"Babeş-Bolyai" Universty of Cluj-Napoca
Faculty of Mathematics and Computer Science

PhD Thesis

Summary

Approximation by Linear and Nonlinear Integral Type Operators of Real and Complex Variables

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Ch. 1

General description of the topic

In this thesis I present my results obtained as co-author or single author, concerning the approximation of functions by integral type operators of real and of complex variable.

Approximation Theory appeared in the 19th century as an important section of Mathematical Analysis. It mainly consists in the approximation of intricated elements of a space (in general functions), with simpler elements from computational point of view (in general algebraic or trigonometric polynomials, piecewise polynomials, splines, etc). Also, deals with quantitative characterizations for the error of approximation, in terms of K-functionals or moduli of smoothness.

Chronologically , in 1895 Karl Weierstrass has obtained the first approximation result expressed by the following theorem.

Theorem I. *For any* $f : [a, b] \to \mathbb{R}$ *continuous on* $[a, b]$*, there exists a sequence of algebraic polynomials,* $P_{m_n}(x) = a_0 x^{m_n} + . + a_{m_n - 1}x + a_{m_n}$*, such that* $\lim_{n\to\infty} P_{m_n}(x) = f(x)$*, uniformly on* $x \in [a, b]$.

Also, Weiesrtrass obtained an analogue for the approximation by trigonometric polynomials.

In 1912, the first constructive proof of Theorem I was obtained by S. N. Bernstein, who proved that the now so-called Bernstein polynomials given by

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f)k/n),$$

converge to any continuous function f, uniformly in $[0, 1]$.

In 1942, T. Popoviciu has obtained the following error estimate

$$|B_n(f)(x) - f(x)| \leq \frac{3}{2}\omega_1(f; 1/\sqrt{n}), \forall x \in [0, 1], n \in \mathbb{N},$$

which is the first quantitative result in approximation by Bernstein polynomials. Here $\omega_1(f;\delta) = \sup\{|f(x)-f(y)|; x, y \in [0, 1], |x-y| \leq \delta\}$ represents the modulus of continuity of f.

In parallel with the approximation by algebraic polynomials, the approximation by trigonometric polynomials of continuous and 2π periodic functions, was developed and in 1900, the first constructive result was obtained by L. Fejér. He showed that if $f : \mathbb{R} \to \mathbb{R}$ is a 2π periodic and continuous function on $\mathbb{R}$, then denoting the Fourier sum of order n by $S_n(f)(x) = \sum_{k=0}^{n} a_k \cos(kx) + b_k \sin(kx)$, were a_k and b_k are the Fourier coefficients of f, it follows that the arithmetic mean of S_n denoted by $T_n(f)(x) = \frac{S_0(f)(x)+\ldots+S_n(f)(x)}{n+1}$, converges uniformly to f on $\mathbb{R}$.

In 1911, D. Jackson obtained the first quantitative result in trigonometric approximation, by proving that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and 2π periodic, then the sequence of trigonometric polynomials now called the Jackson's polynomials and denoted by $J_n(f)(x)$, $n \in \mathbb{N}$, satisfy the error estimate

$$|J_n(f)(x) - f(x)| \leq C\omega_2(f; 1/n), \forall x \in \mathbb{R}, n \in \mathbb{N},$$

where $\omega_2(f;\delta) = \sup\{|f(x+h) - 2f(x) + f(x-h)|; 0 \leq h \leq \delta, x \in \mathbb{R}\}$ denotes the second order modulus of smoothness of f.

As a generalization of the above mentioned results, beginning with 1950 and until even our days, another very important direction in approximation of functions was developed under the name of Korovkin (or Popoviciu-Korovkin, or Bohman-Korovkin) theory, dealing with approximation by various positive and linear operators. Here we can mention the classical contributions of Popoviciu, Bohman, Korovkin, Shisha-Mond and many others. These results say, in essence, that given a sequence of positive and linear operators $(L_n(f))_{n\in\mathbb{N}}$, in order to be uniformly convergent on $[a,b]$ to the continuous function f, it is good enough to check that $L_n(e_k) \to e_k$, for $k = 0, 1$ and 2, uniformly on $[a,b]$, where $e_0(x) = 1$, $e_1(x) = x$ şi $e_2(x) = x^2$.

In the case of approximation of continuous complex functions of complex variables by polynomials or by entire functions, we can mention the classical results of Müntz-Szász, Carleman, Runge, Faber, Walsh, Arakelian, Mergelyan, Dzyadyk and many others.

This thesis is structured in four chapters.

In the present Chapter 1, after the above introduction to approximation Theory, we shortly describe below the contents of the thesis.

In Chapter 2 titled "Approximation by nonlinear integral operators", the basic idea is the replacement of the classical integral in the expressions of some integral linear operators, by more general integrals (which are not linear) and to study the approximation properties of the new obtained operators.

This chapter has three sections.

Thus, in the first section, titled "Quantitative errors in the Durrmeyer-Choquet case", in the expression of the classical Bernstein-Durrmeyer polynomials, the Lebesgue integral is replaced by the nonlinear Choquet integral with respect to a monotone and submodular set function. In this way, we obtain nonlinear approximation operators. For the pointwise and uniform approximation, we obtain quantitative estimates of the approximation error in terms of moduli of continuity. In addition, estimate of the approximation error in L^p-approximation, $1 \leq p < +\infty$, with respect to some L^P K-functional is proved. Many concrete results for particular choices of the submodular set functions are obtained here. It is pointed out that the possibility of choice for various submodular set functions allows to obtain better estimates of the approximation errors.

The second section titled "Approximation in L^p by Kantorovich-Choquet types", deals with quantitative approximation in the L^p norm, with the error estimate obtained in terms of a K-functional for the Bernstein-Kantorovich-Choquet polynomials, completing thus the quantitative estimates for the pointwise and uniform approximation in terms of moduli of continuity obtained in the paper Gal [42].

In the third section of the chapter titled "Approximation by possibilistic integral operators", we reconsider the Feller's scheme which generates linear and positive operators, by replacing the classical Lebesgue integral, with the nonlinear so-called possibilistic integral. This fact allows to generate nonlinear approximation operators having good approximation properties, operators including the max-product operators studied by B. Bede, L. Coroianu and S.G. Gal in numerous papers (see also their research monograph [7] appeared at Springer). Quantitative approximation properties for some convolution possibilistic operators obtained through Feller's scheme are proved.

In Chapter 3 titled "Arbitrary order by linear integral operators on $\mathbb{R}_+$", starting from a sequence $\lambda_n > 0$, $n \in \mathbb{N}$, converging to zero arbitrary fast, we construct sequences of Baskakov-Kantorovich, Szász-Kantorovich, Szász-Durrmeyer, Szász-Durrmeyer-Stancu and Baskakov-Szász-Durrmeyer-Stancu operators, converging to the approximated function $f : [0, \infty) \to \mathbb{R}$ with the order of convergence $\omega_1(f; \sqrt{\lambda_n})$.

We may say that the results in this chapter are of definitive type (that is, the best possible). Also, they have a strong unifying character, namely for various choices of the nodes λ_n, one may reobtain previous results obtained by other authors.

In Chapter 4 titled "Arbitrary order by linear Kantorovich operators in $\mathbb{C}$", we apply the ideas in Chapter 3 to the case of aproximation of analytic functions of complex variable in simply or multiply connected compact subsets in $\mathbb{C}$, by complex Baskakov-Kantorovich-Faber operators, Szász-Kantorovich-Faber operators, Baskakov-Kantorovich-Walsh operators and Szász-Kantorovich-Walsh operators.

Starting again from a sequence $\lambda_n > 0$, $n \in \mathbb{N}$, converging to zero arbitrarily rapid, we construct sequences of Baskakov-Kantorovich-Faber operators, Szász-Kantorovich-Faber operators, Baskakov-Kantorovich-Walsh operators and Szász-Kantorovich-Walsh operators attached to an analytic function of some exponential growth in a simple or multiple connected compact set, which approximate f with the order $O(\lambda_n)$.

This chapter has six sections. The first three sections deal with approximation in simply connected compact sets by Baskakov-Kantorovich-Faber operators and by Szász-Kantorovich-Faber operators, while the next three sections deal with approximation in multiply connected compact sets by Baskakov-Kantorovich-Walsh operators and by Szász-Kantorovich-Walsh operators.

The results presented in this thesis were obtained by the author in collaboration with professor dr. Sorin Gal, Lucian Coroianu and Bogdan Opris, or as a single author, in 6 papers, published by the following journals :

1) Coroianu, Lucian ; Gal, Sorin G. ; Opriş, Bogdan D.; **Trifa, Sorin**, Feller's scheme in approximation by nonlinear possibilistic integral operators, **Numerical Functional Analysis and Optimimization**, 38 (2017), No. 3, 327-343. (IF ISI on 2017 : 0.852 , SRI pe 2017 : 0.623). .

2) Gal, Sorin G. ; **Trifa, Sorin**, Quantitative estimates in uniform and pointwise approximation by Bernstein-Durrmeyer-Choquet operators, **Carpathian Journal of Mathematics**, 33 (2017), no. 1, 49 - 58. (IF ISI on 2017 : 0.788 , SRI on 2017 : 0.351). .

3) Gal, Sorin G. ; **Trifa Sorin**, Quantitative estimates in L^p-approximation by Bernstein-Durrmeyer-Choquet operators with respect to distorted Borel measures, **Results in Mathematics**, 72 (2017), no. 3, 1405-1415. (IF ISI on 2017 : 0.969 , SRI on 2017 : 0.667)

4) **Trifa, Sorin**, Approximation with an arbitrary order by generalized Kantorovich-type and Durrmeyer-type operators on $[0, +\infty)$, **Studia Universitatis "Babes-Bolyai", series mathematics**, vol. 62, no. 4 (2017), 485-500. (B_+ journal, indexed in Mathematical Reviews and Zentralblatt für Mathematik)

5) **Trifa, Sorin**, Approximation of analytic functions with an arbitrary order by Baskakov-Kantorovich-Faber and Szász-Kantorovich-Faber operators in compact sets, **Analele Universitatii din Oradea, fascicola mathematica**, vol. 25 , no. 1, (2018). (B_+ journal, indexed in Mathematical Reviews and Zentralblatt für Mathematik)

6) Gal, Sorin G. ; **Trifa, Sorin**, Quantitative estimates in L^p-approximation by Kantorovich-Choquet operators with respect to distorted Borel mea-

sures, submitted.

The original results obtained in the thesis are the following :

Chapter 2. Theorem 2.1.6 **was published in the paper [51]**.

Theorem 2.1.8 is new and appear for the first time in the thesis. Lemma 2.1.9 and Example 2.1.1 **were published in [51]**.

Examples 2.1.12 and 2.1.13 **were published in the paper [51]**.

Theorem 2.1.16, Remark 2.1.17 and Corollary 2.1.18 **were published in [52]**.

Theorem 2.2.2 and Remark 2.2.3 **were published in [53]**.

Theorems 2.3.3, 2.3.4, 2.3.5 and 2.3.6 **were published in [23]**.

Chapter 3. Lemma 3.2.1, Theorem 3.2.2, Corollary 3.2.3, Lemma 3.3.1, Theorem 3.3.2, Corollary 3.3.3, Lemma 3.4.1, Theorem 3.4.2, Corollary 2.4.3, Lemma 3.4.4, Theorem 3.4.5, Corollary 3.4.6, Lemma 3.5.1, Theorem 3.5.2 and Corollary 3.5.3, **were published in the paper [71]**.

Chapter 4. Definition 4.1.1, Theorem 4.1.2, Lemma 4.2.1, Theorem 4.2.2 and Theorem 4.3.1 **were published in the paper [72]**.

Definition 4.4.2, Theorem 4.5.2 and Theorem 4.6.1 are new and appear for the first time in the thesis.

Key words : monotone and submodular set function, Choquet integral, Bernstein-Durrmeyer-Choquet operator, Bernstein-Durrmeyer-Choquet operator, Kantorovich-Choquet operator, quantitative pointwise, uniform and L^p estimates, moduli of continuity, K-functionals, nonlinear possibilistic integral, possibilistic Picard operators, possibilistic Gauss-Weierstrass operators, possibilistic Poisson-Cauchy operators, generalized Baskakov-Kantorovich, Szász-Kantorovich, Baskakov-Durrmeyer, Szász-Durrmeyer operators of real variable, linear and positive operators, modulus of continuity, arbitrary order of approximation, generalized Baskakov-Kantorovich-

Faber, Szász-Kantorovich-Faber, Baskakov-Kantorovich-Walsh and Szász-Kantorovich-Walsh operators of complex variable, simply connected compact sets, multiple connected compact sets, Faber polynomials, Faber-Walsh polynomials.

I express my deep gratitude to professor dr. Sorin G. Gal, for his important support in elaborating this thesis.

Ch. 2

Approximation by nonlinear integral operators

In this chapter we deal with the study of the approximation properties of the integral operators, in the case when the classical linear integral is replaced with the nonlinear Choquet integral and with the nonlinear possibilistic integral. The chapter consists in three sections : in the first section we deal with the Bernstein-Durrmeyer-Choquet operators, in the second section with the Kantorovich-Choquet operators and in the third section we deal with the possibilistic operators.

2.1 Quantitative errors by Durrmeyer-Choquet type

In this section we study the Bernstein-Durmeyer operators of d-variables, $M_{n,\mu}$, in which the integrals written in terms of a Borel type measure μ (including therefore the Lebesgue measure too) defined on the d-dimensional simplex, are replaced by Choquet integrals with respect to a family of monotone and submodular set function $\Gamma_{n,x}$, $n \in \mathbb{N}$, $x \in S^d$. The new operators are nonlinear and generalize the linear Bernstein-Durrmeyer operators. For these operators, which could be called Bernstein-Durrmeyer-Choquet operators, we obtain uniform, pointwise and L^p quantitative approximation results in terms of moduli of continuity and K-functionals.

Also, in the one dimensional case, some concrete examples improving the classical error estimates are obtained.

2.1.1 Introduction

Let the standard simplex in $\mathbb{R}^d$

$$S^d = \{(x_1, ..., x_d); 0 \leq x_1, ..., x_d \leq 1,\ 0 \leq x_1 + ... + x_d \leq 1\}.$$

Inspired by the paper [11], in the recent papers [8], [9] and [57], uniform, pointwise and L^p convergence (respectively) of $M_{n,\mu}(f)(x)$ to $f(x)$ (as $n \to \infty$) were obtained, where $M_{n,\mu}(f)(x)$ denotes the linear, mixed Bernstein-Durrmeyer operator of d-variables, with respect to a bounded Borel measure $\mu : S^d \to \mathbb{R}_+$, defined by (supposing that f is μ-integrable on S^d)

$$M_{n,\mu}(f)(x)$$

$$= \sum_{|\alpha|=n} \frac{\int_{S^d} f(t)B_\alpha(t)d\mu(t)}{\int_{S^d} B_\alpha(t)d\mu(t)} \cdot B_\alpha(x) := \sum_{|\alpha|=n} c(\alpha, \mu) \cdot B_\alpha(x),\ x \in S^d,\ n \in \mathbb{N}. \tag{2.1}$$

In the above formula (2.1), we used the notations $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$, with $\alpha_j \geq 0$ for all $j = 0, ..., n$, $|\alpha| = \alpha_0 + \alpha_1 + ... + \alpha_n = n$ and

$$B_\alpha(x) = \frac{n!}{\alpha_0! \cdot \alpha_1! \cdot ... \cdot \alpha_n!}(1 - x_1 - x_2 - ... - x_d)^{\alpha_0} \cdot x_1^{\alpha_1} \cdot ... \cdot x_d^{\alpha_d}$$

$$:= \frac{n!}{\alpha_0! \cdot \alpha_1! \cdot \ldots \cdot \alpha_n!} \cdot P_\alpha(x).$$

The qualitative kind results in [8] and [9] on pointwise and uniform convergence, where extended in [50] to the more general setting when μ is only a monotone, bounded and submodular set function on S^d and the integrals appearing in formula (2.1), represent Choquet integrals with respect to μ.

The main goal of this section is to present quantitative pointwise, uniform and L^p estimates in terms of the modulus of continuity and of K-functionals, in approximation by the more general multivariate Bernstein-Durrmeyer-Choquet polynomial operators written in terms of Choquet integrals on the standard d-dimensional simplex, defined by

$$M_{n,\Gamma_{n,x}}(f)(x) = \sum_{|\alpha|=n} c(\alpha, \mu_{n,\alpha,x}) \cdot B_\alpha(x),\ x \in S^d,\ n \in \mathbb{N}, \tag{2.2}$$

where

$$c(\alpha, \mu_{n,\alpha,x}) = \frac{(C)\int_{S^d} f(t)\, B_\alpha(t) d\mu_{n,\alpha,x}(t)}{(C)\int_{S^d} B_\alpha(t) d\mu_{n,\alpha,x}(t)}$$

$$= \frac{(C)\int_{S^d} f(t)\, P_\alpha(t) d\mu_{n,\alpha,x}(t)}{(C)\int_{S^d} P_\alpha(t) d\mu_{n,\alpha,x}(t)}$$

and for every $n \in \mathbb{N}$ and $x \in S^d$, $\Gamma_{n,x} = (\mu_{n,\alpha,x})_{|\alpha|=n}$ is a family of bounded, monotone, submodular and strictly positive set functions on $\mathcal{B}_{S^d}$.

If the family $\Gamma_{n,x}$ reduces to one bounded, monotone, submodular and strictly positive set function (i.e. $\mu_{n,\alpha,x} = \mu$ for all n, x and α), then the operator given by (2.2) reduces to the operator considered in [50].

If $d = 1$ and the Choquet integrals are taken with respect to some concrete possibility measures, the estimates in terms of the modulus of continuity are detailed. Examples improving the estimates given by the classical operators also are presented.

2.1.2 Preliminaries

Firstly, we present a few known concepts in possibility theory useful for the next considerations. For details, see, e.g., [27].

Definition 2.1.3 For the non-empty set Ω, denote by $\mathcal{P}(\Omega)$ the family of all subsets of Ω.

(i) A function $\lambda : \Omega \to [0, 1]$ with the property $\sup\{\lambda(s); s \in \Omega\} = 1$, is called possibility distribution on Ω.

(ii) $P : \mathcal{P}(\Omega) \to [0, 1]$ is called possibility measure, if it satisfies the axioms $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(\bigcup_{i\in I} A_i) = \sup\{P(A_i); i \in I\}$ for all $A_i \subset \Omega$, and any I, an at most countable family of indices. Note that if $A, B \subset \Omega$, $A \subset B$, then the last property easily implies that $P(A) \leq P(B)$ and that $P(A \bigcup B) \leq P(A) + P(B)$.

Any possibility distribution λ on Ω, induces the possibility measure $P_\lambda :$ $\mathcal{P}(\Omega) \to [0, 1]$, $P_\lambda(A) = \sup\{\lambda(s); s \in A\}$, $A \subset \Omega$. Also, if $f : \Omega \to \mathbb{R}_+$, then the possibilistic integral of f on $A \subset \Omega$ with respect to P_λ is defined by $(Pos) \int_A f dP_\lambda = \sup\{f(t) \cdot \lambda(t); t \in A\}$ (see, e.g., [27], Chapter 1).

Some known concepts and results concerning the Choquet integral can be summarized by the following.

Definition 2.1.4 Suppose $\Omega \neq \emptyset$ and let $\mathcal{C}$ be a σ-algebra of subsets in Ω.

(i) (see, e.g., [78], p. 63) The set function $\mu : \mathcal{C} \to [0, +\infty]$ is called a monotone set function (or capacity) if $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{C}$, with $A \subset B$. Also, μ is called bounded if $\mu(\Omega) < +\infty$ and submodular if

$$\mu(A \bigcup B) + \mu(A \bigcap B) \leq \mu(A) + \mu(B), \text{ for all } A, B \in \mathcal{C}.$$

(ii) (see, e.g., [78], p. 233, or [17]) If μ is a monotone set function, normalized on $\mathcal{C}$ and if $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable (i.e., for any Borel subset $B \subset \mathbb{R}$ we have $f^{-1}(B) \in \mathcal{C}$), then for any $A \in \mathcal{C}$, the Choquet integral is defined by

$$(C)\int_A f d\mu = \int_0^{+\infty} \mu\left(F_\beta(f)\bigcap A\right) d\beta + \int_{-\infty}^0 \left[\mu\left(F_\beta(f)\bigcap A\right) - \mu(A)\right] d\beta,$$

with $F_\beta(f) = \{\omega \in \Omega; f(\omega) \geq \beta\}$. If $f \geq 0$ on A, then above we get $\int_{-\infty}^0 = 0$.

The function f will be called Choquet integrable on A if $(C)\int_A f d\mu \in \mathbb{R}$.

In what follows, we list some known properties of the Choquet integral.

Remark 2.1.5 If $\mu : \mathcal{C} \to [0, +\infty]$ is a monotone set function, then the following properties hold :

(i) For all $a \geq 0$ we have $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ (if $f \geq 0$ then see, e.g., [78], Theorem 11.2, (5), p. 228 and if f is of arbitrary sign, then see, e.g., [24], p. 64, Proposition 5.1, (ii)).

(ii) For all $c \in \mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [78], pp. 232-233, or [24], p. 65) $(C)\int_A (f + c) d\mu = (C)\int_A f d\mu + c \cdot \mu(A)$.

If μ is submodular too, then for all f, g of arbitrary sign and lower bounded, we have (see, e.g., [24], p. 75, Theorem 6.3)

$$(C)\int_A (f + g) d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu.$$

(iii) If $f \leq g$ on A then $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ (see, e.g., [78], p. 228, Theorem 11.2, (3) if $f, g \geq 0$ and p. 232 if f, g are of arbitrary sign).

(iv) Let $f \geq 0$. If $A \subset B$ then $(C)\int_A f d\mu \leq (C)\int_B f d\mu$. In addition, if μ is finitely subadditive, then $(C)\int_{A\bigcup B} f d\mu \leq (C)\int_A f d\mu + (C)\int_B f d\mu$.

(v) It is immediate that $(C)\int_A 1 \cdot d\mu(t) = \mu(A)$.

(vi) The formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0, 1] \to [0, 1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and M is a probability measure (or only finitely additive) on a σ-algebra on Ω (that is, $M(\emptyset) = 0$, $M(\Omega) = 1$ and M is countably additive), gives simple examples of normalized, monotone and submodular set functions (see, e.g., [24], pp. 16-17, Example 2.1). For example, we can take $\gamma(t) = \sqrt{t}$.

If the above γ function is increasing, concave and satisfies only $\gamma(0) = 0$, then for any bounded Borel measure $m \leq 1$, $\mu(A) = \gamma(m(A))$ gives a simple example of bounded, monotone and submodular set function.

Note that any possibility measure μ is normalized, monotone and submodular. Indeed, the axiom $\mu(A \bigcup B) = \max\{\mu(A), \mu(B)\}$ implies the monotonicity, while the property $\mu(A \bigcap B) \leq \min\{\mu(A), \mu(B)\}$ implies the submodularity.

(vii) If μ is a countably additive bounded measure, then the Choquet integral $(C)\int_A f d\mu$ reduces to the usual Lebesgue type integral (see, e.g., [24], p. 62, or [78], p. 226).

2.1.3 Pointwise and uniform estimates

The following general quantitative estimates in pointwise and uniform approximation hold.

Theorem 2.1.6 (Gal-Trifa [51]) *For each fixed $n \in \mathbb{N}$ and $x \in S^d$, let $\Gamma_{n,x} = \{\mu_{n,\alpha,x}\}_{|\alpha|=n}$ be a family of bounded, monotone, submodular and strictly positive set functions on $\mathcal{B}_{S^d}$.*

(i) For every $f \in C_+(S^d)$, $x = (x_1, ..., x_d) \in S^d$, $n \in \mathbb{N}$, we have

$$|M_{n,\Gamma_{n,x}}(f)(x) - f(x)| \leq 2\omega_1(f; M_{n,\Gamma_{n,x}}(\varphi_x)(x))_{S^d},$$

where $M_{n,\Gamma_{n,x}}(f)(x)$ is given by (2.2), $\|x\| = \sqrt{\sum_{i=1}^d x_i^2}$, $\varphi_x(t) = \|t - x\|$ and $\omega_1(f; \delta)_{S^d} = \sup\{|f(t) - f(x)|; t, x \in S^d, \|t - x\| \leq \delta\}$.

(ii) Suppose that the family $\Gamma_{n,x}$ does not depend on x. Then, for any $f \in C_+(S^d)$ and $n \in \mathbb{N}$, we get

$$\|M_{n,\Gamma_n}(f) - f\|_{C(S^d)} \le 2K\left(f; \frac{\Delta_n}{2}\right),$$

where $\Delta_n = \sum_{i=1}^{d} \|M_{n,\Gamma_n}(|\varphi_{e_i} - x_i\mathbf{1}|)\|_{C(S^d)}$,

$$K(f;t) = \inf_{g \in C_+^1(S^d)} \{\|f-g\|_{C(S^d)} + t\|\nabla g\|_{C(S^d)}\},$$

$C_+^1(S^d)$ is the subspace of all functions $g \in C_+(S^d)$ with continuous partial derivatives $\partial_i g$, $i \in \{1, ..., d\}$ and $\|\nabla g\|_{C(S^d)} = \max_{i=\{1,...,d\}}\{\|\partial_i g\|_{C(S^d)}\}$, $\varphi_{e_i}(x) = x_i$, $i \in \{1, ..., d\}$, $x = (x_1, ..., x_d)$, $\mathbf{1}(x) = 1$, for all $x \in S^d$.

Remark 2.1.7 (Gal-Trifa [51]) The positivity of function f in Theorem 2.1.6, (i), (ii) is necessary because of the positive homogeneity of the Choquet integral used in its proof. However, if f is of arbitrary sign and lower bounded on S^d with $f(x) - m \ge 0$, for all $x \in S^d$, then the statement of Theorem 2.1.6, (i), (ii) can be restated for the slightly modified Bernstein-Durrmeyer-Choquet operator defined by

$$M^*_{n,\Gamma_{n,x}}(f)(x) = M_{n,\Gamma_{n,x}}(f-m)(x) + m.$$

Indeed, in the case of Theorem 2.1.6, (i), this is immediate from $\omega_1(f - m;\delta)_{S^d} = \omega_1(f;\delta)_{S^d}$ and from $M^*_{n,\Gamma_{n,x}}(f)(x) - f(x) = M_{n,\Gamma_{n,x}}(f-m)(x) - (f(x) - m)$. Note that in the case of Theorem 2.1.6, (ii), since we may consider here that $m < 0$, we immediately get the relations

$$K(f-m;t) = \inf_{g \in C_+^1(S^d)} \{\|f-(g+m)\|_{C(S^d)} + t\|\nabla g\|_{C(S^d)}\}$$

$$= \inf_{g \in C_+^1(S^d)} \{\|f-(g+m)\|_{C(S^d)} + t\|\nabla (g+m)\|_{C(S^d)}\}$$

$$= \inf_{h \in C^1(S^d),\, h \ge m} \{\|f-h\|_{C(S^d)} + t\|\nabla h\|_{C(S^d)}\}.$$

2.1.4 Particular cases of operators

Since the estimates in Theorem 2.1.6, (i), (ii) are of very general and abstract form, involving the apparently difficult to be calculated Choquet integrals, it is of interest to obtain concrete expressions for the orders of approximation.

In this sense, we will apply Theorem 2.1.6, (i) for $d = 1$ and for some special choices of the submodular set functions.

Thus, we will consider the case of the measures of possibility. Denoting $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, let us define $\lambda_{n,k}(t) = \frac{p_{n,k}(t)}{k^k n^{-n}(n-k)^{n-k}\binom{n}{k}} = \frac{t^k(1-t)^{n-k}}{k^k n^{-n}(n-k)^{n-k}}$, $k = 0, ..., n$. Here, by convention we consider $0^0 = 1$, so that the cases $k = 0$ and $k = n$ have sense.

By considering the root $\frac{k}{n}$ of $p'_{n,k}(x)$, it is easy to see that $\max\{p_{n,k}(t); t \in [0,1]\} = k^k n^{-n}(n-k)^{n-k}\binom{n}{k}$, which implies that each $\lambda_{n,k}$ is a possibility distribution on $[0,1]$. Denoting by $P_{\lambda_{n,k}}$ the possibility measure induced by $\lambda_{n,k}$ and $\Gamma_{n,x} := \Gamma_n = \{P_{\lambda_{n,k}}\}_{k=0}^n$ (i.e. Γ is independent of x), the nonlinear Berntein-Durrmeyer polynomial operators given by (2.2), in terms of the Choquet integrals with respect to the set functions in Γ_n, will become

$$D_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_0^1 f(t)t^k(1-t)^{n-k} dP_{\lambda_{n,k}}(t)}{(C)\int_0^1 t^k(1-t)^{n-k} dP_{\lambda_{n,k}}(t)}. \tag{2.3}$$

It is easy to see that any possibility measure $P_{\lambda_{n,k}}$ is bounded, monotone, submodular and strictly positive, $n \in \mathbb{N}$, $k = 0, 1, ..., n$, so that we are under the hypothesis of Theorem 2.1.6.

We can state the following result.

Theorem 2.1.8. *Let $D_{n,\Gamma_n}(f)(x)$ be given by (2.3), $f \in C_+([0,1])$, $x \in [0,1]$ and $n \in \mathbb{N}$, $n \geq 2$. We have :*

(i)

$$|D_{n,\Gamma_n}(f)(x) - f(x)|$$

$$\leq 2\omega_1\left(f;\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{x}}{\sqrt{2n}}+\frac{1+|1-2x|}{2n}\right)_{[0,1]}.$$

(ii)

$$\|D_{n,\Gamma_n}(f)-f\|_{C[0,1]}\leq 6\omega_1\left(f;\frac{1}{\sqrt{n}}\right)_{[0,1]}.$$

2.1.5 Examples improving the classical estimates

This section contains some concrete examples improving the classical estimates.

Example 2.1.11 (Gal-Trifa [51]) Since the Bernstein-Durrmeyer-Choquet type operators in this section can be defined with respect to a family of Borel or Choquet measures, combined in various ways, this fact offers a very high flexibility and generality, allowing to construct operators having even better approximation properties. As a first example, it is clear that $B_n(f)(x)$ can also be viewed as the Bernstein-Durrmeyer-Choquet operators in the case when $\Gamma_{n,x}$ is composed by the Dirac measures $\delta_{k/n}$, $k=0,...,n$. With this occasion, we note that since the Dirac measures are not strictly positive, it is clear that the strict positivity of the set functions in Theorem 2.1.6 is not always necessary.

Example 2.1.12 (Gal-Trifa [51]) In formula (2.3), let us replace the family of measures of possibilities $\Gamma_n=\{P_{\lambda_{n,k}}\}_{k=0}^{n}$, by the family $\Gamma_n=\{\nu_{n,0},\nu_{n,n},\mu_{n-2,k-1},k=1,...,n-1\}$, where the set functions $\mu_{n-2,k-1}$, $k=1,...,n-1$ are the Lebesgue measure, $\nu_{n,0}=\delta_0$ (Dirac measure), $\nu_{n,n}$ is a monotone, submodular and strictly positive set function and define the genuine Bernstein-Durrmeyer-Choquet operators by

$$U_{n,\Gamma_n}(f)(x)=p_{n,0}(x)\cdot\frac{(C)\int_0^1 f(t)(1-t)^n d\nu_{n,0}}{(C)\int_0^1(1-t)^n d\nu_{n,0}}+p_{n,n}(x)\cdot\frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}}{(C)\int_0^1 t^n d\nu_{n,n}}$$

$$+\sum_{k=1}^{n-1} p_{n,k}(x) \cdot \frac{(C)\int_0^1 f(t)p_{n-2,k-1}(t)d\mu_{n-2,k-1}(t)}{(C)\int_0^1 p_{n-2,k-1}(t)d\mu_{n-2,k-1}(t)}.$$

Denoting by $G_n(f)(x)$, the classical genuine Bernstein-Durmeyer operator (see, e.g., [54]), we immediately obtain

$$U_{n,\Gamma_n}(f)(x) - f(x) = G_n(f)(x) - f(x) + x^n \left[\frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}(t)}{(C)\int_0^1 t^n d\nu_{n,n}(t)} - f(1)\right].$$

Let us choose $\nu_{n,n}$ defined by $\nu_{n,n}(A) = \nu(A)$, where for example, $\nu(A) = \sqrt{m(A)}$ or $\nu(A) = \sin[m(A)]$.

Suppose that $f \geq 0$ and strictly increasing on $[0,1]$. Since

$$(C)\int_0^1 f(t)t^n d\nu_{n,n}(t) = \int_0^\infty \nu_{n,n}(\{t \in [0,1]; f(t)t^n \geq \lambda\})d\lambda$$

$$= \int_0^\infty \nu(\{t \in [0,1]; f(t)t^n \geq \lambda\})d\lambda$$

$$= (C)\int_0^1 f(t)t^n d\nu(t)$$

$$\leq f(1) \cdot (C)\int_0^1 t^n d\nu_{n,n}(t),$$

it immediately follows

$$\frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}(t)}{(C)\int_0^1 t^n d\nu_{n,n}(t)} - f(1) \leq f(1) - f(1) = 0.$$

Since the strict convexity of f implies $G_n(f)(x) - f(x) > 0$ for all $x \in (0,1)$ (see, e.g., Lemma 2.1, (iv) in [54]), similar reasonings with those for the previous example show that if $f \geq 0$ is strictly convex and strictly increasing on $[0,1]$ implies

$$|U_{n,\Gamma_n}(f)(x) - f(x)|$$

$$< \max\left\{|G_n(f)(x) - f(x)|, x^n \left|\frac{(C)\int_0^1 f(t)t^n d\mu(t)}{(C)\int_0^1 t^n d\mu(t)} - f(1)\right|\right\}.$$

Therefore, $U_{n,\Gamma_n}(f)(x)$ approximates better f on $(0,1)$ than $G_n(f)(x)$.

Example 2.1.13 (Gal-Trifa [51]) In formula (2.3), let us replace the family Γ_n of measures of possibilities $P_{\lambda_{n,k}}$, $k=0,...,n$, by the family consisting in the Dirac measures $\delta_{k/n}$, $k=0,1,...,n-1$, (which are Borel measures and therefore with the corresponding Choquet integrals reducing to the classical ones) together with a monotone, submodular, strictly positive set function $P_{\lambda_{n,n}} := \nu_{n,n}$ defined by $\nu_{n,n}(A)=\nu(A)$, where for example, $\nu(A)=\sqrt{m(A)}$ or $\nu(A)=\sin[m(A)]$.

Then, denoting by

$$B_n(f)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right)$$

the classical Bernstein operators (see [69]), for D_{n,Γ_n} in (2.3) we get

$$\begin{aligned}&D_{n,\Gamma_n}(f)(x)-f(x)\\ &=\left[\sum_{k=0}^{n-1}p_{n,k}(x)f\left(\frac{k}{n}\right)+x^n\cdot\frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}(t)}{(C)\int_0^1 t^n d\nu_{n,n}(t)}\right]-f(x)\\ &=B_n(f)(x)-f(x)+x^n\left[\frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}(t)}{(C)\int_0^1 t^n d\nu_{n,n}(t)}-f(1)\right],\end{aligned}$$

where by similar reasonings with those from Example 5.2, for f strictly increasing on $[0,1]$ it follows

$$\frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}(t)}{(C)\int_0^1 t^n d\nu_{n,n}(t)}-f(1)\le 0.$$

Suppose now that $f\ge 0$ is strictly increasing and strictly convex on $[0,1]$. The strict convexity of f implies (see [69]) $B_n(f)(x)-f(x)>0$ for all $x\in(0,1)$, so, for $x\in(0,1)$, $D_{n,\Gamma_n}(f)(x)$ approximates better f than $B_n(f)(x)$, since

$$\begin{aligned}&|D_{n,\Gamma_n}(f)(x)-f(x)|\\ &<\max\left\{|B_n(f)(x)-f(x)|,x^n\left|\frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}(t)}{(C)\int_0^1 t^n d\nu_{n,n}(t)}-f(1)\right|\right\}.\end{aligned}$$

2.1.6 Quantitative L^p-approximation

The main aim of the present subsection is to study quantitative L^p-approximation results, $1 \le p < \infty$, for the case when $\Gamma_{n,x}$ reduces to one element μ, which is a particular normalized, monotone and submodular set function called distorted Borel measure , i.e. for the Bernstein-Durrmeyer-Choquet operators given by

$$D_{n,\mu}(f)(x) = \sum_{|\alpha|=n} c(\alpha, \mu) \cdot B_\alpha(x),\ x \in S^d,\ n \in \mathbb{N},$$

where

$$c(\alpha,\mu) = \frac{(C)\int_{S^d} f(t)B_\alpha(t)d\mu(t)}{(C)\int_{S^d} B_\alpha(t)d\mu(t)} = \frac{(C)\int_{S^d} f(t)P_\alpha(t)d\mu(t)}{(C)\int_{S^d} P_\alpha(t)d\mu(t)}.$$

But due to the fact that $(C)\int_0^1 f d\mu$ is not, in general, additive as function of f (it is only subadditive), even in the simple case when, for example $p = 1$ and $d = 1$, for $f \in L^1_\mu$ (meaning that f is $\mathcal{B}_{[0,1]}$-measurable and $\|f\|_{L^1_\mu} = (C)\int_0^1 |f(t)|d\mu(t) < \infty$), we get

$$\|D_{n,\mu}(f)\|_{L^1_\mu} \le \sum_{k=0}^{n} (C)\int_0^1 \binom{n}{k} t^k(1-t)^{n-k}|f(t)|d\mu(t) \le (n+1)\cdot\|f\|_{L^1_\mu},\ n \in \mathbb{N}.$$

This fact implies that in the most general case for μ, quantitative estimates in L^p-approximation by Bernstein-Durrmeyer-Choquet operators, seem to remain an open question.

However, for a large subclass of normalized, monotone and submodular set functions called distorted probability Borel measures, in the present subsection we obtain quantitative L^p-approximation results, $1 \le p < +\infty$, in terms of an appropriate K-functional.

If $\mu : \mathcal{B}_{S^d} \to [0,+\infty)$ is a monotone set function and $1 \le p < +\infty$, then we make the following notations :

$$L^p_\mu(S^d) = \{f : S^d \to \mathbb{R}; f \text{ is } \mathcal{B}_{S^d}\text{-measurable and } (C)\int_{S^d} |f(t)|^p d\mu(t) < +\infty\},$$

$$L^p_{\mu,+}(S^d) = L^p_{\mu}(S^d)\bigcap\{f : S^d \to \mathbb{R}_+\},$$

$$C(S^d) = \{f : S^d \to \mathbb{R}; f \text{ is continuous on } S^d\},$$

endowed with the norm $\|F\|_{C(S^d)} = \sup\{|F(x)|; x \in S^d\}$,

$$C_+(S^d) = \{f \in C(S^d); f \geq 0 \text{ on } S^d\},$$

$C^1_+(S^d)$ is the subspace of all functions $g \in C_+(S^d)$ with continuous partial derivatives $\partial g/\partial x_i$, $i \in \{1, ..., d\}$,

$$\|\nabla g\|_{C(S^d)} = \max_{i=\{1,\ldots,d\}} \left\{ \left\| \frac{\partial g}{\partial x_i} \right\|_{C(S^d)} \right\},$$

$$K\,(f;t)_{L^p_\mu} = \inf_{g\in C^1_+(S^d)} \{\|f-g\|_{L^p_\mu} + t\|\nabla g\|_{C(S^d)}\},\ t \geq 0,$$

$$\text{with the notation } \|F\|_{L^p_\mu} = \left((C)\int_{S^d} |F(t)|^p d\mu(t)\right)^{1/p},$$

$$IC[0,1] = \{g : [0,1] \to [0,1] : g(0) = 0, g(1) = 1, g \text{ is concave and strictly}$$
$$\text{increasing on } [0,1] \text{ and there exists } g'(0) < +\infty\}.$$

Also, denote by $\mathcal{D}(\mathcal{B}_{S^d})$ the class of all set functions $\mu : \mathcal{B}_{S^d} \to [0, +\infty)$ of the form $\mu(A) = g(M(A))$, for all $A \in \mathcal{B}_{S^d}$, where $g \in IC[0,1]$ and M is a strictly positive, probability Borel measure on $\mathcal{B}_{S^d}$. In the words of Remark 2.1.5, (vi), any such a μ will be called distorted probability Borel measure.

Remark 2.1.15 According to Remark 2.1.5, (vi), any $\mu \in \mathcal{D}(\mathcal{B}_{S^d})$ is a normalized, monotone, strictly positive and submodular set function. Simple examples of $\mu \in \mathcal{D}(\mathcal{B}_{S^d})$ are $\mu(A) = \sin[\pi \cdot m(A)/2]$, or $\mu(A) = \arctan[\tan(1) \cdot m(A)]$, or $\mu(A) = \frac{2m(A)}{1+m(A)}$, or $\mu(A) = (1 - e^{-m(A)}) \cdot \frac{e}{e-1}$, or $\mu(A) = \ln[1 + (e-1)m(A)]$, for all $A \in \mathcal{B}_{S^d}$, where m denotes the d-dimensional Lebesgue measure.

The main result of this subsection is the following.

Theorem 2.1.16 (Gal-Trifa [52]) *Let $1 \leq p < \infty$. If $\mu \in \mathcal{D}(\mathcal{B}_{S^d})$, with $\mu = g \circ M$, $g \in IC[0,1]$ and M a strictly positive, probability Borel measure on $\mathcal{B}_{S^d}$, then for all $f \in L^p_{\mu,+}(S^d)$, $n \in \mathbb{N}$, we have*

$$\|f - D_{n,\mu}(f)\|_{L^p_\mu} \leq c \cdot K\left(f; \frac{\Delta_{n,p}}{c}\right)_{L^p_\mu},$$

where $c = 1 + g'(0)^{(p+1)/p}$, $\Delta_{n,p} = \sum_{i=1}^d \|D_{n,\mu}(|\varphi_i(x) - \varphi_i(\cdot)|)\|_{L^p_\mu}$, $\varphi_i(x) = x_i$ for $x = (x_1, ..., x_d) \in S^d$.

Remark 2.1.17 (Gal-Trifa [52]) The positivity of function f in Theorem 2.1.16 is necessary because of the positive homogeneity of the Choquet integral used in the proof. However, if f is of arbitrary sign and lower bounded on S^d with $f(x) - m \geq 0$, for all $x \in S^d$, then the statement of Theorem 2.1.16 can be restated for the slightly modified Bernstein-Durrmeyer-Choquet operator defined by

$$D^*_{n,\mu}(f)(x) = D_{n,\mu}(f - m)(x) + m,$$

where we have $D^*_{n,\mu}(f)(x) - f(x) = D_{n,\mu}(f-m)(x) - (f(x) - m)$. Note that we may consider here that $m < 0$ and we immediately get the relations

$$K(f - m; t)_{L^p_\mu} = \inf_{g \in C^1_+(S^d)} \{\|f - (g + m)\|_{L^p_\mu} + t\|\nabla g\|_{C(S^d)}\}$$
$$= \inf_{g \in C^1_+(S^d)} \{\|f - (g + m)\|_{L^p_\mu} + t\|\nabla(g + m)\|_{C(S^d)}\}$$
$$= \inf_{h \in C^1(S^d), h \geq m} \{\|f - h\|_{L^p_\mu} + t\|\nabla h\|_{C(S^d)}\}.$$

Corollary 2.1.18 (Gal-Trifa [52]) *Under the hypothesis and notations in Theorem 2.1.16, we have the estimate*

$$\|f - D_{n,\mu}(f)\|_{L^p_\mu} \leq c \cdot K\left(f; \frac{d \cdot c_p}{c} \cdot \frac{1}{\sqrt{n}}\right)_{L^p_\mu},$$

where $c_p > 0$ is a constant that depends only p and c is given by Theorem 2.1.16

2.2 L^p approximation by Kantorovich-Choquet type

The aim of the present section is to continue the direction of research in the previous subsection, to L^p approximation by Bernstein-Kantorovich-Choquet operators.

With the notations used in the previous subsection and suggested by the classical forms of the linear and positive operators of Bernstein-Kantorovich, in Gal [42] it was defined the following Bernstein-Kantorovich-Choquet operators/polynomials with respect to $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^{n}$, by the formula

$$K_{n,\Gamma_{n,x}}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1),(k+1)/(n+1)])},$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

In order to be well defined these operators, it is good enough if, for example, we suppose that $f : [0,1] \to \mathbb{R}_+$ is a $\mathcal{B}_{[0,1]}$-measurable function, bounded on $[0,1]$.

Remark 2.2.1 *It is clear that if $\mu_{n,k,x} = M$, for all n, k and x, where M is the Lebesgue measure, then the above polynomials become the classical ones.*

Also, if $\mu_{n,k,x} = \delta_{k/n}$ (the Dirac measures), since $k/n \in (k/(n+1),(k+1)/(n+1))$, it is immediate that $K_{n,\Gamma_{n,x}}(f)(x)$ become the Bernstein polynomials. This fact shows the great flexibility of the formulas of these operators. More exactly, we can generate very many kinds of approximation operators, by choosing for some $\mu_{n,k,x}$ the Lebesgue measure, for some others $\mu_{n,k,x}$, the Dirac measures and for the others $\mu_{n,k,x}$, some Choquet measures.

Note that pointwise and uniform approximation by $K_{n,\Gamma_{n,x}}(f)(x)$ were studied in [42].

In this section we study quantitative L^p-approximation results, $1 \leq p < \infty$, for the Bernstein-Kantorovich polynomials $K_{n,\Gamma_{n,x}}(f)(x)$ when $\Gamma_{n,x} = \{\mu\}$. In this case, we denote it by $K_{n,\mu}$.

But as in the case of Bernstein-Durrmeyer-Choquet polynomials studied in the previous subsection, even in the simple case when, for example $p = 1$, for $f \in L^1_\mu$ (meaning that f is $\mathcal{B}_{[0,1]}$-measurable and $\|f\|_{L^1_\mu} = (C)\int_0^1 |f(t)|d\mu(t) < \infty$), considering for example the operator $K_{n,\mu}$, we easily get

$$\|K_{n,\mu}(f)\|_{L^1_\mu} \leq \sum_{k=0}^{n}(C)\int_0^1 p_{n,k}(x)d\mu(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])}$$

$$\leq \sum_{k=0}^{n}(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t) \leq (n+1) \cdot \|f\|_{L^1_\mu}.$$

This is due to the fact that $(C)\int_0^1 f d\mu$ is is not, in general, additive as function of f (it is only subadditive).

Therefore, quantitative estimates in L^p-approximation by Bernstein-Kantorovich-Choquet polynomials, remain, in the general case, an open question.

However, in what follows, as in the case of L^p- approximation by Bernstein-Durrmeyer-Choquet operators, for a large class of distorted Lebesgue measures, we will be able to prove L^p-approximation results.

With the notations in the previous subsection, we can state the following.

Theorem 2.2.2 *(Trifa [53]) Let $1 \leq p < \infty$. If $\mu \in \mathcal{D}(\mathcal{B}_{[0,1]})$, then for all $f \in L^p_{\mu,+}[0,1]$, $n \in \mathbb{N}$, we have*

$$\|f - K_{n,\mu}(f)\|_{L^p_\mu} \leq c_p \cdot K\left(f; \frac{1}{2\sqrt{n+1}}\right)_{L^p_\mu},$$

where $c_p = 1 + g'(0)^{(p+1)/p}$.

2.3 Approximation by possibilistic integral operators

In this section, we construct sequences of nonlinear approximation operators, by replacing in the classical Feller's probabilistic scheme the Lebesgue integral by the so-called possibilistic integral. In particular, for the discrete case, are reobtained all the so-called max-product Bernstein type operators, together with their qualitative approximation properties. Moreover, we study the convergence of the nonlinear possibilistic operators of Picard, Gauss-Weierstrass and Poisson-Cauchy types.

2.3.1 Introduction

The quantitative approximation properties for the max-product operators of Bernstein, Favard-Szász-Mirakjan, Baskakov, Bleimann-Butzer-Hahn and Meyer-König-Zeller kinds, were deeply studied in, e.g., the papers [5], [6], [19]-[22].

Recently, in the paper [39], by using the Bernstein's idea in [12], (see also [58]) and based on a Chebyshev-type inequality in possibility theory, these kinds of operators were interpreted as possibilistic expectations of some discrete fuzzy variables (with various possibilistic distributions), fact which allowed to obtain qualitative convergence results.

Talking about possibility theory, we can mention that it is a well-developed mathematical theory which deals with some types of uncertainties and which is considered as an alternative to probability theory (see, e.g., [27], [18]) .

The main goal of this section is to develop a possibilistic alternative to the well-known Feller's probabilistic scheme in approximation. This scheme allows to give natural approach to the max-product operators and to introduce and study many new possibilistic approximation operators.

Describing shortly, the Feller's probabilistic scheme for construction linear and positive approximation operators (see, e.g., [32], Chapter 7, or [3], Section 5.2, pp. 283-319), attaches to any continuous, bounded function $f : \mathbb{R} \to \mathbb{R}$, integral operators of the form

$$L_n(f)(x) = \int_{\Omega} f \circ Z(n, x) dP = \int_{\mathbb{R}} f dP_{Z(n,x)}.$$

Here P is a probability, $(\Omega, \mathcal{C})$ is a measurable space, $Z : \mathbb{N} \times I \to \mathcal{M}_2(\Omega)$, I is a subinterval in $\mathbb{R}$, $\mathcal{M}_2(\Omega)$ denotes the space of all square integrable random variables on Ω with respect to P and where $P_{Z(n,x)} = P(Z^{-1}(n, x)(B))$, B-Borel measurable subset of $\mathbb{R}$, represents the distribution of the random variable $Z(n, x)$ with respect to P.

Denoting now by $E(Z(n, x))$ and $Var(Z(n, x))$ the expectance and the variance of $Z(n, x)$, respectively, under the hypothesis

$$\lim_{n\to\infty} E(Z(n, x)) = x, \text{ and } \lim_{n\to\infty} Var(Z(n, x)) = 0, \text{ uniformly on } I,$$

the Feller's scheme states that for all f as above, $L_n(f)$ converges to f uniformly on each compact subinterval of I.

In addition, if for the random variable $Z(n, x)$, its probability density function $\lambda_{n,x}$ is known, then for any f we can write the following important constructive representation formula

$$L_n(f)(x) = \int_{\mathbb{R}} f dP_{Z(n,x)} = \int_{\mathbb{R}} f(t) \cdot \lambda_{n,x}(t) dP(t).$$

It is not without interest to mention here that in the recent paper [40], the classical Feller's scheme was generalized by replacing the above classical integral with the nonlinear Choquet integral.

In what follows, by analogy with the previous ideas, we introduce a Feller kind scheme based on the possibilistic integral, fact which allows to construct various convergent sequences of nonlinear operators. As particular cases, all the so-called max-product Bernstein type operators and their qualitative convergence are reobtained. Also, new nonlinear possibilistic convergent operators of Picard, Gauss-Weierstrass and Poisson-Cauchy type are considered by this Feller's scheme.

2.3.2 Possibilistic Feller's scheme

We begin by summarizing some known concepts in possibility theory, which will be used in what follows (see, e.g., [27] or [18]).

Definition 2.3.1. Let Ω be a non-empty set.

(i) Any application $X : \Omega \to \mathbb{R}$ is called fuzzy variable.

(ii) A function $\lambda : \Omega \to [0,1]$ with the property $\sup\{\lambda(s); s \in \Omega\} = 1$ is called possibility distribution on Ω.

(iii) The possibility expectation of a fuzzy variable X (on Ω), with the possibility distribution λ is defined by $M_{sup}(X) = \sup_{s\in\Omega} X(s)\lambda(s)$. Also, the possibility variance of X is defined by $V_{sup}(X) = \sup\{(X(s) - M_{sup}(X))^2\lambda(s); s \in \Omega\}$.

(iv) Let $\Omega \neq \emptyset$. A possibility measure is a maping $P : \mathcal{P}(\Omega) \to [0,1]$ which satisfies the following axioms : $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(\bigcup_{i\in I} A_i) = \sup\{P(A_i); i \in I\}$ for all $A_i \in \Omega$ and any at most countable family of indices I. This last property immediately implies that if $A \subset B$, then $P(A) \leq P(B)$ and $P(A \bigcup B) \leq P(A) + P(B)$.

Any possibility distribution λ on Ω, induces the possibility measure given by the formula $P_\lambda(A) = \sup\{\lambda(s); s \in A\}$, for all $A \subset \Omega$ (see, e.g., [27]).

To each fuzzy variable $X : \Omega \to \mathbb{R}$, we can attach its so-called distribution measure with respect to a possibility measure P induced by a possibility distribution λ, by the formula

$$P_X : \mathcal{B} \to \mathbb{R}_+,\ P_X(B) = P(X^{-1}(B)) = P(\{\omega \in \Omega; X(\omega) \in B\}),\ B \in \mathcal{B},$$

where $\mathbb{R}_+ = [0, +\infty)$ and $\mathcal{B}$ is the class of all Borel measurable subsets in $\mathbb{R}$.

It is easy to see that P_X is a possibility measure on $\mathcal{B}$, induced by the possibility distribution defined by

$$\lambda_X^* : \mathbb{R} \to [0,1],\ \lambda_X^*(t) = \sup\{\lambda(\omega); \omega \in X^{-1}(t)\}, \quad \text{if } X^{-1}(t) \neq \emptyset,$$

$$\lambda_X^*(t) = 0, \quad \text{if } X^{-1}(t) = \emptyset.$$

(v) (see, e.g., [18]) The possibilistic integral of $f : \Omega \to \mathbb{R}_+$ on $A \subset \Omega$, with respect to the possibilistic measure P_λ induced by the possibilistic distribution λ, is defined by

$$(Pos)\int_A f(t)dP_\lambda(t) = \sup\{f(t)\cdot\lambda(t); t \in A\}.$$

In order to realize our goal, denote by $Var^b(\Omega)$ the family of all bounded $X : \Omega \to \mathbb{R}$ and by $Var_+^b(\Omega)$ the family of all bounded $X : \Omega \to \mathbb{R}_+$. Also, for any interval $I \subset \mathbb{R}$, let us take $Z : \mathbb{N} \times I \to Y$, with $Y = Var^b(\Omega)$ or $Y = Var_+^b(\Omega)$.

Note that if for all $(n, x) \in \mathbb{N} \times I$ it follows $Z(n, x) \in Var_+^b(\Omega)$, then for $M_{sup}(Z(n, x))$ and $V_{sup}(Z(n, x))$ given by Definition 2.3.1, (iii), we can give the formulas

$$M_{sup}(Z(n,x)) = (Pos)\int_\Omega Z(n,x)(t)dP_\lambda(t) := \alpha_{n,x}, \qquad (2.4)$$

$$V_{sup}(Z(n,x)) = (Pos)\int_\Omega (Z(n,x)(t) - \alpha_{n,x})^2 dP_\lambda(t) := \sigma_{n,x}^2. \qquad (2.5)$$

Now, having in mind the classical Feller's scheme, to $f : \mathbb{R} \to \mathbb{R}_+$ we can attach the sequence of nonlinear operators given by

$$L_n(f)(x) := (Pos) \int_{\mathbb{R}} f(t) dP_{Z(n,x)}(t),\ n \in \mathbb{N},\ x \in I, \qquad (2.6)$$

with $P_{Z(n,x)}$ defined as in Definition 2.3.1, (iv).

Firstly, we state that for the operators given by (2.6), the next auxiliary result holds.

Lemma 2.3.3. (Coroianu-Gal-Opriş-Trifa [23]) *If* $Z : \mathbb{N} \times I \to Var^b(\Omega)$ *and* $f : \mathbb{R} \to \mathbb{R}_+$ *is bounded on* $\mathbb{R}$*, then we can write the formula*

$$L_n(f)(x) = (Pos) \int_{\mathbb{R}} f(t) dP_{Z(n,x)}(t) = (Pos) \int_{\Omega} f \circ Z(n,x) dP_\lambda,\ x \in I \quad (2.7)$$

where both possibilistic integrals are finite.

Moreover, if $f : I \to \mathbb{R}_+$ *is bounded on the subinterval* $I \subset \mathbb{R}$ *and* $P_\lambda(\{\omega \in \Omega; Z(n,x)(\omega) \in I\}) = 1$*, then we obtain*

$$L_n(f)(x) = (Pos) \int_{\mathbb{I}} f(t) dP_{Z(n,x)}(t) = (Pos) \int_{\Omega} f \circ Z(n,x) dP_\lambda.$$

Remark. Formula (2.7) can be rewritten as

$$L_n(f)(x) = \sup\{f(t) \cdot \lambda^*_{Z(n,x)}(t); t \in \mathbb{R}\} = \sup\{f[Z(n,x)(t)] \cdot \lambda(t); t \in \Omega\},$$

with $\lambda^*_{Z(n,x)}(t)$ given by Definition 2.3.1, (iv).

In what follows, suppose that $Z(n,x) \in Var^b_+(\Omega)$, imposed by the fact that the next result involves the quantity $\alpha_{n,x}$ given by formula (2.4).

We are now in position to state the following Feller type result.

Theorem 2.3.4. (Coroianu-Gal-Opriş-Trifa [23]) *Suyppose that* $I \subset \mathbb{R}$ *is a subinterval,* $Z(n,x) \in Var^b_+(\Omega)$ *for all* $(n,x) \in \mathbb{N} \times I$ *and* $f : \mathbb{R} \to \mathbb{R}_+$ *is bounded and uniformly continuous on* $\mathbb{R}$*. With the above notations in*

(2.4), (2.5) and the statement of Lemma 2.3.3, under the hypothesis that $\lim_{n\to+\infty}\alpha_{n,x}=x$ *and* $\lim_{n\to+\infty}\sigma_{n,x}^2=0$*, uniformly with respect to* $x\in I$*, we get* $\lim_{n\to\infty}L_n(f)(x)=f(x)$*, uniformly on* I*.*

Remarks. 1) The proof of Theorem 2.3.4, easily implies that the construction of the operators $L_n(f)(x)$ can be generalized by considering that all three Z, λ and P_λ depend on n and x. More exactly, $L_n(f)(x)$ could be written in the more general form

$$L_n(f)(x) := (Pos)\int_{\mathbb{R}} f(t)dP_{Z(n,x)}(t) = (Pos)\int_{\Omega} f\circ Z(n,x)dP_{\lambda_{n,x}},\ x\in I,$$

where $P_{\lambda_{n,x}}:\mathcal{P}(\Omega)\to[0,1]$, $(n,x)\in\mathbb{N}\times I$, is a family of possibility measures induced by the families of distributions $\lambda_{n,x}$, $(n,x)\in\mathbb{N}\times I$. This remark may be used to produce many concrete examples of such operators.

Moreover, it is worth mentioning here that under the hypothesis that $P_\lambda(\{\omega\in\Omega; Z(n,x)(\omega)\in I\}=1$, the operators L_n might be attached to any bounded and continuous function defined on a subinterval $I\subset\mathbb{R}$, $f:I\to\mathbb{R}_+$, by prolonging f to a bounded continuous function on the whole $\mathbb{R}$ given by $f^*:\mathbb{R}\to\mathbb{R}_+$ and based on the relationship

$$(Pos)\int_{\mathbb{R}} f^* dP_{Z(n,x)} = (Pos)\int_I f dP_{Z(n,x)}.$$

2) If we suppose that $f:I\to\mathbb{R}$ is lower bounded but not necessarily positive, then there exists a constant $c>0$ with $f(x)+c\geq 0$, for all $x\in I$. In this case, for all $n\in\mathbb{N}$, we can attach to f the sequence of approximation operators

$$L_n(f)(x)$$
$$= (Pos)\int_{\mathbb{I}}(f(t)+c)dP_{Z(n,x)}(t)-c = (Pos)\int_{\Omega}(f+c)\circ Z(n,x)dP_{\lambda_{n,x}}-c.$$

3) In particular, qualitative approximation properties can be deduced by the Feller's scheme in Theorem 2.3.4, for all the so-called max-product

Bernstein-type operators. For example, taking $\Omega = \{0, 1, ..., n\}$, $I = [0, 1]$, $Z(n, x)(k) = \frac{k}{n}$, $f : [0, 1] \to \mathbb{R}_+$, $\lambda_{n,x}(k) = \frac{p_{n,k}(x)}{\bigvee_{j=0}^{n} p_{n,j}(x)}$, where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $\bigvee_{j=0}^{n} p_{n,j}(x) = \max_{j=\{0,...,n\}}\{p_{n,j}(x)\}$, by Lemma 2.3.3 we get

$$L_n(f)(x) = (Pos)\int_{\Omega} f \circ Z(n, x) dP_{\lambda_{n,x}} = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)},$$

which represent the Bernstein operators $B_n^{(M)}(f)(x)$ of max-product type, to which we can apply now Theorem 2.3.4.

Similarly, if, for another example, we consider $\Omega = \{0, 1, ..., k, ..., \}$ and $P_{\lambda_{n,x}}$ the possibility measure induced by the possibility distribution

$$\lambda_{n,x}(k) = \frac{s_{n,k}(x)}{\bigvee_{k=0}^{\infty} s_{n,k}(x)}, x \in [0, +\infty), k \in \mathbb{N} \bigcup \{0\},$$

with $s_{n,k}(x) = \frac{(nx)^k}{k!}$ and $\bigvee_{k=0}^{\infty} s_{n,k}(x) = \max_{k=\{0,1,...,k,...\}}\{s_{n,k}(x)\}$, then from Lemma 2.3,3 we obtain the Favard-Szász-Mirakjan operators of max-product type.

Also, qualitative approximation results for other max-product type operators, like those of Baskakov, Bleimann-Butzer-Hahn kind and Meyer-König-Zeller kind can be obtained in an analogous way by using Theorem 2.3.4.

Let us mention that quantitative error estimates in approximation by max-product type operators, were obtained by other methods in, e.g., [5], [6], [19]-[22], (see also and their References).

2.3.3 Approximation by convolution possibilistic operators

Based on the previous possibilistic Feller's scheme, in this subsection we define and study the possibilistic variants to the classical convolution operators of Picard, Gauss-Weierstrass and Poisson-Cauchy, which formally are given by

$$P_n(f)(x) = \frac{n}{2}\int_{\mathbb{R}} f(t)e^{-n|x-t|}dt, \quad W_n(f)(x) = \frac{\sqrt{n}}{\sqrt{\pi}}\int_{\mathbb{R}} f(t)e^{-n|t-x|^2}dt,$$

$$Q_n(f)(x) = \frac{n}{\pi}\int_{\mathbb{R}} \frac{f(t)}{n^2(t-x)^2+1},$$

respectively, with $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

For $\Omega = \{0, 1, ..., k, ..., \}$ and $Z(n, x)$ as in the above Remark 3, defining $\lambda_{n,x}(k) = \frac{e^{-n|x-k/n|}}{\bigvee_{k=-\infty}^{\infty} e^{-n|x-k/n|}}$, by Lemma 2.3.3

$$L_n(f)(x) = (Pos)\int_{\Omega} f \circ Z(n, x)dP_{\lambda_{n,x}},$$

we get the following possibilistic (or max-product !) Picard operators

$$P_n^{(M)}(f)(x) = \frac{\bigvee_{k=-\infty}^{+\infty} f(k/n) \cdot e^{-n|x-k/n|}}{\bigvee_{k=-\infty}^{+\infty} e^{-n|x-k/n|}}.$$

Analogously, if $\lambda_{n,x}(k) = \frac{e^{-n(x-k/n)^2}}{\bigvee_{k=-\infty}^{\infty} e^{-n(x-k/n)^2}}$ and $\lambda_{n,x}(k) = \frac{1/(n^2(x-k/n)^2+1)}{\bigvee_{k=0}^{\infty} 1/(n^2(x-k/n)^2+1)}$ we get the following possibilistic operators,

$$W_n^{(M)}(f)(x) = \frac{\bigvee_{k=-\infty}^{+\infty} f(k/n) \cdot e^{-n(x-k/n)^2}}{\bigvee_{k=-\infty}^{+\infty} e^{-n(x-k/n)^2}}, \text{ - of Gauss-Weierstrass kind,}$$

$$Q_n^{(M)}(f)(x) = \frac{\bigvee_{k=-\infty}^{+\infty} f(k/n) \cdot \frac{1}{n^2(x-k/n)^2+1}}{\bigvee_{k=-\infty}^{+\infty} \frac{1}{n^2(x-k/n)^2+1}}, \text{ - of Poisson-Cauchy kind.}$$

Now, denote by $BUC_+(\mathbb{R})$, the space of all bounded, positive and uniformly continuous functions. Qualitative approximation properties for these operators can be obtained from Theorem 2.3.4. More than that, by the next result we can obtain quantitative estimates too, as follows.

Theorem 2.3.5. (Coroianu-Gal-Opriş-Trifa [23]) *If* $f \in BUC_+(\mathbb{R})$, *then we have*

$$|P_n^{(M)}(f)(x) - f(x)| \leq 2 \cdot \omega_1(f; 1/n)_{\mathbb{R}}.$$

Theorem 2.3.7. (Coroianu-Gal-Opriş-Trifa [23]) *If* $f \in BUC_+(\mathbb{R})$ *then we have*

$$|W_n^{(M)}(f)(x) - f(x)| \leq 2 \cdot \omega_1(f; 1/\sqrt{n})_{\mathbb{R}}.$$

Theorem 2.3.9. (Coroianu-Gal-Opriş-Trifa [23]) *If* $f \in BUC_+(\mathbb{R})$, *then we have*

$$|Q_n^{(M)}(f)(x) - f(x)| \leq 2 \cdot \omega_1(f; 1/\,(2n))_{\mathbb{R}}.$$

Ch. 3

Approximation on $\mathbb{R}_+$ by integral operators

Given an arbitrary sequence $\lambda_n > 0$, $n \in \mathbb{N}$, with the property that $\lim_{n\to\infty} \lambda_n = 0$ as fast we want, in this chapter we introduce generalized Szász-Kantorovich, Baskakov-Kantorovich, Szász-Durrmeyer-Stancu and Baskakov-Szász-Durrmeyer-Stancu operators, such that on each compact subinterval in $[0, +\infty)$ the order of uniform approximation is $\omega_1(f; \sqrt{\lambda_n})$. These generalized operators uniformly approximate a Lipschitz 1 function, on each compact subinterval of $[0, \infty)$, with the arbitrary good order of approximation $\sqrt{\lambda_n}$.

3.1 Introduction

It is known that the classical Baskakov operators are given by the formula (see, e.g., [4])

$$V_n(f)(x) = \sum_{j=0}^{\infty} \binom{n+j-1}{j} \frac{x^j}{(1+x)^{n+j}} f(\frac{j}{n})$$

$$= (1+x)^{-n} \sum_{j=0}^{\infty} \frac{(n+j-1)!}{j!(n-1)!} \frac{x^j}{(1+x)^j}$$

$$= (1+x)^{-n} \sum_{j=0}^{\infty} \frac{n(n+1)...(n+j-1)}{j!} \frac{x^j}{(1+x)^j}.$$

In the recent paper [49], this operator was modified by replacing n with $\frac{1}{\lambda_n}$, where $\lim_{n\to\infty} \lambda_n = 0$ as fast we want, and the approximation properties (of arbitrary good order depending on λ_n) of the new obtained Baskakov operator defined by the formula

$$V_n(f;\lambda_n)(x) = (1+x)^{\frac{-1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n}(1+\frac{1}{\lambda_n})...(j-1+\frac{1}{\lambda_n})(\frac{x}{1+x})^j f(j\lambda_n),\ x \geq_0,$$

were obtained. Above by convention, $\frac{1}{j!}\frac{1}{\lambda_n}(1+\frac{1}{\lambda_n})...(j-1+\frac{1}{\lambda_n}) = 1$ for $j=0$. The complex variable case for $V_n(f;\lambda_n)$ was studied in [48]. Also, in [38], the above idea was applied to the Jakimovski-Leviatan-Ismail kind generalization of Szász-Mirakjan operators.

The goal of the present chaper is that based on the above idea, to introduce modified/generalized Szász-Kantorovich, Baskakov-Kantorovich, Szász-Durrmeyer-Stancu and Baskakov-Szász-Durrmeyer-Stancu operators in such a way that on each compact subinterval in $[0,+\infty)$ the order of uniform approximation is $\omega_1(f;\sqrt{\lambda_n})$. These modified operators can uniformly approximate a Lipschitz 1 function, on each compact subinterval of $[0,\infty)$ with the arbitrary good order of approximation $\sqrt{\lambda_n}$ given at the beginning.

In conclusion, it is worth mentioning for these generalized operators that since λ_n ca be chosen with $\lambda_n \searrow 0$ arbitrary fast, in fact it follows that the order of convergence $\omega_1(f;\sqrt{\lambda_n})$ is arbitrary good. For this reason, the results obtained by this chapter have a definitive character (that is they are the best possible). In the same time, the results also have a strong unifying character, in the sense that for various choices of the nodes λ_n one can recapture previous approximation results obtained by many other authors.

3.2 Baskakov-Kantorovich operators

It is known that the classical Baskakov-Kantorovich operators are defined by (see, e.g., [13])

$$K_n(f)(x) = \sum_{j=0}^{\infty} \binom{n+j-1}{j} \frac{x^j}{(1+x)^{n+j}} n \int_{j/n}^{(j+1)/n} f(v)dv$$

$$= (1+x)^{-n} \sum_{j=0}^{\infty} \frac{n(n+1)...(n+j-1)}{j!} \frac{x^j}{(1+x)^j} n \int_{j/n}^{(j+1)/n} f(v)dv.$$

If we replace n with $\frac{1}{\lambda_n}$, then we obtain the generalized Baskakov-Kantorovich operators, defined by the formula

$$K_n(f;\lambda_n)(x)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n}(1+\frac{1}{\lambda_n})...(j-1+\frac{1}{\lambda_n}) \frac{x^j}{(1+x)^j} \frac{1}{\lambda_n} \int_{j\lambda_n}^{(j+1)\lambda_n} f(v)dv.$$

Denote everywhere in the paper $e_k(x) = x^k$, $k = 0, 1, 2, ...,$.

This section deals with the approximation properties of the operator $K_n(f;\lambda_n)(x)$. The main result of this section is the following.

Theorem 3.2.2. (Trifa [71]) *Let $\lambda_n \searrow 0$ (with $n \to \infty$) as fast we want and suppose that $f : [0, +\infty) \to \mathbb{R}$ is uniformly continuous on $[0, +\infty)$ (we write $f \in UC[0, +\infty)$). For all $x \in [0, +\infty)$ and $n \in \mathbb{N}$, we have*

$$|K_n(f;\lambda_n)(x) - f(x)| \leq 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{x^2 + x + \lambda_n/3}),$$

where $\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; x, y \in [0, +\infty), |x - y| \leq \delta\}$ denotes the modulus of continuity of f with the step δ.

As an immediate consequence of Theorem 3.2.2, we get the following.

Corollary 3.2.3. (Trifa [71]) *Let* $\lambda_n \searrow 0$ *as fast we want and suppose that* f *is a Lipschitz function, that is there exists* $M > 0$ *such that* $|f(x) - f(y)| \leq M|x-y|$, *for all* $x, y \in [0, \infty)$. *Then, for all* $x \in [0, +\infty)$ *and* $n \in \mathbb{N}$ *we have*

$$|K_n(f;\lambda_n)(x) - f(x)| \leq 2M\sqrt{\lambda_n} \cdot \sqrt{x + x^2 + \lambda_n/3}.$$

Proof. Since by hypothesis f is a Lipschitz function, we easily get $\omega_1(f;\delta) \leq M\delta$, for all $\delta > 0$. Choosing now $\delta = \sqrt{\lambda_n} \cdot \sqrt{x + x^2 + \lambda_n/3}$ and applying Theorem 3.2.2, we get the desired estimate. □

Remarks. 1) Since $f \in UC[0, +\infty)$, it is well-known that we get $\lim_{\delta \searrow 0} \omega_1(f;\delta) = 0$. Therefore, choosing $\delta = \lambda_n$, we get that for $n \to \infty$, since $\lambda_n \searrow 0$, passing to limit with $n \to \infty$ in the estimate in Theorem 3.2.2, it follows that $K_n(f;\lambda_n)(x) \to f(x)$, pointwise for any $x \in [0, +\infty)$. Now, in order to get uniform convergence in the above results, the expression $\sqrt{x + x^2 + \lambda_n/3}$ must be bounded, fact which holds when x belongs to a compact subinterval of $[0, +\infty)$.

2) If $f \in UC[0, +\infty)$, then $K_n(f;\lambda_n)(x)$ is well defined, that is

$$|K_n(f;\lambda_n)(x)| < +\infty, \text{ for all } x \in [0, +\infty) \text{ and } n \in \mathbb{N}.$$

Indeed, if f is uniformly continuous on $[0, +\infty)$ then it is well known that its growth on $[0, +\infty)$ is linear, i.e. there exist $\alpha, \beta > 0$ such that $|f(x)| \leq \alpha x + \beta$, for all $x \in [0, +\infty)$ (see e.g. [25], p. 48, Problème 4, or [26]). This immediately implies

$$|K_n(f;\lambda_n)(x)| \leq K_n(|f|;\lambda_n)(x) \leq \alpha \cdot K_n(e_1;\lambda_n)(x) + \beta$$

$$= \alpha(x + \lambda_n/2) + \beta < +\infty,$$

for all $x \in [0, +\infty)$, $n \in \mathbb{N}$.

3.3 Szász-Kantorovich operators

The formula for the classic, linear and positive Szász-Kantorovich operators is given by (see, e.g., [73])

$$S_n(f)(x) = e^{-nx}\sum_{j=0}^{\infty}\frac{(nx)^j}{j!}n\int_{\frac{j}{n}}^{\frac{j+1}{n}} f(v)dv = e^{-nx}\sum_{j=0}^{\infty}\frac{(nx)^j}{j!}\int_0^1 f(\frac{t+j}{n})dt.$$

Replacing above n with $\frac{1}{\lambda_n}$, we obtain the generalized Szász-Kantorovich operators, defined by the formula

$$S_n(f;\lambda_n)(x)$$

$$= e^{-\frac{x}{\lambda_n}}\sum_{j=0}^{\infty}\frac{x^j}{\lambda_n^j j!}\frac{1}{\lambda_n}\int_{j\lambda_n}^{(j+1)\lambda_n} f(v)dv = e^{-\frac{x}{\lambda_n}}\sum_{j=0}^{\infty}\frac{x^j}{\lambda_n^j j!}\int_0^1 f(\lambda_n(t+j))dt.$$

We study here the approximation properties of the operator $S_n(f;\lambda_n)(x)$.

The main result of this section is the following.

Theorem 3.3.2. (Trifa [71]) *Let $\lambda_n \searrow 0$ (with $n \to \infty$) as fast we want and suppose that $f : [0,+\infty) \to \mathbb{R}$ is uniformly continuous on $[0,+\infty)$. For all $x \in [0,+\infty)$ and $n \in \mathbb{N}$, we have*

$$|S_n(f;\lambda_n)(x) - f(x)| \le 2\omega_1(f;\sqrt{\lambda_n}\cdot\sqrt{x+\lambda_n/3}),$$

where $\omega_1(f;\delta) = \sup\{|f(x)-f(y)|; x,y \in [0,+\infty),\ |x-y| \le \delta\}$ denotes the modulus of continuity of f with the step δ.

As an immediate consequence of Theorem 3.3.2 we get the following.

Corollary 3.3.3. (Trifa [71]) *Let $\lambda_n \searrow 0$ as fast we want and suppose that f is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \le M|x-y|$, for all $x, y \in [0,\infty)$. Then, for all $x \in [0,+\infty)$ and $n \in \mathbb{N}$ we have*

$$|S_n(f;\lambda_n)(x) - f(x)| \le 2M\sqrt{\lambda_n}\cdot\sqrt{x+\lambda_n/3}.$$

Proof. Since by hypothesis f is a Lipschitz function, we easily get $\omega_1(f;\delta) \leq M\delta$, for all $\delta > 0$. Choosing now $\delta = \sqrt{\lambda_n} \cdot \sqrt{x + \lambda_n/3}$ and applying Theorem 3.3.2, we get the desired estimate. □

3.4 Szász-Durrmeyer type operators

Let us recall that the classical Szász-Durrmeyer operators are given by the formula (see, e.g., [63])

$$SD_n(f)(x) = n\sum_{j=0}^{\infty} s_{n,j}(x)\int_0^{\infty} s_{n,j}(t)f(t)dt,$$

where $s_{n,j}(x) = e^{-nx}\frac{(nx)^j}{j!}$

If we replace n with $\frac{1}{\lambda_n}$, then we obtain the generalized Szász-Durrmeyer operators, defined by the formula

$$SD_n(f;\lambda_n)(x) = \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \cdot \frac{x^j}{\lambda_n^j j!}\int_0^{\infty} e^{-\frac{t}{\lambda_n}} \cdot \frac{t^j}{\lambda_n^j j!}f(t)dt.$$

In the first part of this section we study the approximation properties of the operator $SD_n(f;\lambda_n)(x)$.

The first main result of this section is the following.

Theorem 3.4.2. (Trifa [71]) *Let $\lambda_n \searrow 0$ as fast we want and suppose that $f : [0, +\infty) \to \mathbb{R}$ is uniformly continuous on $[0, +\infty)$. For all $x \in [0, +\infty)$ and $n \in \mathbb{N}$, we have*

$$|SD_n(f;\lambda_n)(x) - f(x)| \leq 2\omega_1(f;\sqrt{\lambda_n} \cdot \sqrt{2x + 2\lambda_n}).$$

Proof. Reasoning exactly as in the proof of Theorem 3.2.2, we can write

$$|SD_n(f;\lambda_n)(x) - f(x)| \leq (1 + \delta^{-1}\sqrt{SD_n(\varphi_x^2;\lambda_n)(x)})\omega_1(f;\delta).$$

Choosing here $\delta = \sqrt{SD_n(\varphi_x^2; \lambda_n)(x)}$ and using Lemma 3.4.1, (ii), we obtain

$$|SD_n(f; \lambda_n)(x) - f(x)| \le 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{2x + 2\lambda_n}),$$

which proves the theorem. □

As an immediate consequence of Theorem 3.4.2 we get the following.

Corollary 3.4.3. (Trifa [71]) *Let* $\lambda_n \searrow 0$ *as fast we want and suppose that* f *is a Lipschitz function, that is there exists* $M > 0$ *such that* $|f(x) - f(y)| \le M|x-y|$, *for all* $x, y \in [0, \infty)$. *Then, for all* $x \in [0, +\infty)$ *and* $n \in \mathbb{N}$ *we have*

$$|SD_n(f; \lambda_n)(x) - f(x)| \le 2M\sqrt{\lambda_n} \cdot \sqrt{2x + 2\lambda_n}.$$

Proof. Since by hypothesis f is a Lipschitz function, we easily get $\omega_1(f; \delta) \le M\delta$, for all $\delta > 0$. Choosing now $\delta = \sqrt{\lambda_n} \cdot \sqrt{2x + 2\lambda_n}$ and applying Theorem 3.4.2, we get the desired estimate. □

In what follows we will introduce and study the generalized Szász-Durrmeyer-Stancu operators. Thus it is well-known that the classical Szász-Durrmeyer-Stancu operators are given by the formula (see, e.g., [44])

$$SD_n^{(\alpha,\beta)}(f)(x) = n \sum_{j=0}^{\infty} s_{n,j}(x) \int_0^{\infty} s_{n,j}(t) f\left(\frac{nt + \alpha}{n + \beta}\right) dt,$$

where $0 \le \alpha \le \beta$ and $s_{n,j}(x) = e^{-nx} \frac{(nx)^j}{j!}$.

If we replace n with $\frac{1}{\lambda_n}$, we obtain:

$$SD_n^{(\alpha,\beta)}(f; \lambda_n)(x) = \frac{1}{\lambda_n} \sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!} \int_0^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!} f\left(\frac{\frac{t}{\lambda_n} + \alpha}{\frac{1}{\lambda_n + \beta}}\right) dt.$$

The main result concerning these operators of Stancu type is the following.

Theorem 3.4.5. (Trifa [71]) *Let* $0 \le \alpha \le \beta$, $\lambda_n \searrow 0$ *as fast we want and suppose that* $f : [0, +\infty) \to \mathbb{R}$ *is uniformly continuous on* $[0, +\infty)$. *For* *all* $x \in [0, +\infty)$ *and* $n \in \mathbb{N}$, *we have*

$$|SD_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \leq 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{E_n^{(\alpha,\beta)}(x)}),$$

where

$$E_n^{(\alpha,\beta)}(x) = \frac{\lambda_n \beta^2}{(1+\lambda_n\beta)^2}x^2 + \frac{1-2\beta(\alpha+1)\lambda_n}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n(\alpha^2+2\alpha+2)}{(1+\lambda_n\beta)^2}$$

.

As an immediate consequence of Theorem 3.4.5, we get the following.

Corollary 3.4.6. (Trifa [71]) *Let* $0 \leq \alpha \leq \beta$, $\lambda_n \searrow 0$ *as fast we want and suppose that* f *is a Lipschitz function, that is there exists* $M > 0$ *such that* $|f(x) - f(y)| \leq M|x-y|$, *for all* $x, y \in [0, \infty)$. *Then, for all* $x \in [0, +\infty)$ *and* $n \in \mathbb{N}$ *we have*

$$|SD_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \leq 2M\sqrt{\lambda_n} \cdot \sqrt{E_n^{(\alpha,\beta)}(x)}.$$

3.5 Baskakov-Szász-Durrmeyer-Stancu operators

For $0 \leq \alpha \leq \beta$, the classical Baskakov- Szász-Durrmeyer-Stancu operators are given by the formula (see, e.g., [56])

$$V_n^{(\alpha,\beta)}(f)(x) = n\sum_{j=0}^{\infty} b_{n,j}(x)\int_0^{\infty} s_{n,j}(t) f(\frac{nt+\alpha}{n+\beta}dt,$$

where, $s_{n,j}(x) = e^{-nx}\frac{(nx)^j}{j!}$ and

$$b_{n,j}(x) = \binom{n+j-1}{j}\frac{x^j}{(1+x)^{n+j}} = (1+x)^{-n}\frac{n(n+1)...(n+j-1)}{j!}\frac{x^j}{(1+x)^j}.$$

If we replace n with $\frac{1}{\lambda_n}$ we obtain the formula:

$$V_n^{(\alpha,\beta)}(f;\lambda_n)(x)$$
$$= \frac{1}{\lambda_n}\sum_{j=0}^{\infty}(1+x)^{-\frac{1}{\lambda_n}}\frac{\frac{1}{\lambda_n}(\frac{1}{\lambda_n}+1)...(\frac{1}{\lambda_n}+j-1)}{j!}\frac{x^j}{(1+x)^j}$$
$$\cdot\int_0^{\infty}e^{-\frac{t}{\lambda_n}}\cdot\frac{(\frac{t}{\lambda_n})^j}{j!}f(\frac{\frac{t}{\lambda_n}+\alpha}{\frac{1}{\lambda_n}+\beta})dt.$$

The main result of this section is the following.

Theorem 3.5.2. (Trifa [71]) *Let $0 \le \alpha \le \beta$, $\lambda_n \searrow 0$ as fast we want and suppose that $f : [0,+\infty) \to \mathbb{R}$ is uniformly continuous on $[0,+\infty)$. For all $x \in [0,+\infty)$ and $n \in \mathbb{N}$, we have*

$$|V_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \le 2\omega_1(f;\sqrt{\lambda_n}\cdot\sqrt{F_n^{(\alpha,\beta)}(x)}),$$

where

$$F_n^{(\alpha,\beta)}(x) = \frac{1+\lambda_n\beta^2}{(1+\lambda_n\beta)^2}x^2 + \frac{2-2\lambda_n\beta-2\lambda_n\alpha\beta}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n\alpha^2+2\lambda_n\alpha+2\lambda_n}{(1+\lambda_n\beta)^2}.$$

As an immediate consequence of Theorem 3.5.2 we get the following.

Corollary 3.5.3. (Trifa [71]) *Let $0 \le \alpha \le \beta$, $\lambda_n \searrow 0$ as fast we want and suppose that f is a Lipschitz function, that is there exists $M > 0$ such that $|f(x)-f(y)| \le M|x-y|$, for all $x, y \in [0,\infty)$. Then, for all $x \in [0,+\infty)$ and $n \in \mathbb{N}$ we have*

$$|V_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \le 2M\sqrt{\lambda_n}\cdot\sqrt{F_n^{(\alpha,\beta)}(x)}.$$

Ch. 4

Approximation by complex Kantorovich type operators

By using a sequence $\lambda_n > 0$, $n \in \mathbb{N}$ with the property that $\lambda_n \to 0$ as fast we want, in this chapter we obtain the approximation order $O(\lambda_n)$ for generalized Baskakov-Kantorovich-Faber operators and for generalized Szász-Kantorovich-Faber operators, respectively, attached to analytic functions of exponential growth in a continuum (i.e. simply connected compact set) $G \subset \mathbb{C}$. Several concrete examples of continuums G are given for which this operator can explicitly be constructed. In addition, we also obtain the approximation order $O(\lambda_n)$ for generalized Baskakov-Kantorovich-Faber-Walsh operators and for generalized Szász-Kantorovich-Faber-Walsh operators attached to analytic functions of exponential growth in a multiply connected compact set $G \subset \mathbb{C}$.

4.1 Simply Connected Compact Sets: Preliminaries

Let $\lambda_n \to 0$ as fast we want, satisfying (without any loss of generality) $0 < \lambda_n \leq \frac{1}{2}$, for all $n \in \mathbb{N}$.

Suggested by the method in the paper [48], where for analytic functions of some exponential growth in simply connected compact sets of the complex plane, approximation with the arbitrary order $O(\lambda_n)$, by the Baskakov-Faber kind operators of the form

$$W_n(f; \lambda_n, G; z)$$
$$= \sum_{k=0}^{\infty} a_k(f) \cdot \sum_{j=0}^{k} (1+\lambda_n) \cdot ... \cdot (1+(j-1)\lambda_n) \cdot [0, \lambda_n, ..., j\lambda_n; e_k] \cdot F_j(z)$$

is obtained. Here $F_j(z)$ represent the Faber polynomials attached to the compact G (called continuum too), $f(z) = \sum_{k=0}^{\infty} a_k(f) F_k(z)$ represents the development in Faber series of f on G and $[0, \lambda_n, ..., j\lambda_n; e_k]$ represents the divided difference of $g(z) = e_k(z) = z^k$, on the $j+1$ knots $0, \lambda_n, ..., j\lambda_n$.

By this method, the approximation order $O(1/n)$ obtained for the classical Baskakov operators (i.e. for $\lambda_n = 1/n$) in compact disks with center at origin in [35], Section 1.9, pp. 124-138, was improved (in [48]) to $O(\lambda_n)$ given by the Baskakov-Faber operators attached to a simply connected compact subset of $\mathbb{C}$.

Also, it is worth mentioning that this topic concerning quantitative estimates in approximation by other complex operators can be found in many other papers, see, e.g., the books [35], [36], [56] and in the papers [16], [34], [45]-[55], [60]-[62].

For our purpose, we briefly recall some basic concepts on Faber polynomials and Faber expansions.

For $G \subset \mathbb{C}$ a compact set such that $\tilde{\mathbb{C}} \setminus G$ is connected, let $A(G)$ be the Banach space of all functions that are continuous on G and analytic in the interior of G, endowed with the norm $\|f\|_G = \sup\{|f(z)|; z \in G\}$. Denoting $\mathbb{D}_r = \{z \in \mathbb{C}; |z| < r\}$, according to the Riemann Mapping Theorem, there exists a unique conformal mapping Ψ of $\tilde{\mathbb{C}} \setminus \overline{\mathbb{D}}_1$ onto $\tilde{\mathbb{C}} \setminus G$ such that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. Then, to G one may attach the polynomial of exact degree n, $F_n(z)$, called *Faber polynomial*, defined by $\frac{\Psi'(w)}{\Psi(w)-z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}$, $z \in G, |w| > 1$.

If $f \in A(G)$ then

$$a_n(f) = \frac{1}{2\pi i} \int_{|u|=1} \frac{f(\Psi(u))}{u^{n+1}} du = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\Psi(e^{it}))e^{-int} dt, n \in \mathbb{N} \cup \{0\}$$

are called the Faber coefficients of f and $\sum_{n=0}^{\infty} a_n(f)F_n(z)$ is called the Faber expansion (series) attached to f on G. It is worth noting that the Faber series represent a natural generalization of the Taylor series, when the unit disk is replaced by an arbitrary simply connected domain bounded by a "nice" curve.

Detailed properties of Faber polynomials and Faber expansions can be found in e.g. [33], [70].

Let G be a connected compact subset in $\mathbb{C}$ (that is a continuum) and suppose that f is analytic on G, that is there exists $R > 1$ such that f is analytic in G_R, given by $f(z) = \sum_{k=0}^{\infty} a_k(f)F_k(z)$, $z \in G_R$. Recall here that G_R denotes the interior of the closed level curve Γ_R given by $\Gamma_R = \{\Psi(w); |w| = R\}$ (and that $G \subset \overline{G_r}$ for all $1 < r < R$).

Let $0 < \lambda_n \leq \frac{1}{2}$, for all $n \in \mathbb{N}$, with $\lambda_n \to 0$ as fast we want.

In what follows, firstly we will introduce the appropriate form for the generalized Baskakov-Kantorovich-Faber operators, by using the method in [48] used to introduce the Baskakov-Faber operators $W_n(f; \lambda_n, G; z)$ mentioned in Introduction.

In this sense, we recall that in [71], we have studied the following form of the Baskakov-Kantorovich operators of real variable, $x \geq 0$,

$$\mathcal{K}_n(\lambda_n; f)(x) = (1+x)^{-1/\lambda_n} \cdot$$

$$\cdot \sum_{j=0}^{\infty} \frac{1}{j!} \cdot \frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right) \cdot \ldots \cdot \left(j-1+\frac{1}{\lambda_n}\right) \cdot \left(\frac{x}{1+x}\right)^j H_n(f)(j\lambda_n),$$

where the function H_n is such that

$$H_n(f)(j\lambda_n) = \frac{1}{\lambda_n} \cdot \int_{j\lambda_n}^{(j+1)\lambda_n} f(t)dt = \frac{1}{\lambda_n} \cdot \int_{j\lambda_n}^{j\lambda_n+\lambda_n} f(t)dt.$$

It is easy to see that we can define $H_n(f)(x) = \frac{1}{\lambda_n} \cdot \int_x^{x+\lambda_n} f(t)dt$.

Now, applying Theorem 2 in [59], we obtain the representation formula

$$\mathcal{K}_n(f; \lambda_n)(x) = \sum_{j=0}^{\infty} (1+\lambda_n) \ldots (1+(j-1)\lambda_n) \cdot [0, \lambda_n, \ldots, j\lambda_n; H_n(f)] x^j,$$

$x \geq 0$, where by convention, $(1+\lambda_n) \ldots (1+(j-1)\lambda_n) = 1$ for $j = 0$.

Supposing that $f(z) = \sum_{k=0}^{\infty} a_k(f) z^k$ is analytic in a compact disk $|z| \leq R$, for $H_n(f)(z)$ we obtain the representation

$$H_n(f)(z) = \frac{1}{\lambda_n} \cdot \int_z^{z+\lambda_n} f(t)dt = \frac{1}{\lambda_n} \cdot \sum_{k=0}^{\infty} a_k(f) \cdot \int_z^{z+\lambda_n} t^k dt$$

$$= \frac{1}{\lambda_n} \cdot \sum_{k=0}^{\infty} \frac{a_k(f)}{k+1}[(z+\lambda_n)^{k+1} - z^{k+1}] = \sum_{k=0}^{\infty} \frac{a_k(f)}{k+1}\left(\sum_{j=0}^{k} \binom{k+1}{j} z^j \lambda_n^{k-j}\right)$$

$$= \sum_{k=0}^{\infty} A_{n,k}(f) z^k,$$

where

$$A_{n,k}(f) = \sum_{j=k}^{\infty} \frac{a_j(f)}{j+1} \lambda_n^{j-k} \binom{j+1}{k}. \tag{4.1}$$

Reasoning as in [48], we can formally introduce the following Baskakov-Kantorovich operators on compact disk centered at origin, by the formula

$$\mathcal{K}_n(f;\lambda_n)(z) = \sum_{k=0}^{\infty} A_{n,k}(f) \cdot \sum_{j=0}^{k} (1+\lambda_n) \cdot \ldots \cdot (1+(j-1)\lambda_n) \cdot [0, \lambda_n, ..., j\lambda_n; e_k] z^j.$$

Finally, proceeding exactly as in Definition 2.1 in [48] (that is replacing z^j by $F_j(z)$), if $f(z) = \sum_{k=0}^{\infty} a_k(f) F_k(z)$, $z \in G \subset \mathbb{C}$, where $G \subset \mathbb{C}$ is a compact and $F_k(z)$ are the Faber polynomials attached to G, we can introduce the following definition.

Definition 4.1.1. (Trifa [72]) The generalized Baskakov-Kantorovich-Faber operators attached to G and f is (formally) defined by

$$\mathcal{K}_n(f;\lambda_n, G; z)$$

$$= \sum_{k=0}^{\infty} A_{n,k}(f) \cdot \sum_{j=0}^{k} (1+\lambda_n) \cdot \ldots \cdot (1+(j-1)\lambda_n) \cdot [0, \lambda_n, ..., j\lambda_n; e_k] F_j(z), \quad (4.2)$$

where for $j = 0$ and $j = 1$, by convention we take

$$(1+\lambda_n) \cdot \ldots \cdot (1+(j-1)\lambda_n) = 1.$$

Remark. For $\lambda_n = 1/n$, $n \in \mathbb{N}$ and $G = \overline{\mathbb{D}}_1$, since $F_j(z) = z^j$, the above generalized Baskakov-Kantorovich-Faber operators reduce to the classical complex Baskakov-Kantorovich operators.

The same formal reasonings can be applied to the generalized Szász-Kantorovich operators, defined in the real case in [71], by the formula

$$K_n(f;\lambda_n)(x) = e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{j!\lambda_n^j} \cdot \frac{1}{\lambda_n} \cdot \int_{j\lambda_n}^{(j+1)\lambda_n} f(v)dv.$$

Indeed, according to the paper [37], page 976 (and denoting there $\frac{b_n}{a_n} := \lambda_n$), the classical generalized Szász operators, $S_n(f;\lambda_n)(z)$, can formally be written by the formula

$$S_n(f;\lambda_n)(x) = e^{-x/\lambda_n} \cdot \sum_{j=0}^{\infty} f(j\lambda_n) \cdot \frac{x^j}{\lambda_n^j j!}.$$

Then, by formula (1), p. 976 in [37], (written again for $\frac{b_n}{a_n} := \lambda_n$) for the generalized Szász operators, we get the form

$$S_n(f)(x) = e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{\lambda_n^j j!} \cdot f(j\lambda_n) = \sum_{j=0}^{\infty} [0, \lambda_n, ..., j\lambda_n; f] \cdot z^j.$$

Keeping the notation for $H_n(f)(x)$, for $K_n(f;\lambda_n)(x)$, we can formally write

$$K_n(f;\lambda_n)(x) = e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{j!\lambda_n^j} \cdot H_n(j\lambda_n) = \sum_{j=0}^{\infty} [0, \lambda_n, ..., j\lambda_n; H_n] \cdot x^j.$$

Replacing now x by z and considering that $f(z) = \sum_{k=0}^{\infty} a_k(f) F_k(z)$ is the Faber expansion of f in the compact G, reasoning as in the case of Definition 4.1.1, we can introduce the following concept.

Definition 4.1.2. (Trifa [72]) The generalized Szász-Kantorovich-Faber operators attached to G and f is (formally) defined by

$$K_n(f;\lambda_n;G)(z) = \sum_{k=0}^{\infty} A_k(f) \sum_{j=0}^{k} [0, \lambda_n, ..., j\lambda_n; e_k] \cdot F_j(z). \tag{4.3}$$

4.2 Baskakov-Kantorovich-Faber Operators

In this section, the method in [48] described in Section 4.1, will be applied to the complex Baskakov-Kantorovich operators studied in the case of real variable in [71].

The main result of this section is the following.

Theorem 4.2.2. (Trifa [72]) *Let* $\lambda_n \searrow 0$, $0 < \lambda_n \leq \frac{1}{2}$, $n \in \mathbb{N}$ *and* f *be analytic on the continuum* G, *that is there exists* $R > 1$ *such that*

f is analytic in G_R, given by $f(z) = \sum_{k=0}^{\infty} a_k(f)F_k(z)$, $z \in G_R$. Also, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{R}, 1\right)$, with $|a_k(f)| \leq M\frac{A^k}{k!}$, for all $k = 0, 1, ...,$ (which implies $|f(z)| \leq C(r)Me^{Ar}$ for all $z \in G_r$, $1 < r < R$). Here G_R denotes the interior of the closed level curve Γ_R given by $\Gamma_R = \{\Psi(w); |w| = R\}$ and $G \subset \overline{G_r}$, for all $1 < r < R$.

Let $1 < r < \frac{1}{A}$ be arbitrary fixed. Then, there exist an index $n_0 \in \mathbb{N}$ and a constant $C''(r, f) > 0$ depending on r and f only, such that for all $z \in \overline{G_r}$ and $n \geq n_0$ we have

$$|\mathcal{K}_n(f; \lambda_n, G; z) - f(z)| \leq C''(r, f) \cdot \lambda_n,$$

where $\mathcal{K}_n(f; \lambda_n, G; z)$ is given by formula (4.2).

Remarks. 1) It is clear that Theorem 4.2.2 holds under the more general hypothesis $|a_k(f)| \leq P_m(k) \cdot \frac{A^k}{k!}$, for all $k \geq 0$, where P_m is an algebraic polynomial of degree m with $P_m(k) > 0$ for all $k \geq 0$.

2) There are many concrete examples for G when the conformal mapping Ψ and the Faber polynomials associated to G, and consequently when the Baskakov-Kantorovich-Faber operators too, can explicitly be written (see, e.g., [36], pp. 81-83, or [37]), as follows : $G = [-1, 1]$, G is the continuum bounded by the m-cusped hypocycloid, G is the regular m-star ($m = 2, 3, ...,$), G is the m-leafed symmetric lemniscate, $m = 2, 3, ...,$, G is a semidisk, or G is a circular lune.

4.3 Szász-Kantorovich-Faber Operators

In this section, the method in [48] described in Section 4.1, will be applied to the complex Szász-Kantorovich operators, studied in the case of real variable in [71].

The main result of this section is the following.

Theorem 4.3.1. (Trifa [72]) *Let* $\lambda_n \searrow 0$, $0 < \lambda_n \leq \frac{1}{2}$, $n \in \mathbb{N}$ *and* f *be analytic on the continuum* G, *that is there exists* $R > 1$ *such that* f *is analytic in* G_R, *given by* $f(z) = \sum_{k=0}^{\infty} a_k(f) F_k(z)$, $z \in G_R$. *Also, suppose that there exist* $M > 0$ *and* $A \in \left(\frac{1}{R}, 1\right)$, *with* $|a_k(f)| \leq M \frac{A^k}{k!}$, *for all* $k = 0, 1, ...,$ *(which implies* $|f(z)| \leq C(r) M e^{Ar}$ *for all* $z \in G_r$, $1 < r < R$*). Here* G_R *denotes the interior of the closed level curve* Γ_R *given by* $\Gamma_R = \{\Psi(w); |w| = R\}$ *and* $G \subset \overline{G_r}$, *for all* $1 < r < R$.

Let $1 < r < \frac{1}{A}$ *be arbitrary fixed. Then, there exist an index* $n_0 \in \mathbb{N}$ *and a constant* $C''(r, f) > 0$ *depending on* r *and* f *only, such that for all* $z \in \overline{G_r}$ *and* $n \geq n_0$ *we have*

$$|K_n(f; \lambda_n, G; z) - f(z)| \leq C''(r, f) \cdot \lambda_n,$$

where $K_n(f; \lambda_n, G; z)$ *is given by formula (4.3).*

Remarks. 1) It is clear that Theorem 4.3.1 holds under the more general hypothesis $|a_k(f)| \leq P_m(k) \cdot \frac{A^k}{k!}$, for all $k \geq 0$, where P_m is an algebraic polynomial of degree m with $P_m(k) > 0$ for all $k \geq 0$.

2) Again, we may indicate many concrete examples for G when the conformal mapping Ψ and the Faber polynomials associated to G, and consequently when the Szász-Kantorovich-Faber operators too, can explicitly be written (see, e.g., [36], pp. 81-83, or [37]), as follows : $G = [-1, 1]$, G is the continuum bounded by the m-cusped hypocycloid, G is the regular m-star ($m = 2, 3, ...,$), G is the m-leafed symmetric lemniscate, $m = 2, 3, ...,$, G is a semidisk, or G is a circular lune.

4.4 Multiply Connected Compact Sets: Preliminaries

Let $f(z) = \sum_{k=0}^{\infty} a_k(f) B_k(z)$ be the development of the analytic function f in Faber-Walsh series on the multiply connected compact G, where $B_k(z)$ represent the so-called Walsh polynomials attached to G. In the paper [41], one define and study the complex Szász-Mirakjan-Faber-Walsh operators, given by the formula (see Definition 3 in [41])

$$M_n(f;\lambda_n;G)(z) = \sum_{k=0}^{\infty} a_k(f) \sum_{j=0}^{k} [0, \lambda_n, ..., j\lambda_n; e_k] \cdot B_j(z).$$

The main goal of the next two sections is to obtain similar approximation properties for the Baskakov-Kantorovich-Faber-Walsh operators in Section 4.5 and for the Szász-Kantorovich-Faber-Walsh operators in Section 4.6.

But firstly, in this section we present some preliminaries on Walsh polynomials.

The Faber polynomials were introduced by Faber in [28] as associated to a simply connected compact set. They allow the expansion of functions analytic on that set into a series with similar properties to the classical power series.

In Walsh [74], were introduced polynomials that generalize the Faber polynomials, attached to compact sets consisting of several components (i.e. whose complement is a multiply connected domain). These generalized Faber polynomials are called Faber-Walsh polynomials and also allow the expansion of an analytic function into a series with properties again similar to the power series.

In what follows, let us briefly recall some basic concepts on Faber-Walsh polynomials and Faber-Walsh expansions we need in the next lines.

Everywhere in Sections 4.5 and 4.6, $G \subset \mathbb{C}$ will be considered a compact set consisting of several components, that is $\tilde{\mathbb{C}} \setminus G$ is multiply connected.

Definition 4.4.1. (see, e.g., Walsh [74]) A lemniscatic domain is a domain of the form $\{w \in \tilde{C}; |U(w)| > \mu\}$, where $\mu > 0$ is some constant and $U(w) = \Pi_{j=1}^{\nu}(w - \alpha_j)^{m_j}$ for some points $\alpha_1, ..., \alpha_\nu \in \mathbb{C}$ and real exponents $m_1, ..., m_\nu > 0$ with $\sum_{j=1}^{\nu} m_j = 1$.

In all what follows, we will consider that the points $\alpha_1, ..., \alpha_\nu$ have the property that from them can be chosen a sequence $(\alpha_j)_{j\in\mathbb{N}}$ such that for any closed set C not containing any of the points $\alpha_1, ..., \alpha_\nu$, there exist constants $A_1(C), A_2(C) > 0$ with

$$A_1(C) < \frac{|u_n(w)|}{|U(w)|^n} < A_2(C),\ n = 0, 1, 2, ...,\ w \in C, \tag{4.4}$$

where $u_n(w) = \Pi_{j=1}^{n}(w - \alpha_j)$.

Let $D_1, ..., D_\nu$ be mutually exterior compact sets (none a single point) of the complex plane such that the complement of $G := \bigcup_{j=1}^{\nu} D_j$ in the extended plane is a ν-times connected region (open and connected set). According to Theorem 3 in Walsh [75], there exists a lemniscatic domain

$$K_1 = \{w \in \tilde{\mathbb{C}}; |U(w)| > \mu\}$$

and a conformal bijection

$$\Phi : \tilde{\mathbb{C}} \setminus G \to \{w \in \tilde{\mathbb{C}}; |U(w)| > \mu\},\ \text{with}\ \Phi(\infty) = \infty,\ \text{and}\ \Phi'(\infty) = 1.$$

Here μ is the logarithmic capacity of G. Further, the inverse conformal bijection satisfies

$$\Psi = \Phi^{-1} = \{w \in \tilde{\mathbb{C}}; |U(w) > \mu\} \to \tilde{\mathbb{C}} \setminus G,\ \text{with}\ \Psi(\infty) = 1\ \text{and}\ \Psi'(\infty) = 1.$$

Consider the Green's functions $H_1(w) = \log(|U(w)|) - \log(\mu)$, $H = H_1 \circ \Phi$ and for $r > 1$ their level curves

$$\Lambda_r = \{w \in \mathbb{C}; H_1(w) = \log(r)\} = \{w \in \mathbb{C}; |U(w)| = r\mu\},$$

$$\Gamma(r) = \{z \in \mathbb{C}; H(z) = \log(r)\}.$$

We have $\Gamma_r = \Psi(\Lambda_r)$. Denote by G_r the interior of Γ_r and by D_r^∞ the exterior of Λ_r (including ∞).

Notice that for $1 < r < \beta < R$ we have $G \subset G_r \subset G_\beta \subset G_R$.

According to Theorem 3 in Walsh [74], for $z \in \Gamma_r$ and $w \in D_r^\infty$ we have

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{B_n(z)}{u_{n+1}(w)}, \text{ with } B_n(z) = \frac{1}{2\pi i} \int_{\Lambda_\lambda} u_n(t) \cdot \frac{\Psi'(t)}{\Psi(t) - z} dt, \lambda > r.$$

The polynomial $B_n(z)$ is called the n-th Faber-Walsh polynomial attached to G and $(\alpha_j)_{j \in \mathbb{N}}$ and according to Lemma 2.5 in Sète [64], the Faber-Walsh polynomials are independent of the lemniscatic domain and the exterior mapping function Ψ.

Remarks. 1) The proof of existence of the above conformal mapping Ψ (and implicitly of the existence of Faber-Walsh polynomials) was obtained in Walsh [75] and it is based on some results on critical points of polynomials obtained in the book of Walsh [76].

2) A nice property of the Faber-Walsh polynomials obtained in Walsh [74], page 31, relation (34), is that

$$\limsup_{k \to \infty} [\|B_k\|_G]^{1/k} = \mu,$$

property which is similar to that for Chebyshev polynomials attached to the multiply connected compact set G and also holds for many sets of polynomials defined by extremal properties (see Fekete-Walsh [30], [31]). Here $\| \cdot \|_G$ denotes the uniform norm on G.

3) Similar to the Faber polynomials, according to Theorem 3 in Walsh [74], the Faber-Walsh polynomials allow the series expansion of functions analytic in compact sets. Namely, if f is analytic on the compact set G (with multiply connected complement), there exists $R > 1$ such that f is

analytic in G_R and inside G_R admits (locally uniformly) the series expansion $f(z) = \sum_{k=0}^{\infty} a_k(f) B_k(z)$, with

$$a_k(f) = \frac{1}{2\pi i} \int_{\Lambda_\beta} \frac{f(\Psi(t))}{u_{k+1}(t)} dt, 1 < \beta < R. \tag{4.5}$$

4) If G is simply connected, then the Faber-Walsh polynomials become the Faber polynomials.

5) In our reasonings, we will also need the following estimate, see, e.g., Walsh [74], p. 29, relation (26)

$$|B_k(z)| \le A_1(r\mu)^k, \text{ for all } z \in \Gamma_r,\ 1 < r < R, k \ge 0, \tag{4.6}$$

where A_1 depends on r only.

Also, by the relationship $\limsup_{n\to\infty} |a_k|^{1/k} \le \frac{1}{\beta\mu}$ in Walsh [74], page 30, we immediately get the estimate

$$|a_k(f)| \le \frac{C(\beta, \mu, f)}{(\beta\mu)^k}, \text{ for all } k = 0, 1, ..., \tag{4.7}$$

where $C(\beta, \mu, f) > 0$ is independent of k. Note that here and in all the next reasonings we will choose $1 < r < \beta < R$.

6) In the past, while the Faber polynomials were studied and used in many previously published papers, the Faber-Walsh polynomials have rarely been studied, excepting the Suetin's book [70], which contains a short section about them. The main reason for neglecting the Faber-Walsh polynomials was the fact that no explicit examples of Walsh's lemniscatic conformal maps were known. But very recently, by the papers [64]-[67], the Faber-Walsh polynomials were bringed again into attention. Thus, the first example of Walsh's lemniscatic conformal maps seems to be mentioned in the very recent paper of Sète-Liesen [66]. Also, the first explicit formulas for the Faber-Walsh polynomials were obtained for the case when G consists in

two disjoint compact intervals in Sète-Liesen [67]. The results in the present work, are also new contributions to the topic of Faber-Walsh polynomials.

For further properties of Faber-Walsh polynomials, see, e.g., Chapter 13 in Suetin [70].

Having as model the definitions in the paper [72], we also can introduce the following.

Definition 4.4.2. Let $D_1, ..., D_\nu$ be mutually exterior compact sets (none a single point) of the complex plane such that the complement of $G := \bigcup_{j=1}^{\nu} D_j$ in the extended plane is a ν-times connected region (open and connected set) and suppose that f is analytic in G, that is there exists $R > 1$ such that f is analytic in G_R, i.e. $f(z) = \sum_{k=0}^{\infty} a_k(f) B_k(z)$ for all $z \in G_R$, where $B_k(z)$ denotes the Faber-Walsh polynomials attached to G and G_R denotes the interior of the closed level curve Γ_R.

The generalized Baskakov-Kantorovich-Faber-Walsh operators and the generalized Szász-Kantorovich-Faber-Walsh operators attached to G and f, will formally be defined by

$$\mathbb{K}_n(f; \lambda_n; G)(z) = \sum_{k=0}^{\infty} A_{n,k}(f) \sum_{j=0}^{k} [0, \lambda_n, ..., j\lambda_n; e_k] \cdot B_j(z), \tag{4.8}$$

and

$$\overline{K}_n(f; \lambda_n; G)(z) = \sum_{k=0}^{\infty} A_{n,k}(f) \sum_{j=0}^{k} [0, \lambda_n, ..., j\lambda_n; e_k] \cdot B_j(z), \tag{4.9}$$

respectively, where

$$A_{n,k}(f) = \sum_{j=k}^{\infty} \frac{a_j(f)}{j+1} \lambda_n^{j-k} \binom{j+1}{k}. \tag{4.10}$$

Everywhere in the next sections, $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of real positive numbers with the property that $\lambda_n \searrow 0$ as fast as we want. Without the loss the generality, we may suppose that $\lambda_n \le \frac{1}{2}$, for all $n \in \mathbb{N}$.

4.5 Baskakov-Kantorovich-Walsh Operators

In this section we obtain approximation properties for the Baskakov-Kantorovich-Faber-Walsh operators.

We are in a position to state the main result of this section.

Theorem 4.5.2. *Let $\mu \geq 1$ and $D_1, ..., D_\nu$ be mutually exterior compact sets (none a single point) of the complex plane, such that the complement of $G := \bigcup_{j=1}^{\nu} D_j$ in the extended plane is a ν-times connected region (open and connected set). Suppose that f is analytic in G, that is there exists $R > 1$ such that f is analytic in G_R, i.e. $f(z) = \sum_{k=0}^{\infty} a_k(f) B_k(z)$ for all $z \in G_R$. Also, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{R\mu}, \frac{1}{\mu}\right)$, with $|a_k(f)| \leq M \frac{A^k}{k!}$, for all $k = 0, 1, ...,$ (which implies $|f(z)| \leq C(r) M e^{\mu A r}$ for all $z \in G_r$, $1 < r < R$). Here μ, G_R and G_r are those defined in Section 4.4.*

Let $1 < r < \frac{1}{A\mu}$ be arbitrary fixed. Then, there exist $n_0 \in \mathbb{N}$ and $C''(r, f, \mu) > 0$ depending on r, μ and f only, such that for all $z \in \overline{G_r}$ and $n \geq n_0$ we have

$$|\mathbb{K}_n(f; \lambda_n, G; z) - f(z)| \leq C''(r, f, \mu) \cdot \lambda_n.$$

Remarks. 1) Since $\sum_{k=0}^{\infty} (k+1) P_m(k) (\mu A r)^k < +\infty$ and $\sum_{k=0}^{\infty} P_m(k+1) \cdot \frac{(\mu A r)^k}{k!} < +\infty$ for any algebraic polynomial P_m of degree $\leq m$ satisfying $P_m(k) > 0$ for all $k \geq 0$, it is immediate from the proof that Theorem 4.5.2 holds under the more general hypothesis $|a_k(f)| \leq P_m(k) \cdot \frac{A^k}{k!}$, for all $k \geq 0$.

2) In the case when the set G is simply connected compact set, Theorem 4.5.2 was obtained in [72].

3) It is worth noting that in fact the condition $\mu \geq 1$ in Theorem 4.5.2 can be dropped and the conditions on A and r in the statement of Theorem

4.5.2 can be written as $A \in (1/R, 1)$, $1 < r < 1/A$ (i.e. independent of the logarithmic capacity μ), simply by suitably normalizing the lemniscatic domain to be $K_1 = \{w : |U(w)| > 1\}$ and choosing $\Phi'(\infty) = 1/\mu > 0$. Indeed, in this case, the attached Faber-Walsh polynomials $\tilde{B}_k(z)$ and the Faber-Walsh coefficients $\tilde{a}_k(f)$ in the expansion $f(z) = \sum_{k=0}^{\infty} \tilde{a}_k(f) \cdot \tilde{B}_k(z)$, satisfy (4.5) and (4.6) without the appearance of μ^k in these estimates (see, e.g., [41]).

4.6 Szász-Kantorovich-Walsh Operators

In this section we obtain approximation properties for the Szász-Kantorovich-Faber-Walsh operators.

The main result of this section is the following.

Theorem 4.6.1. *Let $\mu \geq 1$ and $D_1, ..., D_\nu$ be mutually exterior compact sets (none a single point) of the complex plane, such that the complement of $G := \bigcup_{j=1}^{\nu} D_j$ in the extended plane is a ν-times connected region (open and connected set). Suppose that f is analytic in G, that is there exists $R > 1$ such that f is analytic in G_R, i.e. $f(z) = \sum_{k=0}^{\infty} a_k(f) B_k(z)$ for all $z \in G_R$. Also, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{R\mu}, \frac{1}{\mu}\right)$, with $|a_k(f)| \leq M \frac{A^k}{k!}$, for all $k = 0, 1, ...,$ (which implies $|f(z)| \leq C(r) M e^{\mu A r}$ for all $z \in G_r$, $1 < r < R$). Here μ, G_R and G_r are those defined in Section 4.4.*

Let $1 < r < \frac{1}{A\mu}$ be arbitrary fixed. Then, there exist $n_0 \in \mathbb{N}$ and $C''(r, f, \mu) > 0$ depending on r, μ and f only, such that for all $z \in \overline{G_r}$ and $n \geq n_0$ we have

$$|\overline{K}_n(f; \lambda_n, G; z) - f(z)| \leq C''(r, f, \mu) \cdot \lambda_n.$$

Remarks. 1) Since $\sum_{k=0}^{\infty}(k+1)P_m(k)(\mu Ar)^k < +\infty$ and $\sum_{k=0}^{\infty} P_m(k+1) \cdot \frac{(\mu Ar)^k}{k!} < +\infty$ for any algebraic polynomial P_m of degree $\leq m$ satisfying $P_m(k) > 0$ for all $k \geq 0$, it is immediate from the proof that Theorem 4.6.1 holds under the more general hypothesis $|a_k(f)| \leq P_m(k) \cdot \frac{A^k}{k!}$, for all $k \geq 0$.

2) In the case when the set G is simply connected compact set, Theorem 4.6.1 was proved in [72].

3) It is worth noting that in fact the condition $\mu \geq 1$ in Theorem 4.6.1 can be dropped and the conditions on A and r in the statement of Theorem 4.6.1 can be written as $A \in (1/R, 1)$, $1 < r < 1/A$ (i.e. independent of the logarithmic capacity μ), simply by suitably normalizing the lemniscatic domain to be $K_1 = \{w : |U(w)| > 1\}$ and choosing $\Phi'(\infty) = 1/\mu > 0$. Indeed, in this case, the attached Faber-Walsh polynomials $\tilde{B}_k(z)$ and the Faber-Walsh coefficients $\tilde{a}_k(f)$ in the expansion $f(z) = \sum_{k=0}^{\infty} \tilde{a}_k(f) \cdot \tilde{B}_k(z)$, satisfy (4.5) and (4.6) without the appearance of μ^k in these estimates (see, e.g., [41]).

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