UNIVERSITATEA BABES BOLYAI
Facultatea de Matematica si Informatica
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REZUMAT AL TEZEI DE DOCTORAT

Optimizare bi-criteriala cu aplicatii in economie (Bi-criteria optimization with applications in economy)

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Abstract

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Optimizare bi-criteriala cu aplicatii in economie
(Bi-criteria optimization with applications in economy)

by Traian Ionut LUCA

Contextul global genereaza o serie de provocari ce fac necesara optimizarea productiei de energie electrica. Productia de energie electrica este un proces complex, cu obiective definite pe termen scurt, mediu si lung. Analizand un grafic de productie si consum aferent unei zile, se observa existenta unei fluctuatii. Aceasta fluctuatie ar putea fi privita ca o imprastiere de valori, varful de sarcina fiind valoarea extrema.

Obiectivul nostru, in programul de studii finalizat prin aceasta teza de doctorat, este *de a crea, rezolva si valida un model matematic pentru productia de energie electrica, care sa asigure netezirea varfului de sarcina prin minimizarea fluctuatiei in productia de energie si maximizarea performantelor economice ale producatorului*.

Netezirea varfului de sarcina va genera o supra-productie de energie la alte momente de timp. In functie de strategia producatorului, supra-productia poate fi abordata prin solutii de stocare a energiei, vehicule electrice, managementul cererii sau ajustarea planului de productie. Transferarea productiei dinspre varful de sarcina spre alte momente de timp va genera o genera schimbari in diagrama de productie. Acest aspect poate duce la reducerea performantelor economice ale producatorului. Pentru reducerea acestui risc, vom folosi probleme de optimizare bi-criteriale. Prima componenta a functiei obiectiv vizeaza varful de sarcina, iar a doua componenta vizeaza performanta economica.

Prima parte a tezei (Capitolele 2 si 3) este dedicata analizei si dezvoltarii aparatului matematic necesar. Sunt evaluate masuri ale fluctuatiei si metode de rezolvare a problemelor de optimizare bi-criteriala. Pentru rezolvarea problemelor bi-criteriale vom folosi fie probleme de optimizare parametrica si multiplicatori Kuhn-Tucker, fie vom folosi probleme care sa aproximeze problema bi-criteriala initiala. Capitolul 3 al tezei este dedicat studiului problemelor care aproximeaza problema de optimizare bi-criteriala initiala si conexiunilor dintre cele doua tipuri de probleme. Conditii de convexitate generalizata sunt folosite pentru construirea conditiilor suficiente ca solutia eficienta a problemi initiale sa ramana eficienta si pentru problema care aproximeaza si reciproc.

Cheia in netezirea varfului de sarcina, prin minimizarea fluctuatiei in productia de energie electrica, este de a construi o masura a fluctuatiei care sa vizeze punctul extrem al imprastierii. Mai multe masuri ale fluctuatiei au fost evaluate, concluzia fiind ca maximul abaterii absolute satisface cerintele noastre.

Cifra de afaceri este folosita ca masura a performantelor economice. Ca restrictii ale problemei, vom folosi conditii tehnice ce limiteaza productia de energie electrica.

Masura minimax a fluctuatiei de energie electrica este definita pornind de la maximul abaterii absolute. Astfel, este construit modelul minimax pentru netezirea varfului de sarcina. Solutia eficienta este determinata prin rezolvarea unei probleme parametrice echivalente. Planul de productie astfel construit demonstreaza rezultate bune si excelente, raportat la datele reale referitoare la productia si consumul din 4 Decembrie 2015.

Datele de intrare ale masurii minimax cresc complexitatea acestui model. O noua masura a fluctuatiei de energie electrica (index) a fost creata pentru a reduce complexitatea. Un nou model de netezire a varfului de sarcina (modelul index) este creat. Solutia eficienta este determinata prin rezolvarea unei probleme parametrice echivalente. Planul de productie construit cu modelul index demonstreaza rezultate bune comparativ cu datele reale din 4 Decembrie 2015.

Includerea unor restrictii tehnice mai complexe, folosirea profitului ca masura a performantei economice, o mai buna estimare a datelor de intrare sau o noua abordare a tranzitiei de la perioada de zi la cea de noapte ar putea deschide noi directii de cercetare si ar putea imbunatati acuratetea rezultatelor.

Teza noastra cuprinde 6 capitole.
Primul capitol este dedicat cadrului general al productiei de energie electrica si provocarilor pentru netezirea varfului de sarcina.
Capitolul al doilea este dedicat unor instrumente matematice folosite in cercetarea noastra. Problemele de optimizare bi-criteriala si conditiile Kuhn-Tucker sunt prezentate. De asemenea, sunt prezentate si evaluate cateva masurii ale variatie.
Capitolul al treilea este dedicat studierii unor conditii suficiente astfel incat solutia eficienta a unei probleme bicriteriale sa ramana solutie eficienta si pentru problema care o aproximeaza si reciproc. Teoremele de aproximare construite creaza un mediu propice rezolvarii mai eficiente a problemelor complexe din domeniul energiei.
Capitolul al patrulea este dedicat modelului minimax pentru energie. Sunt prezentate: masura minimax a fluctuatiei de energie, dezvoltarea, rezolvarea si validarea modelului minimax.
Capitolul al cincilea este dedicat modelului indexpentru energie. Crearea acestui model a fost stimulata de complexitatea inputurilor pentru modelul minimax. Este introdusa o noua masura pentru fluctuatia de energie, este dezvoltat, rezolvat si validat modelul index. Modelul index este mai putin complex si mai usor de implementat comparativ cu modelul minimax, dar acuratetea rezultatelor este mai redusa.
Capitolul al saselea prezinta cateva concluzii ale cercetarii noastre, o analiza si comparatie a celor doua modele, precum si cateva directii de extindere si imbunatatire a lor.

Contributiile noastre din aceasta teza ar putea fi centralizate in: o noua metoda de netezire a varfului de sarcina, bazata pe probleme de optimizare bi-criteriale; Teoremele 3.3.1, 3.3.2, 3.3.5, 3.3.6, 3.3.9, 3.3.10, 3.3.11, 3.3.12, 3.4.3, 3.4.4, 3.4.5, 3.4.6, 3.4.7, 3.4.8, 3.4.9, 3.4.10, 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.7, 3.5.8, 3.5.9, 3.5.10, 3.6.1, 3.6.2, 3.6.3, 3.6.4, 3.6.5, 3.6.6, 3.6.7, 3.6.8; Exemplele 3.3.3, 3.3.7, 3.4.11, 3.4.12, 3.5.11, 3.5.12, 3.5.13; definirea masurilor minimax (4.1) si index (5.1) pentru fluctuatia productiei de energie; problemele bicriteriale (4.2) si (5.2) pentru netezirea varfului de sarcina; Lemele 4.4.1 si Lemma 5.4.1 ce transforma problemele bi-criteriale (4.2) si (5.2) in probleme echivalente; Teorema 4.4.11 ce arata ca ordinea scenariilor nu influenteaza solutia; Teoremele 4.4.5, 4.4.12, 5.4.5 si 5.4.6 care determina solutiile eficiente ale problemelor (4.2) si (5.2); clasificarea multiplicatorilor Kuhn-Tucker folosind Definitiile 4.4.6, 4.4.7 si 4.4.8, precum si testarea modelelor minimax si index folosind date rele.

Rezultatele prezentate in aceasta teza au fost diseminate la doua conferinte internationale:

> *Bi-criteria problems for energy optimization*, prezentata la Conferinta: International Conference on Approximation Theory and its Applications, organizata la Sibiu, Romania in perioada 26-29 May 2016.
>
> *Bi-criteria models for energy markets*, prezentata la Conferinta: Management International Conference, organizata la Monastier di Treviso, Italia, in perioada 24-27 May 2017.

si in noua articole:

> *Minimax rule for energy optimization* [98], publicat in Computers and Fluids, un jurnal ISI cu Factorul de Impact de 2.313 si Factorul de Impact pe 5 ani de 2.308.
>
> *Production index for peak load shaving in energy production* [92], trimis spre publicare catre Engineering Optimization, un jurnal ISI cu Factorul de Impact de 1.728.
>
> *Bi-criteria models for peak-load shaving* [90], acceptat pentru publicare de catre Journal of Academy of Business and Economics, un jurnal indexat in EBSCO, EconLit, ProQuest Ulrichsweb, Index Copernicus, Research Bible.
>
> *Approximations of objective function in bi-criteria optimization problem* [96], acceptat pentru publicare de catre European International Journal of Science and Technology, un jurnal indexat in Google Scholar, NewJour, Hochschulbibliothek Reutlingen, CrossRef.
>
> *Approximations of objective function and constraints in bi-criteria optimization problems* [95], trimis spre publicare catre Journal of Numerical Analysis and Approximation Theory, un jurnal indexat in Mathematical Reviews, Zentralblatt MATH.

Approximations of bi-criteria approxiamtion problem [94], trimis spre publicare catre Studia Universitatis Babes-Bolyai Mathematica, un jurnal indexat in Mathematical Reviews, Zentralblatt MATH, EBSCO, ProQuest Ulrichsweb.

Relations between η – approximation problems of a bi-criteria optimization problems [97], trimis spre publicare catre Annals of Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, un jurnal indexat in Mathematical Reviews, Zentralblatt MATH, American Mathematical Society.

Bi-criteria problems for energy optimization [39], publicat in General Mathematics, un jurnal indexat in Zentralblatt MATH, EBSCO, Mathematical Reviews, Index Copernicus.

Portfolio optimization algorithms [93], publicat in Studia Negotia, un jurnal indexat in EBSCO, Index Copernicus, ERIH PLUS.

fiind sustinute de munca noastra anterioara, reflectata in carti, articole si conferinte:

Matematici economice. Elemente de programare liniara si teoria probabilitatilor [19], carte publicata de Presa Universitara Clujeana.

Strict fixed points results for multivalued contractions on gauge spaces [122], publicat in Fixed Point Theory, un jurnal ISI cu factorul de Impact de 1.030 in anul 2010.

Uniqueness algebraic conditions in the study of second order elliptic systems [18], publicat in International Journal of Pure and Applied Mathematics, jurnal indexat in Scopus.

Maximum principles for a class of second order parabolic systems in divergence form [20], publicat in Journal of Nonlinear Functional Analysis and Differential Equations, jurnal indexat in Scopus, Web of Science, Zentralblatt MATH.

Automotive industry and performances of US Economy [125], publicat in International Journal of Finance and Economics, jurnal indexat in EBSCO, Ulrich's, Index Copernicus, EconLit.

Consumer's inflation expectations in Romania [147], publicat in International Journal of Business Research, jurnal indexat in EBSCO, Ulrich's, Index Copernicus, EconLit.

Comparative analysis of low-cost airlines websites [77], publicat in Proceedings of IABE - 2009 Las Vegas - Annual Conference, volum ce este indexat in EBSCO, Ulrich's, Index Copernicus, EconLit.

Economic applications of dynamic optimization, prezentata la Conferinta: 10th International Symposium on Generalized Convexity and Monotonicity, organizata la Cluj Napoca, Romania, in perioada 22-27 August 2011.

A relation between transportation problems and profit, prezentata la Conferinta: Current Issues of Regional Development, organizata la Sec, Republica Ceha, in perioada 26-27 June 2007.

Cuvinte cheie: *teoreme de aproximare, convexitate generalizata, netezirea varfului de sarcina, probleme bi-criteriale, minimizarea fluctuatiei de energie, maximizarea performantelor economice, optimizare parametrica, conditii Kuhn-Tucker*.

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Chapter 1

Contextul general al optimizarii energiei

Evolutia umanitatii este dependenta de energie. Conform *International Energy Agency* (IEA) [66], sursele principale pentru obtinerea energiei sunt: carbuni, petrol si gaze naturale (67.4% din totalul combustibililor folositi). Aceasta genereaza un nivel ridicat al emisiilor de CO_2, cu un puternic impact asupra schimbarilor climatice, reflectate, printre altele, prin cresterea temperaturii medii globale [115]. Adaugand previziunile de crestere a populatiei [150], previziunile de crestere a consumului de energie electrica [151], topologia, capacitatea si costurile asociate retelei de distributie, precum si factorii antropogenici este evidenta existenta a numeroase provocari pentru optimizarea productiei de energie.

Productia de energie este un mediu foarte complex, in care actioneaza trei actori (producatori, consumatori si operatorii independenti), ce au obiective pe termen scurt, mediu si lung [37]. Cerintele specifice si pietele day–ahead [80] fac ca problemele de energie sa fie complexe [37], [118], dificil de rezolvat si cu numeroase provocari.

Rezultate in domeniul optimizarii energiei au fost obtinute, printre altii, de: Frangioni [50], Martinez [102], Philpott [123], Borghetti [13], Belloni [8], Redondo [127], Lemarechal [84], Gollmer [55], Nowak [117], Nürnberg [118], Wen [154], Zhang [161], Gross [58], Conejo [27], [28], Ladurantaye [81], Gonzalez [56], Eichhorn [44], Dicorato [36], Cormio [30], Islam [67], Akella [1], Martins [103], Watson [153], Nakata [114], Dudhani [41], Babu [7], Duic [42], Morais [108], Marcato [100], Mahalov et all [130], [132], [87], [131].

Cresterea consumului de energie electrica, impreuna cu magnitudinea, frecventa si durata prelungita a valurilor de caldura extrema genereaza o crestere a varfurilor de sarcina, creand presiune asupra:

(i) producatorilor [9, 23, 138, 146], (ii) retelelor electrice si (iii) pretului energiei [132].

Abordarea clasica a varfului de sarcina se bazeaza pe metode nefezabile economic [107]. O abordare preferabila, ce a devenit si o importanta tema de cercetare [149] se bazeaza pe netezirea varfului de sarcina. Trei strategii sunt aplicabile pentru netezirea varfului de sarcina: (i) sisteme de stocare a energiei, (ii) vehicule electrice si (iii) managementul cererii.

Contributii importante in netezirea varfului de sarcina pot fi regasite in [15, 23, 24, 26, 33, 47, 49, 51, 57, 68, 70, 72, 87, 82, 88, 111, 130, 113, 116, 120, 128, 134, 139, 141, 144, 145, 149, 156, 157, 158, 162].

Varful de sarcina ce dorim sa il netezim este vizibil in Figura 1.1.

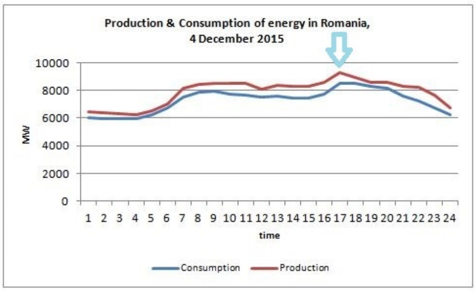

FIGURE 1.1: Evolutia energiei in Romania in 4 decembrie 2015

Aceasi figura arata o fluctuatiea energiei, ce ar putea fi privita ca o imprastiere/variatie.

Obiectivul nostru in aceasta cercetare este de a crea, rezolva si valida un model matematic care sa asigure netezirea varfului de sarcina prin minimizarea fluctuatiei in productia de energie si maximizarea performantelor economice ale producatorului. Obiectivul cercetarii sugereaza utilizarea unor probleme de optimizare bi-criteriale.

Chapter 2

Aparatul matematic

2.1 Introducere

Un rol principal in cercetarea intreprinsa il au problemele de optimizare bi-criteriala. In acest capitol urmarim doua obiective: (i) sa prezentam metode de rezolvare a problemelor de optimzare bi-criteriale si (ii) sa analizam masuri cunoscute ale variatiei (imprastierii) pentru a determina un punct de pornire in construirea masurilor pentru fluctuatiei de energie electrica.

2.2 Probleme de optimizare multi-criteriala si puncte eficiente

Definition 2.2.1 *Fie $X \subseteq \mathbb{R}^n$ si $f = (f_1, f_2, ..., f_m)^T : X \to \mathbb{R}^m$. Problema*

$$\begin{cases} \min f(x) \\ x \in X \end{cases} \tag{2.1}$$

se numeste problema de optimizare multi-criteriala.

Remark 2.2.2 *Daca $m = 2$ problema de optimizare multi-criteriala definita mai sus se numeste problema de optimizare bi-criteriala.*

In studiul nostru, prima componenta a functiei obiectiv se refera la fluctuatia energiei electrice si are un rol cheie in netezirea varfului de sarcina. A doua componenta se refera la performanta economica a producatorului de energie si influenteaza acuratetea solutiei.

Solutia eficienta a unei probleme de optimizare multi-criteriala presupune realizarea unui compromis intre toate componentele functiei obiectiv. Aceasta solutie este cunoscuta ca fiind punct eficient sau solutie eficienta.

Definition 2.2.3 *O solutie admisibila $x^* \in X$ a problemei (2.1) este o solutie eficienta daca $\nexists x \in X$ astfel incat*

$$\begin{aligned} f(x) &\leq f(x^*) \\ f(x) &\neq f(x^*). \end{aligned}$$

Problemele de optimizare multi-criteriala au numeroase aplicatii in domenii precum: investitii financiare [16, 76], inginerie, biologie [69], analiza de date [21] sau logistica [119].

2.3 Rezolvarea problemelor bi-criteriale

Rezolvarea problemelor de optimizare se poate face prin metode analitice (solutia exacta este determinata pe baza unor demonstratii matematice), prin metode metode numerice (solutia este aproximata pe baza unor iteratii) sau cu ajutorul unei probleme care aproximeaza problema initiala.

Metodele de "scalarizare" [62, 106, 142, 17] sunt frecvent folosite in cazul problemelor bi-criteriale. Gradul de importanta al fiecarei componente din functia obiectiv se poate stabili *a priori* sau in timpul procesului de optimizare.

[32] considera ca metodele numerice de rezolvare a problemelor bi-criteriale sunt inspirate din biologie [64, 74, 34, 71], fizica [60, 85], geografie [54, 52] sau cultura sociala [155, 31].

2.3.1 Echivalenta lui Yu

Theorem 2.3.1 (Yu [160]) .

Fie $f_1, f_2, g_1, g_2, ..., g_m : \mathbb{R}^n \to \mathbb{R}$ functii liniare si $X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = \overline{1,m}\}$.
Un punct $x^* \in X$ este solutie eficienta a problemei de optimizare bi-criteriala

$$\begin{cases} \min\left(f_1(x), f_2(x)\right) \\ x \in X \end{cases}$$

daca si numai daca $\exists \lambda \in (0,1)$ astfel incat x^* este solutie optima a problemei parametrice

$$\begin{cases} \min\left(\lambda f_1(x) + (1-\lambda) f_2(x)\right) \\ x \in X \end{cases}$$

Folosind aceasta teorema, problema de optimizare bi-criteriala este transformata in problema de optimizare parametrica.

Una din metodele de rezolvare a problemelor de optimizare se bazeaza pe conditiile Lagrange (restrictiile sunt egalitati) sau conditiile Kuhn-Tucker (restrictiile sunt inegalitati). Multiplicatorii asociati restrictiilor reprezinta pretul ascuns. Ei arata comportamentul marginal al functiei obiectiv in raport cu termenul liber al restrictiei si ofera informatii strategice importante. Acesta este motivul pentru care am ales aceasta metoda de rezolvare a problemlor bi-criteriale de energie, in defavoarea "particle swarm optimization" o metoda foarte folosita in optimizarea energiei electrice.

2.3.2 Teoremele Kuhn-Tucker

Fie problema de optimizare

$$\begin{cases} \min f(x) \\ g_i(x) \leq 0, \quad i = \overline{1,m} \\ x \in X \end{cases} \tag{2.2}$$

unde $X \subseteq \mathbb{R}^n$, $f : X \to \mathbb{R}$ si $g_i : X \to \mathbb{R},\ i = \overline{1,m}$.

Definition 2.3.2 *Restrictia $g_i(x) \leq 0$, cu $i = \overline{1,m}$, este activa in x^0 daca $g_i(x^0) = 0$ si inactiva in x^0 daca $g_i(x^0) < 0$.*

Teorema urmatoare prezinta **conditiile Kuhn-Tucker necesare** pentru existenta unei solutii in cazul problemei de optimizare (2.2).

Theorem 2.3.3 (Kuhn-Tucker [73], [79]: conditii necesare) .
Fie $X \subseteq \mathbb{R}^n$ o multime deschisa si nevida, $x^* \in X$ o solutie admisibila a problemei (2.2) si functiile $f : X \to \mathbb{R}$ si $g_i : X \to \mathbb{R},\ i = \overline{1,m}$ diferentiabile in x^*. Presupunem ca vectorii gradienti $\bigtriangledown g_i(x^*)$ corespunzatori restrictiilor active sunt liniar independenti. Daca $x^* \in X$ este o solutie optima a problemei (2.2), atunci exista multiplicatorii $\lambda_i \in \mathbb{R},\ i = \overline{1,m}$ astfel incat

$$\begin{aligned} &\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0 \\ &\lambda_i g_i(x^*) = 0, \quad i = \overline{1,m} \\ &\lambda_i \geq 0, \quad i = \overline{1,m}. \end{aligned}$$

Este bine cunoscut ca, in general, conditiile necesare Kuhn-Tucker nu sunt si suficiente. Adaugand conditii suplimentare, conditiile necesare devin si suficiente.

Theorem 2.3.4 (Kuhn-Tucker [61], [79]: conditii suficiente) .
Fie $X \subset \mathbb{R}^n$ *o multime deschisa si nevida,* $x^* \in X$ *o solutie admisibila a problemei* (2.2) *si functiile* $f : X \to \mathbb{R}$ *si* $g_i : X \to \mathbb{R}$, $i = \overline{1,m}$ *diferentiabile in* x^* *si convexe. Daca exista multiplicatorii* $\lambda_i \in \mathbb{R}$, $i = \overline{1,m}$ *astfel incat*

$$\begin{aligned}
&\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0 \\
&\lambda_i g_i(x^*) = 0, \quad i = \overline{1,m} \\
&\lambda_i \geq 0, \quad i = \overline{1,m}.
\end{aligned}$$

atunci x^* *este o solutie optima a problemei de optimizare* (2.2).

De-a lungul timpului, matematicienii au fost preocupati sa inlocuiasca convexitatea din conditiile Kuhn-Tucker suficiente cu cerinte mai slabe. Mangasarian [99] si Hanson [61] si-au adus contributia prin introducerea pseudo si cvasi convexitatii, respectiv a invexitatii.

2.3.3 Teorema lui Karush

William Karush a dezvoltat propria versiune a conditiilor Kuhn-Tucker in timpul studiilor la Universitatea din Chicago. Fiind preocupat sa determine conditii necesare si suficiente de existenta a minimului pentru o functie $f(x_1, x_2, ...x_n)$, stiind ca restrictiile $g_\alpha(x_1, x_2, ...x_n) \geq 0$, $\alpha = \overline{1,m}$ sunt indeplinite, iar functiile f si g_α, $\alpha = \overline{1,m}$ sunt continue si diferentiabile, el a formulat urmatoarea teorema:

Theorem 2.3.5 (Karush theorem [73]: rezultat preliminar) .
Daca $f(x_0)$ *este un minim, atunci exista multiplicatorii* l_0, l_α *nu toti zero, astfel incat derivatele* F_{x_i} *ale functiei*

$$F(x) = l_0 f(x) + l_\alpha g_\alpha(x)$$

se anuleaza in x^0. *[Karush, 1939, pp. 12-13]*

Multiplicatorul l_0 este folosit pentru constructia Lagrangianului, iar asupra multiplicatorilor l_α nu se impun restrictii de semn. Pentru a evita aceasta, sunt necesare cerinte suplimentare, numite *conditii de regularitate*. Karush a definit directia admisibila si arcul admisibil.

Definition 2.3.6 *Un vector $\lambda = (\lambda_1, \lambda_2, ...\lambda_n)$ se numeste directie admisibila daca*

$$\sum_{i=1}^{n} \frac{\partial g_\alpha}{\partial x_i} \left(x^0\right) \lambda_i \geq 0.$$

Definition 2.3.7 *Un arc $x : [0, t_0] \to \mathbb{R}^n$ este admisibil, daca $g_\alpha (x (t)) \geq 0$, pentru orice $\alpha = \overline{1, m}$ si $t \in [0, t_0]$.*

Definition 2.3.8 *Un arc $x : [0, t_0] \to \mathbb{R}^n$ porneste din x^0 in directia λ, daca*

$$\begin{aligned} x_i (0) &= x_i^0 \\ x_i^{'} (0) &= \lambda_i. \end{aligned}$$

Folosind aceste conditii de regularitate, Karush formuleaza teorema

Theorem 2.3.9 (Karush [73]) .
Presupunem ca pentru fiecare directie admisibila λ exista un arc admisibil ce porneste din x^0 in directia λ. Atunci, o conditie necesara pentru ca $f\left(x^0\right)$ sa fie un minim este sa existe multiplicatorii $l_\lambda \leq 0$ astfel incat derivatele F_{x_i} functiei

$$F = f + l_\alpha g_\alpha$$

sa se anuleze in x^0. [Karush, 1939, pp.13].

In 1976, Kuhn publica partial teza lui Karush, reliefand contributiile acestuia [78].

2.3.4 Teorema lui John

Fritz John a fost preocupat, printre altele, de extinderea multiplicatorilor Lagrange la cazurile cand restrictiile sunt inegalitati.

Theorem 2.3.10 (John [73]) .
Fie R o multime de puncte in $\mathbb{R}^n$, S o multime de puncte in R si $R^{'}$ multimea tuturor punctelor $x \in R$, care verifica

$$G (x, y) \geq 0, \text{ pentru toti } y \in S$$

unde $G : R \times S \to \mathbb{R}$.
Fie $F : R \to \mathbb{R}$ si x^0 un punct interior pentru R si un punct din $R^{'}$ cu

$$F\left(x^0\right) = \min_{x \in R^{'}} F (x) .$$

Atunci exista o multime finita de puncte $y^1 ... y^s$ in S si numerele $\lambda_0, \lambda_1, ...\lambda_s$ nu toate nule, astfel incat

$$\begin{aligned}
&G\left(x^0, y^r\right) = 0, \quad r = \overline{1, s} \\
&\lambda_0 \geq 0 \\
&\lambda_1 > 0, ...\lambda_s > 0, \quad 0 \leq s \leq n
\end{aligned}$$

si functia

$$\phi\left(x\right) = \lambda_0 F\left(x\right) - \sum_{r=1}^{s} \lambda_r G\left(x, y^r\right)$$

are un punct critic in x^0.

Datorita contextului geometric in care John a formulat aceasta teorema, a folosit parametrii $y \in S$, care nu sunt prezenti in teoremele formulate de Karush si Kuhn-Tucker, iar conditiile de regularitate nu sunt necesare.

2.4 Masuri ale imprastierii

2.4.1 Dispersia

Cea mai cunoscuta masura a imprastierii este dispersia. Aceasta masoara imprastierea valorilor fata de valoarea medie/asteptata, fiind definita astfel:

$$\sigma = E\left[\left(X - \mu\right)^2\right], \tag{2.3}$$

unde X este o variabila aleatoare si μ este valoarea medie/asteptata a acesteia.

Aplicatii ale dispersiei se regasesc in [3, 10, 11, 22, 29, 43, 45, 46, 48, 59, 63, 65, 83, 86, 101, 104, 105, 110, 121, 133, 135, 136, 137, 140, 143]. Aplicarea si interpretarea dispersiei sunt complicate datorita formei patratice. Mai mult, dispersia calculeaza o imprastiere medie, fara a viza in mod direct valoarea extrema, ceea ce o face improprie obiectivului nostru.

2.4.2 Abaterea absoluta medie

Prin inlocuirea formei patratice a dispersiei cu functia modul, se obtine abaterea absoluta medie, definita ca

$$\sigma = E\left[|X-\mu|\right] \tag{2.4}$$

unde X este o variabila aleatoare si μ este valoarea medie/asteptata a acesteia.

[75, 76] au folosit cu succes abaterea absoluta medie, dar pentru obiectivul nostru nu poate fi luata in considerare deoarece calculeaza o imprastiere medie fara a viza direct valoarea extrema.

2.4.3 Abaterea absoluta maxima

Abaterea absoluta maxima este definita ca

$$\sigma = \max\left(|X-\mu|\right), \tag{2.5}$$

unde X este o variabila aleatoare si μ este valoarea medie/asteptata a acesteia.

Aplicatii ale abaterii absolute maxime sunt prezentate in [16, 159]. Prin modul in care este definita, abaterea absoluta maxima vizeaza valoarea extrema a setului de date si astfel ea raspunde cerintelor noastre. Prin urmare, consideram ca abaterea absoluta maxima reprezinta un punct de plecare in definirea masurilor fluctuatiei energiei.

Chapter 3

Teoreme de aproximare pentru probleme de optimizare bi-criteriala

3.1 Introducere

Situatiile practice pot genera probleme de optmizare bi-criteriala foarte complicate si dificil de rezolvat. O metoda eficienta de rezolvare ar putea fi inlocuirea problemelor initiale cu probleme care le aproximeaza si care, in general, sunt mai usor de rezolvat. Antczak [4, 5, 6], Duca [12, 25, 38, 40, 124], Popovici [2, 126] au contribuit, printre altii, la aceasta metoda de rezolvare a problemelor de optimizare bi-criteriala.

3.2 Concepte de baza

Fie X o multime in $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$ si $f : X \to \mathbb{R}$. Daca f este differentiabila in x_0 atunci notam:

$$F^1(x) = f(x_0) + \nabla f(x_0)\, \eta(x, x_0)$$

si o numim prima η - aproximanta a functiei f,
iar daca f este de doua ori diferentiabila in x_0 atunci notam:

$$F^2(x) = f(x_0) + \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T \nabla^2 f(x_0)\, \eta(x, x_0).$$

si o numim a doua η - aproximanta a functiei f.

Definition 3.2.1 *Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior a lui X, $f : X \to \mathbb{R}$ o functie diferentiabila in x_0 si $\eta : X \times X \to X$. Atunci*

functia f este:
***invexa** in x_0 in raport cu η daca pentru toti $x \in X$ avem:*

$$f(x) - f(x_0) \geq \nabla f(x_0)\, \eta(x, x_0)$$

sau echivalent:

$$f(x) \geq F^1(x);$$

***incava** in x_0 in raport cu η daca pentru toti $x \in X$ avem:*

$$f(x) - f(x_0) \leq \nabla f(x_0)\, \eta(x, x_0)$$

sau echivalent

$$f(x) \leq F^1(x);$$

***avexa** in x_0 in raport cu η daca este atat invexa cat si incava in x_0 in raport cu η.*

Definition 3.2.2 *Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $f : X \to \mathbb{R}$ o functie de doua ori diferentiabila in x_0 si $\eta : X \times X \to X$. Atunci functia f este:*
***invexa de ordin doi** in x_0 in raport cu η daca pentru toti $x \in X$ avem:*

$$f(x) - f(x_0) \geq \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\, \nabla^2 f(x_0)\, \eta(x, x_0)$$

sau echivalent:

$$f(x) \geq F^2(x);$$

***incava de ordin doi** in x_0 in raport cu η daca pentru toti $x \in X$ avem:*

$$f(x) - f(x_0) \leq \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\, \nabla^2 f(x_0)\, \eta(x, x_0)$$

sau echivalent:

$$f(x) \leq F^2(x);$$

***avexa de ordin doi** in x_0 in raport cu η daca este atat invexa de ordin doi cat si incava de ordin doi in x_0 in raport cu η.*

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Consideram problema de optimizare bi-criteriala $\left(P_0^{0,0}\right)$

$$\begin{cases} \min\left(f_1, f_2\right)(x) \\ x = \left(x_1, x_2, ...x_n\right) \in X \\ g_t\left(x\right) \leq 0,\ t \in T \\ h_s\left(x\right) = 0,\ s \in S. \end{cases}$$

Presupunand ca functiile f_1, f_2, sunt diferentiabile de ordinul $i,\ j \in \{1,\ 2\}$ iar functiile $g_t,\ (t \in T)\,,\ h_s,\ (s \in S)$ sunt diferentiabile de ordinul $k \in \{0, 1, 2\}$, vom aproxima problema initiala $\left(P_0^{0,0}\right)$ cu problemele $\left(P_k^{i,j}\right)$:

$$\begin{cases} \min\left(F_1^i, F_2^j\right)(x) \\ x = \left(x_1, x_2, ...x_n\right) \in X \\ G_t^k\left(x\right) \leq 0,\ t \in T \\ H_s^k\left(x\right) = 0,\ s \in S \end{cases}$$

unde $(i, j) \in \{(1,0)\,,\ (1,1)\,,\ (2,0)\,,\ (2,1)\,,\ (2,2)\}$, $k \in \{0,1,2\}$ si $F_1^0 = f_1,\ F_2^0 = f_2, G_t^0 = g_t\ \ (t \in T)\,, H_s^0 = h_s\ \ (s \in S)$.
Notam

$$\mathcal{F}^k = \left\{x \in X :\ G_t^k\left(x\right) \leq 0,\ t \in T,\ H_s^k\left(x\right) = 0,\ s \in S\right\},\ k \in \{0,1,2\}$$

multimea solutiilor admisibile ale problemei $\left(P_k^{i,j}\right)$, unde $(i, j) \in \{(1,0)\,,\ (1,1)\,,\ (2,0)\,,\ (2,1)\,,\ (2,2)\}$ si $k \in \{0,1,2\}$.

3.3 Prima si a doua η - aproximare pentru functiile obiectiv si acelasi sistem de restrictii

In acest paragraf sunt studiate conditii suficiente pentru ca solutiile eficiente ale problemelor $\left(P_0^{1,0}\right)$, $\left(P_0^{1,1}\right)$, $\left(P_0^{2,0}\right)$, $\left(P_0^{2,1}\right)$ si $\left(P_0^{2,2}\right)$ sa ramana eficiente si pentru problema initiala $\left(P_0^{0,0}\right)$ si reciproc.

Theorem 3.3.1 (Luca and Duca [96]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) *f_1 este diferentiabila in x_0 si invexa[1] in x_0 in raport cu η,*

b) $\eta(x_0,\ x_0) = 0.$

Daca x_0 este o solutie eficienta pentru $\left(P_0^{1,0}\right)$, atunci x_0 este o solutie eficienta si pentru $\left(P_0^{0,0}\right)$.

Theorem 3.3.2 (Luca and Duca [96]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) *f_1 este diferentiabila in x_0 si incava[1] in x_0 in raport cu η,*

b) $\eta(x_0,\ x_0) = 0.$

Daca x_0 este o solutie eficienta pentru $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si pentru $\left(P_0^{1,0}\right)$.

Example 3.3.3 (Luca and Duca [96]) .
Fie problema initiala de optimizare bi-criteriala $\left(P_0^{0,0}\right)$

$$\begin{cases} \min f(x) = (x_1^2 + x_2^2; x_1 - 2x_2) \\ -x_1 - x_2 + 2 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.1}$$

$x^0 = (1,1) \in \mathcal{F}^0$ este o solutie eficienta a problemei (3.1), iar valoarea functiei obiectiv f este $f(1,1) = (2,-1)$.
Construim problema $\left(P_0^{1,0}\right)$

$$\begin{cases} \min F(x) = (2x_1 + 2x_2 - 2; x_1 - 2x_2) \\ -x_1 - x_2 + 2 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.2}$$

Deoarece $F(1,1) = (2,-1) \geq (2,-4) = F(0,2)$, rezulta ca $x^0 = (1,1) \in \mathcal{F}^0$ nu este solutie eficienta a problemei $\left(P_0^{1,0}\right)$.

Remark 3.3.4 *[40] studiaza conditiile ca solutia eficienta a problemei $\left(P_0^{1,1}\right)$ sa ramana eficienta si pentru problema $\left(P_0^{0,0}\right)$ si reciproc.*

Theorem 3.3.5 (Luca and Duca [96]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) *f_1 este diferentiabila de doua ori in x_0 si invexa[2] in x_0 in raport cu η,*

b) $\eta\left(x_0,\ x_0\right)=0.$

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{2,0}\right)$, atunci x_0 este solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.3.6 (Luca and Duca [96]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ functii.

Presupunem ca:

a) *f_1 este diferentiabila de doua ori in x_0 si incava[2] in x_0 in raport cu η,*

b) $\eta\left(x_0,\ x_0\right)=0.$

Daca x_0 este solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este solutie eficienta si a problemei $\left(P_0^{2,0}\right)$.

Example 3.3.7 (Luca and Duca [96]) .
Fie problema initiala de optimizare bi-criteriala $\left(P_0^{0,0}\right)$

$$\begin{cases} \min\left(x_1^2+x_1x_2+x_2^2-19.25x_1-19.875x_2; x_1+x_2\right) \\ -x_1^2+6x_1-1-x_2 \leq 0 \\ 4x_1+x_2-20 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.3}$$

O solutie eficienta a problemei (3.3) este $x^0 = (3, 8)$. Construim problema $\left(P_0^{2,0}\right)$

$$\begin{cases} \min\left(x_1^2+x_1x_2+x_2^2-19.25x_1-19.875x_2; x_1+x_2\right) \\ -x_1^2+6x_1-1-x_2 \leq 0 \\ 4x_1+x_2-20 \leq 0 \\ x_1; x_2 \geq 0 \end{cases} \tag{3.4}$$

care este identica cu problema initiala (3.3) si astfel ele au aceeasi solutie eficienta.

Remark 3.3.8 *Examplul 3.3.7 arata ca daca incavitatea de ordin doi a functiei f_1 din conditia a) a Teoremei 3.3.6 nu este verificata, este posibil sa fie obtinuta aceeasi solutie eficienta.*

Theorem 3.3.9 (Luca and Duca [96]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) *f_1 este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

b) *f_2 este diferentiabila in x_0 si invexa^1 in x_0 in raport cu η,*

c) $\eta(x_0,\ x_0) = 0$.

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{2,1}\right)$, atunci x_0 este o soultie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.3.10 (Luca and Duca [96]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) *f_1 este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

b) *f_2 este diferentiabila in x_0 si incava^1 in x_0 in raport cu η,*

c) $\eta(x_0,\ x_0) = 0$.

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_0^{2,1}\right)$.

Theorem 3.3.11 (Luca and Duca [96]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) *f_1 este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

b) *f_2 este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

c) $\eta(x_0,\ x_0) = 0$.

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{2,2}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.3.12 (Luca and Duca [96]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) *f_1 este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

b) *f_2 este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

c) $\eta\,(x_0,\ x_0) = 0.$

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_0^{2,2}\right)$.

3.4 Prima si a doua η - aproximare a functiilor obiectiv si prima η - aproximare a sistemului de restrictii

In acest paragraf studiem conditii suficiente pentru ca solutiile eficiente ale problemelor $\left(P_1^{1,0}\right)$, $\left(P_1^{2,0}\right)$, $\left(P_1^{2,1}\right)$ si $\left(P_1^{2,2}\right)$ sa ramana solutii eficiente si pentru problema $\left(P_0^{0,0}\right)$ si reciproc.

Theorem 3.4.1 (Duca and Ratiu [40]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Presupunem ca:

a) *pentru fiecare $t \in T$, functia g_t este diferentiabila in x_0 si invexa^1 in x_0 in raport cu η,*

b) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si avexa^1 in x_0 in raport cu η,*

atunci

$$\mathcal{F}^0 \subseteq \mathcal{F}^1.$$

Theorem 3.4.2 (Duca and Ratiu [40]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Presupunem ca:

a) *pentru fiecare* $t \in T$*, functia* g_t *este diferentiabila in* x_0 *si incava*^{1} *in* x_0 *in raport cu* η*,*

b) *pentru fiecare* $s \in S$*, functia* h_s *este diferentiabila in* x_0 *si avexa*^{1} *in* x_0 *in raport cu* η*,*

atunci

$$\mathcal{F}^1 \subseteq \mathcal{F}^0.$$

Theorem 3.4.3 (Luca and Duca [95]) .
Fie X *o multime nevida din* $\mathbb{R}^n$*,* x_0 *un punct interior al lui* X*,* $\eta : X \times X \to X$*,* T *si* S *multimi index,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functii.*

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$*,*

b) *pentru fiecare* $t \in T$*, functia* g_t *este diferentiabila in* x_0 *si invexa*^{1} *in* x_0 *in raport cu* η*,*

c) *pentru fiecare* $s \in S$*, functia* h_s *este diferentiabila in* x_0 *si avexa*^{1} *in* x_0 *in raport cu* η*,*

d) f_1 *este diferentiabila de doua ori in* x_0 *si invexa*^{2} *in* x_0 *in raport cu* η*,*

e) $\eta\,(x_0,\ x_0) = 0.$

Daca x_0 *este o solutie eficienta a problemei* $\left(P_1^{2,0}\right)$*, atunci* x_0 *este o solutie eficienta si a problemei* $\left(P_0^{0,0}\right)$*.*

Theorem 3.4.4 (Luca and Duca [95]) .
Fie X *o multime nevida din* $\mathbb{R}^n$*,* x_0 *un punct interior al lui* X*,* $\eta : X \times X \to X$*,* T *si* S *multimi index,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *si* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functii.*

Presupunem ca:

a) $x_0 \in \mathcal{F}^1$*,*

b) *pentru fiecare* $t \in T$*, functia* g_t *este diferentiabila in* x_0 *si incava*^{1} *in* x_0 *in raport cu* η*,*

c) *pentru fiecare* $s \in S$*, functia* h_s *este diferentiabila in* x_0 *si avexa*^{1} *in* x_0 *in raport cu* η*,*

d) *f_1 este diferentiabila de doua ori in x_0 si incava[2] in x_0 in raport cu η,*

e) $\eta\left(x_0,\ x_0\right)=0.$

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_1^{2,0}\right)$.

Theorem 3.4.5 (Luca and Duca [95]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f=(f_1,f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila in x_0 si invexa[1] in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si avexa[1] in x_0 in raport cu η,*

d) *f_1 este diferentiabila in x_0 si invexa[1] in x_0 in raport cu η,*

e) $\eta\left(x_0,\ x_0\right)=0.$

Daca x_0 o solutie eficienta a problemei $\left(P_1^{1,0}\right)$, atunci x_0 o solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.4.6 (Luca and Duca [95]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f=(f_1,f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^1$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila in x_0 si incava[1] in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si avexa[1] in x_0 in raport cu η,*

d) *f_1 este diferentiabila in x_0 si incava[1] in x_0 in raport cu η,*

e) $\eta\left(x_0,\ x_0\right)=0.$

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_1^{1,0}\right)$.

Theorem 3.4.7 (Luca and Duca [95]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila in x_0 si invexa^1 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si avexa^1 in x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

e) *f_2 este diferentiabila in x_0 si invexa^1 in x_0 in raport cu η,*

f) $\eta\left(x_0,\ x_0\right) = 0$.

Daca x_0 o solutie eficienta a problemei $\left(P_1^{2,1}\right)$, atunci x_0 o solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.4.8 (Luca and Duca [95]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^1$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila in x_0 si incava^1 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si avexa^1 in x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

e) *f_2 este diferentiabila in x_0 si incava^1 in x_0 in raport cu η,*

f) $\eta\left(x_0,\ x_0\right) = 0$.

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_1^{2,1}\right)$.

Theorem 3.4.9 (Luca and Duca [95]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare $t \in T$, functia g_t etse diferentiabila in x_0 si invexa[1] in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si avexa[1] in x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori in x_0 si invexa[2] in x_0 in raport cu η,*

e) *f_2 este diferentiabila de doua ori in x_0 si invexa[2] in x_0 in raport cu η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

Daca x_0 este o solutie eficienta a problemei $\left(P_1^{2,2}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.4.10 (Luca and Duca [95]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui $X, \eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^1$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila in x_0 si incava[1] in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si avexa[1] in x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori in x_0 si incava[2] in x_0 in raport cu η,*

e) *f_2 este diferentiabila de doua ori in x_0 si incava[2] in x_0 in raport cu η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_1^{2,2}\right)$.

Example 3.4.11 (Luca and Duca [95]) .
Fie problema initiala de optimizare bi-criteriala $\left(P_0^{0,0}\right)$

$$\begin{cases} \min\left(x_1 - 2x_2; x_1 + x_2\right) \\ -x_1 x_2 + 1 \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

O solutie eficienta a problemei $\left(P_0^{0,0}\right)$ este $x_0 = (1,\ 1) \in \mathcal{F}^0$, iar valoarea functiei obiectiv in x_0 este $f(1,\ 1) = (-1,\ 2)$.
Atasam problemele $\left(P_1^{i,j}\right)$, cu $(i, j) \in \{(1,0),(1,1),(2,0),(2,1),(2,2)\}$

$$\begin{cases} \min\left(x_1 - 2x_2; x_1 + x_2\right) \\ -x_1 - x_2 + 2 \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

Valoarea functiei obiectiv a problemelor $\left(P_1^{i,j}\right)$ in $x = (0,\ 2) \in \mathcal{F}^1$ este $\left(F_1^i, F_2^j\right)(0,\ 2) = (-4,\ 2)$.
Prin urmare $x_0 = (1,\ 1) \in \mathcal{F}^1$ nu este solutie eficienta a problemelor $\left(P_1^{i,j}\right)$.

Example 3.4.12 (Luca and Duca [95]) .
Fie problema initiala de optimizare bi-criteriala $\left(P_0^{0,0}\right)$

$$\begin{cases} \min\left(x_1^2 + (x_2 - \pi - 1)^2;\ (x_1 + \frac{1}{10})^2 - \frac{1}{2}(x_2+1)^2\right) \\ -x_1 - \sin x_1 + x_2 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

O solutie eficienta a problemei $\left(P_0^{0,0}\right)$ este $x_0 = (\frac{\pi}{2},\ 1 + \frac{\pi}{2}) \in \mathcal{F}^0$, iar valoarea functiei obiectiv in x_0 este $f(\frac{\pi}{2},\ 1 + \frac{\pi}{2}) = (\frac{\pi^2}{2};\ \frac{\pi^2}{8} - \frac{9\pi}{10} - \frac{199}{100})$.
Atasam problema $\left(P_1^{1,1}\right)$

$$\begin{cases} \min\left(\pi x_1 - \pi x_2 + \pi + \frac{\pi^2}{2};\ (\pi + \frac{1}{5})x_1 - (\frac{\pi}{2} + 2)x_2 - \frac{\pi^2}{8} + \frac{\pi}{2} + \frac{1}{100}\right) \\ -x_1 + x_2 - 1 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0. \end{cases}$$

Valoarea functiei obiectiv a problemei $\left(P_1^{1,1}\right)$ *in* $x = (\frac{5\pi}{2},\ 1+\frac{5\pi}{2}) \in \mathcal{F}^1$ *este* $F^1(\frac{5\pi}{2},\ 1+\frac{5\pi}{2}) = (\frac{\pi^2}{2},\ \frac{9\pi^2}{8} - \frac{9\pi}{2} - \frac{199}{100})$. *Prin urmare,* $x_0 = (\frac{\pi}{2},\ 1+\frac{\pi}{2})$ *nu este solutie eficienta a problemei* $\left(P_1^{1,1}\right)$.

3.5 Prima si a doua η - aproximare a functiilor obiectiv si a doua η - aproximare a sistemului de restrictii

In acest paragraf sunt studiate conditii suficiente pentru ca solutiile eficiente ale problemelor $\left(P_2^{1,0}\right)$, $\left(P_2^{2,0}\right)$, $\left(P_2^{2,1}\right)$ and $\left(P_2^{2,2}\right)$ sa ramana eficiente si pentru problema $\left(P_0^{0,0}\right)$ si reciproc.

Theorem 3.5.1 (Duca and Boncea [12]) .
Fie X *o multime nevida din* $\mathbb{R}^n$, x_0 *un punct interior al lui* X, $\eta : X\times X \to X$, *si* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Presupunem ca:

a) *pentru fiecare* $t \in T$, *functia* g_t *este diferentiabila de doua ori in* x_0 *si invexa*^{2} *in* x_0 *in raport cu* η,

b) *pentru fiecare* $s \in S$, *functia* h_s *este diferentiabila de doua ori in* x_0 *si avexa*^{2} *in* x_0 *in raport cu* η,

atunci

$$\mathcal{F}^0 \subseteq \mathcal{F}^2.$$

Theorem 3.5.2 (Duca and Boncea [12]) .
Fie X *o multime nevida din* $\mathbb{R}^n$, x_0 *un punct interior al lui* X, $\eta : X\times X \to X$, *si* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Presupunem ca:

a) *pentru fiecare* $t \in T$, *functia* g_t *este diferentiabila de doua ori* x_0 *si incava*^{2} *in* x_0 *in raport cu* η,

b) *pentru fiecare* $s \in S$, *functia* h_s *este diferentiabila de doua ori* x_0 *si avexa*^{2} *in* x_0 *in raport cu* η,

atunci

$$\mathcal{F}^2 \subseteq \mathcal{F}^0.$$

Theorem 3.5.3 (Luca and Duca [94]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabile de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabile de doua ori in x_0 si avexa^2 in x_0 in raport cu η,*

d) *f_1 este diferentiabile de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

e) *f_2 este diferentiabila in x_0 si invexa^1 in x_0 in raport cu η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

Daca x_0 este o solutie eficienta a problemei $\left(P_2^{2,1}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.5.4 (Luca and Duca [94]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^2$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si avexa^2 in x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

e) *f_2 este diferentiabila in x_0 si incava^1 in x_0 in raport cu η,*

f) $\eta\left(x_0,\ x_0\right) = 0.$

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_2^{2,1}\right)$.

Theorem 3.5.5 (Luca and Duca [94]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si avexa^2 in x_0 in raport cu η,*

d) *f_1 este diferentiabila in x_0 si invexa^1 in x_0 in raport cu η,*

e) $\eta\left(x_0,\ x_0\right) = 0$.

Daca x_0 este o solutie eficienta a problemei $\left(P_2^{1,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.5.6 (Luca and Duca [94]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^2$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si avexa^2 in x_0 in raport cu η,*

d) *f_1 este diferentiabila in x_0 si incava^1 in x_0 in raport cu η,*

e) $\eta\left(x_0,\ x_0\right) = 0$.

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_2^{1,0}\right)$.

Theorem 3.5.7 (Luca and Duca [94]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si avexa^2 in x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

e) $\eta (x_0,\ x_0) = 0$.

Daca x_0 este o solutie eficienta a problemei $\left(P_2^{2,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.5.8 (Luca and Duca [94]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^2$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si avexa^2 in x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

e) $\eta (x_0,\ x_0) = 0$.

Daca x_0 este o solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este o solutie eficienta si a problemei $\left(P_2^{2,0}\right)$.

Theorem 3.5.9 (Luca and Duca [94]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si avexa^2 at x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

e) *f_2 este diferentiabila de doua ori in x_0 si invexa^2 in x_0 in raport cu η,*

f) $\eta\,(x_0,\ x_0) = 0$.

Daca x_0 este solutie eficienta a problemei $\left(P_2^{2,2}\right)$, atunci x_0 este solutie eficienta si a problemei $\left(P_0^{0,0}\right)$.

Theorem 3.5.10 (Luca and Duca [94]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^2$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si incava^2 in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si avexa^2 in x_0 in raport cu η,*

d) *f_1 este diferentiabila de doua ori x_0 si incava^2 in x_0 in raport cu η,*

e) *f_2 este diferentiabila de doua ori x_0 si incava^2 in x_0 in raport cu η,*

f) $\eta\,(x_0,\ x_0) = 0$.

Daca x_0 este solutie eficienta a problemei $\left(P_0^{0,0}\right)$, atunci x_0 este solutie eficienta si a problemei $\left(P_2^{2,2}\right)$.

Example 3.5.11 (Luca and Duca [94]) .
Fie problema initiala de optimizare bi-criteriala $\left(P_0^{0,0}\right)$

$$\begin{cases} \min\left(-\left(x_1-\frac{3\pi}{5}\right)^2-\left(x_2-\frac{2\pi}{5}-1\right)^2;-x_1+x_2\right) \\ -x_1-\sin x_1+x_2\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0. \end{cases}$$

O solutie eficienta a problemei $\left(P_0^{0,0}\right)$ *este* $x_0=(\frac{\pi}{2},\ 1+\frac{\pi}{2})\in\mathcal{F}^0$.
Atasam problema $\left(P_2^{0,0}\right)$

$$\begin{cases} \min\left(-\left(x_1-\frac{3\pi}{5}\right)^2-\left(x_2-\frac{2\pi}{5}-1\right)^2;-x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0. \end{cases}$$

Deoarece $f(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32})<f(\frac{\pi}{2},\ 1+\frac{\pi}{2})$*, rezulta ca solutia eficienta a problemei* $(P^{0,0})$ *nu este solutie eficienta si pentru problema* $\left(P_2^{0,0}\right)$.

Example 3.5.12 *Consideram aceeasi problema ca si in Examplul 3.5.11.*
Atasam problema $\left(P_2^{1,1}\right)$

$$\begin{cases} \min\left(-\frac{\pi}{5}x_1-\frac{\pi}{5}x_2+\frac{9\pi^2}{50}+\frac{\pi}{5};\ -x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0. \end{cases}$$

Deoarece $F^1(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32})<F^1(\frac{\pi}{2},\ 1+\frac{\pi}{2})$*, rezulta ca solutia eficienta a problemei* $(P^{0,0})$ *nu este solutie eficienta si pentru problema* $\left(P_2^{1,1}\right)$.

Example 3.5.13 (Luca and Duca [94]) .
Consideram aceeasi problema ca si in Examplul 3.5.11.
Atasam problema $\left(P_2^{2,2}\right)$

$$\begin{cases} \min\left(-\frac{\pi}{2}\left(x_1-\frac{\pi}{2}\right)^2-\frac{\pi+2}{2}\left(x_2-1-\frac{\pi}{2}\right)^2-\frac{\pi}{5}x_1-\frac{\pi}{5}x_2+\frac{9\pi^2}{50}+\frac{\pi}{5};\ -x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\leq 0 \\ x_1-\frac{5\pi}{2}\leq 0 \\ x_1;x_2\geq 0. \end{cases}$$

Deoarece $F^2(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}) < F^2(\frac{\pi}{2},\ 1+\frac{\pi}{2})$*, rezulta ca solutia eficienta a problemei* $(P^{0,0})$ *nu este solutie eficienta si pentru problema* $\left(P_2^{2,2}\right)$*.*

3.6 Relatii intre problemele η - aproximante ale problemei $\left(P_0^{0,0}\right)$

3.6.1 Relatii intre problemele $\left(P_0^{i,j}\right)$ si $\left(P_1^{i,j}\right)$

Theorem 3.6.1 (Luca and Duca [97]) .

Fie X *o multime nevida din* $\mathbb{R}^n$, x_0 *un punct interior al lui* X, $\eta : X \times X \to X$*,* T *si* S *multimi index,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *si* $g_t,\ h_s : X \to \mathbb{R}$*,* $(t \in T,\ s \in S)$ *functii.*

Presupunem ca:

a) $x_0 \in \mathcal{F}^1$,

b) *pentru fiecare* $t \in T$*, functia* g_t *este diferentiabila in* x_0 *si incava*[1] *in* x_0 *in raport cu* η*,*

c) *pentru fiecare* $s \in S$*, functia* h_s *este diferentiabila in* x_0 *si avexa*[1] *in* x_0 *in raport cu* η*.*

1. *Daca* f_1 *este diferentiabila in* x_0 *si* x_0 *este solutie eficienta a problemei problem* $\left(P_0^{1,0}\right)$*, atunci* x_0 *este solutie eficienta a problemei* $\left(P_1^{1,0}\right)$*.*

2. *Daca* f_1 *este diferentiabila de doua ori in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_0^{2,0}\right)$*, atunci* x_0 *este solutie eficienta a problemei* $\left(P_1^{2,0}\right)$*.*

3. *Daca* f_1 *este diferentiabila de doua ori in* x_0*,* f_2 *este diferentiabila in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_0^{2,1}\right)$*, atunci* x_0 *este solutie eficienta a problemei* $\left(P_1^{2,1}\right)$*.*

4. *Daca* f_1 *este diferentiabila de doua ori in* x_0*,* f_2 *este diferentiabila de doua ori in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_0^{2,2}\right)$*, atunci* x_0 *este solutie eficienta a problemei* $\left(P_1^{2,2}\right)$*.*

Theorem 3.6.2 (Luca and Duca [97]) .

Fie X *o multime nevida din* $\mathbb{R}^n$, x_0 *un punct interior al lui* X, $\eta : X \times X \to X$*,* T *si* S *multimi index,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *si* $g_t,\ h_s : X \to \mathbb{R}$*,* $(t \in T,\ s \in S)$ *functii.*

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila in x_0 si invexa*[1] *in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si avexa*[1] *in x_0 in raport cu η.*

1. *Daca f_1 este diferentiabila in x_0 si x_0 este solutie eficienta a problemei $\left(P_1^{1,0}\right)$, atunci x_0 este solutie eficienta a problemei $\left(P_0^{1,0}\right)$.*

2. *Daca f_1 este diferentiabila de doua ori in x_0 si x_0 este solutie eficienta a problemei $\left(P_1^{2,0}\right)$, atunci x_0 este solutie eficienta a problemei $\left(P_0^{2,0}\right)$.*

3. *Daca f_1 este diferentiabila de doua ori in x_0, f_2 este diferentiabila in x_0 si x_0 este solutie eficienta a problemei $\left(P_1^{2,1}\right)$, atunci x_0 este solutie eficienta a problemei $\left(P_0^{2,1}\right)$.*

4. *Daca f_1 este diferentiabila de doua ori in x_0, f_2 este diferentiabila de doua ori in x_0 si x_0 este solutie eficienta a problemei $\left(P_1^{2,2}\right)$, atunci x_0 este solutie eficienta a problemei $\left(P_0^{2,2}\right)$.*

3.6.2 Relatii intre problemele $\left(P_0^{i,j}\right)$ si $\left(P_2^{i,j}\right)$

Theorem 3.6.3 (Luca and Duca [97]) .

Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^2$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si incava*[2] *in x_0 in raport cu η,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si avexa*[2] *in x_0 in raport cu η.*

1. *Daca f_1 este diferentiabila in x_0 si x_0 este solutie eficienta si a problemei $\left(P_0^{1,0}\right)$, atunci x_0 este solutie eficienta si a problemei $\left(P_2^{1,0}\right)$.*

2. *Daca f_1 este diferentiabila de doua ori in x_0 si x_0 este solutie eficienta si a problemei $\left(P_0^{2,0}\right)$ atunci x_0 este solutie eficienta a problemei $\left(P_2^{2,0}\right)$.*

3. *Daca* f_1 *este diferentiabila de doua ori in* x_0, f_2 *este diferentiabila in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_0^{2,1}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,1}\right)$.

4. *Daca* f_1 *este diferentiabila de doua ori in* x_0, f_2 *este diferentiabila de doua ori in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_0^{2,2}\right)$, *atunci* x_0 *este solutie eficienta si a problemei* $\left(P_2^{2,2}\right)$.

Theorem 3.6.4 (Luca and Duca [97]) .
Fie X *o multime nevida din* $\mathbb{R}^n$, x_0 *un punct interior al lui* X, $\eta : X \times X \to X$, T *si* S *multimi index,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *si* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functii.*

Presupunem ca:

a) $x_0 \in \mathcal{F}^0$,

b) *pentru fiecare* $t \in T$, *functia* g_t *este diferentiabila de doua ori in* x_0 *si invexa*^{2} *in* x_0 *in raport cu* η,

c) *pentru fiecare* $s \in S$, *functia* h_s *este diferentiabila de doua ori in* x_0 *si avexa*^{2} *in* x_0 *in raport cu* η.

1. *Daca* f_1 *este diferentiabila in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_2^{1,0}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_0^{1,0}\right)$.

2. *Daca* f_1 *este diferentiabila de doua ori in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,0}\right)$, *then* x_0 *este solutie eficienta a problemei* $\left(P_0^{2,0}\right)$.

3. *Daca* f_1 *este diferentiabila de doua ori in* x_0, f_2 *este diferentiabila in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,1}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_0^{2,1}\right)$.

4. *Daca* f_1 *este diferentiabila de doua ori in* x_0, f_2 *este diferentiabila de doua ori in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,2}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_0^{2,2}\right)$.

3.6.3 Relatii intre problemele $\left(P_1^{i,j}\right)$ si $\left(P_2^{i,j}\right)$

Theorem 3.6.5 (Luca and Duca [97]) .
Fie X *o multime nevida din* $\mathbb{R}^n$, x_0 *un punct interior al lui* X, $\eta : X \times X \to X$ *si* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Daca

a) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si $\nabla^2 g_t\left(x_0\right)$ este negativ semi-definita,*

b) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si $\nabla^2 h_s\left(x_0\right)$ este null definita,*

atunci

$$\mathcal{F}^1 \subseteq \mathcal{F}^2.$$

Theorem 3.6.6 (Luca and Duca [97]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.

Daca

a) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si $\nabla^2 g_t\left(x_0\right)$ este positiv semi-definita,*

b) *pentru fiecare $s \in S$, functia h_s este diferentiabila in x_0 si $\nabla^2 h_s\left(x_0\right)$ este null definita,*

atunci

$$\mathcal{F}^2 \subseteq \mathcal{F}^1.$$

Theorem 3.6.7 (Luca and Duca [97]) .
Fie X o multime nevida din $\mathbb{R}^n$, x_0 un punct interior al lui X, $\eta : X \times X \to X$, T si S multimi index, $f = (f_1, f_2) : X \to \mathbb{R}^2$ si $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functii.

Presupunem ca:

a) $x_0 \in \mathcal{F}^2$,

b) *pentru fiecare $t \in T$, functia g_t este diferentiabila de doua ori in x_0 si $\nabla^2 g_t\left(x_0\right)$ este positiv semi-definita,*

c) *pentru fiecare $s \in S$, functia h_s este diferentiabila de doua ori in x_0 si $\nabla^2 h_s\left(x_0\right)$ este null definita.*

1. *Daca f_1 este diferentiabila in x_0 si x_0 este solutie eficienta a problemei $\left(P_1^{1,0}\right)$, atunci x_0 este solutie eficienta a problemei $\left(P_2^{1,0}\right)$.*
2. *Daca f_1 este diferentiabila de doua ori in x_0 si x_0 este solutie eficienta a problemei $\left(P_1^{2,0}\right)$ atunci x_0 este solutie eficienta a problemei $\left(P_2^{2,0}\right)$.*

3. *Daca* f_1 *este diferentiabila de doua ori in* x_0, f_2 *este diferentiabila in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_1^{2,1}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,1}\right)$.

4. *Daca* f_1 *este diferentiabila de doua ori in* x_0, f_2 *este diferentiabila de doua ori in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_1^{2,2}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,2}\right)$.

Theorem 3.6.8 (Luca and Duca [97]) .
Fie X *o multime nevida din* $\mathbb{R}^n$, x_0 *un punct interior al lui* X, $\eta : X \times X \to X$, T *si* S *multimi index,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *si* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functii.*

Presupunem ca:

a) $x_0 \in \mathcal{F}^1$,

b) *pentru fiecare* $t \in T$, *functia* g_t *este diferentiabila de doua ori in* x_0 *si* $\nabla^2 g_t\left(x_0\right)$ *este negativ semi-definita,*

c) *pentru fiecare* $s \in S$, *functia* h_s *este diferentiabila de doua ori in* x_0 *si* $\nabla^2 h_s\left(x_0\right)$ *este null definita.*

1. *Daca* f_1 *este diferentiabila in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_2^{1,0}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_1^{1,0}\right)$.

2. *Daca* f_1 *este diferentiabila de doua ori in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,0}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_1^{2,0}\right)$.

3. *Daca* f_1 *este diferentiabila de doua ori in* x_0, f_2 *este diferentiabila in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,1}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_1^{2,1}\right)$.

4. *Daca* f_1 *este diferentiabila de doua ori in* x_0, f_2 *este diferentiabila de doua ori in* x_0 *si* x_0 *este solutie eficienta a problemei* $\left(P_2^{2,2}\right)$, *atunci* x_0 *este solutie eficienta a problemei* $\left(P_1^{2,2}\right)$.

3.7 Concluzii

Acest capitol a fost dedicat studierii unor conditii suficiente pentru ca solutia eficienta a unei probleme de optimizare bi-criteriala sa ramana solutie eficienta si pentru problemele care o aproximeaza si reciproc. Contributiile noastre la acest capitol constand in 32 de teoreme si 7 contra-exemple au fost diseminate in [94, 95, 96, 97].

Chapter 4

Modelul minimax pentru optimizarea energiei

4.1 Introducere

Acest capitol prezinta model minimax de optimizare a energiei. Este o problema bi-criteriala ce urmareste netezirea varfului de sarcina prin minimizarea fluctuatiei de energie si maximizarea performantelor economice ale producatorului.

Pornind de la abaterea absoluta maxima vom defini o masura pentru fluctuatia de energie. Performanta economica a producatorului va fi masurata prin cifra de afaceri.

4.2 Masura minimax pentru fluctuatia de energie

Abaterea absoluta maxima, prezentata in Capitolul 2, are capacitatea de a adresa valoarea extrema din setul de date. Prin urmare ea va reprezenta punctul de plecare in construirea unei masuri a fluctuatiei energiei. [35, 132, 152] evidentiaza elasticitatea pretului energiei si prin urmare consideram oportuna introducerea pretului in masura fluctuatiei.

Definition 4.2.1 *Masura minimax a fluctuatiei energiei reprezinta maximul, raportat la orizontul de timp, pentru diferenta dintre energia produsa la un anumit moment si nivelul predefinit, inmultita cu pretul energiei la acel moment.*

Remark 4.2.2 *Nivelul predefinit poate fi o valoare aleatoare aleasa de producator sau nivelul mediu al productiei intr-o perioada anterioara.*

Notand $1, 2, ..., i, ..., n$ orizontul de timp pentru care se are in vedere optimizarea productiei de energie si

x_i - energia produsa la momentul de timp i, $i = \overline{1, n}$,

p_i - pretul energiei la momentul de timp i, $i = \overline{1, n}$,

r - nivelul predefinit al energiei,

ε - nivelul minim al energiei asumat de producator ca va fi livrat in retea,

ρ - nivelul maxim al energiei asumat de producator ca va fi livrat in retea,

si considerand ca $\epsilon \leq r \leq \rho$, *fluctuatia minimax a energiei* este

$$\max_{i=\overline{1,n}} |p_i x_i - p_i r| . \tag{4.1}$$

4.3 Formularea problemei

Consideram o centrala electrica ce urmareste netezirea varfului de sarcina prin minimizarea fluctuatiei de energie, fara afectarea performantelor economice. Este evidenta ca ne aflam in fata unei probleme de optimizare bi-criteriala. Prima componenta a functiei obiectiv este masura minimax a fluctuatiei (4.1), iar a doua componenta este cifra de afaceri definita prin

$$\sum_{i=1}^{n} p_i x_i.$$

Folosind restrictii tehnice simple ce limiteaza cantitatea de energie produsa, obtinem modelul minimax

$$\begin{cases} \min \left(\max\limits_{i=\overline{1,n}} |p_i x_i - p_i r| \, , \; -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \varepsilon \leq x_i \leq \rho, \quad i = \overline{1, n}. \end{cases} \tag{4.2}$$

4.4 Determinarea solutiei

Pentru determinarea unei solutii eficiente a problemei (4.2) introducem problema bi-criteriala echivalenta

$$\begin{cases} \min\left(y,\ -\sum_{i=1}^{n} p_i x_i\right)^T \\ |p_i x_i - p_i r| \le y, \quad i = \overline{1,n} \\ \varepsilon \le x_i \le \rho, \quad i = \overline{1,n}. \end{cases} \tag{4.3}$$

echivalenta celor doua probleme fiind demonstrata de Lema

Lemma 4.4.1 *[98] Fie problemele de optimizre bi-criteriala (4.2) si (4.3).*

a) Daca $x \in \mathbb{R}^n$ este o solutie eficienta a problemei (4.2), atunci $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, cu $y = \max_{i=\overline{1,n}} |p_i x_i - p_i r|$ este o solutie eficienta a problemei (4.3).

b) Daca $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, cu $y = \max_{i=\overline{1,n}} |p_i x_i - p_i r|$ este o solutie eficienta a problemei (4.3), atunci $x \in \mathbb{R}^n$ este o solutie eficienta a problemei (4.2).

Folosind rezultate obtinute de Yu [160], Bot et all [14] si Geoffrion [53] problema bi-criteriala (4.3) este echivalenta cu problema parametrica

$$\begin{cases} \min\left\{\lambda y - (1-\lambda)\sum_{i=1}^{n} p_i x_i\right\} \\ |p_i x_i - p_i r| \le y, \quad i = \overline{1,n} \\ \varepsilon \le x_i \le \rho, \quad i = \overline{1,n} \end{cases} \tag{4.4}$$

cu $\lambda \in (0, 1)$ si urmatoarea Lema este adevarata.

Lemma 4.4.2 *[98] $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ este o solutie eficienta a problemei bi-criteriale (4.3) daca si numai daca $\exists\lambda \in (0, 1)$ astfel incat $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ este o solutie optima a problemei parametrice (4.4)*

Remark 4.4.3 *Pentru a determina solutia eficienta a problemei (4.2) trebuie sa determinam solutia optima a problemei (4.4), avand in vedere echivalentele dintre problemele (4.2) si (4.3), respectiv (4.3) si (4.4).*

Remark 4.4.4 *In procesul de determinare a solutiei optime, vom imparti multimea $\{1, 2, ..., n\}$ in submultimi de forma $\{1, 2, ..., l\}$ si $\{l + 1, l + 2, ..., n\}$, sau $\{1, 2, ..., l\}$, $\{l + 1, l + 2, ..., m\}$ si $\{m + 1, m + 2, ..., n\}$. Daca pretul este constant pe o asemenea multime, il vom nota cu $\overline{p}$.*

Teorema urmatoare prezinta o solutie optima a problemei parametrice (4.4).

Theorem 4.4.5 (Luca and Mahalov [98]; *minimax parametric*) .
O solutie optima a problemei parametrice (4.4) este:

1. *Daca* $\lambda < \frac{n}{n+1}$, *atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$

   sau

   - *daca* $\overline{p_1} \leq p_j, \; j = \overline{l+1,n}$, *atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = r + \frac{y^*}{p_j}, \quad j = \overline{l+1,n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$

   unde $\overline{p_1} = p_i, \; i = \overline{1,l}$.

   - *altfel solutia nu exista.*

2. *Daca* $\lambda = \frac{n}{n+1}$, *atunci*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,n} \\ y^* = \min\limits_{i=\overline{1,n}} \{p_i\,(\rho - r)\}\,. \end{cases}$$

3. *Daca* $\lambda > \frac{n}{n+1}$, *atunci*

$$\begin{cases} x_i^* = r, \quad i = \overline{1,n} \\ y^* = 0. \end{cases}$$

4. *Daca* $\lambda < \frac{l}{l+1}$, *atunci*

   - *daca* $p_j < \overline{p}, \; j = \overline{l+1,n}$, *atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p}\,(\rho - r) \end{cases}$$

     unde $\overline{p_1} = p_i, \; i = \overline{1,l}$.

- *altfel solutia nu exista.*

5. *Daca* $\lambda = \frac{l}{l+1}$, *atunci*

   - *daca* $p_j < p_i, i = \overline{1,l}, \ \ j = \overline{l+1,n}$, *atunci*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \min\limits_{i=\overline{1,l}} \left\{ p_i \left(\rho - r \right) \right\}, \ if \ p_j < p_i. \end{cases}$$

   - *altfel solutia nu exista.*

6. *Daca* $\lambda < \frac{l+(n-m)}{l+(n-m)+1}$, *atunci*

   - *daca* $p_j \leq \overline{p} \leq p_i, \ i = \overline{1,l}, \ j = \overline{l+1,m}$, *atunci*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p} \left(\rho - r \right) \end{cases}$$

   unde $\overline{p_3} = p_k, \ k = \overline{m+1,n}$.

   - *altfel solutia nu exista.*

Definition 4.4.6 Combinatiile posibile *sunt combinatiile multiplicatorilor Kuhn-Tucker determiante pentru un* i *fixat in multimea* $\{1, 2, ..., n\}$ *astfel incat conditiile de complementaritate si admisibilitate duala sa fie indeplinite.*

Definition 4.4.7 Combinatiile admisibile *sunt acele combinatii posibile pentru care gradientul Lagrangianului este zero.*

Definition 4.4.8 Combinatiile critice *sunt acele combinatii admisibile care nu pot genera impreuna o solutie.*

Remark 4.4.9 *Deoarece problema (**??**18)) este convexa, conditiile Kuhn-Tucker sunt atat necesare cat si suficiente.*

Remark 4.4.10 *In demonstratia Teoremei 4.4.5 se observa ca pentru scenariul 5 una dintre derivatele partiale ale Lagrangianului nu este egala cu zero, ceea ce inseamna ca scenariul 5 nu va putea genera singur o solutie, dar combinat cu alte scenarii s-ar putea sa genereze solutie.*

Theorem 4.4.11 (Luca and Mahalov [98]; *ordinea scenariilor*) .

Ordinea scenariilor, folosite pentru determinarea unei solutii optime a problemei (4.4), nu influenteaza solutia.

Theorem 4.4.12 (Luca and Mahalov [98]; *minimax energie*) .

O solutie eficienta a problemei bi-criteriale de optimizare (4.2) este

1. *Daca* $\lambda < \frac{n}{n+1}$ *si exista un singur pret pentru energie, atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p}\,(\rho - r) \\ TR = n\overline{p}\rho \end{cases}$$

sau

- *daca exista doua preturi diferite pentru energie in intervalul de timp studiat si* $\overline{p_1} \leq p_j,\ j = \overline{l+1,n}$, *atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = r + \frac{y^*}{p_j}, \quad j = \overline{l+1,n} \\ y^* = \overline{p_1}\,(\rho - r) \\ TR = l\overline{p_1}\rho + r\sum\limits_{j=l+1}^{n} p_j + (n-l)\,y^*. \end{cases}$$

unde $\overline{p_1} = p_i,\ i = \overline{1,l}$.

- *altfel solutia nu exista.*

2. *Daca* $\lambda = \frac{n}{n+1}$, *atunci*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,n} \\ y^* = \min\limits_{i=\overline{1,n}} \{p_i\,(\rho - r)\} \\ TR = r\sum\limits_{i=1}^{n} p_i + ny^*. \end{cases}$$

3. *Daca* $\lambda > \frac{n}{n+1}$, *atunci*

$$\begin{cases} x_i^* = r, \quad i = \overline{1,n} \\ y^* = 0 \\ TR = r\sum\limits_{i=1}^{n} p_i. \end{cases}$$

4. *Daca* $\lambda < \frac{l}{l+1}$*, atunci*

   - *daca* $p_j < \overline{p_1},\ j = \overline{l+1,n}$*, atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p}\left(\rho - r\right) \\ TR = l\overline{p}\rho + \rho \sum\limits_{j=l+1}^{n} p_j. \end{cases}$$

   unde $\overline{p_1} = p_i,\ i = \overline{1,l}.$

   - *altfel solutia nu exista.*

5. *Daca* $\lambda = \frac{l}{l+1}$*, atunci*

   - *daca* $p_j < p_i, i = \overline{1,l},\ \ j = \overline{l+1,n}$*, atunci*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \min\limits_{i=\overline{1,l}} \left\{p_i\left(\rho - r\right)\right\} \\ TR = ly^* + r\sum\limits_{i=1}^{l} p_i + \rho \sum\limits_{j=l+1}^{n} p_j. \end{cases}$$

   - *altfel solutia nu exista.*

6. *Daca* $\lambda < \frac{l+(n-m)}{l+(n-m)+1}$*, atunci*

   - *daca* $p_j \le \overline{p_3} \le p_i,\ i = \overline{1,l}, j = \overline{l+1,m}$*, atunci*

$$\begin{cases} x_i^* = r + \frac{y^*}{p_i}, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p}\left(\rho - r\right) \\ TR = ly^* + r\sum\limits_{i=1}^{l} p_i + \rho \sum\limits_{j=l+1}^{n} p_j + \left(n-m\right)\overline{p}\rho. \end{cases}$$

   unde $\overline{p_3} = p_k,\ k = \overline{m+1,n}.$

   - *altfel solutia nu exista.*

4.5 Validarea modelului si concluzii

4.5.1 Testarea solutiei

Pentru testarea modelului minimax am folosit date referitoare la consumul si productia de energie electrica furnizate de Transelectrica [148] si pretul energiei reglementat de ANRE [129]. In urma validarii datelor folosind teste Box&Whisker si Grubbs, am determinat urmatoarele valori pentru datele de intrare ale modelului minimax: $\varepsilon = 3666$ MW, $\rho = 10808$ MW si $r = 6970$ MW.

Pentru evaluarea performantelor modelului minimax, planul optim de productie generat de Teorema 4.4.5 a fost comparat cu productia reala din data de 4 Decembrie 2015.

Pe langa avantajele economice si cele de planificare asociate netezirii varfului de sarcina, consideram ca modelul minimax genereaza si alte avantaje aditionale, precum: (i) stimularea dezvoltarii unor sisteme de stocare a energiei, (ii) stimularea investitiilor in energii regenerabile, pentru a compensa modificarile in diagrama de productie si pentru a reduce emisiile de CO_2 si costul de productie, (iii) reducerea uzurii echipamentelor si a costurilor de mentenanta si retehnologizare, (iv) cresterea eficientei centralelor electrice sau (v) o retea de distributie mai stabila si mai eficienta.

4.5.2 Concluzii

In urma testarii modelului minimax folosind date reale, a rezultat ca acesta are capacitatea de a netezi varful de sarcina. Astfel, modelul indeplineste obiectivul cercetarii noastre.

Rezultatele obtinute in acest capitol au fost diseminate in doua articole stiintifice [90] si [98].

Chapter 5

Modelul index pentru optimizarea energiei

5.1 Introducere

Nivelul predefinit al productiei de energie electrica creste complexitatea modelului minimax. Mai mult, frontiera eficienta este mult restransa, deoarece un punct situat sub nivelul predefinit nu va fi niciodata eficient Pareto (datorita valorii absolute, punctul va genera aceeasi fluctuatie ca si simetricul sau, dar cifra de afaceri va fi mai mica).

In dorinta de a simplifica modelul minimax si de a-l face mai usor de aplicat, vom cauta sa introducem un nou model pentru netezirea varfului de sarcina - modelul index. In acest sens vom avea nevoie de o noua masura a fluctuatiei de energie. Performanta economica a producatorului va fi evaluata tot prin cifra de afaceri.

O comparatie intre modelele minimax si index, accentuand diferentele conceptuale dintre ele a fost prezentata de autor la conferinta *ICATA 2016* si publicata in [39].

5.2 Masura index pentru fluctuatia de energie

Prin renuntarea la nivelul predefinit de energie, specific masurii minimax, se pierde reperul fata de care masuram fluctuatia. Un alt parametru, mai accesibil practicienilor, trebuie folosit ca reper. In domeniul productiei este natural sa se raporteze gradul de incarcare al unui echipament la capacitatea maxima a acestuia [109, 112].

Folosind notatiile utilizate in Capitolul 4 si ideea prezentata mai sus, definim *masura index pentru fluctuatia de energie* astfel

$$\max_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}. \tag{5.1}$$

Elasticitatea pretului energiei, explicata anterior, ne-a determinat sa pastram pretul si in masura index.

5.3 Formularea problemei

Sa consideram din nou problema bi-criteriala formulata in cazul unei centrale electrice ce urmareste netezirea varfului de sarcina. Prin introducerea masurii index a fluctuatiei de energie (5.1), in locul masurii minimax, obtinem problema

$$\begin{cases} \min \left(\max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}; -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \varepsilon \leq x_i \leq \rho, \ i = \overline{1,n}. \end{cases} \tag{5.2}$$

numita *modelul index*.

5.4 Determinarea solutiei

Pentru determinarea solutiei eficiente a problemei (5.2) introducem problema bi-criteriala de optimizare

$$\begin{cases} \min \left(y; -\sum\limits_{i=1}^{n} p_i x_i \right)^T \\ \frac{x_i}{\rho} p_i \leq y, \ i = \overline{1,n} \\ \varepsilon \leq x_i \leq \rho, \ i = \overline{1,n} \end{cases} \tag{5.3}$$

echivalenta celor doua fiind stabilita prin Lema

Lemma 5.4.1 *[92] Fie problemele de optimizare bi-criteriala (5.2) si (5.3).*

a) Daca $x \in \mathbb{R}^n$ este o solutie eficienta a problemei (5.2), atunci $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, cu $y = \max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}$ este o solutie eficienta a problemei (5.3).

b) Daca $(x, y) \in \mathbb{R}^n \times \mathbb{R}$*, cu* $y = \max\limits_{i=\overline{1,n}} \left\{ \frac{x_i}{\rho} p_i \right\}$ *este o solutie eficienta a problemei* (5.3)*, atunci* $x \in \mathbb{R}^n$ *este o solutie eficienta a problemei* (5.2)*.*

Folosind din nou idea transformarii unei probleme bi-criteriale intr-o problema parametrica si bazandu-ne pe rezultatele obtinute de Yu [160], Bot et all [14] si Geoffrion [53], problema bi-criteriala de optimizare (5.3) este echivalenta cu urmatoarea problema parametrica

$$\begin{cases} \min\limits_{i=\overline{1,n}} \left\{ \lambda y - (1-\lambda) \sum\limits_{i=1}^{n} p_i x_i \right\} \\ \frac{x_i}{\rho} p_i \leq y, \; i = \overline{1,n} \\ \varepsilon \leq x_i, \; i = \overline{1,n} \\ x_i \leq \rho, \; i = \overline{1,n} \end{cases} \tag{5.4}$$

cu $\lambda \in (0, 1)$ si urmatoarea afirmatie este adevarata

Lemma 5.4.2 *[92]* $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ *este o solutie eficienta a problemei bi-criteriale* (5.3) *daca si numai daca* $\exists \lambda \in (0, 1)$ *astfel incat* $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ *este o solutie optima a problemei parametrice* (5.4)*.*

Remark 5.4.3 *Pentru a determina solutia eficienta a problemei* (5.2) *este suficient sa determinam solutia optima a problemei* (5.4)*.*

Remark 5.4.4 *In procesul de determinare a solutiei optime pentru problema* (5.4) *vom imparti multimea* $\{1, 2, ..., n\}$ *in mai multe submultimi, de forma* $\{1, 2, ..., l\}$, $\{l + 1, l + 2, ..., m\}$, $\{m + 1, m + 2, ..., t\}$ *si* $\{t+1, t+2, ..., n\}$*. Daca pe o asemenea submultime, pretul energiei este constant, il vom nota* $\overline{p}$*.*

Teorema urmatoare prezinta o solutie a problemei parametrice (5.4).

Theorem 5.4.5 (Luca and Duca [92]; index parametric) .
O solutie optima $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}$ *a problemei parametrice* (5.4) *este:*

1. *Daca* $\lambda = \frac{\rho n}{1+\rho n}$ *si* $\frac{\min\limits_{i=\overline{1,n}} p_i}{\max\limits_{i=\overline{1,n}} p_i} \geq \frac{\varepsilon}{\rho}$*, atunci*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,n} \\ y^* \in \left[\max\limits_{i=\overline{1,n}} \frac{\varepsilon}{\rho} p_i; \; \min\limits_{i=\overline{1,n}} p_i \right] \end{cases}$$

   altfel solutia nu exista.

2. *Daca* $\lambda > \frac{\rho n}{1+\rho n}$, *atunci*

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p}, \end{cases}$$

altfel solutia nu exista, unde $\overline{p} = p_i,\ i = \overline{1,n}$.

3. *Daca* $\lambda < \frac{\rho n}{1+\rho n}$, *atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,n} \\ y^* = \overline{p}, \end{cases}$$

altfel solutia nu exista, unde $\overline{p} = p_i,\ i = \overline{1,n}$.

4. *Daca* $\lambda = \frac{\rho l}{1+\rho l}$ *si* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \begin{cases} \overline{p_2}, & \text{if } \min\limits_{i=\overline{1,l}} p_i = p_j, \quad j = \overline{l+1,n} \\ \in \left[\max\limits_{j=\overline{l+1,n}} p_j; \min\limits_{i=\overline{1,l}} p_i\right], & \text{if } \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i \leq p_j < \min\limits_{i=\overline{1,l}} p_i, \quad j = \overline{l+1,n} \\ \in \left[\max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i; \min\limits_{i=\overline{1,l}} p_i\right], & \text{if } p_j < \max\limits_{i=\overline{1,l}} \frac{\varepsilon}{\rho} p_i, \quad j = \overline{l+1,n} \end{cases} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_2} = p_j,\ j = \overline{l+1,n}$.

5. *Daca* $\lambda < \frac{\rho l}{1+\rho l}$, *atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p_1},\ \text{if } \frac{\overline{p_1}}{p_j} \geq 1, \quad j = \overline{l+1,n} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_1} = p_i,\ i = \overline{1,l}$.

6. *Daca* $\lambda > \frac{\rho l}{1+\rho l}$, *atunci*

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1},\ \text{if } \frac{\overline{p_1}}{p_j} \geq \frac{\rho}{\varepsilon}, \quad j = \overline{l+1,n} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_1} = p_i,\ i = \overline{1,l}$.

7. *Daca* $\lambda < \frac{\rho n}{1+\rho n}$ *si* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \overline{p_2}, \ \mathit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \frac{\rho}{\varepsilon}\overline{p_2} \\ \overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_2} = p_j, \ j = \overline{l+1,n}$.

8. *Daca* $\lambda > \frac{\rho l}{1+\rho l}$ *si* $\frac{\overline{p_2}}{\overline{p_1}} = \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \varepsilon, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_1} = \overline{p_2} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_1} = p_i, \ i = \overline{1,l}$ *si* $\overline{p_2} = p_j$, $j = \overline{l+1,n}$.

9. *Daca* $\lambda > \frac{\rho n}{1+\rho n}$ *si* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2}, \ \mathit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho}\overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_2} = p_j, \ j = \overline{l+1,n}$.

10. *Daca* $\lambda > \frac{\rho m}{1+\rho m}$, $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$ *si* $\frac{\overline{p_3}}{\overline{p_2}} = \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} = \overline{p_3}, \ \mathit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_2} \\ \frac{\varepsilon}{\rho}\overline{p_2} \leq \min\limits_{i=\overline{1,l}} p_i \end{cases} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_2} = p_j,\ j = \overline{l+1,m}$ *si* $\overline{p_3} = p_k,\ k = \overline{m+1,n}$.

11. *Daca* $\lambda > \frac{\rho l+\rho(n-m)}{1+\rho l+\rho(n-m)}$ *si* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \varepsilon, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3}, \ \textit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq \frac{\varepsilon}{\rho}\overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_3} = p_k,\ k = \overline{m+1,n}$.

12. *Daca* $\lambda < \frac{\rho l+\rho(n-m)}{1+\rho l+\rho(n-m)}$ *si* $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \overline{p_3}, \ \textit{if} \begin{cases} \max\limits_{i=\overline{1,n}} p_i \leq \frac{\rho}{\varepsilon}\overline{p_3} \\ \overline{p_3} \leq \min\limits_{i=\overline{1,n}} p_i \\ p_j \leq \overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_3} = p_k,\ k = \overline{m+1,n}$.

13. *Daca* $\lambda > \frac{\rho(m-l)}{1+\rho(m-l)}$ *si* $\frac{\overline{p_3}}{\overline{p_2}} = \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \rho, \quad i = \overline{1,l} \\ x_j^* = \varepsilon, \quad j = \overline{l+1,m} \\ x_k^* = \rho, \quad k = \overline{m+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_2} = \overline{p_3}, \ \textit{if}\ p_i \leq \frac{\varepsilon}{\rho}\overline{p_2}, \ i = \overline{1,l} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_2} = p_j,\ j = \overline{l+1,m}$ *si* $\overline{p_3} = p_k,\ k = \overline{m+1,n}$.

14. *Daca* $\lambda > \frac{\rho l+\rho(t-m)}{1+\rho l+\rho(t-m)}$, $\frac{\min\limits_{i=\overline{1,l}} p_i}{\max\limits_{i=\overline{1,l}} p_i} \geq \frac{\varepsilon}{\rho}$ *si* $\frac{\overline{p_4}}{\overline{p_3}} = \frac{\varepsilon}{\rho}$, *atunci*

$$\begin{cases} x_i^* = \frac{\rho}{p_i} y^*, \quad i = \overline{1,l} \\ x_j^* = \rho, \quad j = \overline{l+1,m} \\ x_k^* = \varepsilon, \quad k = \overline{m+1,t} \\ x_s^* = \rho, \quad s = \overline{t+1,n} \\ y^* = \frac{\varepsilon}{\rho}\overline{p_3} = \overline{p_4}, \quad \textit{if} \begin{cases} \max\limits_{i=\overline{1,l}} p_i \leq \overline{p_3} \\ \frac{\varepsilon}{\rho}\overline{p_3} \leq \min\limits_{i=\overline{1,l}} p_i \\ p_j \leq \frac{\varepsilon}{\rho}\overline{p_3}, \quad j = \overline{l+1,m} \end{cases} \end{cases}$$

altfel solutia nu exista, unde $\overline{p_3} = p_k,\ k = \overline{m+1,t}$ *si* $\overline{p_4} = p_s,\ s = \overline{t+1,n}$.

In baza Lemei 5.4.1, Teorema 5.4.5 ofera o solutie eficienta $x^* \in \mathbb{R}^n$ a problemei (5.2). Valorile functiei obiectiv din problema (5.2) sunt calculate de: (a) Teorema 5.4.5 pentru fluctuatie, stiind ca $\max\limits_{i=\overline{1,n}} \left\{\frac{x_i}{\rho} p_i\right\} = y^*$ si (b) urmatoarea teorema, pentru cifra de afaceri (TR).

Theorem 5.4.6 (Luca and Duca [92]; index energie) .
Valorile celei de a doua componente a functiei obiectiv din problema (5.2) *sunt:*

1. $TR = n\rho y^*$, *unde* y^* *este definit de Teorema* 5.4.5 *punctul 1;*
2. $TR = \varepsilon\sum\limits_{i=1}^{n} p_i$;
3. $TR = \rho\sum\limits_{i=1}^{n} p_i$;
4. $TR = \rho l y^* + \rho \sum\limits_{j=l+1}^{n} p_j$, *unde* y^* *este definit de Teorema* 5.4.5 *punctul 4;*
5. $TR = l\rho\overline{p_1} + \rho \sum\limits_{j=1+1}^{n} p_j$, *unde* $\overline{p_1} = p_i,\ i = \overline{1,l}$;
6. $TR = l\varepsilon\overline{p_1} + \rho \sum\limits_{j=l+1}^{n} p_j$, *unde* $\overline{p_1} = p_i,\ i = \overline{1,l}$;
7. $TR = n\rho\,\overline{p_2}$, *unde* $\overline{p_2} = p_j,\ j = \overline{l+1,n}$;
8. $TR = n\varepsilon\overline{p_1}$, *unde* $\overline{p_1} = p_i,\ i = \overline{1,l}$;

9. $TR = n\varepsilon\overline{p_2}$, *unde* $\overline{p_2} = p_j,\ j = \overline{l+1, n}$;

10. $TR = n\varepsilon\overline{p_2}$, *unde* $\overline{p_2} = p_j,\ j = \overline{l+1, m}$;

11. $TR = (n - m + l)\varepsilon\overline{p_3} + \rho \sum\limits_{j=l+1}^{m} p_j$, *unde* $\overline{p_3} = p_k,\ k = \overline{m+1, n}$;

12. $TR = (n - m + l)\rho\overline{p_3} + \rho \sum\limits_{j=l+1}^{m} p_j$, *unde* $\overline{p_3} = p_k,\ k = \overline{m+1, n}$;

13. $TR = (n - l)\epsilon\overline{p_2} + \rho\sum\limits_{i=1}^{l} p_i$, *unde* $\overline{p_2} = p_j,\ j = \overline{l+1, m}$;

14. $TR = (n - m + l)\varepsilon\ \overline{p_3} + \rho \sum\limits_{j=l+1}^{m} p_j$, *unde* $\overline{p_3} = p_k,\ k = \overline{m+1, t}$.

5.5 Validarea modelului si concluzii

5.5.1 Testarea solutiei

Testarea solutiilor modelului index se va face folosind aceleasi date ca si in cazul modelului minimax.

Pentru modelul index am efectuat doua teste, ce difera prin datele de intrare folosite ε si ρ.
Pentru prima testare am folosit aceleasi date ca si in cazul modelului minimax. Acestea nu verifica insa multe din conditiile Teoremei 5.4.5 si astfel multe solutii nu pot fi calculate si evaluate.
Pentru a doua testare, datele de intrare au fost estimate si calculate in asa fel incat conditiile Teoremei 5.4.5 sa fie indeplinite. Rezultatele obtinute sunt mult mai bune comparativ cu primul test si sugereaza o sensibilitate a modelului fata de datele de intrare.

5.5.2 Concluzii

Testele efectuate demonstreaza ca modelul index are capacitatea de a netezi varful de sarcina. Astfel obiectivul cercetarii noastre este indeplinit si de acest model.

Mai mult, modelul index este mai accesibil practicienilor si mai usor de implementat, dar acuratetea lui este mai mica comparativ cu modelul minimax.

Rezultatele obtinute in acest capitol sunt diseminate in [90, 92].

Chapter 6

Concluzii

Contextul global al energiei genereaza o serie de provocari pentru optimizarea productiei de energie, o tema de actualitate fiind netezirea varfului de sarcina.

Analiza unui grafic de productie duce la identificarea unor fluctuatii in producerea de energie. Aceste fluctuatii pot fi privite prin prisma imprastierii/variatiei valorilor de productie. Utilizarea unei masuri corespunzatoare pentru fluctuatie (capabila sa vizeze valoarea extrema) si minimzarea acesteia ar duce la netezirea varfului de sarcina. Modificarea diagramei de productie prin netezirea varfului de sarcina genereaza supra-productie la alte momente de timp. In functie de strategia producatorului, supra-productia poate fi tratata prin: (i) sisteme de stocare a energiei, (ii) vehicule electrice, (iii) managementul cererii sau (iv) interventia umana si modificarea planului de productie.

Literatura de specialitate arata o legatura puternica intre componentele tehnice si cele economice. Prin urmare, modificarea diagramei de productie datorita netezirii varfului de sarcina poate genera o scadere a performantelor economice ale producatorului. Pentru evitarea acestei situatii se impune folosirea optimizarii bi-criteriale.

Astfel, obiectivul cercetarii noastre este **crearea, rezolvarea si validarea unui model matematic pentru productia de energie electrica, care sa asigure netezirea varfului de sarcina prin minimizarea fluctuatiei in productia de energie si maximizarea performantelor economice ale producatorului**.

Am reusit construirea a doua modele matematice (modelul minimax (4.2) si modelul index (5.2)) pentru aceasta problema generata de o situatie reala. Pentru rezolvarea acestor modele am recurs la transformarea lor in probleme echivalente de optimizare parametrica. Conditiile Kuhn-Tucker au fost folosite pentru determinarea

solutiei optime a problemelor parametrice, care au furnizat apoi solutiile eficiente pentru problemele bi- criteriale (4.2) si (5.2)).
Utilizarea conditiilor Kuhn-Tucker, in defavoarea “particle swarm optimization“ – metoda de optimizare intens folosita in optimizarea energiei electrice, a fost influentata, printre altele, si de semnificatia economica a multiplicatorilor Kuhn-Tucker ce ofera informatii strategice esentiale factorilor decizionali.

Solutiile obtinute pentru cele doua modele au fost testate folosind date reale. Ambele modele au reusit sa netezeasca varful de sarcina si astfel obiectivul cercetarii noastre este indeplinit.

Utilizarea profitului ca masura a performantei economice, introducerea unor restrictii tehnice suplimentare, estimarea mai precisa a parametrilor de intrare, topologia retelei, ajustarea trecerii de la productia de zi la cea de noapte ar putea genera modele mult mai complexe dar care ar putea avea solutii mai precise.

Cateva referinte privind abordarea noilor provocari de cercetare se regasesc in [125], [77], [147], [122], [18], [20], [91], [89].

Metoda de rezolvare abordata pentru modelele minimax si index s-ar putea sa nu mai fie la fel de eficienta si in cazul acestor modele mai complexe. Pentru aceasta, in Capitolul 3 al tezei noastre, am abordat metoda de rezolvare a unei probleme “initiale“ $\left(P_0^{0,0}\right)$ prin atasarea unor probleme $\left(P_k^{i,j}\right)$ mai usor de rezolvat. Legatura dintre cele doua probleme este data de inlocuirea unor functii prin η - aproximarile lor de ordin 1 sau 2. Capitolul 3 prezinta conditii suficiente pentru ca solutia eficienta a problemei “initiale“ $\left(P_0^{0,0}\right)$ sa ramana eficienta si pentru problemele “care aproximeaza“ $\left(P_k^{i,j}\right)$ si reciproc.

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