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CONTRIBUTIONS TO THE GEOMETRIC FUNCTION THEORY

Ph.D. Thesis Summary

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analytic function, univalent function, starlike function, strongly starlike function, convex function, close-to-convex function, k-parabolic starlikeness, k-uniformly convexity, functions with negative coefficients, composition of functions, functions with varying arguments, extreme points, convolution, differential subordination, Ruscheweyh derivative, Sălăgean derivative, q-derivative, Bernardi-Libera-Livingston integral operator, Noor integral operator, Sălăgean integral operator, Mittag-Leffler function, Laguerre polynomials, Legendre polynomials, radius of starlikeness, radius of convexity, radius of uniform convexity.

Contents

Chapter 1

Introduction

The geometric function theory is an apart branch of the complex analysis which studies the geometric properties of analytic functions. The foundations of this theory can be traced back to the beginning of the 20th century with the appearance of the papers written by P. Koebe [52], T. H. Gronwall [44], J. W. Alexander [7] and L. Bieberbach [22]. In 1916 L. Bieberbach [22] enounced the famous conjecture, which bears his name and which has led to the development of these theories. The most important research methods was the parametric method of K. Löwner [55], the variational methods introduced by M. Schiffer [86] and G. M. Goluzin [41], the integral representation method introduced by G. Herglotz [47], the duality principle for convolutions, developed by S. Ruscheweyh [80], the differential subordination method, developed by S.S. Miller and P.T. Mocanu. In 1984 the Bieberbach conjecture was finally proved by Louis de Branges [26], using the Löwner's parametric method.

In the geometric theory of analytic functions, univalent functions plays an important role. It is well-known that a holomorphic function it's said to be univalent in a domain $\mathbb{D}$, if any of its values are taken once in $\mathbb{D}$.

The most important romanian contributor to the development of this theory is P. T. Mocanu, who created together with S. S. Miller a new study method, the differential subordination method [58]. This method has an important role in the more simpler demonstration of some classical results and also in obtaining much new results. This method is presented detailed in Chapter 4 of the present thesis.

In the last decade the theory of univalent functions had a fast development and new research directions have emerged. In the papers [78], [82], [4], [48], [3] there was introduced the Ruscheweyh differential operator, the Sălăgean differential operator, the Hadamard product of the extended generalized Sălăgean operator and the extended Ruscheweyh operator, the q-differential operator and the Al-Oboudi differential operator. In [82] and [64] there was introduced the Sălăgean integral operator and the Noor integral operator. Using this differential and integral operators many classes of analytic functions were generalized and new classes were also introduced.

An other new direction in the geometric function theory is the determination of the radius of starlikeness, convexity and uniform convexity for some special functions. For example in [18] Á. Baricz, P. A. Kupán and R. Szász have determined the radius of starlikeness of the normalized Bessel functions of the first kind for three different kinds of normalization. In [17] Á. Baricz and R. Szász have determined the radius of convexity for three kinds of normalized Bessel functions of the first kind. In [27] E. Deniz and R. Szász have determined the radius of uniform convexity for three kinds of normalized Bessel functions of the first kind.

This thesis contains eight chapters and a bibliography with 103 titles. The aim of the thesis is to investigate the geometric properties of some newly introduced analytic classes of functions and of some special functions also. In the followings each chapter is summarized, highlighting the author contributions to the thesis.

In **Chapter 1** there is presented the historical background of the geometric function theory.

Chapter 2 is dedicated to basic notations and preliminary results from geometric function theory. The chapter contains three sections. In the first section there is presented some basic concepts from the univalent function theory. For example there are presented the $\mathcal{H}[a,n]$, $\mathcal{A}_n$ and $\mathcal{S}$ classes, where $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$. There is also defined the class of functions with negative coefficients and the Hadamard product of two analytic function. Further we recall five types of differential operators, introduced by Ruscheweyh, Sălăgean, Alb Lupaş, Al-Oboudi and Jackson. After that we present two type of integral operator, introduced by Sălăgean and Noor. Then by the convolution of the Sălăgean and Noor integral operators, we define the Noor-Sălăgean integral operator. The following section deals with the subordination principle. Some properties of the subordination relation is also recalled.

Finally, the aim of the last section is to present some special classes of analytic functions and to recall some analytic characterizations for these, with the addition of the author results. Two new classes of analytic functions there are also introduced. In [32] and [31] for example there is proved that, these classes, namely S^{**} and S^{***}, has the property that the composition of each two functions from S^{**} and S^{***} is starlike, in a disk where the composition is defined. The class S^{***} for functions with negative coefficients is also defined, followed by the coefficient delimitation theorem for this class. This section also contains a new starlike condition, given in Lemma

2.3.1.1, a sharp version of a strongly starlikeness condition, given in Theorem 2.3.2.2 and some consequences of these results can we also found in this sections.

In **Chapter 3** there are given some new classes of analytic functions with varying arguments defined by the Ruscheweyh and by the convolution of the Sălăgean and Ruscheweyh differential operators. There are derived some new results for these classes of analytic functions. The properties of the image of these new classes of analytic functions through the generalized Bernardi-Libera-Livingston integral operator are also studied.

The **Chapter 4** focuses on the differential subordination technique. The first two sections of this chapters deal with the theory of this method, introduced by P. T. Mocanu and S. S. Miller, followed by the third section, which presents some applications for the differential subordination method. In this section, using the differential subordination method, for example there is shown that $S^{**} \subset S^*$ and $S^{***} \subset \mathcal{K}$, where S^* and $\mathcal{K}$ denotes the classes of starlike and convex functions. Besides, for example there is proved also that if $f, g \in S^*$, then $f \circ g$ is starlike in $\mathbb{U}(r_0)$, where $r_0 = \sup\{r \in (0,1] \mid g(\mathbb{U}(r)) \subset \mathbb{U}\}$.

In **Chapter 5** using the q-difference operator, the Noor-Sălăgean and the Sălăgean integral operators there are introduced the $UCC_q(\gamma)$, $C_{NS}(\alpha)$ and $Q_1(m, \lambda, A, B)$ classes. Some geometric properties of these classes are also investigated. The author original contribution in this chapter can be found in the papers [33], [34] and [37].

Chapter 6 is devoted to the presentation of some sufficient conditions regarding to the Mittag-Leffler function. The first section deals with the presentation of the Mittag-Leffler and the generalized Mittag-Leffler function. After that, in the second section there is given some sufficient conditions, so that the generalized Mittag-Leffler function to be in the classes S^*, $\mathcal{K}$, S_p, $\mathcal{UCV}$, $k - S_p(\gamma)$, $k - \mathcal{UCV}(\gamma)$, $k - S_p(\lambda, \gamma)$ and $k - \mathcal{UCV}(\lambda, \gamma)$, where $k \geq 0$ and $\gamma, \lambda \in [0, 1)$.

Chapter 7 is splitted in three sections and there are given some results in connection with integral operators. For example, in Theorem 7.1.1 we give an γ order starlikeness condition for the $F(z) = \int\limits_0^z f(t)dt$ integral operator, where $z \in \mathbb{U}$. In the next two sections we investigate the properties of the images of the classes $UCC_q(g, \gamma)$, $C_{NS}(\alpha)$, S^{**}, S^{***} and TS^{***} trough the Bernardi integral operator and the generalized Bernardi-Libera-Livingston integral operator. Theorem 7.2.1, Theorem 7.2.2, Theorem 7.3.1, Theorem 7.3.2 and Theorem 7.3.3 justify the preserving properties of the Bernardi and the generalized Bernardi-Libera-Livingston integral operators, defined on these classes of functions.

Chapter 8 treats some radius problems for two orthogonal polynomials. In the first section we recall the generalized Laguerre polynomials and some well-known properties of this. Then in the following section we determine the radius of starlikeness and convexity of order β and the radius of uniform convexity of the normalized Laguerre polynomials, where $0 \leq \beta < 1$. Finally, in Example 8.2.1 there is calculated the radius of convexity for the second order Laguerre polynomials. The third section presents the Legendre polynomials. Some related results there are also recalled, followed by the last section, in which there are determined the radius of starlikeness of order β, the radius of convexity of order β and the radius of uniform convexity of the normalized Legendre polynomials of odd degree, where $0 \leq \beta < 1$. The results presented in this section can we found in [24].

The original results presented in the thesis, are contained in the following papers:

1. **O. Engel**, Á.O. Páll-Szabó, P.A. Kupán, *About the radius of convexity of some analytic functions*, Creat. Math. and Inf., 24(2), 155–161, 2015.

2. **O. Engel**, R. Szász, *On a subclass of convex functions*, Stud. Univ. Babeş-Bolyai Math., 59(2), 137–146, 2016.

3. **O. Engel**, *On the composition of two starlike functions*, Acta Univ. Apulensis, 48, 47–53, 2016.

4. **O. Engel**, *On a class of analytic functions defined by the Sălăgean integral operator*, An. Univ. Oradea fasc. Mat., 24(2), 9–14, 2017.

5. **O. Engel**, C. Naicu, *About a generalized class of close-to-convex functions defined by the q-difference operator*, Scient. Bull. of the "Petru Maior" Univ. of Târgu Mureş, 13(1), 30–34, 2016.

6. **O. Engel**, Y.L. Chung, *About a class of analytic functions defined by Noor-Sălăgean integral operator*, J. Math. and Appl., 39, 59–67, 2016.

7. **O. Engel**, Á.O. Páll-Szabó, *The radius of convexity of particular functions and applications to the study of a second order differential subordination*, J. Contemp. Math. Anal., 52(3), 111–120, 2017.

8. Á.O. Páll-Szabó, **O. Engel**, E. Szatmári, *Certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative*, Acta Univ. Apulensis, 51, 61–74, 2017.

9. Á.O. Páll-Szabó, **O. Engel**, *Properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative*, Acta Univ. Sapientiae, 7(2), 278–286, 2015.

10. **O. Engel**, A.R. Juma, *The sharp version of a strongly starlikeness condition*, Acta Univ. Sapientiae, accepted paper.

11. **O. Engel**, G. Murugusundaramoorthy, R. Szász, *The radius of starlikeness, convexity and uniform convexity of the normalized Laguerre polynomials*, submitted paper.

12. **O. Engel**, Y.L. Chung, *Certain properties of the generalized Mittag-Leffler function*, submitted paper.

13. **O. Engel**, Á.O. Páll-Szabó, *Preserving properties of the generalized Bernardi-Libera-Livingston integral operator defined on some subclasses of starlike functions*, Konuralp J. Math., 5(2), 207–215, 2017.

14. S. Bulut, **O. Engel**, *The radius of starlikeness, convexity and uniform convexity of the Legendre polynomials of odd degree*, submitted paper.

A part of the original results, proved in the thesis, were presented at the following international conferences:

1. 5th International Conference on Mathematics and Informatics, September 2-4, 2015, Târgu-Mureş.

2. International Conference on Theory and Applications of Mathematics and Informatics (ICTAMI 2015), September 17-20, 2015, Alba Iulia.

3. International Conference On Sciences, May 13-14, 2016, Oradea.

4. The 15th International Conference On Applied Mathematics and Computer Science, July 5-7, 2016, Cluj-Napoca.

5. 6th Internation Conference on Mathematics and informatics, September 7-9, 2017, Târgu-Mureş.

Chapter 2

Univalent functions in the complex plane

2.1 Basic notations and definitions

Let $\mathbb{U}(r) = \{z \in \mathbb{C} : |z| < r\}$ be a disk in the complex plane $\mathbb{C}$, centered at zero and we note by

$$\mathbb{U} = \mathbb{U}(1) = \{z \in \mathbb{C} : |z| < 1\}$$

the open unit disk.
Let denote by $\mathcal{H}(\mathbb{U}(r))$ the set of all holomorphic functions in a domain $\mathbb{U}(r)$.

For $n \in \mathbb{N}^*$ and $a \in \mathbb{C}$ we consider the following classes

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + ...\}.$$

We remark that $\mathcal{A} = \mathcal{A}_1$.

Univalent functions plays an important role in the geometric function theory. A holomorphic function on an open subset of the complex plane is called univalent if it is injective. Let us denote by

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$$

the class of univalent functions.

In [91] is introduced the class $T \subset \mathcal{S}$, which contains functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0, z \in \mathbb{U}. \tag{2.1}$$

A function $f \in T$ is called a function with negative coefficients.
Let $f, g \in \mathcal{A}$ where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2.2}$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2.3}$$

The convolution or the Hadamard product of f and g is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

In [78] Ruscheweyh defined the derivative $D^{\gamma} : \mathcal{A} \to \mathcal{A}$ by

$$D^{\gamma} f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z),$$

where $\gamma > -1$. In the particular case $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$

$$D^m f(z) = \frac{z(z^{m-1} f(z))^{(m)}}{m!}. \tag{2.4}$$

It is easily seen that

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = z f'(z)$$

and

$$D^m f(z) = z + \sum_{n=2}^{\infty} \delta(m, n) a_n z^n,$$

where $\delta(m, n) = C^m_{m+n-1}$.

In [82] Sălăgean defined the $\mathcal{D}^m$ differential operator. For m positive integer the $\mathcal{D}^m : \mathcal{A} \to \mathcal{A}$ operator is given by

$$\mathcal{D}^0 f(z) = f(z),$$

$$\mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z)$$

and

$$\mathcal{D}^m f(z) = \mathcal{D}(\mathcal{D}^{m-1} f(z)).$$

It is easily seen that for $f \in \mathcal{A}$ and of the form (2.2)

$$\mathcal{D}^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n.$$

Let $m \in \mathbb{N}_0$. Denote by $\mathcal{D}D^m$ the operator given by the Hadamard product (convolution) of the Sălăgean operator $\mathcal{D}^m$ and the Ruscheweyh operator D^m, $\mathcal{D}D^m : \mathcal{A} \to \mathcal{A}$

$$\mathcal{D}D^m f(z) = \mathcal{D}^m \left(\frac{z}{1-z} \right) * D^m f(z), \ z \in \mathbb{U}.$$

If $f \in \mathcal{A}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then

$$\mathcal{D}D^m f(z) = z + \sum_{n=2}^{\infty} C_{m+n-1}^m n^m a_n z^n. \tag{2.5}$$

For a function $f \in \mathcal{A}$, $\lambda \geq 0$ and $m \in \mathbb{N} \cup \{0\}$, the Al-Oboudi differential operator $\mathfrak{D}_\lambda^m$ is defined by [3]

$$\mathfrak{D}^0 f(z) = f(z),$$
$$\mathfrak{D}_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = \mathfrak{D}_\lambda f(z),$$
$$\mathfrak{D}_\lambda^m f(z) = \mathfrak{D}_\lambda(\mathfrak{D}_\lambda^{m-1} f(z)), \ z \in \mathbb{U}.$$

If f has the form (2.2) then we have

$$\mathfrak{D}_\lambda^m f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^m a_n z^n.$$

For $\lambda = 1$ the $\mathfrak{D}_\lambda^m$ differential operator reduces to the Sălăgean differential operator.

For $f \in \mathcal{A}$ and $0 < q < 1$, the q-derivative of the function f is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \tag{2.6}$$

where $z \neq 0$ and $D_q f(0) = f'(0)$.

From (2.6) we can deduce that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q} a_n z^{n-1},$$

where $z \neq 0$.

For $f \in \mathcal{H}(\mathbb{U})$, $f(0) = 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the I_S^n Sălăgean integral operator is defined as follows [82]:

(i) $I_S^0 f(z) = f(z)$,

(ii) $I_S^1 f(z) = If(z) = \int\limits_0^z f(t)t^{-1}dt$,

(iii) $I_S^n f(z) = I_S(I_S^{n-1} f(z))$.

If f has the form (2.1), then

$$I_S^n f(z) = z - \sum_{j=2}^{\infty} \frac{a_j}{j^n} z^j, \tag{2.7}$$

where $n \in \mathbb{N}_0$.

In [64] Noor defined an integral operator $I_N^n : \mathcal{A} \to \mathcal{A}$ as follows

$$I_N^n f(z) = \frac{n+1}{z^n} \int\limits_0^z t^{n-1} I_N^n(f(t))dt, \tag{2.8}$$

where $n \in \mathbb{N}_0$.

We remark that if f has the form (2.1), then

$$I_N^n f(z) = z - \sum_{j=2}^{\infty} \frac{a_j}{C(n,j)} z^j, \tag{2.9}$$

where $C(n,j) = \dfrac{(n+j-1)!}{n!(j-1)!}$.

If $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, using the Noor and Sălăgean integral operators, we define a new operator as follows [34]:

$$I_{NS}^n f(z) = I_N^n f(z) * I_S^n f(z) = z - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)} z^j, \tag{2.10}$$

where $C(n,j) = \dfrac{(n+j-1)!}{n!(j-1)!}$ and $n \in \mathbb{N}_0$.

2.2 Subordination. The Carathéodory class

Definition 2.2.1. *[58][61] Let f and g be analytic functions in $\mathbb{U}$. We say that function f is subordinate to the function g, if there exist a function w, which is analytic in $\mathbb{U}$ and for which $w(0) = 0$, $|w(z)| < 1$ for $z \in \mathbb{U}$, such that $f(z) = g[w(z)]$, for all $z \in \mathbb{U}$.*

We denote by $\prec$ the subordination relation.

The subordination relation has the following properties [61]. For $f, g \in \mathcal{H}(\mathbb{U})$ we have:

1) If $f \prec g$, then $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

2) If $f \prec g$, then $f(\overline{\mathbb{U}(r)}) \subseteq g(\overline{\mathbb{U}(r)})$, $r < 1$. Equality holds if and only if $f(z) = g(\lambda z)$, $|\lambda| = 1$.

3) If $f \prec g$, then $\max\{|f(z)| : |z| \leq r\} \leq \{\max |g(z)| : |z| \leq r\}$, $r < 1$. Equality holds if and only if $f(z) = g(\lambda z)$, $|\lambda| = 1$.

4) If $f \prec g$, then $|f'(0)| \leq |g'(0)|$. Equality holds if and only if $f(z) = g(\lambda z)$, $|\lambda| = 1$.

5) If g is univalent, $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$, then $f \prec g$.

The Carathéodory class of functions:

$$\mathcal{P} = \left\{ p \in \mathcal{H}(\mathbb{U}) : p(z) \prec \frac{1+z}{1-z} \right\}.$$

Theorem 2.2.1. *(**Carathéodory**)[61] If $p(z) = 1 + p_1 z + p_2 z^2 + ... + p_n z^n + ...$ belongs to the class $\mathcal{P}$, then $|p_n| \leq 2$, where $n \geq 1$. Equality holds for the function $p(z) = \dfrac{1+\lambda z}{1-\lambda z}$, $|\lambda| = 1$.*

2.3 Special classes of univalent functions

2.3.1 Starlike functions

Definition 2.3.1.1. *[58, 61] Let $f \in \mathcal{H}(\mathbb{U})$ and $f(0) = 0$. We say that f is starlike in $\mathbb{U}$ with respect to the origin, if the function f is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a starlike domain with respect to origin.*

The class of starlike functions is denoted by S^*.
The analytic characterization of starlike functions is given in the following theorem.

Theorem 2.3.1.1. *[58, 61] Let $f \in \mathcal{A}$. The function f is starlike if and only if*

$$\Re \frac{zf'(z)}{f(z)} > 0, \ z \in \mathbb{U}.$$

Definition 2.3.1.2. *[58, 61] Let $0 \leq \gamma < 1$. We say that $f \in \mathcal{A}$ is starlike of order γ, if and only if*

$$\Re \frac{zf'(z)}{f(z)} > \gamma, \ z \in \mathbb{U}.$$

The class of starlike functions of order γ is denoted by $S^*(\gamma)$.

Theorem 2.3.1.2. *[87] For $f \in T$ the followings are equivalent:*

(i) $\sum\limits_{j=2}^{\infty} j a_j \leq 1$,

(ii) $f \in T \cap \mathcal{S}$,

(iii) $f \in T^*$, *where* $T^* = T \cap S^*$.

Lemma 2.3.1.1. *[38] Let $\gamma \in [0, 1)$, $T \in \mathbb{R}$ and let the function h_T be defined by the power series*

$$h_T(z) = z + \sum_{n=2}^{\infty} \frac{n - \gamma + iT}{1 - \gamma + iT} z^n.$$

The function $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ *is starlike of order γ in $\mathbb{U}$ if and only if*

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0, \; \textit{for all } z \in \mathbb{U} \textit{ and for all } T \in \mathbb{R}.$$

Proof. Since $\Re \frac{zf'(z)}{f(z)}\Big|_{z=0} = 1 > \gamma > 0$, it follows that the condition

$$\Re \frac{zf'(z)}{f(z)} > \gamma, \; z \in \mathbb{U}$$

is equivalent to

$$\frac{zf'(z)}{f(z)} - \gamma \neq -iT, \quad \text{for all } z \in \mathbb{U} \text{ and } T \in \mathbb{R}.$$

This can be rewritten as

$$1 + \sum_{n=2}^{\infty} a_n n z^{n-1} - (\gamma - iT) \left(1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right) \neq 0,$$

and finally we get

$$1 + \sum_{n=2}^{\infty} a_n \frac{n - \gamma + iT}{1 - \gamma + iT} z^{n-1} \neq 0, \quad \text{for all } z \in \mathbb{U} \text{ and } T \in \mathbb{R}.$$

This condition is equivalent to

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0, \; \text{for all } z \in \mathbb{U}, \text{ and for all } T \in \mathbb{R}.$$

□

2.3.2 Strongly starlike functions

Definition 2.3.2.1. *Let $0 \le \gamma < 1$. We say that $f \in \mathcal{A}$ is strongly starlike of order γ if and only if*

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \gamma\frac{\pi}{2},\ z \in \mathbb{U}.$$

The class of strongly starlike functions of order γ we denote by $SS^*(\gamma)$.

In [88] H. Silverman studied the class $\mathcal{G}_b$ of functions where

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} - 1 \right| < b, \quad z \in \mathbb{U} \right\},$$

for some positive b.

Theorem 2.3.2.1. *[66] If the function f belongs to the class $\mathcal{G}_{b(\beta)}$ with*

$$b(\beta) = \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}},$$

where $0 < \beta \le 1$, then $f \in SS^*(\beta)$.

To prove the next results, we need the following lemma.

Lemma 2.3.2.1. *[36] If $f \in \mathcal{A}$, $b \in [0, 1)$ and $p(z) = \dfrac{zf'(z)}{f(z)}$, then the inequality*

$$\left|\frac{zp'(z)}{p^2(z)}\right| < b, \quad z \in \mathbb{U}, \tag{2.11}$$

implies that

$$p(z) \prec \frac{1}{1 - bz}.$$

The result is sharp.

In the following theorem is given the sharp version of Theorem 2.3.2.1.

Theorem 2.3.2.2. *[36] If $\alpha \in (0, 1]$ and $f \in \mathcal{G}_{b(\alpha)}$, where $b(\alpha) = \sin\left(\alpha\dfrac{\pi}{2}\right)$, then $f \in SS^*(\alpha)$. The result is sharp.*

Putting $\alpha = 1$ in Theorem 2.3.2.2, we get the following starlikeness condition.

Corollary 2.3.2.1. *[36] If $f \in \mathcal{A}$ and*

$$\left| \frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} - 1 \right| < 1, \quad z \in \mathbb{U},$$

then $f \in S^*$.

For $\alpha = \dfrac{1}{2}$, we get the the following condition.

Corollary 2.3.2.2. *[36] If $f \in \mathcal{A}$ and*

$$\left| \frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} - 1 \right| < \frac{\sqrt{2}}{2}, \quad z \in \mathbb{U},$$

then $f \in SS^* \left(\dfrac{1}{2}\right)$.

2.3.3 Janowski starlike functions

The class of Janowski starlike functions is defined in [49] and is denoted by $S^*(A, B)$. Let $-1 \le B < A \le 1$. The class $S^*(A, B)$ is defined by the equality

$$S^*(A, B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \ z \in \mathbb{U} \right\}.$$

Theorem 2.3.3.1. *[93] Assume that $-1 \le B < A \le 1$ and $b(1 + |A|)^2 \le |A - B|$. If $f \in \mathcal{G}_b$, then $f \in S^*(A, B)$.*

Theorem 2.3.3.2. *[36] If $f \in \mathcal{G}_b$ and $b(1+A-B+|B|) < A-B$, then $f \in S^*(A, B)$.*

If $0 \le B < A \le 1$, then we get the following corollary.

Corollary 2.3.3.1. *[36] Let $0 \le B < A \le 1$ and $b \in (0, +\infty)$ such that $b(1 + A) \le 1 + B$. If $f \in \mathcal{G}_b$, then $f \in S^*(A, B)$.*

2.3.4 The S^{**} class

Definition 2.3.4.1. *[32] Let $f \in \mathcal{A}$. We say that $f \in S^{**}$ if and only if*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}}, \ z \in \mathbb{U}.$$

The class S^{**} is not empty. It is easily seen that if $f(z) = z - \dfrac{z^2}{100}$, then

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| = \left| \frac{100 - 4z}{100 - 2z} \right| \le \frac{104}{98} < \sqrt{\frac{5}{4}}, \ z \in \mathbb{U}$$

and consequently $f \in S^{**}$.

Remark 2.3.4.1. *[32] The class S^{**} has the property that, the composition of each two functions from S^{**} is starlike on a disk where the composition is defined.*

2.3.5 The S^{***} class

Definition 2.3.5.1. *[31] Let $f \in \mathcal{A}$. We say that $f \in S^{***}$ if and only if*

$$\left|1-\frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z \in \mathbb{U}.$$

Remark 2.3.5.1. *[31] The class S^{***} has the property that, the composition of each two functions from S^{***} is starlike on a disk where the composition is defined.*

In the followings we define the class S^{***} for functions with negative coefficients.

Definition 2.3.5.2. *[40] The function $f \in T$ belongs to the class $TS^{***} = S^{***} \cap T$ if*

$$\left|1-\frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z \in \mathbb{U}.$$

Below is given a coefficient delimitation theorem for the class TS^{***}.

Theorem 2.3.5.1. *[40] The function $f \in T$ belongs to the class TS^{***} if and only if*

$$\sum_{j=2}^{\infty} j\left(j-2+\frac{\sqrt{5}}{2}\right)a_j < \frac{\sqrt{5}}{2}-1. \tag{2.12}$$

2.3.6 Parabolic starlike functions

In [76] Rønning defined the class of parabolic starlike functions by the following way:

$$S_p = \{F \in S^* | F(z) = zf'(z), f \in \mathcal{UCV}\}.$$

Definition 2.3.6.1. *[9] The class S_p of parabolic starlike functions consists of functions $f \in \mathcal{A}$ satisfying*

$$\Re\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)}-1\right|,\ z \in \mathbb{U}.$$

For $-1 < \gamma \le 1$ and $k \ge 0$ a function $f \in \mathcal{A}$ is said to be in the class of k-parabolic starlike functions of order γ, denoted by $k - S_p(\gamma)$ if

$$\Re\left(\frac{zf'(z)}{f(z)}-\gamma\right) > k\left|\frac{zf'(z)}{f(z)}-1\right|,\ z \in \mathbb{U}.$$

In [62] the authors generalized the class of k-parabolic starlike functions, of order γ, for $0 \le \gamma < 1$.

For $0 \le \lambda < 1$, $0 \le \gamma < 1$ and $k \ge 0$ the function $f \in \mathcal{A}$ belongs to the class $k - S_p(\lambda, \gamma)$ if

$$\Re\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-\gamma\right\} > k\left|\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right|,\ z \in \mathbb{U}. \tag{2.13}$$

It is easily seen that, $k - S_p(0,\gamma) = k - S_p(\gamma)$, $k - S_p(0,0) = k - S_p$, where $0 \le \gamma < 1$.

2.3.7 Convex functions

Definition 2.3.7.1. *[58, 61] A function $f \in \mathcal{H}(\mathbb{U})$ is convex in $\mathbb{U}$, if the function f is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a convex domain.*

The class of convex functions is denoted by $\mathcal{K}$.
The analytic characterization of convex functions is given in the below theorem.

Theorem 2.3.7.1. *[58, 61] Let $f \in \mathcal{A}$. Then the function f is convex if and only if*

$$\Re\left(\frac{zf''(z)}{f'(z)}+1\right) > 0, \ \ z \in \mathbb{U}.$$

Definition 2.3.7.2. *[58, 61] Let $0 \leq \gamma < 1$. We say that $f \in \mathcal{A}$ is convex of order γ if and only if*

$$\Re\left(\frac{zf''(z)}{f'(z)}+1\right) > \gamma, \ \ z \in \mathbb{U}.$$

The class of convex functions of order γ is denoted by $\mathcal{K}(\gamma)$.

2.3.8 Alpha convex functions

The class of alpha convex functions was introduced by P. T. Mocanu in 1969, to create a relation between starlikeness and convexity.

Definition 2.3.8.1. *[58, 61] Let $f \in \mathcal{A}$ and α a real number. Then the function f is α convex if and only if*

$$\Re J(\alpha, f; z) > 0,$$

where

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)}+1\right).$$

We denote by M_α the class of alpha convex functions.

2.3.9 Close-to-convex functions

Theorem 2.3.9.1. *[58, 61] The function $f \in \mathcal{A}$ is close-to-convex in $\mathbb{U}$, if there exist a starlike function $g \in S^*$ for which*

$$\Re\frac{zf'(z)}{g(z)} > 0, \ \ z \in \mathbb{U}.$$

The class of close-to-convex functions is denoted by $\mathcal{C}$.

2.3.10 Uniformly convex functions

In [42] Goodman defined the class of uniformly convex functions, denoted by $\mathcal{UCV}$ as follows:

Definition 2.3.10.1. *[42] A function $f \in \mathcal{A}$ is said to be uniformly convex in $\mathbb{U}$ if $f \in \mathcal{K}$ and has the property that for every circular arc γ contained in $\mathbb{U}$, with center ζ, also in $\mathbb{U}$, the arc $f(\gamma)$ is convex.*

Due to the analytic criterion for $f \in \mathcal{UCV}$, given by Rønning [76]:
A function $f \in \mathcal{A}$ is uniformly convex in $\mathbb{U}$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \; z \in \mathbb{U}. \tag{2.14}$$

The class of k-uniformly convex functions was introduced by Kanas and Wisniowska [51], as a generalization of the uniform convexity. The class of k-uniformly convex functions are denoted by $k-\mathcal{UCV}$.
For $-1 < \gamma \leq 1$ and $k \geq 0$, a function $f \in \mathcal{A}$ is said to be in the class of k - uniformly convex functions of order γ if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \; z \in \mathbb{U}.$$

In [62] the authors generalized the class of k-uniformly convex functions, of order γ, for $0 \leq \gamma < 1$.
For $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ and $k \geq 0$, the function $f \in \mathcal{A}$ belongs to the class $k-\mathcal{UCV}(\lambda,\gamma)$ if

$$\Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}-\gamma\right\} > k\left|\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf'(z)}-1\right|, \; z \in \mathbb{U}. \tag{2.15}$$

It is easily seen that, $k-\mathcal{UCV}(0,\gamma) = k-\mathcal{UCV}(\gamma)$ and $k-\mathcal{UCV}(0,0) = k-\mathcal{UCV}$, where $0 \leq \gamma < 1$.

Chapter 3

New results on analytic functions with varying arguments

3.1 Preliminary results

The following preliminary definitions and theorems are required for proving the main results.

In [12] Attiya and Aouf, using the Ruscheweyh operator (2.4), had defined the class $Q(m, \lambda, A, B)$ by this way:

Definition 3.1.1. *[12] For $\lambda > 1; -1 \leq A < B \leq 1; 0 < B \leq 1; m \in \mathbb{N}_0$ let $Q(m, \lambda, A, B)$ denote the subclass of $\mathcal{A}$ which contain functions $f(z)$ of the form (2.2) such that*

$$(1-\lambda)(D^m f(z))' + \lambda(D^{m+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{3.1}$$

Definition 3.1.2. *[72] For $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; m \in \mathbb{N}_0$ let $P(m, \lambda, A, B)$ denote the subclass of $\mathcal{A}$ which contain functions $f(z)$ of the form* (2.2) *such that*

$$(1-\lambda)(\mathcal{D}D^m f(z))' + \lambda(\mathcal{D}D^{m+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{3.2}$$

In 1981 H. Silverman has introduced the class of analytic functions with varying arguments.

Definition 3.1.3. *[89] A function f of the form (2.2) is said to be in the class $V(\theta_n)$ if $f \in \mathcal{A}$ and $arg(a_n) = \theta_n$, for all $n \geq 2$. If further more there exist a real number δ such that $\theta_n + (n-1)\delta \equiv \pi(mod\ 2\pi)$ for all $n \geq 2$, then f is said to be in the class $V(\theta_n, \delta)$. The union of $V(\theta_n, \delta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers δ is denoted by V.*

Let $VQ(m, \lambda, A, B)$ denote the subclass of V consisting of functions $f \in Q(m, \lambda, A, B)$.

Let $VP(m, \lambda, A, B)$ denote the subclass of V consisting of functions $f \in P(m, \lambda, A, B)$.

Theorem 3.1.1. *[29]Let the function f defined by (2.2) be in V. Then $f \in VQ(m, \lambda, A, B)$, if and only if*

$$T(f) = \sum_{n=2}^{\infty} n\delta(m,n)C_n (1+B) |a_n| \leq (B-A)(m+1) \tag{3.3}$$

where

$$\delta(m,n) = \binom{m+n-1}{m} \text{ and } C_n = m + 1 + \lambda(n-1).$$

The extremal functions are

$$f_n(z) = z + \frac{(B-A)(m+1)}{nC_n\delta(m,n)(1+B)} e^{i\theta_n} z^n, (n \geq 2).$$

Let $I(z) = L_c f(z) = \frac{c+1}{z^c} \int_0^z f(t)t^{c-1}dt, c > -1$ be the generalized Bernardi-Libera-Livingston integral operator.

Now we are able to enounce our main results.

3.2 Analytic functions with varying arguments defined by the Ruscheweyh derivative

Let $f, g \in \mathcal{A}$ be two analytic functions of the forms (2.2) and (2.3). In this section there are studied the properties of the image of a class of analytic functions with varying arguments defined by the Ruscheweyh derivative (2.4) through the generalized Bernardi-Libera-Livingston operator.

Theorem 3.2.1. *[70] If $f \in VQ(m, \lambda, 2\alpha - 1, B)$, then $L_c f \in VQ(m, \lambda, 2\beta - 1, B)$, where*

$$\beta = \beta(\alpha) = \frac{B + 1 + 2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

The result is sharp.

Theorem 3.2.2. *[70] If $f \in VQ(m, \lambda, A, B)$, then $L_c f \in VQ(m, \lambda, A^*, B)$, where*

$$A^* = \frac{B + A(c+1)}{c+2} > A.$$

The result is sharp.

Theorem 3.2.3. *[70] If $f \in VQ(m, \lambda, A, B)$, then $L_c f \in VQ(m, \lambda, A, B^*)$, where*

$$B^* = \frac{A(1+B)(c+2) + (B-A)(c+1)}{(1+B)(c+2) - (B-A)(c+1)} < B.$$

The result is sharp.

3.3 Analytic functions with varying arguments defined by the convolution of the Sălăgean and Ruscheweyh derivative

In this section there are given some results for certain new class of analytic functions with varying arguments, defined by the convolution of the Sălăgean and Ruscheweyh derivative (2.5) and there are studied the properties of the image of this classes through the generalized Bernardi-Libera-Livingston integral operator.

3.3.1 Coefficient estimates

Theorem 3.3.1.1. *[72] Let the function f defined by* (2.2) *be in V. Then $f \in VP(m, \lambda, A, B)$, if and only if*

$$T(f) = \sum_{n=2}^{\infty} n^{m+1} C_n (1 + B) |a_n| \leq B - A, \tag{3.4}$$

where

$$C_n = [m + 1 + \lambda (n - 1) (m + n + 1)] \frac{(m + n - 1)!}{(m + 1)! (n - 1)!}.$$

The extremal functions are:

$$f(z) = z + \frac{B - A}{n^{m+1} C_n (1 + B)} e^{i\theta_n} z^n, (n \geq 2).$$

Corollary 3.3.1.1. *[72] Let the function f defined by* (2.2) *be in the class $VP(m, \lambda, A, B)$. Then*

$$|a_n| \leq \frac{B - A}{n^{m+1} C_n (1 + B)}, (n \geq 2).$$

The condition (3.4) *is sharp for the functions*

$$f(z) = z + \frac{B - A}{n^{m+1} C_n (1 + B)} e^{i\theta_n} z^n, (n \geq 2).$$

3.3.2 Distortion theorems

Theorem 3.3.2.1. *[72] Let the function f defined by* (2.2) *be in the class $VP(m, \lambda, A, B)$. Then*

$$|z| - \frac{B - A}{2^{m+1} C_2 (1 + B)} |z|^2 \leq |f(z)| \leq |z| + \frac{B - A}{2^{m+1} C_2 (1 + B)} |z|^2 . \tag{3.5}$$

The result is sharp.

Corollary 3.3.2.1. *[72]Let the function f defined by* (2.2) *be in the class* $VP(m,\lambda,A,B)$. *Then* $f\in\mathbb{U}(0,r_1)$, *where* $r_1=1+\dfrac{B-A}{2^{m+1}C_2\,(1+B)}$.

Theorem 3.3.2.2. *[72] Let the function f defined by* (2.2) *be in the class* $VP(m,\lambda,A,B)$. *Then*

$$1-\frac{B-A}{2^{m}C_2\,(1+B)}\,|z|\le|f'(z)|\le1+\frac{B-A}{2^{m}C_2\,(1+B)}\,|z|\,. \tag{3.6}$$

The result is sharp.

Corollary 3.3.2.2. *[72] Let the function f defined by* (2.2) *be in the class* $VP(m,\lambda,A,B)$. *Then* $f'\in\mathbb{U}(0,r_2)$, *where* $r_2=1+\dfrac{B-A}{2^{m}C_2\,(1+B)}$.

3.3.3 Extreme points

Theorem 3.3.3.1. *[72] Let the function f defined by* (2.2) *be in the class* $VP(m,\lambda,A,B)$, *with* $arg(a_n)=\theta_n$ *where* $\theta_n\equiv\pi,\forall n\ge2$. *Define*

$$f_1(z)=z$$

and

$$f_n(z)=z-\frac{B-A}{n^{m+1}C_n\,(1+B)}z^n,(n\ge2;z\in\mathbb{U}).$$

Then $f\in VP(m,\lambda,A,B)$ *if and only if f can expressed by*

$$f(z)=\sum_{n=1}^{\infty}\mu_nf_n(z),\ \text{where}\ \mu_n\ge0\ \text{and}\ \sum_{n=1}^{\infty}\mu_n=1.$$

If we combine Theorem 3.3.3.1 with Silverman's theorem from [89], we get the following corollary:

Corollary 3.3.3.1. *The closed convex hull of* $VP(m,\lambda,A,B)$ *is*

$$cl\ co\ VP(m,\lambda,A,B)=\left\{f|f\in\mathcal{A},\sum_{n=2}^{\infty}n^{m+1}C_n(1+B)|a_n|\le B-A\right\}.$$

The extreme points of $cl\ co\ VP(m,\lambda,A,B)$ *are*

$$E(cl\ co\ VP(m,\lambda,A,B))=\left\{z+\frac{B-A}{n^{m+1}C_n(1+B)}\xi z^n,|\xi|=1,n\ge2\right\}.$$

Theorem 3.3.3.2. *[72] Let*

$$I(z) = L_c f(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt, \;\; c > -1.$$

If $f \in VP(m, \lambda, 2\alpha - 1, B)$, *then* $I \in VP(m, \lambda, 2\beta - 1, B)$, *where*

$$\beta = \beta(\alpha) = \frac{B + 1 + 2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

The result is sharp.

Theorem 3.3.3.3. *[72] If* $f \in VP(m, \lambda, A, B)$, *then* $I \in VP(m, \lambda, A^*, B)$, *where*

$$A^* = \frac{B + A(c+1)}{c+2} > A.$$

The result is sharp.

Theorem 3.3.3.4. *[72] If* $f \in VP(m, \lambda, A, B)$, *then* $I \in VP(m, \lambda, A, B^*)$, *where*

$$B^* = \frac{A(1+B)(c+2) + (B-A)(c+1)}{(1+B)(c+2) - (B-A)(c+1)} < B.$$

The result is sharp.

Chapter 4

The differential subordination method

4.1 Basic definitions

Let $\Omega, \Delta \subset \mathbb{C}$, $p \in \mathcal{H}(\mathbb{U})$ with $p(0) = a$, $a \in \mathbb{C}$ and let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. With the differential subordination method there are studied problems of the form:

$$\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{U}\} \subset \Omega \Rightarrow p(\mathbb{U}) \subset \Delta. \tag{4.1}$$

Definition 4.1.1. *[59] Let $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$, $p \in \mathcal{H}[a, n]$ and h univalent in $\mathbb{U}$. The differential subordination of the form*

$$\psi(p(z), zp'(z)) \prec h(z)$$

is called a first-order differential subordination.

Definition 4.1.2. *[61] Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$, $p \in \mathcal{H}[a, n]$ and h univalent in $\mathbb{U}$. The differential subordination of the form*

$$\psi(p(z), zp'(z), z^2p''(z)) \prec h(z) \tag{4.2}$$

is called a second-order differential subordination and p is called an (a, n) - solution of the differential subordination.

A particular differential subordination is the Briot-Bouquet differential subordination.

Definition 4.1.3. *[58, 61] Let h be a univalent function in $\mathbb{U}$, with $h(0) = a$, and let $p \in \mathcal{H}[a, n]$ satisfy*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \tag{4.3}$$

where $\beta, \gamma \in \mathbb{C}$ and $\beta \neq 0$. This first-order differential subordination is called the Briot-Bouquet differential subordination.

4.2 Basic lemmas

Definition 4.2.1. *[61] Let $\mathcal{Q}$ be the class of analytic functions q in $\mathbb{U}$ which has the property that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$, where*

$$E(q) = \{\zeta \in \partial\mathbb{U} : \lim_{z \longrightarrow \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$.

Lemma 4.2.1. ***(S.S. Miller, P. T. Mocanu)****[60, 61] Let $q \in \mathcal{Q}$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \ldots$ be analytic in $\mathbb{U}$ with $p(z) \not\equiv a$ and $n \geq 1$. If $p \not\prec q$, then there are two points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(q)$, and a number $m \geq n \geq 1$, for which $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$ and*
(i) $\quad p(z_0) = q(\zeta_0)$
(ii) $\quad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$
(iii) $\quad \Re \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \Re \left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$

The following two lemmas are particular cases of Lemma 4.2.1. In the first case $q(\mathbb{U})$ is a disk and in the second case $q(\mathbb{U})$ is a half-plane.

Lemma 4.2.2. *[58, 61] Let $p \in \mathcal{H}[a, n]$, with $p(z) \not\equiv a$ and $n \geq 1$. If $z_0 \in \mathbb{U}$ and*

$$|p(z_0)| = \max\{|p(z)| : |z| \leq |z_0|\},$$

then
(i) $\quad \dfrac{z_0 p'(z_0)}{p(z_0)} \geq n \dfrac{|p(z_0) - a|^2}{|p(z_0)|^2 - |a|^2}$ *and*
(ii) $\quad \Re \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq n \dfrac{|p(z_0) - a|^2}{|p(z_0)|^2 - |a|^2}.$

Lemma 4.2.3. *[58, 61] Let $p \in \mathcal{H}[a, n]$, with $p(z) \not\equiv a$ and $n \geq 1$. If $z_0 \in \mathbb{U}$, and*

$$\Re\, p(z_0) = \min\{\Re\, p(z) : |z| \leq |z_0|\},$$

then
(i) $\quad z_0 p'(z_0) \leq -\dfrac{n}{2} \dfrac{|p(z_0) - a|^2}{\Re[a - p(z_0)]}$ *and*
(ii) $\quad \Re[z_0^2 p''(z_0)] + z_0 p'(z_0) \leq 0.$

Lemma 4.2.4. *[58](Hallenbeck and Ruscheweyh) Let h be a convex function with $h(0) \equiv a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $\Re\, \gamma \geq 0$. If $p \in \mathcal{H}[\mathbb{U}]$ with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z)$$

then

$$p(z) \prec g(z) \prec h(z)$$

where

$$g(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function g is convex and is the best dominant.

Lemma 4.2.5. *[59](Miller and Mocanu) Let g be a convex function in $\mathbb{U}$ and let*

$$h(z) = g(z) + n\alpha z g'(z),$$

where $\alpha > 0$ and n is a positive integer. If $p(z) = g(0) + p_n z^n + ...$ is holomorphic in $\mathbb{U}$ and

$$p(z) + \alpha z p'(z) \prec h(z),$$

then

$$p(z) \prec g(z)$$

and the result is sharp.

4.3 Applications

The differential subordination method plays an important role in the study of functions which are differentiable in complex sense. Using this method the geometric properties of holomorphic functions can be established in a more simply way.

Theorem 4.3.1. *[32] We have $S^{**} \subset S^*$.*

Theorem 4.3.2. *[32] If $f \in S^{**}$, then $\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}$, $z \in \mathbb{U}$.*

Theorem 4.3.3. *[32] If $f \in S^{**}$, then $|\arg f'(z)| < \frac{\pi}{4}$, $z \in \mathbb{U}$.*

Using Theorem 4.3.2 and Theorem 4.3.3, now we are able to prove the following result for composition of functions.

Theorem 4.3.4. *[32] If $f, g \in S^{**}$ and $r_0 = \sup\{r \in (0,1] \big| g\big(\mathbb{U}(r)\big) \subset \mathbb{U}\}$, then $f \circ g$ is starlike in $\mathbb{U}(r_0)$.*

Theorem 4.3.5. *[31] If $f \in \mathcal{A}$ and*

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{7}, \ z \in \mathbb{U},$$

then it follows that $f \in S^*$.

Remark 4.3.1. *[31] The result of Theorem 4.3.5 shows that the inclusion* $S^{***} \subset S^{*}$ *holds.*

Using the differential subordination method we can easily proof that the class S^{***} is a subclass of convex functions.

Theorem 4.3.6. *[31] We have* $S^{***} \subset \mathcal{K}$.

Theorem 4.3.7. *[31] If* $f \in S^{***}$, *then* $\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4},\ z \in \mathbb{U}.$

Theorem 4.3.8. *[31] If* $f \in S^{***}$, *then* $|\arg f'(z)| < \frac{\pi}{4},\ z \in \mathbb{U}.$

Next is proved the result regarding the composition of functions.

Theorem 4.3.9. *[31] If* $f, g \in S^{***}$ *and* $r_0 = \sup\{r \in (0,1] \big| f(\mathbb{U}(r)) \subset \mathbb{U}\}$, *then* $f \circ g$ *will be starlike in* $\mathbb{U}(r_0)$.

Chapter 5

New classes of analytic functions defined by some differential and integral operators

5.1 The class $UCC_q(\gamma)$

In this section are given some generalizations of the class of close-to-convex functions, in the case of functions with negative coefficients, using the q-difference operator, defined by (2.6).

Definition 5.1.1. *[33] A function $f \in T$ is said to be in the generalized class of close-to-convex functions of order γ, denoted by $UCC_q(\gamma)$, if*

$$\Re \frac{zD_q f(z)}{g(z)} \geq \gamma,$$

where $0 \leq \gamma < 1$ and $g \in T^*$.

Remark 5.1.1. *[33] If $\gamma = 0$, then $UCC_q(0) = UCC_q$.*

Definition 5.1.2. *[33] A function $f \in T$ is said to be in the generalized class of close-to-convex functions of order γ, relative to a fixed function $g \in T^*$, denoted by $UCC_q(g, \gamma)$, if*

$$\Re \frac{zD_q f(z)}{g(z)} \geq \gamma,$$

where $0 \leq \gamma < 1$.

Theorem 5.1.1. *[33] Let $f(z) = z - \sum_{j=2}^{\infty} a_j z^j, g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $g \in T^*$, where $a_j, b_j \geq 0$, $j \in \{2, 3, ..\}$ and $\gamma \in [0, 1)$.*

If $f \in UCC_q(g,\gamma)$, then

$$\sum_{j=2}^{\infty}\left(\frac{1-q^j}{1-q}a_j - \gamma b_j\right) < 1-\gamma. \tag{5.1}$$

If

$$\sum_{j=2}^{\infty}\left[\frac{1-q^j}{1-q}a_j + (2-\gamma)b_j\right] < 1-\gamma, \tag{5.2}$$

then $f \in UCC_q(g,\gamma)$.
In the particular case when

$$\frac{1-q^j}{1-q}a_j \geq b_j,\ \ j \in \{2,3,..\},$$

then (5.1) *is necessary and sufficient condition for f to belong $UCC_q(g,\gamma)$.*

Theorem 5.1.2. *[33] Let $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \in \{2,3,...\}$ and $0 \leq \gamma < 1$. If $f \in UCC_q(\gamma)$, then there exist $g \in T^*$, $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, such that*

$$\sum_{j=2}^{\infty}\left(\frac{1-q^j}{1-q}a_j - \gamma b_j\right) < 1-\gamma. \tag{5.3}$$

If

$$\sum_{j=2}^{\infty}\frac{1-q^j}{1-q}a_j < 1-\gamma, \tag{5.4}$$

then $f \in UCC_q(\gamma)$.
In the particular case when

$$\frac{1-q^j}{1-q}a_j \geq b_j,\ \ j \in \{2,3,...\}$$

the inequality (5.3) *implies that $f \in UCC_q(\gamma)$.*

Remark 5.1.2. *[33] When $f_2(z) = z - \dfrac{z^2}{1+q} \in UCC_q(g_2,\gamma)$, where $g_2(z) = z - \dfrac{z^2}{2} \in T^*$ we have*

$$\Re\frac{zD_q f_2(z)}{g_2(z)} = \Re\frac{z(1-z)}{z\left(1-\frac{z}{2}\right)} = 2\Re\frac{1-z}{2-z} > 0.$$

But

$$\sum_{j=2}^{\infty}\frac{1-q^j}{1-q}a_j + (2-\gamma)b_j = 1 + \frac{2-\gamma}{2} = 2 - \frac{\gamma}{2} \not< 1.$$

This show that (5.2) is only a sufficient condition.

In [79] the authors proved that the convolution of a starlike and a convex function is starlike, we can give the following theorem.

Theorem 5.1.3. *[33] Let* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$ *and let* $\phi(z) = z - \sum_{j=2}^{\infty} c_j z^j \in T^*$, *where* $a_j, b_j, c_j \geq 0$, $j \in \{2, 3, ..\}$. *If* $f \in UCC_q(g, \gamma)$, *where* $\frac{1-q^j}{1-q} a_j \geq b_j$ *for* $j \in \{2, 3, ..\}$, *then* $f * \phi \in UCC_q(g, \gamma)$.

5.2 The class $C_{NS}(\alpha)$

In this section, using the Noor-Sălăgean integral operator (2.10) is defined the $C_{NS}(\alpha)$ class and is studying some properties of this class.

Definition 5.2.1. *[34] A function* $f \in T$ *belongs to the class* $C_{NS}(\alpha)$, *if*

$$\Re \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} > \alpha, \tag{5.5}$$

where $\alpha \in [0, 1)$ *and* $z \in \mathbb{U}$.

For $\alpha = 0$ we obtain the following definition.

Definition 5.2.2. *A function* $f \in T$ *belongs to the class* C_{NS}, *if*

$$\Re \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} > 0, \tag{5.6}$$

where $z \in \mathbb{U}$.

Theorem 5.2.1. *[34] Let* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$. *Then* $f \in C_{NS}(\alpha)$, *if and only if*

$$\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} \left[1 - \frac{\alpha}{j}\right] < 1 - \alpha. \tag{5.7}$$

If $\alpha = 0$ then we obtain the following result.

Corollary 5.2.1. *Let* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$. *Then* $f \in C_{NS}$, *if and only if*

$$\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} < 1. \tag{5.8}$$

Let $E_{NS}(\alpha)$ be a subclass of $C_{NS}(\alpha)$. The class is defined as follows [34]:

$$E_{NS}(\alpha) = \left\{ f \in T : \left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right| < 1 - 2\alpha \text{ and } \alpha \in \left(0, \frac{1}{2}\right) \right\}. \tag{5.9}$$

Theorem 5.2.2. *[34] Let $f \in T$. If $f \in E_{NS}(\alpha)$, then $\Re \dfrac{I_{NS}^n f(z)}{z} > 0$.*

5.3 The class $Q_1(m, \lambda, A, B)$

In [12] Attiya and Aouf defined the class $Q(m, \lambda, A, B)$ as follows:

$$(1-\lambda)(D^m f(z))' + \lambda (D^{m+1} f(z))' \prec \frac{1+Az}{1+Bz}, \tag{5.10}$$

where $\lambda \geq 0, -1 \leq A < B \leq 1$, $m \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

In this section we define a similar class to $Q(m, \lambda, A, B)$, by the convex combination of the Sălăgean integral operator $I^m f(z)$, for $\lambda \in [0, 1]$ and we study some properties of this class, in the case when $A = -1$ and $B = 1$.

Definition 5.3.1. *[37] A function $f \in \mathcal{A}$ of the form (2.2) belongs to the class $Q_1(m, \lambda, A, B)$ if*

$$(1-\lambda)[I^m f(z)]' + \lambda [I^{m+1} f(z)]' \prec \frac{1+Az}{1+Bz}, \tag{5.11}$$

where $\lambda \geq 0, -1 \leq A < B \leq 1$, $m \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

A particular case of the Definition 5.3.1 is the following:

Definition 5.3.2. *[37] A function $f \in \mathcal{A}$ of the form (2.2) belongs to the class $Q_1(m, \lambda, -1, 1)$ if*

$$\Re(1-\lambda)[I^m f(z)]' + \lambda [I^{m+1} f(z)]' > 0, \tag{5.12}$$

where $\lambda \in [0, 1]$, $m \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Remark 5.3.1. *[37] If we note $p(z) = [I^{m+1} f(z)]'$, then (5.12) is equivalent to*

$$\Re[p(z) + (1-\lambda) z p'(z)] > 0, \tag{5.13}$$

where $0 \leq \lambda \leq 1$ and $z \in \mathbb{U}$.

Theorem 5.3.1. *[37] Let $f \in \mathcal{A}$ be with real coefficients and of the form (2.2). If $f \in Q_1(m, \lambda, -1, 1)$ then*

$$\sum_{j=2}^{\infty} [j(1-\lambda) + \lambda] j^{-m} a_j > -1, \tag{5.14}$$

where $m \in \mathbb{N}_0$ and $\lambda \in [0, 1]$.

In [20] C. M. Bălăeţi defined the class $\mathcal{I}^m(\alpha)$ for analytic functions which satisfy the inequality:

$$\Re[I^m f(z)]' > \alpha,$$

where $\alpha \in [0,1)$.

Remark 5.3.2. *[37] If we put $\lambda = 0$ in Definition 5.3.2 we obtain the class $\mathcal{I}^m(0)$, defined in [20].*

Putting $\lambda = 0$ in Theorem 5.3.1 we obtain the following corollary.

Corollary 5.3.1. *[37] Let $f \in \mathcal{A}$ be with real coefficients and of the form (2.2). If $f \in \mathcal{I}^m(0)$ then*

$$\sum_{j=2}^{\infty} j^{1-m} a_j > -1, \tag{5.15}$$

where $m \in \mathbb{N}_0$.

In the following theorem we proved an inclusion result for the classes $\mathcal{I}^{m+1}(0)$ and $Q_1(m, \lambda, -1, 1)$.

Theorem 5.3.2. *[37] Let $f \in \mathcal{A}$ be of the form (2.2). If $f \in Q_1(m, \lambda, -1, 1)$, then $f \in \mathcal{I}^{m+1}(0)$.*

Theorem 5.3.3. *[37] Let $f \in \mathcal{A}$ be of the form (2.2). If $f \in Q_1(m, \lambda, -1, 1)$, then $\Re\dfrac{I^{m+1}f(z)}{z} > 0$.*

In (5.13) if we substitute $p(z)$ by any other analytic function, we obtain the following results.

Theorem 5.3.4. *[37] Let u be a convex function, such that $u(0) = 1$ and*

$$h(z) = u(z) + (1-\lambda)zu'(z),\ \ z \in \mathbb{U}.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$[I^m f(z)]' \prec h(z), \tag{5.16}$$

then

$$[I^{m+1} f(z)]' \prec u(z),$$

and the result is sharp.

Theorem 5.3.5. *[37] Let u be a function, such that $u(0) = 1$ and*

$$h(z) = u(z) + (1-\lambda)zu'(z),\ z \in \mathbb{U}$$

is convex. If $f \in \mathcal{A}$ verifies the differential subordination

$$[I^m f(z)]' \prec h(z), \tag{5.17}$$

then

$$\frac{I^m f(z)}{z} \prec u(z),$$

where

$$u(z) = \frac{1-\lambda}{z^{1-\lambda}} \int_0^z h(t)t^{-\lambda}dt,\ z \in \mathbb{U}$$

and the result is sharp.

Chapter 6

Some new properties of the generalized Mittag-Leffler function

6.1 The generalized Mittag-Leffler function

The function of the form

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

where $\Re(\alpha) > 0$ and $z \in \mathbb{C}$, was introduced by Mittag-Leffler in 1903 and is called the Mittag-Leffler function.

The generalized Mittag-Leffer function has the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{6.1}$$

where $z, \alpha, \beta \in \mathbb{C}$ and $\Re(\alpha) > 0$ was studied by Wiman [103].
Because the generalized Mittag-Leffler function $E_{\alpha,\beta}$ does not belong to the family $\mathcal{A}$, it is natural to consider the following normalization:

$$\mathcal{E}_{\alpha,\beta}(z) = \frac{zE_{\alpha,\beta}(z)}{E_{\alpha,\beta}(0)} = z + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}z^2 + \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}z^3 + ...,$$

which is equivalent to

$$\mathcal{E}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}z^n.$$

6.2 Sufficient includeness conditions for the generalized Mittag-Leffler function in several special analytic classes

In this section we find sufficient conditions so that, the generalized Mittag-Leffler function $\mathcal{E}_{\alpha,\beta}$ to be in the classes S^*, $\mathcal{K}$, S_p, $\mathcal{UCV}$, $k-S_p(\gamma)$, $k-\mathcal{UCV}(\gamma)$, respectively in $k-S_p(\lambda,\gamma)$ and $k-\mathcal{UCV}(\lambda,\gamma)$.

Theorem 6.2.1. *[39] Let $\alpha,\beta>0$, $k\geq 0$ and $0<\gamma\leq 1$. If*

$$\sum_{n=2}^{\infty}\frac{(n-1)(k+1)+1-\gamma}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1-\gamma}{\Gamma(\beta)}, \tag{6.2}$$

then $\mathcal{E}_{\alpha,\beta}\in k-S_p(\gamma)$.

Theorem 6.2.2. *[39] Let $\alpha,\beta>0$, $k\geq 0$ and $0<\gamma\leq 1$. If*

$$\sum_{n=2}^{\infty}n\frac{(n-1)(k+1)+1-\gamma}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1-\gamma}{\Gamma(\beta)}, \tag{6.3}$$

then $\mathcal{E}_{\alpha,\beta}\in k-\mathcal{UCV}(\gamma)$.

For $k=1$ and $\gamma=0$ we obtain the following characterization properties for the classes S_p and $\mathcal{UCV}$.

Corollary 6.2.1. *[39] Let $\alpha,\beta>0$. If*

$$\sum_{n=2}^{\infty}\frac{2n-1}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1}{\Gamma(\beta)},$$

then $\mathcal{E}_{\alpha,\beta}\in S_p$.

Corollary 6.2.2. *[39] Let $\alpha,\beta>0$. If*

$$\sum_{n=2}^{\infty}\frac{n(2n-1)}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1}{\Gamma(\beta)}, \tag{6.4}$$

then $\mathcal{E}_{\alpha,\beta}\in\mathcal{UCV}$.

Theorem 6.2.3. *[39] Let $\alpha,\beta>0$, $k\geq 0$ and $0<\gamma\leq 1$. If*

$$\sum_{n=2}^{\infty}\frac{n-1}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1-\gamma}{\Gamma(\beta)[1-\lambda\gamma+k(1-\lambda)]}, \tag{6.5}$$

where $0\leq\lambda<1$ then $\mathcal{E}_{\alpha,\beta}\in k-S_p(\lambda,\gamma)$.

Putting $k = \lambda = \gamma = 0$ in the above theorem we obtain the analytic criteria for the class S^*.

Corollary 6.2.3. *[39] Let $\alpha, \beta > 0$. If*

$$\sum_{n=2}^{\infty} \frac{n-1}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1}{\Gamma(\beta)}, \tag{6.6}$$

then $\mathcal{E}_{\alpha,\beta} \in S^*$.

We give a similary theorem for the class $k - \mathcal{UCV}(\lambda, \gamma)$, without proof.

Theorem 6.2.4. *[39] Let $\alpha, \beta > 0$, $k \geq 0$ and $0 < \gamma \leq 1$. If*

$$\sum_{n=2}^{\infty} n \frac{(n-1)[1-\lambda\gamma + k(1-\lambda)] + 1 - \gamma}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1-\gamma}{\Gamma(\beta)}, \tag{6.7}$$

where $0 \leq \lambda < 1$, then $\mathcal{E}_{\alpha,\beta} \in k - \mathcal{UCV}(\lambda, \gamma)$.

Putting $k = \lambda = \gamma = 0$ in the Theorem 6.2.4 we obtain the anlytic criteria for the class $\mathcal{K}$.

Corollary 6.2.4. *[39] Let $\alpha, \beta > 0$. If*

$$\sum_{n=2}^{\infty} \frac{n^2}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1}{\Gamma(\beta)},$$

then $\mathcal{E}_{\alpha,\beta} \in \mathcal{K}$.

Chapter 7

Integral operators

In mathematics, those integral mappings which preservs the geometric properties of the domain in which are defined, has a great importance. We begin this chapter with presenting some well known integral operators in the specialty literature.

In 1915 J.W. Alexander has introduced the following integral operator [7]:

$$I : \mathcal{A} \to \mathcal{A},$$

$$A(f)(z) = \int_0^z \frac{f(t)}{t} dt.$$

In [7] is showed that the operator A maps S^* onto $\mathcal{K}$.

In [54] R.J. Libera has defined the

$$L : \mathcal{A} \to \mathcal{A},$$

$$L(f)(z) = \frac{2}{z} \int_0^z f(t) dt$$

integral operator. It is known that this operator maps S^* onto S^*. In 1969 Bernardi has introduced the following operator [21]

$$F : \mathcal{A} \to \mathcal{A},$$

$$F(f)(z) = I_c f(z) = \frac{1+c}{z^c} \int_0^z f(t) t^{c-1} dt, \text{ where } c = 1, 2, ...,$$

as an extension of the previous result. Is also known, that this operator maps S^* onto S^* if $c = 1, 2, ...$.

In many years, numerous generalizations of these operators have been studied. For example let us recall the generalized Bernardi - Libera - Livingston integral operator

$$I(z) = L_c f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \tag{7.1}$$

where $f \in \mathcal{A}$ and $c > -1$. This operator was studied by Bernardi for $c \in \{1, 2, 3, ...\}$ and for $c = 1$ by Libera.

7.1 The γ order starlikeness of an integral operator

Lemma 7.1.1. *[38] For the dual set of the class $\mathcal{P} = \{f \in A_0 |\ \Re f(z) > 0,\ z \in \mathbb{U}\}$ we have*

$$\{f \in A_0 |\ \Re f(z) > \frac{1}{2},\ z \in \mathbb{U}\} \subset \mathcal{P}^d.$$

Lemma 7.1.2. *[38] For $\theta \in [0, 2\pi]$, the following equalities holds:*

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)^2} = \int_0^1 \int_0^1 \frac{(1-x)y(\cos\theta - xy)}{1 + x^2y^2 - 2xy\cos\theta} dxdy + i \int_0^1 \int_0^1 \frac{(1-x)y\sin\theta}{1 + x^2y^2 - 2xy\cos\theta} dxdy$$

and

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{(n+1)^2} = \int_0^1 \int_0^1 \frac{xy(\cos\theta - xy)}{1 + x^2y^2 - 2xy\cos\theta} dxdy + i \int_0^1 \int_0^1 \frac{xy\sin\theta}{1 + x^2y^2 - 2xy\cos\theta} dxdy.$$

Lemma 7.1.3. *[38]If $\gamma = \dfrac{2 - \ln 4}{3 - \ln 6 - \frac{\pi^2}{12}}$, then the inequalities*

$$(1-\gamma) \int_0^1 \int_0^1 (1-x)y \frac{(1-xy)(1+\cos\theta)}{(1+xy)(1+x^2y^2 - 2xy\cos\theta)} dxdy \geq$$
$$\frac{1}{6} \int_0^1 \int_0^1 xy \frac{(1-xy)(1+\cos\theta)}{(1+xy)(1+x^2y^2-2xy\cos\theta)} dxdy, \quad \theta \in \left[\frac{\pi}{2}, \pi\right]$$

and

$$\int_0^1 \int_0^1 \frac{xy\sin\theta}{1 + x^2y^2 - 2xy\cos\theta} dxdy \leq \int_0^1 \int_0^1 \frac{xy\sqrt{2(1+\cos\theta)}}{(1+xy)\sqrt{1+x^2y^2-2xy\cos\theta}} dxdy,\ \theta \in [0,\pi]$$

holds.

Lemma 7.1.4. *[38] For $\theta \in [0, \pi]$, the following inequality holds:*

$$\int_0^1 \int_0^1 \frac{2xy}{1-x^2y^2} dxdy \int_0^1 \int_0^1 xy \frac{(1-xy)(1+\cos\theta)}{(1+x^2y^2-2xy\cos\theta)(1+xy)} dxdy \leq$$
$$4(1-\gamma)\left(1 + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n(n+1)^2}\right) \left[(1-\alpha) \cdot \left(1 + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n(n+1)^2}\right) + \sum_{n=1}^{\infty} \frac{\cos n\theta}{(n+1)^2}\right].$$

Theorem 7.1.1. *[38] If $f(0) = 1$ and*

$$\Re\big(1 + 4zf'(z) + 2z^2 f''(z)\big) > 0, \quad z \in \mathbb{U} \tag{7.2}$$

then the function F defined by $F(z) = \int_0^z f(t)dt$ is starlike of order $\gamma = \dfrac{2 - \ln 4}{3 - \ln 4 - \frac{\pi^2}{12}} =$ $0.7756\ldots$, that is

$$\Re\frac{zF'(z)}{F(z)} > \frac{2 - \ln 4}{3 - \ln 4 - \frac{\pi^2}{12}}, \quad z \in \mathbb{U}. \tag{7.3}$$

The result is sharp.

7.2 Preserving properties of the Bernardi integral operator defined on the $UCC_q(g, \gamma)$ and $C_{NS}(\alpha)$ classes

Theorem 7.2.1. *[33] Let $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j \in T^*$ and*

$$F(z) = I_c f(z) = \frac{c+1}{z^c} \int_0^z f(t)t^{c-1}dt, \ c \in \mathbb{N}^*.$$

If $f \in UCC_q(g, \gamma)$, where $\dfrac{1-q^j}{1-q} a_j \le b_j$ for $j \in \{2, 3, ..\}$, then $F \in UCC_q(g, \gamma)$.

Theorem 7.2.2. *[34] Let*

$$F(z) = I_c f(z) = \frac{c+1}{z^c} \int_0^z f(t)t^{c-1}dt, \ c \in \mathbb{N}^*.$$

If $f \in C_{NS}(\alpha)$, then $F = I_c(f) \in C_{NS}(\beta)$, where

$$\beta = \beta(\alpha, 2) = 1 - \frac{(1-\alpha)(c+1)^2}{(c+2)^2(2-\alpha) - (c+1)^2(1-\alpha)} \tag{7.4}$$

and $\beta > \alpha$, $\alpha \in [0, 1)$.

7.3 Preserving properties of the generalized Bernardi-Libera-Livingston integral operator defined on the classes S^{**} and S^{***}

In this section we study the properties of the image of some subclasses of starlike functions, trough the generalized Bernardi-Libera-Livingston integral operator.

Theorem 7.3.1. *[40] Let*

$$I(z) = L_c f(z) = \frac{c+1}{z^c} \int\limits_0^z t^{c-1} f(t) dt.$$

If $c \geq \sqrt{\frac{5}{4}}$ *and* $f \in S^{**}$, *then* $I \in S^{**}$.

Theorem 7.3.2. *[40] Let*

$$I(z) = L_c f(z) = \frac{c+1}{z^c} \int\limits_0^z t^{c-1} f(t) dt, \ c > -2.$$

If $f \in S^{***}$, *then* $I \in S^{***}$.

Theorem 7.3.3. *[40] Let*

$$I(z) = L_c f(z) = \frac{c+1}{z^c} \int\limits_0^z t^{c-1} f(t) dt, \ c \in (-1, 0].$$

If $f \in TS^{***}$, *then* $I \in TS^{***}$.

Chapter 8

Radius problems of starlikeness, convexity and uniform convexity of orthogonal polynomials

The real numbers

$$r^*(f) = \sup\left\{ r > 0 \middle| \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \text{ for all } z \in \mathbb{U}(r) \right\}$$

and

$$r^*_\beta(f) = \sup\left\{ r > 0 \middle| \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \text{ for all } z \in \mathbb{U}(r) \right\}$$

are called the radius of starlikeness respectively the radius of starlikeness of order β of the function f. We note that $r^*(f)$ is the largest radius such that, $f(\mathbb{U}(r^*(f)))$ is a starlike domain with respect to 0.

Similarly, let

$$r^c(f) = \sup\left\{ r > 0 \middle| \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \text{ for all } z \in \mathbb{U}(r) \right\}$$

be the radius of convexity of the function f and the radius of convexity of order β, is

$$r^c_\beta = \sup\left\{ r > 0 \middle| \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \text{ for all } z \in \mathbb{U}(r) \right\}.$$

The radius of uniform convexity of the function f is given by

$$r^{uc}(f) = \sup\left\{ r > 0 \middle| \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \text{ for all } z \in \mathbb{U}(r) \right\}.$$

8.1 The generalized Laguerre polynomials

Let us consider the generalized Laguerre polynomials, that satisfy a second order linear differential equation

$$xy'' + (\lambda + 1 - x)y' + ny = 0,$$

for arbitrary real λ and $n \in \mathbb{Z}_+$ which is often called the Laguerre equation.
The Rodrigues formula of the Laguerre polynomials is

$$L_n^\lambda(z) = \frac{1}{n!} z^{-\lambda} e^z \frac{d^n}{dz^n}(z^{\lambda+n} e^{-z}), \tag{8.1}$$

where $n \in \mathbb{Z}_+$. The formula (8.6) and the Rolle theorem implies that every root of $L_n^\lambda(z) = 0$ is real and positive, provided that $\lambda + 1 > 0$.

Let z_i, $i \in \{1, 2, ..., n\}$ be the roots, where $0 < z_1 < z_2 < ... < z_n$ and let us consider the following normalized form of the L_n^λ polynomial:

$$\mathcal{L}_{n+1}^\lambda(z) = z \frac{L_n^\lambda(z)}{L_n^\lambda(0)} = z + a_1 z^2 + ... + a_n z^{n+1}.$$

The product representation of the polynomial $\mathcal{L}_{n+1}^\lambda$ is

$$\mathcal{L}_{n+1}^\lambda(z) = a_n z(z - z_1)(z - z_2)...(z - z_n). \tag{8.2}$$

8.2 The radius of starlikeness, convexity and uniform convexity of the normalized Laguerre polynomials

Before we determine the radius of starlikeness, convexity and uniform convexity of the normalized Laguerre polynomials we need the following result.

Lemma 8.2.1. *[35] If $|z| \le r < \beta$, then we have*

$$\Re \frac{z}{\beta - z} \le \frac{r}{\beta - r}, \tag{8.3}$$

$$\left| \frac{z}{\beta - z} \right| \le \frac{r}{\beta - r}, \tag{8.4}$$

$$\left| \frac{z}{(\beta - z)^2} \right| \le \frac{r}{(\beta - r)^2}. \tag{8.5}$$

Theorem 8.2.1. *[35] Let $0 \leq \beta < 1$, then the radius of starlikeness of order β of the normalized Laguerre polynomial $\mathcal{L}_{n+1}^{\lambda}$ is $r_{\beta}^{*}(\mathcal{L}_{n+1}^{\lambda}) = r_0$, where r_0 denotes the smallest positive root of the equation $r(\mathcal{L}_{n+1}^{\lambda})'(r) - \beta(\mathcal{L}_{n+1}^{\lambda}(r)) = 0$.*

Theorem 8.2.2. *[35] Let $0 \leq \beta < 1$, then the radius of convexity of order β of the normalized Laguerre polynomial $\mathcal{L}_{n+1}^{\lambda}$ is $r_{\beta}^{c}(\mathcal{L}_{n+1}^{\lambda}) = r_1$, where r_1 denotes the smallest positive root of the equation $r(\mathcal{L}_{n+1}^{\lambda})''(r) - (\beta - 1)(\mathcal{L}_{n+1}^{\lambda})'(r) = 0$.*

Theorem 8.2.3. *[35] The radius of uniform convexity of the normalized Laguerre polynomial is $r^{uc}(\mathcal{L}_{n+1}^{\lambda}) = r_2$, where r_2 denotes the smallest positive root of the equation*

$$1 + 2\frac{r(\mathcal{L}_{n+1}^{\lambda})''(r)}{(\mathcal{L}_{n+1}^{\lambda})'(r)} = 0.$$

In the following theorem we will determine the largest disk, centered in the origin, which is mapped on a convex domain by the L_n^{λ} polynomial.

Theorem 8.2.4. *[35] Let r_3 be the smallest positive root of the equation*

$$1 + \frac{r(L_n^{\lambda})''(r)}{(L_n^{\lambda})'(r)} = 0.$$

Then $\mathbb{U}(r_3)$ is the largest disk which is mapped by L_n^{λ} on a convex domain, where $r_3 = r^c(L_n^{\lambda})$ is the radius of convexity of the polynomial L_n^{λ}.

It is easily seen that the image $L_n^{\lambda}\big(\mathbb{U}(r^c(L_n^{\lambda}))\big)$ is symmetric with respect to the axis OX and the boundary of $L_n^{\lambda}\big(\mathbb{U}(r^c(L_n^{\lambda}))\big)$ intersects this axis at the points $L_n^{\lambda}(r^c(L_n^{\lambda}))$ and $L_n^{\lambda}(-r^c(L_n^{\lambda}))$. Thus we have the following corollary.

Corollary 8.2.1. *[35] The following inequality holds*

$$L_n^{\lambda}\big(r^c(L_n^{\lambda})\big) \geq L_n^{\lambda}\big(r\big) \geq \Re\big(L_n^{\lambda}(re^{i\theta})\big) \geq L_n^{\lambda}\big(-r\big) \geq L_n^{\lambda}\big(-r^c(L_n^{\lambda})\big),$$

for every $r \in (0, r^c(L_n^{\lambda}))$, $z \in \mathbb{U}(r)$ and $\theta \in [0, 2\pi]$. The number $r^c(L_n^{\lambda})$ is the biggest positive real number with this property.

Example 8.2.1. *[35] The radius of convexity of the Laguerre polynomial L_2^{λ}, defined by the equality*

$$L_2^{\lambda}(z) = \frac{z^2}{2} - (\lambda + 2)z + \frac{(\lambda + 2)(\lambda + 1)}{2},$$

is the smallest positive root of the equation

$$r[L_2^{\lambda}(r)]'' + [L_2^{\lambda}(r)]' = 0.$$

Because $[L_2^{\lambda}(r)]' = r - \lambda - 2$ and $[L_2^{\lambda}(r)]'' = 1$, then we get $r = \dfrac{\lambda + 2}{2}$, which is the radius of convexity of the Laguerre polynomial L_2^{λ}, where $\lambda > -2$.

8.3 The Legendre polynomials

The Legendre polynomials are solutions of the Legendre differential equation:

$$\frac{d}{dz}\left[(1-z^2)\frac{d}{dz}P_n(z)\right]+n(n+1)P_n(z)=0,$$

for $n \in \mathbb{Z}_+$.
The Rodrigues formula of the Legendre polynomials is

$$P_n(z)=\frac{1}{2^n n!}\frac{d^n}{dz^n}[(z^2-1)^n], \tag{8.6}$$

where $n \in \mathbb{Z}_+$. Bonnet's recursion formula for Legendre polynomials are:

$$(n+1)P_{n+1}(z)-(2n+1)zP_n(z)+nP_{n-1}(z)=0.$$

Legendre polynomials are symmetric or antisymmetric, that is

$$P_n(-z)=(-1)^n P_n(z).$$

Let us consider the following normalized form of the Legendre polynomial of odd degree P_{2n-1}:

$$\mathcal{P}_{2n-1}(z)=\frac{P_{2n-1}(z)}{P'_{2n-1}(0)}=z+a_2z^2+...+a_{2n-1}z^{2n-1}.$$

The roots of $P_{2n-1}(z)=0$ are $0=z_0<z_1<...<z_{n-1}$ and $-z_1,-z_2,...,-z_{n-1}$.
The product representation of the polynomial $\mathcal{P}_{2n-1}(z)$ is

$$\mathcal{P}_{2n-1}(z)=a_{2n-1}z(z^2-z_1^2)(z^2-z_2^2)...(z^2-z_{n-1}^2). \tag{8.7}$$

8.4 The radius of starlikeness, convexity and uniform convexity of the normalized Legendre polynomials of odd degree

Theorem 8.4.1. *[24] The radius of starlikeness of order β of the normalized Legendre polynomials $\mathcal{P}_{2n-1}$ of odd degree is $r^*_\beta(\mathcal{P}_{2n-1})=r_0$, where r_0 denotes the smallest positive root of the equation $r(\mathcal{P}_{2n-1})'(r)-\beta(\mathcal{P}_{2n-1}(r))=0$, where $0\le\beta<1$.*

Theorem 8.4.2. *[24] The radius of convexity of order β of the normalized Legendre polynomial of odd degree $\mathcal{P}_{2n-1}$ is $r^c_\beta(\mathcal{P}_{2n-1})=r_1$, where r_1 denotes the smallest positive root of the equation $r(\mathcal{P}_{2n-1})''(r)-(\beta-1)(\mathcal{P}_{2n-1})'(r)=0$, where $0\le\beta<1$.*

Theorem 8.4.3. *[24] The radius of uniform convexity of the normalized Legendre polynomial of odd degree is $r^{uc}(\mathcal{P}_{2n-1}) = r_2$, where r_2 denotes the smallest positive root of the equation*

$$1 + 2\frac{r(\mathcal{P}_{2n-1})''(r)}{(\mathcal{P}_{2n-1})'(r)} = 0.$$

In the following theorem we will determine the largest disk, centered in the origin, which is mapped on a convex domain by the P_{2n-1} polynomial.

Theorem 8.4.4. *[24] Let r_3 be the smallest positive root of the equation*

$$1 + \frac{r(P_{2n-1})''(r)}{(P_{2n-1})'(r)} = 0.$$

Then $\mathbb{U}(r_3)$ is the largest disk, which is mapped by P_{2n-1} on a convex domain, where $r_3 = r^c(P_{2n-1})$ is the radius of convexity of the polynomial P_{2n-1}.

It is easily seen that the image $P_{2n-1}\big(\mathbb{U}(r^c(P_{2n-1}))\big)$ is symmetric with respect to the axis OX and the boundary of $P_{2n-1}\big(\mathbb{U}(r^c(P_{2n-1}))\big)$ intersects this axis at the points $P_{2n-1}(r^c(P_{2n-1}))$ and $P_{2n-1}(-r^c(P_{2n-1}))$. Thus we have the following corollary.

Corollary 8.4.1. *[24] The following inequality holds*

$$P_{2n-1}\big(r^c(P_{2n-1})\big) \geq P_{2n-1}\big(r\big) \geq \Re\big(P_{2n-1}(re^{i\theta})\big) \geq P_{2n-1}\big(-r\big) \geq P_{2n-1}\big(-r^c(P_{2n-1})\big),$$

for every $r \in (0, r^c(P_{2n-1}))$, $z \in \mathbb{U}(r)$ and $\theta \in [0, 2\pi]$. The number $r^c(P_{2n-1})$ is the biggest positive real number with this property.

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