BABES BOLYAI UNIVERSITY
Faculty of Mathematics and Computer Science
Department of Mathematics

SUMMARY OF DOCTORAL THESIS

Bi-criteria optimization with applications in economy (Optimizare bi-criteriala cu aplicatii in economie)

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Abstract

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by Traian Ionut LUCA

Global context is generating challenges to optimize production of energy. Production of energy is a complex process with long, medium and short term objectives, regulated and deregulated markets. Analyzing a daily production and consumption chart, a fluctuation is visible. We compare this fluctuation with a spread, the peak load being the most extreme point.

Our objective, for the research study which has generated this thesis, is: *To create, solve and validate a mathematical model which will shave the peak load by minimizing fluctuation of energy and maximizing the economic performance.*

Shaving the peak load will generate a supra-production of energy which, according to the strategy of power plant, might be addressed for example by energy storage, demand side management, electric vehicles strategy or by human interference to adjust the production plan. Shifting the production will generate changes in the production diagram, which might reduce the economic performance of the power plant. To mitigate this risk, a bi-criteria optimization problem might be used, the first component of the objective function being focused on peak-load, while the second component being focused on economic performance.

First part of our thesis (Chapters 2 and 3) is dedicated to analysis and development of mathematical tools necessary during our research. Measures for fluctuation and methods to solve bi-criteria optimization problems are evaluated and developed. To solve bi-criteria optimization problems we have used equivalent parametric problems with Kuhn-Tucker conditions and approximate problems. Chapter 3 is dedicated to approximate problems and their connection to the initial problem. Invexity, incavity and avexity are used to prove conditions such that an efficient solution of an initial bi-criteria optimization problem remain efficient also for the approximate problem and reciprocally.

Key to shave the peak load, by minimizing fluctuation, is to find a proper measure of spread able to target directly the most extreme point. Few measures of spread have been evaluated, conclusion being that maximum absolute deviation satisfies our request.

Turnover is employed as measure for economic performance and simple technical constraints, which limit the amount of energy to be produced, are used.

Minimax measure for fluctuation of energy was defined starting from maximum absolute deviation. Energy price is included in the measure. Minimax measure for fluctuation is generating minimax model for energy. Using some equivalent bi-criteria and parametric problems, efficient solution is computed and verified using real data. Efficient production plan proved to have good and excellent behavior compared to real production data.

Input data required for minimax measure are increasing the complexity of the model. To reduce complexity, we have created a new measure for fluctuation of energy and called it index measure for fluctuation. A new model for peak-load shaving, index model, is generated. Using again some equivalent bi-criteria and parametric problems we manage to compute the efficient solution. Efficient production plan proved to have good behavior compared to real production data.

Minimax and index models use only simple technical constraints. Inclusion of more complex technical constraints, profit as measure of economic performance, a better estimation for input data and a new approach for dealing with the transition from night period to day period will open new research directions and might improve the accuracy of solution.

Our thesis is structured in six chapters.
First chapter is dedicated to the general context of energy which generates challenges for optimization.
Second chapter is dedicated to evaluate some mathematical tools used in our research. Bi-criteria optimization problems and Kuhn-Tucker conditions are presented. Some measures of spread are presented and evaluated for finding a starting point in defining our measures for fluctuation of energy.
Chapter 3 is dedicated to development of conditions such that efficient solution of initial bi-criteria optimization problem will remain efficient also for the approximate problem and reciprocally. Approximation is creating a proper environment for solving in a more efficient way complex energy problems which might arise due to technical constraints.
Chapter 4 is dedicated to the minimax energy model. It presents the minimax measure for fluctuation of energy, development, solving and validation of minimax model.
Chapter 5 is dedicated to index energy model. Its development is stimulated by the complexity of input data required by minimax model. A new measure for fluctuation of energy is introduced. Development, solving and validation of index model are presented. Index is less complex than minimax and easier to be applied, but performances are less accurate.
Chapter 6 presents some conclusions of our research, an analysis and comparison for the two models and some possibilities to extend and improve the two models developed.

Our contributions to this thesis might be summarized as: a new approach, based on bi-criteria optimization problems, for shaving the peak load of energy; Theorems 3.3.1, 3.3.2, 3.3.5, 3.3.6, 3.3.9, 3.3.10, 3.3.11, 3.3.12, 3.4.3, 3.4.4, 3.4.5, 3.4.6, 3.4.7, 3.4.8, 3.4.9, 3.4.10, 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.7, 3.5.8, 3.5.9, 3.5.10, 3.6.1, 3.6.2, 3.6.3, 3.6.4, 3.6.5, 3.6.6, 3.6.7, 3.6.8; Examples 3.3.3, 3.3.7, 3.4.11, 3.4.12, 3.5.11, 3.5.12, 3.5.13; definition of minimax measure for fluctuation (4.1) and index measure for fluctuation (5.1); bi-criteria problems (4.2) and (5.2) used for shaving the peak load; Lemma 4.4.1 and Lemma 5.4.1 used to transform energy problems (4.2) and (5.2) in equivalent problems easier to be solved; Theorem 4.4.11 which proves that order of scenarios does not change the solution; Theorem 4.4.5 which computes the optimal solution for parametric model (4.4); Theorem 4.4.12 which computes the efficient solution for minimax model (4.2); Theorem 5.4.5 which computes the optimal solution for parametric model (5.4); Theorem 5.4.6 which computes the efficient solution for index energy model (5.2); tests for minimax and index models performed using real data and some economic analysis for Kuhn-Tucker multipliers; classification, using Definitions 4.4.6, 4.4.7 and 4.4.8, of Kuhn-Tucker multipliers based on to their capacity to generate feasible and optimal solutions for parametric optimization problem.

Results presented in this thesis were disseminated at two international conferences:

> *Bi-criteria problems for energy optimization*, presented at Conference: International Conference on Approximation Theory and its Applications, organized in Sibiu, Romania during 26-29 May 2016.

> *Bi-criteria models for energy markets*, presented at Conference: Management International Conference, organized in Monastier di Treviso, Italy, during 24-27 May 2017.

and in nine articles:

> *Minimax rule for energy optimization* [98], published in Computers and Fluids, an ISI journal with an Impact Factor of 2.221 and a 5-years Impact Factor of 2.610.

> *Index model for peak-load shaving in energy production* [92], submitted to Engineering Optimization, an ISI Journal with an Impact Factor of 1.728.

> *Bi-criteria models for peak-load shaving* [90], accepted for publication by Journal of Academy of Business and Economics, a journal indexed in EBSCO, EconLit, Ulrich's, Index Copernicus, Research Bible.

> *Approximations of objective function in bi-criteria optimization problems* [96], accepted for publication by European International Journal of Science and Technology, a journal indexed in Google Scholar, NewJour, Hochschulbibliothek Reutlingen, CrossRef.

Approximations of objective function and constraints in bi-criteria optimization problems [95], submitted to Journal of Numerical Analysis and Approximation Theory, a journal indexed in Mathematical Reviews, Zentralblatt MATH.

Approximations of bi-criteria optimization problem [94], submitted to Studia Universitatis Babes-Bolyai Mathematica, a journal indexed in Mathematical Reviews, Zentralblatt MATH, EBSCO, ProQuest Ulrichsweb.

Relations between η - approximation problems of a bi-criteria optimization problem [97], submitted to Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, a journal indexed in Mathematical Reviews, Zentralblatt MATH, American Mathematical Society.

Bi-criteria problems for energy optimization [39], published in General Mathematics, a journal indexed in Zentralblatt MATH, EBSCO, Mathematical Reviews, Index Copernicus.

Portfolio optimization algorithms [93], published in Studia Universitatis Babes-Bolyai Negotia, a journal indexed in EBSCO, Index Copernicus, ERIH PLUS.

and are supported by our previous work reflected in books, articles and conferences:

Matematici economice. Elemente de programare liniara si teoria probabilitatilor [19], a book published by Presa Universitara Clujeana.

Strict fixed points results for multivalued contractions on gauge spaces [122], published in Fixed Point Theory, an ISI journal with an Impact Factor of 1.030 in 2010.

Uniqueness algebraic conditions in the study of second order elliptic systems [18], published in International Journal of Pure and Applied Mathematics, a journal indexed in Scopus.

Maximum principles for a class of second order parabolic systems in divergence form [20], published in Journal of Nonlinear Functional Analysis and Differential Equations, a journal indexed in Scopus, Web of Science, Zentralblatt MATH.

Automotive industry and performances of US Economy, published in International Journal of Finance and Economics, a journal indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Consumer's inflation expectations in Romania [147], published in International Journal of Business Research, a journal indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Comparative analysis of low-cost airlines websites [77], published in Proceedings of IABE - 2009 Las Vegas - Annual Conference, a volume indexed in EBSCO, Ulrich's, Index Copernicus, EconLit.

Economic applications of dynamic optimization, presented at Conference: 10th International Symposium on Generalized Convexity and Monotonicity, organized in Cluj Napoca, Romania, during 22-27 August 2011.

A relation between transportation problems and profit, presented at Conference: Current Issues of Regional Development, organized in Sec, Czech Republic, during 26-27 June 2007.

Key words: *approximation theorems, generalized convexity, peak load shaving, bi-criteria problems, energy fluctuation minimization, economic performance maximization, parametric optimization, Kuhn-Tucker conditions*.

Contents

Chapter 1

General context of energy optimization

Evolution of humanity is highly dependent on energy. According to *International Energy Agency* (IEA) [66] the main energy sources are coal, oil and gas (67.4% of total fuels used). This generates high CO_2 emissions, with impact on climate change, transposed among others by an increase of global average temperature [115].
Adding the projected increase of population [150], the projected increase of energy consumption [151], the topology, capacity and associated costs of power grids and the anthropogenic factors, it is obvious that there are a lot of challenges related to production of energy.

Production of energy is a highly complex environment with three main actors (producers, consumers and independent system operators) and long, medium and short term objectives [37]. Specific requirements and day-ahead markets [80], make energy problems complex [37], [118], difficult to solve and with a lot of challenges.

Results in this field have been obtained among others in: [50, 102, 123, 13, 8, 127, 84, 55, 117, 118, 154, 161, 58, 27, 28, 81, 56, 44, 36, 30, 67, 1, 103, 153, 114, 41, 7, 42, 108, 100, 130, 132, 87, 131].

Increased consumption of electricity, combined with increased magnitude, frequency and duration of extreme heat events, due to climate change, will result in higher peak loads, creating pressure on: (i) producers [9, 23, 138, 146], (ii) power grids and (iii) price of electricity [132].

Conventional approach for dealing with peak load is based on non-economically feasible solutions. A more preferable approach, which has become an important research area is peak load shaving [149]. Three main strategies might be applied to realize a peak load shaving: (i) energy storage systems – ESS, (ii) electric vehicles – EV and (iii) demand side management – DSM.

Important contributions to peak load shaving are presented in [15, 23, 24, 26, 33, 47, 49, 51, 57, 68, 70, 72, 87, 82, 88, 111, 130, 113, 108, 107, 116, 120, 128, 134, 139, 141, 144, 145, 149, 156, 157, 158, 162].

The peak load of energy which we aim to shave, is visible in Figure 1.1.

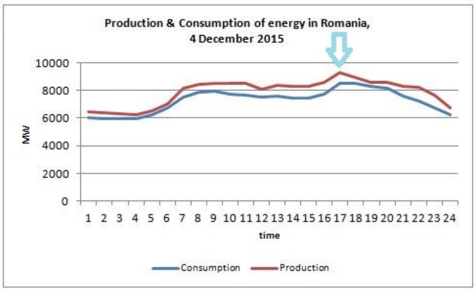

FIGURE 1.1: Evolution of electricity in Romania in 4th of December 2015

Same figure shows a fluctuation of energy, which might be regarded as a spread.

Our objective for this research is *to create, solve and validate a mathematical model which will shave the peak load by minimizing fluctuation of energy and maximizing the economic performance.*

Chapter 2

Mathematic tools

2.1 Introduction

Bi-criteria optimization problems are playing a central role in our research. For this Chapter we define two objectives: (i) to present methods used for solving bi-criteria optimization problems and (ii) to analyze known measures for spread of data in order to identify a starting point for defining measures for fluctuation of energy.

2.2 Multi-criteria optimization problem and efficient points

Definition 2.2.1 *Let* $X \subseteq \mathbb{R}^n$ *and* $f = (f_1, f_2, ...f_m)^T : X \to \mathbb{R}^m$*. The optimization problem*

$$\begin{cases} \min f(x) \\ x \in X \end{cases} \tag{2.1}$$

is called a multi-criteria optimization problem.

Remark 2.2.2 *If* $m = 2$ *the multi-criteria optimization problem defined above is called bi-criteria optimization problem.*

In our study, the first component of the objective function is fluctuation of energy, with a key role in shaving the peak load. The second component of the objective function is economic performance of the power plant, influencing accuracy of solution.

Solution of a multi-criteria optimization problem has to realize a trade off between all components of objective. This solution is called efficient point or efficient solution.

Definition 2.2.3 *A feasible solution $x^* \in X$ of problem (2.1) is an efficient solution if $\nexists x \in X$ such that*

$$\begin{aligned} f(x) &\leq f(x^*) \\ f(x) &\neq f(x^*). \end{aligned}$$

Multi-criteria optimization problems have several applications in fields like: financial investments [16, 76], engineering, biology [69], data analysis [21] or logistics [119].

2.3 Solving a bi-criteria problem

Optimization problems might be solved by analytical methods (compute the exact solution based on mathematical proofs), numerical methods (approximate the solution using appropriate iterations) or using an approximation problem for the initial optimization problem.

"Scalarization" methods [62, 106, 142, 17] are frequently used for bi-criteria optimization problems. The importance degree of each component of the objective function might be established *a priori* or during the process.

[32] shows that numerical methods for solving bi-criteria optimization problems are inspired from biology [64, 74, 34, 71], physics [60, 85], geography [54, 52] or from social culture [155, 31].

2.3.1 Equivalence of Yu

Theorem 2.3.1 (Yu [160]) .
Let $f_1, f_2, g_1, g_2, ..., g_m : \mathbb{R}^n \to \mathbb{R}$ linear functions and $X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = \overline{1,m}\}$.
A point $x^* \in X$ is an efficient solution for bi-criteria optimization problem

$$\begin{cases} \min\left(f_1(x), f_2(x)\right) \\ x \in X \end{cases}$$

if and only if $\exists\lambda \in (0,1)$ *such that* x^* *is an optimal solution for parametric optimization problem*

$$\begin{cases} \min\left(\lambda f_1(x) + (1-\lambda) f_2(x)\right) \\ x \in X \end{cases}$$

Using this Theorem, the bi-criteria optimization problem is transformed into an equivalent parametric optimization problem.

One of the methods used to solve optimization problems is based on Lagrange conditions (when constraints are equalities) or Kuhn-Tucker conditions (when constraints are inequalities). Associated multipliers are representing the shadow price. They provide the marginal behavior of the objective function with respect to the constant value of the corresponding constraint and offer important strategical information. It is the reason for choosing this method for solving bi-criteria optimization problems, instead of “ particle swarm optimization“ a common method for energy optimization.

2.3.2 Kuhn-Tucker Theorems

Let us consider the following nonlinear optimization problem

$$\begin{cases} \min f(x) \\ g_i(x) \leq 0, \quad i = \overline{1,m} \\ x \in X \end{cases} \tag{2.2}$$

where $X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}$ and $g_i : X \to \mathbb{R},\ i = \overline{1,m}$.

Definition 2.3.2 *The constraint* $g_i(x) \leq 0$, *with* $i = \overline{1,m}$, *is called active at* x^0 *if* $g_i(x^0) = 0$ *and inactive at* x^0 *if* $g_i(x^0) < 0$.

The following *Theorem* is presenting **necessary Kuhn-Tucker conditions** for existence of solution for nonlinear optimization problem (2.2).

Theorem 2.3.3 (Kuhn-Tucker [73], [79]: necessary conditions) .
Let $X \subseteq \mathbb{R}^n$ *be an open and nonempty set,* $x^* \in X$ *a feasible solution for problem* (2.2) *and functions* $f : X \to \mathbb{R}$ *and* $g_i : X \to \mathbb{R},\ i = \overline{1,m}$ *differentiable at* x^*. *Suppose that gradient vectors* $\nabla g_i(x^*)$ *corresponding to active constraints are linear independent. If* $x^* \in X$ *is an optimal solution*

for problem (2.2), then there exists multipliers $\lambda_i \in \mathbb{R},\ i = \overline{1,m}$ *such that*

$$\begin{aligned}
&\nabla f\left(x^*\right) + \sum_{i=1}^{m} \lambda_i \nabla g_i\left(x^*\right) = 0 \\
&\lambda_i g_i\left(x^*\right) = 0, \quad i = \overline{1,m} \\
&\lambda_i \geq 0, \quad i = \overline{1,m}.
\end{aligned}$$

It is well known that, in general, the necessary Kuhn-Tucker conditions for optimum are not sufficient too. Adding some additional requirements, the necessary conditions become sufficient too.

Theorem 2.3.4 (Kuhn-Tucker [61], [79]: sufficient conditions) .
Let $X \subseteq \mathbb{R}^n$ *be an open and nonempty set,* $x^* \in X$ *a feasible solution for problem (2.2) and functions* $f : X \to \mathbb{R}$ *and* $g_i : X \to \mathbb{R},\ i = \overline{1,m}$ *differentiable in* x^* *and convex. If there exists multipliers* $\lambda_i \in \mathbb{R},\ i = \overline{1,m}$ *such that*

$$\begin{aligned}
&\nabla f\left(x^*\right) + \sum_{i=1}^{m} \lambda_i \nabla g_i\left(x^*\right) = 0 \\
&\lambda_i g_i\left(x^*\right) = 0, \quad i = \overline{1,m} \\
&\lambda_i \geq 0, \quad i = \overline{1,m}.
\end{aligned}$$

then x^* *is an optimal solution for optimization problem (2.2).*

During time, mathematicians have been preoccupied to replace convexity in sufficient Kuhn-Tucker conditions, with a weaker one. Mangasarian [99] and Hanson [61] have contributed by introducing pseudo and cvasi convexity, respectively invexity.

2.3.3 Karush Theorem

While studying at University of Chicago, William Karush has developed his own version for Kuhn-Tucker conditions. Being preoccupied to determine necessary and sufficient conditions for existence of minimum for a function $f = f\left(x_1, x_2, ...x_n\right)$, knowing that constraints $g_\alpha\left(x_1, x_2, ...x_n\right) \geq 0,\ \alpha = \overline{1,m}$ are fulfilled and functions f and $g_\alpha,\ \alpha = \overline{1,m}$ are subject to continuity and differentiability, he has formulated the following Theorem:

Theorem 2.3.5 (Karush theorem [73]: preliminary result) .
If $f\left(x_0\right)$ *is a minimum then there exists multipliers* l_0*,* l_α *not all zero such*

that the derivatives F_{x_i} of the function

$$F(x) = l_0 f(x) + \sum_{\alpha=1}^{m} l_\alpha g_\alpha(x)$$

all vanish at x^0. [Karush, 1939, pp. 12-13]

Multiplier l_0 is used to compute the Lagrangian and no sign restrictions are imposed for multipliers l_α. To avoid this situation, additional restrictions called *regularity conditions* are necessary. Karush has defined admissible direction and admissible arc.

Definition 2.3.6 *A vector $\lambda = (\lambda_1, \lambda_2, ...\lambda_n)$ is called admissible direction if*

$$\sum_{i=1}^{n} \frac{\partial g_\alpha}{\partial x_i}\left(x^0\right)\lambda_i \geq 0, \quad \alpha = \overline{1, m}.$$

Definition 2.3.7 *An arc $x : [0, t_0] \to \mathbb{R}^n$ is called admissible, if $g_\alpha(x(t)) \geq 0$, for any $\alpha = \overline{1, m}$ and $t \in [0, t_0]$.*

Definition 2.3.8 *An arc $x : [0, t_0] \to \mathbb{R}^n$ starts from x^0 to direction λ, if*

$$\begin{aligned} x_i(0) &= x_i^0, \quad i = \overline{1, n} \\ x_i^{'}(0) &= \lambda_i, \quad i = \overline{1, n}. \end{aligned}$$

Using these regularity conditions, Karush is formulating the following Theorem:

Theorem 2.3.9 (Karush [73]) .
Suppose that for each admissible direction λ there is an admissible arc issuing from x^0 in the direction λ. Then a first necessary condition for x^0 to be a minimum is that there exist multipliers $l_\alpha \leq 0, \quad \alpha = \overline{1, m}$ such that the derivatives F_{x_i} of the function

$$F = f + \sum_{\alpha=1}^{m} l_\alpha g_\alpha$$

all vanish at x^0. [Karush, 1939, pp.13].

In 1976 Kuhn is partially publishing Karush's dissertation, emphasizing his contribution to the development of nonlinear programming [78].

2.3.4 John Theorem

Fritz John was preoccupied, among others, by extending Lagrange multipliers to cases when constraints are inequalities.

Theorem 2.3.10 (John [73]) .
Let R be a set of points in $\mathbb{R}^n$, S a set of points in R and R' the set of all points $x \in R$, which satisfy

$$G(x, y) \geq 0, \textit{for all } y \in S$$

where $G : R \times S \to \mathbb{R}$.
Let $F : R \to \mathbb{R}$ and x^0 be an interior point of R and a point of R' with

$$F\left(x^0\right) = \min_{x \in R'} F(x).$$

Then there exists a finite set of points $y^1 ... y^s$ in S and numbers $\lambda_0, \lambda_1, ... \lambda_s$ which do not all vanish, such that

$$\begin{aligned}
&G\left(x^0, y^r\right) = 0, \quad r = \overline{1, s} \\
&\lambda_0 \geq 0 \\
&\lambda_1 > 0, ... \lambda_s > 0, \quad 0 \leq s \leq n
\end{aligned}$$

and the function

$$\phi(x) = \lambda_0 F(x) - \sum_{r=1}^{s} \lambda_r G(x, y^r)$$

has a critical point at x^0.

Due to geometrical context in which John has developed his theorem, he was using parameters $y \in S$ which are not present in the theorems formulated by Karush or Kuhn-Tucker.

2.4 Measures of spread

2.4.1 Variance

The best known measure of spread is variance. It measures the spread of some values around their average or expected value, being defined as:

$$\sigma = E\left[(X - \mu)^2\right], \tag{2.3}$$

where X is a random variable and μ is its average or expected value.

Variance has several application, some of them being available in [3, 10, 11, 22, 29, 43, 45, 46, 48, 59, 63, 65, 83, 86, 101, 104, 105, 110, 121, 133, 135, 136, 137, 140, 143]. Quadratic form of variance makes it very complex and difficult to be applied and explained. Variance is measuring how far the values are spread around the expected value, offering an average distance of spread. Thus variance is not targeting directly the most extreme value and we consider it not suitable for our objective.

2.4.2 Mean absolute deviation

Mean absolute deviation is obtained by replacing quadratic form of variance with module. It is defined as:

$$\sigma = E\left[|X - \mu|\right] \tag{2.4}$$

where X is a random variable and μ is its average or expected value.

[75, 76] have successfully applied mean absolute deviation in some analysis, but for our objective we can't consider it, due to the fact that an average spread is calculated, without targeting directly the most extreme value.

2.4.3 Maximum absolute deviation

Maximum absolute deviation is defined as

$$\sigma = \max\left(|X - \mu|\right), \tag{2.5}$$

where X is a random variable and μ is its average or expected value.

It was successfully applied in [16, 159]. From its definition, maximum absolute deviation is addressing the most extreme point, satisfying our requests. Thus we might consider it as a starting point for developing a proper measure for fluctuation of energy.

Chapter 3

Approximation theorems for bi-criteria optimization problems

3.1 Introduction

Practical problems arising from different fields of activity might generate highly complex multi-criteria optimization problems. Approximation problems are representing one of the method used to solve these complex practical problems. Antczak [4, 5, 6], Duca [12, 25, 38, 40, 124], Popovici [126, 2] have contributed, among others, to this method of solving optimization problems.

3.2 Basic concepts

Let X be a set in $\mathbb{R}^n$, x_0 an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}$. If f is differentiable at x_0 then we denote:

$$F^1(x) = f(x_0) + \nabla f(x_0)\, \eta(x, x_0)$$

and call it first $\eta-$approximation of f
and if f is twice differentiable at x_0 then we denote:

$$F^2(x) = f(x_0) + \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T \nabla^2 f(x_0)\, \eta(x, x_0).$$

and call it second $\eta-$approximation of f.

Definition 3.2.1 *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function differentiable at x_0 and $\eta : X \times X \to X$. Then*

function f is:
invex *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \geq \nabla f(x_0)\, \eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^1(x);$$

incave *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \leq \nabla f(x_0)\, \eta(x, x_0)$$

or equivalently

$$f(x) \leq F^1(x);$$

avex *at x_0 with respect to η if it is both invex and incave at x_0 w.r.t. η.*

Definition 3.2.2 *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function twice differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:*
second order invex *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \geq \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2} \eta(x, x_0)^T \nabla^2 f(x_0)\, \eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^2(x);$$

second order incave *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \leq \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2} \eta(x, x_0)^T \nabla^2 f(x_0)\, \eta(x, x_0)$$

or equivalently:

$$f(x) \leq F^2(x);$$

second order avex *at x_0 with respect to η if it is both second order invex and second order incave at x_0 w.r.t. η.*

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

We consider the bi-criteria optimization problem $\left(P_0^{0,0}\right)$, defined as:

$$\begin{cases} \min\left(f_1, f_2\right)(x) \\ x = \left(x_1, x_2, ...x_n\right) \in X \\ g_t\left(x\right) \leq 0,\ t \in T \\ h_s\left(x\right) = 0,\ s \in S. \end{cases}$$

Assuming that functions f_1, f_2, are differentiable of order $i,\ j \in \{1,\ 2\}$ and functions $g_t,\ (t \in T)\ ,\ h_s,\ (s \in S)$ are differentiable of order $k \in \{0, 1, 2\}$, we will approximate original problem $\left(P_0^{0,0}\right)$ by problems $\left(P_k^{i,j}\right)$:

$$\begin{cases} \min\left(F_1^i, F_2^j\right)(x) \\ x = \left(x_1, x_2, ...x_n\right) \in X \\ G_t^k\left(x\right) \leq 0,\ t \in T \\ H_s^k\left(x\right) = 0,\ s \in S \end{cases}$$

where $(i, j) \in \{(1, 0)\ ,\ (1, 1)\ ,\ (2, 0)\ ,\ (2, 1)\ ,\ (2, 2)\}$, $k \in \{0, 1, 2\}$ and $F_1^0 = f_1,\ F_2^0 = f_2, G_t^0 = g_t\ \ (t \in T)\ , H_s^0 = h_s\ \ (s \in S)$.
We denote by

$$\mathcal{F}^k = \left\{x \in X :\ G_t^k\left(x\right) \leq 0,\ t \in T,\ H_s^k\left(x\right) = 0,\ s \in S\right\},\ k \in \{0, 1, 2\}$$

the set of feasible solutions for bi-criteria optimization problem $\left(P_k^{i,j}\right)$, where $(i, j) \in \{(1, 0)\ ,\ (1, 1)\ ,\ (2, 0)\ ,\ (2, 1)\ ,\ (2, 2)\}$ and $k \in \{0, 1, 2\}$.

3.3 First and second η - approximations for components of objective function on the same feasible set

In this section we will study conditions such that efficient solutions of approximated problems $\left(P_0^{1,0}\right)$, $\left(P_0^{1,1}\right)$, $\left(P_0^{2,0}\right)$, $\left(P_0^{2,1}\right)$ and $\left(P_0^{2,2}\right)$ will remain efficient also for original problem $\left(P_0^{0,0}\right)$ and reciprocally.

Theorem 3.3.1 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is differentiable at x_0 and invex[1] at x_0 with respect to η,*

b) $\eta\left(x_0,\ x_0\right)=0.$

If x_0 is an efficient solution for $\left(P_0^{1,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.3.2 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is differentiable at x_0 and incave[1] at x_0 with respect to η,*

b) $\eta\left(x_0,\ x_0\right)=0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{1,0}\right)$.

Example 3.3.3 (Luca and Duca [96]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min f\left(x\right)=\left(x_1^2+x_2^2; x_1-2x_2\right) \\ -x_1-x_2+2\leq 0 \\ x_1; x_2\geq 0 \end{cases} \tag{3.1}$$

$x^0=(1,1)\in\mathcal{F}^0$ is an efficient solution for problem (3.1) and the value of function f is $f\left(1,1\right)=(2,-1)$.
The approximate problem $\left(P_0^{1,0}\right)$ will be

$$\begin{cases} \min F\left(x\right)=\left(2x_1+2x_2-2; x_1-2x_2\right) \\ -x_1-x_2+2\leq 0 \\ x_1; x_2\geq 0 \end{cases} \tag{3.2}$$

Because $F\left(1,1\right)=\left(2,-1\right)\geq\left(2,-4\right)=F\left(0,2\right)$, it follows that $x^0=(1,1)\in\mathcal{F}^0$ is not an efficient solution for the problem $\left(P_0^{1,0}\right)$.

Remark 3.3.4 *Conditions such that efficient solution of problem $\left(P_0^{1,1}\right)$ will remain efficient for problem $\left(P_0^{0,0}\right)$ and reciprocally have been studied in [40].*

Theorem 3.3.5 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to$

X, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

b) $\eta\,(x_0,\ x_0) = 0.$

If x_0 is an efficient solution for $\left(P_0^{2,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.3.6 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

b) $\eta\,(x_0,\ x_0) = 0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{2,0}\right)$.

Example 3.3.7 (Luca and Duca [96]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min\left(x_1^2 + x_1x_2 + x_2^2 - 19.25x_1 - 19.875x_2; x_1 + x_2\right) \\ -x_1^2 + 6x_1 - 1 - x_2 \leq 0 \\ 4x_1 + x_2 - 20 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.3}$$

An efficient solution for problem (3.3) is $x^0 = (3, 8)$. The corresponding approximate problem $\left(P_0^{2,0}\right)$ is:

$$\begin{cases} \min\left(x_1^2 + x_1x_2 + x_2^2 - 19.25x_1 - 19.875x_2; x_1 + x_2\right) \\ -x_1^2 + 6x_1 - 1 - x_2 \leq 0 \\ 4x_1 + x_2 - 20 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.4}$$

which is identical with initial problem (3.3) and thus they have the same efficient solution.

Remark 3.3.8 *Example 3.3.7 shows that if second order incavity of f_1 from condition a) of Theorem 3.3.6 is not satisfied it might be possible to obtain the same efficient solution.*

Theorem 3.3.9 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

b) *f_2 is differentiable at x_0 and invex^1 at x_0 with respect to η,*

c) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_0^{2,1}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.3.10 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

b) *f_2 is differentiable at x_0 and incave^1 at x_0 with respect to η,*

c) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{2,1}\right)$.

Theorem 3.3.11 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

b) *f_2 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) $\eta\left(x_0,\ x_0\right)=0.$

If x_0 is an efficient solution for $\left(P_0^{2,2}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.3.12 (Luca and Duca [96]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X\times X\to X$, T and S index sets, $f=(f_1,f_2) : X\to\mathbb{R}^2$ and $g_t,\ h_s : X\to\mathbb{R}, (t\in T,\ s\in S)$ functions.

Assume that:

a) *f_1 is twice differentiable at x_0 and incave[2] at x_0 with respect to η,*

b) *f_2 is twice differentiable at x_0 and incave[2] at x_0 with respect to η,*

c) $\eta\left(x_0,\ x_0\right)=0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{2,2}\right)$.

3.4 First and second η-approximations for components of objective function on the first η -approximation of feasible set

In this section we will study conditions such that efficient solutions of approximated problems $\left(P_1^{1,0}\right)$, $\left(P_1^{2,0}\right)$, $\left(P_1^{2,1}\right)$ and $\left(P_1^{2,2}\right)$ will remain efficient also for initial problem $\left(P_0^{0,0}\right)$ and reciprocally.
Conditions for the relation $\left(P_0^{0,0}\right)$ vs. $\left(P_1^{1,1}\right)$ have been studied in [40] so we will not analyze them anymore.

Theorem 3.4.1 (Duca and Ratiu [40]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X\times X\to X$, and $g_t,\ h_s : X\to\mathbb{R}, (t\in T,\ s\in S)$.

Assume that:

a) *for each $t\in T$, the function g_t is differentiable at x_0 and invex[1] at x_0 with respect to η,*

b) *for each $s\in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η,* *then*

$$\mathcal{F}^0 \subseteq \mathcal{F}^1.$$

Theorem 3.4.2 (Duca and Ratiu [40]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that

a) *for each $t \in T$, the function g_t is differentiable at x_0 and incave^1 at x_0 with respect to η,*

b) *for each $s \in S$, the function h_s is differentiable at x_0 and avex^1 at x_0 with respect to η,*

then

$$\mathcal{F}^1 \subseteq \mathcal{F}^0.$$

Theorem 3.4.3 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex^1 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex^1 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

e) $\eta\,(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_1^{2,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.4.4 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and incave*^{1} *at* x_0 *with respect to* η,

c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*^{1} *at* x_0 *with respect to* η,

d) f_1 *is twice differentiable at* x_0 *and incave*^{2} *at* x_0 *with respect to* η,

e) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 *is an efficient solution for* $\left(P_0^{0,0}\right)$, *then* x_0 *is an efficient solution for* $\left(P_1^{2,0}\right)$.

Theorem 3.4.5 (Luca and Duca [95]) .
Let X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and invex*^{1} *at* x_0 *with respect to* η,

c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*^{1} *at* x_0 *with respect to* η,

d) f_1 *is differentiable at* x_0 *and invex*^{1} *at* x_0 *with respect to* η,

e) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 *is an efficient solution for* $\left(P_1^{1,0}\right)$, *then* x_0 *is an efficient solution for* $\left(P_0^{0,0}\right)$.

Theorem 3.4.6 (Luca and Duca [95]) .
Let X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave*^{1} *at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*^{1} *at x_0 with respect to η,*

d) *f_1 is differentiable at x_0 and incave*^{1} *at x_0 with respect to η,*

e) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_1^{1,0}\right)$.

Theorem 3.4.7 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex*^{1} *at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*^{1} *at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex*^{2} *at x_0 with respect to η,*

e) *f_2 is differentiable at x_0 and invex*^{1} *at x_0 with respect to η,*

f) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $\left(P_1^{2,1}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.4.8 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave*^{1} *at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*^{1} *at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave*^{2} *at x_0 with respect to η,*

e) *f_2 is differentiable at x_0 and incave*^{1} *at x_0 with respect to η,*

f) $\eta(x_0, x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_1^{2,1}\right)$.

Theorem 3.4.9 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex*^{1} *at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*^{1} *at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex*^{2} *at x_0 with respect to η,*

e) *f_2 is twice differentiable at x_0 and invex*^{2} *at x_0 with respect to η,*

f) $\eta(x_0, x_0) = 0$.

If x_0 is an efficient solution for $\left(P_1^{2,2}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.4.10 (Luca and Duca [95]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave[1] at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave[2] at x_0 with respect to η,*

e) *f_2 is twice differentiable at x_0 and incave[2] at x_0 with respect to η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_1^{2,2}\right)$.

Example 3.4.11 (Luca and Duca [95]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min\left(x_1 - 2x_2; x_1 + x_2\right) \\ -x_1x_2 + 1 \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

An efficient solution of problem $\left(P_0^{0,0}\right)$ is $x_0 = (1,\ 1) \in \mathcal{F}^0$ and the value of the objective function in x_0 is $f(1,\ 1) = (-1,\ 2)$.
The approximate problems $\left(P_1^{i,j}\right)$, with $(i,j) \in \{(1,0),(1,1),$ $(2,0),(2,1),(2,2)\}$ are:

$$\begin{cases} \min\left(x_1 - 2x_2; x_1 + x_2\right) \\ -x_1 - x_2 + 2 \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

The value of objective function for problem $\left(P_1^{i,j}\right)$ in $x = (0,\ 2) \in \mathcal{F}^1$ is $\left(F_1^i, F_2^j\right)(0,\ 2) = (-4,\ 2)$. Thus $x_0 = (1,\ 1) \in \mathcal{F}^1$ is not an efficient solution for approximate problem $\left(P_1^{i,j}\right)$.

Example 3.4.12 (Luca and Duca [95]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min\left(x_1^2 + (x_2 - \pi - 1)^2;\ (x_1 + \frac{1}{10})^2 - \frac{1}{2}(x_2 + 1)^2\right) \\ -x_1 - \sin x_1 + x_2 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

An efficient solution of problem $\left(P_0^{0,0}\right)$ *is* $x_0 = (\frac{\pi}{2},\ 1+\frac{\pi}{2}) \in \mathcal{F}^0$ *and the value of the objective function in* x_0 *is* $f(\frac{\pi}{2},\ 1+\frac{\pi}{2}) = (\frac{\pi^2}{2};\ \frac{\pi^2}{8} - \frac{9\pi}{10} - \frac{199}{100})$. *The approximate problem* $\left(P_1^{1,1}\right)$ *is:*

$$\begin{cases} \min\left(\pi x_1 - \pi x_2 + \pi + \frac{\pi^2}{2};\ (\pi+\frac{1}{5})x_1 - (\frac{\pi}{2}+2)x_2 - \frac{\pi^2}{8} + \frac{\pi}{2} + \frac{1}{100}\right) \\ -x_1 + x_2 - 1 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

The value for the objective function of problem $\left(P_1^{1,1}\right)$ *in* $x = (\frac{5\pi}{2},\ 1+\frac{5\pi}{2}) \in \mathcal{F}^1$ *is* $F^1(\frac{5\pi}{2},\ 1+\frac{5\pi}{2}) = (\frac{\pi^2}{2},\ \frac{9\pi^2}{8} - \frac{9\pi}{2} - \frac{199}{100})$ *Thus* $x_0 = (\frac{\pi}{2},\ 1+\frac{\pi}{2})$ *is not an efficient solution for problem* $\left(P_1^{1,1}\right)$.

3.5 First and second η - approximations for components of objective function on the second η - approximation of feasible set

In this section we will study conditions such that efficient solutions of approximated problems $\left(P_2^{1,0}\right)$, $\left(P_2^{2,0}\right)$, $\left(P_2^{2,1}\right)$ and $\left(P_2^{2,2}\right)$ will remain efficient also for original problem $\left(P_0^{0,0}\right)$ and reciprocally.

Theorem 3.5.1 (Duca and Boncea [12]) .
Let X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that:

a) *for each* $t \in T$, *the function* g_t *is twice differentiable at* x_0 *and invex*^{2} *at* x_0 *with respect to* η,

b) *for each* $s \in S$, *the function* h_s *is twice differentiable at* x_0 *and avex*^{2} *at* x_0 *with respect to* η,

then

$$\mathcal{F}^0 \subseteq \mathcal{F}^2.$$

Theorem 3.5.2 (Duca and Boncea [12]) .
Let X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that

a) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

b) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

then

$$\mathcal{F}^2 \subseteq \mathcal{F}^0.$$

Theorem 3.5.3 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

e) *f_2 is differentiable at x_0 and invex^1 at x_0 with respect to η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_2^{2,1}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.5.4 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

e) *f_2 is differentiable at x_0 and incave^1 at x_0 with respect to η,*

f) $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,1}\right)$.

Theorem 3.5.5 (Luca and Duca [94]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is differentiable at x_0 and invex^1 at x_0 with respect to η,*

e) $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{1,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.5.6 (Luca and Duca [94]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is differentiable at x_0 and incave^1 at $x_0$with respect to η,*

e) $\eta\left(x_0,\ x_0\right)=0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{1,0}\right)$.

Theorem 3.5.7 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

e) $\eta\left(x_0,\ x_0\right)=0.$

If x_0 is an efficient solution for $\left(P_2^{2,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.5.8 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

e) $\eta\left(x_0,\ x_0\right)=0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,0}\right)$.

Theorem 3.5.9 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

e) *f_2 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_2^{2,2}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Theorem 3.5.10 (Luca and Duca [94]) .

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

e) *f_2 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,2}\right)$.

Example 3.5.11 (Luca and Duca [94]) .
Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min\left(-\left(x_1 - \frac{3\pi}{5}\right)^2 - \left(x_2 - \frac{2\pi}{5} - 1\right)^2; -x_1 + x_2\right) \\ -x_1 - \sin x_1 + x_2 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

An efficient solution of problem $\left(P_0^{0,0}\right)$ is $x_0 = \left(\frac{\pi}{2},\ 1 + \frac{\pi}{2}\right) \in \mathcal{F}^0$.
The approximate problem $\left(P_2^{0,0}\right)$ is:

$$\begin{cases} \min\left(-\left(x_1 - \frac{3\pi}{5}\right)^2 - \left(x_2 - \frac{2\pi}{5} - 1\right)^2; -x_1 + x_2\right) \\ -x_1 + x_2 + \frac{1}{2}\left(x_1 - \frac{\pi}{2}\right)^2 - 1 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

Because $f\left(\frac{3\pi}{4};\ \frac{3\pi}{4} + 1 - \frac{\pi^2}{32}\right) < f\left(\frac{\pi}{2},\ 1 + \frac{\pi}{2}\right)$, it follows that efficient solution of the problem $\left(P^{0,0}\right)$ is not efficient also for the problem $\left(P_2^{0,0}\right)$.

Example 3.5.12 *Let's consider the same initial problem as in Example 3.5.11. The approximate problem $\left(P_2^{1,1}\right)$ is:*

$$\begin{cases} \min\left(-\frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5};\ -x_1 + x_2\right) \\ -x_1 + x_2 + \frac{1}{2}\left(x_1 - \frac{\pi}{2}\right)^2 - 1 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

Because $F^1\left(\frac{3\pi}{4};\ \frac{3\pi}{4} + 1 - \frac{\pi^2}{32}\right) < F^1\left(\frac{\pi}{2},\ 1 + \frac{\pi}{2}\right)$, it follows that efficient solution of the problem $\left(P^{0,0}\right)$ is not efficient also for the problem $\left(P_2^{1,1}\right)$.

Example 3.5.13 (Luca and Duca [94]) .
Let's consider the same initial problem as in Example 3.5.11. The approximate problem $\left(P_2^{2,2}\right)$ is:

$$\begin{cases} \min\left(-\frac{\pi}{2}\left(x_1 - \frac{\pi}{2}\right)^2 - \frac{\pi+2}{2}\left(x_2 - 1 - \frac{\pi}{2}\right)^2 - \frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5};\ -x_1 + x_2\right) \\ -x_1 + x_2 + \frac{1}{2}\left(x_1 - \frac{\pi}{2}\right)^2 - 1 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

Because $F^2(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}) < F^2(\frac{\pi}{2},\ 1+\frac{\pi}{2})$, *it follows that efficient solution of the problem* $(P^{0,0})$ *is not efficient also for the problem* $\left(P_2^{2,2}\right)$.

3.6 Relations between η - approximation problems for $\left(P_0^{0,0}\right)$

3.6.1 Relations between $\left(P_0^{i,j}\right)$ and $\left(P_1^{i,j}\right)$

Theorem 3.6.1 (Luca and Duca [97]) .

Let X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and incave*[1] *at* x_0 *with respect to* η,

c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*[1] *at* x_0 *with respect to* η.

1. *If* f_1 *is differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_0^{1,0}\right)$, *then* x_0 *is an efficient solution for problem* $\left(P_1^{1,0}\right)$.

2. *If* f_1 *is twice differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_0^{2,0}\right)$, *then* x_0 *is an efficient solution for problem* $\left(P_1^{2,0}\right)$.

3. *If* f_1 *is twice differentiable at* x_0, f_2 *is differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_0^{2,1}\right)$, *then* x_0 *is an efficient solution for problem* $\left(P_1^{2,1}\right)$.

4. *If* f_1 *is twice differentiable at* x_0, f_2 *is twice differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_0^{2,2}\right)$, *then* x_0 *is an efficient solution for problem* $\left(P_1^{2,2}\right)$.

Theorem 3.6.2 (Luca and Duca [97]) .

Let X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and invex*[1] *at* x_0 *with respect to* η,

c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*[1] *at* x_0 *with respect to* η.

1. *If* f_1 *is differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_1^{1,0}\right)$ *then* x_0 *is an efficient solution for problem* $\left(P_0^{1,0}\right)$.
2. *If* f_1 *is twice differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_1^{2,0}\right)$ *then* x_0 *is an efficient solution for problem* $\left(P_0^{2,0}\right)$.
3. *If* f_1 *is twice differentiable at* x_0, f_2 *is differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_1^{2,1}\right)$ *then* x_0 *is an efficient solution for problem* $\left(P_0^{2,1}\right)$.
4. *If* f_1 *is twice differentiable at* x_0, f_2 *is twice differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_1^{2,2}\right)$ *then* x_0 *is an efficient solution for problem* $\left(P_0^{2,2}\right)$.

3.6.2 Relations between $\left(P_0^{i,j}\right)$ and $\left(P_2^{i,j}\right)$

Theorem 3.6.3 (Luca and Duca [97]) .

Let X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*

Assume that:

a) $x_0 \in \mathcal{F}^2$,

b) *for each* $t \in T$, *the function* g_t *is twice differentiable at* x_0 *and incave*[2] *at* x_0 *with respect to* η,

c) *for each* $s \in S$, *the function* h_s *is twice differentiable at* x_0 *and avex*[2] *at* x_0 *with respect to* η.

1. *If* f_1 *is differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_0^{1,0}\right)$, *then* x_0 *is an efficient solution for problem* $\left(P_2^{1,0}\right)$.
2. *If* f_1 *is twice differentiable at* x_0 *and* x_0 *is an efficient solution for problem* $\left(P_0^{2,0}\right)$, *then* x_0 *is an efficient solution for problem* $\left(P_2^{2,0}\right)$.

3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{2,1}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,1}\right)$.*

4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_0^{2,2}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,2}\right)$.*

Theorem 3.6.4 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{1,0}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{1,0}\right)$.*

2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,0}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,0}\right)$.*

3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,1}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,1}\right)$.*

4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,2}\right)$ then x_0 is an efficient solution for problem $\left(P_0^{2,2}\right)$.*

3.6.3 Relations between $\left(P_1^{i,j}\right)$ and $\left(P_2^{i,j}\right)$

Theorem 3.6.5 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Assume that:

a) *for each $t \in T$, the function g_t is twice differentiable at x_0 and $\nabla^2 g_t\left(x_0\right)$ is negative semi-definite,*

b) *for each $s \in S$, the function h_s is twice differentiable at x_0 and $\nabla^2 h_s\left(x_0\right)$ is null definite,*

then

$$\mathcal{F}^1 \subseteq \mathcal{F}^2.$$

Theorem 3.6.6 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.
If

a) *for each $t \in T$, the function g_t is twice differentiable at x_0 and $\nabla^2 g_t\left(x_0\right)$ is positive semi-definite,*

b) *for each $s \in S$, the function h_s is differentiable at x_0 and $\nabla^2 h_s\left(x_0\right)$ is null definite,*

then

$$\mathcal{F}^2 \subseteq \mathcal{F}^1.$$

Theorem 3.6.7 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.
Assume that:

a) $x_0 \in \mathcal{F}^2$*,*

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and $\nabla^2 g_t\left(x_0\right)$ is positive semi-definite,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and $\nabla^2 h_s\left(x_0\right)$ is null definite.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{1,0}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{1,0}\right)$.*
2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,0}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,0}\right)$.*

3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,1}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,1}\right)$.*

4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_1^{2,2}\right)$, then x_0 is an efficient solution for problem $\left(P_2^{2,2}\right)$.*

Theorem 3.6.8 (Luca and Duca [97]) .
Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each $t \in T$, the function g_t is twice differentiable at x_0 and $\nabla^2 g_t\left(x_0\right)$ is negative semi-definite,*

c) *for each $s \in S$, the function h_s is twice differentiable at x_0 and $\nabla^2 h_s\left(x_0\right)$ is null definite.*

1. *If f_1 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{1,0}\right)$ then x_0 is an efficient solution for problem $\left(P_1^{1,0}\right)$.*

2. *If f_1 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,0}\right)$ then x_0 is an efficient solution for problem $\left(P_1^{2,0}\right)$.*

3. *If f_1 is twice differentiable at x_0, f_2 is differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,1}\right)$ then x_0 is an efficient solution for problem $\left(P_1^{2,1}\right)$.*

4. *If f_1 is twice differentiable at x_0, f_2 is twice differentiable at x_0 and x_0 is an efficient solution for problem $\left(P_2^{2,2}\right)$ then x_0 is an efficient solution for problem $\left(P_1^{2,2}\right)$.*

3.7 Conclusions

This chapter was dedicated to study of some conditions such that efficient solution of a bi-criteria optimization problem remains efficient also for the approximate problem and reciprocally. Our contributions to this Chapter, consisting of 32 Theorems and 7 counter-examples, were disseminated in [96], [95], [94] and [97].

Chapter 4

Minimax model for energy optimization

4.1 Introduction

This chapter is minimax model for energy optimization. It is a bi-criteria optimization problem which aims to shave the peak load by minimizing fluctuation of energy and maximizing economic performance of the power plant.

Starting from maximum absolute deviation we will define a measure for fluctuation of energy production. Economic performance of the power plant will be measured by turnover.

4.2 Minimax measure for fluctuation of energy

Maximum absolute deviation, presented in Chapter 2, is targeting directly the most extreme value. Thus, it will represent the starting point in developing a measure for fluctuation of energy. [35, 132, 152] emphasize elasticity of energy price. Consequently we consider adequate to introduce energy price in the measure of fluctuation.

Definition 4.2.1 *Minimax fluctuation of energy is the maximum, over all time periods, for difference between energy produced at certain time moments and a predefined level of energy, multiplied with price of energy at the corresponding time moment.*

Remark 4.2.2 *Predefined level of energy might be for example a random value chosen by energy plant or the average amount of energy produced* *during a certain period from the past. Of course, when the average is employed, it might be necessary to adjust it with a factor to cover the projected increase of demand.*

Denoting by $1, 2, ..., i, ..., n$ the time horizon considered and

x_i - energy produced at time moment i, $i = \overline{1, n}$,

p_i - price of energy at time moment i, $i = \overline{1, n}$,

r - predefined level of energy,

ε - minimum level of energy assumed by the power plant to be delivered in the power grid,

ρ - maximum level of energy assumed by the power plant to be delivered in the power grid,

and considering that $\epsilon \leq r \leq \rho$, the *minimax fluctuation of energy* is

$$\max_{i=\overline{1,n}} |p_i x_i - p_i r| . \tag{4.1}$$

4.3 Problem formulation

Let's consider a power plant which aims to shave the peak load by minimizing fluctuation of energy, without decreasing its economic performances. It is obvious that we are dealing with a bi-criteria optimization problem. Its first component is minimax measure of fluctuation (4.1) and the second is turnover defined as

$$\sum_{i=1}^{n} p_i x_i .$$

Considering simple technical constraints which limit the amount of energy produced, the minimax model is defined as

$$\begin{cases} \min \left(\max\limits_{i=\overline{1,n}} |p_i x_i - p_i r| , \ -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \varepsilon \leq x_i \leq \rho, \quad i = \overline{1, n}. \end{cases} \tag{4.2}$$

4.4 Computing the solution

To determine the efficient solution for problem (4.2) we will introduce the following bi-criteria equivalent problem

$$\begin{cases} \min \left(y, \ -\sum_{i=1}^{n} p_i x_i\right)^T \\ |p_i x_i - p_i r| \leq y, \quad i = \overline{1,n} \\ \varepsilon \leq x_i \leq \rho, \quad i = \overline{1,n}. \end{cases} \tag{4.3}$$

their equivalence being proved by the following Lemma

Lemma 4.4.1 *[98] Let's consider the bi-criteria optimization problems (4.2) and (4.3).*

a) If $x \in \mathbb{R}^n$ is an efficient solution for problem (4.2), then $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max_{i=\overline{1,n}} |p_i x_i - p_i r|$ is an efficient solution for problem (4.3).

b) If $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max_{i=\overline{1,n}} |p_i x_i - p_i r|$ is an efficient solution for problem (4.3), then $x \in \mathbb{R}^n$ is an efficient solution for problem (4.2).

Using results of Yu [160], Bot et all [14] and Geoffrion [53] the bi-criteria problem (4.3) is equivalent to the following parametric optimization problem

$$\begin{cases} \min \left\{ \lambda y - (1-\lambda) \sum_{i=1}^{n} p_i x_i \right\} \\ |p_i x_i - p_i r| \leq y, \quad i = \overline{1,n} \\ \varepsilon \leq x_i \leq \rho, \quad i = \overline{1,n}. \end{cases} \tag{4.4}$$

with $\lambda \in (0, 1)$ and the following Lemma holds.

Lemma 4.4.2 *[98] $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ is an efficient solution for bi-criteria problem (4.3) if and only if $\exists \lambda \in (0, 1)$ such that $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ is an optimal solution for parametric optimization problem (4.4)*

Remark 4.4.3 *Considering the equivalence between problems (4.2) and (4.3), respectively problems (4.3) and (4.4), it follows from transitivity that problems (4.2) and (4.4) are equivalent. This means that in order to compute the efficient solution for (4.2) we have to determine the optimal solution for (4.4).*

Remark 4.4.4 *In the process of computing the optimal solution, we will split the set* $\{1, 2, ..., n\}$ *in subsets like* $\{1, 2, ..., l\}$ *and* $\{l+1, l+2, ..., n\}$*, or* $\{1, 2, ..., l\}$*,* $\{l+1, l+2, ..., m\}$ *and* $\{m+1, m+2, ..., n\}$*. If price is constant on such an interval, it will be denoted by* $\overline{p}$*.*

The following theorem is computing an optimal solution for parametric optimization problem (4.4).

Theorem 4.4.5 (Luca and Mahalov [98]; *parametric minimax*) .
The optimal solutions for parametric optimization problem (4.4) *are:*

1. *If* $\lambda < \frac{n}{n+1}$*, then*
$$\begin{cases} x_i^* = \rho, \quad i = \overline{1, n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$
or
   - *if* $\overline{p_1} \leq p_j, \; j = \overline{l+1, n}$*, then*
$$\begin{cases} x_i^* = \rho, \quad i = \overline{1, l} \\ x_j^* = r + \frac{y^*}{p_j}, \quad j = \overline{l+1, n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$
where $\overline{p_1} = p_i, \; i = \overline{1, l}$*.*
   - *else problem has no solution.*

2. *If* $\lambda = \frac{n}{n+1}$*, then*
$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1, n} \\ y^* = \min\limits_{i=\overline{1,n}} \{p_i\,(\rho - r)\}\,. \end{cases}$$

3. *If* $\lambda > \frac{n}{n+1}$*, then*
$$\begin{cases} x_i^* = r, \quad i = \overline{1, n} \\ y^* = 0. \end{cases}$$

4. *If* $\lambda < \frac{l}{l+1}$*, then*
   - *if* $p_j < \overline{p}, \; j = \overline{l+1, n}$*, then*
$$\begin{cases} x_i^* = \rho, \quad i = \overline{1, l} \\ x_j^* = \rho, \quad j = \overline{l+1, n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$

where $\overline{p_1} = p_i, \ i = \overline{1, l}$.

- *else problem has no solution.*

5. *If* $\lambda = \frac{l}{l+1}$, *then*

- *if* $p_j < p_i, i = \overline{1, l}, \ \ j = \overline{l+1, n}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1, l} \\ x_j^* = \rho, \quad j = \overline{l+1, n} \\ y^* = \min\limits_{i=\overline{1,l}} \{p_i (\rho - r)\}, \ if \ p_j < p_i. \end{cases}$$

- *else problem has no solution*

6. *If* $\lambda < \frac{l+(n-m)}{l+(n-m)+1}$, *then*

- *if* $p_j \leq \overline{p} \leq p_i, \ i = \overline{1, l}, \ j = \overline{l+1, m}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1, l} \\ x_j^* = \rho, \quad j = \overline{l+1, m} \\ x_k^* = \rho, \quad k = \overline{m+1, n} \\ y^* = \overline{p} (\rho - r) \end{cases}$$

where $\overline{p_3} = p_k, \ k = \overline{m+1, n}$.

- *else problem has no solution.*

Definition 4.4.6 Possible combinations *are the combinations of Kuhn-Tucker multipliers, determined for a fixed* $i \in \{1, 2, ..., n\}$ *for which complementarity slackness and dual feasibility conditions are fulfilled.*

Definition 4.4.7 Feasible combinations *are those possible combinations for which the gradient of Lagrangian is zero.*

Definition 4.4.8 Critical combinations *are those feasible combinations for which a solution does not exist if they are combined.*

Remark 4.4.9 *Due to the fact that the optimization problem (**??**18)) is a convex one, it follows that Kuhn-Tucker conditions are both necessary and sufficient.*

Remark 4.4.10 *During proof of Theorem 4.4.5 it can be noticed that for scenario 5 one of the partial derivatives of Lagrangian is not equal to zero. This means that scenario 5 will not generate a solution by its own, but combined with other scenarios, it might generate a solution.*

Theorem 4.4.11 (Luca and Mahalov [98]; *order of scenarios*) .
The order of scenarios, used to compute an optimal solution for parametric optimization problem (4.4), does not influence the solution.

Theorem 4.4.12 (Luca and Mahalov [98]; *energy minimax*) .
The efficient solution for bi-criteria energy optimization problem (4.2) is

1. *If* $\lambda < \frac{n}{n+1}$ *and there is a single price for energy, then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p}\,(\rho - r) \\ TR = n\overline{p}\rho \end{cases}$$

or

- *if energy is sold against two different prices during 24 hours and* $\overline{p_1} \leq p_j,\ j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = r + \frac{y^*}{p_j}, \quad j = \overline{l+1,n} \\ y^* = \overline{p_1}\,(\rho - r) \\ TR = l\overline{p_1}\rho + r\sum\limits_{j=l+1}^{n} p_j + (n-l)\,y^*. \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$.

- *else problem has no solution.*

2. *If* $\lambda = \frac{n}{n+1}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,n} \\ y^* = \min\limits_{i=\overline{1,n}} \{p_i\,(\rho - r)\} \\ TR = r\sum\limits_{i=1}^{n} p_i + ny^*. \end{cases}$$

3. *If* $\lambda > \frac{n}{n+1}$, *then*

$$\begin{cases} x_i^* = r, \quad i = \overline{1,n} \\ y^* = 0 \\ TR = r\sum\limits_{i=1}^{n} p_i. \end{cases}$$

4. *If* $\lambda < \frac{l}{l+1}$, *then*

- *if* $p_j < \overline{p_1},\ j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p}\left(\rho - r\right) \\ TR = l\overline{p}\rho + \rho \sum\limits_{j=l+1}^{n} p_j. \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$.

- *else problem has no solution.*

5. *If* $\lambda = \frac{l}{l+1}$, *then*

- *if* $p_j < p_i, i = \overline{1,l},\ \ j = \overline{l+1,n}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \min\limits_{i=\overline{1,l}} \left\{p_i\left(\rho - r\right)\right\} \\ TR = ly^* + r\sum\limits_{i=1}^{l} p_i + \rho \sum\limits_{j=l+1}^{n} p_j. \end{cases}$$

- *else problem has no solution.*

6. *If* $\lambda < \frac{l+(n-m)}{l+(n-m)+1}$, *then*

- *if* $p_j \le \overline{p_3} \le p_i,\ i = \overline{1,l}, j = \overline{l+1,m}$, *then*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p}\left(\rho - r\right) \\ TR = ly^* + r\sum\limits_{i=1}^{l} p_i + \rho \sum\limits_{j=l+1}^{n} p_j + \left(n - m\right)\overline{p}\rho. \end{cases}$$

where $\overline{p_3} = p_k,\ k = \overline{m+1,n}$.

- *else problem has no solution.*

4.5 Model validation and conclusions

4.5.1 Testing of solution

To test the minimax model we have used data regarding consumption and production of electricity provided by Transelectrica [148] and energy price established by Romanian Energy Regulatory Authority [129]. Using Box&Whisker and Grubbs tests data were validated and used to generate the input for minimax model: $\varepsilon = 3666$ MW, $\rho = 10808$ MW si $r = 6970$ MW.

To evaluate performances of the minimax model, the optimal production plan computed with Theorem 4.4.5 was compared with real data from 4th of December 2015. Results obtained have been disseminated in [90].

Besides planning and economic advantages associated to peak load shaving, we consider that minimax model generates several additional advantages like: (i) incentives for power plants to develop storage systems, (ii) incentives to invest in green energy to compensate the production shift, to reduce CO_2 emissions and to reduce production cost, (iii) reduced fatigue, maintenance and development costs for equipments of power plant (iv) increased efficiency for power plant or (v) a more stable and efficient power grid.

4.5.2 Conclusions

Tests performed for minimax model prove its capacity to shave the peak load. Thus, the objective of our research is realized.

Results presented during this chapter have been disseminated in [90] and [98].

Chapter 5

Index model for energy optimization

5.1 Introduction

Predefined level specific for minimax measure of fluctuation is increasing the complexity of the model. Moreover, the efficient frontier is limited, because a point situated under the predefined level will never be Pareto efficient (due to absolute value, the point will generate the same fluctuation as its symmetric but turnover will be smaller).

To simplify minimax model and to make it more friendly for practitioners we will define a new model for peak load shaving – index model. We will need a new measure for fluctuation of energy. Economic performance of the power plant will be evaluated by turnover.

A comparison between minimax and index models, emphasizing the conceptual differences between them was presented at *ICATA 2016* and published in [39].

5.2 Index measure for fluctuation of energy

Skipping the predefined level, specific to minimax, the reference point for fluctuation is lost. Another parameter, more friendly for practitioners has to be used as reference point. In production it is natural to express loading of an equipment as percentage from its maximum capacity [109, 112].

Using the notations introduced in Chapter 4 and the above mentioned idea, the *index measure for fluctuation of energy* is defined as

$$\max_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}. \tag{5.1}$$

Elasticity of energy price, explained during previous Chapter, has determined us to keep price also in the index measure for fluctuation.

5.3 Problem formulation

Let's consider again the bi-criteria optimization problem developed for a power plant which aims to shave the peak load . By replacing minimax measure of fluctuation with index measure (5.1), the following problem is obtained for index model

$$\begin{cases} \min \left(\max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\} ; -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \varepsilon \leq x_i \leq \rho,\ i = \overline{1,n}. \end{cases} \tag{5.2}$$

5.4 Computing the solution

To determine the efficient solution for the problem (5.2) we introduce the following bi-criteria optimization problem

$$\begin{cases} \min \left(y; -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \frac{x_i}{\rho} p_i \leq y,\ i = \overline{1,n} \\ \varepsilon \leq x_i \leq \rho,\ i = \overline{1,n} \end{cases} \tag{5.3}$$

their equivalency being established by the following Lemma.

Lemma 5.4.1 *[92] Let's consider the bi-criteria optimization problems (5.2) and (5.3).*

a) If $x \in \mathbb{R}^n$ is an efficient solution for problem (5.2), then $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}$ is an efficient solution for problem (5.3).

b) If $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, with $y = \max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}$ is an efficient solution for problem (5.3), then $x \in \mathbb{R}^n$ is an efficient solution for problem (5.2).

Using again the idea to transform the bi-criteria problem into a parametric one and based on Theorem 2.3.1 of Yu [160] and similar results of Bot et all [14] and Geoffrion [53] the bi-criteria optimization problem (5.3) is equivalent to the following parametric optimization problem

$$\begin{cases} \min\left\{\lambda y-(1-\lambda)\sum\limits_{i=1}^{n}p_i x_i\right\} \\ \frac{x_i}{\rho}p_i \le y,\ i=\overline{1,n} \\ \varepsilon \le x_i,\ i=\overline{1,n} \\ x_i \le \rho,\ i=\overline{1,n} \end{cases} \tag{5.4}$$

with $\lambda \in (0,1)$ and the following is true

Lemma 5.4.2 *[92] $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ is an efficient solution for bi-criteria problem (5.3) if and only if $\exists\lambda \in (0,1)$ such that $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ is an optimal solution for parametric optimization problem (5.4).*

Remark 5.4.3 *To determine the efficient solution for problem (5.2) it is sufficient to determine the optimal solution for problem (5.4).*

Remark 5.4.4 *In the process of computing the optimal solution for parametric optimization problem (5.4) we will split the set $\{1,2,...,n\}$ in several subsets like $\{1,2,...l\}$, $\{l+1,l+2,...m\}$, $\{m+1,m+2,...t\}$ and $\{t+1,t+2,...n\}$. If on such a set or subset the price is constant we will denote it by $\overline{p_*}$.*

The following *Theorem* presents an optimal solution for the parametric optimization problem (5.4).

Theorem 5.4.5 (Luca and Duca [92]; parametric index) .
An optimal solution $(x^*,y^*) \in \mathbb{R}^n \times \mathbb{R}$ for parametric optimization problem (5.4) is:

1. *If $\lambda = \frac{\rho n}{1+\rho n}$ and $\frac{\min\limits_{i=\overline{1,n}} p_i}{\max\limits_{i=\overline{1,n}} p_i} \ge \frac{\varepsilon}{\rho}$, then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i}y^*, \quad i=\overline{1,n} \\ y^* \in \left[\max\limits_{i=\overline{1,n}}\frac{\varepsilon}{\rho}p_i;\ \min\limits_{i=\overline{1,n}}p_i\right] \end{cases}$$

else solution does not exist.

2. *If* $\lambda > \frac{\rho n}{1+\rho n}$, *then*
$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p}, \end{cases}$$
where $\overline{p} = p_i,\ i = \overline{1,n}$,
else solution does not exist.

3. *If* $\lambda < \frac{\rho n}{1+\rho n}$, *then*
$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p}, \end{cases}$$
where $\overline{p} = p_i,\ i = \overline{1,n}$,
else solution does not exist.

4. *If* $\lambda = \frac{\rho l}{1+\rho l}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*
$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \begin{cases} \overline{p_2}, & \text{if } \min\limits_{i=\overline{1,l}} p_i = p_j, \quad j = \overline{l+1,n} \\ \in \left[\max\limits_{j=\overline{l+1,n}} p_j; \min\limits_{i=\overline{1,l}} p_i\right], & \text{if } \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq p_j < \min\limits_{i=\overline{1,l}} p_i, \quad j = \overline{l+1,n} \\ \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i; \min\limits_{i=\overline{1,l}} p_i\right], & \text{if } p_j < \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i, \quad j = \overline{l+1,n} \end{cases} \end{cases}$$
where $\overline{p_2} = p_j,\ j = \overline{l+1,n}$,
else solution does not exist.

5. *If* $\lambda < \frac{\rho l}{1+\rho l}$, *then*
$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p_1},\ \text{if } \frac{\overline{p_1}}{p_j} \geq 1, \quad j = \overline{l+1,n} \end{cases}$$
where $\overline{p_1} = p_i,\ i = \overline{1,l}$,
else solution does not exist.

6. *If* $\lambda > \frac{\rho l}{1+\rho l}$, *then*
$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1}, \quad \text{if } \frac{\overline{p_1}}{p_j} \geq \frac{\rho}{\varepsilon}, \quad j = \overline{l+1,n} \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$,
else solution does not exist.

7. *If* $\lambda < \frac{\rho n}{1+\rho n}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p_2},\ \textit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \frac{\rho}{\varepsilon}\overline{p_2} \\ \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

where $\overline{p_2} = p_j,\ j = \overline{l+1,n}$,
else solution does not exist.

8. *If* $\lambda > \frac{\rho l}{1+\rho l}$ *and* $\frac{\overline{p_2}}{\overline{p_1}} = \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1} = \overline{p_2} \end{cases}$$

where $\overline{p_1} = p_i,\ i = \overline{1,l}$ *and* $\overline{p_2} = p_j,\ j = \overline{l+1,n}$,
else solution does not exist,

9. *If* $\lambda > \frac{\rho n}{1+\rho n}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2},\ \textit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho}\overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

where $\overline{p_2} = p_j,\ j = \overline{l+1,n}$,
else solution does not exist.

10. *If* $\lambda > \frac{\rho m}{1+\rho m}$, $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$ *and* $\frac{\overline{p_3}}{\overline{p_2}} = \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} = \overline{p_3}, \ \textit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho}\overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

where $\overline{p_2} = p_j, \ j = \overline{l+1,m}$ *and* $\overline{p_3} = p_k, \ k = \overline{m+1,n}$, *else solution does not exist.*

11. *If* $\lambda > \frac{\rho l + \rho(n-m)}{1+\rho l+\rho(n-m)}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \varepsilon, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3}, \ \textit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq \frac{\varepsilon}{\rho}\overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

where $\overline{p_3} = p_k, \ k = \overline{m+1,n}$, *else solution does not exist.*

12. *If* $\lambda < \frac{\rho l + \rho(n-m)}{1+\rho l+\rho(n-m)}$ *and* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p_3}, \ \textit{if} \begin{cases} \max\limits_{i=\overline{1,n}} p_i \leq \frac{\rho}{\varepsilon}\overline{p_3} \\ \overline{p_3} \leq \min\limits_{i=\overline{1,n}} p_i \\ p_j \leq \overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

where $\overline{p_3} = p_k, \ k = \overline{m+1,n}$, *else solution does not exist.*

13. *If* $\lambda > \frac{\rho(m-l)}{1+\rho(m-l)}$ *and* $\frac{\overline{p_3}}{\overline{p_2}} = \frac{\varepsilon}{\rho}$*, then*

$$\begin{cases} x_i^* = \rho, & i = \overline{1,l} \\ x_j^* = \varepsilon, & j = \overline{l+1,m} \\ x_k^* = \rho, & k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} = \overline{p_3}, & \text{if } p_i \leq \frac{\varepsilon}{\rho}\overline{p_2},\ i = \overline{1,l} \end{cases}$$

where $\overline{p_2} = p_j,\ j = \overline{l+1,m}$ *and* $\overline{p_3} = p_k,\ k = \overline{m+1,n}$*,*
else solution does not exist.

14. *If* $\lambda > \frac{\rho l+\rho(t-m)}{1+\rho l+\rho(t-m)}$, $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$ *and* $\frac{\overline{p_4}}{\overline{p_3}} = \frac{\varepsilon}{\rho}$*, then*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, & i = \overline{1,l} \\ x_j^* = \rho, & j = \overline{l+1,m} \\ x_k^* = \varepsilon, & k = \overline{m+1,t} \\ x_s^* = \rho, & s = \overline{t+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3} = \overline{p_4}, & \text{if } \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq \frac{\varepsilon}{\rho}\overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

where $\overline{p_3} = p_k,\ k = \overline{m+1,t}$ *and* $\overline{p_4} = p_s,\ s = \overline{t+1,n}$*,*
else solution does not exist.

Based on Lemma 5.4.1, Theorem 5.4.5 is providing an efficient solution $x^* \in \mathbb{R}^n$ for problem (5.2).

The values for the objective function of problem (5.2) are provided by: (a) Theorem 5.4.5 in case of fluctuation and (b) the following Theorem in case of turnover, where TR denotes the turnover of the power plant.

Theorem 5.4.6 (Luca and Duca [92]; energy index) .
The values for second member of the objective function from problem (5.2) *are:*

1. $TR = n\rho y^*$*, where* y^* *is defined by Theorem 5.4.5 item 1;*

2. $TR = \varepsilon \sum\limits_{i=1}^{n} p_i$*;*

3. $TR = \rho \sum\limits_{i=1}^{n} p_i$*;*

4. $TR = \rho l y^* + \rho \sum\limits_{j=l+1}^{n} p_j$, *where* y^* *is defined by Theorem 5.4.5 item 4;*

5. $TR = l\rho\overline{p_1} + \rho \sum\limits_{j=1+1}^{n} p_j$, *where* $\overline{p_1} = p_i,\ i = \overline{1,l}$;

6. $TR = l\varepsilon\overline{p_1} + \rho \sum\limits_{j=l+1}^{n} p_j$, *where* $\overline{p_1} = p_i,\ i = \overline{1,l}$;

7. $TR = n\rho\,\overline{p_2}$, *where* $\overline{p_2} = p_j,\ j = \overline{l+1,n}$;

8. $TR = n\varepsilon\overline{p_1}$, *where* $\overline{p_1} = p_i,\ i = \overline{1,l}$;

9. $TR = n\varepsilon\overline{p_2}$, *where* $\overline{p_2} = p_j,\ j = \overline{l+1,n}$;

10. $TR = n\varepsilon\overline{p_2}$, *where* $\overline{p_2} = p_j,\ j = \overline{l+1,m}$;

11. $TR = (n-m+l)\varepsilon\overline{p_3} + \rho \sum\limits_{j=l+1}^{m} p_j$, *where* $\overline{p_3} = p_k,\ k = \overline{m+1,n}$;

12. $TR = (n-m+l)\rho\overline{p_3} + \rho \sum\limits_{j=l+1}^{m} p_j$, *where* $\overline{p_3} = p_k,\ k = \overline{m+1,n}$;

13. $TR = (n-l)\epsilon\overline{p_2} + \rho\sum\limits_{i=1}^{l} p_i$, *where* $\overline{p_2} = p_j,\ j = \overline{l+1,m}$;

14. $TR = (n-m+l)\varepsilon\,\overline{p_3} + \rho \sum\limits_{j=l+1}^{m} p_j$, *where* $\overline{p_3} = p_k,\ k = \overline{m+1,t}$.

5.5 Model validation and conclusions

5.5.1 Testing of solution

Index model is tested using the same date as for minimax model.

Two test were performed for the index model, difference being generated by input data ε si ρ.
Same inputs as for minimax model were used during the first test. Existence and no-critical conditions specified by Theorem 5.4.5 were not satisfied in case of several solutions. For the second test, inputs were estimated and calculated such that conditions specified by Theorem 5.4.5 are satisfied for as much solutions as possible. Results obtained are better compared to the first test and sugest a sensitivity of the model to input data.

5.5.2 Concluzii

Tests performed prove the capacity of index model to shave the peak load. Thus, the objective of our research is realized. Additional advantages associated to peak load shaving, like: (i) incentives for power plants to develop storage systems, (ii) incentives to invest in green energy to compensate the production shift, to reduce CO_2 emissions and to reduce production cost, (iii) reduced fatigue, maintenance and development costs for equipments of power plant (iv) increased efficiency for power plant or (v) a more stable and efficient power grid. remain valid also for index model, remain valid also for index model.

Index model is more friendly to practitioners and easyer to be implemented, but has a lower accuracy compared to minimax model.

Results presented during this chapter are disseminated in [90, 92].

Chapter 6

Conclusions

Global context is generating challenges for energy optimization, a current research topic being peak load shaving.

Analyzing a production chart reveals fluctuation in energy production. These fluctuations might be regarded as a spread of values. Defining a proper measure for fluctuation of energy (a measure able to target the most extreme value) and minimizing it might generate a peak load shaving. Power diagram modification due to peak load shaving is generating supra-production at other time moments. According to the strategy of the power plant, supra-production might be addressed by: (i) Energy Storage Systems, (ii) electric vehicles, (iii) demand side management or (iv) human interferences for planning alteration.

Literature is presenting a strong connection between technical and economical components. Thus power diagram modification (due to peak load shaving) might generate a decrease of economic performances. A bi-criteria optimization problem is required to mitigate this risk.

Thus, the objective of our research might be formulated as *to create, solve and validate a mathematical model which will shave the peak load by minimizing fluctuation of energy and maximizing the economic performance*.

We have developed two mathematical models (minimax and index model) for this problem generated by a real life context. To solve the models we transformed them in equivalent parametric optimization problems. Kuhn-Tucker conditions were used to compute the optimal solutions for parametric problems, which have generated the efficient solutions for problems (4.2) si (5.2)).

Despite popularity of “particle swarm optimization“ in energy field, Kuhn-Tucker conditions were used due to economic meaning of multipliers, which offer strategic information to decision makers.

Solutions provided by both models were tested using real data. Both model have achieved the objective to shave the peak load.

Profit as measure for economic performance, additional technical constraints, a more accurate prediction for input data, power grid topology and a better transition from day to night periods might generate more complex model able to provide more accurate solutions. A few references for dealing with identified challenges are available in [125], [77], [147], [122], [18], [20], [91], [89].

The method used to solve minimax and index models might have a low efficiency for more complex models. Chapter 3 of our thesis is presenting a method for solving an initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ by attaching approximate problems $\left(P_k^{i,j}\right)$. Link between the two sets of problems is created by the replacement of some functions with their η - approximations of first or second order. Chapter 3 is presenting sufficient conditions such that efficient solution of initial problem $\left(P_0^{0,0}\right)$ will remain efficient also for the approximate problem $\left(P_k^{i,j}\right)$ and reciprocally.

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