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Properties of certain classes of analytic or harmonic functions

Summary

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1.1 Introduction

Complex analysis is a branch of mathematics that has various scientific and technical applications.

From the beginning of the XX-th century both real and complex analysis have gone through major development due to natural internal progress or to satisfy the needs of different branches of mathematics or even other sciences.

The basics of geometric theory of functions of one complex variable, with roots in the XIX-th century or even earlier, were set by the work of P. Koebe (1907) and L. Bieberbach, whose famous conjecture from 1916 was proved only in 1984 by Louis de Branges. The analytic functions of one complex variable are the ideal model for geometric transformations of the plane.

One of the most important centers in the field of geometric theory of functions was in Cluj, where G. Călugăreanu obtained some important results in 1931, establishing the first necessary and sufficient conditions of univalency expressed by coefficients.

One of the fields that aroused the interest of a large number of mathematicians all over the world was that of the geometric theory of functions of one or several complex variables, a special branch of complex analysis. In this field there are some very interesting problems to study as those referring at differential subordinations, integral or differential operators for some classes of functions, other properties of some classes of analytic, univalent or multivalent functions, some of them with negative coefficients.

The school of complex analysis from Cluj, under the leadership of professor Petru T. Mocanu contributed with some important results. Mocanu, together with S. S. Miller, introduced the method of differential subordination also known as method of admissible function. Their method was further developed by other mathematicians who demonstrated in a much simpler way some classical results in this area, adding expansions, and even new results to the study of several integral operators, conditions for starlikeness and convexity, preserving of some geometric and analytic properties of several integral and differential operators. At this moment the theory of differential subordinations is studied and successfully used by mathematicians all over the world, from U.S.A to Germany, Poland, Turkey, India, China, Japan, Canada, Malaysia or Egypt.

Miller and Mocanu introduced also the notion of differential superordination, dual notion of the differential subordination. The theory of subordination chains is one of the most modern theories of complex analysis that give new research directions for some important problems related to univalent functions.

Another recent research method is based on the use of some conditions of coefficients developed in power series of analytic functions (with positive and negative coefficients). Using this method some well appreciated results were obtained, cited by mathematicians all over the world. Nowadays the complex analysis is well represented in Cluj through the activity of scientists as G. Ş. Sălăgean, G. Kohr or T. Bulboacă, as well as in other university centers of Romania.
The present doctoral thesis contains six chapters.

In **Chapter 1** we present the historical background of the geometric function theory and we give some basic notations and preliminary results.

Chapter 2 (Analytic functions with varying arguments) contains four subchapters. In this chapter we study some properties of certain class of analytic functions with varying arguments defined by Ruscheweyh and Sălăgean derivative and the combination of these derivatives.

Chapter 3 (Analytic functions) In this chapter we study coefficient bounds and Fekete-Szegő problem for new classes of analytic functions with varying arguments defined by Sălăgean integro-differential operator.

In **Chapter 4** (Differential subordinations and superordinations), which contains six subchapters.

In the first subchapter we consider the $\mathscr{D}I^n : \mathscr{A} \to \mathscr{A}$, $\mathscr{D}I^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z)$ linear operator, where $\mathscr{D}^n$ is the Sălăgean differential operator and I^n is the Sălăgean integral operator. We study several differential subordinations generated by $\mathscr{D}I^n$ and we introduce a class of holomorphic functions $L_n^m(\beta)$, and obtain some subordination results.

In the second subchapter we define the operator $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} : \mathscr{A} \to \mathscr{A}$, given by $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = (1-\alpha-\beta)\mathscr{R}^\nu\mathscr{D}^n f(z) + \alpha\mathscr{R}^\nu\Omega_z^\lambda f(z) + \beta\mathscr{D}^n\Omega_z^\lambda f(z)$, for $z \in U$, where $\mathscr{R}^\nu$ is the Ruscheweyh derivative, $\mathscr{D}^n$ is the Sălăgean operator, Ω_z^λ is a fractional differintegral operator introduced by S. Owa and H. M. Srivastava, $\mathscr{A} = \{f \in \mathscr{H}(U) : f(z) = z + a_2 z^2 + a_3 z^3 + ..., z \in U\}$, $\alpha, \beta \geq 0, \nu > -1, n \in \mathbb{N}_0 = \{0,1,2,3,...\}, -\infty < \lambda < 2$. A certain subclass of analytic functions in the open unit disk, $\mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$, where $0 \leq \delta \leq 1$, is introduced using the new operator. We obtain some properties of the class $\mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$ and some differential subordinations using the operator $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}$.

In the third subchapter we use the $\mathscr{D}I^n$ operator defined in the first subchapter. We give some results and applications for differential subordinations and superordinations for analytic functions and we will determine some properties on admissible functions defined with the new operator.

In the next part we present Loewner chains and their utility in obtaining new univalence criteria.

In the fifth subchapter we study the properties of the image of some subclasses of starlike functions, through the generalized Bernardi - Libera - Livingston integral operator. A new subclass of functions with negative coefficients is introduced and we study some properties of this class.

In the sixth subchapter we determine the radius of convexity of particular functions. The obtained results will be used to deduce sharp estimations regarding functions which satisfy a second order differential subordination. A lemma regarding starlikeness is deduced which involves the notion of convolution. This lemma is used in order to obtain a sharp starlikeness condition.

In **Chapter 5** (Bi-univalent functions), which contains three subchapters, we define new subclasses of bi-univalent functions for which we obtain estimates of coefficients a_2, a_3 and a_4. The results in this chapter are original and are presented in [79], [80] and [83].

In **Chapter 6** (Harmonic functions), which contains four subchapters, we investigate several classes of harmonic functions with varying argument of coefficients which are defined by means of the principle of subordination between harmonic functions. Such properties as the coefficient estimates, distortion theorems, convolution properties, radii of convexity, starlikeness and the closure properties of these classes under the generalized Bernardi-Libera-Livingston integral operator $\mathscr{L}_c(f)$, $(c > -1)$ which is defined by $\mathscr{L}_c(f) = \mathscr{L}_c(h) + \overline{\mathscr{L}_c(g)}$ where

$$\mathscr{L}_c(h)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt \text{ and } \mathscr{L}_c(g)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt$$

are investigated.

We also investigate some generalizations of classes of harmonic functions defined by Sălăgean and Ruscheweyh derivative. By using the extreme points theory we obtain coefficients estimates distortion theorems and integral mean inequalities in these classes of functions.

The bibliography contains 130 titles. The original results were published or were submitted for publication in 26 articles 16 as single author and 10 in collaboration.
3 articles are published in Web of Science indexed journals.

The following articles contain the original results:

1. **Á. O. PÁLL-SZABÓ**, *Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative*, Miskolc Mathematical Notes, 18 (2017), pp. 397–406.

2. **Á. O. PÁLL-SZABÓ** AND G. S. SĂLĂGEAN, *A unified class of harmonic functions with varying argument of coefficients*, accepted, Filomat.

3. O. ENGEL AND **Á. PÁLL-SZABÓ**, *The radius of convexity of particular functions and applications to the study of a second order differential inequality*, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 52 (2017), pp. 118–127.

4. O. Engel, P. Kupán and **Á. Páll-Szabó**, *About the radius of convexity of some analytic functions*, Creative Mathematics and Informatics, 24 (2015), pp. 157 – 163.

5. **Á. O. Páll-Szabó**, O. Engel, and E. Szatmári, *Certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative*, Acta Universitatis Apulensis, 51 (2017), pp. 61–74.

6. **Á. O. Páll-Szabó**, *Coefficient bounds and Fekete-Szegő problem for new classes of analytic functions defined by Sălăgean integro-differential operator*, submitted, (-).

7. **Á. O. Páll-Szabó**, *Coefficient estimates and Fekete-Szegő problem for new classes of bi-univalent functions defined by Sălăgean integro-differential operator*, submitted, (-).

8. **Á. O. Páll-Szabó**, *Coefficient estimates for some new classes of bi-Bazilevič functions of Ma-Minda type involving the Sălăgean integro-differential operator*, submitted, (-).

9. **Á. O. Páll-Szabó**, *Differential subordinations and superordinations for analytic functions defined by Sălăgean integro-differential operator*, submitted, (-).

10. **Á. O. Páll-Szabó** and E. Szatmari, *Differential subordination results obtained by using a new operator*, General Mathematics, Vol. 25, No. 1-2 (2017), 119–131.

11. **Á. O. Páll-Szabó**, *Extensions of coefficient estimates for new classes of bi-univalent functions defined by Sălăgean integro-differential operator*, submitted, (-).

12. **Á. O. Páll-Szabó**, *Generalizations of starlike harmonic functions defined by Sălăgean and Ruscheweyh derivative*, submitted, (-).

13. **Á. O. Páll-Szabó**, *Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by the convolution of Ruscheweyh and Sălăgean derivative*, submitted, (-).

14. **Á. O. Páll-Szabó**, *On a class of univalent functions defined by Sălăgean integro-differential operator*, submitted, (-).

15. **Á. O. Páll-Szabó**, and O. Engel, *Properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative*, Acta Universitatis Sapientiae, Mathematica, 7 (2015), pp. 278–286.

16. **Á. O. Páll-Szabó**, *Univalence criteria related with the generalised Sălăgean and Ruscheweyh operator*, submitted, (-).

17. **Á. O. Páll-Szabó**, *Integral properties of certain class of analytic functions with varying arguments defined by Sălăgean derivative*, Annals of Oradea University-Mathematics Fascicola, 23 (2016), pp. 177 – 182.

18. **Á. O. PÁLL-SZABÓ**, *Certain class of analytic functions with varying arguments defined by Sălăgean and Ruscheweyh derivative*, Mathematica (Cluj), 59 (82) (2017), pp. 80–88.

19. **Á. O. PÁLL-SZABÓ**, *Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Sălăgean and Ruscheweyh derivative.*, Studia Universitatis Babes-Bolyai, Mathematica, 62 (2017), pp. 465–472.

20. **Á. O. PÁLL-SZABÓ**, *Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Sălăgean derivative*, Automation, Computers, Applied Mathematics (ACAM), Vol. 25 (2016), No. 1, pp. 85–91.

21. **Á. O. PÁLL-SZABÓ** AND O. ENGEL, *Certain class of analytic functions with varying arguments defined by Sălăgean derivative*, in Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics, Ictami, 2015, pp. 113–120.

22. O. ENGEL AND **Á. PÁLL-SZABÓ**, *Preserving properties of the generalized Bernardi-Libera-Livingston integral operator defined on some subclasses of starlike functions*, Konuralp Journal of Mathematics, 5 (2017), pp. 207–215.

23. **Á. O. PÁLL-SZABÓ** AND G. S. SĂLĂGEAN, *On the order of convolution consistence of the harmonic functions with varying arguments*, submitted, (-).

24. **Á. O. PÁLL-SZABÓ** AND G. S. SĂLĂGEAN, *On a certain class of harmonic functions and the generalized Bernardi-Libera-Livingston integral operator*, submitted, (-).

25. **Á. O. PÁLL-SZABÓ**, *Where Are the Quadratic's Complex Roots ?,* Acta Didactica Napocensia, Volume 8, Number 1, 2015, pp. 37–48.

26. **Á. O. PÁLL-SZABÓ**, *Visualizing roots of a cubic equation,* The Electronic Journal of Mathematics & Technology, Volume 11 (2017), nr. 1, Research Journal of Mathematics & Technology, RJMT Vol. 6, Nr. 1 , 2017, pp. 1–8.

1.2 Definitions and notations

Let $\mathbb{R}$ be the field of real numbers. Also let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ while $\mathbb{Z}$ is the set of integers.

Let $\mathbb{C}$ be the complex plane, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. Let $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ be the open disk of radius $r > 0$, centered at $z_0 \in \mathbb{C}$.
The closure of $U(z_0, r)$ will be denoted by $\overline{U}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ and its boundary by $\partial U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$. Let $\dot{U}(z_0, r) = U(z_0, r) \setminus \{z_0\}$.
The disk $U(0, r)$ is denoted by U_r and the unit disk is denoted by $U_1 = U = \{z \in \mathbb{C} : |z| = r < 1\}$.

Definition 1.1. [29] A complex-valued function $f : G \to \mathbb{C}$ (G open set) of a complex variable is differentiable at a point $z_0 \in G$ if it possesses the complex derivative at z_0:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}.$$

Definition 1.2. A function is analytic if and only if its Taylor series expansion about z_0 converges to the function in some neighborhood for every z_0 in its domain:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \text{ where } a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Definition 1.3. A complex-valued function $f : G \to \mathbb{C}$ (G open set) of a complex variable is called a holomorphic function at z_0 if it is differentiable at every point in some neighborhood of z_0. Let $G \subseteq \mathbb{C}$ be an open set. We denote by $H(G)$ the set of holomorphic functions defined on G with values in $\mathbb{C}$. Holomorphic functions on the whole complex plane are called entire functions.

Let $H(U)$ be the set of holomorphic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$H[a,n] = \{f \in H(U) : f(z) = a + a_n z^n + ...\}$$

and

$$\mathcal{A}_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + ...\}, \text{ with } \mathcal{A} = \mathcal{A}_1.$$

The Taylor series expansion of a function $f \in \mathcal{A}$ is :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

1.3 Univalent functions

Definition 1.4. [29] A single-valued function $f \in H(D)$ is said to be univalent (or schlicht) in a domain $D \subset \mathbb{C}$ if f is injective (it never takes the same value twice). We denote by $H_u(D)$ the class of univalent functions in D.

Theorem 1.1. *[43] If $f \in H_u(D)$ then $f'(z) \neq 0$, $\forall z \in D$.*

Corollary 1.1. *[61],[40]*
If D is a convex domain and $f \in H(D)$ such that $\Re f'(z) > 0$, for any $z \in D$, then $f \in H_u(D)$.

We denote by $S = \{f \in \mathcal{A} : f \in H_u(U)\}$. the class of univalent function in the unit disk U normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Denote with S^* the class of starlike functions in U:

$$S^* = \left\{ f \in \mathcal{A} : \Re \frac{z f'(z)}{f(z)} > 0 \ , \ z \in U \right\}, \ S^* \subset S. \tag{1.2}$$

Denote with K the class of convex functions in U:

$$K = \left\{ f \in \mathcal{A} : \Re \frac{z f''(z)}{f'(z)} + 1 > 0 \ , \ z \in U \right\}, \ K \subset S^* \subset S \tag{1.3}$$

Definition 1.5. Let r_f be the radius of convergence of the function f. The radius of convexity of the function f by the equality

$$r_f^c = \sup\left\{ r \in (0, r_f) \middle| \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ \ \forall\, z \in U_r \right\}. \tag{1.4}$$

Definition 1.6. ([61], $def.$ 3.5.1) Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exists a function w, which is analytic in U and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z))$; $\forall z \in U$. We denote by $\prec$ the subordination relation. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

1.4 Differential and integral operators

Definition 1.7. [106]

For $f \in \mathscr{A}$, $n \in \mathbb{N}_0$, the Sălăgean differential operator $\mathscr{D}^n$ is defined by $\mathscr{D}^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{D}^0 f(z) = f(z),$$

$$\mathscr{D}^1 f(z) = zf'(z),$$

$$\mathscr{D}^{n+1} f(z) = z\left(\mathscr{D}^n f(z)\right)', z \in U$$

Remark 1.1. *If* $f \in \mathscr{A}$ *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then*

$$\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U. \tag{1.5}$$

Definition 1.8. [106]

For $f \in \mathscr{A}, n \in \mathbb{N}_0$, the Sălăgean integral operator I^n is defined by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt, \ldots$$

$$I^{n+1} f(z) = I\left(I^n f(z)\right), z \in U$$

The I^1 is the Alexander operator used for the first time in [7], the I^n operator is called the generalized Alexander operator.

Remark 1.2. *If* $f \in \mathscr{A}$ *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then*

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k, \tag{1.6}$$

$z \in U$, $n \in \mathbb{N}_0$ *and* $z\left(I^{n+1} f(z)\right)' = I^n f(z)$.

Remark 1.3. *We have* $\mathscr{D}^n I^n f(z) = I^n \mathscr{D}^n f(z) = f(z)$, $f \in \mathscr{A}$, $z \in U$.

Definition 1.9. Let $\lambda \geq 0$, $n \in \mathbb{N}_0$. Denote by $\mathscr{D}I^n$ the operator given by $\mathscr{D}I^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{D}I^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z), z \in U.$$

Remark 1.4. *If* $f \in \mathscr{A}$ *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then*

$$\mathscr{D}I^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n} \right] a_k z^k, z \in U. \tag{1.7}$$

Definition 1.10. Let $f, g \in H(U)$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. The convolution (or Hadamard product) of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.8}$$

The modified Hadamard product is

$$(f \otimes g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g \otimes f)(z). \tag{1.9}$$

Definition 1.11. Ruscheweyh [102] defined the derivative $\mathscr{R}^\gamma : \mathscr{A} \to \mathscr{A}$ by

$$\mathscr{R}^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z), (\gamma \geq -1),\ f \in \mathscr{A},\ z \in U. \tag{1.10}$$

In the particular case $n \in \mathbb{N}_0$

$$\mathscr{R}^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}. \tag{1.11}$$

The symbol $\mathscr{R}^n f(z)$ ($n \in \mathbb{N}_0$) was called the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [2]. It is easy to see that

$$\mathscr{R}^0 f(z) = f(z),$$

$$\mathscr{R}^1 f(z) = z f'(z), \ldots$$

$$(n+1)\mathscr{R}^{n+1} f(z) = z\left(\mathscr{R}^n f(z)\right)' + n\mathscr{R}^n f(z), z \in U.$$

Remark 1.5. *If* $f \in \mathscr{A}$ *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then*

$$\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} a_k z^k = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)} a_k z^k,\ n > -1,\ z \in U, \tag{1.12}$$

or

$$\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \quad \textit{where}\ \delta(n,k) = \binom{n+k-1}{n}\ z \in U, \tag{1.13}$$

Definition 1.12. [5] Let $\lambda \geq 0, n \in \mathbb{N}_0$. Denote by $\mathscr{R}\mathscr{D}^n$ the operator given by $\mathscr{R}\mathscr{D}^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{R}\mathscr{D}^n f(z) = (1-\lambda)\mathscr{R}^n f(z) + \lambda \mathscr{D}^n f(z), z \in U.$$

Remark 1.6. *If $f \in \mathscr{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{R}\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} \left\{ (1-\lambda)\frac{(n+k-1)!}{n!(k-1)!} + \lambda k^n \right\} a_k z^k, z \in U. \tag{1.14}$$

Definition 1.13. [4]

For $f \in \mathscr{A}, \lambda \geq 0$ and $n \in \mathbb{N}_0$, the operator $\mathscr{D}_\lambda^n$ is defined by $\mathscr{D}_\lambda^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{D}_\lambda^0 f(z) = f(z),$$

$$\mathscr{D}_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = \mathscr{D}_\lambda f(z), \ldots$$

$$\mathscr{D}_\lambda^{n+1} f(z) = (1-\lambda)\mathscr{D}_\lambda^n f(z) + \lambda z \left(\mathscr{D}_\lambda^n f(z)\right)' = \mathscr{D}_\lambda \left(\mathscr{D}_\lambda^n f(z)\right), z \in U$$

Remark 1.7. *If $f \in \mathscr{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{D}_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k, \ z \in U. \tag{1.15}$$

Remark 1.8. *For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [106].*

Definition 1.14. Let $\gamma, \lambda \geq 0, n \in \mathbb{N}_0$. Denote by $\mathscr{R}\mathscr{D}_\lambda{}^n$ the operator given by $\mathscr{R}\mathscr{D}_\lambda{}^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{R}\mathscr{D}_\lambda{}^n f(z) = \left(1-\gamma\right)\mathscr{R}^n f(z) + \gamma \mathscr{D}_\lambda^n f(z), z \in U.$$

Remark 1.9. *If $f \in \mathscr{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{R}\mathscr{D}_\lambda{}^n f(z) = z + \sum_{k=2}^{\infty} \left\{ \gamma [1 + (k-1)\lambda]^n + \left(1-\gamma\right)\frac{(n+k-1)!}{n!(k-1)!} \right\} a_k z^k, \ z \in U. \tag{1.16}$$

Definition 1.15. Let $n \in \mathbb{N}$. Denote by $\mathscr{S}\mathscr{R}^n$ the operator given by the Hadamard product (convolution) of the Sălăgean operator $\mathscr{S}^n$ and the Ruscheweyh operator $\mathscr{R}^n$, $\mathscr{S}\mathscr{R}^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{S}\mathscr{R}^n f(z) = \mathscr{S}^n \left(\frac{z}{1-z}\right) * \mathscr{R}^n f(z), z \in U.$$

Remark 1.10. *If $f \in \mathscr{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{S}\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \frac{k^n (n+k-1)!}{n!(k-1)!} a_k z^k, z \in U. \tag{1.17}$$

Definition 1.16. The Bernardi integral operator $L_c : \mathscr{A} \to \mathscr{A}$ is defined by

$$L_c f(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt, \quad c > -1. \tag{1.18}$$

Definition 1.17. In [71] are defined the following operators:

the fractional integral operator $D_z^{-\mu}$ of order μ, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt, \ z \in U, \ f \in \mathscr{A}, \ \mu > 0, \tag{1.19}$$

where the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$, and the fractional derivative operator D_z^{λ} of order λ, by

$$D_z^{\lambda} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{d}{dz} \displaystyle\int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt, & 0 \le \lambda < 1 \\ \dfrac{d^n}{dz^n} D_z^{\lambda-n} f(z), & n \le \lambda < n+1 \end{cases}, n \in \mathbb{N}_0, f \in \mathscr{A}, \lambda \ge 0, \tag{1.20}$$

where the multiplicity of $(z-t)^{-\lambda}$ is likewise understood.

Definition 1.18. In [72] is defined the fractional differintegral operator $\Omega_z^{\lambda} : \mathscr{A} \to \mathscr{A}$, by

$$\Omega_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z), z \in U, -\infty < \lambda < 2, \tag{1.21}$$

where $D_z^{\lambda} f(z)$ is the fractional integral of order$\lambda, -\infty < \lambda < 0$, and a fractional derivative of order λ, $0 \le \lambda < 2$.

The series expression of the operator Ω_z^{λ} for the function $f \in \mathscr{A}$ is given by

$$\Omega_z^{\lambda} f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(k+2)}{\Gamma(k+2-\lambda)} a_{k+1} z^{k+1}, -\infty < \lambda < 2, z \in U. \tag{1.22}$$

Definition 1.19. In [110] is defined the fractional operator $\mathbb{D}_{\lambda}^{\nu,n} : \mathscr{A} \to \mathscr{A}$ for $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$ as a composition of fractional differintegral operator, the Sălăgean operator and the Ruscheweyh operator:

$$\mathbb{D}_{\lambda}^{\nu,n} f(z) = \mathscr{R}^{\nu} \mathscr{D}^n \Omega_z^{\lambda} f(z). \tag{1.23}$$

The series expression of $\mathbb{D}_{\lambda}^{\nu,n} f(z)$ for $f \in \mathscr{A}$ is given by

$$\mathbb{D}_{\lambda}^{\nu,n} f(z) = z + \sum_{k=1}^{\infty} \frac{(\nu+1)_k}{(2-\lambda)_k} (k+1)^{n+1} a_{k+1} z^{k+1}, \tag{1.24}$$

$-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, z \in U$, where the symbol $(\gamma)_k$ denotes the usual Pochhammer symbol, for $\gamma \in \mathbb{C}$, defined by

$$(\gamma)_k = \begin{cases} 1, k = 0 \\ \gamma(\gamma+1)...(\gamma+k-1), k \in \mathbb{N} \end{cases} = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

Chapter 2

Analytic functions with varying arguments

Let $f, g \in \mathscr{A}$ two analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{2.1}$$

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{2.2}$$

Definition 2.1. [111]A function $f(z)$ of the form (2.1) is said to be in the class $V(\theta_k)$ if $f \in \mathscr{A}$ and $arg(a_k) = \theta_k$,$\forall k \geq 2$. If $\exists \delta \in \mathbb{R}$ such that $\theta_k + (k-1)\delta \equiv \pi(mod\ 2\pi), \forall k \geq 2$ then $f(z)$ is said to be in the class $V(\theta_k, \delta)$. The union of $V(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V.

2.1 Properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative

We recall the Ruscheweyh differential operator defined in (1.13).
Attiya and Aouf defined in [12] the class $Q(n, \lambda, A, B)$ this way:

Definition 2.2. [12][36] For $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ let $Q(n, \lambda, A, B)$ denote the subclass of $\mathscr{A}$ which contain functions $f(z)$ of the form (2.1) such that

$$(1-\lambda)(\mathscr{R}^n f(z))' + \lambda(\mathscr{R}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{2.3}$$

Let $VQ(n, \lambda, A, B)$ denote the subclass of V consisting of functions $f(z) \in Q(n, \lambda, A, B)$.

Theorem 2.1. *[36]Let the function f defined by* (2.1) *be in V. Then $f \in VQ(n,\lambda,A,B)$, if and only if*

$$T(f) = \sum_{k=2}^{\infty} k\delta(n,k)C_k(1+B)|a_k| \le (B-A)(n+1) \tag{2.4}$$

where

$$C_k = n+1+\lambda(k-1).$$

The extremal functions are

$$f_k(z) = z + \frac{(B-A)(n+1)}{kC_k\delta(n,k)(1+B)} e^{i\theta_k} z^k, \forall k \ge 2.$$

Let $L_c f(z) = \dfrac{c+1}{z^c} \displaystyle\int_0^z f(t)t^{c-1}dt, c > -1$ be the well-known Bernardi operator.

Theorem 2.2. *[87] If $f \in VQ(n,\lambda,2\alpha-1,B)$ then $L_c f \in VQ(n,\lambda,2\beta-1,B)$, where*

$$\beta = \beta(\alpha) = \frac{B+1+2\alpha(c+1)}{2(c+2)} \ge \alpha.$$

The result is sharp.

Theorem 2.3. *[87] If $f \in VQ(n,\lambda,A,B)$ then $L_c f \in VQ(n,\lambda,A^*,B)$, where $A^* = \dfrac{B+A(c+1)}{c+2} > A$. The result is sharp.*

Theorem 2.4. *[87] If $f \in VQ(n,\lambda,A,B)$ then $L_c f \in VQ(n,\lambda,A,B^*)$,where*

$$B^* = \frac{A(1+B)(c+2)+(B-A)(c+1)}{(1+B)(c+2)-(B-A)(c+1)} < B.$$

The result is sharp.

The modified Hadamard product of two functions f and g of the form (2.1) and (2.2), and which belong to $V(\theta_k,\delta)$ is defined by (see also [44, 104, 108])

$$(f \otimes g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g \otimes f)(z). \tag{2.5}$$

Theorem 2.5. *[73] If $f \in VQ(n,\lambda,A_1,B), g \in VQ(n,\lambda,A_2,B)$ then $f \otimes g \in VQ(n,\lambda,A^*,B)$, where*

$$A^* = B - \frac{(B-A_1)(B-A_2)(n+1)}{2C_2(1+B)\delta(n,2)}.$$

The result is sharp.

Corollary 2.1. *[73] If $f,g \in VQ(n,\lambda,A,B)$ then $f \otimes g \in VQ(n,\lambda,A^*,B)$, where*

$$A^* = B - \frac{(B-A)^2(n+1)}{2C_2(1+B)\delta(n,2)}.$$

The result is sharp.

Theorem 2.6. *[73] If $f \in VQ(n,\lambda,A,B_1), g \in VQ(n,\lambda,A,B_2)$ then $f \otimes g \in VQ(n,\lambda,A,B^*)$, where*

$$B^* - A = \frac{(A+1)(n+1)(B_1-A)(B_2-A)}{2C_2\delta(n,2)(1+B_1)(1+B_2)-(n+1)(B_1-A)(B_2-A)}.$$

The result is sharp.

Corollary 2.2. *[73] If $f,g \in VQ(n,\lambda,A,B)$ then $f \otimes g \in VQ(n,\lambda,A,B^*)$, where*

$$B^* = A + \frac{(A+1)(n+1)(B-A)^2}{2C_2\delta(n,2)(1+B)^2-(n+1)(B-A)^2}.$$

The result is sharp.

Theorem 2.7. *[73] If $f_j \in VQ(n,\lambda,A_j,B), j=\overline{1,s}, s \in \{2,3,4,\dots\}$ then $f_1 \otimes f_2 \otimes \dots \otimes f_s \in VQ(n,\lambda,A^{(s-1)*},B)$, where $A^{(s-1)*} = B - \dfrac{(n+1)^{s-1}\prod_{j=1}^{s}(B-A_j)}{2^{s-1}C_2^{s-1}(1+B)^{s-1}[\delta(n,2)]^{s-1}}$. The result is sharp.*

Theorem 2.8. *[73] If $f_j \in VQ(n,\lambda,A,B_j), j=\overline{1,s}, s \in \{2,3,4,\dots\}$ then $f_1 \otimes f_2 \otimes \dots \otimes f_s \in VQ(n,\lambda,A,B^{(s-1)*})$, where*

$$B^{(s-1)*} = A + \frac{(A+1)(n+1)^{s-1}\prod_{j=1}^{s}(B_j-A)}{2^{s-1}C_2^{s-1}[\delta(n,2)]^{s-1}\prod_{j=1}^{s}(B_j-A)-(n+1)^{s-1}\prod_{j=1}^{s}(B_j-A)}.$$

The result is sharp.

2.2 Properties of certain class of analytic functions with varying arguments defined by Sălăgean derivative

Definition 2.3. For $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ let $S(n,\lambda,A,B)$ denote the subclass of $\mathscr{A}$ which contain functions f of the form (2.1) such that

$$(1-\lambda)(\mathscr{D}^n f(z))' + \lambda(\mathscr{D}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{2.6}$$

Attiya and Aouf defined in [12] the class $\mathscr{R}(n,\lambda,A,B)$ with a condition like (2.4), but there instead of the operator $\mathscr{D}$ they used the Ruscheweyh operator $\mathscr{R}$, where

$$\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \binom{n+k-1}{n} a_k z^k.$$

Let $VS(n,\lambda,A,B)$ denote the subclass of V consisting of functions $f(z) \in S(n,\lambda,A,B)$.

Particular case 2.1. *Let* $n = 0, \lambda = 0, A = -1, B = 1$. *If* $f \in VS(0,0,-1,1)$ *then* $f'(z) \prec \frac{1-z}{1+z}$. *If* $Re\ f'(z) > 0$ *then* f *is univalent (Noshiro-Warschawski-Wolff theorem).*

Particular case 2.2. *Let* $n = 0, \lambda = 1, A = -1, B = 1$. *If* $f \in VS(0,1,-1,1)$ *then* $(zf'(z))' \prec \frac{1-z}{1+z}$. *If* $Re\ (zf'(z))' > 0$ *then* $Re\ \left(f'(z) + zf''(z)\right) > 0 \Leftrightarrow Re \frac{zf'(z)}{f(z)} > 0 \Rightarrow f$ *is starlike.*

Aouf et al. previously studied this theorem in [10] :

Theorem 2.9. *Let the function* f *defined by* (2.1) *be in V. Then* $f \in VS(n,\lambda,A,B)$, *if and only if*

$$T(f) = \sum_{k=2}^{\infty} k^{n+1} C_k (1+B)|a_k| \leq B - A \tag{2.7}$$

where

$$C_k = 1 - \lambda + \lambda k.$$

The extremal functions are:

$$f(z) = z + \frac{B-A}{k^{n+1}C_k(1+B)} e^{i\theta_k} z^k, \forall k \geq 2.$$

Theorem 2.10. *[91] If* $f \in VS(n,\lambda,2\alpha-1,B)$ *then* $L_c f \in VS(n,\lambda,2\beta-1,B)$, *where*

$$\beta = \beta(\alpha) = \frac{B+1+2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

The result is sharp.

Theorem 2.11. *[91] If* $f \in VS(n,\lambda,A,B)$ *then* $L_c f \in VS(n,\lambda,A^*,B)$, *where* $A^* = \frac{B+A(c+1)}{c+2} > A$. *The result is sharp.*

Theorem 2.12. *[91] If* $f \in VS(n,\lambda,A,B)$ *then* $L_c f \in VS(n,\lambda,A,B^*)$,*where*

$$B^* = \frac{A(1+B)(c+2)+(B-A)(c+1)}{(1+B)(c+2)-(B-A)(c+1)} < B.$$

The result is sharp.

Theorem 2.13. *[94] If* $f \in VS(n,\lambda,A_1,B), g \in VS(n,\lambda,A_2,B)$ *then* $f \otimes g \in VS(n,\lambda,A^*,B)$, *where*

$$A^* = B - \frac{(B-A_1)(B-A_2)}{2^{n+1}C_2(1+B)}.$$

The result is sharp.

Corollary 2.3. *[94] If* $f,g \in VS(n,\lambda,A,B)$ *then* $f \otimes g \in VS(n,\lambda,A^*,B)$, *where*

$$A^* = B - \frac{(B-A)^2}{2^{n+1}C_2(1+B)}.$$

The result is sharp.

Theorem 2.14. *[94] If $f \in VS(n,\lambda,A,B_1), g \in VS(n,\lambda,A,B_2)$ then $f \otimes g \in VS(n,\lambda,A,B^*)$, where*

$$B^* = A + \frac{(B_1 - A)(B_2 - A)(A+1)}{2C_2(1+B_1)(1+B_2) - (B_1 - A)(B_2 - A)}.$$

The result is sharp.

Corollary 2.4. *[94] If $f, g \in VS(n,\lambda,A,B)$ then $f \otimes g \in VS(n,\lambda,A,B^*)$, where*

$$B^* = A + \frac{(B-A)^2(A+1)}{2C_2(1+B)^2 - (B-A)^2}.$$

The result is sharp.

Theorem 2.15. *[94] If $f_j \in VS(n,\lambda,A_j,B), j = \overline{1,s}, s \in \{2,3,4,\ldots\}$ then $f_1 * f_2 * \ldots * f_s \in VS(n,\lambda,A^{(s-1)*},B)$, where $A^{(s-1)*} = B - \dfrac{\prod_{j=1}^{s}(B - A_j)}{\left(2^{n+1}\right)^{s-1} C_2^{s-1}(1+B)^{s-1}}$. The result is sharp.*

Theorem 2.16. *[94]If $f_j \in VS(n,\lambda,A,B_j), j = \overline{1,s}, s \in \{2,3,4,\ldots\}$ then $f_1 \otimes f_2 \otimes \ldots \otimes f_s \in VS(n,\lambda,A,B^{(s-1)*})$, where*

$$B^{(s-1)*} = A + \frac{(A+1)\prod_{j=1}^{s}(B_j - A)}{2^{s-1}C_2^{s-1}\prod_{j=1}^{s}(1+B_j) - \prod_{j=1}^{s}(B_j - A)}.$$

The result is sharp.

2.3 Properties of certain class of analytic functions with varying arguments defined by Sălăgean and Ruscheweyh derivative

Definition 2.4. For $\widetilde{\lambda} \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ let $L(n,\widetilde{\lambda},A,B)$ denote the subclass of $\mathscr{A}$ which contain functions $f(z)$ of the form (2.1) such that

$$(1-\widetilde{\lambda})(\mathscr{R}\mathscr{D}_{\lambda}{}^{n} f(z))' + \widetilde{\lambda}(\mathscr{R}\mathscr{D}_{\lambda}{}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{2.8}$$

Let $VL(n,\widetilde{\lambda},A,B)$ denote the subclass of V consisting of functions $f(z) \in L(n,\widetilde{\lambda},A,B)$.

Theorem 2.17. *[92] Let the function $f(z)$ defined by (2.1) be in V. Then $f(z) \in VL(n,\widetilde{\lambda},A,B)$, if and only if*

$$T(f) = \sum_{k=2}^{\infty} kC_k(1+B)|a_k| \leq B - A \tag{2.9}$$

where

$$C_k = \gamma[1+(k-1)\lambda]^n \left[1+\widetilde{\lambda}\lambda(k-1)\right] + \frac{(n+k-1)!}{n!(k-1)!}\left(1-\gamma\right)\left[1+\widetilde{\lambda}\frac{k-1}{n+1}\right].$$

The extremal functions are:

$$f(z) = z + \frac{B-A}{kC_k(1+B)}e^{i\theta_k}z^k, \forall k \geq 2.$$

Corollary 2.5. *[92]Let the function $f(z)$ defined by* (2.1) *be in the class $VL(n,\widetilde{\lambda},A,B)$. Then*

$$|a_k| \leq \frac{B-A}{kC_k(1+B)}, \forall k \geq 2.$$

The result (2.9) *is sharp for the functions*

$$f(z) = z + \frac{B-A}{kC_k(1+B)}e^{i\theta_k}z^k, \forall k \geq 2.$$

Theorem 2.18. *[92]Let the function $f(z)$ defined by* (2.1) *be in the class $VL(n,\widetilde{\lambda},A,B)$. Then*

$$|z| - \frac{B-A}{2C_2(1+B)}|z|^2 \leq |f(z)| \leq |z| + \frac{B-A}{2C_2(1+B)}|z|^2. \tag{2.10}$$

Corollary 2.6. *[92]Let the function $f(z)$ defined by* (2.1) *be in the class $VL(n,\widetilde{\lambda},A,B)$. Then $f(z) \in U(0,r_1)$, where $r_1 = 1 + \frac{B-A}{2C_2(1+B)}$.*

Theorem 2.19. *[92]Let the function $f(z)$ defined by* (2.1) *be in the class $VL(n,\widetilde{\lambda},A,B)$. Then*

$$1 - \frac{B-A}{C_2(1+B)}|z| \leq \left|f'(z)\right| \leq 1 + \frac{B-A}{C_2(1+B)}|z|. \tag{2.11}$$

The result is sharp.

Corollary 2.7. *[92]Let the function $f(z)$ defined by* (2.1) *be in the class $VL(n,\widetilde{\lambda},A,B)$. Then $f'(z) \in U(0,r_2)$, where $r_2 = 1 + \frac{B-A}{C_2(1+B)}$.*

Theorem 2.20. *[92]Let the function $f(z)$ defined by* (2.1) *be in the class $VL(n,\widetilde{\lambda},A,B)$, with $arg(a_k) = \theta_k$, where $\theta_k \equiv \pi, \forall k \geq 2$. Define*

$$f_1(z) = z$$

and

$$f_k(z) = z - \frac{B-A}{kC_k(1+B)}z^k, \forall k \geq 2; z \in U.$$

Then $f(z) \in VL(n,\widetilde{\lambda},A,B)$ if and only if $f(z)$ can expressed by $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$, where $\mu_k \geq 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

If we combine the previous theorem with Silverman's theorem 5 from [111] we get the following corollary:

Corollary 2.8. *The closed convex hull of* $VL(n,\widetilde{\lambda},A,B)$ *is*

$$cl\ co\ VL(n,\widetilde{\lambda},A,B)=\left\{f\ |f\in\mathscr{A},\ \sum_{k=2}^{\infty}k^nC_k(1+B)|a_k|\le B-A\right\}.$$

The extreme points of $cl\ co\ VL(n,\widetilde{\lambda},A,B)$ *are*

$$\boldsymbol{E}(cl\ co\ VL(n,\lambda,A,B))=\left\{z+\frac{B-A}{k^nC_k(1+B)}\xi z^k,\ |\xi|=1,\ k\ge 2\right\}.$$

Theorem 2.21. *[92]If* $f\in VL(n,\widetilde{\lambda},A,B)$*, then* $L_cf\in VL(n,\widetilde{\lambda},A^*,B)$*, where*

$$A^*=\frac{B+A(c+1)}{c+2}>A.$$

The result is sharp.

Corollary 2.9. *[92] If* $f\in VL(n,\widetilde{\lambda},2\alpha-1,B)$ *then* $L_cf\in VL(n,\widetilde{\lambda},2\beta-1,B)$*, where*

$$\beta=\beta(\alpha)=\frac{B+1+2\alpha(c+1)}{2(c+2)}\ge\alpha.$$

The result is sharp.

Theorem 2.22. *[92]If* $f\in VL(n,\widetilde{\lambda},A,B)$*, then* $L_cf\in VL(n,\widetilde{\lambda},A,B^*)$*,where*

$$B^*=\frac{A(1+B)(c+2)+(B-A)(c+1)}{(1+B)(c+2)-(B-A)(c+1)}<B.$$

The result is sharp.

Theorem 2.23. *[93] If* $f\in VL(n,\widetilde{\lambda},A_1,B),g\in VL(n,\widetilde{\lambda},A_2,B)$ *then* $f\otimes g\in VL(n,\widetilde{\lambda},A^*,B)$*, where* $A^*=B-\dfrac{(B-A_1)(B-A_2)}{2C_2(1+B)}$.
The result is sharp.

Corollary 2.10. *[93] If* $f,g\in VL(n,\widetilde{\lambda},A,B)$ *then* $f\otimes g\in VL(n,\widetilde{\lambda},A^*,B)$*, where* $A^*=B-\dfrac{(B-A)^2}{2C_2(1+B)}$.
The result is sharp.

Theorem 2.24. *[93] If* $f\in VL(n,\widetilde{\lambda},A,B_1),g\in VL(n,\widetilde{\lambda},A,B_2)$ *then* $f\otimes g\in VL(n,\widetilde{\lambda},A,B^*)$*, where*

$$B^*=A+\frac{(B_1-A)(B_2-A)(A+1)}{2C_2(1+B_1)(1+B_2)-(B_1-A)(B_2-A)}.$$

The result is sharp.

Corollary 2.11. *[93] If* $f,g\in VL(n,\widetilde{\lambda},A,B)$ *then* $f\otimes g\in VL(n,\widetilde{\lambda},A,B^*)$*, where*

$$B^*=A+\frac{(B-A)^2(A+1)}{2C_2(1+B)^2-(B-A)^2}.$$

The result is sharp.

Theorem 2.25. *[93] If* $f_j \in VL(n,\widetilde{\lambda},A_j,B)$, $j=\overline{1,m}$, $m \in \{2,3,4,\ldots\}$ *then* $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VL(n,\widetilde{\lambda},A^{(m-1)*},B)$*, where* $A^{(m-1)*} = B - \dfrac{\prod_{j=1}^{m}(B-A_j)}{2^{m-1}C_2^{m-1}(1+B)^{m-1}}$. *The result is sharp.*

Theorem 2.26. *[93]If* $f_j \in VL(n,\widetilde{\lambda},A,B_j), j=\overline{1,m}, m \in \{2,3,4,\ldots\}$ *then* $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VL(n,\widetilde{\lambda},A,B^{(m-1)*})$*, where*

$$B^{(m-1)*} = A + \frac{(A+1)\prod_{j=1}^{m}(B_j-A)}{2^{m-1}C_2^{m-1}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j-A)}.$$

The result is sharp.

2.4 Properties of certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative

Definition 2.5. For $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ let $P(n,\lambda,A,B)$ denote the subclass of $\mathscr{A}$ which contain functions $f(z)$ of the form (2.1) such that

$$(1-\lambda)(\mathscr{S}\mathscr{R}^n f(z))' + \lambda(\mathscr{S}\mathscr{R}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{2.12}$$

Let $VP(n,\lambda,A,B)$ denote the subclass of V consisting of functions $f(z) \in P(n,\lambda,A,B)$.

Theorem 2.27. *[77] Let the function* $f(z)$ *defined by* (2.1) *be in V. Then* $f(z) \in VP(n,\lambda,A,B)$*, if and only if*

$$T(f) = \sum_{k=2}^{\infty} k^{n+1} C_k (1+B)|a_k| \leq B-A, \tag{2.13}$$

where

$$C_k = [n+1+\lambda(k-1)(n+k+1)]\frac{(n+k-1)!}{(n+1)!(k-1)!}.$$

The extremal functions are:

$$f(z) = z + \frac{B-A}{k^{n+1}C_k(1+B)} e^{i\theta_k} z^k, \forall k \geq 2.$$

Corollary 2.12. *[77]Let the function* $f(z)$ *defined by* (2.1) *be in the class* $VP(n,\lambda,A,B)$*. Then*

$$|a_k| \leq \frac{B-A}{k^{n+1}C_k(1+B)}, \forall k \geq 2.$$

The result (2.13) *is sharp for the functions*

$$f(z) = z + \frac{B-A}{k^{n+1}C_k(1+B)} e^{i\theta_k} z^k, \forall k \geq 2.$$

Theorem 2.28. *[77] Let the function $f(z)$ defined by* (2.1) *be in the class $VP(n,\lambda,A,B)$. Then*

$$|z| - \frac{B-A}{2^{n+1}C_2(1+B)}|z|^2 \le |f(z)| \le |z| + \frac{B-A}{2^{n+1}C_2(1+B)}|z|^2. \tag{2.14}$$

The result is sharp.

Corollary 2.13. *[77]Let the function $f(z)$ defined by* (2.1) *be in the class $VP(n,\lambda,A,B)$. Then $f(z) \in U(0,r_1)$, where $r_1 = 1 + \frac{B-A}{2^{n+1}C_2(1+B)}$.*

Theorem 2.29. *[77] Let the function $f(z)$ defined by* (2.1) *be in the class $VP(n,\lambda,A,B)$. Then*

$$1 - \frac{B-A}{2^{n}C_2(1+B)}|z| \le \left|f'(z)\right| \le 1 + \frac{B-A}{2^{n}C_2(1+B)}|z|. \tag{2.15}$$

The result is sharp.

Corollary 2.14. *[77]Let the function $f(z)$ defined by* (2.1) *be in the class $VP(n,\lambda,A,B)$. Then $f'(z) \in U(0,r_2)$, where $r_2 = 1 + \frac{B-A}{2^{n}C_2(1+B)}$.*

Theorem 2.30. *[77] Let the function $f(z)$ defined by* (2.1) *be in the class $VP(n,\lambda,A,B)$, with $arg(a_k) = \theta_k$ where $\theta_k \equiv \pi, \forall k \ge 2$. Define*

$$f_1(z) = z$$

and

$$f_k(z) = z - \frac{B-A}{k^{n+1}C_k(1+B)}z^k, (k \ge 2; z \in U).$$

Then $f(z) \in VP(n,\lambda,A,B)$ if and only if $f(z)$ can expressed by $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$, where $\mu_k \ge 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Corollary 2.15. *[77] Let $VP_\pi(n,\lambda,A,B) = VP(n,\lambda,A,B) \cap V(\pi,0)$. The extreme points of $VP_\pi(n,\lambda,A,B)$ are*

$$f_1(z) = z \ \text{ and } \ f_k(z) = z - \frac{B-A}{k^{n+1}C_k(1+B)}z^k, \ (k \ge 2, \ z \in U).$$

If we combine theorem 2.30 with Silverman's theorem 5 from [111] we get the following corollary:

Corollary 2.16. *[77] The closed convex hull of $VP(n,\lambda,A,B)$ is*

$$cl\ co\ VP(n,\lambda,A,B) = \left\{ f\ |f \in \mathcal{A},\ \sum_{k=2}^{\infty} k^{n+1}C_k(1+B)|a_k| \le B-A \right\}.$$

The extreme points of $cl\ co\ VP(n,\lambda,A,B)$ are

$$\boldsymbol{E}(cl\ co\ VP(n,\lambda,A,B)) = \left\{ z + \frac{B-A}{k^{n+1}C_k(1+B)}\xi z^k,\ |\xi| = 1,\ k \ge 2 \right\}.$$

Theorem 2.31. *[77] If $f \in VP(n,\lambda,2\alpha-1,B)$ then $L_c f \in VP(n,\lambda,2\beta-1,B)$, where*

$$\beta = \beta(\alpha) = \frac{B+1+2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

The result is sharp.

Theorem 2.32. *[77] If $f \in VP(n,\lambda,A,B)$ then $L_c f \in VP(n,\lambda,A^*,B)$, where*

$$A^* = \frac{B+A(c+1)}{c+2} > A.$$

The result is sharp.

Theorem 2.33. *[77] If $f \in VP(n,\lambda,A,B)$ then $L_c f \in VP(n,\lambda,A,B^*)$,where*

$$B^* = \frac{A(1+B)(c+2)+(B-A)(c+1)}{(1+B)(c+2)-(B-A)(c+1)} < B.$$

The result is sharp.

Theorem 2.34. *[85] If $f \in VP(n,\lambda,A_1,B), g \in VP(n,\lambda,A_2,B)$ then $f \otimes g \in VP(n,\lambda,A^*,B)$, where*

$$A^* = B - \frac{(B-A_1)(B-A_2)}{2^{n+1}C_2(1+B)}.$$

The result is sharp.

Corollary 2.17. *[85] If $f, g \in VP(n,\lambda,A,B)$ then $f \otimes g \in VP(n,\lambda,A^*,B)$, where*

$$A^* = B - \frac{(B-A)^2}{2^{n+1}C_2(1+B)}.$$

The result is sharp.

Theorem 2.35. *[85] If $f \in VP(n,\lambda,A,B_1), g \in VP(n,\lambda,A,B_2)$ then $f \otimes g \in VP(n,\lambda,A,B^*)$, where*

$$B^* = A + \frac{(B_1-A)(B_2-A)(A+1)}{2^{n+1}C_2(1+B_1)(1+B_2)-(B_1-A)(B_2-A)}.$$

The result is sharp.

Corollary 2.18. *[85] If $f, g \in VP(n,\lambda,A,B)$ then $f \otimes g \in VP(n,\lambda,A,B^*)$, where*

$$B^* = A + \frac{(B-A)^2(A+1)}{2^{n+1}C_2(1+B)^2-(B-A)^2}.$$

The result is sharp.

Theorem 2.36. *[85] If $f_j \in VP(n,\lambda,A_j,B), j = \overline{1,s}, s \in \{2,3,4,\ldots\}$ then $f_1 \otimes f_2 \otimes \ldots \otimes f_s \in VP(n,\lambda,A^{(s-1)*},B)$, where*

$$A^{(s-1)*} = B - \frac{\prod_{j=1}^{s}(B-A_j)}{2^{(n+1)(s-1)}C_2^{s-1}(1+B)^{s-1}}.$$

The result is sharp.

Theorem 2.37. *[85] If* $f_j \in VP(n,\lambda,A,B_j), j=\overline{1,s}, s \in \{2,3,4,\dots\}$ *then* $f_1 \otimes f_2 \otimes \dots \otimes f_s \in VP(n,\lambda,A,B^{(s-1)*})$*, where*

$$B^{(s-1)*} = A + \frac{(A+1)\prod_{j=1}^{s}(B_j - A)}{2^{(s-1)(n+1)}C_2^{s-1}\prod_{j=1}^{s}(1+B_j) - \prod_{j=1}^{s}(B_j - A)}.$$

The result is sharp.

Chapter 3

Analytic functions

3.1 Coefficient bounds and Fekete-Szegő problem for new classes of analytic functions defined by Sălăgean integro-differential operator

In the following definitions, new classes of analytic functions containing the new operator (1.7) are introduced:

Definition 3.1. Let $f \in \mathscr{A}$. Then $f(z)$ is in the class $S^n(\mu)$ if and only if

$$\Re\left(\frac{z(\mathscr{D}I^n f(z))'}{\mathscr{D}I^n f(z)}\right) > \mu, \quad 0 \le \mu < 1, z \in U. \tag{3.1}$$

Definition 3.2. Let $f \in \mathscr{A}$. Then $f(z)$ is in the class $C^n(\mu)$ if and only if

$$\Re\left(\frac{\left[z(\mathscr{D}I^n f(z))'\right]'}{(\mathscr{D}I^n f(z))'}\right) > \mu, \quad 0 \le \mu < 1, z \in U. \tag{3.2}$$

Definition 3.3. [23] Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ be an univalent starlike function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$. The class $S^*(\varphi)$ consists of all functions $f \in \mathscr{A}$ satisfying the following subordination:

$$\frac{z(\mathscr{D}I^n f(z))'}{\mathscr{D}I^n f(z)} \prec \varphi(z), \tag{3.3}$$

and $C(\varphi)$ be the class of functions $f \in \mathscr{A}$ for which

$$\frac{\left[z(\mathscr{D}I^n f(z))'\right]'}{(\mathscr{D}I^n f(z))'} \prec \varphi(z). \tag{3.4}$$

Remark 3.1. *If* $\varphi_\mu(z) = \dfrac{1+(1-2\mu)z}{1-z}$ *then* $\mathscr{S}^n(\mu) = S^*(\varphi_\mu)$ *and* $\mathscr{C}^n(\mu) = C(\varphi_\mu)$.

Theorem 3.1. *[78] Let the function $f(z)$ defined by* (2.1) *be in $\mathscr{A}$. If*

$$\sum_{k=2}^{\infty}\left(k-\mu\right)\left[k^{n}\left(1-\lambda\right)+\lambda\frac{1}{k^{n}}\right]|a_k|\leq 1-\mu, \tag{3.5}$$

then $f(z)\in\mathscr{S}^{n}\left(\mu\right)$. The result (3.5) *is sharp.*

Corollary 3.1. *If* (3.5) *holds true, then*

$$|a_k|\leq\frac{1-\mu}{\left(k-\mu\right)\left[k^{n}\left(1-\lambda\right)+\lambda\frac{1}{k^{n}}\right]},\forall k\geq 2. \tag{3.6}$$

Theorem 3.2. *[78] Let the function $f(z)$ defined by* (2.1) *be in $\mathscr{A}$. If*

$$\sum_{k=2}^{\infty}\left(k-\mu\right)\left[k^{n+1}\left(1-\lambda\right)+\lambda\frac{1}{k^{n-1}}\right]|a_k|\leq 1-\mu, \tag{3.7}$$

then $f(z)\in\mathscr{C}^{n}\left(\mu\right)$. The result (3.7) *is sharp.*

Corollary 3.2. *If* (3.7) *holds true, then*

$$|a_k|\leq\frac{1-\mu}{\left(k-\mu\right)\left[k^{n+1}\left(1-\lambda\right)+\lambda\frac{1}{k^{n-1}}\right]},\forall k\geq 2. \tag{3.8}$$

Theorem 3.3. *[78] If* (3.5) *holds true, then*

$$|z|-\frac{1-\mu}{2-\mu}|z|^2\leq\left|\mathscr{D}I^{n}f(z)\right|\leq|z|+\frac{1-\mu}{2-\mu}|z|^2,\quad\forall z\in U,0\leq\mu<1.$$

Theorem 3.4. *[78] If* (3.7) *holds true, then*

$$|z|-\frac{1-\mu}{2\left(2-\mu\right)}|z|^2\leq\left|\mathscr{D}I^{n}f(z)\right|\leq|z|+\frac{1-\mu}{2\left(2-\mu\right)}|z|^2,\quad\forall z\in U,0\leq\mu<1.$$

Theorem 3.5. *[78] If* (3.5) *holds true, then*

$$|z|-\frac{1-\mu}{\left(2-\mu\right)\left[2^{n}\left(1-\lambda\right)+\lambda\frac{1}{2^{n}}\right]}|z|^2\leq|f(z)|\leq|z|+\frac{1-\mu}{\left(2-\mu\right)\left[2^{n}\left(1-\lambda\right)+\lambda\frac{1}{2^{n}}\right]}|z|^2,$$

$\forall z\in U,0\leq\mu<1.$

Theorem 3.6. *[78] If* (3.7) *holds true, then*

$$|z|-\frac{1-\mu}{2\left(2-\mu\right)\left[2^{n}\left(1-\lambda\right)+\lambda\frac{1}{2^{n}}\right]}|z|^2\leq|f(z)|\leq|z|+\frac{1-\mu}{2\left(2-\mu\right)\left[2^{n}\left(1-\lambda\right)+\lambda\frac{1}{2^{n}}\right]}|z|^2,$$

$\forall z\in U,0\leq\mu<1.$

Many authors obtained Fekete-Szegő inequalities for different classes of functions (see [100],[23],[32],[109],[11]).

Next we determine the upper bound for $|a_2|$ for the classes $\mathcal{S}^n(\mu)$ and $\mathcal{C}^n(\mu)$, that is sharp. Also, we calculate the Fekete-Szegő $|a_3 - \xi a_2|$ functional for the above classes.

Lemma 3.1. *[29] Let $p \in \mathcal{P}$ (the class $\mathcal{P}$ is a Carathéodory class of functions which are analytic with positive real part in U) be of the form $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2} \text{ and } |c_k| \leq 2, \quad \forall k \in \mathbb{N}.$$

Lemma 3.2. *[56] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots, z \in U$ is a function with positive real part in U and ξ is a complex number, then*

$$\left| c_2 - \xi c_1^2 \right| \leq 2 \max\{1; |2\xi - 1|\}.$$

The result is sharp for the function given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}, \ z \in U.$$

Theorem 3.7. *[78] Let $0 \leq \mu < 1$ and $\varphi = \varphi_\mu$. If $f(z)$ given by (2.1) belongs to the class $\mathcal{S}^n(\mu)$, then*

$$|a_2| \leq \frac{B_1}{2^n (1-\lambda) + \lambda \frac{1}{2^n}}$$

and $\forall \xi \in \mathbb{C}$

$$\left| a_3 - \xi a_2^2 \right| \leq$$

$$\leq \frac{B_1}{4\left[3^n (1-\lambda) + \lambda \frac{1}{3^n}\right]} max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{B_1}{2^n (1-\lambda) + \lambda \frac{1}{2^n}} \left(2 \cdot \frac{3^n (1-\lambda) + \lambda \frac{1}{3^n}}{2^n (1-\lambda) + \lambda \frac{1}{2^n}} \xi - 1 \right) \right| \right\}.$$

The result is sharp.

Theorem 3.8. *[78] Let $0 \leq \mu < 1$ and $\varphi = \varphi_\mu$. If $f(z)$ given by (2.1) belongs to the class $\mathcal{C}^n(\mu)$, then*

$$|a_2| \leq \frac{B_1}{2^{n+2} (1-\lambda) + \lambda \frac{1}{2^{n-2}}}$$

and $\forall \xi \in \mathbb{C}$

$$\left| a_3 - \xi a_2^2 \right| \leq$$

$$\leq \frac{B_1}{4\left[3^{n+1} (1-\lambda) + \lambda \frac{1}{3^{n-1}}\right]} max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{B_1}{2^{n+1} (1-\lambda) + \lambda \frac{1}{2^{n-1}}} \left(2 \cdot \frac{3^{n+1} (1-\lambda) + \lambda \frac{1}{3^{n-1}}}{2^{n+1} (1-\lambda) + \lambda \frac{1}{2^{n-1}}} \xi - 1 \right) \right| \right\}.$$

The result is sharp.

Chapter 4

Differential subordinations and superordinations

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be a function and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

(i) $\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z)$, $(z \in U)$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (i).
A dominant $\widetilde{q}$, which satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). The best dominant is unique up to a rotation of U. In order to prove the original results we use the following lemmas.

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\}$$

and $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ be a function and let h be univalent in U and $q \in \mathcal{Q}$. In article [57] it is studied the problem of determining conditions on admissible function ψ such that

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \ (z \in U) \tag{4.1}$$

(second-order) differential -subordination, implies $p(z) \prec q(z)$, $\forall p \in \mathcal{H}[a, n]$. The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (4.1).
A dominant $\widetilde{q}$, which is the "smallest" function with this property and satisfies $\widetilde{q} \prec q$ for all dominants q of (4.1) is said to be the best dominant of (4.1). The best dominant is unique up to a rotation of U.

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ be a function and let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a,n]$. If p and $\varphi\left(p(z), zp'(z), z^2p''(z); z\right)$ are univalent in U and satisfy the (second-order) differential superordination

$$h(z) \prec \varphi\left(p(z), zp'(z), z^2p''(z); z\right),\ (z \in U) \tag{4.2}$$

then p is called a solution of the differential superordination. In [59] the authors studied the dual problem of determining properties of functions p that satisfy the differential superordination (4.2). The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all p satisfying (4.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (4.2) is said to be the best subordinant of (4.2) and is the "largest" function with this property. The best subordinant is unique up to a rotation of U.

Lemma 4.1. *[41] (Hallenbeck and Ruscheweyh) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\Re\gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z),\ \ z \in U$$

then

$$p(z) \prec q(z) \prec h(z), z \in U$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt,\ \ z \in U.$$

Lemma 4.2. *[58] (Miller and Mocanu) Let q be a convex function in U and let*

$$h(z) = q(z) + n\alpha z q'(z),\ \ z \in U$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = q(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots,\ \ z \in U$$

is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z),\ \ z \in U$$

then

$$p(z) \prec q(z)$$

and this result is sharp.

Lemma 4.3. *[24]Let q be an univalent function in U and $\gamma \in \mathbb{C}^*$ such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If p is an analytic function in U, with $p(0) = q(0)$ and

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z), \tag{4.3}$$

then $p(z) \prec q(z)$ and q is the best dominant of (4.3).

Lemma 4.4. *[24]Let q be convex function in U, with $q(a) = 0$ and $\gamma \in \mathbb{C}$ such that $\Re\, \gamma > 0$. If $p \in \mathscr{H}[a,1] \cap \mathscr{Q}$ and $p(z)+\gamma z p'(z)$ is univalent in U, then*

$$q(z)+\gamma z q'(z) \prec p(z)+\gamma z p'(z) \Rightarrow q(z) \prec p(z)$$

and q is the best subordinant.

S. S. Miller and P. T. Mocanu obtained special results related to differential subordinations in [58] .

Lemma 4.5. *[4] If $p(z)$ is analytic in $U, p(0) = 1$ and $\Re\big(p(z)\big) > \frac{1}{2}, z \in U$, then for any function F analytic in U, the function $p * F$ takes its values in the convex hull of $F(U)$.*

4.1 On a class of univalent functions defined by Sălăgean integro-differential operator

We recall the $\mathscr{D}I^n$ Sălăgean integro-differential operator defined in (1.7).

Theorem 4.1. *[86] Let q be a convex function, $q(0) = 1$ and let h be the function*

$$h(z) = q(z)+zq'(z), z \in U.$$

If $f \in \mathscr{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ and satisfies the differential subordination

$$\left[\mathscr{D}I^n f(z)\right]' \prec h(z), \quad z \in U \tag{4.4}$$

then

$$\frac{\mathscr{D}I^n f(z)}{z} \prec q(z), \quad z \in U$$

and this result is sharp.

Remark 4.1. *If $\lambda = 0$ we get Theorem 4 from Oros [70] and for $\lambda = 1$ we get Theorem 4 from Bălăeţi [15].*

Theorem 4.2. *[86] Let q be a convex function, $q(0) = 1$ and let h be the function*

$$h(z) = q(z)+zq'(z), z \in U.$$

If $f \in \mathscr{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ and satisfies the differential subordination

$$\left(\frac{z\mathscr{D}I^{n+1} f(z)}{\mathscr{D}I^n f(z)}\right)' \prec h(z), \quad z \in U \tag{4.5}$$

then

$$\frac{\mathscr{D}I^{n+1} f(z)}{\mathscr{D}I^n f(z)} \prec q(z), \quad z \in U$$

and this result is sharp.

Theorem 4.3. *[86] Let q be a convex function, $q(0)=1$ and let h be the function*

$$h(z)=q(z)+zq'(z), z\in U.$$

If $f\in\mathscr{A}$, $\lambda\geq 0$, $n\in\mathbb{N}$ and satisfies the differential subordination

$$\left(\mathscr{D}I^{n+1}f(z)\right)'+\lambda\left[\left(I^{n-1}f(z)\right)'-\left(I^{n+1}f(z)\right)'\right]\prec h(z),\quad z\in U \tag{4.6}$$

then

$$\left[\mathscr{D}I^{n}f(z)\right]'\prec q(z),\quad z\in U$$

and this result is sharp.

Remark 4.2. *If $\lambda=0$ we get Theorem 2 from Oros [70] and for $\lambda=1$ we get Theorem 2 from Bălăeţi [15].*

Theorem 4.4. *[86]Let $h\in\mathscr{H}(U)$ such that $h(0)=1$ and*

$$\Re\left[1+\frac{zh''(z)}{h'(z)}\right]>-\frac{1}{2},\quad z\in U.$$

If $f\in\mathscr{A}$ satisfies the differential subordination

$$\left(\mathscr{D}I^{n+1}f(z)\right)'+\lambda\left[\left(I^{n-1}f(z)\right)'-\left(I^{n+1}f(z)\right)'\right]\prec h(z),\quad z\in U \tag{4.7}$$

then

$$\left[\mathscr{D}I^{n}f(z)\right]'\prec q(z),\quad z\in U$$

where q is given by $q(z)=\frac{1}{z}\int_0^z h(t)dt$. The function q is convex and is the best dominant.

Remark 4.3. *If $\lambda=0$ we get Theorem 3 from Oros [70].*

Theorem 4.5. *[86]Let $h\in\mathscr{H}(U)$ such that $h(0)=1$ and*

$$\Re\left[1+\frac{zh''(z)}{h'(z)}\right]>-\frac{1}{2},\quad z\in U.$$

If $f\in\mathscr{A}$ satisfies the differential subordination

$$\left[\mathscr{D}I^{n}f(z)\right]'\prec h(z),\quad z\in U \tag{4.8}$$

then

$$\frac{\mathscr{D}I^{n}f(z)}{z}\prec q(z),\quad z\in U$$

where q is given by $q(z)=\frac{1}{z}\int_0^z h(t)dt$. The function q is convex and is the best dominant.

Remark 4.4. *If $\lambda=0$ we get Theorem 5 from Oros [70] and for $\lambda=1$ we get Theorem 5 from Bălăeţi [15].*

Definition 4.1. [69], [125], [15], [70]If $0 \leq \beta < 1$ and $n \in \mathbb{N}$, we let $L_n^m(\beta)$ stand for the class of functions $f \in \mathscr{A}_m$, which satisfy the inequality

$$\Re\left[\mathscr{D}I^n f(z)\right]' > \beta, \quad (z \in U).$$

Remark 4.5. *For $n = 0$ we obtain $\Re f'(z) > \beta$.*

Theorem 4.6. *[86]The set $L_n^m(\beta)$ is convex.*

Theorem 4.7. *[86]If $0 \leq \beta < 1$ and $m, n \in \mathbb{N}$ then we have*

$$L_n^m(\beta) \subset L_{n+1}^m(\delta),$$

where $\delta(\beta, m) = 2\beta - 1 + 2(1-\beta)\frac{1}{m}\sigma\left(\frac{1}{m}\right)$ and $\sigma(x) = \int_0^z \frac{t^{x-1}}{1+t}dt$. The result is sharp.

Remark 4.6. *If $\lambda = 0$ we get Theorem 1 from Oros [70] and for $\lambda = 1$ we get Theorem 1 from Bălăeţi [15].*

Theorem 4.8. *[86]Let q be a convex function in U with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{1}{c+2} z q'(z), z \in U,$$

where c is a complex number, with $\Re c > -2$.
If $f \in L_n^m(\beta)$ and $F = I_c(f)$, where

$$F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt, \quad \Re c > -2, \tag{4.9}$$

then

$$\left[\mathscr{D}I^n f(z)\right]' \prec h(z), \quad z \in U, \tag{4.10}$$

implies

$$\left[\mathscr{D}I^n F(z)\right]' \prec q(z), \quad z \in U,$$

and this result is sharp.

Remark 4.7. *If $\lambda = 0$ we get Theorem 2.2 from Tăut et al. [125].*

4.2 Differential subordination results obtained by using a new operator

Next we use the Definitions 1.17, 1.18, 1.19 from Chapter 1.

Definition 4.2. Let $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, \alpha, \beta \geq 0$. Denote by $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}$ the operator given by $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = (1-\alpha-\beta)\mathscr{R}^\nu \mathscr{D}^n f(z) + \alpha \mathscr{R}^\nu \Omega_z^\lambda f(z) + \beta \mathscr{D}^n \Omega_z^\lambda f(z),$$

for $z \in U$.

Remark 4.8. *$\mathscr{R}^\nu \mathscr{D}^n f(z)$ is the composition of the Sălăgean operator and the Ruscheweyh derivative, $\mathscr{R}^\nu \Omega_z^\lambda f(z)$ is the composition of fractional differintegral operator and the Ruscheweyh derivative, and $\mathscr{D}^n \Omega_z^\lambda f(z)$ is the composition of fractional differintegral operator and the Sălăgean operator.*

Remark 4.9. *If $f \in \mathscr{A}, f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}$, then*

(4.11)
$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = z + \sum_{k=1}^{\infty} \left((1-\alpha-\beta)\frac{(\nu+1)_k}{(2)_k}(k+1)^{n+1} + \alpha\frac{(\nu+1)_k}{(2-\lambda)_k}(k+1) + \beta\frac{(1)_k}{(2-\lambda)_k}(k+1)^{n+1}\right) a_{k+1} z^{k+1},$$

for $z \in U$.

Remark 4.10. *$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = (1-\alpha-\beta)\mathbb{D}_0^{\nu,n} f(z) + \alpha \mathbb{D}_\lambda^{\nu,0} f(z) + \beta \mathbb{D}_\lambda^{0,n} f(z)$, for $z \in U$, where $\mathbb{D}_\lambda^{\nu,n}$ is defined in (1.23).*

Remark 4.11. *For $\alpha = 0$ and $\beta = 0$, we obtain $\mathscr{D}_{0,0}^{\lambda,\nu,n} f(z) = \mathscr{R}^\nu \mathscr{D}^n f(z)$, where $z \in U$.*

For $\alpha = 1$ and $\beta = 0$, we obtain $\mathscr{D}_{1,0}^{\lambda,\nu,n} f(z) = \mathscr{R}^\nu \Omega_z^\lambda f(z)$, where $z \in U$.

For $\alpha = 0$ and $\beta = 1$, we obtain $\mathscr{D}_{0,1}^{\lambda,\nu,n} f(z) = \mathscr{D}^n \Omega_z^\lambda f(z)$, where $z \in U$.

For $\beta = 0$ and $\nu = 0$, we obtain $\mathscr{D}_{\alpha,0}^{\lambda,0,n} f(z) = (1-\alpha)\mathscr{D}^n f(z) + \alpha \Omega_z^\lambda f(z)$, where $z \in U$.

For $\alpha = 0$ and $n = 0$, we obtain $\mathscr{D}_{0,\beta}^{\lambda,\nu,0} f(z) = (1-\beta)\mathscr{R}^\nu f(z) + \beta \Omega_z^\lambda f(z)$, where $z \in U$.

For $\alpha + \beta = 1$ and $\lambda = 0$, we obtain $\mathscr{D}_{1-\beta,\beta}^{0,\nu,n} f(z) = (1-\beta)\mathscr{R}^\nu f(z) + \beta \mathscr{D}^n f(z)$, where $z \in U$.

For $\alpha + \beta = 1, \lambda = 0$ and $\nu = n$, we obtain $\mathscr{D}_{1-\beta,\beta}^{0,n,n} f(z) = (1-\beta)\mathscr{R}^n f(z) + \beta \mathscr{D}^n f(z), z \in U$. This operator was introduced and studied in [6].

For $\alpha = \beta = n = 0$, we obtain $\mathscr{D}_{0,0}^{\lambda,\nu,0} f(z) = \mathscr{R}^\nu f(z)$, and for $\beta = \lambda = n = 0$, we obtain $\mathscr{D}_{\alpha,0}^{0,\nu,0} f(z) = \mathscr{R}^\nu f(z)$, where $z \in U$.

For $\alpha = \beta = \nu = 0$, we obtain $\mathscr{D}_{0,0}^{\lambda,0,n} f(z) = \mathscr{D}^n f(z)$, and for $\alpha = \lambda = \nu = 0$, we obtain $\mathscr{D}_{0,\beta}^{0,0,n} f(z) = \mathscr{D}^n f(z)$, where $z \in U$.

For $\alpha = 0$ and $\lambda = \nu = 1$, we obtain $\mathscr{D}_{0,\beta}^{1,1,n} f(z) = \mathscr{D}^{n+1} f(z)$, where $z \in U$.

For $\alpha = 1$ and $\beta = \nu = 0$, we obtain $\mathscr{D}_{1,0}^{\lambda,0,n} f(z) = \Omega_z^\lambda f(z)$ and for $\alpha = n = 0$ and $\beta = 1$, we obtain $\mathscr{D}_{0,1}^{\lambda,\nu,0} f(z) = \Omega_z^\lambda f(z)$, where $z \in U$.

For $\lambda = \nu = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{0,0,n} f(z) = (1-\alpha)\mathscr{D}^n f(z) + \alpha f(z)$, where $z \in U$.

For $\lambda = n = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{0,\nu,0} f(z) = (1-\beta)\mathscr{R}^\nu f(z) + \beta f(z)$, where $z \in U$.

For $\nu = n = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{\lambda,0,0} f(z) = (1-\alpha-\beta) f(z) + (\alpha+\beta)\Omega_z^\lambda f(z)$, where $z \in U$.

For $\lambda = 0$ and $\nu = 1$, we obtain $\mathscr{D}_{\alpha,\beta}^{0,1,n} f(z) = (1-\alpha-\beta)\mathscr{D}^{n+1} f(z) + \alpha \mathscr{D}^1 f(z) + \beta \mathscr{D}^n f(z)$, where $z \in U$.

For $\lambda = 1$ and $\nu = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{1,0,n} f(z) = (1-\alpha-\beta)\mathscr{D}^n f(z) + \alpha\mathscr{D}^1 f(z) + \beta\mathscr{D}^{n+1} f(z)$, where $z \in U$.

For $\lambda = \nu = 1$, we obtain $\mathscr{D}_{\alpha,\beta}^{1,1,n} f(z) = (1-\alpha)\mathscr{D}^{n+1} f(z) + \alpha\mathscr{D}^2 f(z)$, where $z \in U$.

For $\lambda = \nu = n = 0$, we obtain $\mathscr{D}_{\alpha,\beta}^{0,0,0} f(z) = f(z)$, for $\alpha = \beta = \nu = n = 0$, we obtain $\mathscr{D}_{0,0}^{\lambda,0,0} f(z) = f(z)$, for $\alpha = 1$ and $\lambda = \nu = 0$, we obtain $\mathscr{D}_{1,\beta}^{0,0,n} f(z) = f(z)$, and for $\beta = 1$ and $\lambda = n = 0$, we obtain $\mathscr{D}_{\alpha,1}^{0,\nu,0} f(z) = f(z)$, for $z \in U$.

Definition 4.3. Let $f \in \mathscr{A}$. We say that the function f is in the class $\mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$, where $0 \le \delta \le 1, \alpha, \beta \ge 0, -\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$, if f satisfies the condition

$$\Re(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z))' > \delta, z \in U. \tag{4.12}$$

Theorem 4.9. *[82] Let $f \in \mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$ and $g \in K$, where K denote the class of convex functions. Then $f * g \in \mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$.*

Theorem 4.10. *[82] The set $\mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$ is convex.*

Theorem 4.11. *[82] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathscr{A}$ verifies the differential subordination

$$\left(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)\right)' \prec h(z), z \in U \tag{4.13}$$

then

$$\frac{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)}{z} \prec g(z), z \in U.$$

The result is sharp.

Theorem 4.12. *[82] Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathscr{A}$ verifies the differential subordination

$$\left(\frac{z\mathscr{D}_{\alpha,\beta}^{\lambda,\nu+1,n} f(z)}{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)}\right)' \prec h(z), z \in U, \tag{4.14}$$

then

$$\frac{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu+1,n} f(z)}{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)} \prec g(z), z \in U.$$

The result is sharp.

Theorem 4.13. *[82] Let g be a convex function,* $g(0)=1$ *and let h be a function such that*

$$h(z)=g(z)+zg'(z), z\in U.$$

If $f\in\mathscr{A}$ *verifies the differential subordination*

$$\left(\frac{z\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)}{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)}\right)'\prec h(z), z\in U, \tag{4.15}$$

then

$$\frac{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)}{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)}\prec g(z), z\in U.$$

The result is sharp.

Theorem 4.14. *[82] Let g be a convex function,* $g(0)=0$ *and let h be a function such that*

$$h(z)=g(z)+zg'(z), z\in U.$$

If $f\in\mathscr{A}$ *verifies the differential subordination*

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)+\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)+\alpha\left(\mathscr{D}\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)-\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)\right)\prec h(z), z\in U, \tag{4.16}$$

then

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)\prec g(z), z\in U.$$

The result is sharp.

Theorem 4.15. *[82] Let* $h(z)=\dfrac{1+(2\delta-1)z}{1+z}$ *be a convex function in U, where* $0\le\delta<1$. *If* $f\in\mathscr{A}$ *satisfies the differential subordination*

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)+\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)+\alpha\left(\mathscr{D}\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)-\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)\right)\prec h(z), z\in U, \tag{4.17}$$

then

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)\prec g(z), z\in U,$$

where g is given by $g(z)=2\delta-1+2(1-\delta)\dfrac{\ln(1+z)}{z}$, $z\in U$. *The function g is convex and is the best dominant.*

Theorem 4.16. *[82] Let g be a convex function,* $g(0)=1$ *and let h be a function such that*

$$h(z)=g(z)+zg'(z), z\in U.$$

If $f\in\mathscr{A}$ *verifies the differential subordination*

$$\frac{1}{z}\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+2}f(z)+\frac{1}{z}\alpha\left(\mathscr{D}^2\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)-\mathscr{D}_{1,0}^{\lambda,\nu,n}f(z)\right)\prec h(z), z\in U, \tag{4.18}$$

then

$$\left(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)\right)'\prec g(z), z\in U.$$

The result is sharp.

Theorem 4.17. *[82] Let* $h(z) = \dfrac{1+(2\delta-1)z}{1+z}$ *be a convex function in* U*, where* $0 \le \delta < 1$*. If* $f \in \mathscr{A}$ *satisfies the differential subordination*

$$\frac{1}{z}\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+2} f(z) + \frac{1}{z}\alpha\left(\mathscr{D}^2\mathscr{D}_{1,0}^{\lambda,\nu,n} f(z) - \mathscr{D}_{1,0}^{\lambda,\nu,n} f(z)\right) \prec h(z), \tag{4.19}$$

then

$$\left(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)\right)' \prec g(z), z \in U,$$

where g *is given by* $g(z) = 2\delta - 1 + 2(1-\delta)\dfrac{\ln(1+z)}{z}$*,* $z \in U$*. The function* g *is convex and is the best dominant.*

4.3 Differential subordinations and superordinations for analytic functions defined by Sălăgean integro-differential operator

Theorem 4.18. *[81]Let* q *be an univalent function in* U *with* $q(0) = 1$*,* $\gamma \in \mathbb{C}^*$ *such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \ge \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathscr{A}$ *and*

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^n f(z)} + \gamma\left\{1 - \frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z) + \lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^n f(z)]^2} + \right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z) - \mathscr{D}^n f(z)\right]}{\mathscr{D}I^n f(z)}\right\} \prec q(z) + \gamma z q'(z),$$

(4.20)

then

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^n f(z)} \prec q(z) \tag{4.21}$$

and q *is the best dominant of* (4.20).

In the particular case $\lambda = 0$ and $n = 0$ we obtain:

Corollary 4.1. *[81] Let* q *be an univalent function in* U *with* $q(0) = 1$*,* $\gamma \in \mathbb{C}^*$ *such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \ge \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathscr{A}$ *and*

$$(1+\gamma)\frac{zf'(z)}{f(z)} + \gamma\left[\frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2\right] \prec q(z) + \gamma z q'(z)$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

In the particular case $\lambda = 0$ and $n = 1$, we obtain:

Corollary 4.2. *[81] Let q be an univalent function in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathscr{A}$ and

$$1 + \left(1 + 3\gamma\right)\frac{zf''(z)}{f'(z)} + \gamma\left[1 - \left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + \frac{z^2 f'''(z)}{f'(z)}\right] \prec q(z) + \gamma z q'(z)$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec q(z)$$

and q is the best dominant.

When $\lambda = 1$ we get the Cotîrlă's result [27]:

We select in Theorem 4.18 a particular dominant q.

Corollary 4.3. *[81] Let $A, B, \gamma \in \mathbb{C}, A \neq B$ such that $|B| \leq 1$ and $\Re\,\gamma > 0$. If for $f \in \mathscr{A}$*

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^n f(z)} + \gamma\left\{1 - \frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z) + \lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^n f(z)]^2} + \right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z) - \mathscr{D}^n f(z)\right]}{\mathscr{D}I^n f(z)}\right\} \prec \frac{1 + Az}{1 + Bz} + \gamma\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{\mathscr{D}I^n f(z)}{\mathscr{D}I^{n+1}f(z)} \prec \frac{1 + Az}{1 + Bz}$$

and $q(z) = \dfrac{1 + Az}{1 + Bz}$ is the best dominant.

Theorem 4.19. *[81] Let q be a convex function in U with $q(0) = 1$ and $\gamma \in \mathbb{C}$ such that $\Re\,\gamma > 0$. If $f \in \mathscr{A}$,*

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^n f(z)} \in \mathscr{H}[1,1] \cap \mathscr{Q},$$

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^n f(z)} + \gamma\left\{1 - \frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z) + \lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^n f(z)]^2} + \right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z) - \mathscr{D}^n f(z)\right]}{\mathscr{D}I^n f(z)}\right\}$$

is univalent in U and

$$q(z)+\gamma zq'(z)\prec \frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^{n}f(z)]^2}+\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{D}I^{n}f(z)}\right\},$$

(4.22)

then $q(z)\prec \frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}$ *and q is the best subordinant .*

From the combination of Theorem 4.18 and Theorem 4.19 we get the following "sandwich-type theorem".

Theorem 4.20. *[81] Let q_1 and q_2 be convex functions in U with $q_1(0)=q_2(0)=1$, $\gamma\in\mathbb{C}$ such that $\Re\,\gamma>0$. If $f\in\mathscr{A}$,*

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}\in\mathscr{H}[1,1]\cap\mathscr{Q},$$

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^{n}f(z)]^2}+\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{D}I^{n}f(z)}\right\}$$

is univalent in U and

$$q_1(z)+\gamma zq_1'(z)\prec \frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^{n}f(z)]^2}+\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{D}I^{n}f(z)}\right\}\prec q_2(z)+\gamma zq_2'(z),$$

(4.23)

then $q_1(z)\prec \frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}\prec q_2(z)$*, q_1 is the best subordinant and $q_2(z)$ is the best dominant.*

Theorem 4.21. *[81] Let q be a convex function in U with $q(0)=1$, $\gamma\in\mathbb{C}^*$ such that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}\geq\max\left\{0,-\Re\frac{1}{\gamma}\right\}.$$

If $f\in\mathscr{A}$ and

$$(1+\gamma)z\frac{\mathscr{D}I^{n}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}-$$

$$-2\gamma z\frac{\mathscr{D}I^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{D}I^{n+1}f(z)\right]^3}\prec q(z)+\gamma z q'(z), \tag{4.24}$$

then

$$z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}\prec q(z),$$

q is the best dominant.

We consider $n=0$ and $\lambda=0$.

Corollary 4.4. *[81] Let q be univalent in U with $q(0)=1$, $\gamma\in\mathbb{C}^*$ such that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}\geq\max\left\{0,-\Re\frac{1}{\gamma}\right\}.$$

If $f\in\mathscr{A}$ and

$$\left(1-\gamma\right)\frac{f(z)}{z[f'(z)]^2}+\gamma\left[\frac{1}{f'(z)}-\left(\frac{2f(z)\cdot f''(z)}{[f'(z)]^3}\right)^2\right]\prec q(z)+\gamma z q'(z)$$

then

$$\frac{f(z)}{z[f'(z)]^2}\prec q(z)$$

and q is the best dominant.

Corollary 4.5. *[81] Let $A,B,\gamma\in\mathbb{C},A\neq B$ such that $|B|\leq 1$ and $\Re\,\gamma>0$. If for $f\in\mathscr{A}$*

$$\left(1+\gamma\right)z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}-$$

$$-2\gamma z\frac{\mathscr{D}I^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{D}I^{n+1}f(z)\right]^3}\prec\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2}, \tag{4.25}$$

then

$$z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}\prec\frac{1+Az}{1+Bz}$$

and $q(z)=\dfrac{1+Az}{1+Bz}$ is the best dominant.

Theorem 4.22. *[81] Let q be a convex function in U with $q(0)=1$, $\gamma\in\mathbb{C}$ such that $\Re\,\gamma>0$. If $f\in\mathscr{A}$*

$$z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}\in\mathscr{H}[1,1]\cap\mathscr{Q},$$

$$\left(1+\gamma\right)z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}-$$

$$-2\gamma z\frac{\mathscr{D}I^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{D}I^{n+1}f(z)\right]^3}$$

is univalent in U and

$$q(z)+\gamma zq'(z)\prec\left(1+\gamma\right)z\frac{\mathscr{D}I^{n}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}}-$$

$$-2\gamma z\frac{\mathscr{D}I^{n}f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^{n}f(z)\right]}{\left[\mathscr{D}I^{n+1}f(z)\right]^{3}}, \tag{4.26}$$

then

$$q(z)\prec z\frac{\mathscr{D}I^{n}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}},$$

q is the best subordinant.

From Theorem 4.21 and Theorem 4.22 we get the following "sandwich-type theorem".

Theorem 4.23. *[81] Let q_1 and q_2 be convex functions in U with $q_1(0)=q_2(0)=1$, $\gamma\in\mathbb{C}$ such that $\Re\,\gamma>0$. If $f\in\mathscr{A}$*

$$z\frac{\mathscr{D}I^{n}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}}\in\mathscr{H}[1,1]\cap\mathscr{Q},$$

$$\left(1+\gamma\right)z\frac{\mathscr{D}I^{n}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}}-$$

$$-2\gamma z\frac{\mathscr{D}I^{n}f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^{n}f(z)\right]}{\left[\mathscr{D}I^{n+1}f(z)\right]^{3}}$$

is univalent in U and

$$q_1(z)+\gamma zq_1'(z)\prec\left(1+\gamma\right)z\frac{\mathscr{D}I^{n}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}}-$$

$$-2\gamma z\frac{\mathscr{D}I^{n}f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^{n}f(z)\right]}{\left[\mathscr{D}I^{n+1}f(z)\right]^{3}}\prec q_2(z)+\gamma zq_2'(z), \tag{4.27}$$

then

$$q_1(z)\prec z\frac{\mathscr{D}I^{n}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^{2}}\prec q_2(z),$$

and q_1 is the best subordinant and $q_2(z)$ is the best dominant.

4.4 Univalence criteria related with the generalised Sălăgean and Ruscheweyh operator

We recall the $\mathscr{R}\mathscr{D}^n$ Ruscheweyh and Sălăgean differential operator defined in (1.14).

In order to prove our main result we need the theory of Loewner chains.

Let $U_r=\{z\in\mathbb{C}:|z|<r,\ \ r\in(0,1]\}$, $I=[0,\infty)$ and $p\in\mathscr{P}$ (the class $\mathscr{P}$ is a Carathéodory class of functions which are analytic with positive real part in U) be of the form $p(z)=1+c_1z+c_2z^2+\ldots$.

A function $L(z,t):U\times I\to\mathbb{C}$ is said to be a Loewner chain if the following conditions are satisfied:

i) $L(z,t)$ is analytic and univalent in U, $\forall t \in I$

ii) $L(z,t) \prec L(z,s)$, $\forall \ \ 0 \le t \le s < \infty$.

Lemma 4.6. *[99]*
Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots, \quad a_1(t) \neq 0$ *be an analytic function in* U_r, $\forall t \in I$. *Suppose that:*

i) $L(z,t)$ *is a locally absolutely continuous function in* I *and locally uniformly with respect to* U_r.

ii) $a_1(t)$ *is a complex valued continuous function on* I *such that* $\lim\limits_{t\to\infty} |a_1(t)| = \infty$ *and* $\left\{\dfrac{L(z,t)}{a_1(t)}\right\}_{t\in I}$ *is a normal family of functions in* U_r.

iii) *There exists an analytic function* $p : U_r \times I \to \mathbb{C}$ *satisfying* $\Re p(z,t) > 0$, $\forall (z,t) \in U \times I$ *and*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \ \ z \in U_r, \ \ t \in I. \tag{4.28}$$

Then, for each $t \in I$, *the function* $L(z,t)$ *has an analytic and univalent extension to the whole unit disk* U, *i.e* $L(z,t)$ *is a Loewner chain.*

The equation (4.28) is called the generalized Loewner differential equation.
If $a_1(t) = e^t$ then we say that $L(z,t)$ is a standard Loewner chain.
We follow Nistor [68], and we generalise her results.

Theorem 4.24. *[88] Let* $f \in \mathcal{A}$ *and* p *an analytic function with* $p(0) = 1$. *If the inequalities*

$$\left|\frac{2}{p(z)+1}\cdot\frac{zf'(z)}{\mathcal{RD}^{n+1}f(z) + (1-\gamma)\,n\left(\mathcal{R}^{n+1}f(z) - \mathcal{R}^n f(z)\right)} - 1\right| \le 1 \tag{4.29}$$

and

$$\left|\left(\frac{2}{p(z)+1}\cdot\frac{zf'(z)}{\mathcal{RD}^{n+1}f(z) + (1-\gamma)\,n\left(\mathcal{R}^{n+1}f(z) - \mathcal{R}^n f(z)\right)} - 1\right)|z|^2 + \right.$$
$$+\left(1-|z|^2\right)\left(\frac{\mathcal{RD}^{n+2}f(z)+}{\mathcal{RD}^{n+1}f(z) + (1-\gamma)\,n\left(\mathcal{R}^{n+1}f(z) - \mathcal{R}^n f(z)\right)} - \right.$$
$$\left.\left.\frac{(1-\gamma)\left[\left(n^2+3n+1\right)\mathcal{R}^{n+2}f(z) - \left(2n^2+3n+1\right)\mathcal{R}^{n+1}f(z) + n^2\mathcal{R}^n f(z)\right]}{\mathcal{RD}^{n+1}f(z) + (1-\gamma)\,n\left(\mathcal{R}^{n+1}f(z) - \mathcal{R}^n f(z)\right)} - 1 + \frac{zp'(z)}{p(z)+1}\right)\right| \le 1$$

(4.30)

holds true for $z \in U$, *then the function* f *is univalent in* U.

Remark 4.12. *If* $\gamma = 1$ *we get Theorem 1 from Nistor [68] and for* $\gamma = 0$ *we get Theorem 3 from the same article.*

Setting $n = 0$ in Theorem 4.24 we obtain the following corollary due to Lewandowski [54] :

Corollary 4.6. *[88] Let $f \in \mathcal{A}$ and $p \in \mathcal{P}$. If*

$$\left|\frac{1-p(z)}{1+p(z)}|z|^2 + \left(1-|z|^2\right)\left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{1+p(z)}\right)\right| \le 1, \quad z \in U,$$

then the function f is univalent in U.

For $p = 1$ the following criterion reduces to a well-known criterion found by Becker [20] (see also Duren et al. [31]).

Corollary 4.7. *[88] Let $f \in \mathcal{A}$. If*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \quad z \in U,$$

then the function f is univalent in U.

For $n = 1$, Theorem 4.24 results

Corollary 4.8. *[88] Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If*

$$\left|\frac{2}{p(z)+1} \cdot \frac{f'(z)}{f'(z)+zf''(z)} - 1\right| \le 1$$

and

$$\left|\left(\frac{2}{p(z)+1} \cdot \frac{f'(z)}{f'(z)+zf''(z)} - 1\right)|z|^2 + \left(1-|z|^2\right)\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)} + \frac{zp'(z)}{p(z)+1}\right)\right| \le 1$$

holds true for $z \in U$ then the function f is univalent in U.

For the Loewner chain

$$L(z,t) := f\left(e^{-t}z\right) + \left(e^t z - e^{-t}z\right)\frac{p\left(e^{-t}z\right)+1}{2} \cdot \frac{\mathcal{RD}^{n+1}f(e^{-t}z)}{\mathcal{RD}^n f(e^{-t}z)} \tag{4.31}$$

identically with the proof of Theorem 4.24, we get:

Theorem 4.25. *[88] Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left|\frac{2}{p(z)+1} \cdot f'(z) \cdot \frac{\mathcal{RD}^n f(z)}{\mathcal{RD}^{n+1} f(z)} - 1\right| \le 1 \tag{4.32}$$

and

$$\left|\left(\frac{2f'(z)}{p(z)+1} \cdot \frac{\mathcal{RD}^n f(z)}{\mathcal{RD}^{n+1} f(z)} - 1\right)|z|^2 + \left(1-|z|^2\right)\left(\frac{\mathcal{RD}^{n+2} f(z)}{\mathcal{RD}^{n+1} f(z)} - \frac{\mathcal{RD}^{n+1} f(z)}{\mathcal{RD}^n f(z)} + (1-\gamma)\right.\right.$$
$$\left.\left.\frac{\left[(n+1)\mathcal{RD}^n f(z)\left(\mathcal{R}^{n+2}f(z) - \mathcal{R}^{n+1}f(z)\right) - n\mathcal{RD}^{n+1}f(z)\left(\mathcal{R}^{n+1}f(z) - \mathcal{R}^n f(z)\right)\right]}{\mathcal{RD}^n f(z) \cdot \mathcal{RD}^{n+1} f(z)} + \frac{zp'(z)}{p(z)+1}\right)\right| \le 1$$

(4.33)

holds true for $z \in U$, then the function f is univalent in U.

Remark 4.13. *If $\gamma = 1$ we get Theorem 2 from Nistor [68] and for $\gamma = 0$ we get Theorem 4 from the same article.*

Setting $n = 0$ in Theorem 4.25 the result is:

Corollary 4.9. *[88] Let $f \in \mathscr{A}$ and p an analytic function with $p(0) = 1$. If*

$$\left| \frac{2}{p(z)+1} \cdot \frac{f'(z)}{z} - 1 \right| \le 1$$

and

$$\left| \left(\frac{2}{p(z)+1} \cdot \frac{f'(z)}{z} - 1 \right) |z|^2 + \left(1 - |z|^2\right) \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)+1} \right) \right| \le 1$$

holds true for $z \in U$, then the function f is univalent in U.

If we put $p = 1$ in the corollary above we get the result of Kanas and Lecko [51].

Setting $p(z) = \dfrac{f(z)}{z}$ we obtain:

Corollary 4.10. *[88] Let $f \in \mathscr{A}$ with $\Re \dfrac{f(z)}{z} > 0$. If*

$$\left| \left(\frac{f(z)}{z} - 1 \right) |z|^2 + \left(1 - |z|^2\right) \left[1 + \frac{zf''(z)}{f'(z)} \left(\frac{f(z)}{z} + 1 \right) - \frac{zf'(z)}{f(z)} \right] \right| \le \left| \frac{f(z)}{z} + 1 \right|$$

holds true for $z \in U$, then the function f is univalent in U.

For $p(z) = \dfrac{zf'(z)}{f(z)}$ in Corollary 4.9, the result is:

Corollary 4.11. *[88] Let $f \in \mathscr{A}$. If*

$$\left| 2\frac{f(z)}{z} - \frac{zf'(z)}{f(z)} - 1 \right| \le \left| 1 + \frac{zf'(z)}{f(z)} \right|$$

and

$$\left| \left(2\frac{f(z)}{z} - \frac{zf'(z)}{f(z)} - 1 \right) |z|^2 + \left(1 - |z|^2\right) \frac{2zf'(z)}{f(z)+1} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right| \le \left| 1 + \frac{zf'(z)}{f(z)} \right|$$

holds true for $z \in U$, then the function f is univalent in U.

4.5 Preserving properties of the generalized Bernardi-Libera-Livingston integral operator defined on some subclasses of starlike functions

Let T denote a subclass of $\mathscr{A}$, consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \tag{4.34}$$

where $a_j \ge 0$, $j = 2, 3, \ldots$ and $z \in U$. A function $f \in T$ is called a function with negative coefficients. For the class T, the followings are equivalent [112]:

i) $\sum_{j=2}^{\infty} ja_j \le 1$,

ii) $f \in T \cap S$,

iii) $f \in T^*$, where $T^* = T \cap S^*$.

In [38] the authors introduced the following subclass of analytic functions

$$S^{**} = \left\{ f \in \mathscr{A} : \left| 1 + \frac{zf''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}},\ z \in U \right\}. \tag{4.35}$$

In the same paper the authors has shown that the class S^{**} is a subclass of S^* and this class has the property that the composition of two starlike functions from S^{**} is in the class S^* of starlike functions.

In [37] the authors studied the following subclass of convex functions

$$S^{***} = \left\{ f \in \mathscr{A} : \left| 1 - \frac{zf''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}},\ z \in U \right\}. \tag{4.36}$$

In the same paper the authors has shown that the class S^{***} is a subclass of K, has determined the order of starlikeness of the class S^{***} and have shown that if $f, g \in S^{***}$ then $f \circ g$ is starlike in U_{r_0}, where $r_0 = \sup\{r > 0 | g(U_r) \subset U\}$.

Now we consider the generalized Bernardi-Libera-Livingston integral operator

$$F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t)dt, \tag{4.37}$$

where $f \in \mathscr{A}$ and $p > -1$. This operator was studied by Bernardi for $p \in \{1, 2, 3, ...\}$ and for $p = 1$ by Libera.

We study the properties of the image of the classes S^{**} and S^{***} by the generalized Bernardi-Libera-Livingston integral operator $L_p f(z)$. The subclass S^{***} is defined also for functions with negative coefficients and some other results are derived for this class.

Definition 4.4. [61] Let Q be the class of analytic functions q in U which has the property that are analytic and injective on $\overline{U} \setminus E(q)$, where

$$E(q) = \{\zeta \in \partial U : \lim_{z \longrightarrow \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

Lemma 4.7. *[Miller-Mocanu] Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \ldots$ be analytic in U with $p(z) \not\equiv a$ and $n \ge 1$. If $p \not\prec q$, then there are two points $z_0 = r_0 e^{i\theta_0} \in U$, and $\zeta_0 \in \partial U \setminus E(q)$ and a real number $m \in [n, \infty)$ for which $p(U_{r_0}) \subset q(U)$,* *i)* $p(z_0) = q(\zeta_0)$

ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

iii) $\Re \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\Re\left(\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right)$.

The following result is a particular case of Lemma 4.7.

Lemma 4.8. *[Miller-Mocanu] Let $p(z) = 1 + a_n z^n + \ldots$ be analytic in U with $p(z) \not\equiv 1$ and $n \geq 1$. If $p(z) \not\prec q(z) = M\dfrac{Mz+1}{M+z}$ then there is a point $z_0 \in U$, and $\zeta_0 \in \partial U \setminus E(q)$ and a real number $m \in [n, \infty)$ for which $p(U_{r_0}) \subset q(U)$, such that*

i) $p(z_0) = q(\zeta_0)$, *where* $\zeta_0 = e^{i\theta}$

ii) $z_0 p'(z_0) = me^{i\theta} M \dfrac{M^2 - 1}{(M + e^{i\theta})^2}$,

iii) $\Re z_0^2 p''(z_0) + z_0 p'(z_0) \leq 0$.

Theorem 4.26. *[96] Let*

$$F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t) dt.$$

If $p \geq \sqrt{\dfrac{5}{4}}$ and $f \in S^{**}$, then $F \in S^{**}$.

Theorem 4.27. *[96] Let $F(z) = L_p f(z) = \dfrac{p+1}{z^p} \displaystyle\int_0^z t^{p-1} f(t) dt$, $p > -2$. If $f \in S^{***}$ then $F \in S^{***}$.*

In the followings we define the class S^{***} for functions with negative coefficients.

Definition 4.5. [96] The function $f \in T$ belongs to the class $TS^{***} = S^{***} \cap T$ if

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}}, z \in U.$$

Below we give a coefficient delimitation theorem for the class TS^{***}.

Theorem 4.28. *[96] The function $f \in T$ belongs to the class TS^{***} if and only if*

$$\sum_{j=2}^{\infty} j\left(j - 2 + \frac{\sqrt{5}}{2}\right) a_j < \frac{\sqrt{5}}{2} - 1. \tag{4.38}$$

Next we prove that the class TS^{***} is closed under convolution with convex functions.

Theorem 4.29. *[96] Let $f \in T$ be of the form (**??**) and $\phi(z) = z - \sum_{j=2}^{\infty} b_j z^j$ convex in U, where $b_j \geq 0$ for $j \in \{2, 3, \ldots\}$. If $f \in TS^{***}$ then $f * \phi \in TS^{***}$.*

Theorem 4.30. *[96] Let $F(z) = L_p f(z) = \dfrac{p+1}{z^p} \displaystyle\int_0^z t^{p-1} f(t) dt$, $p \in (-1, 0]$. If $f \in TS^{***}$, then $F \in TS^{***}$.*

4.6 The radius of convexity of particular functions and applications to the study of a second order differential inequality

Let the function f be defined by $f(z)=\sum\limits_{n=0}^{\infty} a_n z^n$.

Next we determine the radius of convexity of particular functions in order to determine sharp bounds regarding functions which satisfy a differential inequality and we deduce a sharp starlikeness condition. Papers concerning these problems are [16],[17] and [123].
In [61] the following problem is proposed (pg.243): if $f(0)=a$ with $\Re\, a>0$, and

$$\Re\big(a+4zf'(z)+2z^2f''(z)\big)>0,\ z\in U, \tag{4.39}$$

then $\Re f(z)>0,\ z\in U$. This implication is easy to prove using the theory of differential subordinations presented in [58] and [61]. We will determine the best upper and lower bound of $\Re f(z)$ provided that $a=1$ and the condition (4.39) holds. The basic tool in the proofs will be the convexity of a particular function.
Differential inequalities of type (4.39) are studied in [118] and [120], where the theory of extreme points developed in [42] is used.

We define the classes of functions A_0 and $\mathscr{P}$ by the equalities

$$A_0=\{f\in H(U)|\ f(0)=1\}\ \text{ and }\ \mathscr{P}=\{f\in A_0|\ \operatorname{Re} f(z)>0,\ z\in U\}.$$

Lemma 4.9. *([42] p. 27 Herglotz).*
The function f belongs to the class $\mathscr{P}$ if and only if there is a probability measure μ on $[0,2\pi]$ such that

$$f(z)=\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t).$$

Lemma 4.10. *[75] If $\theta\in[-\pi,\pi]$, then*

$$2(1-\cos\theta)\Big(\int_0^1 t\frac{1}{\sqrt{1+t^2-2t\cos\theta}}dt\Big)^2\le\int_0^1 t\frac{(1+t)(1-\cos\theta)}{1+t^2-2t\cos\theta}dt.$$

For $\widetilde{V}\subset A_0$ the dual set of $\widetilde{V}$ is defined by

$$\widetilde{V}^d=\big\{g\in A_0|(f*g)(z)\neq 0,\ \text{ for all }\ f\in\widetilde{V}\ \text{and for all}\ z\in U\big\}.$$

Lemma 4.11. *[75] Let α be a real number with $\alpha\in[0,1)$, and let the function h_T be defined by the power series $h_T(z)=z+\sum\limits_{n=2}^{\infty}\frac{n-\alpha+iT}{1-\alpha+iT}z^n$. The function $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ is starlike of order α in U if and only if*

$$\frac{f(z)}{z}*\frac{h_T(z)}{z}\neq 0,\ \textit{for all}\ z\in U,\ \textit{and}\ \forall T\in\mathbb{R}.$$

Lemma 4.12. *[75] For the dual set of the class $\mathscr{P} = \{f \in A_0 | \ \Re f(z) > 0,\ z \in U\}$ we have*

$$\mathscr{L}^d = \{f \in A_0 | \ \Re f(z) > \frac{1}{2},\ z \in U\} \subset \mathscr{P}^d.$$

Lemma 4.13. *([58], p.64)([61], p.236) Let K be the class of functions of the form $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, and satisfying the condition*

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0,\quad \forall z \in U.$$

If $\boldsymbol{L}$ denotes the operator of Libera defined by $\boldsymbol{L}(f)(z) = \dfrac{2}{z}\displaystyle\int_0^z f(t)dt$, then

$$\boldsymbol{L}(K) \subset K.$$

Lemma 4.14. *[75] The following equalities hold*

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)^2} = \int_0^1\int_0^1 \frac{(1-x)y(\cos\theta - xy)}{1+x^2y^2-2xy\cos\theta}dxdy + i\int_0^1\int_0^1 \frac{(1-x)y\sin\theta}{1+x^2y^2-2xy\cos\theta}dxdy,$$

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{(n+1)^2} = \int_0^1\int_0^1 \frac{xy(\cos\theta - xy)}{1+x^2y^2-2xy\cos\theta}dxdy + i\int_0^1\int_0^1 \frac{xy\sin\theta}{1+x^2y^2-2xy\cos\theta}dxdy.$$

Lemma 4.15. *[75] If $\alpha = \dfrac{2-\ln 4}{3-\ln 6 - \frac{\pi^2}{12}}$, then the following inequalities hold:*

$$\begin{aligned}&(1-\alpha)\int_0^1\int_0^1 (1-x)y\frac{(1-xy)(1+\cos\theta)}{(1+xy)(1+x^2y^2-2xy\cos\theta)}dxdy\\ \geq\ &\frac{1}{6}\int_0^1\int_0^1 xy\frac{(1-xy)(1+\cos\theta)}{(1+xy)(1+x^2y^2-2xy\cos\theta)}dxdy,\quad \theta \in [\frac{\pi}{2},\pi],\end{aligned}$$

$$\int_0^1\int_0^1 \frac{xy\sin\theta}{1+x^2y^2-2xy\cos\theta}dxdy \leq \int_0^1\int_0^1 \frac{xy\sqrt{2(1+\cos\theta)}}{(1+xy)\sqrt{1+x^2y^2-2xy\cos\theta}}dxdy,\ \theta \in [0,\pi].$$

Lemma 4.16. *[75] The following inequality holds*

$$\begin{aligned}&\int_0^1\int_0^1 \frac{2xy}{1-x^2y^2}dxdy\int_0^1\int_0^1 xy\frac{(1-xy)(1+\cos\theta)}{(1+x^2y^2-2xy\cos\theta)(1+xy)}dxdy \leq\\ &\leq 4(1-\alpha)\Big(1+\sum_{n=1}^{\infty}\frac{\cos n\theta}{n(n+1)^2}\Big)\Big[(1-\alpha)\cdot\Big(1+\sum_{n=1}^{\infty}\frac{\cos n\theta}{n(n+1)^2}\Big)+\sum_{n=1}^{\infty}\frac{\cos n\theta}{(n+1)^2}\Big],\\ &\theta \in [0,\pi].\end{aligned} \tag{4.40}$$

Theorem 4.31. *[75] The functions ψ and φ defined by the power series*

$$\psi(z) = 1 + \sum_{n=1}^{\infty}\frac{z^n}{n(n+1)},\qquad \varphi(z) = 1 + \sum_{n=1}^{\infty}\frac{z^n}{n(n+1)^2}$$

are convex in the unit disk U, the radius of convexity of ψ and φ are $r_{\psi}^c = r_{\varphi}^c = 1$.

Corollary 4.12. *[75] If $f(0) = 1$ and if*

$$\Re\big(1 + 4zf'(z) + 2z^2 f''(z)\big) > 0, \quad \forall z \in U \tag{4.41}$$

then

$$2 - \frac{1+r}{r}\ln(1+r) < \Re\big(f(z)\big) < 2 + \frac{1-r}{r}\ln(1-r), \quad z \in U_r \tag{4.42}$$

for every $r \in (0,1)$, and

$$2 - \ln 4 < \Re\big(f(z)\big) < 2, \quad \forall z \in U. \tag{4.43}$$

The bounds are the best possible.

Other results regarding the radius of starlikeness and the radius of convexity of particular functions can be found in [16], [19],[18],[17] and [123].

Remark 4.14. *[75] The restriction $a = 1$ does not detract the generality. Indeed, if $a = \alpha + i\beta$ with $\alpha > 0$, then condition (4.39) is equivalent to*

$$\Re\big(1 + \frac{4}{\alpha}zf'(z) + \frac{2}{\alpha}z^2 f''(z)\big) > 0, \ \forall \ z \in U$$

and a similar calculation to the proof of Corollary 4.12 leads to

$$f(z) = \int_0^{2\pi}\Big(\alpha + \sum_{n=1}^{\infty}\frac{\alpha}{n(n+1)}z^n e^{-int}\Big)d\mu(t).$$

Thus we get

$$\Re f(z) = \alpha\Re\int_0^{2\pi}\Big(1 + \sum_{n=1}^{\infty}\frac{1}{n(n+1)}z^n e^{-int}\Big)d\mu(t),$$

and

$$\alpha\big[2 - \frac{1+r}{r}\ln(1+r)\big] < \Re\big(f(z)\big) < \alpha\big[2 + \frac{1-r}{r}\ln(1-r)\big], \quad z \in U_r.$$

Corollary 4.13. *[75] If $f(0) = 1$ and (4.41) holds, then*

$$\big|f(z)\big| < 2 + \frac{1-r}{r}\ln(1-r), \quad z \in U(r) \tag{4.44}$$

for every $r \in (0,1)$, and

$$\big|f(z)\big| < 2, \quad z \in U. \tag{4.45}$$

The bounds are the best possible.

Theorem 4.32. *[75] If $f(0) = 1$ and (4.41) holds, then the function F defined by $F(z) = \int_0^z f(t)dt$ is starlike of order $\alpha = \dfrac{2 - \ln 4}{3 - \ln 4 - \frac{\pi^2}{12}} = 0.7756\ldots$, that is*

$$\Re\frac{zF'(z)}{F(z)} > \frac{2 - \ln 4}{3 - \ln 4 - \frac{\pi^2}{12}}, \quad z \in U. \tag{4.46}$$

The result is sharp.

Remark 4.15. *As far as we know the result presented in Lemma 4.11 is a new form of starlikeness condition which involves convolution. The idea of use integral representations of Fourier series in order to deduce sharp inequalities which lead to sharp starlikeness results has been used many times. Regarding these questions we mention the papers [119]-[121]. In [122] geometric properties of a particular function are studied. We mention that [103] is a basic work in applications of convolutions in geometric function theory.*

Chapter 5

Bi-univalent functions

The Koebe One-Quarter Theorem [30] ensures that the image of the unit disk under every $f \in S$ univalent function contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z, \quad (z \in U),$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}\right), \tag{5.1}$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \tag{5.2}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in U if $U \subset f(U)$ and if both f and f^{-1} are univalent in U.

Let Σ denotes the class of bi-univalent functions in U given by (1.1). Some examples of functions in the class Σ are given in [130]. See also [55], [22] , [124], [39],[128], [129] , [8] , [9],[47].

Lemma 5.1. *[99] If $h \in P$ then $|c_k| \leq 2$, $\forall k$, where P is the family of all functions h analytic in U for which $\Re h(z) > 0$, where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in U$.*

In [56] Ma and Minda unified some subclasses of starlike and convex functions for which both of the functions

$$\frac{zf'(z)}{f(z)} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)}$$

is subordinate to a more general superordinate functions. To accomplish this, they used an analytic function Φ, with

$$\Re\, \Phi > 0 \quad ,\Phi(0) = 1 \quad \text{and} \quad \Phi'(0) > 0,$$

which maps the unit disk onto a starlike region with respect to 1 and symmetric with respect to the real axis. A function $f \in \mathscr{A}$ belongs to the class of Ma-Minda starlike respectively convex functions if

$$\frac{zf'(z)}{f(z)} \prec \Phi(z) \quad \text{respectively} \quad 1+\frac{zf''(z)}{f'(z)} \prec \Phi(z), \;\; z \in U.$$

A function f is said to be bi-starlike respectively bi-convex of Ma-Minda type in U if both f and f^{-1} are Ma-Minda starlike respectively Ma-Minda convex in U.

We consider the $\mathscr{D}I^n$ operator defined in Definition 1.9.

5.1 Coefficient estimates and Fekete-Szegő problem for new classes of bi-univalent functions defined by Sălăgean integro-differential operator

Definition 5.1. [115], [13], [79] For $0<\alpha\le 1$, $0\le\widetilde{\lambda}\le 1$ a function $f(z)$ given by (1.1) is said to be in the class $\mathscr{P}^{\alpha}_{\Sigma}(\widetilde{\lambda})$ if the following conditions are satisfied:

$$\left|arg\left(\frac{z(\mathscr{D}I^n f(z))'+\widetilde{\lambda}z^2(\mathscr{D}I^n f(z))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n f(z)+\widetilde{\lambda}z(\mathscr{D}I^n f(z))'}\right)\right|<\frac{\alpha\pi}{2} \tag{5.3}$$

and

$$\left|arg\left(\frac{w(\mathscr{D}I^n g(w))'+\widetilde{\lambda}w^2(\mathscr{D}I^n g(w))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n g(w)+\widetilde{\lambda}w(\mathscr{D}I^n g(w))'}\right)\right|<\frac{\alpha\pi}{2} \tag{5.4}$$

where $z,w\in U$ and the function g is given by (5.2).

Remark 5.1. *If $\widetilde{\lambda}=n=0$ we have the well-known class of strongly bi-starlike functions of order α and if $\widetilde{\lambda}=1$ and $n=0$ we have the class of strongly bi-convex functions of order α.*

Definition 5.2. [79] For $0<\alpha\le 1$, $0\le\widetilde{\lambda}\le 1$ a function $f(z)$ given by (1.1) is said to be in the class $\mathscr{Q}^{\beta}_{\Sigma}(\widetilde{\lambda})$ if the following conditions are satisfied:

$$\Re\left(\frac{z(\mathscr{D}I^n f(z))'+\widetilde{\lambda}z^2(\mathscr{D}I^n f(z))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n f(z)+\widetilde{\lambda}z(\mathscr{D}I^n f(z))'}\right)>\beta \tag{5.5}$$

and

$$\Re\left(\frac{w(\mathscr{D}I^n g(w))'+\widetilde{\lambda}w^2(\mathscr{D}I^n g(w))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n g(w)+\widetilde{\lambda}w(\mathscr{D}I^n g(w))'}\right)>\beta \tag{5.6}$$

where $z,w\in U$ and the function g is given by (5.2).

Definition 5.3. [79] Let $h, l : U \to \mathbb{C}$ be analytic functions and

$$min\{\Re(h(z)), \Re(l(z))\} > 0, \quad (z \in U) \quad h(0) = l(0) = 1.$$

A function $f(z)$ given by (1.1) is said to be in the class $\mathscr{P}_\Sigma^{h,l}$ if the following conditions are satisfied:

$$\frac{z(\mathscr{D}I^n f(z))' + \widetilde{\lambda} z^2 (\mathscr{D}I^n f(z))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n f(z) + \widetilde{\lambda} z (\mathscr{D}I^n f(z))'} \in h(U) \tag{5.7}$$

and

$$\frac{w(\mathscr{D}I^n g(w))' + \widetilde{\lambda} w^2 (\mathscr{D}I^n g(w))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n g(w) + \widetilde{\lambda} w (\mathscr{D}I^n g(w))'} \in l(U) \tag{5.8}$$

where $z, w \in U$ and the function g is given by (5.2).

Theorem 5.1. *[79] Let $0 < \alpha \le 1$, $0 \le \widetilde{\lambda} \le 1$ and let $f(z)$ given by* (1.1) *be in the class $\mathscr{P}_\Sigma^\alpha(\widetilde{\lambda})$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\left|4\alpha\Gamma_3\left(1+2\widetilde{\lambda}\right) + \Gamma_2^2\left(1+\widetilde{\lambda}\right)^2(1-3\alpha)\right|}}, \tag{5.9}$$

$$|a_3| \le \frac{\alpha}{\Gamma_3\left(1+2\widetilde{\lambda}\right)} + \frac{4\alpha^2}{\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2} \tag{5.10}$$

and

$$|a_4| \le \frac{2\alpha\left(2\alpha^2+1\right)}{9\Gamma_4\left(1+3\widetilde{\lambda}\right)} - \frac{10\alpha(2\alpha-1)}{3\left[2\Gamma_2\Gamma_3\left(1+\widetilde{\lambda}\right)\left(1+2\widetilde{\lambda}\right) - 5\Gamma_4\left(1+3\widetilde{\lambda}\right)\right]} + $$
$$+ \frac{8\alpha^3\left[3\left(1+2\widetilde{\lambda}\right)\Gamma_3 - \left(1+\widetilde{\lambda}\right)^2\Gamma_2^2\right]}{3\Gamma_2\Gamma_4\left(1+\widetilde{\lambda}\right)\left(1+3\widetilde{\lambda}\right)\sqrt{\left|4\alpha\Gamma_3\left(1+2\widetilde{\lambda}\right) + \Gamma_2^2\left(1+\widetilde{\lambda}\right)^2(1-3\alpha)\right|}}.$$

(5.11)

Theorem 5.2. *[79] Let f of the form* (1.1) *be in the class $\mathscr{P}_\Sigma^\alpha(\widetilde{\lambda})$. Then*

$$\left|a_3 - \xi a_2^2\right| \le \begin{cases} \frac{\alpha}{\Gamma_3(1+2\widetilde{\lambda})}; & \left|\frac{\alpha(1-\xi)}{4\alpha\Gamma_3(1+2\widetilde{\lambda})+\Gamma_2^2(1+\widetilde{\lambda})^2(1-3\alpha)}\right| \le \frac{1}{4\Gamma_3(1+2\widetilde{\lambda})} \\ \left|\frac{4\alpha^2(1-\xi)}{4\alpha\Gamma_3(1+2\widetilde{\lambda})+\Gamma_2^2(1+\widetilde{\lambda})^2(1-3\alpha)}\right|; & \left|\frac{\alpha(1-\xi)}{4\alpha\Gamma_3(1+2\widetilde{\lambda})+\Gamma_2^2(1+\widetilde{\lambda})^2(1-3\alpha)}\right| \ge \frac{1}{4\Gamma_3(1+2\widetilde{\lambda})} \end{cases}$$

Theorem 5.3. *[79] Let $0 < \beta \le 1$, $0 \le \widetilde{\lambda} \le 1$ and let $f(z)$ given by* (1.1) *be in the class $\mathscr{Q}_\Sigma^\beta(\widetilde{\lambda})$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{\left|2\left(1+2\widetilde{\lambda}\right)\Gamma_3 - \Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}}, \tag{5.12}$$

(5.13)
$$|a_3| \le \frac{1-\beta}{\Gamma_3\left(1+2\widetilde{\lambda}\right)} + \frac{4\left(1-\beta\right)^2}{\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}$$

and

$$|a_4| \le \frac{2\left(1-\beta\right)}{3\Gamma_4\left(1+3\widetilde{\lambda}\right)} - \frac{10\left(1-\beta\right)}{3\Gamma_4\left(1+3\widetilde{\lambda}\right)\left[2\Gamma_2\Gamma_3\left(1+\widetilde{\lambda}\right)\left(1+2\widetilde{\lambda}\right)-5\Gamma_4\left(1+3\widetilde{\lambda}\right)\right]} + \\ + \frac{2\Gamma_2\left(1-\beta\right)\left(1+\widetilde{\lambda}\right)\left[3\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right]}{3\Gamma_4\left(1+3\widetilde{\lambda}\right)\left[2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right]}\sqrt{\frac{2\left(1-\beta\right)}{\left|2\left(1+2\widetilde{\lambda}\right)\Gamma_3-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}}.$$

(5.14)

Theorem 5.4. *[79] Let f of the form* (1.1) *be in the class $\mathcal{Q}_{\Sigma}^{\beta}(\widetilde{\lambda})$. Then*

$$\left|a_3-\xi a_2^2\right| \le \begin{cases} \frac{1-\beta}{\Gamma_3\left(1+2\widetilde{\lambda}\right)}; & \left|\frac{1-\xi}{4\alpha\Gamma_3\left(1+2\widetilde{\lambda}\right)-2\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}\right| \le \frac{1}{4\Gamma_3\left(1+2\widetilde{\lambda}\right)} \\ \left|\frac{4(1-\beta)(1-\xi)}{4\alpha\Gamma_3\left(1+2\widetilde{\lambda}\right)-2\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}\right|; & \left|\frac{1-\xi}{4\alpha\Gamma_3\left(1+2\widetilde{\lambda}\right)-2\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}\right| \ge \frac{1}{4\Gamma_3\left(1+2\widetilde{\lambda}\right)} \end{cases}$$

Theorem 5.5. *[79] Let $0<\alpha\le 1$, $0\le\widetilde{\lambda}\le 1$ and let $f(z)$ given by* (1.1) *be in the class $\mathcal{P}_{\Sigma}^{h,l}$. Then*

(5.15)
$$|a_2| \le \min\left\{\sqrt{\frac{|h'(0)|^2+|l'(0)|^2}{2\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}}, \sqrt{\frac{|h''(0)|+|l''(0)|}{4\left|2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}}\right\}$$

$$|a_3| \le \min\left\{\frac{\left|h'(0)\right|^2+\left|l'(0)\right|^2}{2\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2} + \frac{\left|h''(0)\right|+\left|l''(0)\right|}{8\Gamma_3\left(1+2\widetilde{\lambda}\right)}, \right. \\ \left. \frac{\left|h''(0)\right|\left|4\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|+\left|l''(0)\right|\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}{8\Gamma_3\left(1+2\widetilde{\lambda}\right)\left|2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}\right\}$$

(5.16)

and

$$|a_4| \le \min\left\{\frac{\left|h'''(0)\right|+\left|l'''(0)\right|}{36}\left|\frac{1}{\Gamma_4\left(1+3\widetilde{\lambda}\right)} - \frac{5}{2\Gamma_2\Gamma_3\left(1+\widetilde{\lambda}\right)\left(1+2\widetilde{\lambda}\right)-5\Gamma_4\left(1+3\widetilde{\lambda}\right)}\right| + \right. \\ + \frac{\left|h'(0)\right|^2+\left|l'(0)\right|^2}{\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}\sqrt{\frac{|h'(0)|^2+|l'(0)|^2}{2}}\frac{\left|3\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}{6\Gamma_4\left(1+3\widetilde{\lambda}\right)}, \\ \frac{\left|h'''(0)\right|+\left|l'''(0)\right|}{36}\left|\frac{1}{\Gamma_4\left(1+3\widetilde{\lambda}\right)} - \frac{5}{2\Gamma_2\Gamma_3\left(1+\widetilde{\lambda}\right)\left(1+2\widetilde{\lambda}\right)-5\Gamma_4\left(1+3\widetilde{\lambda}\right)}\right| + \\ \left. \frac{\left|h''(0)\right|+\left|l''(0)\right|}{\left|2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}\sqrt{\frac{|h''(0)|+|l''(0)|}{\left|2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}}\frac{\Gamma_2\left(1+\widetilde{\lambda}\right)\left|3\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}{24\Gamma_4\left(1+3\widetilde{\lambda}\right)}\right\}$$

(5.17)

5.2 Extensions of coefficient estimates for new classes of bi-univalent functions defined by Sălăgean integro-differential operator

In the sequel, it is assumed that φ, ψ are analytic functions with positive real part in the unit disk U, satisfying $\varphi(0) = \psi(0) = 1$, $\varphi'(0) > 0$, $\psi'(0) > 0$ and $\varphi(U), \psi(U)$ are symmetric with respect to the real axis (see [127]). Assume also that:

$$\varphi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots, \quad (A_1 > 0) \tag{5.18}$$

and

$$\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{5.19}$$

Definition 5.4. [83] A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{M}_\Sigma(\varphi, \psi)$ if the following conditions are satisfied:

$$\left(\mathscr{D}I^n f(z)\right)' \prec \varphi(z) \tag{5.20}$$

and

$$\left(\mathscr{D}I^n g(w)\right)' \prec \psi(w) \tag{5.21}$$

where $z, w \in U$ and the function g is given by (5.2).

Definition 5.5. [83] For $0 < \alpha$, a function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\varphi, \psi, \alpha)$ if the following conditions are satisfied:

$$\left(\frac{z(\mathscr{D}I^n f(z))'}{\mathscr{D}I^n f(z)}\right)^\alpha \left(1 + \frac{z(\mathscr{D}I^n f(z))''}{(\mathscr{D}I^n f(z))'}\right)^{1-\alpha} \prec \varphi(z) \tag{5.22}$$

and

$$\left(\frac{w(\mathscr{D}I^n g(w))'}{\mathscr{D}I^n g(w)}\right)^\alpha \left(1 + \frac{w(\mathscr{D}I^n g(w))''}{(\mathscr{D}I^n g(w))'}\right)^{1-\alpha} \prec \psi(w) \tag{5.23}$$

where $z, w \in U$ and the function g is given by (5.2).

Definition 5.6. [83] Let $h, l : U \to \mathbb{C}$ be analytic functions and

$$\min\{\Re(h(z)), \Re(l(z))\} > 0, \quad (z \in U) \quad h(0) = l(0) = 1.$$

A function $f(z)$ given by (1.1) is said to be in the class $\mathscr{H}_{\Sigma}^{h,l}(\alpha)$ if the following conditions are satisfied:

$$\left(\frac{z(\mathscr{D}I^n f(z))'}{\mathscr{D}I^n f(z)}\right)^{\alpha}\left(1+\frac{z(\mathscr{D}I^n f(z))''}{(\mathscr{D}I^n f(z))'}\right)^{1-\alpha} \in h(U) \tag{5.24}$$

and

$$\left(\frac{w(\mathscr{D}I^n g(w))'}{\mathscr{D}I^n g(w)}\right)^{\alpha}\left(1+\frac{w(\mathscr{D}I^n g(w))''}{(\mathscr{D}I^n g(w))'}\right)^{1-\alpha} \in l(U) \tag{5.25}$$

where $z,w \in U$ and the function g is given by (5.2).

In particular, taking $n=0,\alpha=1$ and $n=0,\alpha=0$ in Definition 5.6, we can obtain te subclasses of bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type functions.

Theorem 5.6. *[83] Let $f(z)$ given by* (1.1) *be in the class $\mathscr{M}_{\Sigma}(\varphi,\psi)$. Then*

$$|a_2| \le \sqrt{\frac{1}{6\Gamma_3}\left|A_2+B_1-\frac{A_1^2}{B_1}\right|}, \tag{5.26}$$

$$|a_3| \le \frac{A_1}{3\Gamma_3}+\frac{1}{6\Gamma_3}\left|2A_2-2A_1-\frac{B_2A_1^2}{B_1^2}\right| \tag{5.27}$$

and

$$|a_4| \le \frac{B_3}{4\Gamma_4} \tag{5.28}$$

Theorem 5.7. *[83] Let $f(z)$ given by* (1.1) *be in the class $\mathscr{H}_{\Sigma}(\varphi,\psi,\alpha)$. Then*

$$|a_2| \le \sqrt{\frac{A_2+B_1+\frac{A_1^2}{B_1^2}|B_2-B_1|}{\left|\alpha^2+5\alpha-8\right|\Gamma_2^2+4|3-2\alpha|\Gamma_3}}, \tag{5.29}$$

$$|a_3| \le \frac{A_2+B_1+\frac{A_1^2}{B_1^2}|B_2-B_1|}{\left|\alpha^2+5\alpha-8\right|\Gamma_2^2+4|3-2\alpha|\Gamma_3}+\frac{|A_2-B_2|}{4|3-2\alpha|\Gamma_3} \tag{5.30}$$

and

$$|a_4| \le \left|\frac{B_3-A_3}{6\Gamma_4(3\alpha-4)}-\frac{5}{2}\frac{A_3+B_3}{15\Gamma_4(3\alpha-4)-2\Gamma_2\Gamma_3\left(4\alpha^2+11\alpha-18\right)}+\right.$$
$$\left.+\frac{A_1^3}{6(2-\alpha)^3\Gamma_2^2\Gamma_4(3\alpha-4)}\left[2\Gamma_3\left(4\alpha^2+11\alpha-18\right)-\frac{1}{3}\Gamma_2^2\left(\alpha^3+21\alpha^2+20\alpha-48\right)\right]\right|$$

(5.31)

Theorem 5.8. *[83] Let $f(z)$ given by* (1.1) *be in the class $\mathcal{H}_{\Sigma}^{h,l}(\alpha)$. Then*

$$|a_2| \le \min\left\{\sqrt{\frac{|h'(0)|^2+|l'(0)|^2}{2\Gamma_2^2(2-\alpha)^2}};\sqrt{\frac{|h''(0)|+|l''(0)|}{2\left|\alpha^2+5\alpha-8\right|\Gamma_2^2+8\Gamma_3|3-2\alpha|}}\right\} \tag{5.32}$$

$$|a_3| \le \min\left\{\frac{|h''(0)|+|l''(0)|}{8\Gamma_3(3-2\alpha)}+\frac{|h'(0)|^2+|l'(0)|^2}{2\Gamma_2^2(2-\alpha)^2},\right.$$
$$\left.\frac{|h''(0)|\left|\alpha^2+5\alpha-8\right|\Gamma_2^2+|l''(0)|\left|(\alpha^2+5\alpha-8)\Gamma_2^2+8\Gamma_3(3-2\alpha)\right|}{8\Gamma_3\left|(3-2\alpha)\left[(\alpha^2+5\alpha-8)\Gamma_2^2+4\Gamma_3(3-2\alpha)\right]\right|}\right\}$$

(5.33)

and

$$|a_4| \le \frac{1}{18\Gamma_4|3\alpha-4|}\min\left\{|h'''(0)|+\left|\alpha^3+21\alpha^2+20\alpha-48\right|\left(\frac{|h'(0)|^2+|l'(0)|^2}{2(2-\alpha)^2}\right)^{\frac{3}{2}};\right.$$
$$\left.|h'''(0)|+\left|\alpha^3+21\alpha^2+20\alpha-48\right|\Gamma_2^3\left(\frac{|h''(0)|+|l''(0)|}{2\left|\alpha^2+5\alpha-8\right|\Gamma_2^2+8\Gamma_3|3-2\alpha|}\right)^{\frac{3}{2}}\right\}$$

(5.34)

5.3 Coefficient estimates for some new classes of bi-Bazilevič functions of Ma-Minda type involving the Sălăgean integro-differential operator

Definition 5.7. Let $h: U \to \mathbb{C}$ be a convex univalent function in U such that

$$h(0)=1 \quad \text{and} \quad \Re\,(h(z))>0, \quad (z\in U).$$

Suppose that $h(z)=1+\sum_{k=1}^{\infty}B_k z^k$, $(z\in U)$.

Motivated by the work of Strivastava et al. (see [116]) we introduced new subclasses of bi-univalent functions. A function $f\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{M}_{\Sigma}^{n}(\beta,\widetilde{\lambda};h)$ if the following conditions are satisfied:

$$e^{i\beta}\left(\frac{z^{1-\widetilde{\lambda}}(\mathcal{D}I^n f(z))'}{[\mathcal{D}I^n f(z)]^{1-\widetilde{\lambda}}}\right)\prec h(z)\cos\beta+i\sin\beta \tag{5.35}$$

and

$$e^{i\beta}\left(\frac{w^{1-\widetilde{\lambda}}(\mathcal{D}I^n g(w))'}{[\mathcal{D}I^n g(w)]^{1-\widetilde{\lambda}}}\right)\prec h(w)\cos\beta+i\sin\beta, \tag{5.36}$$

where

$$\beta\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right),\ \ \widetilde{\lambda}\ge 0;\quad z,w\in U \quad \text{and the function } g \text{ is given by (5.2).}$$

Lemma 5.2. *([101], [30]) Let the function $\Psi(z)$ given by $\Psi(z)=\sum_{k=1}^{\infty}B_kz^k$, $(z\in U)$ be convex in U. Suppose that the function $h(z)=\sum_{k=1}^{\infty}h_kz^k$ is holomorphic in U. If $h(z)\prec\Psi(z)$, $(z\in U)$ then $|h_k|\le|B_1|$, $(n\in\mathbb{N})$.*

Theorem 5.9. *[80] Let the function $f(z)$ given by (1.1) be in the class $\mathscr{M}_{\Sigma}^{n}(\beta,\tilde{\lambda};h)$. Then*

$$|a_2|\le\sqrt{\frac{2|B_1|\cos\beta}{\left(\tilde{\lambda}+2\right)\left|\left(\tilde{\lambda}-1\right)\Gamma_2^2+2\Gamma_3\right|}},\quad \left(\tilde{\lambda}-1\right)\Gamma_2^2+2\Gamma_3\neq0 \tag{5.37}$$

$$|a_3|\le\frac{|B_1|\cos\beta}{\left(\tilde{\lambda}+2\right)\Gamma_3}+\left(\frac{|B_1|\cos\beta}{\left(1+\tilde{\lambda}\right)\Gamma_2}\right)^2 \tag{5.38}$$

and

$$|a_4|\le|B_1|\cos\beta\left\{\frac{\Gamma_2}{3\Gamma_4}\cdot\frac{\left(1-\tilde{\lambda}\right)\left[\left(\tilde{\lambda}-2\right)\Gamma_2^2+6\Gamma_3\right]}{\left(\tilde{\lambda}+2\right)\left[\left(\tilde{\lambda}-1\right)\Gamma_2^2+2\Gamma_3\right]}\sqrt{\frac{2|B_1|\cos\beta}{\left(\tilde{\lambda}+2\right)\left|\left(\tilde{\lambda}-1\right)\Gamma_2^2+2\Gamma_3\right|}}+\right.$$
$$\left.+\frac{1}{\left(3+\tilde{\lambda}\right)\Gamma_4}+\frac{5}{\left(\tilde{\lambda}+3\right)\left|2\left(\tilde{\lambda}-1\right)\Gamma_2\Gamma_3+5\Gamma_4\right|}\right\},\quad \left(\tilde{\lambda}-1\right)\Gamma_2^2+2\Gamma_3\neq0$$

and

$$2\left(\tilde{\lambda}-1\right)\Gamma_2\Gamma_3+5\Gamma_4\neq0. \tag{5.39}$$

Corollary 5.1. *[80] Let the function $f(z)$ given by (1.1) be in the class $\mathscr{M}_{\Sigma}^{n}(\beta,0;h)$. Then*

$$|a_2|\le\sqrt{\frac{|B_1|\cos\beta}{2\Gamma_3-\Gamma_2^2}},\qquad |a_3|\le\frac{|B_1|\cos\beta}{2\Gamma_3}+\left(\frac{|B_1|\cos\beta}{\Gamma_2}\right)^2,\quad\text{and}$$

$$|a_4|\le\frac{|B_1|\cos\beta}{3}\left\{\frac{\Gamma_2\left(-\Gamma_2^2+3\Gamma_3\right)}{\Gamma_4\left(-\Gamma_2^2+2\Gamma_3\right)}\sqrt{\frac{|B_1|\cos\beta}{-\Gamma_2^2+2\Gamma_3}}+\frac{1}{\Gamma_4}-\frac{5}{2\Gamma_2\Gamma_3-5\Gamma_4}\right\}.$$

Corollary 5.2. *[80] Let the function $f(z)$ given by (1.1) be in the class $\mathscr{M}_{\Sigma}^{n}(\beta,1;h)$. Then*

$$|a_2|\le\sqrt{\frac{|B_1|\cos\beta}{3\Gamma_3}},\qquad |a_3|\le\frac{|B_1|\cos\beta}{3\Gamma_3}+\left(\frac{|B_1|\cos\beta}{2\Gamma_2}\right)^2,\quad\text{and}$$

$$|a_4|\le|B_1|\cos\beta\frac{1}{2\Gamma_4}.$$

Corollary 5.3. *[80] Let the function $f(z)$ given by (1.1) be in the class $\mathscr{M}_{\Sigma}^{0}(\beta,0;h)$. Then*

$$|a_2|\le\sqrt{|B_1|\cos\beta},\qquad |a_3|\le|B_1|\cos\beta\left(|B_1|\cos\beta+\frac{1}{2}\right),\quad\text{and}$$

$$|a_4|\le\frac{2|B_1|\cos\beta}{3}\left(\sqrt{|B_1|\cos\beta}+\frac{4}{3}\right).$$

Corollary 5.4. *[80] Let the function $f(z)$ given by* (1.1) *be in the class $\mathscr{M}_{\Sigma}^{0}(\beta, 1; h)$. Then*

$$|a_2| \leq \sqrt{\frac{|B_1|\cos\beta}{3}}, \qquad |a_3| \leq \frac{|B_1|\cos\beta}{3} + \left(\frac{|B_1|\cos\beta}{2}\right)^2, \quad \text{and}$$

$$|a_4| \leq \frac{|B_1|\cos\beta}{2}.$$

Chapter 6

Harmonic functions

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain G if both u and v are real and harmonic in G. In any simply-connected domain $D \subset G$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [26]).

We denote with $\mathscr{H}$ the family of continuous complex-valued functions that are harmonic in the open unit disk U, with $\mathscr{H}_u$ the harmonic univalent functions and with $\mathscr{H}_{op}$ the harmonic orientation preserving functions.
Let $\mathscr{H}_{u,op}$ denote the family of functions

$$f = h + \overline{g} \tag{6.1}$$

which are harmonic, univalent and orientation preserving in the open unit disc U so that f is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathscr{H}_{u,op}$, the functions h and g analytic in U can be expressed in the following forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m \quad (|b_1| < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \quad (|b_1| < 1). \tag{6.2}$$

For functions $f \in \mathscr{H}_{u,op}$ given by (6.2) and $F \in \mathscr{H}_{u,op}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{m=2}^{\infty} A_m z^m + \overline{\sum_{m=1}^{\infty} B_m z^m}, \tag{6.3}$$

we denote the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \overline{\sum_{m=1}^{\infty} b_m B_m z^m} \quad (z \in U). \tag{6.4}$$

Let $V_{\mathscr{H}}$ be the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [46], consisting of functions f of the form (6.1) in $\mathscr{H}_{u,op}$ for which there exists a real number ξ such that

(6.5) $$\eta_m + (m-1)\xi \equiv \pi(mod\ 2\pi), \quad \delta_m + (m+1)\xi \equiv 0(mod\ 2\pi), \quad m \geq 2,$$

where $\eta_m = arg(a_m)$ and $\delta_m = arg(b_m)$.

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathscr{H}_{u,op}$ such that $(f * F)(z)$ exists for any $f \in \mathscr{H}_{u,op}$, and for various choices of F we get different linear operators which have been studied in recent past.
In [65] it is defined and studied a subclass of $\mathscr{H}_{u,op}$ denoted by $S_{\mathscr{H}}(F;\gamma)$, for $0 \leq \gamma < 1$, which involves the convolution (6.4) and consist of functions of the form (6.1) satisfying the inequality:

(6.6) $$\frac{\partial}{\partial\theta}(arg[(f * F)(z)]) > \gamma$$

$0 \leq \theta < 2\pi$ and $z = re^{i\theta}$. Equivalently

(6.7) $$Re\left\{\frac{z(h(z) * H(z))' - \overline{z(g(z) * G(z))'}}{h(z) * H(z) + \overline{g(z) * G(z)}}\right\} \geq \gamma$$

where $z \in U$. We also let $\mathcal{V}_{\mathscr{H}}(F;\gamma) = S_{\mathscr{H}}(F;\gamma) \bigcap V_{\mathscr{H}}$. Some of the function classes emerge from the function class $S_{\mathscr{H}}(F;\gamma)$ defined above. Indeed, if we specialize the function $F(z)$ we can obtain, respectively, (see [65]) the class of functions defined using: the Wright's generalized operator on harmonic functions ([66],[126]), the Dzioc-Srivastava operator on harmonic functions ([3]), the Carlson-Shaffer operator ([25]), the Ruscheweyh derivative operator on harmonic functions ([45],[64],[102]) , the Srivastava-Owa fractional derivative operator ([117]), the Salagean derivative operator for harmonic functions ([49], [106]).

Let

(6.8) $$f(z) = h(z) + \overline{g(z)} = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m}.$$

A function f of the form (6.8) is said to be in $S^*_{\mathscr{H}}(\alpha)$ if and only if (see [26], [34], [30])

(6.9) $$\frac{\partial}{\partial\theta}\left(arg\ f\left(re^{i\theta}\right)\right) > \alpha, \quad 0 \leq \theta < 2\pi, \quad |z| = r < 1.$$

Similarly, a function f of the form (6.8) is said to be in $S^c_{\mathscr{H}}(\alpha)$ if and only if

(6.10) $$\frac{\partial}{\partial\theta}\left(arg\frac{\partial}{\partial\theta}\left(f\left(re^{i\theta}\right)\right)\right) > \alpha, \quad 0 \leq \theta < 2\pi, \quad |z| = r < 1.$$

We note that (see [48]) a harmonic function $f \in S^*_{\mathscr{H}}(\alpha)$ if and only if

$$\Re\frac{J_{\mathscr{H}} f(z)}{f(z)} > \alpha, \quad |z| = r < 1,$$

or

$$\left|\frac{J_{\mathscr{H}}f(z)-(1+\alpha)f(z)}{J_{\mathscr{H}}f(z)+(1-\alpha)f(z)}\right|<1, |z|=r<1,$$

where

$$J_{\mathscr{H}}f(z)=zh'(z)-\overline{zg'(z)}.$$

Definition 6.1. Let $\mathscr{B}\subseteq\mathscr{H}$. We define the radius of starlikeness and the radius of convexity of the class $\mathscr{B}$:

$$R^*_\alpha(\mathscr{B}):=\inf_{f\in\mathscr{B}}(\sup\{r\in(0,1]: f \text{ is starlike of order } \alpha \text{ in } U(r)\}),$$

$$R^c_\alpha(\mathscr{B}):=\inf_{f\in\mathscr{B}}(\sup\{r\in(0,1]: f \text{ is convex of order } \alpha \text{ in } U(r)\}).$$

Definition 6.2. The generalized Bernardi-Libera-Livingston integral operator for harmonic functions is $\mathscr{L}_c(f)$, $(c>-1)$: which is defined by $\mathscr{L}_c(f)=\mathscr{L}_c(h)+\overline{\mathscr{L}_c(g)}$ where

$$\mathscr{L}_c(h)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt \text{ and } \mathscr{L}_c(g)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt$$

(see [65]).

6.1 On a certain class of harmonic functions and the generalized Bernardi-Libera-Livingston integral operator

In the following we suppose that $F(z)$ is of the form

(6.11)
$$F(z)=H(z)+\overline{G(z)}=z+\overline{z}+\sum_{m=2}^{\infty}C_m\left(z^m+\overline{z^m}\right),$$

where $C_m\geq 0\,(m\geq 2)$.

In [65] the following characterization theorem is proved

Theorem 6.1. *Let $f=h+\overline{g}$ be given by* (6.2) *with restrictions* (6.5) *and* $0\leq b_1<\frac{1-\gamma}{1+\gamma}, 0\leq\gamma<1$. *Then $f\in\mathcal{V}_{\mathscr{H}}(F;\gamma)$ if and only if the inequality*

(6.12)
$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_m|+\frac{m+\gamma}{1-\gamma}|b_m|\right)C_m\leq 1-\frac{1+\gamma}{1-\gamma}b_1$$

holds true.

Theorem 6.2. *[65] Set $\lambda_m=\frac{1-\gamma}{(m-\gamma)C_m}$ and $\mu_m=\frac{1-\gamma}{(m+\gamma)C_m}$. Then for b_1 fixed, $0\leq b_1<\frac{1-\gamma}{1+\gamma}$ the extreme points for $\mathcal{V}_{\mathscr{H}}(F;\gamma)$, $0\leq\gamma<1$ are*

$$\left\{z+\lambda_m xz^m+\overline{b_1z}\right\}\cup\left\{z+\overline{b_1z+\mu_m xz^m}\right\}$$

where $m\geq 2$ and $x=1-\frac{1+\gamma}{1-\gamma}b_1$.

The closure properties of the class $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_c(f)$, $(c>-1)$:

Theorem 6.3. *[98] Let $f\in\mathcal{V}_{\mathcal{H}}(F;\gamma)$. Then $\mathcal{L}_c(f)\in\mathcal{V}_{\mathcal{H}}(F;\delta(\gamma))$ where*

$$\delta(\gamma)=\frac{(2+\gamma)(c+2)(1-b_1)-2(c+1)\left[(1-\gamma)-(1+\gamma)b_1\right]}{(2+\gamma)(c+2)(1+b_1)+(c+1)\left[(1-\gamma)-(1+\gamma)b_1\right]}>\gamma.$$

The result is sharp.

6.2 On the order of convolution consistence of the harmonic functions with varying arguments

Let us consider the integral operator (for the analytic case see [21], [14], [106])
$\mathcal{I}^s : f\in\mathcal{V}_{\mathcal{H}}(F,\gamma)\to\mathcal{V}_{\mathcal{H}}(F,\gamma), s\in\mathbb{R}$, such that

$$\mathcal{I}^s f(z)=\mathcal{I}^s\left(z+\sum_{m=2}^{\infty}a_m z^m+\overline{\sum_{m=1}^{\infty}b_m z^m}\right)=z+\sum_{m=2}^{\infty}\frac{a_m}{m^s}z^m+\overline{\sum_{m=1}^{\infty}\frac{b_m}{m^s}z^m}. \tag{6.13}$$

Definition 6.3. The modified Hadamard product or $\circledast$-convolution of two functions f_1 and f_2 in $\mathcal{V}_{\mathcal{H}}$ of the form

$$f_1(z)=z+\sum_{m=2}^{\infty}a_{1,m}z^m+\overline{\sum_{m=1}^{\infty}b_{1,m}z^m}\text{ and }f_2(z)=z+\sum_{m=2}^{\infty}a_{2,m}z^m+\overline{\sum_{m=1}^{\infty}b_{2,m}z^m} \tag{6.14}$$

is the function $(f\circledast g)$ defined as

$$(f_1\circledast f_2)(z)=z-\sum_{m=2}^{\infty}a_{1,m}a_{2,m}z^m+\overline{\sum_{m=1}^{\infty}b_{1,m}b_{2,m}z^m}.$$

We note that $(f\circledast g)$ also belongs to $\mathcal{V}_{\mathcal{H}}$.

Definition 6.4. ([21], [107]) Let $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ be subsets of $\mathcal{V}_{\mathcal{H}}(F;\gamma)$. We say that the three $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is $S_{\circledast}$-closed under the convolution if there exists a number $S_{\circledast}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ such that

$$S_{\circledast}(\mathcal{X},\mathcal{Y},\mathcal{Z})=min\left\{s\in\mathbb{R}:\mathcal{I}^s(f\circledast g)\in\mathcal{Z},\forall f\in\mathcal{X},\forall g\in\mathcal{Y}\right\} \tag{6.15}$$

The number $S_{\circledast}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is called the order of $\circledast$-convolution consistence of the three $(\mathcal{X},\mathcal{Y},\mathcal{Z})$

U. Bednarz and J. Sokol in [21] obtained the order of convolution consistence concerning certain classes of univalent functions (starlike, convex, uniform-starlike or uniform-convex functions) and in [107] it is obtained the order of $\circledast$-convolution consistence for certain classes of analytic functions with negative coefficients. In this paper we obtain similar results, but concerning the class $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ and for $\circledast$-convolution.

Let denote by $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$ the subset of $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ consisting of functions of the form (6.2) which satisfy $|a_m|\le 1, |b_m|\le 1, \forall m\ge 2$.

Theorem 6.4. *[97] Let f_1, f_2 be two functions in $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$ of the form* (6.1)*; then $(f_1 \circledast f_2)$also belongs to $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$.*

Remark 6.1. *[97] Let the function $F = F_{m_0}, (m_0 \geq 2)$ be of the form* (6.11) *with $C_{m_0} = \dfrac{1-\gamma}{m_0-\gamma}$; then if*

$$f_1(z) = f_2(z) = z - \frac{z^{m_0}}{C_0 \dfrac{m_0-\gamma}{1-\gamma}} \tag{6.16}$$

then the condition (6.12) *for f_1 becomes $\dfrac{m_0-\gamma}{1-\gamma}\left(\left|a_{1,m_0}\right| + \left|b_{1,m_0}\right|\right) C_{m_0} = 1$ and similar for f_2 and this shows that f_1, f_2 belong to $\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$. For the function $(f_1 \circledast f_2)$ we have*

$$\frac{m_0-\gamma}{1-\gamma}\left(\left|a_{1,m_0}\right|\left|a_{2,m_0}\right| + \left|b_{1,m_0}\right|\left|b_{2,m_0}\right|\right) C_{m_0} = \frac{m_0-\gamma}{1-\gamma}\frac{1}{C^2_{m_0}}\left(\frac{1-\gamma}{m_0-\gamma}\right)^2 C_{m_0} = 1$$

and this imply that also $(f_1 \circledast f_2) \in \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$. This shows that the result in Theorem 6.4 is sharp when $F = F_{m_0}, (m_0 \geq 2)$.

Corollary 6.1. *[97] The order of $\circledast$-convolution consistence for the classes $\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$ is*

$$S_{\circledast}\left(\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma), \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma), \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)\right) = 0 \tag{6.17}$$

Theorem 6.5. *[97] Let $f_1 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_1)$, $f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_2)$ be two functions of the form* (6.1) *then $(f_1 \circledast f_2)$ belongs to $\mathcal{V}^1_{\mathcal{H}}(F;\gamma^*)$, where*
$\gamma^* = \dfrac{(2+\gamma_1)(2+\gamma_2)(1-b_{1,1}b_{2,1}) - 2\left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2+\gamma_1)(2+\gamma_2)(1+b_{1,1}b_{2,1}) + \left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$, *if*

$$1 - b_{1,1}b_{2,1} - (1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) > 0$$

or $\gamma^* = \dfrac{(2-\gamma_1)(2-\gamma_2)(1-b_{1,1}b_{2,1}) - 2\left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2-\gamma_1)(2-\gamma_2)(1+b_{1,1}b_{2,1}) - \left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$, *if*

$$1 - b_{1,1}b_{2,1} - (1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) < 0.$$

6.3 A unified class of harmonic functions with varying argument of coefficients

Let k, A and B be real parameters such that

$$k \geq 0,\ 0 \leq B \leq 1 \text{ and } -1 \leq A < B.$$

Also let $\varphi, \phi \in \mathcal{H}$. Motivated by J.Dziok [35] ,we denote by $\mathcal{W}_{\mathcal{H}}(\phi, \varphi; A, B; k)$, $0 \leq B < 1$ the class of functions $f \in \mathcal{H}$ such that

$$(\varphi * f)(z) \neq 0, z \in U \setminus \{0\}$$

and

$$\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - k\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| - \frac{1-AB}{1-B^2}\right| < \frac{B-A}{1-B^2} \quad (z \in U). \tag{6.18}$$

If $B = 1$, then we have

$$\Re\left(\frac{(\phi * f)(z)}{(\varphi * f)(z)}\right) - k\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| > \frac{1+A}{2} \quad (z \in U). \tag{6.19}$$

Let us define

$$\mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k) := \mathscr{W}_{\mathscr{H}}(\phi,\varphi;A,B;k) \cap V_{\mathscr{H}}.$$

We assume that φ and ϕ are the functions of the following forms:

$$\varphi(z) = z + \sum_{m=2}^{\infty} c_m z^m + \overline{\sum_{m=1}^{\infty} d_m z^m} \ \text{ and } \ \phi(z) = z + \sum_{m=2}^{\infty} e_m z^m + \overline{\sum_{m=1}^{\infty} f_m z^m} \tag{6.20}$$

where

$$0 \le c_m \le e_m \ \text{ and } \ 0 \le d_m \le f_m. \tag{6.21}$$

Theorem 6.6. *[74] Let* $0 \le B \le 1$, $-1 \le A < B$ and $(\varphi * f)(z) \neq 0, z \in U \setminus \{0\}$. *If*

$$\sum_{m=2}^{\infty} \left(|a_m|\alpha_m + |b_m|\beta_m\right) \le B - A \tag{6.22}$$

then $f \in \mathscr{W}_{\mathscr{H}}(\phi,\varphi;A,B;k)$ *where*

$$\alpha_m = (k+1)(1+B)e_m + [(B-A)-(k+1)(1+B)]c_m,$$

$$\beta_m = (k+1)(1+B)f_m + [(B-A)-(k+1)(1+B)]d_m.$$

Theorem 6.7. *[74] Let f be a function of the form* (6.2) *satisfying the argument property* (6.5). *Then* $f \in \mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)$ *if and only if condition* (6.22) *holds true.*

Corollary 6.2. *[74] If a function f of the form* (6.2) *belongs to the class* $\mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)$, *then*

$$|a_m| \le \frac{B-A}{\alpha_m}, \ \ |b_m| \le \frac{B-A}{\beta_m}, \quad (m \in \{2,3,\ldots\}) \tag{6.23}$$

where α_m and β_m are defined by (6.22). *The result is sharp and the extremal functions are*

$$f_{1,m} = z - \frac{B-A}{\alpha_m} e^{i(1-m)\xi} z^m, \tag{6.24}$$

and

$$f_{2,m} = z + \frac{B-A}{\beta_m} e^{i(1+m)\xi} \overline{z}^m, \ m \in \{2,3,\cdots\}. \tag{6.25}$$

Theorem 6.8. *[74] Let f be a function in the class $\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)$ then*

$$(1+|b_1|)r-\frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}r^2\le|f(z)|\le$$

$$\le(1+|b_1|)r+\frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2}r^2 \tag{6.26}$$

where $\alpha_2\le\alpha_m$, $\beta_2\le\beta_m$ $(m\in\mathbb{N}\setminus\{1\})$.

Theorem 6.9. *[74] Let $0\le\alpha<1$ and α_k and β_k be defined by (6.22). Then*

$$R_\alpha^*\left(\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)\right)=\inf_{m\ge2}\left(\frac{1-\alpha}{B-A}min\left\{\frac{\alpha_m}{m-\alpha},\frac{\beta_m}{m+\alpha}\right\}\right)^{\frac{1}{m-1}}.$$

Theorem 6.10. *[74] Let $0\le\alpha<1$ and α_k and β_k be defined by (6.22). Then*

$$R_\alpha^c\left(\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)\right)=\inf_{m\ge2}\left(\frac{1-\alpha}{B-A}min\left\{\frac{\alpha_m}{m(m-\alpha)},\frac{\beta_m}{m(m+\alpha)}\right\}\right)^{\frac{1}{m-1}}.$$

Theorem 6.11. *[74] If $f,F\in\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)$ and $|a_m|,|b_m|,|A_m|,|B_m|\in[0,1]$ then $f*F\in\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)$.*

Theorem 6.12. *[74] Let $f\in\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)$. Then $\mathscr{L}_c(f)\in\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A^*,B;k\right)$ where $A^*=min\left\{A_1^*,A_2^*\right\}>A$,*

$$A_1^*=B-\frac{(B-A)(k+1)(1+B)(c+1)(e_m-c_m)}{(B-A)(m-1)c_m+(k+1)(1+B)(c+m)(e_m-c_m)},$$

$$A_2^*=B-\frac{(B-A)(k+1)(1+B)(c+1)(f_m-d_m)}{(B-A)(m-1)d_m+(k+1)(1+B)(c+m)(f_m-d_m)}.$$

Theorem 6.13. *[74] Let $f\in\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)$. Then $\mathscr{L}_c(f)\in\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B^*;k\right)$ where $B^*=min\left\{B_1^*,B_2^*\right\}<B$,*

$$B_1^*=A+\frac{(B-A)(k+1)(1+A)(c+1)(e_m-c_m)}{(B-A)(m-1)c_m+(k+1)(e_m-c_m)[(1+B)(c+m)-(c+1)(B-A)]},$$

$$B_2^*=A+\frac{(B-A)(k+1)(1+A)(c+1)(f_m-d_m)}{(B-A)(m-1)d_m+(k+1)(f_m-d_m)[(1+B)(c+m)-(c+1)(B-A)]}.$$

6.4 Generalizations of starlike harmonic functions defined by Sălăgean and Ruscheweyh derivative

We denote with $\mathscr{L}^n$ the operator defined in Definition 1.12.
We consider the linear operator $\mathscr{L}_{\mathscr{H}}^n:\mathscr{H}\to\mathscr{H}$ defined for a function $f=h+\overline{g}\in\mathscr{H}$ by $\mathscr{L}_{\mathscr{H}}^n f:=\mathscr{L}_{\mathscr{H}}^n h+(-1)^n\overline{\mathscr{L}_{\mathscr{H}}^n g}$. For a function $f\in\mathscr{H}$ of the form (6.1), we have

$$\mathscr{L}_{\mathscr{H}}^n f(z)=z+\sum_{k=2}^{\infty}\left[\gamma\eta(k,n,\lambda)+\left(1-\gamma\right)\mu(k,n)\right]a_kz^k+(-1)^n\sum_{k=2}^{\infty}\left[\gamma\eta(k,n,\lambda)+\left(1-\gamma\right)\mu(k,n)\right]\overline{b_k}\overline{z}^k,z\in U,$$

where $\eta(k,n,\lambda)=[1+(k-1)\lambda]^n$ and $\mu(k,n)=\dfrac{(n+k-1)!}{n!(k-1)!}$.

Definition 6.5. For $-B \le A < B \le 1$ *and* $n \in \mathbb{N}$ let $\widetilde{S}^n_{\mathcal{H}}(A,B)$ denote the class of functions $f \in \mathcal{H}$ such that

$$\left| \frac{\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - \mathcal{L}^{n}_{\mathcal{H}} f(z)}{B\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - A\mathcal{L}^{n}_{\mathcal{H}} f(z)} \right| < 1 \quad (z \in U). \tag{6.27}$$

Remark 6.2. *Dziok et al. studied the case* $\gamma = 0$ *in [33] while the case* $\gamma = 1$ *and* $\lambda = 1$ *was studied in [34].*
Note that the classes $\widetilde{S}^0_{\mathcal{H}}(A,B)$ *for the analytic case, i.e.* $g \equiv 0$, *were introduced by Janowski [50]. Jahangiri [47], [48] and Silverman [113] studied the classes* $S^*_{\mathcal{H}}(\alpha) = \widetilde{S}^0_{\mathcal{H}}(2\alpha - 1, 1)$ *and* $S^c_{\mathcal{H}}(\alpha) = \widetilde{S}^1_{\mathcal{H}}(2\alpha - 1, 1)$ *for the harmonic case.*

Theorem 6.14. *[84] A function* $f \in \mathcal{H}$ *of the form* (6.1) *belongs to the class* $\widetilde{S}^n_{\mathcal{H}}(A,B)$ *if*

$$\sum_{k=2}^{\infty} \left(\alpha_k |a_k| + \beta_k |b_k| \right) \le B - A, \tag{6.28}$$

where

$$\alpha_k = \sigma\left(A,B,n,\gamma,\lambda,k\right) + \sigma\left(1,1,n,\gamma,\lambda,k\right),$$

$$\beta_k = \delta\left(A,B,n,\gamma,\lambda,k\right) + \delta\left(1,1,n,\gamma,\lambda,k\right),$$

$$\sigma\left(A,B,n,\gamma,\lambda,k\right) = \gamma\eta(k,n,\lambda)[(k-1)\lambda B + B - A] + \left(1-\gamma\right)\mu(k,n)\frac{(B-A)n + Bk - A}{n+1},$$

$$\delta\left(A,B,n,\gamma,\lambda,k\right) = \gamma\eta(k,n,\lambda)[(k-1)\lambda B + B + A] + \left(1-\gamma\right)\mu(k,n)\frac{(B+A)n + Bk + A}{n+1}.$$

Theorem 6.15. *[84] If* $f \in \widetilde{S}^n_{\mathcal{H}}(A,B)$ *then* $f \in \mathcal{H}_u$.

Let $\mathcal{N}$ denote the class of functions $f = h + \overline{g} \in \mathcal{H}$ of the form (see [113])

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=2}^{\infty} |b_k| \overline{z}^k, \tag{6.29}$$

and let denote by $\widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)$ the class $\mathcal{N} \cap \widetilde{S}^n_{\mathcal{H}}(A,B)$.

Theorem 6.16. *[84] Let* $f = h + \overline{g}$ *be defined by* (6.29). *Then* $f \in \widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)$ *if and only if the condition* (6.28) *holds true.*

Definition 6.6. We say that a class $\mathcal{F}$ is *convex* if $\eta f + (1-\eta) g \in \mathcal{F}$ for all f and g in $\mathcal{F}$ and $0 \le \eta \le 1$. The *closed convex* hull of $\mathcal{F}$, denoted by $\overline{co}\mathcal{F}$, is the intersection of all closed convex subsets of $\mathcal{H}$ (with respect to the topology of locally uniform convergence) that contain $\mathcal{F}$.

Definition 6.7. Let $\mathcal{F}$ be a convex set. A function $f \in \mathcal{F} \subset \mathcal{H}$ is called *an extreme point* of $\mathcal{F}$ if $f = \eta f_1 + (1-\eta) f_2$ implies $f_1 = f_2 = f$ for all f_1 and f_2 in $\mathcal{F}$ and $0 < \eta < 1$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.

Lemma 6.1. *[33], [34] Let $\mathscr{F}$ be a non-empty compact convex subclass of the class $\mathscr{H}$ and $\mathscr{J}:\mathscr{H}\to\mathbb{R}$ be a real-valued, continuous, and convex functional on $\mathscr{F}$. Then*

$$max\left\{\mathscr{J}(f): f\in\mathscr{F}\right\} = max\left\{\mathscr{J}(f): f\in E\mathscr{F}\right\}$$

.

If $\mathscr{H}$ is a complete metric space, we can use Montel's theorem [63].

Lemma 6.2. *[33], [34] A class $\mathscr{F}\subset\mathscr{H}$ is compact if and only if $\mathscr{F}$ is closed and locally uniformly bounded.*

Theorem 6.17. *[84] The class $\widetilde{S}^n_{\mathscr{H}\mathscr{N}}(A,B)$ is a convex and compact subset of $\mathscr{H}$.*

Theorem 6.18. *[84] The set of extreme points of the class $\widetilde{S}^n_{\mathscr{H}\mathscr{N}}(A,B)$ is*
$E\widetilde{S}^n_{\mathscr{H}\mathscr{N}}(A,B) = \{h_k : k\in\mathbb{N}\}\cup\{g_k : k\in\{2,3,\dots\}\}$

$$h_1 = z,\ \ h_k(z) = z - \frac{B-A}{\alpha_k}z^k,$$

$$g_k(z) = z + (-1)^n\frac{B-A}{\beta_k}\overline{z}^k, (z\in U, k\in\{2,3,\dots\}) \tag{6.30}$$

If the class $\mathscr{F} = \{f_k\in\mathscr{H} : k\in\mathbb{N}\}$ is locally uniformly bounded, then its closed convex hull is $\overline{co}\mathscr{F} = \left\{\sum_{k=1}^{\infty}\eta_k f_k : \sum_{k=1}^{\infty}\eta_k = 1, \eta_k\ge 0 (k\in\mathbb{N})\right\}$.

Corollary 6.3. *[84] Let h_k, g_k be defined by* (6.30)*, then*

$$\widetilde{S}^n_{\mathscr{H}\mathscr{N}}(A,B) = \left\{\sum_{k=1}^{\infty}\left(\eta_k h_k + \delta_k g_k\right) : \sum_{k=1}^{\infty}\left(\eta_k+\delta_k\right) = 1, \delta_1 = 0, \eta_k, \delta_k\ge 0 (k\in\mathbb{N})\right\}.$$

For each fixed value of $k\in\mathbb{N}, z\in U$, the following real-valued functionals are continuous and convex on $\mathscr{H}$:

$$\mathscr{J}(f) = |a_k|, \mathscr{J}(f) = |b_k|, \mathscr{J}(f) = |f(z)|, \mathscr{J}(f) = \left|\mathscr{L}^k_{\mathscr{H}}f(z)\right|, (f\in\mathscr{H}),$$

The real-valued functional

$$\mathscr{J}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\gamma}d\theta\right)^{1/\gamma}\ \left(f\in\mathscr{H}, \gamma\ge 1, 0<r<1\right)$$

is continuous on $\mathscr{H}$. For $\gamma\ge 1$ it is also convex on $\mathscr{H}$ (Minkowski's inequality).

Corollary 6.4. *[84] Let $f\in\widetilde{S}^n_{\mathscr{H}\mathscr{N}}(A,B)$ be a function of the form* (6.29)*. Then*

$$|a_k|\le\frac{B-A}{\alpha_k},\ |b_k|\le\frac{B-A}{\beta_k}, (k=2,3,\dots)$$

where α_k, β_k are defined by (6.28)*. The result is sharp. The extremal functions are h_k, g_k of the form* (6.30)*.*

Theorem 6.19. *[84] Let $f \in \widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)$ and $|z| = r < 1$. Then*

$$r - \frac{B-A}{\alpha_2} r^2 \le |f(z)| \le r + \frac{B-A}{\alpha_2} r^2$$

$$r - \frac{(B-A)\left[\gamma(1+\lambda)^n + \left(1-\gamma\right)(n+1)\right]}{\alpha_2} r^2 \le \left|\mathcal{L}^n_{\mathcal{H}} f(z)\right| \le r + \frac{(B-A)\left[\gamma(1+\lambda)^n + \left(1-\gamma\right)(n+1)\right]}{\alpha_2} r^2$$

The result is sharp. The extremal functions are h_2 of the form (6.30).

Corollary 6.5. *[84] If $f \in \widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)$ then $U(r) \subset f(U(r))$ where*

$$r = 1 - \frac{B-A}{\alpha_2}$$

and

$$U(r) := \{z \in \mathbb{C} : |z| < r \le 1\}.$$

Corollary 6.6. *[84] Let $0 < r < 1$ and $\xi \ge 1$. If $f \in \widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)$ then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\xi} d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|h_2(re^{i\theta})\right|^{\xi} d\theta,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{L}^k_{\mathcal{H}} f(re^{i\theta})\right|^{\xi} d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{L}^k_{\mathcal{H}} h_2(re^{i\theta})\right|^{\xi} d\theta \ (\xi = 1,2,\ldots).$$

Theorem 6.20. *[84] Let $0 \le \alpha < 1$ and α_k and β_k be defined by* (6.28). *Then*

$$R^*_{\alpha}\left(\widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)\right) = \inf_{k\ge 2}\left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_k}{k-\alpha}, \frac{\beta_k}{k+\alpha}\right\}\right)^{\frac{1}{k-1}}$$

Similarly, we get:

Theorem 6.21. *[84] Let $0 \le \alpha < 1$ and α_k and β_k be defined by* (6.28). *Then*

$$R^c_{\alpha}\left(\widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)\right) = \inf_{k\ge 2}\left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_k}{k(k-\alpha)}, \frac{\beta_k}{k(k+\alpha)}\right\}\right)^{\frac{1}{k-1}}.$$

Theorem 6.22. *[84] Let $f \in \widetilde{S}^n_{\mathcal{H}}(A,B)$. Then $\mathcal{L}_c(f) \in \widetilde{S}^n_{\mathcal{H}}(A,B)$.*

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