Contents

Introduction and a description of the contents

In linear optimization, the important steps were made by the simplex method, published by Dantzig in 1947, and the duality theorem, given by Gale, Kuhn and Tucker in 1951 (see [50]). Then, Fenchel [49], Brøndsted [29], Moreau [66, 67] and Rockafellar [72, 73] investigated in their works the theory of convex functions, conjugate functions and duality in convex optimization. The convex analysis in finite-dimensional spaces was studied by Borwein and Lewis [9], Hiriart-Urruty and Lemaréchal [53, 54, 55] and Rockafellar [71], while, the infinite-dimensional case was studied by Ekeland and Temam [46], Rockafellar [70] and Zălinescu [80].

To solve an optimization problem one can attach to it a dual problem. The most used approaches in the literature are Fenchel and Lagrange duality. For the primal and dual problems *weak duality* holds, which means that the optimal objective value of the dual is less than or equal to the optimal objective value of the primal problem. In duality theory, finding regularity conditions to assure *strong duality* represents an important problem. The strong duality is the case when the optimal objective values of the two problems are equal and the dual has an optimal solution.

In this work one tried to extend and generalize the existing results from the literature giving new regularity conditions by using epigraphs and ε-subdifferentials.

In the first chapter one presented some preliminary notions and results which are well-known from the many books and monographies.

In the second chapter one presented different regularity conditions that equivalently characterize various ε-duality gap statements (with $\varepsilon \geq 0$) for constrained optimization problems and their Lagrange, Fenchel and Fenchel-Lagrange duals in separated locally convex spaces, respectively. Between a primal problem (P) and its dual problem (D) one always has *weak duality*, i.e. $v(P) \geq v(D)$. When $v(P) = v(D)$ one says that there is *zero duality gap* between (P) and (D) and if (D) has moreover an optimal solution, the situation is called *strong duality*. If $v(P) - v(D) \leq \varepsilon$, with $\varepsilon \geq 0$, one has ε-*duality gap* for (P) and (D). If one of these situations holds for (P_{x^*}) and (D_{x^*}) for all $x^* \in X^*$, it will be called *stable*. These regularity conditions are formulated by using epigraphs and ε-subdifferentials. When $\varepsilon = 0$ one rediscovers recent results on stable strong and total duality and zero duality gap from the literature.

Motivated by recent results on stable strong and total duality for constrained convex optimization problems in [19, 18, 48, 47, 61] and the ones on zero duality gap in [59, 60] one introduces in this chapter several regularity conditions which characterize ε-duality gap statements (with $\varepsilon \geq 0$) for a constrained optimization problem and its Lagrange, Fenchel and Fenchel-Lagrange dual problems, respectively. One extends many of the results in the mentioned papers, which are recovered as special cases when $\varepsilon = 0$, delivering thus generalizations of the classical *Farkas-Minkowski* and *basic* constraint qualifications. Moreover some statements in [19, 18, 12, 59, 60], which arise from our results in the special case $\varepsilon = 0$, are extended by removing convexity and topological hypotheses, while various assertions from [59, 60] are improved by working in locally convex spaces instead of Banach spaces and removing the continuity and nonempty domain interior assumptions of the involved functions. Also, in this chapter one presents results concerning ε-optimality conditions and ε-Farkas-type. The author's contributions are presented in the following theorems 2.1.4, 2.1.10, 2.1.12, 2.1.15, 2.1.16, 2.1.19, 2.1.21, 2.1.22, 2.1.27, 2.1.29, 2.1.31, 2.1.32, 2.1.36, 2.1.40, 2.1.42, 2.1.43, 2.1.49, 2.1.52, 2.1.54, 2.2.1, 2.2.4, 2.2.8, 2.2.9, 2.2.10, 2.2.12, 2.2.14, 2.2.15, 2.2.19, 2.2.22, 2.2.23, 2.2.27, 2.2.29, 2.2.30, 2.2.34, 2.2.38, 2.2.41; corollaries: 2.1.6, 2.1.8, 2.1.11, 2.1.13, 2.1.18, 2.1.20, 2.1.25, 2.1.28, 2.1.30, 2.1.34, 2.1.37, 2.1.41, 2.1.45, 2.1.46, 2.1.47, 2.1.50, 2.1.51, 2.1.53, 2.2.7, 2.2.37; remarks: 2.1.5, 2.1.9, 2.1.14, 2.1.17, 2.1.23, 2.1.24, 2.1.33, 2.1.38, 2.1.44, 2.2.2, 2.2.5, 2.2.6, 2.2.11, 2.2.13, 2.2.16, 2.2.17, 2.2.20, 2.2.21, 2.2.24, 2.2.25, 2.2.28, 2.2.31, 2.2.32, 2.2.35, 2.2.36, 2.2.39, 2.2.40, 2.2.42 and Lemma 2.1.3. Some of these results can be found in [7].

The third chapter is devoted to present different regularity conditions that equivalently characterize ε-duality gap statements for optimization problems consisting of the sum of a function and the precomposition of cone-increasing function with a vector function. Farkas-type results for inequality systems involving convex functions using approaches based on conjugate duality were given in [26, 27]. Then, these results were extended at convex problems involving composed convex functions in [22]. The regularity conditions given in this part are formulated by using epigraphs and ε-subdifferentials. When $\varepsilon = 0$ one rediscovers recent results on stable strong and total duality and zero duality gap from the literature. Also, there are given ε-optimality conditions and ε-Farkas-type results using the results presented along of the chapter. In order to characterize the solutions of an optimization problem involving composed convex functions, it is important to provide a formula with ε-subdifferentials. If the reader is interested in more such results he/she can consult papers like [11, 22, 24, 33, 34, 35, 64]. The author's contributions are presented in the following theorems: 3.1.2, 3.1.8, 3.1.15, 3.1.19, 3.1.24, 3.1.25, 3.1.27, 3.1.29, 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.5, 3.2.6, 3.3.1, 3.3.6, 3.3.8, 3.3.13, 3.4.1, 3.4.2, 3.4.5, 3.4.6, 3.4.8, 3.4.9, 3.4.10, 3.4.11; corollaries: 3.1.4, 3.1.6, 3.1.10, 3.1.12, 3.1.17, 3.1.18, 3.1.20, 3.1.21, 3.3.2, 3.3.9; remarks: 3.1.3, 3.1.7, 3.1.9, 3.1.13, 3.1.14, 3.1.16, 3.1.22, 3.1.23, 3.1.26, 3.1.28, 3.3.3, 3.3.4, 3.3.5, 3.3.7, 3.3.10, 3.3.11, 3.3.12, 3.3.14, 3.3.15, 3.4.3, 3.4.4, 3.4.7. Some of these results can be found in [6].

In the fourth chapter one dealt with a modern research area, namely, entropy optimization, which has various backgrounds: mathematicians, physycists, engineers, etc. One gave some generalizations for the existing problems concerning usual entropy optimization and one catalogued them into five cases. For each problem from every case one attached the Lagrange and Fenchel-Lagrange duals, then one gave regularity conditions that assure the strong duality. Also, one presented results concerning optimality conditions for each case. The author's contributions are presented in the following theorems: 4.2.1, 4.2.3, 4.2.4, 4.2.5, 4.2.7, 4.2.8, 4.2.9, 4.2.11, 4.2.12, 4.2.13, 4.2.15, 4.2.16, 4.2.17, 4.2.19, 4.2.20; corollaries: 4.2.2, 4.2.6, 4.2.10, 4.2.14, 4.2.18 and Proposition 4.1.1.These results can be found in [8].

Motivated by [1, 2, 3], in the fifth chapter one attached to a given optimization problem, an η-*approximated problem* and one obtained results concerning relations between the solutions of those problems. The mathematical programming problems were studied by many authors (see [63, 71]). The solutions of optimization problems had been given by means of the Lagrange multipliers and the saddle points of the Lagrange function associated to the optimization problem. In order to give weaker conditions for solving the optimization problems, various classes of generalized convex functions have been introduced. An example is given by the invex functions, introduced and used first by Hanson [52] and Craven [36]. Later, Antczak gave a new method for solving a nonlinear mathematical programming problem and was called *η-approximated method*. It consists in a construction of an *η-approximated problem* by modifying both the objective and constraint functions of the primary problem at an arbitrary, but fixed, point $\overline{x}$. Also, one obtained results concerning relations between the solutions of the mentioned problems and the saddle points of the Lagrangian functions attached to them. The author's contributions are presented in the following theorems: 5.2.1, 5.2.6, 5.2.7, 5.2.8, 5.3.1, 5.3.2, 5.3.3, 5.3.4; examples: 5.2.2, 5.2.3, 5.2.5 and Remark 5.2.4. Most of these results can be found in [5].

Keywords

Optimization problem, conjugate functions, conjugate duality, ε-duality gap, constraint qualifications, Lagrange dual problem, Fenchel dual problem, Fenchel-Lagrange dual problem, optimality conditions, ε-Farkas-type results, composed convex functions, entropy optimization, optimal solutions, saddle point, (ε, η)-saddle point, $(1, 2)$-η-approximated optimization problem.

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Chapter 1

Preliminary notions and results

In this chapter one presents some basic notions and results in convex analysis, concerning to sets and functions, which one needs and uses in this work. These notions can be found in many books and monographes such as [9, 12, 80].

Chapter 2

Characterizations of ε-duality gap statements for constrained optimization problems

For this part one considers two separated locally convex vector spaces X and Y and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$ respectively. Let the nonempty closed convex cone $C \subseteq Y$ and its dual cone C^*.

Let U be a nonempty subset of X and $h : X \to Y^\bullet$ a proper vector function. Denote $\mathcal{A} = \{x \in U : h(x) \in -C\}$ and assume this set non-empty. For a proper function $f : X \to \overline{\mathbb{R}}$ fulfilling $\mathcal{A} \cap dom(f) \neq \emptyset$ consider the optimization problem

$$\inf_{x \in \mathcal{A}} f(x). \tag{P}$$

One denotes by $v(P)$ the optimal objective value of the optimization problem (P). In the following one will write min(max) instead of inf(sup) when the corresponding infimum (supremum) is attained.

For $x^* \in X^*$ one also considers the linearly perturbed optimization problem

$$\inf_{x \in \mathcal{A}} \left[f(x) + \langle x^*, x\rangle\right]. \tag{P_{x^*}}$$

The Fenchel dual problem for the problem (P_{x^*}) is

$$\sup_{\beta \in X^*} \left\{-f^*(\beta) - \sigma_U(-x^* - \beta)\right\}. \tag{$D^F_{x^*}$}$$

To (P_{x^*}) one can attach the Lagrange dual problem

$$\sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + \langle x^*, x \rangle + (\lambda h)(x)], \tag{$D^L_{x^*}$}$$

which can be equivalently written as

$$\sup_{\lambda \in C^*} -(f + (\lambda h))^*_U(-x^*). \tag{$D^L_{x^*}$}$$

For a $\lambda \in C^*$, the inner minimization problem that appears in the first formulation of $(D^L_{x^*})$ can be rewritten as

$$\inf_{x \in X} [f(x) + \langle x^*, x \rangle + \delta_U(x) + (\lambda h)(x)].$$

To this problem one can attach different Fenchel type dual problems, obtaining via $(D^L_{x^*})$ some Fenchel-Lagrange type dual problems to (P_{x^*}). The name Fenchel-Lagrange is given to the folowing dual problems because they are thus "combinations" of the classical Fenchel and Lagrange dual problems. Keeping together δ_U and (λh) one gets the following Fenchel-Lagrange type dual problem to (P_{x^*})

$$\sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \left\{ -f^*(\beta) - (\lambda h)^*_U(-x^* - \beta) \right\}. \tag{$\overline{D}_{x^*}$}$$

When f and δ_U are put together, one can obtain

$$\sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \left\{ -f^*_U(\beta) - (\lambda h)^*(-x^* - \beta) \right\}. \tag{$\widetilde{D}_{x^*}$}$$

When f, (λh) and δ_U are separated, the following Fenchel-Lagrange-type dual problem to (P_{x^*}) is obtained

$$\sup_{\substack{\lambda \in C^*, \\ \beta, \alpha \in X^*}} \left\{ -f^*(\beta) - (\lambda h)^*(\alpha) - \sigma_U(-x^* - \alpha - \beta) \right\} \tag{D_{x^*}}$$

When $x^* = 0$ these duals to (P) are denoted simply by $(D^L), (D^F)(\overline{D}), (\widetilde{D})$ and (D), respectively.

2.1 $\varepsilon-$duality gap statements involving epigraphs for Lagrange, Fenchel and Fenchel-Lagrange duality

Motivated by the characterizations of the stable strong duality from [18, 19] one gives in this section several equivalent representations of different instances of ε-duality gap for

(P) and its duals by means of epigraphs. Inspired by [59, 60] one also gives regularity conditions which characterize ε-duality gap statements for (P) and its duals by means of functions $h^{\diamond}$ and $h_U^{\diamond}$ defined as $h^{\diamond}, h_U^{\diamond} : X^* \to \overline{\mathbb{R}}$ by $h^{\diamond}(x^*) = \inf_{\lambda \in C^*} (\lambda h)^*(x^*)$ and $h_U^{\diamond}(x^*) = \inf_{\lambda \in C^*} (\lambda h)_U^*(x^*)$, for $x^* \in X^*$. From the definitions it follows that $\cup_{\lambda \in C^*} epi(\lambda h)^* \subseteq epi(h^{\diamond})$, respectively $\cup_{\lambda \in C^*} epi(\lambda h)_U^* \subseteq epi(h_U^{\diamond})$. Moreover, in this section, one gives some ε-Farkas-type results obtained from the regularity conditions presented.

2.1.1 Lagrange duality

One gives the results concerning Lagrange duality.

Theorem 2.1.4 *(H.-V. Boncea, S.-M. Grad, [7]) Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi\,(f + \delta_{\mathcal{A}})^* \subseteq \bigcup_{\lambda \in C^*} epi(f + (\lambda h))_U^* - (0, \varepsilon) \qquad (RCE^L)$$

holds if and only if for all $x^* \in X^*$ there exists $\overline{\lambda} \in C^*$ such that

$$v(P_{x^*}) \leq -\left(f + (\overline{\lambda} h)\right)_U^* (-x^*) + \varepsilon. \qquad (2.1.1)$$

Remark 2.1.5 *The quantity in the right-hand side of* (2.1.1) *is not necessarily $v(D_{x^*}^L) + \varepsilon$, as the suprema in $(D_{x^*}^L)$ are not shown to be attained at $\overline{\lambda}$. Though,* (2.1.1) *implies $v(P_{x^*}) \leq v(D_{x^*}^L) + \varepsilon$.*

If we take $\varepsilon = 0$ in Theorem 2.1.4 we obtain the next result.

Corollary 2.1.6 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$. The condition*

$$epi\,(f + \delta_{\mathcal{A}})^* = \bigcup_{\lambda \in C^*} epi(f + (\lambda h))_U^*$$

holds if and only if for all $x^* \in X^*$ there exists $\overline{\lambda} \in C^*$ such that

$$v(P_{x^*}) = -\left(f + (\overline{\lambda} h)\right)_U^* (-x^*).$$

If we take $f(x) = 0$ for all $x \in X$, (RCE^L) becomes

$$epi(\sigma_{\mathcal{A}}) \subseteq \bigcup_{\lambda \in C^*} epi(\lambda h)_U^* - (0, \varepsilon) \qquad (RCE_0^L)$$

Corollary 2.1.8 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition (RCE_0^L) holds if and only if for each $x^* \in X^*$ there exits $\overline{\lambda} \in C^*$ such that*

$$\inf_{x \in \mathcal{A}} \langle x^*, x \rangle \leq -(\lambda h)_U^*(-x^*) + \varepsilon. \tag{2.1.2}$$

Remark 2.1.9 *If we take $\varepsilon = 0$ in Corollary 2.1.8 we get equality in both the condition (RCE_0^L) and the relation* (2.1.2).

Theorem 2.1.10 *(H.-V. Boncea, S.-M. Grad, [7]) Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi(f + \delta_{\mathcal{A}})^* \subseteq epi \inf_{\lambda \in C^*} (f + \lambda h)_U^* - (0, \varepsilon) \tag{RCI^L}$$

holds if and only if there is stable $\varepsilon-$duality gap for the problems (P) and (D^L), i.e. one has ε-duality gap for the pair of problems (P_{x^*}) and $(D_{x^*}^L)$ for all $x^* \in X^*$.

If we take $\varepsilon = 0$ in Theorem 2.1.10 we get the following result.

Corollary 2.1.11 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$. The condition*

$$epi(f + \delta_{\mathcal{A}})^* = epi \inf_{\lambda \in C^*} (f + \lambda h)_U^*$$

holds if and only if there is stable zero duality gap for the problems (P) and (D^L), i.e. one has zero duality gap for the pair of problems (P_{x^*}) and $(D_{x^*}^L)$ for all $x^* \in X^*$.

We consider the folowing regularity condition for f and $\mathcal{A}$:

$$epi(f + \delta_{\mathcal{A}})^* \subseteq epi(f^* \Box h_U^\diamond) - (0, \varepsilon). \tag{$\overline{RCI}$}$$

Theorem 2.1.12 *Let $\varepsilon \geq 0$. Let the condition $0 \in sqri(dom(f) - dom(h) \cap U)$ holds. The set A and the function f satisfy the condition $(\overline{RCI})$ if and only if there is stable $\varepsilon-$duality gap for the problems (P) and (D^L), i.e. one has $\varepsilon-$duality gap for the pair of problems (P_{x^*}) and $(D_{x^*}^L)$, for all $x^* \in X^*$.*

If we take $\varepsilon = 0$ in Theorem 2.1.12 we obtain the following result.

Corollary 2.1.13 *Let the condition $0 \in sqri(dom(f) - dom(h) \cap U)$ holds. The set A and the function f satisfy the condition*

$$epi(f + \delta_{\mathcal{A}})^* = epi(f^* \Box h_U^\diamond)$$

if and only if there is stable zero duality gap for the problems (P) and (D^L), i.e. one has zero duality gap for the pair of problems (P_{x^*}) and $(D_{x^*}^L)$, for all $x^* \in X^*$.

Using (RCE^L) one can obtain the following ε-Farkas-type result.

Theorem 2.1.15 *(i) Suppose that (RCE^L) holds. If $f(x) + \langle x^*, x\rangle \geq \varepsilon/2$ for all $x \in X$ then there exists $\overline{\lambda} \in C^*$ such that $\left(f + (\overline{\lambda}h)\right)^*_U(-x^*) \leq \varepsilon/2$,*

(ii) If there exists $\overline{\lambda} \in C^*$ such that $\left(f + (\overline{\lambda}h)\right)^*_U(-x^*) \leq -\varepsilon/2$, then $f(x) + \langle x^*, x\rangle \geq \varepsilon/2$ for all $x \in X$.

2.1.2 Fenchel duality

The results concerning Fenchel duality follow like in the case of Lagrange duality.

Theorem 2.1.16 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi\,(f + \delta_{\mathcal{A}})^* \subseteq epi(f^*) + epi(\sigma_U) - (0, \varepsilon) \qquad (RCE^F)$$

holds if and only if for all $x^* \in X^*$ there exists $\overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq -f^*(\overline{\beta}) - \sigma_U(-x^* - \overline{\beta}) + \varepsilon. \qquad (2.1.3)$$

We consider the folowing regularity condition for f and $\mathcal{A}$:

$$epi(f + \delta_{\mathcal{A}})^* \subseteq epi(f^* \Box \sigma_U) - (0, \varepsilon). \qquad (RCI^F)$$

Theorem 2.1.19 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition (RCI^F) holds if and only if and there is stable $\varepsilon-$duality gap for the problems (P) and (D^F), i.e. one has $\varepsilon-$duality gap for the pair of problems (P_{x^*}) and $(D^F_{x^*})$ for all $x^* \in X^*$.*

From (RCE^F) we can obtain the following ε-Farkas-type result.

Theorem 2.1.21 *(i) Suppose that (RCE^F) holds. If $f(x) + \langle x^*, x\rangle \geq \varepsilon/2$ for all $x \in X$ then there exists $\overline{\beta} \in X^*$ such that $f^*(\overline{\beta}) + \sigma_U(-x^* - \overline{\beta}) \leq \varepsilon/2$,*

(ii) If there exists $\overline{\beta} \in X^*$ such that $f^*(\overline{\beta}) + \sigma_U(-x^* - \overline{\beta}) \leq -\varepsilon/2$, then $f(x) + \langle x^*, x\rangle \geq \varepsilon/2$ for all $x \in X$.

2.1.3 Fenchel-Lagrange duality

In this subsection one will give regularity conditions using epigraphs for all the three types of Fenchel-Lagrange duals.

Fenchel-Lagrange dual of type I $(\overline{D}_{x^*})$

One starts with the results for the first type of Fenchel-Lagrange duality.

Theorem 2.1.22 *(H.-V. Boncea, S.-M. Grad, [7]) Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi\,(f + \delta_{\mathcal{A}})^* \subseteq \bigcup_{\lambda \in C^*} (epi(f^*) + epi(\lambda h)^*_U) - (0, \varepsilon) \tag{$\overline{RCE}$}$$

holds if and only if for all $x^* \in X^*$ there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq -f^*(\overline{\beta}) - (\overline{\lambda} h)^*_U(-x^* - \overline{\beta}) + \varepsilon. \tag{2.1.4}$$

Remark 2.1.24 *If we take $f(x) = 0$, the condition $(\overline{RCE})$ becomes (RCE^L_0) and we rediscover Corollary 2.1.8.*

Theorem 2.1.27 *(H.-V. Boncea, S.-M. Grad, [7]) Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi(f + \delta_{\mathcal{A}})^* \subseteq epi(f^* \square h^\diamond_U) - (0, \varepsilon) \tag{$\overline{RCI}$}$$

holds if and only if there is stable $\varepsilon-$duality gap for the problems (P) and $(\overline{D})$, i.e. one has $\varepsilon-$duality gap for the pair of problems (P_{x^*}) and $(\overline{D}_{x^*})$ for all $x^* \in X^*$.

Theorem 2.1.29 *(H.-V. Boncea, S.-M. Grad, [7]) Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi\,(f + \delta_{\mathcal{A}})^* \subseteq epi(f^*) + epi(h^\diamond_U) - (0, \varepsilon) \tag{$\overline{RCP}$}$$

holds if and only if for all $x^* \in X^*$ there exists $\overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq \sup_{\lambda \in C^*} \left\{ -f^*(\overline{\beta}) - (\lambda h)^*_U(-x^* - \overline{\beta}) \right\} + \varepsilon.$$

From $(\overline{RCE})$ one can obtain the following ε-Farkas-type result.

Theorem 2.1.31 *(i) Suppose that $(\overline{RCE})$ holds. If $f(x) + \langle x^*, x\rangle \geq \varepsilon/2$ for all $x \in X$ then there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that $f^*(\overline{\beta}) + (\overline{\lambda} h)^*_U(-x^* - \overline{\beta}) \leq \varepsilon/2$,*

(ii) If there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that $f^*(\overline{\beta}) + (\overline{\lambda} h)^*_U(-x^* - \overline{\beta}) \leq -\varepsilon/2$, then $f(x) + \langle x^*, x\rangle \geq \varepsilon/2$ for all $x \in X$.

Fenchel-Lagrange dual of type II $(\widetilde{D}_{x^*})$

Now are given the results concerning the second type of Fenchel-Lagrange duality.

Theorem 2.1.32 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi\,(f+\delta_{\mathcal{A}})^* \subseteq \bigcup_{\lambda \in C^*} (epi(f_U^*) + epi(\lambda h)^*) - (0,\varepsilon) \qquad (\widetilde{RCE})$$

holds if and only if for all $x^* \in X^*$ there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq -f_U^*(\overline{\beta}) - (\overline{\lambda} h)^*(-x^* - \overline{\beta}) + \varepsilon. \qquad (2.1.5)$$

Theorem 2.1.36 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi(f+\delta_{\mathcal{A}})^* \subseteq epi(f_U^* \Box h^\diamond) - (0,\varepsilon) \qquad (\widetilde{RCI})$$

holds if and only if there is stable $\varepsilon-$duality gap for the problems (P) and $(\widetilde{D})$, i.e. one has $\varepsilon-$duality gap for the pair of problems (P_{x^*}) and $(\widetilde{D}_{x^*})$ for all $x^* \in X^*$.

Theorem 2.1.40 *Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi\,(f+\delta_{\mathcal{A}})^* \subseteq epi(f_U^*) + epi(h^\diamond) - (0,\varepsilon) \qquad (\widetilde{RCP})$$

holds if and only if for all $x^* \in X^*$ there exists $\overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq \sup_{\lambda \in C^*} \left\{ -f_U^*(\overline{\beta}) - (\lambda h)^*(-x^* - \overline{\beta}) \right\} + \varepsilon.$$

From $(\widetilde{RCE})$ one can obtain the following ε-Farkas-type result.

Theorem 2.1.42 *(i) Suppose that $(\widetilde{RCE})$ holds. If $f(x) + \langle x^*, x \rangle \geq \varepsilon/2$ for all $x \in X$ then there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that $f_U^*(\overline{\beta}) + (\overline{\lambda} h)^*(-x^* - \overline{\beta}) \leq \varepsilon/2$,*

(ii) If there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that $f_U^*(\overline{\beta}) + (\overline{\lambda} h)^*(-x^* - \overline{\beta}) \leq -\varepsilon/2$, then $f(x) + \langle x^*, x \rangle \geq \varepsilon/2$ for all $x \in X$.

Fenchel-Lagrange dual of type III (D_{x^*})

The results concerning the third type of Fenchel-Lagrange duality follow.

Theorem 2.1.43 *(H.-V. Boncea, S.-M. Grad, [7]) Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. Then the condition*

$$epi\,(f+\delta_{\mathcal{A}})^* \subseteq \bigcup_{\lambda \in C^*} (epi(f^*) + epi(\lambda h)^* + epi(\sigma_U)) - (0,\varepsilon) \qquad (RCE)$$

holds if and only if for all $x^* \in X^*$ there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq -f^*(\overline{\beta}) - (\overline{\lambda}h)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) + \varepsilon. \tag{2.1.6}$$

Theorem 2.1.49 *(H.-V. Boncea, S.-M. Grad, [7]) Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi(f + \delta_{\mathcal{A}})^* \subseteq epi(f^* \square h^\diamond \square \sigma_U) - (0, \varepsilon) \tag{RCI}$$

holds if and only if and there is stable $\varepsilon-$duality gap for the problems (P) and (D), i.e. one has $\varepsilon-$duality gap for the pair of problems (P_{x^*}) and (D_{x^*}) for all $x^* \in X^*$.

Theorem 2.1.52 *(H.-V. Boncea, S.-M. Grad, [7]) Let $f : X \to \overline{\mathbb{R}}$ be proper and fulfilling $A \cap dom(f) \neq \emptyset$ and $\varepsilon \geq 0$. The condition*

$$epi\,(f + \delta_{\mathcal{A}})^* \subseteq epi(f^*) + epi(h^\diamond) + epi(\sigma_U) - (0, \varepsilon) \tag{RCP}$$

holds if and only if for all $x^* \in X^*$ there exist $\overline{\alpha}, \overline{\beta} \in X^*$ such that

$$v(P_{x^*}) \leq \sup_{\lambda \in C^*} \left\{ -f^*(\overline{\beta}) - (\lambda h)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) \right\} + \varepsilon.$$

From (RCE) one can obtain the following ε-Farkas-type result.

Theorem 2.1.54 *(i) Suppose that (RCE) holds. If $f(x) + \langle x^*, x \rangle \geq \varepsilon/2$ for all $x \in X$ then there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that $f^*(\overline{\beta}) + (\overline{\lambda}h)^*(\overline{\alpha}) + \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) \leq \varepsilon/2$,*

(ii) If there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that $f^*(\overline{\beta}) + (\overline{\lambda}h)^*(\overline{\alpha}) + \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) \leq -\varepsilon/2$, then $f(x) + \langle x^*, x \rangle \geq \varepsilon/2$ for all $x \in X$.

2.2 $\varepsilon-$duality gap statements involving $\varepsilon-$subdifferentials for Lagrange, Fenchel and Fenchel-Lagrange duality

One introduces regularity conditions to characterize ε-duality gap statements, using ε-subdifferentials, too, when the existence of an ε-optimal solution to the primal problem is assumed. Recall that, for $x^* \in X^*$, $\overline{x} \in \mathcal{A} \cap dom(f)$ is an ε-*optimal solution* to (P_{x^*}) if and only if $0 \in \partial_\varepsilon(f + x^* + \delta_{\mathcal{A}})(\overline{x})$, which is equivalent to $-x^* \in \partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x})$. From the results presented in this and previous section one can derive other useful statements concerning ε-optimality conditions.

2.2.1 Lagrange duality

One gives the results concerning Lagrange duality.

Theorem 2.2.1 *(H.-V. Boncea, S.-M. Grad, [7]) Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ and $\varepsilon \geq 0$. Then*

$$\partial_\varepsilon (f + \delta_{\mathcal{A}})(\overline{x}) = \bigcup_{\lambda \in C^*} \partial_{\varepsilon + (\lambda h)(\overline{x})} (f + \delta_U + (\lambda h))(\overline{x}) \qquad (RCL^L)$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) there exists $\overline{\lambda} \in C^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq -(f + (\overline{\lambda} h))^*_U(-x^*) + \varepsilon. \tag{2.2.1}$$

Remark 2.2.2 *The quantity in the left-hand side of* (2.2.1) *is not necessarily $v(P_{x^*})$, while in the right-hand side one have something smaller than $v(D^L_{x^*}) + \varepsilon$. However,* (2.2.1) *implies $v(P_{x^*}) \leq v(D^L_{x^*}) + \varepsilon$.*

Theorem 2.2.4 *(H.-V. Boncea, S.-M. Grad, [7]) Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ and $\varepsilon \geq 0$. Then*

$$\partial_\varepsilon (f + \delta_{\mathcal{A}})(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\lambda \in C^*} \partial_{\varepsilon + \eta + (\lambda h)(\overline{x})} (f + \delta_U + (\lambda h))(\overline{x}) \qquad (RCS^L)$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) it holds

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq \sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + \langle x^*, x \rangle + (\lambda h)(x)] + \varepsilon. \tag{2.2.2}$$

Remark 2.2.5 *Relation* (2.2.2) *implies $v(P_{x^*}) \leq v(D^L_{x^*}) + \varepsilon$, without being a consequence of it in general.*

In the following we give a result concerning ε-optimality conditions.

Theorem 2.2.9 *Suppose that the condition (RCI^L) is fulfilled.*

(a) Let $\varepsilon, \eta \geq 0$. If $\overline{x}$ is an ε-optimal solution of the problem (P), then there exists $\overline{\lambda} \in C^*$ such that

$$(f + (\overline{\lambda} h))^*_U(0) + (f + (\overline{\lambda} h))_U(\overline{x}) \leq \varepsilon + \eta + (\overline{\lambda} h)(\overline{x}). \tag{2.2.3}$$

Moreover, $\overline{\lambda}$ is an $(\varepsilon + \eta)$-optimal solution of the problem (D^L).

(b) If there exists $\overline{\lambda} \in C^*$ such that the relation (2.2.3) *holds for $\overline{x} \in X$ and $\overline{\lambda} \in C^*$ then $\overline{x}$ is an ε-optimal solution of the problem (P). Moreover, $\overline{\lambda}$ is an $(\varepsilon + \eta)$-optimal solution of the problem (D^L).*

2.2.2 Fenchel duality

Further one gives the results for the Fenchel duality.

Theorem 2.2.10 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ and $\varepsilon \geq 0$. Then*

$$\partial_\varepsilon (f + \delta_{\mathcal{A}})(\overline{x}) = \bigcup_{\substack{\varepsilon_i \geq 0, i=1,2 \\ \varepsilon_1+\varepsilon_2=\varepsilon}} (\partial_{\varepsilon_1} f(\overline{x}) + \partial_{\varepsilon_2} \delta_U(\overline{x})) \qquad (RCL^F)$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) there exists $\overline{\beta} \in X^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq -f^*(\overline{\beta}) - \sigma_U(-x^* - \overline{\beta}) + \varepsilon. \qquad (2.2.4)$$

Theorem 2.2.12 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ and $\varepsilon \geq 0$. Then*

$$\partial_\varepsilon (f + \delta_{\mathcal{A}})(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\substack{\varepsilon_i \geq 0, i=1,2 \\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta}} (\partial_{\varepsilon_1} f(\overline{x}) + \partial_{\varepsilon_2} \delta_U(\overline{x})) \qquad (RCS^F)$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) it holds

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq \sup_{\beta \in X^*} \{-f^*(\beta) - \sigma_U(-x^* - \beta)\} + \varepsilon. \qquad (2.2.5)$$

Now, we give a result concerning ε-optimality conditions.

Theorem 2.2.14 *Suppose that the condition (RCI^F) is fulfilled.*

(a) Let $\varepsilon, \eta \geq 0$. If $\overline{x}$ is an ε-optimal solution of the problem (P), then there exists $\overline{\beta} \in X^*$ such that

(i) $f(\overline{x}) + f^*(\overline{\beta}) \leq \langle \overline{\beta}, \overline{x} \rangle + \varepsilon_1$,

(ii) $\sigma_U(-\overline{\beta}) + \delta_U(\overline{x}) \leq \langle -\overline{\beta}, \overline{x} \rangle + \varepsilon_2$,

(iii) $\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta$.

Moreover, $\overline{\beta}$ is an $(\varepsilon + \eta)$-optimal solution of the problem (D^F).

(b) If there exist $\varepsilon_1, \varepsilon_2 \geq 0$ and $\overline{\beta} \in X^*$ such that the relations (i)-(iii) hold for $\overline{x} \in X$ and $\overline{\beta} \in X^*$ then $\overline{x}$ is an ε-optimal solution of the problem (P). Moreover, $\overline{\beta}$ is an $(\varepsilon + \eta)$-optimal solution of the problem (D^F).

2.2.3 Fenchel-Lagrange duality

In this subsection one will give regularity conditions using ε-subdifferentials for all the three types of Fenchel-Lagrange duals.

Fenchel-Lagrange dual of type I $(\overline{D}_{x^*})$

One starts with the first type of Fenchel-Lagrange duality.

Theorem 2.2.15 *(H.-V. Boncea, S.-M. Grad, [7]) Let the proper function* $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ *and* $\varepsilon \geq 0$. *Then*

$$\partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcup_{\substack{\lambda \in C^* \\ \varepsilon_i \geq 0,\ i=1,2 \\ \varepsilon_1+\varepsilon_2=\varepsilon+(\lambda h)(\overline{x})}} (\partial_{\varepsilon_1} f(\overline{x}) + \partial_{\varepsilon_2}(\delta_U + (\lambda h))(\overline{x})) \qquad (\overline{RCL})$$

holds if and only if for each $x^* \in X^*$ *for which* $\overline{x}$ *is an* ε*-optimal solution to* (P_{x^*}) *there exist* $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that*

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq -f^*(\overline{\beta}) - (\overline{\lambda}h)^*_U(-x^* - \overline{\beta}) + \varepsilon. \qquad (2.2.6)$$

Theorem 2.2.19 *(H.-V. Boncea, S.-M. Grad, [7]) Let the proper function* $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ *and* $\varepsilon \geq 0$. *Then*

$$\partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\substack{\lambda \in C^* \\ \varepsilon_i \geq 0,\ i=1,2 \\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta+(\lambda h)(\overline{x})}} \left(\partial_{\varepsilon_1} f(\overline{x}) + \partial_{\varepsilon_2}\left(\delta_U + (\lambda h)\right)(\overline{x})\right) \qquad (\overline{RCS})$$

holds if and only if for each $x^* \in X^*$ *for which* $\overline{x}$ *is an* ε*-optimal solution to* (P_{x^*}) *it holds*

$$f(\overline{x}) + \langle x^*, \overline{x}\rangle \leq \sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f^*(\beta) - (\lambda h)^*_U(-x^* - \beta)\} + \varepsilon. \qquad (2.2.7)$$

Further, we give the following result concerning ε-optimality conditions.

Theorem 2.2.22 *Suppose that the condition* $(\overline{RCI})$ *is fulfilled.*

(a) Let $\varepsilon, \eta \geq 0$. *If* $\overline{x}$ *is an* ε*-optimal solution of the problem* (P), *then there exist* $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that*

(i) $f(\overline{x}) + f^*(\overline{\beta}) \leq \langle \overline{\beta}, \overline{x}\rangle + \varepsilon_1$,

(ii) $(\overline{\lambda}h)^*_U(-\overline{\beta}) + (\overline{\lambda}h)_U(\overline{x}) \leq \langle -\overline{\beta}, \overline{x}\rangle + \varepsilon_2.$,

(iii) $\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta + (\overline{\lambda}h)(\overline{x})$.

Moreover, $(\overline{\lambda}, \overline{\beta})$ *is an* $(\varepsilon + \eta)$*-optimal solution of the problem* $(\overline{D})$.

(b) If there exist $\varepsilon_1, \varepsilon_2 \geq 0$, $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that the relations (i)-(iii) hold for* $\overline{x} \in X$, $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *then* $\overline{x}$ *is an* ε*-optimal solution of the problem* (P). *Moreover,* $(\overline{\lambda}, \overline{\beta})$ *is an* $(\varepsilon + \eta)$*-optimal solution of the problem* $(\overline{D})$.

Fenchel-Lagrange dual of type II $(\widetilde{D}_{x^*})$

Further one gives the results concerning the second type of Fenchel-Lagrange duality.

Theorem 2.2.23 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ and $\varepsilon \geq 0$. Then*

$$\partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcup_{\substack{\lambda \in C^* \\ \varepsilon_i \geq 0,\ i=1,2 \\ \varepsilon_1+\varepsilon_2=\varepsilon+(\lambda h)(\overline{x})}} (\partial_{\varepsilon_1} f_U(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(\overline{x})) \qquad (\widetilde{RCL})$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq -f_U^*(\overline{\beta}) - (\overline{\lambda} h)^*(-x^* - \overline{\beta}) + \varepsilon. \qquad (2.2.8)$$

Theorem 2.2.27 *Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ and $\varepsilon \geq 0$. Then*

$$\partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\substack{\lambda \in C^* \\ \varepsilon_i \geq 0,\ i=1,2 \\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta+(\lambda h)(\overline{x})}} (\partial_{\varepsilon_1} f_U(\overline{x}) + \partial_{\varepsilon_2}(\lambda h)(\overline{x})) \qquad (\widetilde{RCS})$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) it holds

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq \sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f_U^*(\beta) - (\lambda h)^*(-x^* - \beta)\} + \varepsilon. \qquad (2.2.9)$$

Further, we give the following result concerning ε-optimality conditions.

Theorem 2.2.29 *Suppose that the condition $(\widetilde{RCI})$ is fulfilled.*

(a) Let $\varepsilon, \eta \geq 0$. If $\overline{x}$ is an ε-optimal solution of the problem (P), then there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that

(i) $f_U(\overline{x}) + f_U^*(\overline{\beta}) \leq \langle \overline{\beta}, \overline{x} \rangle + \varepsilon_1$,

(ii) $(\overline{\lambda} h)^*(-\overline{\beta}) + (\overline{\lambda} h)(\overline{x}) \leq \langle -\overline{\beta}, \overline{x} \rangle + \varepsilon_2$.,

(iii) $\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta + (\overline{\lambda} h)(\overline{x})$.

Moreover, $(\overline{\lambda}, \overline{\beta})$ is an $(\varepsilon + \eta)$-optimal solution of the problem $(\widetilde{D})$.

(b) If there exist $\varepsilon_1, \varepsilon_2 \geq 0$, $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that the relations (i)-(iii) hold for $\overline{x} \in X$, $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ then $\overline{x}$ is an ε-optimal solution of the problem (P). Moreover, $(\overline{\lambda}, \overline{\beta})$ is an $(\varepsilon + \eta)$-optimal solution of the problem $(\widetilde{D})$.

Fenchel-Lagrange dual of type III (D_{x^*})

The results concerning the third type of Fenchel-Lagrange duality follow.

Theorem 2.2.30 *(H.-V. Boncea, S.-M. Grad, [7]) Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ and $\varepsilon \geq 0$. Then*

$$\partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcup_{\substack{\lambda \in C^* \\ \varepsilon_i \geq 0,\ i=1,2,3 \\ \varepsilon_1+\varepsilon_2+\varepsilon_3 = \varepsilon + (\lambda h)(\overline{x})}} (\partial_{\varepsilon_1} f(\overline{x}) + N_U^{\varepsilon_2}(\overline{x}) + \partial_{\varepsilon_3}(\lambda h)(\overline{x})) \qquad (RCL)$$

holds if and only if for all $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq -f^*(\overline{\beta}) - (\overline{\lambda} h)^*(\overline{\alpha}) - \sigma_U(-x^* - \overline{\beta} - \overline{\alpha}) + \varepsilon. \qquad (2.2.10)$$

Remark 2.2.31 (*H.-V. Boncea, S.-M. Grad, [7]) The quantity in the left-hand side of* (2.2.10) *is not necessarily $v(P)$, while in the right-hand side one have something smaller than $v(D_{x^*}) + \varepsilon$. However,* (2.2.10) *implies $v(P_{x^*}) \leq v(D_{x^*}) + \varepsilon$.*

Theorem 2.2.34 *(H.-V. Boncea, S.-M. Grad, [7]) Let the proper function $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in A \cap dom(f)$ and $\varepsilon \geq 0$. Then*

$$\partial_\varepsilon(f + \delta_{\mathcal{A}})(\overline{x}) = \bigcap_{\eta > 0} \bigcup_{\substack{\lambda \in C^* \\ \varepsilon_i \geq 0,\ i=1,2,3 \\ \varepsilon_1+\varepsilon_2+\varepsilon_3 = \varepsilon + \eta + (\lambda h)(\overline{x})}} (\partial_{\varepsilon_1} f(\overline{x}) + N_U^{\varepsilon_2}(\overline{x}) + \partial_{\varepsilon_3}(\lambda h)(\overline{x})) \qquad (RCS)$$

holds if and only if for each $x^* \in X^*$ for which $\overline{x}$ is an ε-optimal solution to (P_{x^*}) it holds

$$f(\overline{x}) + \langle x^*, \overline{x} \rangle \leq \sup_{\substack{\lambda \in C^*, \\ \alpha, \beta \in X^*}} \{-f^*(\beta) - \sigma_U(-x^* - \beta - \alpha) - (\lambda h)^*(\alpha)\} + \varepsilon. \qquad (2.2.11)$$

We give now the following result concerning ε-optimality conditions.

Theorem 2.2.41 *Suppose that the condition (RCI) is fulfilled.*

(a) Let $\varepsilon, \eta \geq 0$. If $\overline{x}$ is an ε-optimal solution of the problem (P), then there exist $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that

(i) $f(\overline{x}) + f^*(\overline{\beta}) \leq \langle \overline{\beta}, \overline{x} \rangle + \varepsilon_1$,

(ii) $(\overline{\lambda} h)(\overline{x}) + (\overline{\lambda} h)^*(\overline{\alpha}) \leq \langle \overline{\alpha}, \overline{x} \rangle + \varepsilon_2$,

(iii) $\delta_U(\overline{x}) + \sigma_U(-\overline{\alpha} - \overline{\beta}) \leq \langle -\overline{\alpha} - \overline{\beta}, \overline{x} \rangle + \varepsilon_3$,

(iv) $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + \eta + (\overline{\lambda} h)(\overline{x})$.

Moreover, $(\overline{\lambda}, \overline{\alpha}, \overline{\beta})$ is an $(\varepsilon + \eta)$-optimal solution of the problem (D).

(b) If there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ such that the relations (i)-(iv) hold for $\overline{x} \in X$, $\overline{\lambda} \in C^*$ and $\overline{\alpha}, \overline{\beta} \in X^*$ then $\overline{x}$ is an ε-optimal solution of the problem (P). Moreover, $(\overline{\lambda}, \overline{\alpha}, \overline{\beta})$ is an $(\varepsilon + \eta)$-optimal solution of the problem (D).

Remark 2.2.42 *Similar statements concerning ε-optimality conditions for (P) and its considered dual problems, were also obtained in [18, 19]. The results about ε-optimality conditions presented in this chapter extend the results obtained in the mentioned papers.*

Chapter 3

Characterizations of ε-duality gap statements for composed optimization problems

Consider two separated locally convex vector spaces X and Y and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$ respectively. On Y one considers the partial ordering "$\leqq_C$" induced by the convex cone C.

Let $f : X \to \overline{\mathbb{R}}$ be a proper function, $g : Y \to \overline{\mathbb{R}}$ be a proper function, which is also C-increasing and $h : X \to Y^\bullet$ be a proper vector function fulfilling $dom g \cap (h(dom f) + C) \neq \emptyset$. Unless otherwise stated, these hypotheses remain valid throught the entire chapter. Consider the optimization problem

$$\inf_{x \in X} [f(x) + (g \circ h)(x)]. \tag{P^C}$$

For $x^* \in X^*$ one also considers the linearly perturbed optimization problem

$$\inf_{x \in X} \left[f(x) + (g \circ h)(x) - \langle x^*, x \rangle \right]. \tag{$P^C_{x^*}$}$$

To this problem one can attach different dual Fenchel-Lagrange-type problems. If f and (λh) are taken together one gets the following dual to $(P^C_{x^*})$

$$\sup_{\lambda \in C^*} \{-g^*(\lambda) - (f + (\lambda h))^*(x^*)\}. \tag{$D^C_{x^*}$}$$

When f and (λh) are separated, one gets the following dual problem

$$\sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-g^*(\lambda) - f^*(\beta) - (\lambda h)^*(x^* - \beta)\}. \tag{$\overline{D^C_{x^*}}$}$$

One denotes by $v(P^C)$ the optimal objective value of the optimization problem (P^C). Note that $v(\overline{D^C_{x^*}}) \leq v(D^C_{x^*}) \leq v(P^C_{x^*})$ for all $x^* \in X^*$. When $x^* = 0$ these duals to (P^C) are denoted simply by (D^C) and $(\overline{D^C})$, respectively.

Between (P^C) and its duals one always has *weak duality*, i.e. $v(P^C) \geq v(D^C)$, respectively, $v(P^C) \geq v(\overline{D^C})$. When $v(P^C) = v(D^C)$ one says that there is *zero duality gap* between (P^C) and (D^C) and if (D^C) has moreover an optimal solution, the situation is called *strong duality*. If $v(P^C) - v(D^C) \leq \varepsilon$, with $\varepsilon \geq 0$, one has an ε*-duality gap* for (P^C) and (D^C). If one of these situations holds for $(P^C_{x^*})$ and $(D^C_{x^*})$ for all $x^* \in X^*$, it will be called *stable*.

In the following one writes min(max) instead of inf(sup) when the corresponding infimum (supremum) is attained.

3.1 ε-duality gap statements using epigraphs

Let $\varepsilon \geq 0$. Consider the regularity conditions

$$\left| \begin{array}{l} \{(x^*, 0, r) : (x^*, r) \in epi(f + g \circ h)^*\} \subseteq [\{0\} \times epi(g^*) + \bigcup\limits_{\lambda \in C^*} \{(a, -\lambda, r) : \\ (a, r) \in epi((f + (\lambda h))^*)\}] \cap (X^* \times \{0\} \times \mathbb{R}) - (0, 0, \varepsilon) \end{array} \right. \tag{RC}$$

and

$$\left| \begin{array}{l} \{(x^*, 0, r) : (x^*, r) \in epi(f + g \circ h)^*\} \subseteq [\{0\} \times epi(g^*) + \{(x^*, 0, r) : \\ (x^*, r) \in epi(f^*)\} + \bigcup\limits_{\lambda \in C^*} \{(a, -\lambda, r) : (a, r) \in epi((\lambda h)^*)\}] \cap \\ (X^* \times \{0\} \times \mathbb{R}) - (0, 0, \varepsilon) \end{array} \right. \tag{$\overline{RC}$}$$

They are inspired by the closedness type regularity conditions from [11], but unlike there, we do not use convexity and topological hypotheses for most of the proven statements.

Theorem 3.1.2 *(H.-V. Boncea, S.-M. Grad, [6]) The condition (RC) is fulfilled if and only if for any $x^* \in X^*$ there exists a $\overline{\lambda} \in C^*$ such that*

$$(f + g \circ h)^*(x^*) \geq g^*(\overline{\lambda}) + (f + (\overline{\lambda} h))^*(x^*) - \varepsilon. \tag{3.1.1}$$

Remark 3.1.3 *(H.-V. Boncea, S.-M. Grad, [6]) In the left-hand side of* (3.1.1) *one can easily recognize* $-v(P^C_{x^*})$. *The quantity in the right-hand side of* (3.1.1) *is not necessarily* $-v(D^C_{x^*})-\varepsilon$, *as the supremum in* $(D^C_{x^*})$ *is not shown to be attained at* $\overline{\lambda}$. *Though,* (3.1.1) *implies* $v(P^C_{x^*}) \leq v(D^C_{x^*})+\varepsilon$, *which actually means that for* $(P^C_{x^*})$ *and* $(D^C_{x^*})$ *there is* ε-*duality gap. Thus,* (RC) *yields that there is stable* ε-*duality gap for* (P^C) *and* (D^C). *Note also that* $\overline{\lambda} \in C^*$ *obtained in Theorem 3.1.2 is an* ε-*optimal solution of* $(D^C_{x^*})$.

Similar results can be obtained for $(\overline{D^C})$ by making use of $(\overline{RC})$ as follows.

Theorem 3.1.8 *(H.-V. Boncea, S.-M. Grad, [6]) The condition* $(\overline{RC})$ *is fulfilled if and only if for any* $x^* \in X^*$ *there exist some* $\overline{\lambda} \in C^*$ *and* $\overline{\beta} \in X^*$ *such that*

$$(f + g \circ h)^*(x^*) \geq g^*(\overline{\lambda}) + f^*(\overline{\beta}) + (\overline{\lambda}h)^*(x^* - \overline{\beta}) - \varepsilon. \tag{3.1.2}$$

In order to characterize formulae similar to (3.1.1) and (3.1.2), where appear actually the optimal values of (D^C) and $(\overline{D^C})$, let us consider the following regularity conditions

$$epi(f + g \circ h)^* \subseteq epi \inf_{\lambda \in C^*} [g^*(\lambda) + (f + (\lambda h))^*(\cdot)] - (0, \varepsilon) \tag{RCI}$$

and

$$epi(f + g \circ h)^* \subseteq epi \inf_{\substack{\lambda \in C^* \\ \beta \in X^*}} [g^*(\lambda) + f^*(\beta) + (\lambda h)^*(\cdot - \beta)] - (0, \varepsilon). \tag{$\overline{RCI}$}$$

Theorem 3.1.15 *(H.-V. Boncea, S.-M. Grad, [6]) The condition* (RCI) *is fulfilled if and only if for any* $x^* \in X^*$ *we have*

$$(f + g \circ h)^*(x^*) \geq \inf_{\lambda \in C^*} [g^*(\lambda) + (f + (\lambda h))^*(x^*)] - \varepsilon. \tag{3.1.3}$$

Remark 3.1.16 *(H.-V. Boncea, S.-M. Grad, [6]) Relation* (3.1.3) *means actually* $v(P^C_{x^*}) \leq v(D^C_{x^*}) + \varepsilon$, *i.e. we have stable* ε-*duality gap for* (P^C) *and* (D^C).

Theorem 3.1.19 *(H.-V. Boncea, S.-M. Grad, [6]) The condition* $(\overline{RCI})$ *is fulfilled if and only if for any* $x^* \in X^*$ *we have*

$$(f + g \circ h)^*(x^*) \geq \inf_{\substack{\lambda \in C^* \\ \beta \in X^*}} [g^*(\lambda) + f^*(\beta) + (\lambda h)^*(x^* - \beta)] - \varepsilon. \tag{3.1.4}$$

Remark 3.1.22 *(H.-V. Boncea, S.-M. Grad, [6]) Taking into consideration Theorem 3.1.2, Theorem 3.1.8, Theorem 3.1.15 and Theorem 3.1.19 we get the following implications:* $(\overline{RC}) \Rightarrow (RC) \Rightarrow (RCI)$ *and* $(\overline{RC}) \Rightarrow (\overline{RCI}) \Rightarrow (RCI)$. *Using, for instance, [11, Example 3.10] one can construct examples that show that the opposite implications are not valid in general.*

3.2 Special cases

The results one gave for composed functions can be particularized for combinations of functions that appear often in both theoretical and practical problems.

3.3 ε-duality gap statements using subdifferentials

In this section we show that the relations $(3.1.1), (3.1.2), (3.1.3), (3.1.4)$ can be characterized by regularity conditions involving subdifferentials, too.

Theorem 3.3.1 *(H.-V. Boncea, S.-M. Grad, [6]) One has*

$$\partial(f+g\circ h)(x) \subseteq \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_{1,2}\geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta\\ \lambda\in C^*\cap\partial_{\varepsilon_2}g(h(x))}} \partial_{\varepsilon_1}(f+(\lambda h))(x) \tag{$RCSC$}$$

for all $x\in X$ if and only if (3.1.3) *holds for all $x^*\in R(\partial(f+g\circ h))$.*

In the following result we give another characterization for relation (3.1.1), this time by making use of (ε)-subdifferentials.

Theorem 3.3.6 *(H.-V. Boncea, S.-M. Grad, [6]) One has*

$$\partial(f+g\circ h)(x) \subseteq \bigcup_{\substack{\varepsilon_{1,2}\geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon\\ \lambda\in C^*\cap\partial_{\varepsilon_2}g(h(x))}} \partial_{\varepsilon_1}(f+(\lambda h))(x) \tag{$RCLC$}$$

for all $x\in X$ if and only if for all $x^*\in R(\partial(f+g\circ h))$, there exists $\overline{\lambda}\in C^*$ such that (3.1.1) *holds.*

The following result characterizes the relation (3.1.4).

Theorem 3.3.8 *(H.-V. Boncea, S.-M. Grad, [6]) One has*

$$\partial(f+g\circ h)(x) \subseteq \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_{1,2}\geq 0\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon+\eta\\ \lambda\in C^*\cap\partial_{\varepsilon_3}g(h(x))}} \partial_{\varepsilon_1}f(x)+\partial_{\varepsilon_2}(\lambda h)(x) \tag{$\overline{RCSC}$}$$

for all $x\in X$ if and only if for all $x^*\in R(\partial(f+g\circ h))$, (3.1.4) *holds.*

In the following result we characterize relation (3.1.2).

Theorem 3.3.13 *(H.-V. Boncea, S.-M. Grad, [6]) One has*

$$\partial(f+g\circ h)(x) \subseteq \bigcup_{\substack{\varepsilon_{1,2}\geq 0\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon+\eta\\ \lambda\in C^*\cap\partial_{\varepsilon_3}g(h(x))}} \partial_{\varepsilon_1}f(x)+\partial_{\varepsilon_2}(\lambda h)(x) \tag{$\overline{RCLC}$}$$

for all $x \in X$ if and only if for all $x^* \in R(\partial(f + g \circ h))$, there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that (3.1.2) *holds.*

Remark 3.3.15 *(H.-V. Boncea, S.-M. Grad, [6]) Looking at the conditions* (RC) *and* $(RCLC)$ *one can observe that* (RC) *implies* (3.1.1) *for all $x^* \in X^*$ and* $(RCLC)$ *implies* (3.1.1) *for all $x^* \in R(\partial(f + g \circ h))$, which means that* (RC) *implies* $(RCLC)$. *Analogously,* $(\overline{RC})$ *implies* (3.1.2) *for all $x^* \in X^*$ and* $(\overline{RCLC})$ *implies* (3.1.2) *for all $x^* \in R(\partial(f + g \circ h))$, which means that* $(\overline{RC})$ *implies* $(\overline{RCLC})$.

3.4 Results concerning ε-optimality conditions, ε-Farkas statements and (ε, η)-saddle points

From the results presented in the previous sections one can derive other useful statements concerning ε-optimality conditions, ε-Farkas assertions and characterizations for (ε, η)-saddle points as follows. We begin with the ε-optimality conditions.

One considers the following conditions

$$(epi(f+g\circ h)^*)\cap(\{0\}\times\mathbb{R}) \subseteq (epi \inf_{\lambda \in C^*}[g^*(\lambda)+(f+(\lambda h))^*(\cdot)])\cap(\{0\}\times\mathbb{R})-(0,\varepsilon), \quad (RCI^0)$$

$$\left| \begin{array}{l} (epi(f + g \circ h)^*) \cap (\{0\} \times \mathbb{R}) \subseteq (epi \inf\limits_{\substack{\lambda \in C^* \\ \beta \in X^*}} [g^*(\lambda) + f^*(\beta) + (\lambda h)^*(\cdot - \beta)]) \\ \cap(\{0\} \times \mathbb{R}) - (0, \varepsilon). \end{array} \right. \quad (\overline{RCI}^0)$$

Theorem 3.4.1 *(H.-V. Boncea, S.-M. Grad, [6]) (a) Let $\varepsilon, \eta \geq 0$. Suppose that the condition (RCI^0) is fulfilled. If $\overline{x}$ is an ε-optimal solution of the problem (P^C), then there exist $\varepsilon_1, \varepsilon_2 \geq 0$, and $\overline{\lambda} \in C^*$ such that*

(i) $g^*(\overline{\lambda}) + g(h(\overline{x})) \leq (\overline{\lambda}h)(\overline{x}) + \varepsilon_2$,

(ii) $(f + (\overline{\lambda}h))^*(0) + (f + (\overline{\lambda}h))(\overline{x}) \leq \varepsilon_1$,

(iii) $\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta$.

Moreover, $\overline{\lambda}$ is an $(\varepsilon + \eta)$-optimal solution of the problem (D^C).

(b) If there exist $\varepsilon_1, \varepsilon_2 \geq 0$ and $\overline{\lambda} \in C^*$ such that the relations (i)-(iii) hold for $\overline{x} \in X$ and $\overline{\lambda} \in C^*$ then $\overline{x}$ is an $(\varepsilon+\eta)$-optimal solution of the problem (P^C). Moreover, $\overline{\lambda}$ is an $(\varepsilon + \eta)$-optimal solution of the problem (D^C).

The similar statement for $(\overline{D^C})$ can be proven analogously.

Theorem 3.4.2 *(H.-V. Boncea, S.-M. Grad, [6]) (a) Let $\varepsilon, \eta \geq 0$. Suppose that the condition $(\overline{RCI}^0)$ is fulfilled. If $\overline{x}$ is an ε-optimal solution of the problem (P^C), then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that*

(i) $g^*(\overline{\lambda}) + g(h(\overline{x})) \leq (\overline{\lambda}h)(\overline{x}) + \varepsilon_3$,

(ii) $f^*(\overline{\beta}) + f(\overline{x}) \leq \langle \overline{\beta}, \overline{x} \rangle + \varepsilon_1$,

(iii) $(\overline{\lambda}h)^*(-\overline{\beta}) + (\overline{\lambda}h)(\overline{x}) \leq \langle -\overline{\beta}, \overline{x} \rangle + \varepsilon_2$,

(iv) $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + \eta$.

Moreover, $(\overline{\lambda}, \overline{\beta})$ is an $(\varepsilon + \eta)$-optimal solution of the problem $(\overline{D^C})$.

(b) If there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that the relations (i)-(iv) hold for $\overline{x} \in X$, $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ then $\overline{x}$ is an $(\varepsilon + \eta)$-optimal solution of the problem (P^C). Moreover, $(\overline{\lambda}, \overline{\beta})$ is an $(\varepsilon + \eta)$-optimal solution of the problem $(\overline{D^C})$.

In the following we give ε-Farkas-type results for (P^C) and its duals, too. Let us consider the following regularity conditions

$$\left| \begin{array}{l} \{(0,0,r) : (0,r) \in epi(f + g \circ h)^*\} \subseteq [\{0\} \times epi(g^*) + \bigcup\limits_{\lambda \in C^*} \{(a, -\lambda, r) : \\ (a,r) \in epi((f + (\lambda h))^*)\}] \cap (\{0\} \times \{0\} \times \mathbb{R}) - (0,0,\varepsilon), \end{array} \right. \qquad (RC^0)$$

$$\left| \begin{array}{l} \{(0,0,r) : (0,r) \in epi(f + g \circ h)^*\} \subseteq [\{0\} \times epi(g^*) + \{(0,0,r) : \\ (0,r) \in epi(f^*)\} + \bigcup\limits_{\lambda \in C^*} \{(a, -\lambda, r) : (a,r) \in epi((\lambda h)^*)\}] \cap \\ (\{0\} \times \{0\} \times \mathbb{R}) - (0,0,\varepsilon). \end{array} \right. \qquad (\overline{RC}^0)$$

Theorem 3.4.5 *(H.-V. Boncea, S.-M. Grad, [6]) (i) Suppose that (RC^0) holds. If $f(x) + (g \circ h)(x) \geq \varepsilon/2$ for all $x \in X$ then there exists $\overline{\lambda} \in C^*$ such that $g^*(\overline{\lambda}) + (f + \overline{\lambda}h)^*(0) \leq \varepsilon/2$.*

(ii) If there exists $\overline{\lambda} \in C^*$ such that $g^*(\overline{\lambda}) + (f + \overline{\lambda}h)^*(0) \leq -\varepsilon/2$, then $f(x) + (g \circ h)(x) \geq \varepsilon/2$ for all $x \in X$.

Analogously, one can prove the following statements for (P^C) and $(\overline{D^C})$, too.

Theorem 3.4.6 *(H.-V. Boncea, S.-M. Grad, [6]) (i) Suppose that $(\overline{RC}^0)$ holds. If $f(x) + (g \circ h)(x) \geq \varepsilon/2$ for all $x \in X$ then there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that $f^*(\overline{\beta}) + g^*(\overline{\lambda}) + (\overline{\lambda}h)^*(-\overline{\beta}) \leq \varepsilon/2$.*

(ii) If there exist $\overline{\lambda} \in C^*$ and $\overline{\beta} \in X^*$ such that $f^*(\overline{\beta}) + g^*(\overline{\lambda}) + (\overline{\lambda}h)^*(-\overline{\beta}) \leq -\varepsilon/2$, then $f(x) + (g \circ h)(x) \geq \varepsilon/2$ for all $x \in X$.

Nevertheless, one can extend the investigations from this section also towards generalized saddle points.

The Lagrangian function assigned to $(P^C) - (D^C)$ is $L^C : X \times Y^* \to \overline{\mathbb{R}}$, defined by (cf. [12])

$$L^C(x, \lambda) = \begin{cases} f(x) + (\lambda h)(x) - g^*(\lambda), & \text{if } \lambda \in C^* \\ -\infty, & \text{otherwise.} \end{cases}$$

Let $\eta \geq 0$. We say that $(\overline{x}, \overline{\lambda}) \in X \times Y^*$ is (η, ε)-*saddle point* of the Lagrangian L^C if

$$L^C(\overline{x}, \lambda) - \eta \leq L^C(\overline{x}, \overline{\lambda}) \leq L^C(x, \overline{\lambda}) + \varepsilon, \text{ for all } (x, \lambda) \in X \times Y^*.$$

Theorem 3.4.10 *(H.-V. Boncea, S.-M. Grad, [6]) Assume that g is a convex and lower semicontinuous function fulfilling $g(y) > -\infty$ for all $y \in Y$. If $(\overline{x}, \overline{\lambda})$ is an (η, ε)-saddle point of L^C then $\overline{x} \in X$ is an $(\varepsilon + \eta)$-optimal solution to (P^C), $\overline{\lambda} \in C^*$ is an $(\varepsilon + \eta)$-optimal solution to (D^C) and there is $(\varepsilon + \eta)$-duality gap for the pair of problems (P^C) and (D^C), i.e. $v(P^C) \leq (D^C) + \varepsilon + \eta$.*

An analogous result with Theorem **??** can be formulated for the pair of problems (P^C) and $(\overline{D^C})$ with the corresponding Lagrangian function given by (cf. [12]) $\overline{L^C} : X \times X^* \times Y^* \to \overline{\mathbb{R}}$

$$\overline{L^C}(x, \beta, \lambda) = \begin{cases} \langle \beta, x \rangle + (\lambda h)(x) - f^*(\beta) - g^*(\lambda), & \text{if } \lambda \in C^* \\ -\infty, & \text{otherwise.} \end{cases}$$

Theorem 3.4.11 *(H.-V. Boncea, S.-M. Grad, [6]) Assume that g is a convex and lower semicontinuous function fulfilling $g(y) > -\infty$ for all $y \in Y$. If $(\overline{x}, \overline{\lambda})$ is an (η, ε)-saddle point of $\overline{L^C}$ then $\overline{x} \in X$ is an $(\varepsilon + \eta)$-optimal solution to (P^C), $\overline{\lambda} \in C^*$ is an $(\varepsilon + \eta)$-optimal solution to $(\overline{D^C})$ and there is $(\varepsilon + \eta)$-duality gap for the pair of problems (P^C) and $(\overline{D^C})$, i.e. $v(P^C) \leq (\overline{D^C}) + \varepsilon + \eta$.*

Chapter 4

Convex optimization problems with entropy-like objective functions

4.1 Regularity conditions for strong duality when the primal problem has entropy-like objective functions

Consider the non-empty convex set $X \subseteq \mathbb{R}^n$. Motivated by [13, 37, 38, 39, 40] we introduce the following problem

$$\inf_{\substack{x \in X \\ h(x) \leqq 0}} \left\{ \sum_{j=1}^{k} g_j(x) \Phi_j \left(\frac{f_j(x)}{g_j(x)} \right) \right\}, \tag{P^{Φ}}$$

where $f = (f_1, \ldots, f_k)^T : X \to \mathbb{R}^k$, $g = (g_1, \ldots, g_k)^T : X \to \mathbb{R}^k$, $h = (h_1, \ldots, h_m)^T : X \to \mathbb{R}^m$, $\Phi = (\Phi_1, \ldots, \Phi_k) : \mathbb{R}_+ \to (\mathbb{R}^k)^\bullet$, fulfilling $f_j(x) \geq 0$ and $g_j(x) > 0$ for all $x \in X$ such that $h(x) \leq 0$.

So far no assumption of convexity or concavity regarding the functions involved in (P^{Φ}) has been made. As we are trying to cover a large number of different problems as special cases of (P^{Φ}), the properties of the respective functions will be flexible.

As the dual obtained directly from (P^{Φ}) seems rather unattractive for computational issues, we could resort to the method used in [13]. It consists in attaching a problem whose Lagrange and Fenchel-Lagrange duals are easier computable than the one to the primal problem. The most important connection between the two problems stands in the equality between their optimal objective values. In order to deal with the problem (P^{Φ}) by means of duality we try to bring it in the form

$$\inf_{(x,s,t)\in\mathcal{A}} \left\{ \sum_{j=1}^{k} t_j \Phi_j \left(\frac{s_j}{t_j} \right) \right\}, \tag{P'}$$

where

$$\mathcal{A} = \{(x,s,t) \in X \times \mathbb{R}^k_+ \times int(\mathbb{R}^k_+) : h(x) \leqq 0, \Psi(f(x), s) = 0,$$

$$\Omega(g(x), t) = 0\}.$$

Additional properties of the functions f and g, as well as the functions $\Psi : \mathbb{R}^k_+ \times \mathbb{R}^k_+ \to \mathbb{R}^k$ and $\Omega : int(\mathbb{R}^k_+) \times int(\mathbb{R}^k_+) \to \mathbb{R}^k$ are introduced in order to assure the convexity of the problem (P') and to obtain

$$\inf(P^{\Phi}) = \inf(P').$$

After considering some convexity or concavity properties for f and g we determine the Lagrange and Fenchel-Lagrange dual problems to the corresponding attached problems. As the strong duality statements we use require the convexity of the primal problem, we consider further the functions $h_j, j = 1, \ldots, m$, convex and the inequalities in $\mathcal{A}$ concerning $f(x)$ and $g(x)$ are chosen in order to assure the convexity of the set. Because of the following assertion we consider further the functions $\Phi_j, j = 1, \ldots, k$, convex too.

Proposition 4.1.1 *If Φ_j is convex, then $(s_j, t_j) \longmapsto t_j \Phi_j \left(\frac{s_j}{t_j} \right)$ is convex, $j = 1, ..., k$.*

The proof of the previous Proposition can be seen as an extended case of [4, Lemma 2.1] to $\overline{\mathbb{R}}^k_+ \times \overline{\mathbb{R}}^k_+$.

This last result, alongside the convexity of $\mathcal{A}$, guaranteed by construction, yields that (P') is a convex optimization problem. Therefore it is suitable for the application of the Lagrange and Fenchel-Lagrange duality.

The expresion from the problem (P^{Φ}) can be seen as a composition of two functions.

Let us consider $C \subseteq \mathbb{R}^{2k}$ a closed convex cone and $S \subseteq \mathbb{R}^m$ a convex cone. On $\mathbb{R}^{2k}$ we consider the partial order induced by C, "$\leqq_C$", defined by $z \leqq_C y \Leftrightarrow y - z \in C$ and let $(\mathbb{R}^{2k}_+)^{\bullet} = \mathbb{R}^{2k}_+ \cup \{\infty_{\mathbb{R}^{2k}_+}\}$. Consider, also, $X \subseteq \mathbb{R}^n$, the convex and C-increasing function $u : (\mathbb{R}^{2k}_+)^{\bullet} \to \overline{\mathbb{R}}$, the C-convex function $v : X \to (\mathbb{R}^{2k}_+)^{\bullet}$ and the convex function $h : \mathbb{R}^n \to \mathbb{R}^m$.

The problem (P^{Φ}) can be seen as a composed convex optimization problem, i.e.

$$\inf_{\substack{x\in X\\ h(x)\in -S}} (u\circ v)(x). \tag{P^{CC}}$$

Inspired by [12], we can attach to this problem a Fenchel-Lagrange type dual problem, namely

$$\sup_{\substack{\alpha\in S^*,\beta\in C^*\\ p\in X^*}} \{-u^*(\beta)-(\beta^T v)^*(p)-(\alpha^T h)^*(-p)\}. \tag{D^{CC}}$$

For the pair of problems $(P^{CC})-(D^{CC})$ there is strong duality if one of the regularity conditions $(RC_i^{CC}), i=1,\ldots,4$ (cf. [12]) is fulfilled. These regularity conditions are

$$\left| \begin{array}{l} \exists x'\in X\cap v^{-1}(\mathbb{R}^{2k}_+) \text{ such that } u \text{ is continuous at } v(x') \text{ and}\\ h(x')\in -int(S) \end{array}\right. \tag{RC_1^{CC}}$$

$$\left| \begin{array}{l} u \text{ is lower semicontinuous, } v \text{ is star } C\text{-lower semicontinuous}\\ h \text{ is } S\text{-epi closed and } 0\in sqri(\mathbb{R}^{2k}_+ - v(X)), \end{array}\right. \tag{RC_2^{CC}}$$

$$\left| \begin{array}{l} u \text{ is lower semicontinuous, } v \text{ is star } C\text{-lower semicontinuous}\\ h \text{ is } S\text{-epi closed and } 0\in core(\mathbb{R}^{2k}_+ - v(X)), \end{array}\right. \tag{$RC_{2'}^{CC}$}$$

$$\left| \begin{array}{l} u \text{ is lower semicontinuous, } v \text{ is star } C\text{-lower semicontinuous}\\ h \text{ is } S\text{-epi closed and } 0\in int(\mathbb{R}^{2k}_+ - v(X)), \end{array}\right. \tag{$RC_{2''}^{CC}$}$$

$$\dim(lin(\mathbb{R}^{2k}_+ - v(X)))<\infty \text{ and } int(\mathbb{R}^{2k}_+)\cap ri(v(X))\neq\emptyset, \tag{RC_3^{CC}}$$

$$\left| \begin{array}{l} u \text{ is lower semicontinuous, } v \text{ is star } C\text{-lower semicontinuous}\\ h \text{ is } S\text{-epi closed and } \bigcup\limits_{\lambda\in C^*}(epi(\lambda v)^* + (0,u^*(\lambda))) \text{ is}\\ \text{closed in the topology } w(\mathbb{R}^n_+,\mathbb{R}^n)\times\mathbb{R}. \end{array}\right. \tag{RC_4^{CC}}$$

Further, let us consider $u(s,t)=\sum_{j=1}^k t_j\Phi_j\left(\frac{s_j}{t_j}\right)$ and $v(x)=(g(x),f(x))$. We note with $u^+(\beta)$, the monotone conjugate of u defined as (cf. [71]) $u^+(\beta)=\sup\limits_{x\geqq 0}\{\beta^T x-u(x)\}$. Calculating it, we get that $u^+(\beta)=\sup\limits_{\substack{s\geq 0\\ t>0}}[a^s s+a^t t-t\Phi\left(\frac{s}{t}\right)]=\sup\limits_{t>0}[a^t t+\sup\limits_{s\geq 0}(a^s s-t\Phi\left(\frac{s}{t}\right))]$. If we take $b:=\frac{s}{t}$ we get that $\sup\limits_{s\geq 0}[a^s s-t\Phi\left(\frac{s}{t}\right)]=\sup\limits_{b\geq 0}[a^s bt-t\Phi\left(b\right)]=t\Phi^+(a^s)$. So,

$$u^+(\beta)=\sup_{t>0} t[a^t+\Phi^+(a^s)]=\begin{cases} 0, & a^t+\Phi^+(a^s)\leqq 0\\ +\infty, & \text{else.}\end{cases}$$

4.2 Some particular cases for the cone

Taking some particular cases for the cone C, we can obtain the following problems. We attach them the Lagrange and Fenchel-Lagrange dual problems and in order to obtain strong duality we give the adequate regularity conditions.

First case. $C = (-\mathbb{R}^k_+) \times \{0\}$, Φ_j -decreasing, f_j -concave and g_j -affine for all $j = 1, ..., k$. In this case consider

$$\mathcal{A} = \left\{(x, s, t) \in X \times \mathbb{R}^k_+ \times int(\mathbb{R}^k_+) : h(x) \leqq 0, f(x) \geqq s, g(x) = t\right\}.$$

We have $\inf(P^\Phi) = \inf(P')$.

The Lagrange dual problem to (P^Φ) in the first case, becomes

$$\sup_{\substack{\alpha \leqq 0, \beta \in \mathbb{R}^k, \gamma \geqq 0, \\ \Phi_j^+(\alpha_j) + \beta_j \leq 0, j=1,...,k}} \inf_{x \in X} \left[\alpha^T f(x) + \beta^T g(x) + \gamma^T h(x)\right] \tag{D^Φ_{L1}}$$

and its Fenchel-Lagrange dual problem is:

$$\sup_{\substack{\alpha \leqq 0, \beta \in \mathbb{R}^k, \gamma \geqq 0 \\ \Phi_j^+(\alpha_j) + \beta_j \leq 0, j=\overline{1,k} \\ p,q \in X^*}} \left\{-(\alpha^T f)^*(p) - (\beta^T g)^*(q) - (\gamma^T h)^*(-p-q)\right\}. \tag{D^Φ_{FL1}}$$

We know that weak duality always holds. In order to get strong duality we give the following regularity condition.

$$\exists (x', s', t') \in ri(X) \times int(\mathbb{R}^k_+) \times int(\mathbb{R}^k_+) : \begin{cases} f(x') > s', \\ g(x') = t', \\ h_j(x') \leq 0, \text{ if } j \in L, \\ h_j(x') < 0, \text{ if } j \in N, \end{cases} \tag{CQ_1}$$

where we have divided the set $\{1, \ldots, m\}$ into two disjunctive sets as follows

$L = \{j \in \{1, \ldots, m\} : h_j$ is the restriction to X of an affine function$\}$

and $N = \{1, \ldots, m\} \setminus L$.

Theorem 4.2.1 *(H.-V. Boncea, S.-M. Grad, [8]) If the constraint qualification (CQ_1), is fulfilled then there is strong duality between problems (P') and (D^Φ_{FL1}), i.e. (D^Φ_{FL1}) has an optimal solution and $v(P') = v(P^\Phi) = v(D^\Phi_{FL1})$.*

Taking into consideration that $v(D^\Phi_{FL1}) \leq v(D^\Phi_{L1}) \leq v(P^\Phi)$ we obtain the following corollary.

Corollary 4.2.2 *(H.-V. Boncea, S.-M. Grad, [8]) If the constraint qualification* (CQ_1)*, is fulfilled then there is strong duality between problems* (P') *and* (D^{Φ}_{L1})*, i.e.* (D^{Φ}_{L1}) *has an optimal solution and* $v(P') = v(P^{\Phi}) = v(D^{\Phi}_{L1})$*.*

Theorem 4.2.3 *(H.-V. Boncea, S.-M. Grad, [8]) (a) Let the constraint qualification* (CQ_1) *be fulfilled and assume that the primal problem* (P^{Φ}) *has an optimal solution* $\overline{x}$*. Then the dual problem* (D^{Φ}_{FL1}) *has an optimal solution, too, let it be* $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{p}, \overline{q})$*, and the following optimality conditions are true,*

(i) $\sum_{j=1}^{k} g_j(\overline{x})\Phi_j\left(\frac{f_j(\overline{x})}{g_j(\overline{x})}\right) = -(\overline{\alpha}^T f)^*(\overline{p}) - (\overline{\beta}^T g)^*(\overline{q}) - (\overline{\gamma}^T h)^*(-\overline{p} - \overline{q}),\ j = 1, \ldots, k,$

(ii) $\Phi_j^+(\overline{\alpha}_j) + \overline{\beta}_j \leq 0, j = 1, ..., k$

(iii) $h(\overline{x}) \leqq 0.$

(b) If $\overline{x}$ *is a feasible point to* (P^{Φ}) *and* $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{p}, \overline{q})$ *is feasible to* (D^{Φ}_{FL1}) *fulfilling the optimality conditions (i)-(iii), then there is strong duality between* (P^{Φ}) *and* (D^{Φ}_{FL1})*.*

Moreover, $\overline{x}$ *is an optimal solution to the primal problem and* $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{p}, \overline{q})$ *an optimal solution to the dual.*

Theorem 4.2.4 *(H.-V. Boncea, S.-M. Grad, [8]) (a) Let the constraint qualification* (CQ_1) *be fulfilled and assume that the primal problem* (P^{Φ}) *has an optimal solution* $\overline{x}$*. Then the dual problem* (D^{Φ}_{L1}) *has an optimal solution, too, let it be* $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$*, and the following optimality conditions are true,*

(i) $\sum_{j=1}^{k} g_j(\overline{x})\Phi_j\left(\frac{f_j(\overline{x})}{g_j(\overline{x})}\right) = \inf_{x \in X}\left[\overline{\alpha}^T f(x) + \overline{\beta}^T g(x) + \overline{\gamma}^T h(x)\right],\ j = 1, \ldots, k,$

(ii) $\Phi_j^+(\overline{\alpha}_j) + \overline{\beta}_j \leq 0, j = 1, ..., k$

(iii) $h(\overline{x}) \leqq 0.$

(b) If $\overline{x}$ *is a feasible point to* (P^{Φ}) *and* $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ *is feasible to* (D^{Φ}_{L1}) *fulfilling the optimality conditions (i)-(iii), then there is strong duality between* (P^{Φ}) *and* (D^{Φ}_{L1})*.*

Moreover, $\overline{x}$ *is an optimal solution to the primal problem and* $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ *an optimal solution to the dual.*

Special case. Consider $\Phi_j(x) = -\ln(x), x > 0$ for all $j = 1, ..., k$. The problem (P^{Φ}) turns out to be the problem treatead in [13].

In the following we present the other four cases without exposing the duals and the results concerning the strong duality and optimality conditions, which are similar to the first case.

Second case: $C = \mathbb{R}^k_+ \times \{0\}$, Φ_j -increasing, f_j -convex and g_j -affine for all $j = 1, ..., k$.

In this case consider

$$\mathcal{A} = \left\{(x,s,t) \in X \times \mathbb{R}^k_+ \times int(\mathbb{R}^k_+) : h(x) \leqq 0, f(x) \leqq s, g(x) = t\right\}.$$

Special case. Taking $\Phi_j(x) = x$, and $g_j(x) = 1$, for all $j = 1, ..., k$, and considering $f_j(x) = k_j(x - y^j)$ we obtain the Steiner-Fermat problem in [51], to which we may add some constraints by properly choosing h.

Third case. $C = \mathbb{R}^k_+ \times (-\mathbb{R}^k_+)$, Φ_j -increasing, f_j -convex and g_j -concave for all $j = 1, ..., k$. Moreover assume $\Phi_j(y) \leq 0$, $\forall y \in \left\{\frac{f(x)}{g(x)} : x \text{ feasible to } (P^\Phi)\right\}$. The feasible set of (P') is in this case

$$\mathcal{A} = \left\{(x,s,t) \in X \times \mathbb{R}^k_+ \times int(\mathbb{R}^k_+) : h(x) \leqq 0, f(x) \leqq s, g(x) \geqq t\right\}.$$

Special case. For $\Phi_j(x) = -1$, and $g_j(x) = \ln x_j, x > 0$ for all $j = 1, ..., k$, $X = int(\mathbb{R}^n_+)$ and $h(x) = Ax - b$ we obtain the Burg entropy problem in [32].

Fourth case. $C = (-\mathbb{R}^k_+) \times \mathbb{R}^k_+$, Φ_j -decreasing, f_j -concave and g_j -convex for all $j = 1, ..., k$, with the additional assumption $\Phi_j(y) \geq 0$, $\forall y \in \{\frac{f(x)}{g(x)} : x \text{ feasible to } (P^\Phi)\}$. In this case

$$\mathcal{A} = \left\{(x,s,t) \in X \times \mathbb{R}^k_+ \times int(\mathbb{R}^k_+) : h(x) \leqq 0, f(x) \geqq s, g(x) \leqq t\right\}.$$

Special case. Set $\Phi_j(x) = \lambda_j \frac{1}{x}, x > 0, \lambda_j > 0$ for all $j = 1, ..., k$. The objective function of the problem (P^Φ) is the one in the scalarized problem treatead in [77].

Fifth case. $C = \{0\} \times \{0\}$, f_j and g_j affine for all $j = 1, ..., k$. In this case we have

$$\mathcal{A} = \left\{(x,s,t) \in X \times \mathbb{R}^k_+ \times int(\mathbb{R}^k_+) : h(x) \leqq 0, f(x) = s, g(x) = t\right\}.$$

Special case. For $f_j(x) = x_j$ and $g_j(x) = d_j$, for all $j = 1, ..., k$ and $h_j(x) = \Psi_j(A_j x + \alpha_j) + b_j^T x + c_j$, $j = 1, ..., m$, we obtain the problem in [75].

Chapter 5

On the $\eta - (1,2)$ approximated optimization problems

For this section one considers X a nonempty subset of $\mathbb{R}^n$, x^0 an interior point of X, $f : X \to \mathbb{R}$ a differentiable function at x^0, $g : X \to \mathbb{R}^m$ a twice differentiable function at x^0 and let $\eta : X \times X \to \mathbb{R}^n$ be a function.

One considers the optimization problem:

$$\begin{cases} \min f(x) \\ x \in X \\ g(x) \leqq 0, \end{cases} \tag{P_η}$$

One denotes by $\mathcal{F}(P_\eta) := \{x \in X : g(x) \leqq 0\}$ the set of all feasible solutions of problem (P_η). For solving the optimization problem (P_η), there are various manners to approach (see [63, 65]). One of these manners is that for problem (P_η) one can attache another optimization problem, whose solutions gives us the (information about) optimal solutions of the initial problem (P_η) (see [1, 3, 43, 44, 45]).

Further, one attaches to problem (P_η), the problem:

$$\begin{cases} \min F(x) := f(x^0) + \langle \nabla f(x^0), \eta(x, x^0) \rangle \\ x \in X \\ G(x) := g(x^0) + [\nabla g(x^0)](\eta(x, x^0)) + \\ +\frac{1}{2} \left\langle [\nabla^2 g(x^0)](\eta(x, x^0)), \eta(x, x^0) \right\rangle \leqq 0, \end{cases} \tag{AP_η}$$

where

$$\left\langle [\nabla^2 g(x^0)](\eta(x,x^0)), \eta(x,x^0) \right\rangle \ : \ = (\left\langle [\nabla^2 g_1(x^0)](\eta(x,x^0)), \eta(x,x^0) \right\rangle, ..., \left\langle [\nabla^2 g_m(x^0)](\eta(x,x^0)), \eta(x,x^0) \right\rangle).$$

Let us denote with $\mathcal{F}(AP_\eta) := \{x \in X : G(x) \leqq 0\}$ the set of all feasible solutions of Problem (AP_η).

5.1 Preliminary notions and results

One gives here some known notions and results, which needed in this chapter and which can be found in books and monographies like [63, 65].

5.2 $\eta-$ Approximated optimization problem

In this section one considers X a nonempty subset of $\mathbb{R}^n$, x^0 an interior point of X, $f : X \to \mathbb{R}$ a differentiable function at x^0, $g : X \to \mathbb{R}^m$ twice differentiable functions at x^0 and $\eta : X \times X \to \mathbb{R}^n$ a function.

Theorem 5.2.1 *(H.-V. Boncea, D. Duca, [5])Let g be a second order invex function at x^0 w.r.t. η. If $\widetilde{x}$ is a feasible solution of the problem (P_η), then $\widetilde{x}$ is a feasible solution of the problem (AP_η), i.e. $\mathcal{F}(P_\eta) \subseteq \mathcal{F}(AP_\eta)$.*

Example 5.2.2 *(H.-V. Boncea, D. Duca, [5]) We consider the following optimization problem:*

$$\begin{cases} \min f(x) = \ln(x+1) \\ x \in (-1, \infty) \subseteq \mathbb{R} \\ g(x) = x^2 - 2x \leqq 0. \end{cases} \qquad (\overline{P})$$

The set of feasible solutions of problem $(\overline{P})$ is $\mathcal{F}(\overline{P}) = [0,2]$. For $x^0 = 0$ and $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by $\eta(x,x^0) = x - x^0 = x$, the problem $(A\overline{P})$ is

$$\begin{cases} \min F(x) = x \\ G(x) = x^2 - 2x \leqq 0. \end{cases} \qquad (A\overline{P})$$

We have $\mathcal{F}(A\overline{P}) = \mathcal{F}(\overline{P})$.

Example 5.2.3 *(H.-V. Boncea, D. Duca, [5]) We consider the following optimization problem:*

$$\begin{cases} \min f(x) = x_1 \\ x = (x_1, x_2) \in \mathbb{R}^2 \\ g(x) = -x_1 + x_2^4 \leqq 0. \end{cases} \tag{P^*}$$

The set of feasible solutions of the problem (P^*) *is* $\mathcal{F}(P^*) = \{x \in \mathbb{R}^2 : g(x) \leqq 0\}$. *For* $x^0 = (0,0)$ *and* $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by*

$$\eta(x,y) = x - y, \text{ for all } (x,y) \in \mathbb{R} \times \mathbb{R},$$

the problem (AP^*) *is*

$$\begin{cases} \min F(x) = x_1 \\ x = (x_1, x_2) \in \mathbb{R}^2 \\ G(x) = -x_1 \leqq 0. \end{cases} \tag{AP^*}$$

We have $(1,5) \in \mathcal{F}(AP^*)$, *but* $(1,5) \notin \mathcal{F}(P^*)$. *So* $\mathcal{F}(P^*) \subsetneqq \mathcal{F}(AP^*)$.

Theorem 5.2.6 *(H.-V. Boncea, D. Duca, [5]) Let* g *be a second order incave function at* x^0 *w.r.t.* η. *If* x *is a feasible solution of the problem* (AP_η), *then* x *is a feasible solution of the problem* (P_η), *i.e.* $\mathcal{F}(AP_\eta) \subseteq \mathcal{F}(P_\eta)$.

Theorem 5.2.7 *(H.-V. Boncea, D. Duca, [5]) Let* f *be a quasi-incave function at* x^0 *w.r.t.* η, g *be a second order avex function at* x^0 *w.r.t.* η *and* $\eta(x^0, x^0) = 0$. *If* $x^0 \in X$ *is an optimal solution of the original problem* (P_η), *then* x^0 *is an optimal solution of the problem* (AP_η).

Theorem 5.2.8 *(H.-V. Boncea, D. Duca, [5]) Let* f *be a pseudo-invex function at* x^0 *w.r.t.* η, g *be a second order invex function at* x^0 *w.r.t.* η *and* $\eta(x^0, x^0) = 0$.

If x^0 *is an optimal solution of Problem* (AP_η), *then* x^0 *is an optimal solution of Problem* (P_η).

5.3 Equivalence between saddle points of the $\eta-$approximated problem and of the original problem

In this section one considers X a nonempty subset of $\mathbb{R}^n$, x^0 an interior point of X, $f : X \to \mathbb{R}$ a differentiable function at x^0, $g : X \to \mathbb{R}^m$ twice differentiable function at x^0 and $\eta : X \times X \to \mathbb{R}^n$ a function. One denotes the lagrangian of the problem (AP_η), $L^\eta_{AP} : X \times \mathbb{R}^m_+ \to \mathbb{R}$, defined by

$$\begin{aligned} L^\eta_{AP}(x,v) \ :\ &= f(x^0) + \left\langle \nabla f(x^0), \eta(x,x^0) \right\rangle + \left\langle v, g(x^0) \right\rangle + \\ &+ \left\langle v, [\nabla g(x^0)](\eta(x,x^0)) + \frac{1}{2} \left\langle [\nabla^2 g(x^0)](\eta(x,x^0)), \eta(x,x^0) \right\rangle \right\rangle , \end{aligned}$$

for all $(x,v) \in X \times \mathbb{R}^m_+$.

Theorem 5.3.1 *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X , $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$ be differentiable functions at x^0. If there exists $v^0 \in \mathbb{R}^m_+$ such that $(x^0, v^0) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L^η_P, then x^0 is an optimal solution of the original problem (P_η).*

Theorem 5.3.2 *Let X be a convex subset of $\mathbb{R}^n$, x^0 be an interior point of X, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$ are convex functions. If $x^0 \in X$ is an optimal solution of the problem (P_η) then there exists $v^0 \in \mathbb{R}^m_+$ such that $(x^0, v^0) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L^η_P.*

Theorem 5.3.3 *(H.-V. Boncea, D. Duca, [5]) Let the problem (P_η) be $(1,2)$- order KT invex at x^0 w.r.t. η, g be a second order invex function at x^0 w.r.t. η and $\eta(x^0, x^0) = 0$. If $(x^0, v^0) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L^η_{AP}, then x^0 is an optimal solution of the original problem (P_η).*

Theorem 5.3.4 *(H.-V. Boncea, D. Duca, [5]) Assume that $\eta(x^0, x^0) = 0$, $\left\langle \left\langle [\nabla^2 g(x^0)](\eta(x,x^0)), \eta(x,x^0) \right\rangle , v^0 \right\rangle \geqq 0$, for all $x \in X$ and a suitable constraint qualification for the problem (P_η) is satisfied at x^0. If x^0 is an optimal solution of the problem (P_η), then there exists a point $v^0 \in \mathbb{R}^m_+$ such that $(x^0, v^0) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L^η_{AP}.*

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