nu}(t) = p_\mu(u_{\mu\nu}(t)) + \int_0^t b(t-s)\, u_{\mu\nu}^+(s)\, ds \qquad \text{on} \quad \Gamma_3, \tag{8.20}$$

$$\boldsymbol{\sigma}_{\mu\tau}(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3, \tag{8.21}$$

where $p_\mu : \mathbb{R} \to \mathbb{R}$ defined by (6.15) satisfies condition (6.16).

The difference between problems $\mathcal{M}$ and $\mathcal{M}_\mu$ arises in the fact that here the contact condition with normal compliance, memory term and unilateral constraint (8.5) is replaced with the contact condition with normal compliance and memory term (8.20). In this condition μ represents a penalization parameter, as discussed in Chapter 6.

Using similar arguments to those used in the study of Problem $\mathcal{M}$ the following variational formulation of Problem $\mathcal{M}_\mu$ is obtained.

Problem $\mathcal{M}_\mu^V$. *Find a displacement field $\boldsymbol{u}_\mu : [0,T] \to V$ such that the equality below holds, for all $t \in [0,T]$:*

$$\begin{aligned}
&(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \Big(\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) \Big)_Q \\
&\qquad + (p_\mu(u_{\mu\nu}(t)), v_\nu)_{L^2(\Gamma_3)} + \Big(\int_0^t b(t-s) u_{\mu\nu}^+(s)\, ds, v_\nu \Big)_{L^2(\Gamma_3)} \\
&\qquad = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2)^d} \quad \forall\, \boldsymbol{v} \in V.
\end{aligned} \tag{8.22}$$

In the rest of the section the following existence, uniqueness and convergence result is proved.

Theorem 8.3. *Assume that* (5.13), (5.15), (7.8), (7.7), (8.7) *and* (6.14) *hold. Then:*

1) *For each $\mu > 0$ Problem $\mathcal{M}_\mu^V$ has a unique solution which satisfies $\boldsymbol{u}_\mu \in C([0,T]; V)$.*

2) *The solution $\boldsymbol{u}_\mu$ of the Problem $\mathcal{M}_\mu^V$ converges to the solution $\boldsymbol{u}$ of the Problem $\mathcal{M}^V$, that is*

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \to 0 \tag{8.23}$$

as $\mu \to 0$, for all $t \in [0,T]$.

The convergence result in Theorem 8.3 is extended to the weak solution of the corresponding contact problems $\mathcal{M}$ and $\mathcal{M}_\mu$. Thus, it is showed that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_{Q_1} \to 0 \qquad \text{as} \quad \mu \to 0. \tag{8.24}$$

In addition to the mathematical interest in the convergence result (8.23), (8.24), it is important from the mechanical point of view, since it shows that the weak solution of the viscoelastic contact problem with normal compliance memory term and unilateral constraint may be approached as closely as one wishes by the solution of the viscoelastic contact problem with normal compliance and memory term, with a sufficiently small deformability coefficient.

8.5 Numerical Example

The chapter ends with numerical simulations which validate the convergence results obtained in Theorem 8.2 and Theorem 8.3.

The material presented in the sections of this chapter is original and has made the object of our paper [97]. Other references on quasistatic contact problems involving viscoelastic materials with long memory include [85, 86, 87].

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