Facultatea de Matematică şi Informatică
Universitatea Babeş-Bolyai

Studiul proprietăţilor unor clase de funcţii analitice sau armonice

Rezumat

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1.1 Introducere

Analiza complexă este o ramură a matematicii cu largi aplicaţii în diferite domenii ale ştiinţei şi tehnicii.

Începând cu prima parte a secolului XX analiza reală cât şi cea complexă au cunoscut un nou avânt datorat atât necesităţilor interne de dezvoltare, cât şi solicitărilor venite din partea altor domenii ale matematicii sau ale altor ştiinţe.

Bazele teoriei geometrice a funcţiilor de o variabilă complexă, cu rădăcini în secolul 19 şi chiar mai devreme, au fost puse odată cu lucrările lui P. Koebe (1907) şi L. Bieberbach, care în 1916 enunţa celebra conjectură demonstrată doar în anul 1984 de către Louis de Branges. Funcţiile analitice de o variabilă complexă constituie modelul ideal al transformărilor geometrice din plan.

Una dintre centrele importante în domeniul teoriei geometrice a funcţiilor s-a dezvoltat la Cluj, acolo unde G. Călugăreanu a obţinut rezultate semnificative în 1931, stabilind primele condiţii necesare şi suficiente de univalenţă exprimate cu ajutorul coeficienţilor.

Unul dintre domeniile care a stârnit interesul unui număr mare de matematicieni din întreaga lume este cel al teoriei geometrice a funcţiilor complexe de una sau mai multe variabile, o ramură aparte a analizei complexe. Se abordează probleme cum ar fi cele referitoare la subordonările diferenţiale, operatorii integrali sau diferenţiali ce acţionează în diferite clase de funcţii, alte proprietăţi ale unor clase de funcţii analitice, univalente sau multivalente, unele cu coeficienţi negativi.

La acestea a contribuit cu rezultate importante şcoala din Cluj, desăvârşit prin activitatea regretatului profesor Petru T. Mocanu. Acesta, împreună cu S. S. Miller a iniţiat "metoda funcţiilor admisibile" sau metoda subordonărilor diferenţiale. Dezvoltând această metodă de către alţi matematicieni, s-a realizat demonstrarea mult mai simplă a anumitor rezultate clasice din acest domeniu , ajungându-se chiar la extinderi ale acestora şi rezultate noi referitoare la studiul diverşilor operatori integrali pe spaţii de funcţii, criterii simple de stelaritate şi convexitate, conservarea unor proprietăţi geometrice şi analitice prin intermediul unor operatori diferenţiali şi integrali. în prezent teoria subordonărilor diferenţiale este studiată şi utilizată cu succes de matematicieni valoroşi din întreaga lume, cum ar fi S.U.A, Germania, Polonia, Turcia, India, China, Japonia, Canada sau Egipt.

Tot lui Mocanu şi Miller se datorează introducerea noţiunii de superordonare diferenţială, noţiune duală a celei de subordonare diferenţială. Teoria lanţurilor de subordonare diferenţială este una din cele mai moderne teorii din analiza complexă, care oferă noi direcţii de cercetare ale unor probleme importante referitoare la funcţiile univalente.

O altă metodă de cercetare descoperită şi utilizată recent este cea bazată pe utilizarea unor condiţii asupra coeficienţilor dezvoltării în serii de puteri ale funcţiilor analitice (funcţii analitice cu coeficienţi negativi, pozitivi). Folosind această metodă s-au obţinut rezultate apreciate şi citate de matematicieni din diverse ţări ale lumii.

Astăzi analiza complexă este prezentă atât la Cluj, prin activitatea unor cercetători de talie internaţională cum ar fi G. Ş. Sălăgean, G. Kohr sau T. Bulboacă, cât şi în alte centre universitare din România.

Teza de doctorat actuală cuprinde şase capitole şi o bibliografie conţinând 130 titluri, dintre care 26 semnate de autoarea tezei de doctorat, 16 ca singur autor, iar 10 în colaborare.

Rezultatele originale sunt prezentate în următoarele articole:

1. **Á. O. PÁLL-SZABÓ**, *Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative*, Miskolc Mathematical Notes, 18 (2017), pp. 397–406.

2. **Á. O. PÁLL-SZABÓ** AND G. S. SĂLĂGEAN, *A unified class of harmonic functions with varying argument of coefficients*, accepted, Filomat.

3. O. ENGEL AND **Á. PÁLL-SZABÓ**, *The radius of convexity of particular functions and applications to the study of a second order differential inequality*, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 52 (2017), pp. 118–127.

4. O. ENGEL, P. KUPÁN AND **Á. PÁLL-SZABÓ**, *About the radius of convexity of some analytic functions*, Creative Mathematics and Informatics, 24 (2015), pp. 157 – 163.

5. **Á. O. PÁLL-SZABÓ**, O. ENGEL, AND E. SZATMÁRI, *Certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative*, Acta Universitatis Apulensis, 51 (2017), pp. 61–74.

6. **Á. O. PÁLL-SZABÓ**, *Coefficient bounds and Fekete-Szegő problem for new classes of analytic functions defined by Sălăgean integro-differential operator*, submitted, (-).

7. **Á. O. PÁLL-SZABÓ**, *Coefficient estimates and Fekete-Szegő problem for new classes of bi-univalent functions defined by Sălăgean integro-differential operator*, submitted, (-).

8. **Á. O. PÁLL-SZABÓ**, *Coefficient estimates for some new classes of bi-Bazilevič functions of Ma-Minda type involving the Sălăgean integro-differential operator*, submitted, (-).

9. **Á. O. PÁLL-SZABÓ**, *Differential subordinations and superordinations for analytic functions defined by Sălăgean integro-differential operator*, submitted, (-).

10. **Á. O. PÁLL-SZABÓ** AND E. SZATMARI, *Differential subordination results obtained by using a new operator*, General Mathematics, Vol. 25, No. 1-2 (2017), 119–131.

11. **Á. O. PÁLL-SZABÓ**, *Extensions of coefficient estimates for new classes of bi-univalent functions defined by Sălăgean integro-differential operator*, submitted, (-).

12. **Á. O. PÁLL-SZABÓ**, *Generalizations of starlike harmonic functions defined by Sălăgean and Ruscheweyh derivative*, submitted, (-).

13. **Á. O. PÁLL-SZABÓ**, *Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by the convolution of Ruscheweyh and Sălăgean derivative*, submitted, (-).

14. **Á. O. PÁLL-SZABÓ**, *On a class of univalent functions defined by Sălăgean integro-differential operator*, submitted, (-).

15. **Á. O. PÁLL-SZABÓ**, AND O. ENGEL, *Properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative*, Acta Universitatis Sapientiae, Mathematica, 7 (2015), pp. 278–286.

16. **Á. O. PÁLL-SZABÓ**, *Univalence criteria related with the generalised Sălăgean and Ruscheweyh operator*, submitted, (-).

17. **Á. O. PÁLL-SZABÓ**, *Integral properties of certain class of analytic functions with varying arguments defined by Sălăgean derivative*, Annals of Oradea University-Mathematics Fascicola, 23 (2016), pp. 177 – 182.

18. **Á. O. PÁLL-SZABÓ**, *Certain class of analytic functions with varying arguments defined by Sălăgean and Ruscheweyh derivative*, Mathematica (Cluj), 59 (82) (2017), pp. 80–88.

19. **Á. O. PÁLL-SZABÓ**, *Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Sălăgean and Ruscheweyh derivative.*, Studia Universitatis Babes-Bolyai, Mathematica, 62 (2017), pp. 465–472.

20. **Á. O. PÁLL-SZABÓ**, *Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Sălăgean derivative*, Automation, Computers, Applied Mathematics (ACAM), Vol. 25 (2016), No. 1, pp. 85–91.

21. **Á. O. PÁLL-SZABÓ** AND O. ENGEL, *Certain class of analytic functions with varying arguments defined by Sălăgean derivative*, in Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics, Ictami, 2015, pp. 113–120.

22. O. ENGEL AND **Á. PÁLL-SZABÓ**, *Preserving properties of the generalized Bernardi-Libera-Livingston integral operator defined on some subclasses of starlike functions*, Konuralp Journal of Mathematics, 5 (2017), pp. 207–215.

23. **Á. O. PÁLL-SZABÓ** AND G. S. SĂLĂGEAN, *On the order of convolution consistence of the harmonic functions with varying arguments*, submitted, (-).

24. **Á. O. PÁLL-SZABÓ** AND G. S. SĂLĂGEAN, *On a certain class of harmonic functions and the generalized Bernardi-Libera-Livingston integral operator*, submitted, (-).

25. **Á. O. PÁLL-SZABÓ**, *Where Are the Quadratic's Complex Roots ?,* Acta Didactica Napocensia, Volume 8, Number 1, 2015, pp. 37–48.

26. **Á. O. PÁLL-SZABÓ**, *Visualizing roots of a cubic equation,* The Electronic Journal of Mathematics & Technology, Volume 11 (2017), nr. 1, Research Journal of Mathematics & Technology, RJMT Vol. 6, Nr. 1 , 2017, pp. 1–8.

1.2 Definiţii şi notaţii

Fie $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ discul unitate de raza $r > 0$, centrat în $z_0 \in \mathbb{C}$. Discul $U(0, r)$ este notat cu U_r şi $U_1 = U = \{z \in \mathbb{C} : |z| = r < 1\}$.

Fie $H(U)$ mulţimea funcţiilor olomorfe în U.

Pentru $a \in \mathbb{C}$ şi $n \in \mathbb{N}$, fie

$$H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + ...\}$$

şi

$$\mathscr{A}_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + ...\}, \text{ cu } \mathscr{A} = \mathscr{A}_1.$$

O funcţie $f \in \mathscr{A}$ are următoarea dezvoltare în serie Taylor:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

1.3 Funcţii univalente

Definiţia 1.1. [29]

O funcţie olomorfă pe un interval deschis al planului complex se numeşte univalentă dacă este injectivă.

Notăm cu $S = \{f \in \mathscr{A} : f \text{ este univalentă în } U\}$ clasa funcţiilor univalente pe discul unitate U, normalizate prin condiţiile $f(0) = f'(0) - 1 = 0$.

Clasa funcţiilor stelate se notează cu S^*

$$S^* = \left\{ f \in \mathscr{A} : \Re \frac{zf'(z)}{f(z)} > 0 \ , \ z \in U \right\}, \ S^* \subset S. \tag{1.2}$$

Clasa funcţiilor convexe se notează cu

$$K = \left\{ f \in \mathscr{A} : \Re \frac{zf''(z)}{f'(z)} + 1 > 0 \ , \ z \in U \right\}, \ K \subset S^* \subset S \tag{1.3}$$

Definiţia 1.2. ([61], def. 3.5.1) Fie f şi g funcţii analitice în U.Spunem că funcţia f este subordonată funcţiei g, dacă există o funcţie analitică în U w şi $w(0) = 0; |w(z)| < 1; z \in U$, astfel încât $f(z) = g(w(z)); \forall z \in U$. Notăm cu $\prec$ relaţia de subordonare. Dacă g este univalentă, atunci $f \prec g$ dacă şi numai dacă $f(0) = g(0)$ şi $f(U) \subseteq g(U)$.

1.4 Operatori diferenţiali şi integrali

Definiţia 1.3. [106]

Pentru $f \in \mathscr{A}$, $n \in \mathbb{N}_0$, definim operatorul diferenţial Sălăgean $\mathscr{D}^n$: $\mathscr{D}^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{D}^0 f(z) = f(z),$$

$$\mathscr{D}^1 f(z) = zf'(z),$$

$$\mathscr{D}^{n+1} f(z) = z\left(\mathscr{D}^n f(z)\right)', z \in U$$

Observaţia 1.1. *Dacă* $f \in \mathscr{A}$ *şi* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *atunci*

$$\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U. \tag{1.4}$$

Definiţia 1.4. [106] Pentru $f \in \mathscr{A}, n \in \mathbb{N}_0$, definim operatorul integral Sălăgean I^n :

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = If(z) = \int_0^z f(t) t^{-1} dt, \ldots$$

$$I^{n+1} f(z) = I\left(I^n f(z)\right), z \in U$$

Observaţia 1.2. *Dacă* $f \in \mathscr{A}$ *şi* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *atunci*

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k, \tag{1.5}$$

$z \in U$, $n \in \mathbb{N}_0$ *şi* $z\left(I^{n+1} f(z)\right)' = I^n f(z)$.

Definiţia 1.5. Pentru $\lambda \geq 0$, $n \in \mathbb{N}_0$. Notăm cu $\mathscr{D}I^n$ operatorul $\mathscr{D}I^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{D}I^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z), z \in U.$$

Observaţia 1.3. *Dacă* $f \in \mathscr{A}$ *şi* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *atunci*

$$\mathscr{D}I^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n} \right] a_k z^k, z \in U. \tag{1.6}$$

Definiţia 1.6. Fie $f, g \in H(U)$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. Convoluţia (sau produsul Hadamard) a funcţiilor f şi g este definit:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.7}$$

Produsul modificat Hadamard :

$$(f \otimes g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g \otimes f)(z). \tag{1.8}$$

Definiţia 1.7. Ruscheweyh [102] a definit operatorul $\mathscr{R}^\gamma : \mathscr{A} \to \mathscr{A}$

$$\mathscr{R}^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z), (\gamma \geq -1),\ f \in \mathscr{A},\ z \in U. \tag{1.9}$$

În caz particular $n \in \mathbb{N}_0$

$$\mathscr{R}^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}. \tag{1.10}$$

$$\mathscr{R}^0 f(z) = f(z),$$

$$\mathscr{R}^1 f(z) = z f'(z), \ldots$$

$$(n+1)\mathscr{R}^{n+1} f(z) = z\left(\mathscr{R}^n f(z)\right)' + n\mathscr{R}^n f(z), z \in U.$$

Observaţia 1.4. *Dacă* $f \in \mathscr{A}$ *şi* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *atunci*

$$\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} a_k z^k = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)} a_k z^k,\ n > -1,\ \ z \in U, \tag{1.11}$$

sau

$$\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \quad \textit{unde}\ \ \delta(n,k) = \begin{pmatrix} n+k-1 \\ n \end{pmatrix}\ z \in U, \tag{1.12}$$

Definiţia 1.8. [5] Fie $\lambda \geq 0, n \in \mathbb{N}_0$. Notăm cu $\mathscr{R}\mathscr{D}^n : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{R}\mathscr{D}^n f(z) = (1-\lambda)\mathscr{R}^n f(z) + \lambda \mathscr{D}^n f(z), z \in U.$$

Observaţia 1.5. *Dacă* $f \in \mathcal{A}$ *şi* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *atunci*

$$\mathcal{RD}^n f(z) = z + \sum_{k=2}^{\infty} \left\{ (1-\lambda) \frac{(n+k-1)!}{n!(k-1)!} + \lambda k^n \right\} a_k z^k, z \in U. \tag{1.13}$$

Definiţia 1.9. [4] Fie $f \in \mathcal{A}, \lambda \geq 0$ şi $n \in \mathbb{N}_0$, definim operatorul $\mathcal{D}_\lambda^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{D}_\lambda^0 f(z) = f(z),$$

$$\mathcal{D}_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = \mathcal{D}_\lambda f(z), \ldots$$

$$\mathcal{D}_\lambda^{n+1} f(z) = (1-\lambda) \mathcal{D}_\lambda^n f(z) + \lambda z \left(\mathcal{D}_\lambda^n f(z)\right)' = \mathcal{D}_\lambda \left(\mathcal{D}_\lambda^n f(z)\right), z \in U$$

Observaţia 1.6. *Dacă* $f \in \mathcal{A}$ *şi* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *atunci*

$$\mathcal{D}_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k, \ z \in U. \tag{1.14}$$

Definiţia 1.10. Fie $\gamma, \lambda \geq 0, n \in \mathbb{N}_0$. Definim operatorul $\mathcal{RD}_\lambda{}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{RD}_\lambda{}^n f(z) = \left(1-\gamma\right) \mathcal{R}^n f(z) + \gamma \mathcal{D}_\lambda^n f(z), z \in U.$$

Observaţia 1.7. *Dacă* $f \in \mathcal{A}$ *şi* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *atunci*

$$\mathcal{RD}_\lambda{}^n f(z) = z + \sum_{k=2}^{\infty} \left\{ \gamma [1 + (k-1)\lambda]^n + \left(1-\gamma\right) \frac{(n+k-1)!}{n!(k-1)!} \right\} a_k z^k, \ z \in U. \tag{1.15}$$

Definiţia 1.11. Fie $n \in \mathbb{N}$. Notăm cu $\mathcal{SR}^n$ convoluţia operatorilor Sălăgean şi Ruscheweyh , $\mathcal{SR}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{SR}^n f(z) = \mathcal{S}^n \left(\frac{z}{1-z} \right) * \mathcal{R}^n f(z), z \in U.$$

Observaţia 1.8. *Dacă* $f \in \mathcal{A}$ *şi* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *atunci*

$$\mathcal{SR}^n f(z) = z + \sum_{k=2}^{\infty} \frac{k^n (n+k-1)!}{n!(k-1)!} a_k z^k, z \in U. \tag{1.16}$$

Definiţia 1.12. Definim operatorul integral Bernardi $L_c : \mathcal{A} \to \mathcal{A}$

$$L_c f(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt, \ \ c > -1. \tag{1.17}$$

Definiţia 1.13. În [71] sunt definite următorii operatori: operatorul integral fracţional $D_z^{-\mu}$ de ordinul μ:

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt, \ z \in U, \ f \in \mathcal{A}, \ \mu > 0, \tag{1.18}$$

şi operatorul diferenţial fracţional D_z^λ de ordinul λ,

$$D_z^\lambda f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{d}{dz} \displaystyle\int_0^z \dfrac{f(t)}{(z-t)^\lambda} dt, & 0 \le \lambda < 1 \\ \dfrac{d^n}{dz^n} D_z^{\lambda-n} f(z), & n \le \lambda < n+1 \end{cases}, n \in \mathbb{N}_0, f \in \mathscr{A}, \lambda \ge 0, \tag{1.19}$$

Definiţia 1.14. În [72] este definit operatorul differintegral fracţional $\Omega_z^\lambda : \mathscr{A} \to \mathscr{A}$,

$$\Omega_z^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), z \in U, -\infty < \lambda < 2, \tag{1.20}$$

unde $\lambda, -\infty < \lambda < 0$, λ, $0 \le \lambda < 2$.

Dezvoltarea în serie a operatorului Ω_z^λ pentru funcţia $f \in \mathscr{A}$ este dată de

$$\Omega_z^\lambda f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(k+2)}{\Gamma(k+2-\lambda)} a_{k+1} z^{k+1}, -\infty < \lambda < 2, z \in U. \tag{1.21}$$

Definiţia 1.15. În [110] este definit operatorul fracţional $\mathbb{D}_\lambda^{\nu,n} : \mathscr{A} \to \mathscr{A}$ pentru $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$ ca o compoziţie a operatorilor fracţional differintegral Sălăgean şi Ruscheweyh :

$$\mathbb{D}_\lambda^{\nu,n} f(z) = \mathscr{R}^\nu \mathscr{D}^n \Omega_z^\lambda f(z). \tag{1.22}$$

Dezvoltarea în serie a operatorului $\mathbb{D}_\lambda^{\nu,n} f(z)$ pentru $f \in \mathscr{A}$ este dată de

$$\mathbb{D}_\lambda^{\nu,n} f(z) = z + \sum_{k=1}^{\infty} \frac{(\nu+1)_k}{(2-\lambda)_k} (k+1)^{n+1} a_{k+1} z^{k+1}, \tag{1.23}$$

$-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, z \in U$, unde prin $(\gamma)_k$ notăm simbolul Pochhammer, pentru $\gamma \in \mathbb{C}$, este definită prin

$$(\gamma)_k = \begin{cases} 1, k = 0 \\ \gamma(\gamma+1)...(\gamma+k-1), k \in \mathbb{N} \end{cases} = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

Capitolul 2

Funcţii analitice cu argument variabil

Fie $f, g \in \mathscr{A}$ două funcţii analitice de forma:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{2.1}$$

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{2.2}$$

Definiţia 2.1. [111]O funcţie f de forma (2.1) se spune că aparţine clasei $V(\theta_k)$ dacă $f \in \mathscr{A}$ şi $arg(a_k) = \theta_k$,$\forall k \geq 2$. Dacă $\exists \delta \in \mathbb{R}$ astfel încât $\theta_k + (k-1)\delta \equiv \pi(mod\ 2\pi), \forall k \geq 2$ atunci se spune că f aparţine clasei $V(\theta_k, \delta)$. Reuniunea tuturor mulţimilor $V(\theta_k, \delta)$, în raport cu toate secvenţele posibile $\{\theta_k\}$ şi toate numere reale posibile δ, este notată cu V.

2.1 Clasă de funcţii analitice cu argument variabil definită cu ajutorul operatorului Ruscheweyh

Attiya şi Aouf au definit în [12] clasa $Q(n, \lambda, A, B)$:

Definiţia 2.2. [12][36] Pentru $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ fie $Q(n, \lambda, A, B)$ subclasa funcţiilor $\mathscr{A}$, care conţine funcţii f de forma (2.1) astfel încât

$$(1-\lambda)(\mathscr{R}^n f(z))' + \lambda(\mathscr{R}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{2.3}$$

Se notează cu $VQ(n, \lambda, A, B)$ subclasa lui V unde $f(z) \in Q(n, \lambda, A, B)$.

Teorema 2.1. *[36]Fie funcţia f de forma* (2.1)*, ce aparţine clasei V. Atunci $f \in VQ(n, \lambda, A, B)$, dacă şi numai dacă*

$$T(f) = \sum_{k=2}^{\infty} k\delta(n,k) C_k (1+B)|a_k| \leq (B-A)(n+1) \tag{2.4}$$

unde

$$C_k = n+1+\lambda(k-1).$$

Funcţiile extremale sunt

$$f_k(z) = z + \frac{(B-A)(n+1)}{kC_k\delta(n,k)(1+B)}e^{i\theta_k}z^k, \forall k \geq 2.$$

Fie $L_c f(z) = \frac{c+1}{z^c}\int_0^z f(t)t^{c-1}dt, c > -1$ operatorul integral Bernardi.

Teorema 2.2. *[87] Dacă $f \in VQ(n,\lambda,2\alpha-1,B)$ atunci $L_c f \in VQ(n,\lambda,2\beta-1,B)$, unde*

$$\beta = \beta(\alpha) = \frac{B+1+2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

Rezultatul este exact.

Teorema 2.3. *[87] Dacă $f \in VQ(n,\lambda,A,B)$ atunci $L_c f \in VQ(n,\lambda,A^*,B)$, unde $A^* = \frac{B+A(c+1)}{c+2} > A$. Rezultatul este exact.*

Teorema 2.4. *[87] Dacă $f \in VQ(n,\lambda,A,B)$ atunci $L_c f \in VQ(n,\lambda,A,B^*)$,unde*

$$B^* = \frac{A(1+B)(c+2)+(B-A)(c+1)}{(1+B)(c+2)-(B-A)(c+1)} < B.$$

Rezultatul este exact.

Convoluţia sau produsul Hadamard modificat al funcţiilor f şi g de forme (2.1) şi (2.2) din $V(\theta_k,\delta)$ este dată de ([44, 104, 108])

$$(f \otimes g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g \otimes f)(z). \tag{2.5}$$

Teorema 2.5. *[73] Dacă $f \in VQ(n,\lambda,A_1,B), g \in VQ(n,\lambda,A_2,B)$ atunci $f \otimes g \in VQ(n,\lambda,A^*,B)$, unde*

$$A^* = B - \frac{(B-A_1)(B-A_2)(n+1)}{2C_2(1+B)\delta(n,2)}.$$

Rezultatul este exact.

Corolarul 2.1. *[73] Dacă $f,g \in VQ(n,\lambda,A,B)$ atunci $f \otimes g \in VQ(n,\lambda,A^*,B)$, unde*

$$A^* = B - \frac{(B-A)^2(n+1)}{2C_2(1+B)\delta(n,2)}.$$

Rezultatul este exact.

Teorema 2.6. *[73] Dacă $f \in VQ(n,\lambda,A,B_1), g \in VQ(n,\lambda,A,B_2)$ atunci*
$f \otimes g \in VQ(n,\lambda,A,B^*)$, unde

$$B^* - A = \frac{(A+1)(n+1)(B_1-A)(B_2-A)}{2C_2\delta(n,2)(1+B_1)(1+B_2)-(n+1)(B_1-A)(B_2-A)}.$$

Rezultatul este exact.

Corolarul 2.2. *[73] Dacă $f,g \in VQ(n,\lambda,A,B)$ atunci $f \otimes g \in VQ(n,\lambda,A,B^*)$, unde*

$$B^* = A + \frac{(A+1)(n+1)(B-A)^2}{2C_2\delta(n,2)(1+B)^2-(n+1)(B-A)^2}.$$

Rezultatul este exact.

Teorema 2.7. *[73] Dacă $f_j \in VQ(n,\lambda,A_j,B), j=\overline{1,s}, s \in \{2,3,4,\dots\}$ atunci $f_1 \otimes f_2 \otimes \dots \otimes f_s \in VQ(n,\lambda,A^{(s-1)*},B)$, unde $A^{(s-1)*} = B - \dfrac{(n+1)^{s-1}\prod\limits_{j=1}^{s}(B-A_j)}{2^{s-1}C_2^{s-1}(1+B)^{s-1}[\delta(n,2)]^{s-1}}$. Rezultatul este exact.*

Teorema 2.8. *[73] Dacă $f_j \in VQ(n,\lambda,A,B_j), j=\overline{1,s}, s \in \{2,3,4,\dots\}$ atunci $f_1 \otimes f_2 \otimes \dots \otimes f_s \in VQ(n,\lambda,A,B^{(s-1)*})$, unde*

$$B^{(s-1)*} = A + \frac{(A+1)(n+1)^{s-1}\prod\limits_{j=1}^{s}(B_j-A)}{2^{s-1}C_2^{s-1}[\delta(n,2)]^{s-1}\prod\limits_{j=1}^{s}(B_j-A)-(n+1)^{s-1}\prod\limits_{j=1}^{s}(B_j-A)}.$$

Rezultatul este exact.

2.2 Clasă de funcţii analitice cu argument variabil definită cu ajutorul operatorului Sălăgean

Definiţia 2.3. Pentru $\lambda \ge 0; -1 \le A < B \le 1; 0 < B \le 1; n \in \mathbb{N}_0$ fie $S(n,\lambda,A,B)$ subclasa funcţiilor $\mathscr{A}$, care conţine funcţii f de forma (2.1) astfel încât

$$(1-\lambda)(\mathscr{D}^n f(z))' + \lambda(\mathscr{D}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{2.6}$$

Se notează cu $VS(n,\lambda,A,B)$ subclasa lui V unde $f(z) \in S(n,\lambda,A,B)$.

Teorema 2.9. *[10] Fie funcţia f de forma* (2.1)*, ce aparţine clasei V. Atunci $f \in VS(n,\lambda,A,B)$, dacă şi numai dacă*

$$T(f) = \sum_{k=2}^{\infty} k^{n+1} C_k (1+B)|a_k| \le B - A \tag{2.7}$$

unde

$$C_k = 1 - \lambda + \lambda k.$$

Funcţiile extremale sunt

$$f(z) = z + \frac{B-A}{k^{n+1}C_k(1+B)} e^{i\theta_k} z^k, \forall k \ge 2.$$

Teorema 2.10. *[91] Dacă $f \in VS(n,\lambda,2\alpha-1,B)$ atunci $L_c f \in VS(n,\lambda,2\beta-1,B)$, unde*

$$\beta = \beta(\alpha) = \frac{B+1+2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

Rezultatul este exact.

Teorema 2.11. *[91] Dacă $f \in VS(n,\lambda,A,B)$ atunci $L_c f \in VS(n,\lambda,A^*,B)$, unde $A^* = \dfrac{B+A(c+1)}{c+2} > A$. Rezultatul este exact.*

Teorema 2.12. *[91] Dacă $f \in VS(n,\lambda,A,B)$ atunci $L_c f \in VS(n,\lambda,A,B^*)$,unde*

$$B^* = \frac{A(1+B)(c+2)+(B-A)(c+1)}{(1+B)(c+2)-(B-A)(c+1)} < B.$$

Rezultatul este exact.

Teorema 2.13. *[94] Dacă $f \in VS(n,\lambda,A_1,B), g \in VS(n,\lambda,A_2,B)$ atunci $f \otimes g \in VS(n,\lambda,A^*,B)$, unde*

$$A^* = B - \frac{(B-A_1)(B-A_2)}{2^{n+1}C_2(1+B)}.$$

Rezultatul este exact.

Corolarul 2.3. *[94] Dacă $f,g \in VS(n,\lambda,A,B)$ atunci $f \otimes g \in VS(n,\lambda,A^*,B)$, unde*

$$A^* = B - \frac{(B-A)^2}{2^{n+1}C_2(1+B)}.$$

Rezultatul este exact.

Teorema 2.14. *[94] Dacă $f \in VS(n,\lambda,A,B_1), g \in VS(n,\lambda,A,B_2)$ atunci $f \otimes g \in VS(n,\lambda,A,B^*)$, unde*

$$B^* = A + \frac{(B_1-A)(B_2-A)(A+1)}{2C_2(1+B_1)(1+B_2)-(B_1-A)(B_2-A)}.$$

Rezultatul este exact.

Corolarul 2.4. *[94] Dacă $f,g \in VS(n,\lambda,A,B)$ atunci $f \otimes g \in VS(n,\lambda,A,B^*)$, unde*

$$B^* = A + \frac{(B-A)^2(A+1)}{2C_2(1+B)^2-(B-A)^2}.$$

Rezultatul este exact.

Teorema 2.15. *[94] Dacă $f_j \in VS(n,\lambda,A_j,B), j=\overline{1,s}, s \in \{2,3,4,\ldots\}$ atunci*

$f_1 * f_2 * \ldots * f_s \in VS(n,\lambda,A^{(s-1)*},B)$, unde $A^{(s-1)*} = B - \dfrac{\prod_{j=1}^{s}(B-A_j)}{\left(2^{n+1}\right)^{s-1} C_2^{s-1}(1+B)^{s-1}}$. Rezultatul este exact.

Teorema 2.16. *[94]Dacă* $f_j \in VS(n,\lambda,A,B_j), j=\overline{1,s}, s\in\{2,3,4,\dots\}$ *atunci* $f_1\otimes f_2\otimes\dots\otimes f_s \in VS(n,\lambda,A,B^{(s-1)*})$*, unde*

$$B^{(s-1)*}=A+\frac{(A+1)\prod\limits_{j=1}^{s}(B_j-A)}{2^{s-1}C_2^{s-1}\prod\limits_{j=1}^{s}(1+B_j)-\prod\limits_{j=1}^{s}(B_j-A)}.$$

Rezultatul este exact.

2.3 Clasă de funcţii analitice cu argument variabil definită cu ajutorul operatorului Sălăgean şi Ruscheweyh

Definiţia 2.4. Pentru $\widetilde{\lambda}\ge 0; -1\le A<B\le 1; 0<B\le 1; n\in\mathbb{N}_0$ fie $L(n,\widetilde{\lambda},A,B)$ subclasa funcţiilor $\mathscr{A}$, care conţine funcţii f de forma (2.1) astfel încât

$$(1-\widetilde{\lambda})(\mathscr{R}\mathscr{D}_\lambda{}^n f(z))'+\widetilde{\lambda}(\mathscr{R}\mathscr{D}_\lambda{}^{n+1}f(z))'\prec\frac{1+Az}{1+Bz}. \tag{2.8}$$

Se notează cu $VL(n,\widetilde{\lambda},A,B)$ subclasa lui V unde $f(z)\in L(n,\widetilde{\lambda},A,B)$.

Teorema 2.17. *[92] Fie funcţia f de forma* (2.1)*, ce aparţine clasei V. Atunci* $f(z)\in VL(n,\widetilde{\lambda},A,B)$*, dacă şi numai dacă*

$$T(f)=\sum_{k=2}^{\infty}kC_k(1+B)|a_k|\le B-A \tag{2.9}$$

unde

$$C_k=\gamma[1+(k-1)\lambda]^n\left[1+\widetilde{\lambda}\lambda(k-1)\right]+\frac{(n+k-1)!}{n!(k-1)!}(1-\gamma)\left[1+\widetilde{\lambda}\frac{k-1}{n+1}\right].$$

Funcţiile extremale sunt

$$f(z)=z+\frac{B-A}{kC_k(1+B)}e^{i\theta_k}z^k, \forall k\ge 2.$$

Corolarul 2.5. *[92]Fie funcţia f de forma* (2.1)*, ce aparţine clasei* $VL(n,\widetilde{\lambda},A,B)$*. Atunci*

$$|a_k|\le\frac{B-A}{kC_k(1+B)}, \forall k\ge 2.$$

Rezultatul (2.9) *este exact pentru funcţiile*

$$f(z)=z+\frac{B-A}{kC_k(1+B)}e^{i\theta_k}z^k, \forall k\ge 2.$$

Teorema 2.18. *[92]Fie funcţia f de forma* (2.1)*, ce aparţine clasei* $VL(n,\widetilde{\lambda},A,B)$*. Atunci*

$$|z|-\frac{B-A}{2C_2(1+B)}|z|^2\le|f(z)|\le|z|+\frac{B-A}{2C_2(1+B)}|z|^2. \tag{2.10}$$

Corolarul 2.6. *[92]Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VL(n,\widetilde{\lambda},A,B)$. Atunci $f(z)\in U(0,r_1)$, unde $r_1=1+\dfrac{B-A}{2C_2(1+B)}$.*

Teorema 2.19. *[92]Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VL(n,\widetilde{\lambda},A,B)$. Atunci*

$$1-\frac{B-A}{C_2(1+B)}|z|\le\left|f'(z)\right|\le 1+\frac{B-A}{C_2(1+B)}|z|. \tag{2.11}$$

Rezultatul este exact.

Corolarul 2.7. *[92]Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VL(n,\widetilde{\lambda},A,B)$. Atunci $f'(z)\in U(0,r_2)$, unde $r_2=1+\dfrac{B-A}{C_2(1+B)}$.*

Teorema 2.20. *[92]Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VL(n,\widetilde{\lambda},A,B)$, cu $arg(a_k)=\theta_k$, unde $\theta_k\equiv\pi,\forall k\ge 2$. Definim*

$$f_1(z)=z$$

şi

$$f_k(z)=z-\frac{B-A}{kC_k(1+B)}z^k,\forall k\ge 2; z\in U.$$

Atunci $f(z)\in VL(n,\widetilde{\lambda},A,B)$ dacă şi numai dacă $f(z)$ poate fi exprimat prin $f(z)=\sum\limits_{k=1}^{\infty}\mu_k f_k(z)$, unde $\mu_k\ge 0$ şi $\sum\limits_{k=1}^{\infty}\mu_k=1$.

Corolarul 2.8. *Învelitoarea convexă şi închisă al clasei $VL(n,\widetilde{\lambda},A,B)$ este*

$$cl\ co\ VL(n,\widetilde{\lambda},A,B)=\left\{f\ |f\in\mathscr{A},\ \sum_{k=2}^{\infty}k^n C_k(1+B)|a_k|\le B-A\right\}.$$

Punctele extreme $cl\ co\ VL(n,\widetilde{\lambda},A,B)$ sunt

$$\boldsymbol{E}(cl\ co\ VL(n,\lambda,A,B))=\left\{z+\frac{B-A}{k^n C_k(1+B)}\xi z^k,\ |\xi|=1,\ k\ge 2\right\}.$$

Teorema 2.21. *[92]Dacă $f\in VL(n,\widetilde{\lambda},A,B)$, atunci $L_c f\in VL(n,\widetilde{\lambda},A^*,B)$, unde*

$$A^*=\frac{B+A(c+1)}{c+2}>A.$$

Rezultatul este exact.

Corolarul 2.9. *[92] Dacă $f\in VL(n,\widetilde{\lambda},2\alpha-1,B)$ atunci $L_c f\in VL(n,\widetilde{\lambda},2\beta-1,B)$, unde*

$$\beta=\beta(\alpha)=\frac{B+1+2\alpha(c+1)}{2(c+2)}\ge\alpha.$$

Rezultatul este exact.

Teorema 2.22. *[92]Dacă $f \in VL(n,\widetilde{\lambda},A,B)$, atunci $L_c f \in VL(n,\widetilde{\lambda},A,B^*)$,unde*

$$B^* = \frac{A(1+B)(c+2)+(B-A)(c+1)}{(1+B)(c+2)-(B-A)(c+1)} < B.$$

Rezultatul este exact.

Teorema 2.23. *[93] Dacă $f \in VL(n,\widetilde{\lambda},A_1,B), g \in VL(n,\widetilde{\lambda},A_2,B)$ atunci $f \otimes g \in VL(n,\widetilde{\lambda},A^*,B)$, unde $A^* = B - \dfrac{(B-A_1)(B-A_2)}{2C_2(1+B)}$.*
Rezultatul este exact.

Corolarul 2.10. *[93] Dacă $f,g \in VL(n,\widetilde{\lambda},A,B)$ atunci $f \otimes g \in VL(n,\widetilde{\lambda},A^*,B)$, unde $A^* = B - \dfrac{(B-A)^2}{2C_2(1+B)}$. Rezultatul este exact.*

Teorema 2.24. *[93] Dacă $f \in VL(n,\widetilde{\lambda},A,B_1), g \in VL(n,\widetilde{\lambda},A,B_2)$ atunci $f \otimes g \in VL(n,\widetilde{\lambda},A,B^*)$, unde*

$$B^* = A + \frac{(B_1-A)(B_2-A)(A+1)}{2C_2(1+B_1)(1+B_2)-(B_1-A)(B_2-A)}.$$

Rezultatul este exact.

Corolarul 2.11. *[93] Dacă $f,g \in VL(n,\widetilde{\lambda},A,B)$ atunci $f \otimes g \in VL(n,\widetilde{\lambda},A,B^*)$, unde*

$$B^* = A + \frac{(B-A)^2(A+1)}{2C_2(1+B)^2-(B-A)^2}.$$

Rezultatul este exact.

Teorema 2.25. *[93] Dacă $f_j \in VL(n,\widetilde{\lambda},A_j,B)$, $j=\overline{1,m}$, $m \in \{2,3,4,\ldots\}$ atunci $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VL(n,\widetilde{\lambda},A^{(m-1)*},B)$, unde $A^{(m-1)*} = B - \dfrac{\prod_{j=1}^{m}(B-A_j)}{2^{m-1}C_2^{m-1}(1+B)^{m-1}}$. Rezultatul este exact.*

Teorema 2.26. *[93]Dacă $f_j \in VL(n,\widetilde{\lambda},A,B_j), j=\overline{1,m}, m \in \{2,3,4,\ldots\}$ atunci $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VL(n,\widetilde{\lambda},A,B^{(m-1)*})$, unde*

$$B^{(m-1)*} = A + \frac{(A+1)\prod_{j=1}^{m}(B_j-A)}{2^{m-1}C_2^{m-1}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j-A)}.$$

Rezultatul este exact.

2.4 Clasă de funcţii analitice cu argument variabil definită de convoluţia operatorilor Sălăgean şi Ruscheweyh

Definiţia 2.5. Pentru $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ fie $P(n,\lambda,A,B)$ subclasa funcţiilor $\mathscr{A}$, care conţine funcţii f de forma (2.1) astfel încât

$$(1-\lambda)(\mathscr{S}\mathscr{R}^n f(z))' + \lambda(\mathscr{S}\mathscr{R}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{2.12}$$

Se notează cu $VP(n,\lambda,A,B)$ subclasa lui V unde $f(z) \in P(n,\lambda,A,B)$.

Teorema 2.27. *[77] Fie funcţia f de forma* (2.1)*, ce aparţine clasei V. Atunci $f(z) \in VP(n,\lambda,A,B)$, dacă şi numai dacă*

$$T(f) = \sum_{k=2}^{\infty} k^{n+1} C_k (1+B)|a_k| \leq B - A, \tag{2.13}$$

unde

$$C_k = [n+1+\lambda(k-1)(n+k+1)] \frac{(n+k-1)!}{(n+1)!(k-1)!}.$$

Funcţiile extremale sunt

$$f(z) = z + \frac{B-A}{k^{n+1}C_k(1+B)} e^{i\theta_k} z^k, \forall k \geq 2.$$

Corolarul 2.12. *[77]Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VP(n,\lambda,A,B)$. Atunci*

$$|a_k| \leq \frac{B-A}{k^{n+1}C_k(1+B)}, \forall k \geq 2.$$

Rezultatul (2.13) *este exact pentru funcţiile*

$$f(z) = z + \frac{B-A}{k^{n+1}C_k(1+B)} e^{i\theta_k} z^k, \forall k \geq 2.$$

Teorema 2.28. *[77] Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VP(n,\lambda,A,B)$. Atunci*

$$|z| - \frac{B-A}{2^{n+1}C_2(1+B)}|z|^2 \leq |f(z)| \leq |z| + \frac{B-A}{2^{n+1}C_2(1+B)}|z|^2. \tag{2.14}$$

Rezultatul este exact.

Corolarul 2.13. *[77]Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VP(n,\lambda,A,B)$. Atunci $f(z) \in U(0,r_1)$, unde $r_1 = 1 + \dfrac{B-A}{2^{n+1}C_2(1+B)}$.*

Teorema 2.29. *[77] Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VP(n,\lambda,A,B)$. Atunci*

$$1 - \frac{B-A}{2^n C_2(1+B)}|z| \leq \left|f'(z)\right| \leq 1 + \frac{B-A}{2^n C_2(1+B)}|z|. \tag{2.15}$$

Rezultatul este exact.

Corolarul 2.14. *[77]Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VP(n,\lambda,A,B)$. Atunci $f'(z) \in U(0,r_2)$, unde $r_2 = 1 + \dfrac{B-A}{2^n C_2(1+B)}$.*

Teorema 2.30. *[77] Fie funcţia f de forma* (2.1)*, ce aparţine clasei $VP(n,\lambda,A,B)$, cu $arg(a_k)=\theta_k$ unde $\theta_k \equiv \pi, \forall k \geq 2$. Definim*

$$f_1(z)=z$$

şi

$$f_k(z)=z-\frac{B-A}{k^{n+1}C_k(1+B)}z^k, (k \geq 2; z \in U).$$

Atunci $f(z) \in VP(n,\lambda,A,B)$ dacă şi numai dacă $f(z)$ poate fi exprimat $f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z)$, unde $\mu_k \geq 0$ şi $\sum_{k=1}^{\infty}\mu_k=1$.

Corolarul 2.15. *[77] Fie $VP_\pi(n,\lambda,A,B)=VP(n,\lambda,A,B)\cap V(\pi,0)$. Punctele extreme ale lui $VP_\pi(n,\lambda,A,B)$ sunt*

$$f_1(z)=z \text{ and } f_k(z)=z-\frac{B-A}{k^{n+1}C_k(1+B)}z^k, \ (k \geq 2, \ z \in U).$$

Corolarul 2.16. *[77] Învelitoarea convexă şi închisă al clasei $VP(n,\lambda,A,B)$ este*

$$cl \ co \ VP(n,\lambda,A,B)=\left\{f \ |f \in \mathscr{A}, \ \sum_{k=2}^{\infty}k^{n+1}C_k(1+B)|a_k| \leq B-A\right\}.$$

Punctele extreme $cl \ co \ VP(n,\lambda,A,B)$ sunt

$$\boldsymbol{E}(cl \ co \ VP(n,\lambda,A,B))=\left\{z+\frac{B-A}{k^{n+1}C_k(1+B)}\xi z^k, \ |\xi|=1, \ k \geq 2\right\}.$$

Teorema 2.31. *[77] Dacă $f \in VP(n,\lambda,2\alpha-1,B)$ atunci $L_c f \in VP(n,\lambda,2\beta-1,B)$, unde*

$$\beta=\beta(\alpha)=\frac{B+1+2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

Rezultatul este exact.

Teorema 2.32. *[77] Dacă $f \in VP(n,\lambda,A,B)$ atunci $L_c f \in VP(n,\lambda,A^*,B)$, unde*

$$A^*=\frac{B+A(c+1)}{c+2}>A.$$

Rezultatul este exact.

Teorema 2.33. *[77] Dacă $f \in VP(n,\lambda,A,B)$ atunci $L_c f \in VP(n,\lambda,A,B^*)$,unde*

$$B^*=\frac{A(1+B)(c+2)+(B-A)(c+1)}{(1+B)(c+2)-(B-A)(c+1)}<B.$$

Rezultatul este exact.

Teorema 2.34. *[85] Dacă $f \in VP(n,\lambda,A_1,B), g \in VP(n,\lambda,A_2,B)$ atunci $f \otimes g \in VP(n,\lambda,A^*,B)$, unde*

$$A^*=B-\frac{(B-A_1)(B-A_2)}{2^{n+1}C_2(1+B)}.$$

Rezultatul este exact.

Corolarul 2.17. *[85] Dacă $f,g \in VP(n,\lambda,A,B)$ atunci $f \otimes g \in VP(n,\lambda,A^*,B)$, unde*

$$A^* = B - \frac{(B-A)^2}{2^{n+1}C_2(1+B)}.$$

Rezultatul este exact.

Teorema 2.35. *[85] Dacă $f \in VP(n,\lambda,A,B_1), g \in VP(n,\lambda,A,B_2)$ atunci $f \otimes g \in VP(n,\lambda,A,B^*)$, unde*

$$B^* = A + \frac{(B_1-A)(B_2-A)(A+1)}{2^{n+1}C_2(1+B_1)(1+B_2)-(B_1-A)(B_2-A)}.$$

Rezultatul este exact.

Corolarul 2.18. *[85] Dacă $f,g \in VP(n,\lambda,A,B)$ atunci $f \otimes g \in VP(n,\lambda,A,B^*)$, unde*

$$B^* = A + \frac{(B-A)^2(A+1)}{2^{n+1}C_2(1+B)^2-(B-A)^2}.$$

Rezultatul este exact.

Teorema 2.36. *[85] Dacă $f_j \in VP(n,\lambda,A_j,B), j=\overline{1,s}, s \in \{2,3,4,\dots\}$ atunci $f_1 \otimes f_2 \otimes \dots \otimes f_s \in VP(n,\lambda,A^{(s-1)*},B)$, unde*

$$A^{(s-1)*} = B - \frac{\prod_{j=1}^{s}(B-A_j)}{2^{(n+1)(s-1)}C_2^{s-1}(1+B)^{s-1}}.$$

Rezultatul este exact.

Teorema 2.37. *[85] Dacă $f_j \in VP(n,\lambda,A,B_j), j=\overline{1,s}, s \in \{2,3,4,\dots\}$ atunci $f_1 \otimes f_2 \otimes \dots \otimes f_s \in VP(n,\lambda,A,B^{(s-1)*})$, unde*

$$B^{(s-1)*} = A + \frac{(A+1)\prod_{j=1}^{s}(B_j-A)}{2^{(s-1)(n+1)}C_2^{s-1}\prod_{j=1}^{s}(1+B_j)-\prod_{j=1}^{s}(B_j-A)}.$$

Rezultatul este exact.

Capitolul 3

Funcţii analitice

3.1 Estimarea coeficienţilor şi problema Fekete-Szegő pentru noi clase de funcţii analitice definite de operatorul integro-diferenţial Sălăgean

Definiţia 3.1. Fie $f \in \mathscr{A}$. Atunci f aparţine clasei $S^n(\mu)$ dacă şi numai dacă

$$\Re\left(\frac{z(\mathscr{D}I^n f(z))'}{\mathscr{D}I^n f(z)}\right) > \mu, \quad 0 \le \mu < 1, z \in U. \tag{3.1}$$

Definiţia 3.2. Fie $f \in \mathscr{A}$. f aparţine clasei $C^n(\mu)$ dacă şi numai dacă

$$\Re\left(\frac{\left[z(\mathscr{D}I^n f(z))'\right]'}{(\mathscr{D}I^n f(z))'}\right) > \mu, \quad 0 \le \mu < 1, z \in U. \tag{3.2}$$

Definiţia 3.3. [23] Fie $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ o funcţie univalentă şi stelată relativ la 1 care transformă discul unitate în semiplanul drept care este simetric în raport cu axa reală, cu $\varphi(0) = 1$ şi $\varphi'(0) > 0$. Clasa $S^*(\varphi)$ conţine toate funcţiile $f \in \mathscr{A}$ care satisfac următoarea subordonare:

$$\frac{z(\mathscr{D}I^n f(z))'}{\mathscr{D}I^n f(z)} \prec \varphi(z), \tag{3.3}$$

şi $C(\varphi)$ este clasa funcţiilor $f \in \mathscr{A}$ pentru care

$$\frac{\left[z(\mathscr{D}I^n f(z))'\right]'}{(\mathscr{D}I^n f(z))'} \prec \varphi(z). \tag{3.4}$$

Observaţia 3.1. *Dacă* $\varphi_\mu(z) = \dfrac{1 + (1-2\mu)z}{1-z}$ *atunci* $\mathscr{S}^n(\mu) = S^*(\varphi_\mu)$ *şi* $\mathscr{C}^n(\mu) = C(\varphi_\mu)$.

Teorema 3.1. *[78] Fie funcţia $f(z)$ definit în (2.1) din $\mathscr{A}$. Dacă*

$$\sum_{k=2}^{\infty}(k-\mu)\left[k^n(1-\lambda) + \lambda\frac{1}{k^n}\right]|a_k| \le 1-\mu, \tag{3.5}$$

atunci $f(z) \in S^n(\mu)$. Rezultatul (3.5) este exact.

Corolarul 3.1. *Dacă* (3.5) *este adevărat, atunci*

$$|a_k| \le \frac{1-\mu}{(k-\mu)\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]}, \forall k \ge 2. \tag{3.6}$$

Teorema 3.2. *[78] Fie funcţia* $f(z)$ *definit în* (2.1) *din* $\mathscr{A}$*. Dacă*

$$\sum_{k=2}^{\infty}(k-\mu)\left[k^{n+1}(1-\lambda)+\lambda\frac{1}{k^{n-1}}\right]|a_k| \le 1-\mu, \tag{3.7}$$

atunci $f(z) \in \mathscr{C}^n(\mu)$*. Rezultatul* (3.7) *este exact.*

Corolarul 3.2. *Dacă* (3.7) *este adevărat, atunci*

$$|a_k| \le \frac{1-\mu}{(k-\mu)\left[k^{n+1}(1-\lambda)+\lambda\frac{1}{k^{n-1}}\right]}, \forall k \ge 2. \tag{3.8}$$

Teorema 3.3. *[78] Dacă* (3.5) *este adevărat, atunci*

$$|z| - \frac{1-\mu}{2-\mu}|z|^2 \le \left|\mathscr{D}I^n f(z)\right| \le |z| + \frac{1-\mu}{2-\mu}|z|^2, \quad \forall z \in U, 0 \le \mu < 1.$$

Teorema 3.4. *[78] Dacă* (3.7) *este adevărat, atunci*

$$|z| - \frac{1-\mu}{2(2-\mu)}|z|^2 \le \left|\mathscr{D}I^n f(z)\right| \le |z| + \frac{1-\mu}{2(2-\mu)}|z|^2, \quad \forall z \in U, 0 \le \mu < 1.$$

Teorema 3.5. *[78] Dacă* (3.5) *este adevărat, atunci*

$$|z| - \frac{1-\mu}{(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2 \le |f(z)| \le |z| + \frac{1-\mu}{(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2,$$

$\forall z \in U, 0 \le \mu < 1.$

Teorema 3.6. *[78] Dacă* (3.7) *este adevărat, atunci*

$$|z| - \frac{1-\mu}{2(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2 \le |f(z)| \le |z| + \frac{1-\mu}{2(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2,$$

$\forall z \in U, 0 \le \mu < 1.$

Teorema 3.7. *[78] Fie* $0 \le \mu < 1$ *şi* $\varphi = \varphi_\mu$*. Dacă* $f(z)$ *definit în* (2.1) *aparţine clasei* $\mathscr{S}^n(\mu)$*, atunci*

$$|a_2| \le \frac{B_1}{2^n(1-\lambda)+\lambda\frac{1}{2^n}}$$

şi $\forall \xi \in \mathbb{C}$

$$\left|a_3 - \xi a_2^2\right| \le$$

$$\le \frac{B_1}{4\left[3^n(1-\lambda)+\lambda\frac{1}{3^n}\right]} max\left\{1, \left|\frac{B_2}{B_1} - \frac{B_1}{2^n(1-\lambda)+\lambda\frac{1}{2^n}}\left(2\cdot\frac{3^n(1-\lambda)+\lambda\frac{1}{3^n}}{2^n(1-\lambda)+\lambda\frac{1}{2^n}}\xi - 1\right)\right|\right\}.$$

Rezultatul este exact.

Teorema 3.8. *[78] Fie* $0 \leq \mu < 1$ *şi* $\varphi = \varphi_\mu$. *Dacă* $f(z)$ *definit în* (2.1) *aparţine clasei* $\mathscr{C}^n(\mu)$, *atunci*

$$|a_2| \leq \frac{B_1}{2^{n+2}(1-\lambda) + \lambda \frac{1}{2^{n-2}}}$$

şi $\forall \xi \in \mathbb{C}$

$$\left|a_3 - \xi a_2^2\right| \leq$$

$$\leq \frac{B_1}{4\left[3^{n+1}(1-\lambda) + \lambda \frac{1}{3^{n-1}}\right]} max\left\{1, \left|\frac{B_2}{B_1} - \frac{B_1}{2^{n+1}(1-\lambda) + \lambda \frac{1}{2^{n-1}}}\left(2 \cdot \frac{3^{n+1}(1-\lambda) + \lambda \frac{1}{3^{n-1}}}{2^{n+1}(1-\lambda) + \lambda \frac{1}{2^{n-1}}}\xi - 1\right)\right|\right\}.$$

Rezultatul este exact.

Capitolul 4

Subordonări şi superordonări diferenţiale

Notăm cu $\mathcal{Q}$ clasa funcţiilor f care sunt analitice şi injective în $\overline{U} \setminus E(f)$, unde

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

$f'(\zeta) \neq 0$, $\zeta \in \partial U \setminus E(f)$.

Lema 4.1. *[41] (Hallenbeck and Ruscheweyh) Fie h o funcţie convexă $h(0) = a$, şi fie $\gamma \in \mathbb{C}^*$ un număr complex $\Re\gamma \geq 0$. Dacă $p \in \mathcal{H}[a,n]$ şi*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \quad z \in U$$

atunci

$$p(z) \prec q(z) \prec h(z), z \in U$$

unde

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt, \quad z \in U.$$

Lema 4.2. *[58] (Miller and Mocanu) Fie q o funcţie convexă în U şi fie*

$$h(z) = q(z) + n\alpha z q'(z), \quad z \in U$$

unde $\alpha > 0$ şi n este un număr întreg pozitiv. Dacă

$$p(z) = q(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \quad z \in U$$

este olomorfă în U şi

$$p(z) + \alpha z p'(z) \prec h(z), \quad z \in U$$

atunci

$$p(z) \prec q(z)$$

Rezultatul este exact.

Lema 4.3. *[24]Fie q o funcţie convexă univalentă în U şi $\gamma \in \mathbb{C}^*$ astfel încât*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

Dacă p este o funcţie analitică în U, cu $p(0) = q(0)$ şi

$$p(z)+\gamma zp'(z) \prec q(z)+\gamma zq'(z), \tag{4.1}$$

atunci $p(z) \prec q(z)$ şi q este cea mai bună dominantă a (4.1)*.*

Lema 4.4. *[24]Fie q o funcţie convexă în U, cu $q(a) = 0$ şi $\gamma \in \mathbb{C}$ astfel încât $\Re\, \gamma > 0$. If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$ şi $p(z)+\gamma zp'(z)$ is univalent in U, atunci*

$$q(z)+\gamma zq'(z) \prec p(z)+\gamma zp'(z) \Rightarrow q(z) \prec p(z)$$

şi q este cea mai bună subordonantă.

4.1 Clase de funcţii univalente definite prin operatorul integro-diferenţial Sălăgean

Teorema 4.1. *[86] Fie q o funcţie convexă, $q(0) = 1$ şi fie*

$$h(z) = q(z)+zq'(z), z \in U.$$

Dacă $f \in \mathcal{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ şi are loc următoarea subordonare

$$\left[\mathcal{D}I^n f(z)\right]' \prec h(z), \quad z \in U \tag{4.2}$$

atunci

$$\frac{\mathcal{D}I^n f(z)}{z} \prec q(z), \quad z \in U$$

şi rezultatul este exact.

Teorema 4.2. *[86] Fie q o funcţie convexă, $q(0) = 1$ şi fie*

$$h(z) = q(z)+zq'(z), z \in U.$$

Dacă $f \in \mathcal{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ şi are loc următoarea subordonare

$$\left(\frac{z\mathcal{D}I^{n+1} f(z)}{\mathcal{D}I^n f(z)}\right)' \prec h(z), \quad z \in U \tag{4.3}$$

atunci

$$\frac{\mathcal{D}I^{n+1} f(z)}{\mathcal{D}I^n f(z)} \prec q(z), \quad z \in U$$

şi rezultatul este exact.

Teorema 4.3. *[86] Fie q o funcţie convexă, $q(0) = 1$ şi fie*

$$h(z) = q(z) + zq'(z), z \in U.$$

Dacă $f \in \mathscr{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ şi are loc următoarea subordonare

$$\left(\mathscr{D}I^{n+1}f(z)\right)' + \lambda\left[\left(I^{n-1}f(z)\right)' - \left(I^{n+1}f(z)\right)'\right] \prec h(z), \quad z \in U \tag{4.4}$$

atunci

$$\left[\mathscr{D}I^n f(z)\right]' \prec q(z), \quad z \in U$$

şi rezultatul este exact.

Teorema 4.4. *[86]Fie $h \in \mathscr{H}(U)$ astfel încât $h(0) = 1$ şi*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

Dacă $f \in \mathscr{A}$ are loc următoarea subordonare

$$\left(\mathscr{D}I^{n+1}f(z)\right)' + \lambda\left[\left(I^{n-1}f(z)\right)' - \left(I^{n+1}f(z)\right)'\right] \prec h(z), \quad z \in U \tag{4.5}$$

atunci

$$\left[\mathscr{D}I^n f(z)\right]' \prec q(z), \quad z \in U$$

unde q este dat de $q(z) = \frac{1}{z}\int_0^z h(t)dt$. Funcţia q este convexă şi este cea mai bună dominantă.

Teorema 4.5. *[86]Fie $h \in \mathscr{H}(U)$ astfel încât $h(0) = 1$ şi*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

Dacă $f \in \mathscr{A}$ are loc următoarea subordonare

$$\left[\mathscr{D}I^n f(z)\right]' \prec h(z), \quad z \in U \tag{4.6}$$

atunci

$$\frac{\mathscr{D}I^n f(z)}{z} \prec q(z), \quad z \in U$$

unde q este dat de $q(z) = \frac{1}{z}\int_0^z h(t)dt$. Funcţia q este convexă şi este cea mai bună dominantă.

Definiţia 4.1. [69], [125], [15], [70]Dacă $0 \leq \beta < 1$ şi $n \in \mathbb{N}$, fie $L_n^m(\beta)$ clasa funcţiilor $f \in \mathscr{A}_m$, care satisfac inegalitatea:

$$\Re\left[\mathscr{D}I^n f(z)\right]' > \beta, \quad (z \in U).$$

Teorema 4.6. *[86]Clasa de funcţii univalente $L_n^m(\beta)$ este convexă.*

Teorema 4.7. *[86]Dacă $0 \le \beta < 1$ şi $m, n \in \mathbb{N}$ atunci avem*

$$L_n^m(\beta) \subset L_{n+1}^m(\delta),$$

unde $\delta(\beta, m) = 2\beta - 1 + 2(1-\beta)\dfrac{1}{m}\sigma\left(\dfrac{1}{m}\right)$ *şi* $\sigma(x) = \displaystyle\int_0^z \frac{t^{x-1}}{1+t}dt$. *Rezultatul este exact.*

Teorema 4.8. *[86]Fie q o funcţie convexă în U, $q(0) = 1$ şi fie*

$$h(z) = q(z) + \frac{1}{c+2} z q'(z), z \in U,$$

unde c este un număr complex, $\Re c > -2$.
Dacă $f \in L_n^m(\beta)$ şi $F = I_c(f)$, unde

$$F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt, \quad \Re c > -2, \tag{4.7}$$

atunci

$$[\mathscr{D} I^n f(z)]' \prec h(z), \quad z \in U, \tag{4.8}$$

implică

$$[\mathscr{D} I^n F(z)]' \prec q(z), \quad z \in U.$$

Rezultatul este exact.

4.2 Rezultate de subordonare diferenţială obţinute prin folosirea unui nou operator

Definiţia 4.2. Fie $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, \alpha, \beta \ge 0$. Fie $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} : \mathscr{A} \to \mathscr{A}$,

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = (1-\alpha-\beta)\mathscr{R}^\nu \mathscr{D}^n f(z) + \alpha \mathscr{R}^\nu \Omega_z^\lambda f(z) + \beta \mathscr{D}^n \Omega_z^\lambda f(z), z \in U.$$

Observaţia 4.1. *Dacă $f \in \mathscr{A}, f(z) = z + \sum\limits_{k=1}^{\infty} a_{k+1} z^{k+1}$, atunci*

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = z + \sum_{k=1}^{\infty} \left((1-\alpha-\beta) \frac{(\nu+1)_k}{(2)_k} (k+1)^{n+1} + \alpha \frac{(\nu+1)_k}{(2-\lambda)_k} (k+1) + \beta \frac{(1)_k}{(2-\lambda)_k} (k+1)^{n+1} \right) a_{k+1} z^{k+1}, \tag{4.9}$$

pentru $z \in U$.

Observaţia 4.2. $\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) = (1-\alpha-\beta)\mathbb{D}_0^{\nu,n} f(z) + \alpha \mathbb{D}_\lambda^{\nu,0} f(z) + \beta \mathbb{D}_\lambda^{0,n} f(z)$, *$z \in U$, unde $\mathbb{D}_\lambda^{\nu,n}$ este definit în (1.22).*

Observaţia 4.3. *Pentru $\alpha = 0$ şi $\beta = 0$, obţinem $\mathscr{D}_{0,0}^{\lambda,\nu,n} f(z) = \mathscr{R}^\nu \mathscr{D}^n f(z)$, unde $z \in U$.*

Pentru $\alpha = 1$ şi $\beta = 0$, obţinem $\mathscr{D}_{1,0}^{\lambda,\nu,n} f(z) = \mathscr{R}^\nu \Omega_z^\lambda f(z)$, unde $z \in U$.

Pentru $\alpha = 0$ şi $\beta = 1$, obţinem $\mathscr{D}_{0,1}^{\lambda,\nu,n} f(z) = \mathscr{D}^n \Omega_z^\lambda f(z)$, unde $z \in U$.

Pentru $\beta = 0$ *şi* $\nu = 0$, *obţinem* $\mathscr{D}_{\alpha,0}^{\lambda,0,n} f(z) = (1-\alpha)\mathscr{D}^n f(z) + \alpha\Omega_z^\lambda f(z)$, *unde* $z \in U$.

Pentru $\alpha = 0$ *şi* $n = 0$, *obţinem* $\mathscr{D}_{0,\beta}^{\lambda,\nu,0} f(z) = (1-\beta)\mathscr{R}^\nu f(z) + \beta\Omega_z^\lambda f(z)$, *unde* $z \in U$.

Pentru $\alpha + \beta = 1$ *şi* $\lambda = 0$, *obţinem* $\mathscr{D}_{1-\beta,\beta}^{0,\nu,n} f(z) = (1-\beta)\mathscr{R}^\nu f(z) + \beta\mathscr{D}^n f(z)$, *unde* $z \in U$.

Pentru $\alpha + \beta = 1, \lambda = 0$ *şi* $\nu = n$, *obţinem* $\mathscr{D}_{1-\beta,\beta}^{0,n,n} f(z) = (1-\beta)\mathscr{R}^n f(z) + \beta\mathscr{D}^n f(z), z \in U$. *[6].*

Pentru $\alpha = \beta = n = 0$, *obţinem* $\mathscr{D}_{0,0}^{\lambda,\nu,0} f(z) = \mathscr{R}^\nu f(z)$, *şi pentru* $\beta = \lambda = n = 0$, *obţinem* $\mathscr{D}_{\alpha,0}^{0,\nu,0} f(z) = \mathscr{R}^\nu f(z)$, *unde* $z \in U$.

Pentru $\alpha = \beta = \nu = 0$, *obţinem* $\mathscr{D}_{0,0}^{\lambda,0,n} f(z) = \mathscr{D}^n f(z)$, *şi pentru* $\alpha = \lambda = \nu = 0$, *obţinem* $\mathscr{D}_{0,\beta}^{0,0,n} f(z) = \mathscr{D}^n f(z)$, *unde* $z \in U$.

Pentru $\alpha = 0$ *şi* $\lambda = \nu = 1$, *obţinem* $\mathscr{D}_{0,\beta}^{1,1,n} f(z) = \mathscr{D}^{n+1} f(z)$, *unde* $z \in U$.

Pentru $\alpha = 1$ *şi* $\beta = \nu = 0$, *obţinem* $\mathscr{D}_{1,0}^{\lambda,0,n} f(z) = \Omega_z^\lambda f(z)$ *şi pentru* $\alpha = n = 0$ *şi* $\beta = 1$, *obţinem* $\mathscr{D}_{0,1}^{\lambda,\nu,0} f(z) = \Omega_z^\lambda f(z)$, *unde* $z \in U$.

Pentru $\lambda = \nu = 0$, *obţinem* $\mathscr{D}_{\alpha,\beta}^{0,0,n} f(z) = (1-\alpha)\mathscr{D}^n f(z) + \alpha f(z)$, *unde* $z \in U$.

Pentru $\lambda = n = 0$, *obţinem* $\mathscr{D}_{\alpha,\beta}^{0,\nu,0} f(z) = (1-\beta)\mathscr{R}^\nu f(z) + \beta f(z)$, *unde* $z \in U$.

Pentru $\nu = n = 0$, *obţinem* $\mathscr{D}_{\alpha,\beta}^{\lambda,0,0} f(z) = (1-\alpha-\beta) f(z) + (\alpha+\beta)\Omega_z^\lambda f(z)$, *unde* $z \in U$.

Pentru $\lambda = 0$ *şi* $\nu = 1$, *obţinem* $\mathscr{D}_{\alpha,\beta}^{0,1,n} f(z) = (1-\alpha-\beta)\mathscr{D}^{n+1} f(z) + \alpha\mathscr{D}^1 f(z) + \beta\mathscr{D}^n f(z)$, *unde* $z \in U$.

Pentru $\lambda = 1$ *şi* $\nu = 0$, *obţinem* $\mathscr{D}_{\alpha,\beta}^{1,0,n} f(z) = (1-\alpha-\beta)\mathscr{D}^n f(z) + \alpha\mathscr{D}^1 f(z) + \beta\mathscr{D}^{n+1} f(z)$, *unde* $z \in U$.

Pentru $\lambda = \nu = 1$, *obţinem* $\mathscr{D}_{\alpha,\beta}^{1,1,n} f(z) = (1-\alpha)\mathscr{D}^{n+1} f(z) + \alpha\mathscr{D}^2 f(z)$, *unde* $z \in U$.

Pentru $\lambda = \nu = n = 0$, *obţinem* $\mathscr{D}_{\alpha,\beta}^{0,0,0} f(z) = f(z)$, *pentru* $\alpha = \beta = \nu = n = 0$, *obţinem* $\mathscr{D}_{0,0}^{\lambda,0,0} f(z) = f(z)$, *pentru* $\alpha = 1$ *şi* $\lambda = \nu = 0$, *obţinem* $\mathscr{D}_{1,\beta}^{0,0,n} f(z) = f(z)$, *şi pentru* $\beta = 1$ *şi* $\lambda = n = 0$, *obţinem* $\mathscr{D}_{\alpha,1}^{0,\nu,0} f(z) = f(z)$, $z \in U$.

Definiţia 4.3. Fie $f \in \mathscr{A}$. Spunem că funcţia f aparţine clasei $\mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$, unde $0 \le \delta \le 1, \alpha, \beta \ge 0, -\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$, dacă f satisface condiţia

$$\Re(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z))' > \delta, z \in U. \tag{4.10}$$

Teorema 4.9. *[82] Fie* $f \in \mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$ *şi* $g \in K$. *Atunci* $f * g \in \mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$.

Teorema 4.10. *[82] Mulţimea* $\mathscr{R}_{\alpha,\beta}^{\lambda,\nu,n}(\delta)$ *este convexă.*

Teorema 4.11. *[82] Fie* g *o funcţie convexă,* $g(0) = 1$ *şi fie* h *o funcţie astfel încât*

$$h(z) = g(z) + zg'(z), z \in U.$$

Dacă $f \in \mathscr{A}$ *şi are loc*

$$\left(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)\right)' \prec h(z), z \in U \tag{4.11}$$

atunci

$$\frac{\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)}{z} \prec g(z), z \in U.$$

Rezultatul este exact.

Teorema 4.12. *[82] Fie g o funcţie convexă, $g(0)=1$ şi fie h o funcţie astfel încât*

$$h(z)=g(z)+zg'(z), z\in U.$$

Dacă $f\in\mathcal{A}$ şi are loc

$$\left(\frac{z\mathcal{D}_{\alpha,\beta}^{\lambda,\nu+1,n}f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)}\right)'\prec h(z), z\in U, \tag{4.12}$$

atunci

$$\frac{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu+1,n}f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)}\prec g(z), z\in U.$$

Rezultatul este exact.

Teorema 4.13. *[82] Fie g o funcţie convexă, $g(0)=1$ şi fie h o funcţie astfel încât*

$$h(z)=g(z)+zg'(z), z\in U.$$

Dacă $f\in\mathcal{A}$ şi are loc

$$\left(\frac{z\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)}\right)'\prec h(z), z\in U, \tag{4.13}$$

atunci

$$\frac{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)}{\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)}\prec g(z), z\in U.$$

Rezultatul este exact.

Teorema 4.14. *[82] Fie g o funcţie convexă, $g(0)=1$ şi fie h o funcţie astfel încât*

$$h(z)=g(z)+zg'(z), z\in U.$$

Dacă $f\in\mathcal{A}$ şi are loc

$$\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)+\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)+\alpha\left(\mathcal{D}\mathcal{D}_{1,0}^{\lambda,\nu,n}f(z)-\mathcal{D}_{1,0}^{\lambda,\nu,n}f(z)\right)\prec h(z), z\in U, \tag{4.14}$$

atunci

$$\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)\prec g(z), z\in U.$$

Rezultatul este exact.

Teorema 4.15. *[82] Fie $h(z)=\dfrac{1+(2\delta-1)z}{1+z}$ o funcţie convexă în U, $0\le\delta<1$. Dacă $f\in\mathcal{A}$ şi are loc următoarea subordonare*

$$\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n+1}f(z)+\mathcal{D}_{\alpha,\beta}^{\lambda,\nu,n}f(z)+\alpha\left(\mathcal{D}\mathcal{D}_{1,0}^{\lambda,\nu,n}f(z)-\mathcal{D}_{1,0}^{\lambda,\nu,n}f(z)\right)\prec h(z), z\in U, \tag{4.15}$$

atunci

$$\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z) \prec g(z), z \in U,$$

unde g este dat de $g(z) = 2\delta - 1 + 2(1-\delta)\dfrac{\ln(1+z)}{z}$, $z \in U$. *Funcţia g este convexă şi este cea mai bună dominantă.*

Teorema 4.16. *[82] Fie g o funcţie convexă,* $g(0) = 1$ *şi let h o funcţie astfel încât*

$$h(z) = g(z) + zg'(z), z \in U.$$

Dacă $f \in \mathscr{A}$ *şi are loc*

$$\frac{1}{z}\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+2} f(z) + \frac{1}{z}\alpha\left(\mathscr{D}^2\mathscr{D}_{1,0}^{\lambda,\nu,n} f(z) - \mathscr{D}_{1,0}^{\lambda,\nu,n} f(z)\right) \prec h(z), z \in U, \tag{4.16}$$

atunci

$$\left(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)\right)' \prec g(z), z \in U.$$

Rezultatul este exact.

Teorema 4.17. *[82] Fie* $h(z) = \dfrac{1+(2\delta-1)z}{1+z}$ *o funcţie convexă în U,* $0 \le \delta < 1$. *Dacă* $f \in \mathscr{A}$ *şi are loc următoarea subordonare*

$$\frac{1}{z}\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n+2} f(z) + \frac{1}{z}\alpha\left(\mathscr{D}^2\mathscr{D}_{1,0}^{\lambda,\nu,n} f(z) - \mathscr{D}_{1,0}^{\lambda,\nu,n} f(z)\right) \prec h(z), \tag{4.17}$$

atunci

$$\left(\mathscr{D}_{\alpha,\beta}^{\lambda,\nu,n} f(z)\right)' \prec g(z), z \in U,$$

unde g is given by $g(z) = 2\delta - 1 + 2(1-\delta)\dfrac{\ln(1+z)}{z}$, $z \in U$. *Funcţia g este convexă şi este cea mai bună dominantă.*

4.3 Subordonări şi superordonări diferenţiale pentru funcţii analitice definite prin operatorul integro-diferenţial Sălăgean

Teorema 4.18. *[81]Fie q o funcţie univalentă în discul unitate U cu* $q(0) = 1$, $\gamma \in \mathbb{C}^*$ *astfel încât are loc*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \ge \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

Dacă $f \in \mathscr{A}$ *şi*

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^n f(z)} + \gamma\left\{1 - \frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z) + \lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^n f(z)]^2} + \right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z) - \mathscr{D}^n f(z)\right]}{\mathscr{D}I^n f(z)}\right\} \prec q(z) + \gamma z q'(z),$$

(4.18)

atunci are loc

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)} \prec q(z) \tag{4.19}$$

şi q este cea mai bună dominantă a subordonării (4.18).

În cazul particular $\lambda = n = 0$ obţinem:

Corolarul 4.1. *[81] Fie q o funcţie univalentă în U cu $q(0)=1$, $\gamma \in \mathbb{C}^*$ astfel încât are loc*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

Dacă $f \in \mathscr{A}$ şi

$$\left(1+\gamma\right)\frac{zf'(z)}{f(z)}+\gamma\left[\frac{z^2f''(z)}{f(z)}-\left(\frac{zf'(z)}{f(z)}\right)^2\right] \prec q(z)+\gamma zq'(z)$$

atunci are loc

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

şi q este cea mai bună dominantă.

În cazul particular $\lambda = 0$ şi $n = 1$, obţinem:

Corolarul 4.2. *[81] Fie q o funcţie univalentă în U cu $q(0)=1$, $\gamma \in \mathbb{C}^*$ astfel încât are loc*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

Dacă $f \in \mathscr{A}$ şi

$$1+\left(1+3\gamma\right)\frac{zf''(z)}{f'(z)}+\gamma\left[1-\left(1+\frac{zf''(z)}{f'(z)}\right)^2+\frac{z^2f'''(z)}{f'(z)}\right] \prec q(z)+\gamma zq'(z)$$

atunci are loc

$$1+\frac{zf''(z)}{f'(z)} \prec q(z)$$

şi q este cea mai bună dominantă.

Corolarul 4.3. *[81] Fie $A, B, \gamma \in \mathbb{C}, A \neq B$ astfel încât $|B| \leq 1$ şi $\Re\, \gamma > 0$. Dacă $f \in \mathscr{A}$ şi*

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^{n}f(z)]^2}+\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{D}I^{n}f(z)}\right\} \prec \frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2},$$

atunci are loc

$$\frac{\mathscr{D}I^{n}f(z)}{\mathscr{D}I^{n+1}f(z)} \prec \frac{1+Az}{1+Bz}$$

şi $q(z)=\dfrac{1+Az}{1+Bz}$ este cea mai bună dominantă.

Teorema 4.19. *[81] Fie q o funcţie convexă în U cu $q(0)=1$ şi $\gamma\in\mathbb{C}$ astfel încât $\Re\,\gamma>0$. Dacă $f\in\mathscr{A}$,*

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}\in\mathscr{H}[1,1]\cap\mathscr{Q},$$

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^{n}f(z)]^{2}}+\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{D}I^{n}f(z)}\right\}$$

este univalentă în U şi

$$q(z)+\gamma zq'(z)\prec\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^{n}f(z)]^{2}}+\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{D}I^{n}f(z)}\right\},$$

(4.20)

atunci $q(z)\prec\dfrac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}$ şi q este cea mai bună subordonantă .

Combinând rezultatele obţinute în teoremele 4.18 şi 4.19 putem da un rezultat de tip ”sandwich”.

Teorema 4.20. *[81] Fie q_1 şi q_2 două funcţii convexe în U cu $q_1(0)=q_2(0)=1$, $\gamma\in\mathbb{C}$ astfel încât $\Re\,\gamma>0$. Dacă $f\in\mathscr{A}$,*

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}\in\mathscr{H}[1,1]\cap\mathscr{Q},$$

$$\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^{n}f(z)]^{2}}+\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{D}I^{n}f(z)}\right\}$$

este univalentă în U şi

$$q_1(z)+\gamma zq_1'(z)\prec\frac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{D}I^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{[\mathscr{D}I^{n}f(z)]^{2}}+\right.$$
$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{D}I^{n}f(z)}\right\}\prec q_2(z)+\gamma zq_2'(z),$$

(4.21)

atunci $q_1(z) \prec \dfrac{\mathscr{D}I^{n+1}f(z)}{\mathscr{D}I^n f(z)} \prec q_2(z)$, *iar* q_1 *şi* $q_2(z)$ *sunt cea mai bună subordonantă, respectiv cea mai bună dominantă a (4.21).*

Teorema 4.21. *[81] Fie* q *o funcţie convexă în* U *cu* $q(0)=1$, $\gamma \in \mathbb{C}^*$ *astfel încât*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

Dacă $f \in \mathscr{A}$ *şi*

$$\left(1+\gamma\right)z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}-$$

$$-2\gamma z\frac{\mathscr{D}I^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{D}I^{n+1}f(z)\right]^3} \prec q(z)+\gamma z q'(z), \tag{4.22}$$

atunci

$$z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2} \prec q(z),$$

q cea mai bună dominantă.

Corolarul 4.4. *[81] Fie* q *o funcţie univalentă în* U *cu* $q(0)=1$, $\gamma \in \mathbb{C}^*$ *astfel încât*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

Dacă $f \in \mathscr{A}$ *şi*

$$\left(1-\gamma\right)\frac{f(z)}{z[f'(z)]^2}+\gamma\left[\frac{1}{f'(z)}-\left(\frac{2f(z)\cdot f''(z)}{[f'(z)]^3}\right)^2\right] \prec q(z)+\gamma z q'(z)$$

atunci

$$\frac{f(z)}{z[f'(z)]^2} \prec q(z)$$

şi q cea mai bună dominantă.

Corolarul 4.5. *[81] Fie* $A, B, \gamma \in \mathbb{C}, A \neq B$ *astfel încât* $|B| \leq 1$ *şi* $\Re\, \gamma > 0$. *Dacă* $f \in \mathscr{A}$

$$\left(1+\gamma\right)z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2}-$$

$$-2\gamma z\frac{\mathscr{D}I^n f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{D}I^{n+1}f(z)\right]^3} \prec \frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2}, \tag{4.23}$$

atunci

$$z\frac{\mathscr{D}I^n f(z)}{\left[\mathscr{D}I^{n+1}f(z)\right]^2} \prec \frac{1+Az}{1+Bz}$$

şi $q(z)=\dfrac{1+Az}{1+Bz}$ *cea mai bună dominantă*

Teorema 4.22. *[81] Fie q o funcţie convexă în U cu $q(0)=1$, $\gamma\in\mathbb{C}$ astfel încât $\Re\gamma>0$. Dacă $f\in\mathcal{A}$*

$$z\frac{\mathcal{D}I^nf(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}\in\mathcal{H}[1,1]\cap\mathcal{Q},$$

$$\left(1+\gamma\right)z\frac{\mathcal{D}I^nf(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathcal{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}-$$
$$-2\gamma z\frac{\mathcal{D}I^nf(z)\left[(1-\lambda)\mathcal{D}^{n+2}f(z)+\lambda I^nf(z)\right]}{\left[\mathcal{D}I^{n+1}f(z)\right]^3}$$

este univalentă în U şi

$$q(z)+\gamma zq'(z)\prec\left(1+\gamma\right)z\frac{\mathcal{D}I^nf(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathcal{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}-$$

$$-2\gamma z\frac{\mathcal{D}I^nf(z)\left[(1-\lambda)\mathcal{D}^{n+2}f(z)+\lambda I^nf(z)\right]}{\left[\mathcal{D}I^{n+1}f(z)\right]^3},\tag{4.24}$$

atunci

$$q(z)\prec z\frac{\mathcal{D}I^nf(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2},$$

q este cea mai bună subordonantă .

Combinând rezultatele obţinute în teoremele 4.21 şi 4.22 putem da un rezultat de tip "sandwich".

Teorema 4.23. *[81] Fie q_1 şi q_2 două funcţii convexe în U cu $q_1(0)=q_2(0)=1$, $\gamma\in\mathbb{C}$ astfel încât $\Re\gamma>0$. Dacă $f\in\mathcal{A}$*

$$z\frac{\mathcal{D}I^nf(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}\in\mathcal{H}[1,1]\cap\mathcal{Q},$$

$$\left(1+\gamma\right)z\frac{\mathcal{D}I^nf(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathcal{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}-$$
$$-2\gamma z\frac{\mathcal{D}I^nf(z)\left[(1-\lambda)\mathcal{D}^{n+2}f(z)+\lambda I^nf(z)\right]}{\left[\mathcal{D}I^{n+1}f(z)\right]^3}$$

este univalentă în U şi

$$q_1(z)+\gamma zq_1'(z)\prec\left(1+\gamma\right)z\frac{\mathcal{D}I^nf(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathcal{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}-$$

$$-2\gamma z\frac{\mathcal{D}I^nf(z)\left[(1-\lambda)\mathcal{D}^{n+2}f(z)+\lambda I^nf(z)\right]}{\left[\mathcal{D}I^{n+1}f(z)\right]^3}\prec q_2(z)+\gamma zq_2'(z),\tag{4.25}$$

atunci

$$q_1(z)\prec z\frac{\mathcal{D}I^nf(z)}{\left[\mathcal{D}I^{n+1}f(z)\right]^2}\prec q_2(z),$$

iar q_1 este cea mai bună subordonantă şi $q_2(z)$ este cea mai bună dominantă.

4.4 Criteriu de univalenţă definit cu ajutorul operatorului generalizat Sălăgean şi Ruscheweyh

$\mathscr{RD}^n$ este operatorul diferenţial Ruscheweyh şi Sălăgean definit în (1.13).

Fie $U_r = \{z \in \mathbb{C} : |z| < r, \;\; r \in (0,1]\}$, $I = [0, \infty)$ şi $p \in \mathscr{P}$ ($\mathscr{P}$ este clasa Carathéodory) de forma $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$.

O funcţie $L(z,t) : U \times I \to \mathbb{C}$ se numeşte lanţ Loewner dacă satisface următoarele condiţii:

$i)$ $L(z,t)$ este analitică şi univalentă în U, $\forall t \in I$

$ii)$ $L(z,t) \prec L(z,s)$, $\forall \;\; 0 \le t \le s < \infty$.

Teorema 4.24. *[88] Fie $f \in \mathscr{A}$ şi p o funcţie analitică cu $p(0) = 1$. Dacă inegalităţile*

$$\left| \frac{2}{p(z)+1} \cdot \frac{zf'(z)}{\mathscr{RD}^{n+1}f(z) + (1-\gamma)\, n \left(\mathscr{R}^{n+1}f(z) - \mathscr{R}^n f(z)\right)} - 1 \right| \le 1 \tag{4.26}$$

şi

$$\Bigg| \left(\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{\mathscr{RD}^{n+1}f(z) + (1-\gamma)\, n \left(\mathscr{R}^{n+1}f(z) - \mathscr{R}^n f(z)\right)} - 1 \right) |z|^2 + \\ + \left(1 - |z|^2\right) \left(\frac{\mathscr{RD}^{n+2}f(z) +}{\mathscr{RD}^{n+1}f(z) + (1-\gamma)\, n \left(\mathscr{R}^{n+1}f(z) - \mathscr{R}^n f(z)\right)} - \right. \\ \left. \frac{(1-\gamma)\left[\left(n^2+3n+1\right)\mathscr{R}^{n+2}f(z) - \left(2n^2+3n+1\right)\mathscr{R}^{n+1}f(z) + n^2\mathscr{R}^n f(z)\right]}{\mathscr{RD}^{n+1}f(z) + (1-\gamma)\, n \left(\mathscr{R}^{n+1}f(z) - \mathscr{R}^n f(z)\right)} - 1 + \frac{zp'(z)}{p(z)+1} \right) \Bigg| \le 1$$

(4.27)

sunt adevărate pentru $z \in U$, atunci funcţia f este univalentă în U.

Observaţia 4.4. *Dacă $\gamma = 1$ obţinem Teorema 1 şi pentru $\gamma = 0$ Theorem 3 din [68].*

Pentru $n = 0$ în Teorema 4.24 obţinem rezultatul lui Lewandowski [54] :

Corolarul 4.6. *[88] Fie $f \in \mathscr{A}$ şi $p \in \mathscr{P}$. Dacă*

$$\left| \frac{1-p(z)}{1+p(z)} |z|^2 + \left(1-|z|^2\right) \left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{1+p(z)} \right) \right| \le 1, \;\; z \in U,$$

atunci funcţia f este univalentă în U.

Pentru $p = 1$ criteriul se reduce la binecunoscutul criteriu al lui Becker [20] (see also Duren et al. [31]).

Corolarul 4.7. *[88] Fie $f \in \mathscr{A}$. Dacă*

$$\left(1-|z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \le 1, \;\; z \in U,$$

atunci funcţia f este univalentă în U.

Pentru $n = 1$, Teorema 4.24 ne conduce la rezultatul următor

Corolarul 4.8. *[88] Fie $f \in \mathscr{A}$ şi p o funcţie analitică cu $p(0) = 1$. Dacă*

$$\left|\frac{2}{p(z)+1}\cdot\frac{f'(z)}{f'(z)+zf''(z)}-1\right|\le 1$$

şi

$$\left|\left(\frac{2}{p(z)+1}\cdot\frac{f'(z)}{f'(z)+zf''(z)}-1\right)|z|^2+\left(1-|z|^2\right)\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}+\frac{zp'(z)}{p(z)+1}\right)\right|\le 1$$

sunt adevărate pentru $z \in U$ atunci funcţia f este univalentă în U.

Pentru lanţul Loewner de forma

$$L(z,t) := f\left(e^{-t}z\right)+\left(e^tz-e^{-t}z\right)\frac{p\left(e^{-t}z\right)+1}{2}\cdot\frac{\mathscr{R}\mathscr{D}^{n+1}f(e^{-t}z)}{\mathscr{R}\mathscr{D}^{n}f(e^{-t}z)} \tag{4.28}$$

obţinem:

Teorema 4.25. *[88] Fie $f \in \mathscr{A}$ şi p o funcţie analitică cu $p(0) = 1$. Dacă inegalităţile*

$$\left|\frac{2}{p(z)+1}\cdot f'(z)\cdot\frac{\mathscr{R}\mathscr{D}^{n}f(z)}{\mathscr{R}\mathscr{D}^{n+1}f(z)}-1\right|\le 1 \tag{4.29}$$

şi

$$\left|\left(\frac{2f'(z)}{p(z)+1}\cdot\frac{\mathscr{R}\mathscr{D}^{n}f(z)}{\mathscr{R}\mathscr{D}^{n+1}f(z)}-1\right)|z|^2+\left(1-|z|^2\right)\left(\frac{\mathscr{R}\mathscr{D}^{n+2}f(z)}{\mathscr{R}\mathscr{D}^{n+1}f(z)}-\frac{\mathscr{R}\mathscr{D}^{n+1}f(z)}{\mathscr{R}\mathscr{D}^{n}f(z)}+\left(1-\gamma\right)\right.\right.$$
$$\left.\left.\frac{\left[(n+1)\mathscr{R}\mathscr{D}^{n}f(z)\left(\mathscr{R}^{n+2}f(z)-\mathscr{R}^{n+1}f(z)\right)-n\mathscr{R}\mathscr{D}^{n+1}f(z)\left(\mathscr{R}^{n+1}f(z)-\mathscr{R}^{n}f(z)\right)\right]}{\mathscr{R}\mathscr{D}^{n}f(z)\cdot\mathscr{R}\mathscr{D}^{n+1}f(z)}+\frac{zp'(z)}{p(z)+1}\right)\right|\le 1$$

(4.30)

sunt adevărate pentru $z \in U$ atunci funcţia f este univalentă în U.

Corolarul 4.9. *[88] Fie $f \in \mathscr{A}$ şi p o funcţie analitică cu $p(0) = 1$. Dacă*

$$\left|\frac{2}{p(z)+1}\cdot\frac{f'(z)}{z}-1\right|\le 1$$

şi

$$\left|\left(\frac{2}{p(z)+1}\cdot\frac{f'(z)}{z}-1\right)|z|^2+\left(1-|z|^2\right)\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\frac{zp'(z)}{p(z)+1}\right)\right|\le 1$$

sunt adevărate pentru $z \in U$ atunci funcţia f este univalentă în U.

Pentru $p = 1$ în Corolarul anterior obţinem rezultatul lui Kanas şi Lecko [51].

Corolarul 4.10. *[88] Fie $f \in \mathscr{A}$ cu $\Re \frac{f(z)}{z} > 0$. Dacă*

$$\left| \left(\frac{f(z)}{z} - 1 \right) |z|^2 + \left(1 - |z|^2\right) \left[1 + \frac{zf''(z)}{f'(z)} \left(\frac{f(z)}{z} + 1 \right) - \frac{zf'(z)}{f(z)} \right] \right| \le \left| \frac{f(z)}{z} + 1 \right|$$

sunt adevărate pentru $z \in U$ atunci funcţia f este univalentă în U.

Corolarul 4.11. *[88] Fie $f \in \mathscr{A}$. Dacă*

$$\left| 2\frac{f(z)}{z} - \frac{zf'(z)}{f(z)} - 1 \right| \le \left| 1 + \frac{zf'(z)}{f(z)} \right|$$

şi

$$\left| \left(2\frac{f(z)}{z} - \frac{zf'(z)}{f(z)} - 1 \right) |z|^2 + \left(1 - |z|^2\right) \frac{2zf'(z)}{f(z)+1} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right| \le \left| 1 + \frac{zf'(z)}{f(z)} \right|$$

sunt adevărate pentru $z \in U$ atunci funcţia f este univalentă în U.

4.5 Proprietatea de conservare al operatorului integral generalizat Bernardi-Libera-Livingston, definită pe clase de funcţii stelate

Fie T subclasa lui $\mathscr{A}$, formată din funcţii f de forma

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \tag{4.31}$$

unde $a_j \ge 0$, $j = 2, 3, \ldots$ şi $z \in U$. Pentru clasa T, următoarele sunt echivalente [112]:

i) $\sum_{j=2}^{\infty} j a_j \le 1$,

ii) $f \in T \cap S$,

iii) $f \in T^*$, unde $T^* = T \cap S^*$.

$$S^{**} = \left\{ f \in \mathscr{A} : \left| 1 + \frac{zf''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}},\ z \in U \right\}. \tag{4.32}$$

$$S^{***} = \left\{ f \in \mathscr{A} : \left| 1 - \frac{zf''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}},\ z \in U \right\}. \tag{4.33}$$

Teorema 4.26. *[96] Fie*

$$F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t)\,dt.$$

Dacă $p \ge \sqrt{\frac{5}{4}}$ şi $f \in S^{**}$, atunci $F \in S^{**}$.

Teorema 4.27. *[96] Fie* $F(z) = L_p f(z) = \dfrac{p+1}{z^p}\displaystyle\int_0^z t^{p-1} f(t)dt$, $p > -2$. *Dacă* $f \in S^{***}$ *atunci* $F \in S^{***}$.

Definiţia 4.4. [96] Funcţia $f \in T$ aparţine clasei $TS^{***} = S^{***} \cap T$ dacă

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}}, z \in U.$$

Teorema 4.28. *[96] Funcţia* $f \in T$ *aparţine clasei* TS^{***} *dacă şi numai dacă*

$$\sum_{j=2}^{\infty} j\left(j - 2 + \frac{\sqrt{5}}{2}\right)a_j < \frac{\sqrt{5}}{2} - 1. \tag{4.34}$$

Teorema 4.29. *[96] Fie* $F(z) = L_p f(z) = \dfrac{p+1}{z^p}\displaystyle\int_0^z t^{p-1} f(t)dt$, $p \in (-1,0]$. *Dacă* $f \in TS^{***}$, *atunci* $F \in TS^{***}$.

4.6 Raza de convexitate ale unor funcţii particulare şi aplicaţii in studiul a unor inegalităţi diferenţiale de gradul doi

Fie f o funcţie definită astfel: $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$.

În [61](pg.243) este propusă : Dacă $f(0) = a$ cu $\Re\, a > 0$, şi

$$\Re\big(a + 4zf'(z) + 2z^2 f''(z)\big) > 0,\ z \in U, \tag{4.35}$$

atunci $\Re f(z) > 0,\ z \in U$.

$$A_0 = \{f \in H(U) |\ f(0) = 1\} \text{ şi } \mathscr{P} = \{f \in A_0 |\ \mathrm{Re} f(z) > 0,\ z \in U\}.$$

Lema 4.5. *[75] Dacă* $\theta \in [-\pi, \pi]$, *atunci*

$$2(1-\cos\theta)\left(\int_0^1 t\frac{1}{\sqrt{1+t^2-2t\cos\theta}}dt\right)^2 \le \int_0^1 t\frac{(1+t)(1-\cos\theta)}{1+t^2-2t\cos\theta}dt.$$

Pentru $\widetilde{V} \subset A_0$ mulţimea duală $\widetilde{V}$ este definită

$$\widetilde{V}^d = \left\{g \in A_0 | (f * g)(z) \ne 0,\ \text{for all } f \in \widetilde{V} \text{ şi } \forall z \in U\right\}.$$

Lema 4.6. *[75] Pentru mulţimea duală a clasei* $\mathscr{P} = \{f \in A_0 |\ \Re f(z) > 0,\ z \in U\}$ *avem*

$$\mathscr{L}^d = \{f \in A_0 |\ \Re f(z) > \frac{1}{2},\ z \in U\} \subset \mathscr{P}^d.$$

Lema 4.7. *[75]*

$$\sum_{n=1}^{\infty}\frac{e^{in\theta}}{n(n+1)^2}=\int_0^1\int_0^1\frac{(1-x)y(\cos\theta-xy)}{1+x^2y^2-2xy\cos\theta}dxdy+i\int_0^1\int_0^1\frac{(1-x)y\sin\theta}{1+x^2y^2-2xy\cos\theta}dxdy,$$

$$\sum_{n=1}^{\infty}\frac{e^{in\theta}}{(n+1)^2}=\int_0^1\int_0^1\frac{xy(\cos\theta-xy)}{1+x^2y^2-2xy\cos\theta}dxdy+i\int_0^1\int_0^1\frac{xy\sin\theta}{1+x^2y^2-2xy\cos\theta}dxdy.$$

Lema 4.8. *[75] Dacă* $\alpha=\dfrac{2-\ln 4}{3-\ln 6-\frac{\pi^2}{12}}$, *atunci :*

$$\begin{aligned}&(1-\alpha)\int_0^1\int_0^1(1-x)y\frac{(1-xy)(1+\cos\theta)}{(1+xy)(1+x^2y^2-2xy\cos\theta)}dxdy\\ \geq\ &\frac{1}{6}\int_0^1\int_0^1 xy\frac{(1-xy)(1+\cos\theta)}{(1+xy)(1+x^2y^2-2xy\cos\theta)}dxdy,\quad \theta\in[\frac{\pi}{2},\pi],\end{aligned}$$

$$\int_0^1\int_0^1\frac{xy\sin\theta}{1+x^2y^2-2xy\cos\theta}dxdy\leq\int_0^1\int_0^1\frac{xy\sqrt{2(1+\cos\theta)}}{(1+xy)\sqrt{1+x^2y^2-2xy\cos\theta}}dxdy,\ \theta\in[0,\pi].$$

Lema 4.9. *[75]*

$$\begin{aligned}&\int_0^1\int_0^1\frac{2xy}{1-x^2y^2}dxdy\int_0^1\int_0^1 xy\frac{(1-xy)(1+\cos\theta)}{(1+x^2y^2-2xy\cos\theta)(1+xy)}dxdy\leq\\ &\leq 4(1-\alpha)\Big(1+\sum_{n=1}^{\infty}\frac{\cos n\theta}{n(n+1)^2}\Big)\Big[(1-\alpha)\cdot\Big(1+\sum_{n=1}^{\infty}\frac{\cos n\theta}{n(n+1)^2}\Big)+\sum_{n=1}^{\infty}\frac{\cos n\theta}{(n+1)^2}\Big],\\ &\theta\in[0,\pi].\end{aligned}\tag{4.36}$$

Teorema 4.30. *[75] Funcţiile* ψ *şi* φ *definite*

$$\psi(z)=1+\sum_{n=1}^{\infty}\frac{z^n}{n(n+1)},\qquad \varphi(z)=1+\sum_{n=1}^{\infty}\frac{z^n}{n(n+1)^2}$$

sunt convexe în U, *razele de convexitate* $r_\psi^c=r_\varphi^c=1$.

Corolarul 4.12. *[75] Dacă* $f(0)=1$ *şi*

$$\Re\big(1+4zf'(z)+2z^2f''(z)\big)>0,\quad \forall z\in U\tag{4.37}$$

atunci

$$2-\frac{1+r}{r}\ln(1+r)<\Re\big(f(z)\big)<2+\frac{1-r}{r}\ln(1-r),\quad z\in U_r\tag{4.38}$$

pentru orice $r\in(0,1)$, *şi*

$$2-\ln 4<\Re\big(f(z)\big)<2,\quad \forall z\in U.\tag{4.39}$$

Corolarul 4.13. *[75] Dacă* $f(0) = 1$ *şi* (4.37) *sunt adevărate, atunci*

$$\left|f(z)\right| < 2 + \frac{1-r}{r}\ln(1-r), \quad z \in U(r) \tag{4.40}$$

pentru orice $r \in (0,1)$, *şi*

$$\left|f(z)\right| < 2, \quad z \in U. \tag{4.41}$$

Teorema 4.31. *[75] Dacă* $f(0) = 1$ *şi* (4.37) *sunt adevărate, atunci funcţia* F *definit în* $F(z) = \int_0^z f(t)dt$ *este stelată de ordinul* $\alpha = \dfrac{2-\ln 4}{3-\ln 4 - \frac{\pi^2}{12}} = 0.7756\ldots$, *şi*

$$\Re \frac{zF'(z)}{F(z)} > \frac{2-\ln 4}{3-\ln 4-\frac{\pi^2}{12}}, \quad z \in U. \tag{4.42}$$

Rezultatul este exact.

Capitolul 5

Funcţii bi-univalente

Fiecare funcţie univalentă f are o inversă f^{-1} :

$$f^{-1}(f(z)) = z, \quad (z \in U),$$

şi

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}\right), \tag{5.1}$$

unde

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots \tag{5.2}$$

O funcţie $f \in \mathscr{A}$ este bi-univalentă în U dacă $U \subset f(U)$ şi dacă f şi f^{-1} sunt univalente în U.

Lema 5.1. *[99] Dacă $h \in P$ atunci $|c_k| \leq 2$, $\forall k$, unde P este familia tuturor funcţiilor h analitice în U pentru care $\Re h(z) > 0$, unde $h(z) = 1 + c_1z + c_2z^2 + \cdots$, $z \in U$.*

5.1 Estimarea coeficienţilor şi problema Fekete-Szegő pentru noi clase de funcţii bi-univalente definite de operatorul integro-diferenţial Sălăgean

Definiţia 5.1. [115], [13], [79] Pentru $0 < \alpha \leq 1$, $0 \leq \widetilde{\lambda} \leq 1$ o funcţie $f(z)$ definită în (1.1) aparţine clasei $\mathscr{P}_{\Sigma}^{\alpha}(\widetilde{\lambda})$ dacă următoarele condiţii sunt satisfăcute:

$$\left| arg\left(\frac{z(\mathscr{D}I^n f(z))' + \widetilde{\lambda}z^2(\mathscr{D}I^n f(z))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n f(z) + \widetilde{\lambda}z(\mathscr{D}I^n f(z))'}\right)\right| < \frac{\alpha\pi}{2} \tag{5.3}$$

şi

$$\left|arg\left(\frac{w(\mathscr{D}I^n g(w))' + \widetilde{\lambda}w^2(\mathscr{D}I^n g(w))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n g(w) + \widetilde{\lambda}w(\mathscr{D}I^n g(w))'}\right)\right| < \frac{\alpha\pi}{2} \tag{5.4}$$

unde $z, w \in U$ şi funcţia g este definită în (5.2).

Definiţia 5.2. [79] Pentru $0 < \alpha \leq 1$, $0 \leq \widetilde{\lambda} \leq 1$ funcţia $f(z)$ definită în (1.1) aparţine clasei $\mathscr{Q}_{\Sigma}^{\beta}(\widetilde{\lambda})$ dacă următoarele condiţii sunt satisfăcute:

$$\Re\left(\frac{z(\mathscr{D}I^n f(z))' + \widetilde{\lambda}z^2(\mathscr{D}I^n f(z))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n f(z) + \widetilde{\lambda}z(\mathscr{D}I^n f(z))'}\right) > \beta \tag{5.5}$$

şi

$$\Re\left(\frac{w(\mathscr{D}I^n g(w))' + \widetilde{\lambda}w^2(\mathscr{D}I^n g(w))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n g(w) + \widetilde{\lambda}w(\mathscr{D}I^n g(w))'}\right) > \beta \tag{5.6}$$

unde $z, w \in U$ şi funcţia g este definită în (5.2).

Definiţia 5.3. [79] Fie $h, l : U \to \mathbb{C}$ funcţii analitice şi

$$min\{\Re(h(z)), \Re(l(z))\} > 0, \quad (z \in U) \quad h(0) = l(0) = 1.$$

O funcţie $f(z)$ definită în (1.1) aparţine clasei $\mathscr{P}_{\Sigma}^{h,l}$ dacă următoarele condiţii sunt satisfăcute:

$$\frac{z(\mathscr{D}I^n f(z))' + \widetilde{\lambda}z^2(\mathscr{D}I^n f(z))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n f(z) + \widetilde{\lambda}z(\mathscr{D}I^n f(z))'} \in h(U) \tag{5.7}$$

şi

$$\frac{w(\mathscr{D}I^n g(w))' + \widetilde{\lambda}w^2(\mathscr{D}I^n g(w))''}{\left(1-\widetilde{\lambda}\right)\mathscr{D}I^n g(w) + \widetilde{\lambda}w(\mathscr{D}I^n g(w))'} \in l(U) \tag{5.8}$$

unde $z, w \in U$ şi funcţia g este definită în (5.2).

Teorema 5.1. *[79] Pentru* $0 < \alpha \leq 1$, $0 \leq \widetilde{\lambda} \leq 1$ *şi fie* $f(z)$ *definită în* (1.1) *din clasa* $\mathscr{P}_{\Sigma}^{\alpha}(\widetilde{\lambda})$. *Atunci*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left|4\alpha\Gamma_3\left(1+2\widetilde{\lambda}\right) + \Gamma_2^2\left(1+\widetilde{\lambda}\right)^2(1-3\alpha)\right|}}, \tag{5.9}$$

$$|a_3| \leq \frac{\alpha}{\Gamma_3\left(1+2\widetilde{\lambda}\right)} + \frac{4\alpha^2}{\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2} \tag{5.10}$$

şi

$$|a_4| \le \frac{2\alpha\left(2\alpha^2+1\right)}{9\Gamma_4\left(1+3\widetilde{\lambda}\right)} - \frac{10\alpha(2\alpha-1)}{3\left[2\Gamma_2\Gamma_3\left(1+\widetilde{\lambda}\right)\left(1+2\widetilde{\lambda}\right)-5\Gamma_4\left(1+3\widetilde{\lambda}\right)\right]} + \\ + \frac{8\alpha^3\left[3\left(1+2\widetilde{\lambda}\right)\Gamma_3-\left(1+\widetilde{\lambda}\right)^2\Gamma_2^2\right]}{3\Gamma_2\Gamma_4\left(1+\widetilde{\lambda}\right)\left(1+3\widetilde{\lambda}\right)\sqrt{\left|4\alpha\Gamma_3\left(1+2\widetilde{\lambda}\right)+\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2(1-3\alpha)\right|}}.$$

(5.11)

Teorema 5.2. *[79] Fie f de forma* (1.1) *din clasa $\mathscr{P}_\Sigma^\alpha(\widetilde{\lambda})$. Atunci*

$$\left|a_3-\xi a_2^2\right| \le \begin{cases} \frac{\alpha}{\Gamma_3(1+2\widetilde{\lambda})}; & \left|\frac{\alpha(1-\xi)}{4\alpha\Gamma_3(1+2\widetilde{\lambda})+\Gamma_2^2(1+\widetilde{\lambda})^2(1-3\alpha)}\right| \le \frac{1}{4\Gamma_3(1+2\widetilde{\lambda})} \\ \left|\frac{4\alpha^2(1-\xi)}{4\alpha\Gamma_3(1+2\widetilde{\lambda})+\Gamma_2^2(1+\widetilde{\lambda})^2(1-3\alpha)}\right|; & \left|\frac{\alpha(1-\xi)}{4\alpha\Gamma_3(1+2\widetilde{\lambda})+\Gamma_2^2(1+\widetilde{\lambda})^2(1-3\alpha)}\right| \ge \frac{1}{4\Gamma_3(1+2\widetilde{\lambda})} \end{cases}$$

Teorema 5.3. *[79] Pentru $0<\beta\le 1$, $0\le\widetilde{\lambda}\le 1$ şi fie $f(z)$ definită în* (1.1) *din clasa $\mathscr{Q}_\Sigma^\beta(\widetilde{\lambda})$. Atunci*

(5.12)
$$|a_2| \le \sqrt{\frac{2(1-\beta)}{\left|2\left(1+2\widetilde{\lambda}\right)\Gamma_3-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}},$$

(5.13)
$$|a_3| \le \frac{1-\beta}{\Gamma_3\left(1+2\widetilde{\lambda}\right)} + \frac{4(1-\beta)^2}{\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}$$

şi

$$|a_4| \le \frac{2(1-\beta)}{3\Gamma_4\left(1+3\widetilde{\lambda}\right)} - \frac{10(1-\beta)}{3\Gamma_4\left(1+3\widetilde{\lambda}\right)\left[2\Gamma_2\Gamma_3\left(1+\widetilde{\lambda}\right)\left(1+2\widetilde{\lambda}\right)-5\Gamma_4\left(1+3\widetilde{\lambda}\right)\right]} + \\ + \frac{2\Gamma_2(1-\beta)\left(1+\widetilde{\lambda}\right)\left[3\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right]}{3\Gamma_4\left(1+3\widetilde{\lambda}\right)\left[2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right]}\sqrt{\frac{2(1-\beta)}{\left|2\left(1+2\widetilde{\lambda}\right)\Gamma_3-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}}.$$

(5.14)

Teorema 5.4. *[79] Fie f de forma* (1.1) *din clasa $\mathscr{Q}_\Sigma^\beta(\widetilde{\lambda})$. Atunci*

$$\left|a_3-\xi a_2^2\right| \le \begin{cases} \frac{1-\beta}{\Gamma_3(1+2\widetilde{\lambda})}; & \left|\frac{1-\xi}{4\alpha\Gamma_3(1+2\widetilde{\lambda})-2\Gamma_2^2(1+\widetilde{\lambda})^2}\right| \le \frac{1}{4\Gamma_3(1+2\widetilde{\lambda})} \\ \left|\frac{4(1-\beta)(1-\xi)}{4\alpha\Gamma_3(1+2\widetilde{\lambda})-2\Gamma_2^2(1+\widetilde{\lambda})^2}\right|; & \left|\frac{1-\xi}{4\alpha\Gamma_3(1+2\widetilde{\lambda})-2\Gamma_2^2(1+\widetilde{\lambda})^2}\right| \ge \frac{1}{4\Gamma_3(1+2\widetilde{\lambda})} \end{cases}$$

Teorema 5.5. *[79] Pentru* $0 < \alpha \leq 1$, $0 \leq \widetilde{\lambda} \leq 1$ *şi fie* $f(z)$ *definită în* (1.1) *din clasa* $\mathscr{P}_{\Sigma}^{h,l}$. *Atunci*

(5.15)
$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|l'(0)|^2}{2\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}}, \sqrt{\frac{|h''(0)|+|l''(0)|}{4\left|2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}}\right\}$$

$$|a_3| \leq \min\left\{\frac{\left|h'(0)\right|^2+\left|l'(0)\right|^2}{2\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}+\frac{\left|h''(0)\right|+\left|l''(0)\right|}{8\Gamma_3\left(1+2\widetilde{\lambda}\right)},\right.$$
$$\left.\frac{\left|h''(0)\right|\left|4\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|+\left|l''(0)\right|\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}{8\Gamma_3\left(1+2\widetilde{\lambda}\right)\left|2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}\right\}$$

(5.16)

şi

$$|a_4| \leq \min\left\{\frac{\left|h'''(0)\right|+\left|l'''(0)\right|}{36}\left|\frac{1}{\Gamma_4\left(1+3\widetilde{\lambda}\right)}-\frac{5}{2\Gamma_2\Gamma_3\left(1+\widetilde{\lambda}\right)\left(1+2\widetilde{\lambda}\right)-5\Gamma_4\left(1+3\widetilde{\lambda}\right)}\right|+\right.$$
$$+\frac{\left|h'(0)\right|^2+\left|l'(0)\right|^2}{\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2}\sqrt{\frac{|h'(0)|^2+|l'(0)|^2}{2}}\frac{\left|3\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}{6\Gamma_4\left(1+3\widetilde{\lambda}\right)},$$
$$\frac{\left|h'''(0)\right|+\left|l'''(0)\right|}{36}\left|\frac{1}{\Gamma_4\left(1+3\widetilde{\lambda}\right)}-\frac{5}{2\Gamma_2\Gamma_3\left(1+\widetilde{\lambda}\right)\left(1+2\widetilde{\lambda}\right)-5\Gamma_4\left(1+3\widetilde{\lambda}\right)}\right|+$$
$$\left.\frac{\left|h''(0)\right|+\left|l''(0)\right|}{\left|2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}\sqrt{\frac{|h''(0)|+|l''(0)|}{\left|2\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}}\frac{\Gamma_2\left(1+\widetilde{\lambda}\right)\left|3\Gamma_3\left(1+2\widetilde{\lambda}\right)-\Gamma_2^2\left(1+\widetilde{\lambda}\right)^2\right|}{24\Gamma_4\left(1+3\widetilde{\lambda}\right)}\right\}$$

(5.17)

5.2 Extensia unor estimări ale coeficienţilor pentru noi clase de funcţii bi-univalente definite de operatorul integro-diferenţial Sălăgean

Fie φ, ψ funcţii analitice cu parte reală pozitivă în discul unitate U, satisfăcând $\varphi(0) = \psi(0) = 1$, $\varphi'(0) > 0$, $\psi'(0) > 0$ şi $\varphi(U), \psi(U)$ sunt simetrice faţă de axa reală (see [127]). Presupunem că:

(5.18)
$$\varphi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots, \quad (A_1 > 0)$$

şi

(5.19)
$$\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0).$$

Definiţia 5.4. [83] O funcţie $f(z)$ definită în (1.1) aparţine clasei $\mathscr{M}_{\Sigma}(\varphi,\psi)$ dacă următoarele condiţii sunt satisfăcute:

$$\left(\mathscr{D}I^n f(z)\right)' \prec \varphi(z) \tag{5.20}$$

şi

$$\left(\mathscr{D}I^n g(w)\right)' \prec \psi(w) \tag{5.21}$$

unde $z,w \in U$ şi funcţia g este definită în (5.2).

Definiţia 5.5. [83] Pentru $0<\alpha$, o funcţie $f(z)$ definită în (1.1) aparţine clasei $\mathscr{H}_{\Sigma}(\varphi,\psi,\alpha)$ dacă următoarele condiţii sunt satisfăcute:

$$\left(\frac{z(\mathscr{D}I^n f(z))'}{\mathscr{D}I^n f(z)}\right)^{\alpha}\left(1+\frac{z(\mathscr{D}I^n f(z))''}{(\mathscr{D}I^n f(z))'}\right)^{1-\alpha} \prec \varphi(z) \tag{5.22}$$

şi

$$\left(\frac{w(\mathscr{D}I^n g(w))'}{\mathscr{D}I^n g(w)}\right)^{\alpha}\left(1+\frac{w(\mathscr{D}I^n g(w))''}{(\mathscr{D}I^n g(w))'}\right)^{1-\alpha} \prec \psi(w) \tag{5.23}$$

unde $z,w \in U$ şi funcţia g este definită în (5.2).

Definiţia 5.6. [83] Fie $h,l : U \to \mathbb{C}$ funcţii analitice şi

$$\min\{\Re(h(z)),\Re(l(z))\} > 0, \ \ (z \in U) \quad h(0)=l(0)=1.$$

A funcţie $f(z)$ definită în (1.1) aparţine clasei $\mathscr{H}_{\Sigma}^{h,l}(\alpha)$ dacă următoarele condiţii sunt satisfăcute:

$$\left(\frac{z(\mathscr{D}I^n f(z))'}{\mathscr{D}I^n f(z)}\right)^{\alpha}\left(1+\frac{z(\mathscr{D}I^n f(z))''}{(\mathscr{D}I^n f(z))'}\right)^{1-\alpha} \in h(U) \tag{5.24}$$

şi

$$\left(\frac{w(\mathscr{D}I^n g(w))'}{\mathscr{D}I^n g(w)}\right)^{\alpha}\left(1+\frac{w(\mathscr{D}I^n g(w))''}{(\mathscr{D}I^n g(w))'}\right)^{1-\alpha} \in l(U) \tag{5.25}$$

unde $z,w \in U$ şi funcţia g este definită în (5.2).

Teorema 5.6. *[83] Fie $f(z)$ definită în* (1.1) *din clasa $\mathscr{M}_{\Sigma}(\varphi,\psi)$. Atunci*

$$|a_2| \le \sqrt{\frac{1}{6\Gamma_3}\left|A_2+B_1-\frac{A_1^2}{B_1}\right|}, \tag{5.26}$$

(5.27)
$$|a_3| \leq \frac{A_1}{3\Gamma_3} + \frac{1}{6\Gamma_3}\left|2A_2 - 2A_1 - \frac{B_2A_1^2}{B_1^2}\right|$$

şi

(5.28)
$$|a_4| \leq \frac{B_3}{4\Gamma_4}$$

Teorema 5.7. *[83] Fie $f(z)$ definită în* (1.1) *din clasa $\mathscr{H}_\Sigma(\varphi,\psi,\alpha)$. Atunci*

(5.29)
$$|a_2| \leq \sqrt{\frac{A_2 + B_1 + \frac{A_1^2}{B_1^2}|B_2 - B_1|}{\left|\alpha^2 + 5\alpha - 8\right|\Gamma_2^2 + 4|3-2\alpha|\Gamma_3}},$$

(5.30)
$$|a_3| \leq \frac{A_2 + B_1 + \frac{A_1^2}{B_1^2}|B_2 - B_1|}{\left|\alpha^2 + 5\alpha - 8\right|\Gamma_2^2 + 4|3-2\alpha|\Gamma_3} + \frac{|A_2 - B_2|}{4|3-2\alpha|\Gamma_3}$$

şi

$$|a_4| \leq \left|\frac{B_3 - A_3}{6\Gamma_4(3\alpha-4)} - \frac{5}{2}\frac{A_3+B_3}{15\Gamma_4(3\alpha-4) - 2\Gamma_2\Gamma_3\left(4\alpha^2+11\alpha-18\right)} + \right.$$
$$\left. + \frac{A_1^3}{6(2-\alpha)^3\Gamma_2^2\Gamma_4(3\alpha-4)}\left[2\Gamma_3\left(4\alpha^2+11\alpha-18\right) - \frac{1}{3}\Gamma_2^2\left(\alpha^3+21\alpha^2+20\alpha-48\right)\right]\right|$$

(5.31)

Teorema 5.8. *[83] Fie $f(z)$ definită în* (1.1) *din clasa $\mathscr{H}_\Sigma^{h,l}(\alpha)$. Atunci*

(5.32)
$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2 + |l'(0)|^2}{2\Gamma_2^2(2-\alpha)^2}}; \sqrt{\frac{|h''(0)| + |l''(0)|}{2\left|\alpha^2+5\alpha-8\right|\Gamma_2^2 + 8\Gamma_3|3-2\alpha|}}\right\}$$

$$|a_3| \leq \min\left\{\frac{\left|h''(0)\right| + \left|l''(0)\right|}{8\Gamma_3(3-2\alpha)} + \frac{\left|h'(0)\right|^2 + \left|l'(0)\right|^2}{2\Gamma_2^2(2-\alpha)^2},\right.$$
$$\left.\frac{\left|h''(0)\right|\left|\alpha^2+5\alpha-8\right|\Gamma_2^2 + \left|l''(0)\right|\left|\left(\alpha^2+5\alpha-8\right)\Gamma_2^2 + 8\Gamma_3(3-2\alpha)\right|}{8\Gamma_3\left|(3-2\alpha)\left[\left(\alpha^2+5\alpha-8\right)\Gamma_2^2 + 4\Gamma_3(3-2\alpha)\right]\right|}\right\}$$

(5.33)

şi

$$|a_4| \leq \frac{1}{18\Gamma_4|3\alpha-4|}\min\left\{\left|h'''(0)\right| + \left|\alpha^3+21\alpha^2+20\alpha-48\right|\left(\frac{\left|h'(0)\right|^2 + \left|l'(0)\right|^2}{2(2-\alpha)^2}\right)^{\frac{3}{2}};\right.$$
$$\left.\left|h'''(0)\right| + \left|\alpha^3+21\alpha^2+20\alpha-48\right|\Gamma_2^3\left(\frac{\left|h''(0)\right| + \left|l''(0)\right|}{2\left|\alpha^2+5\alpha-8\right|\Gamma_2^2 + 8\Gamma_3|3-2\alpha|}\right)^{\frac{3}{2}}\right\}$$

(5.34)

5.3 Estimări ale coeficienţilor pentru noi clase de funcţii bi-Bazilevič de tipul Ma-Minda definite de operatorul integro-diferenţial Sălăgean

Definiţia 5.7. Fie $h : U \to \mathbb{C}$ o funcţie convexă şi univalentă în U astfel încât

$$h(0) = 1 \qquad \text{şi} \qquad \Re\,(h(z)) > 0, \quad (z \in U).$$

$h(z) = 1 + \sum\limits_{k=1}^{\infty} B_k z^k$, $(z \in U)$.

O funcţie $f \in \Sigma$ definită în (1.1) aparţine clasei $\mathscr{M}_{\Sigma}^{n}(\beta, \widetilde{\lambda}; h)$ dacă următoarele condiţii sunt satisfăcute:

$$e^{i\beta}\left(\frac{z^{1-\widetilde{\lambda}}(\mathscr{D}I^n f(z))'}{[\mathscr{D}I^n f(z)]^{1-\widetilde{\lambda}}}\right) \prec h(z)\cos\beta + i\sin\beta \tag{5.35}$$

şi

$$e^{i\beta}\left(\frac{w^{1-\widetilde{\lambda}}(\mathscr{D}I^n g(w))'}{[\mathscr{D}I^n g(w)]^{1-\widetilde{\lambda}}}\right) \prec h(w)\cos\beta + i\sin\beta, \tag{5.36}$$

unde

$$\beta \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right), \ \ \widetilde{\lambda} \geq 0; \quad z, w \in U \quad \text{şi} \quad \text{funcţia } g \text{ este dată de (5.2)}.$$

Teorema 5.9. *[80] Fie funcţia $f(z)$ definită în (1.1) din clasa $\mathscr{M}_{\Sigma}^{n}(\beta, \widetilde{\lambda}; h)$. Atunci*

$$|a_2| \leq \sqrt{\frac{2|B_1|\cos\beta}{(\widetilde{\lambda}+2)\left|(\widetilde{\lambda}-1)\Gamma_2^2 + 2\Gamma_3\right|}}, \ \ (\widetilde{\lambda}-1)\Gamma_2^2 + 2\Gamma_3 \neq 0 \tag{5.37}$$

$$|a_3| \leq \frac{|B_1|\cos\beta}{(\widetilde{\lambda}+2)\Gamma_3} + \left(\frac{|B_1|\cos\beta}{(1+\widetilde{\lambda})\Gamma_2}\right)^2 \tag{5.38}$$

şi

$$\begin{aligned}|a_4| \leq |B_1|\cos\beta \Bigg\{ & \frac{\Gamma_2}{3\Gamma_4} \cdot \frac{(1-\widetilde{\lambda})\left[(\widetilde{\lambda}-2)\Gamma_2^2 + 6\Gamma_3\right]}{(\widetilde{\lambda}+2)\left[(\widetilde{\lambda}-1)\Gamma_2^2 + 2\Gamma_3\right]}\sqrt{\frac{2|B_1|\cos\beta}{(\widetilde{\lambda}+2)\left|(\widetilde{\lambda}-1)\Gamma_2^2 + 2\Gamma_3\right|}} + \\ & + \frac{1}{(3+\widetilde{\lambda})\Gamma_4} + \frac{5}{(\widetilde{\lambda}+3)\left|2(\widetilde{\lambda}-1)\Gamma_2\Gamma_3 + 5\Gamma_4\right|}\Bigg\}, \ \ (\widetilde{\lambda}-1)\Gamma_2^2 + 2\Gamma_3 \neq 0\end{aligned}$$

şi

$$2(\widetilde{\lambda}-1)\Gamma_2\Gamma_3 + 5\Gamma_4 \neq 0. \tag{5.39}$$

Corolarul 5.1. *[80] Fie funcţia $f(z)$ definită în* (1.1) *din clasa* $\mathscr{M}_{\Sigma}^{n}(\beta,0;h)$. *Atunci*

$$|a_2| \le \sqrt{\frac{|B_1|\cos\beta}{2\Gamma_3-\Gamma_2^2}}, \qquad |a_3| \le \frac{|B_1|\cos\beta}{2\Gamma_3} + \left(\frac{|B_1|\cos\beta}{\Gamma_2}\right)^2, \quad \text{şi}$$

$$|a_4| \le \frac{|B_1|\cos\beta}{3}\left\{\frac{\Gamma_2\left(-\Gamma_2^2+3\Gamma_3\right)}{\Gamma_4\left(-\Gamma_2^2+2\Gamma_3\right)}\sqrt{\frac{|B_1|\cos\beta}{-\Gamma_2^2+2\Gamma_3}} + \frac{1}{\Gamma_4} - \frac{5}{2\Gamma_2\Gamma_3-5\Gamma_4}\right\}.$$

Corolarul 5.2. *[80] Fie funcţia $f(z)$ definită în* (1.1) *din clasa* $\mathscr{M}_{\Sigma}^{n}(\beta,1;h)$. *Atunci*

$$|a_2| \le \sqrt{\frac{|B_1|\cos\beta}{3\Gamma_3}}, \qquad |a_3| \le \frac{|B_1|\cos\beta}{3\Gamma_3} + \left(\frac{|B_1|\cos\beta}{2\Gamma_2}\right)^2, \quad \text{şi}$$

$$|a_4| \le |B_1|\cos\beta\frac{1}{2\Gamma_4}.$$

Corolarul 5.3. *[80] Fie funcţia $f(z)$ definită în* (1.1) *din clasa* $\mathscr{M}_{\Sigma}^{0}(\beta,0;h)$. *Atunci*

$$|a_2| \le \sqrt{|B_1|\cos\beta}, \qquad |a_3| \le |B_1|\cos\beta\left(|B_1|\cos\beta + \frac{1}{2}\right), \quad \text{şi}$$

$$|a_4| \le \frac{2|B_1|\cos\beta}{3}\left(\sqrt{|B_1|\cos\beta} + \frac{4}{3}\right).$$

Corolarul 5.4. *[80] Fie funcţia $f(z)$ definită în* (1.1) *din clasa* $\mathscr{M}_{\Sigma}^{0}(\beta,1;h)$. *Atunci*

$$|a_2| \le \sqrt{\frac{|B_1|\cos\beta}{3}}, \qquad |a_3| \le \frac{|B_1|\cos\beta}{3} + \left(\frac{|B_1|\cos\beta}{2}\right)^2, \quad \text{şi}$$

$$|a_4| \le \frac{|B_1|\cos\beta}{2}.$$

Capitolul 6

Funcţii armonice

Se noteaază cu $\mathscr{H}$ clasa funcţiilor $f = h + \overline{g}$ care sunt armonice în discul unitate U. Se noteaază cu $\mathscr{H}_u$ clasa funcţiilor armonic-univalente în discul unitate U şi cu $\mathscr{H}_{op}$ cele care sunt armonic-univalente şi păstrează orientarea în discul unitate U, sunt normate şi $f(0) = h(0) = f_z'(0) - 1 = 0$.

Fie $V_{\mathscr{H}}$ clasa funcţiilor armonice cu argument variabil introdus de Jahangiri and Silverman [46], $f \in \mathscr{H}_{u,op}$ $\exists\ \xi \in \mathbb{R}$:

$$\eta_m + (m-1)\xi \equiv \pi (mod\ 2\pi), \quad \delta_m + (m+1)\xi \equiv 0 (mod\ 2\pi), \quad m \geq 2, \tag{6.1}$$

unde $\eta_m = arg(a_m)$ şi $\delta_m = arg(b_m)$.

Fie

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m}. \tag{6.2}$$

Definiţia 6.1. Operatorul integral generalizat Bernardi-Libera-Livingston pentru funcţii armonice este $\mathscr{L}_c(f)$, $(c > -1)$: $\mathscr{L}_c(f) = \mathscr{L}_c(h) + \overline{\mathscr{L}_c(g)}$ unde

$$\mathscr{L}_c(h)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t)dt \ \text{ şi } \ \mathscr{L}_c(g)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} g(t)dt$$

([65]).

6.1 Clase de funcţii armonice şi operatorul integral generalizat Bernardi-Libera-Livingston

Teorema 6.1. *[65] Fie $f = h + \overline{g}$ cu* (6.1) *şi* $0 \leq b_1 < \dfrac{1-\gamma}{1+\gamma}, 0 \leq \gamma < 1$. *Atunci $f \in \mathscr{V}_{\mathscr{H}}(F;\gamma)$ dacă şi numai dacă*

$$\sum_{m=2}^{\infty} \left(\frac{m-\gamma}{1-\gamma} |a_m| + \frac{m+\gamma}{1-\gamma} |b_m| \right) C_m \leq 1 - \frac{1+\gamma}{1-\gamma} b_1 \tag{6.3}$$

Teorema 6.2. *[65] Fie* $\lambda_m = \dfrac{1-\gamma}{(m-\gamma)C_m}$ *şi* $\mu_m = \dfrac{1-\gamma}{(m+\gamma)C_m}$. *Atunci pentru* b_1 *fixat,* $0 \le b_1 < \dfrac{1-\gamma}{1+\gamma}$ *punctele extreme pentru* $\mathcal{V}_{\mathcal{H}}(F;\gamma)$, $0 \le \gamma < 1$ *sunt*

$$\left\{z + \lambda_m x z^m + \overline{b_1 z}\right\} \cup \left\{z + \overline{b_1 z + \mu_m x z^m}\right\}$$

unde $m \ge 2$ *and* $x = 1 - \dfrac{1+\gamma}{1-\gamma} b_1$.

Teorema 6.3. *[98] Fie* $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$. *Atunci* $\mathcal{L}_c(f) \in \mathcal{V}_{\mathcal{H}}(F;\delta(\gamma))$ *unde*

$$\delta(\gamma) = \frac{(2+\gamma)(c+2)(1-b_1) - 2(c+1)\left[(1-\gamma) - (1+\gamma)b_1\right]}{(2+\gamma)(c+2)(1+b_1) + (c+1)\left[(1-\gamma) - (1+\gamma)b_1\right]} > \gamma.$$

Rezultatul este exact.

6.2 Ordinul de consistenţă al convoluţiei funcţiilor armonice cu argument variabil

Considerăm operatorul integral (pentru cazul analitic [21], [14], [106])
$\mathcal{I}^s : f \in \mathcal{V}_{\mathcal{H}}(F,\gamma) \to \mathcal{V}_{\mathcal{H}}(F,\gamma), s \in \mathbb{R}$, astfel încât

$$\mathcal{I}^s f(z) = \mathcal{I}^s\left(z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}\right) = z + \sum_{m=2}^{\infty} \frac{a_m}{m^s} z^m + \overline{\sum_{m=1}^{\infty} \frac{b_m}{m^s} z^m}. \tag{6.4}$$

Definiţia 6.2. ([21], [107]) Fie $\mathcal{X},\mathcal{Y}$ şi $\mathcal{Z}$ subclase ale lui $\mathcal{V}_{\mathcal{H}}(F;\gamma)$. Spunem că tripletul $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ este $S_{\circledast}$–închis în raport cu convoluţia dacă există numărul $S_{\circledast}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ astfel încât

$$S_{\circledast}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = min\left\{s \in \mathbb{R} : \mathcal{I}^s(f \circledast g) \in \mathcal{Z}, \forall f \in \mathcal{X}, \forall g \in \mathcal{Y}\right\} \tag{6.5}$$

Numărul $S_{\circledast}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ se numeşte ordinul de consistenţă al convoluţiei pentru tripletul $(\mathcal{X},\mathcal{Y},\mathcal{Z})$

Notăm cu $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$ subclasa lui $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ unde $|a_m| \le 1, |b_m| \le 1, \forall m \ge 2$.

Teorema 6.4. *[97] Fie* f_1, f_2 *două funcţii din* $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$ *; atunci* $(f_1 \circledast f_2) \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma)$.

Teorema 6.5. *[97] Fie* $f_1 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_1)$, $f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_2)$ *atunci* $(f_1 \circledast f_2)$ *aparţine* $\mathcal{V}^1_{\mathcal{H}}(F;\gamma^*)$, *unde* $\gamma^* = \dfrac{(2+\gamma_1)(2+\gamma_2)(1-b_{1,1}b_{2,1}) - 2\left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2+\gamma_1)(2+\gamma_2)(1+b_{1,1}b_{2,1}) + \left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$, *dacă*

$$1 - b_{1,1}b_{2,1} - (1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) > 0$$

sau $\gamma^* = \dfrac{(2-\gamma_1)(2-\gamma_2)(1-b_{1,1}b_{2,1}) - 2\left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2-\gamma_1)(2-\gamma_2)(1+b_{1,1}b_{2,1}) - \left[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$, *if*

$$1 - b_{1,1}b_{2,1} - (1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) < 0.$$

6.3 Clase unificate de funcţii armonice cu argument variabil

Fie k, A şi B numere reale,

$$k \geq 0,\ 0 \leq B \leq 1 \text{ şi } -1 \leq A < B.$$

Fie $\varphi, \phi \in \mathscr{H}$. Notăm cu $\mathscr{W}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)$, $0 \leq B < 1$ clasa funcţiilor $f \in \mathscr{H}$ astfel încât

$$\left(\varphi * f\right)(z) \neq 0, z \in U \setminus \{0\}$$

şi

$$\left|\frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} - k\left|\frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} - 1\right| - \frac{1-AB}{1-B^2}\right| < \frac{B-A}{1-B^2} \quad (z \in U). \tag{6.6}$$

Dacă $B = 1$, atunci

$$\Re\left(\frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)}\right) - k\left|\frac{\left(\phi * f\right)(z)}{\left(\varphi * f\right)(z)} - 1\right| > \frac{1+A}{2} \quad (z \in U). \tag{6.7}$$

Definim

$$\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right) := \mathscr{W}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right) \cap V_{\mathscr{H}}.$$

Presupunem că φ şi ϕ sunt funcţii de forma:

$$\varphi(z) = z + \sum_{m=2}^{\infty} c_m z^m + \overline{\sum_{m=1}^{\infty} d_m z^m} \text{ şi } \phi(z) = z + \sum_{m=2}^{\infty} e_m z^m + \overline{\sum_{m=1}^{\infty} f_m z^m} \tag{6.8}$$

unde

$$0 \leq c_m \leq e_m \text{ şi } 0 \leq d_m \leq f_m. \tag{6.9}$$

Teorema 6.6. *[74] Fie $0 \leq B \leq 1$, $-1 \leq A < B$ şi $\left(\varphi * f\right)(z) \neq 0, z \in U \setminus \{0\}$. Dacă*

$$\sum_{m=2}^{\infty}\left(|a_m|\,\alpha_m + |b_m|\,\beta_m\right) \leq B - A \tag{6.10}$$

atunci $f \in \mathscr{W}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)$ unde

$$\alpha_m = (k+1)(1+B)e_m + [(B-A)-(k+1)(1+B)]c_m,$$

$$\beta_m = (k+1)(1+B)f_m + [(B-A)-(k+1)(1+B)]d_m.$$

Corolarul 6.1. *[74] Dacă f aparţine clasei $\mathscr{W}\mathscr{V}_{\mathscr{H}}\left(\phi,\varphi;A,B;k\right)$, atunci*

$$|a_m| \leq \frac{B-A}{\alpha_m},\quad |b_m| \leq \frac{B-A}{\beta_m},\quad (m \in \{2,3,\ldots\}). \tag{6.11}$$

Rezultatul este exact şi funcţiile extremale sunt

$$f_{1,m} = z - \frac{B-A}{\alpha_m} e^{i(1-m)\xi} z^m, \tag{6.12}$$

şi

$$f_{2,m} = z + \frac{B-A}{\beta_m} e^{i(1+m)\xi} \overline{z}^m, \; m \in \{2,3,\cdots\}. \tag{6.13}$$

Teorema 6.7. *[74] Fie f din clasa $\mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)$ atunci*

$$(1+|b_1|)r - \frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2} r^2 \le |f(z)| \le$$

$$\le (1+|b_1|)r + \frac{B-A}{(k+1)(1+B)(e_2-c_2)+(B-A)c_2} r^2 \tag{6.14}$$

unde $\alpha_2 \le \alpha_m$, $\beta_2 \le \beta_m$ ($m \in \mathbb{N} \setminus \{1\}$).

Teorema 6.8. *[74] Fie $0 \le \alpha < 1$. Atunci*

$$R_\alpha^*\left(\mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)\right) = \inf_{m\ge 2}\left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_m}{m-\alpha}, \frac{\beta_m}{m+\alpha}\right\}\right)^{\frac{1}{m-1}}.$$

Teorema 6.9. *[74] Fie $0 \le \alpha < 1$. Atunci*

$$R_\alpha^c\left(\mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)\right) = \inf_{m\ge 2}\left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_m}{m(m-\alpha)}, \frac{\beta_m}{m(m+\alpha)}\right\}\right)^{\frac{1}{m-1}}.$$

Teorema 6.10. *[74] Dacă $f, F \in \mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)$ şi $|a_m|, |b_m|, |A_m|, |B_m| \in [0,1]$ atunci $f * F \in \mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)$.*

Teorema 6.11. *[74] Fie $f \in \mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)$. Atunci $\mathscr{L}_c(f) \in \mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A^*,B;k)$ unde $A^* = \min\{A_1^*, A_2^*\} > A$,*

$$A_1^* = B - \frac{(B-A)(k+1)(1+B)(c+1)(e_m-c_m)}{(B-A)(m-1)c_m + (k+1)(1+B)(c+m)(e_m-c_m)},$$

$$A_2^* = B - \frac{(B-A)(k+1)(1+B)(c+1)(f_m-d_m)}{(B-A)(m-1)d_m + (k+1)(1+B)(c+m)(f_m-d_m)}.$$

Teorema 6.12. *[74] Fie $f \in \mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B;k)$. Atunci $\mathscr{L}_c(f) \in \mathscr{W}\mathscr{V}_{\mathscr{H}}(\phi,\varphi;A,B^*;k)$ unde $B^* = \min\{B_1^*, B_2^*\} < B$,*

$$B_1^* = A + \frac{(B-A)(k+1)(1+A)(c+1)(e_m-c_m)}{(B-A)(m-1)c_m + (k+1)(e_m-c_m)[(1+B)(c+m)-(c+1)(B-A)]},$$

$$B_2^* = A + \frac{(B-A)(k+1)(1+A)(c+1)(f_m-d_m)}{(B-A)(m-1)d_m + (k+1)(f_m-d_m)[(1+B)(c+m)-(c+1)(B-A)]}.$$

6.4 Generalizări ale unor funcţii armonice stelate definite prin operatorii Sălăgean şi Ruscheweyh

Considerăm operatorul linear $\mathcal{L}^n_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ definit pentru o funcţie $f = h + \overline{g} \in \mathcal{H}$: $\mathcal{L}^n_{\mathcal{H}} f := \mathcal{L}^n_{\mathcal{H}} h + (-1)^n \overline{\mathcal{L}^n_{\mathcal{H}} g}$. Pentru o funcţie $f \in \mathcal{H}$, avem

$$\mathcal{L}^n_{\mathcal{H}} f(z) = z + \sum_{k=2}^{\infty} \left[\gamma\eta(k,n,\lambda) + (1-\gamma)\mu(k,n)\right] a_k z^k + (-1)^n \sum_{k=2}^{\infty} \left[\gamma\eta(k,n,\lambda) + (1-\gamma)\mu(k,n)\right] \overline{b_k} \overline{z}^k, z \in U,$$

unde $\eta(k,n,\lambda) = [1 + (k-1)\lambda]^n$ şi $\mu(k,n) = \dfrac{(n+k-1)!}{n!(k-1)!}$.

Definiţia 6.3. Pentru $-B \le A < B \le 1$ şi $n \in \mathbb{N}$ notăm cu $\widetilde{S}^n_{\mathcal{H}}(A,B)$ clasa funcţiilor $f \in \mathcal{H}$ astfel încât

$$\left| \frac{\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - \mathcal{L}^n_{\mathcal{H}} f(z)}{B\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - A\mathcal{L}^n_{\mathcal{H}} f(z)} \right| < 1 \quad (z \in U). \tag{6.15}$$

Teorema 6.13. *[84] O funcţie $f \in \mathcal{H}$ aparţine clasei $\widetilde{S}^n_{\mathcal{H}}(A,B)$ dacă*

$$\sum_{k=2}^{\infty} \left(\alpha_k |a_k| + \beta_k |b_k|\right) \le B - A, \tag{6.16}$$

unde

$$\alpha_k = \sigma(A,B,n,\gamma,\lambda,k) + \sigma(1,1,n,\gamma,\lambda,k),$$

$$\beta_k = \delta(A,B,n,\gamma,\lambda,k) + \delta(1,1,n,\gamma,\lambda,k),$$

$$\sigma(A,B,n,\gamma,\lambda,k) = \gamma\eta(k,n,\lambda)[(k-1)\lambda B + B - A] + (1-\gamma)\mu(k,n)\frac{(B-A)n + Bk - A}{n+1},$$

$$\delta(A,B,n,\gamma,\lambda,k) = \gamma\eta(k,n,\lambda)[(k-1)\lambda B + B + A] + (1-\gamma)\mu(k,n)\frac{(B+A)n + Bk + A}{n+1}.$$

Teorema 6.14. *[84] Dacă $f \in \widetilde{S}^n_{\mathcal{H}}(A,B)$ atunci f este univalentă.*

Fie $\mathcal{N}$ clasa funcţiilor $f = h + \overline{g} \in \mathcal{H}$ de forma ([113])

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=2}^{\infty} |b_k| \overline{z}^k, \tag{6.17}$$

şi fie $\widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)$ unde $\mathcal{N} \cap \widetilde{S}^n_{\mathcal{H}}(A,B)$.

Teorema 6.15. *[84] Fie $f = h + \overline{g}$ definit în* (6.17). *Atunci $f \in \widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)$ dacă şi numai dacă condiţia* (6.16) *este adevărat.*

Teorema 6.16. *[84] Clasa $\widetilde{S}^n_{\mathcal{H}\mathcal{N}}(A,B)$ este convexă şi mulţime compactă a lui $\mathcal{H}$.*

Teorema 6.17. *[84] Mulţimea punctelor extreme $\widetilde{S}^n_{\mathcal{HN}}(A,B)$ este*
$E\widetilde{S}^n_{\mathcal{HN}}(A,B) = \{h_k : k \in \mathbb{N}\} \cup \{g_k : k \in \{2,3,\ldots\}\}$

$$h_1 = z,\;\; h_k(z) = z - \frac{B-A}{\alpha_k} z^k,$$

$$g_k(z) = z + (-1)^n \frac{B-A}{\beta_k} \overline{z}^k, (z \in U, k \in \{2,3,\ldots\}) \tag{6.18}$$

Teorema 6.18. *[84] Fie $f \in \widetilde{S}^n_{\mathcal{HN}}(A,B)$ şi $|z| = r < 1$. Atunci*

$$r - \frac{B-A}{\alpha_2} r^2 \le |f(z)| \le r + \frac{B-A}{\alpha_2} r^2$$

$$r - \frac{(B-A)\left[\gamma(1+\lambda)^n + (1-\gamma)(n+1)\right]}{\alpha_2} r^2 \le \left|\mathcal{L}^n_{\mathcal{H}} f(z)\right| \le r + \frac{(B-A)\left[\gamma(1+\lambda)^n + (1-\gamma)(n+1)\right]}{\alpha_2} r^2$$

Rezultatul este exact. Funcţiile extremale sunt h_2 de forma (6.18).

Corolarul 6.2. *[84] Dacă $f \in \widetilde{S}^n_{\mathcal{HN}}(A,B)$ atunci $U(r) \subset f(U(r))$ unde*

$$r = 1 - \frac{B-A}{\alpha_2}$$

şi

$$U(r) := \{z \in \mathbb{C} : |z| < r \le 1\}.$$

Corolarul 6.3. *[84] Fie $0 < r < 1$ şi $\xi \ge 1$. Dacă $f \in \widetilde{S}^n_{\mathcal{HN}}(A,B)$ atunci*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^\xi d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|h_2(re^{i\theta})\right|^\xi d\theta,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{L}^k_{\mathcal{H}} f(re^{i\theta})\right|^\xi d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{L}^k_{\mathcal{H}} h_2(re^{i\theta})\right|^\xi d\theta \;\; (\xi = 1,2,\ldots).$$

Teorema 6.19. *[84] Fie $0 \le \alpha < 1$ şi α_k şi β_k definit în* (6.16)*. Atunci*

$$R^*_\alpha\left(\widetilde{S}^n_{\mathcal{HN}}(A,B)\right) = \inf_{k\ge2} \left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_k}{k-\alpha}, \frac{\beta_k}{k+\alpha}\right\}\right)^{\frac{1}{k-1}}$$

Teorema 6.20. *[84] Fie $0 \le \alpha < 1$ şi α_k şi β_k definit în* (6.16)*. Atunci*

$$R^c_\alpha\left(\widetilde{S}^n_{\mathcal{HN}}(A,B)\right) = \inf_{k\ge2} \left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_k}{k(k-\alpha)}, \frac{\beta_k}{k(k+\alpha)}\right\}\right)^{\frac{1}{k-1}}.$$

Teorema 6.21. *[84] Fie $f \in \widetilde{S}^n_{\mathcal{H}}(A,B)$. Atunci $\mathcal{L}_c(f) \in \widetilde{S}^n_{\mathcal{H}}(A,B)$.*

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