"Babeş-Bolyai" Universty of Cluj-Napoca
Faculty of Mathematics and Computer Science

PhD Thesis

Summary

Contributions to the Approximation Theory of Functions of Real and Complex Variable

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Contents

Keywords : monotone and submodular set functions, Choquet integral, approximation by Bernstein-Durrmeyer-Choquet operators, Feller's scheme, posibilistic integral, approximation by posibilistic operators, approximation by generalized Baskakov operators of a real variable, approximation in compact disks of $\mathbb{C}$ by complex generalized Szász, Szász-Kantorovich and Baskakov, approximation in compact sets in $\mathbb{C}$ by Baskakov-Faber generalized operators, arbitrary order of approximation.

Ch. 1

General Introduction

This thesis contains the results I have obtained in the topic of approximation of functions of real and complex variable.

Approximation Theory is a part of Mathematical Analysis, having its roots in the 19th century. It deals, in essence, with the approximation of some complicated elements of a space (most of the time functions), with simpler elements (most of the time algebraic polynomials, trigonometric polynomials, spline functions, so on). Moreover, quantitative characterizations of this approximation are obtained, most of the time in terms of the so-called moduli of continuity (smoothness).

From historical point of view, in the case of approximation of functions of real variable, probably that the first result was obtained by the German mathematician K. Weierstrass in 1895, who proved the following result.

Theorem A. *If* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$*, then there exists a sequence of algebraic polynomials with real coefficients,* $P_{m_n}(x) = a_0 x^{m_n} + . + a_{m_n-1}x + a_{m_n}$*, such that* $\lim_{n\to\infty} P_{m_n}(x) = f(x)$*, uniformly with respect to* $x \in [a, b]$.

A constructive proof of the above result was obtained by the Russian mathematician S.N. Bernstein in 1912, who proved that the sequence of algebraic polynomials (called in our days "Bernstein polynomials"), $B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f)k/n)$, converges uniformly to the continuous function f.

The first quantitative result in the Weierstrass' and Bernstein's result was obtained by the Romanian mathematician Tiberiu Popoviciu in 1942, who proved

$$|B_n(f)(x) - f(x)| \le \frac{3}{2}\omega_1(f; 1/\sqrt{n}), \forall x \in [0,1], n \in \mathbb{N},$$

where $\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,1], |x-y| \le \delta\}$ denotes the modulus of continuity of f.

In the case of approximation of continuous and 2π periodic functions, the first constructive result was obtained by the Hungarian mathematician L. Fejér in 1900, who proved that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous and 2π periodic function on $\mathbb{R}$, denoting $S_n(f)(x) = \sum_{k=0}^{n} a_k \cos(kx) + b_k \sin(kx)$, were a_k and b_k are the Fourier coefficients of f, then $T_n(f)(x) = \frac{S_0(f)(x)+\ldots+S_n(f)(x)}{n+1}$ represents a sequence of trigonometric polynomials converging uniformly to f on $\mathbb{R}$.

The first quantitative and constructive result in approximation by trigonometric polynomials was obtained by the American mathematician D. Jackson in the doctoral thesis in 1911, who proved that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and 2π periodic, then a sequence of trigonometric polynomials can be constructed, $J_n(f)(x)$, $n \in \mathbb{N}$, with the property that

$$|J_n(f)(x) - f(x)| \le C\omega_2(f; 1/n), \forall x \in \mathbb{R}, n \in \mathbb{N},$$

where $\omega_2(f;\delta) = \sup\{|f(x+h) - 2f(x) + f(x-h)|; 0 \le h \le \delta, x \in \mathbb{R}\}$ represents the second order modulus of smoothness of f.

An important direction in approximation of functions is represented by the theory of approximation by positive and linear operators, having its roots between 1950 and 1970, by the classical results of Tiberiu Popoviciu, Bohman, Korovkin, Shisha-Mond and others. In essence, these results state (see Korovkin's results) that in order that a sequence of positive and linear operators, $(L_n(f))_{n\in\mathbb{N}}$, be uniformly convergent on $[a, b]$ to a continuous function f, is that $L_n(e_k) \to e_k$, for $k = 0, 1$ and 2, where $e_0(x) = 1$, $e_1(x) = x$ şi $e_2(x) = x^2$.

In the case of complex approximation, the roots of this theory can be found, for the approximation of continuous functions by polynomials or entire functions in the Müntz-Szász Carleman's papers, while for approximation of analytic functions of a complex variable by polynomials or rational functions, can be mentioned the results obtained by Runge, Walsh, Faber, Mergelyan, Arakelyan and Dzyadyk.

This thesis is structured in four chapters.

In the present Chapter 1, we make a general introduction in Approximation Theory and we shortly describe the thesis.

In Chapter 2 titled "Approximation by nonlinear integral operators", the basic idea is the replacement of the classical integral in the expressions of some integral linear operators, by more general integrals (which are not linear) and to study the approximation properties of the new obtained operators.

The chapter has two sections.

Thus, in the first section, titled "Approximation by Durrmeyer-Choquet operators", in the expression of the classical Bernstein-Durrmeyer operators, the Lebesgue integral is replaced by the nonlinear Choquet integral with respect to a monotone and submodular set function. We show that the new obtained nonlinear operators remain uniformly convergent to the approximated function.

In the second section, in the classical Feller's scheme of generation of linear and positive operators with good approximation properties, we replace the classical (linear) integral with respect to the Lebesgue measure, with the nonlinear possibilistic integral. In this way, we generate new (nonlinear) operators with good approximation properties, including the so-called max-product operators studied in a long series of papers by B. Bede, L. Coroianu and S.G. Gal (which culminates with the research monograph [10] published at Springer).

In the same section, we study the quantitative approximation properties of the convolution possibilistic operators obtained by the Feller's scheme.

In Chapter 3 titled "Arbitrary order by Szász and Baskakov operators", starting from a sequence $\lambda_n > 0$, $n \in \mathbb{N}$, converging to zero as fast we want (that is, arbitrary fast), we construct sequences of Baskakov, q-Baskakov, Szász-Stancu and Baskakov-Stancu operators, converging to the approximated function $f : [0, \infty) \to \mathbb{R}$ with the order of convergence $\omega_1(f; \sqrt{\lambda_n})$ (in fact, arbitrary good, because λ_n can be chosen to converge to zero, arbitrarily rapid).

For this reason, the results in this chapter can be considered of definitive type (that is, the best possible). In the same time, the results obtained have also a strong unifying character, in the sense that one can recapture from them all the results previously obtained by other authors, for various choices of the nodes λ_n.

In Chapter 4 titled "Complex Szász and Baskakov operators", we apply the ideas in Chapter 3 to the case of aproximation of analytic functions of complex variable, by complex Szász, Szász-Kantorovich and Baskakov.

In the first section of the chapter, starting again from a sequence $\lambda_n > 0$, $n \in \mathbb{N}$, converging to zero as fat we want (arbitrarily rapid), we construct sequences of Szász, Szász-Kantorovich and Baskakov operators attached to an analytic function of exponential growth in a compact disk centered at origin, which approximate f with the order $O(\lambda_n)$ and for which quantitative Voronovskaja type results with the order $O(\lambda_n^2)$ are obtained.

In the second section of the chapter, we consider the same problem as in the previous section, for the so-called complex Baskakov-Faber operators, attached through the Faber polynomials to an analytic function of exponential growth in a compact set of $\mathbb{C}$ (not necessarily a disk).

The results presented in this thesis were obtained by the author in collaboration with professor dr. Sorin Gal, Nazim Mahmodov, Lucian Coroianu, Sorin Trifa, or as a single author, in 6 papers, published by the following journals :

1) Gal, Sorin G.; **Opriş, Bogdan D.**, *Approximation with an arbitrary order by modified Baskakov type operators.* **Appl. Math. Comput.**, 265 (2015), 329-332 (Impact Factor ISI (IF)on 2015 : 1.345, Relative Score of Influence (RSI) on 2016 : 0.733)

2) Gal, Sorin G.; **Opriş, Bogdan D.**, *Uniform and pointwise convergence of Bernstein-Durrmeyer operators with respect to monotone and submodular set functions.* **J. Math. Anal. Appl.** 424 (2015), no. 2, 1374-1379 (IF on 2015 : 1.014, RSI on 2016 : 1.125)

3) Gal, Sorin G.; **Opriş, Bogdan D.**, *Approximation of analytic functions with an arbitrary order by generalized Baskakov-Faber operators in compact sets.* **Complex Anal. Oper. Theory** 10 (2016), no. 2, 369-377 (IF on 2015 : 0.663, RSI on 2016 : 0.724)

4) Coroianu, Lucian ; Gal, Sorin G. ; **Opriş, Bogdan D.**; Trifa, Sorin,

Feller's scheme in approximation by nonlinear possibilistic integral operators, **Numer. Funct. Anal. and Optim.**, 38 (2017), No. 3, 327-343 (IF on 2015 : 0.649, RSI on 2016 : 0.540).

5) Gal, Sorin G.; Mahmudov, Nazim I.; **Opriş, Bogdan D.**, *Approximation with an arbitrary order of Szász, Szász-Kantorovich and Baskakov complex operators in compact disks.* **Azerb. J. Math.** 6 (2016), no. 2, 3-12 (indexed in Mathematical Reviews and Zentralblatt für Mathematik)

6) **Opriş, Bogdan, D.**, *Approximation with an arbitrary order by generalized Szász-Stancu and Baskakov-Stancu type operators*, **Anal. Univ. Oradea, fasc. math.**, XXIV (2017), No. 1, 75-81 (B+ journal, indexed in Mathematical Reviews and Zentralblatt für Mathematik).

The original results obtained in the thesis are the following :

Chapter 2. Section 2.1 : Lemma 2.1.2, Theorem 2.1.3, Theorem 2.1.4 ; **The results were published in the paper [44];**

Section 2.2 : Theorem 2.2.2, Lemma 2.2.3, Theorem 2.2.4, Theorem 2.2.5, Corollary 2.2.6, Theorem 2.2.7, Corollary 2.2.8, Theorem 2.2.9, Corollary 2.2.9 ; **The results were published in the paper [21]** ;

Chapter 3. Section 3.1 : Lemma 3.1.1, Corollary 3.1.2, Theorem 3.1.3, Corollary 3.1.4, Lemma 3.1.5, Theorem 3.1.6, Corollary 3.1.7, Corollary 3.1.8 ; **The results were published in the paper [43];**

Section 3.2 : Lemma 3.2.1, Theorem 3.2.2 and Corollary 3.2.3 ; **The results were published in the paper [59];**

Section 3.3 : Lemma 3.3.1, Theorem 3.3.2, Corollary 3.3.3 ; **The results were published in the paper [59];**

Chapter 4. Section 4.1 : Theorem 4.1.1, Theorem 4.1.2, Theorem 4.1.3 ; **The results were published in the paper [46];**

Section 4.2 : Definition 4.2.1, Lemma 4.2.2, Lemma 4.2.3, Theorem

4.2.4. **The results were published by the paper [45].**

Key words : monotone and submodular set function, Choquet integral, Bernstein-Durrmeyer operator, uniform convergence, pointwise convergence ; theory of possibility, Feller's scheme, Chebyshev type inequality, nonlinear possibilistic integral, possibilistic Picard operators, possibilistic Gauss-Weierstrass operators, possibilistic Poisson-Cauchy operators, max-product (possibilistic) Bernstein kind operators ; generalized Baskakov operator of real variable, linear and positive operators, modulus of continuity, order of approximation, q-calculus ; generalized Szász, Szász-Kantorovich and Baskakov complex operators, Voronovskaja-type results ; compact sets, Faber polynomials, generalized Baskakov-Faber operator.

I want to express my deep gratitude to professor dr. Sorin Gal for his constant support in the elaboration of this thesis.

Ch. 2

Approximation by nonlinear integral operators

In this chapter we deal with the study of the approximation properties of the integral operators, in the case when the classical linear integral is replaced with the nonlinear Choquet integral and the nonlinear possibilistic integral. The chapter consists in two sections : in the first section we deal with the Durrmeyer-Choquet operators and in the second section we deal with the possibilistic operators.

2.1 Approximation by Durrmeyer-Choquet operators

In this section we study the Bernstein-Durmeyer type operator of d-variables, $M_{n,\mu}$, in which the integrals written in terms of a Borel type measure μ (including therefore the Lebesgue measure too) defined on the d-dimensional simplex, are replaced by Choquet integrals with respect to μ supposed to be only monotone and submodular. The new operator is nonlinear and generalizes the linear Bernstein-Durrmeyer. For this operator which could be called of Durrmeyer-Choque type, we prove the uniform and pointwise convergence to $f(x)$. As a consequence, the results obtained generalize those in the recent papers [11] and [12].

2.1.1 Introduction

Let the standard simplex in $\mathbb{R}^d$

$$S^d = \{(x_1, ..., x_d); 0 \leq x_1, ..., x_d \leq 1,\ 0 \leq x_1 + ... + x_d \leq 1\}.$$

Inspired by the paper [13], in the recent papers [11], [12] and [52], uniform, pointwise and L^p convergence (respectively) of $M_{n,\mu}(f)(x)$ to $f(x)$ (as $n \to \infty$) were obtained, where $M_{n,\mu}(f)(x)$ denotes the linear, mixed Bernstein-Durrmeyer operator of d-variables, with respect to a bounded Borel measure $\mu : S^d \to \mathbb{R}_+$, defined by (supposing that f is μ-integrable on S^d)

$$M_{n,\mu}(f)(x)$$

$$= \sum_{|\alpha|=n} \frac{\int_{S^d} f(t) B_\alpha(t) d\mu(t)}{\int_{S^d} B_\alpha(t) d\mu(t)} \cdot B_\alpha(x) := \sum_{|\alpha|=n} c(\alpha, \mu) \cdot B_\alpha(x),\ x \in S^d,\ n \in \mathbb{N}. \tag{2.1}$$

In the above formula (2.1), we used the notations $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$, with $\alpha_j \geq 0$ for all $j = 0, ..., n$, $|\alpha| = \alpha_0 + \alpha_1 + ... + \alpha_n = n$ and

$$B_\alpha(x) = \frac{n!}{\alpha_0! \cdot \alpha_1! \cdot ... \cdot \alpha_n!}(1 - x_1 - x_2 - ... - x_d)^{\alpha_0} \cdot x_1^{\alpha_1} \cdot ... \cdot x_d^{\alpha_d}$$

$$:= \frac{n!}{\alpha_0! \cdot \alpha_1! \cdot ... \cdot \alpha_n!} \cdot P_\alpha(x).$$

We will prove that the results in [11] and [12] on pointwise and uniform convergence, remain valid in the more general setting when μ is only a monotone, bounded and submodular set function on S^d and the integrals appearing in formula (2.1), represent Choquet integrals with respect to μ.

2.1.2 Preliminaries

In this subsection, by Definition 2.1.1 and the Remarks after this definition, we present known concepts and results useful in the next subsections.

Definition 2.1.1. Consider that Ω is a nonempty set, $\mathcal{C}$ is a σ-algebra of subsets in Ω and $(\Omega, \mathcal{C})$ i a measurable space.

(i) (see, e.g., [63], p. 63) The set function $\mu : \mathcal{C} \to [0, +\infty]$ will be called a monotone set function (or capacity) if $\mu(\emptyset) = 0$ and $A, B \in \mathcal{C}$, with $A \subset B$, implies $\mu(A) \leq \mu(B)$. If

$$\mu(A \bigcup B) + \mu(A \bigcap B) \leq \mu(A) + \mu(B), \text{ for all } A, B \in \mathcal{C},$$

then μ is called submodular. Then, μ will be called normalized, if $\mu(\Omega) = 1$

(ii) (see [16], or [63], p. 233) Let $\mu : \mathcal{C} \to \mathbb{R}_+$ be a normalized, monotone set function. The function $f : \Omega \to \mathbb{R}$ is called $\mathcal{C}$-measurable if for any Borel subset $B \subset \mathbb{R}$, it holds $f^{-1}(B) \in \mathcal{C}$.

If $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable, then for any $A \in \mathcal{C}$, the Choquet integral will be defined by the formula

$$(C) \int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f) \bigcap A) d\beta + \int_{-\infty}^0 [\mu(F_\beta(f) \bigcap A) - \mu(A)] d\beta,$$

where $F_\beta(f) = \{\omega \in \Omega; f(\omega) \geq \beta\}$. If $(C) \int_A f d\mu$ exists in $\mathbb{R}$, then f is called Choquet integrable on A. We observe that if $f \geq 0$ on A, then the term integral $\int_{-\infty}^0$ in the above formula becomes equal to zero.

When μ is the Lebesgue measure (i.e. countably additive), then the Choquet integral $(C) \int_A f d\mu$ reduces to the Lebesgue integral.

In the lines of the following remarks, we list without proofs some known properties which will be used in the next subsections.

Remarks. Let $\mu : \mathcal{C} \to [0, +\infty]$ be a monotone set function. Then, the following properties hold :

(i) $(C)\int_A$ is positively homogeneous, i.e. for $a \geq 0$ we have $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ (for $f \geq 0$ see, e.g., [63], Theorem 11.2, (5), p. 228 and for f of arbitrary sign, see, e.g., [23], p. 64, Proposition 5.1, (ii)).

(ii) In the general case for $f$and g, we have $(C)\int_A (f+g)d\mu \neq (C)\int_A f d\mu + (C)\int_A g d\mu$. If μ is submodular too, then the Choquet integral is sublinear, that is

$$(C)\int_A (f+g)d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu,$$

for all f, g of arbitrary sign and lower bounded (see, e.g., [23], p. 75, Theorem 6.3).

Then, for all $c \in \mathbb{R}$ and f of arbitrary sign we have

$$(C)\int_A (f+c)d\mu = (C)\int_A f d\mu + c \cdot \mu(A),$$

(see, e.g., [63], pp. 232-233, or [23], p. 65).

(iii) If $f \leq g$ on A then $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ (see, e.g., [63], p. 228, Theorem 11.2, (3) for $f, g \geq 0$ and p. 232 for f, g of arbitrary sign).

(iv) Let $f \geq 0$. By the definition of the Choquet integral, it is immediate that if $A \subset B$ then

$$(C)\int_A f d\mu \leq (C)\int_B f d\mu$$

and if, in addition, μ is finitely subadditive, then

$$(C)\int_{A\bigcup B} f d\mu \leq (C)\int_A f d\mu + (C)\int_B f d\mu.$$

(v) By the definition of the Choquet integral, it is immediate that

$$(C)\int_A 1 \cdot d\mu(t) = \mu(A).$$

(vi) Simple concrete examples of monotone and submodular set functions μ, can be obtained from a probability measure M on a σ-algebra on Ω (i.e. $M(\emptyset) = 0$, $M(\Omega) = 1$ and M is countably additive), by the formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0, 1] \to [0, 1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ (see, e.g., [23], pp. 16-17, Example 2.1). Note that in fact if M is only finitely additive, then $\mu(A) = \gamma(M(A))$ still is submodular.

Recall here that a set function $\mu : \mathcal{P}(\Omega) \to [0, 1]$ ($\mathcal{P}(\Omega)$ denotes the family of all subset of Ω) is called a possibility measure on the non-empty set Ω, if it satisfies the axioms $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$ and $\mu(\bigcup_{i\in I} A_i) = \sup\{\mu(A_i); i \in I\}$ for all $A_i \in \Omega$, and any I, family of indices.

Concerning this concept, it is known that any possibility measure μ is monotone and submodular. Indeed, we observe that the monotonicity and the submodularity are immediate from the axioms (respectively)

$$\mu(A \bigcup B) = \max\{\mu(A), \mu(B)\},\ \mu(A \bigcap B) \leq \min\{\mu(A), \mu(B)\}.$$

It is also known that any given possibility distribution (on Ω), that is a function $\lambda : \Omega \to [0, 1]$, such that $\sup\{\lambda(s); s \in \Omega\} = 1$, induces a possibility measure $\mu_\lambda : \mathcal{P}(\Omega) \to [0, 1]$, given by the formula $\mu_\lambda(A) = \sup\{\lambda(s); s \in A\}$, for all $A \subset \Omega$, $A \neq \emptyset$, $\mu_\lambda(\emptyset) = 0$ (for the definition and the properties of the measures of possibility, see, e.g., [27], Chapter 1).

2.1.3 Main Results

Let $\mathcal{B}_{S^d}$ be the sigma algebra of all Borel measurable subsets in $\mathcal{P}(S^d)$ and $\mu : \mathcal{B}_{S^d} \to [0, +\infty)$ be a normalized, monotone and submodular set function on $\mathcal{B}_{S^d}$.

We say that μ is strictly positive if $\mu(A \cap S^d) > 0$, for every open set $A \subset \mathbb{R}^n$ with $A \cap S^d \neq \emptyset$.

Also, by definition, the support of μ, denoted by *supp*(μ), is the set of all $x \in S^d$ with the property that for every open neighborhood $N_x \in \mathcal{B}_{S^d}$ of x, we have $\mu(N_x) > 0$.

Denote by $C_+(S^d)$ the space of all positive-valued continuous functions on S^d and by $L^\infty_\mu(S^d)$ the space of all real-valued $\mathcal{B}_{S^d}$-measurable functions f, such that there exists a set $E \subset S^d$ (depending on f) with $\mu(E) = 0$ and f is bounded on $S^d \setminus E$.

Denote

$$M_{n,\mu}(f)(x) = \sum_{|\alpha|=n} c(\alpha,\mu) \cdot B_\alpha(x),\ x \in S^d,\ n \in \mathbb{N},$$

where applying Remark 2.2, (i), we easily get

$$c(\alpha,\mu) = \frac{(C)\int_{S^d} f(t)B_\alpha(t)d\mu(t)}{(C)\int_{S^d} B_\alpha(t)d\mu(t)} = \frac{(C)\int_{S^d} f(t)P_\alpha(t)d\mu(t)}{(C)\int_{S^d} P_\alpha(t)d\mu(t)}.$$

It is worth noting here that we did not loose any generality by the normalization condition on the set valued function μ and that the condition *supp*$(\mu) \setminus \partial S^d \neq \emptyset$, guarantees that $(C)\int_{S^d} B_\alpha(t)d\mu(t) > 0$, for all B_α.

For the proof of the main results, we need the following auxiliary result.

Lemma 2.1.2. (Gal-Opriş [44]) *Let us suppose that μ is a normalized, monotone and submodular set function. If we define $T_n : C_+(S^d) \to \mathbb{R}_+$ by*

$$T_n(f) = (C)\int_{S^d} f(t)P_\alpha(t)d\mu(t), f \in C_+(S^d), n \in \mathbb{N},\ |\alpha| = n,$$

then for all $f, g \in C_+(S^d)$, we have

$$|T_n(f) - T_n(g)| \leq T_n(|f-g|) = (C)\int_{S^d} |f(t)-g(t)| \cdot P_\alpha(t)d\mu(t).$$

The first main result is an analogous result to Theorem 1 in [11] and refers to uniform approximation.

Theorem 2.1.3. (Gal-Opriş [44]) *Let μ be a normalized, monotone, submodular and strictly positive set function on $\mathcal{B}_{S^d}$, such that supp$(\mu) \setminus \partial S^d \neq \emptyset$. For every $f \in C_+(S^d)$ we have*

$$\lim_{n\to\infty} \|M_{n,\mu}(f) - f\|_{C(S^d)} = 0,$$

where $\|F\|_{C(S^d)} = \max\{|F(x)|; x \in S^d\}$.

The second main result is an analogue result to Theorem 1 in [12] and refers to pointwise convergence. In this sense, analysing the reasonings in the proof of Theorem 1 in [12] and using the same properties of the Choquet integral as in the proof of the above Theorem 2.1.3, we easily get the following.

Theorem 2.1.4. (Gal-Opriş [44]) *Let μ be a normalized, monotone, submodular set function on $\mathcal{B}_{S^d}$, such that supp$(\mu) \setminus \partial S^d \neq \emptyset$. If $f \in L^\infty_\mu(S^d)$ and $f(x) \geq 0$, for all $x \in S^d$, then at any point $x \in$ supp(μ) where f is continuous, we have*

$$\lim_{n\to\infty} |M_{n,\mu}(f)(x) - f(x)| = 0.$$

Remarks. 1) According to the previous Remark, (vi), an example of submodular set function μ satisfying all the requirements in the statements of Theorems 2.1.3 and 2.1.4, can simply be defined by $\mu(A) = \sqrt{\nu(A)}$, where ν is a Borel probability measure as in [11] and [12]. Also, it is worth noting that due to the nonlinearity of the Choquet integral (see Remark (ii)), unlike the case in [11], [12], the Bernstein-Durrmeyer operator in the present paper is nonlinear.

2) The positivity of function f in Theorems 2.1.3 and 2.1.4 is necessary because of the positive homogeneity of the Choquet integral applied in the proof. However, if f is of arbitrary sign on S^d, then it is immediate that the statements of Theorems 2.1.3 and 2.1.4 can be restated for the slightly modified Bernstein-Durrmeyer operator defined by

$$M^*_{n,\mu}(f)(x) = M_{n,\mu}(f - m)(x) + m,$$

where $m \in \mathbb{R}$ is a lower bound for f, that is $f(x) \geq m$, for all $x \in S^d$.

2.2 Approximation by possibilistic integral operators

By analogy with the Feller's general probabilistic scheme used in the construction of many classical convergent sequences of linear operators, in this paper we consider a Feller-kind scheme based on the possibilistic integral, for the construction of convergent sequences of nonlinear operators. As particular cases, in the discrete case, all the so-called max-product Bernstein type operators and their qualitative convergence properties are recovered. In addition, discrete non-discrete nonlinear possibilistic convergent operators of Picard type, Gauss-Weierstrass type and Poisson-Cauchy type are considered.

2.2.1 Introduction

In the very recent paper [32], the so-called max-product operators of Bernstein, of Favard-Szász-Mirakjan kind, of Baskakov kind, of Bleimann-Butzer-Hahn kind and of Meyer-König-Zeller kind (whose quantitative approximation properties were intensively studied in many previously published papers, see, e.g., [8], [9], [17]-[20] and the References in [32]), were naturally interpreted as possibilistic expectations of particular discrete fuzzy variables having various possibilistic distributions. By using the Bernstein's idea in [14], (see also the more accessible paper [51]), but based on a Chebyshev-type inequality in possibility theory, these interpretations allowed to obtain for them qualitative convergence results.

It is worth mentioning here that possibility theory is a well-established mathematical theory dealing with certain types of uncertainties and is considered as an alternative to probability theory (see, e.g., [27], [22]) .

The main aim of this section is to present the well-known Feller's probabilistic scheme in approximation, in the setting of possibility theory. In particular, this scheme will allow not just another natural approach of the max-product operators, but also to introduce and study many other possibilistic approximation operators too.

Firstly, let us recall that a classical scheme in constructing linear and positive approximation operators, is the Feller's probabilistic scheme (see [29], Chapter 7, or more detailed, [3], Section 5.2, pp. 283-319). Described shortly, it consists in attaching to a continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$, approximation operators of the form

$$L_n(f)(x) = \int_{\Omega} f \circ Z(n,x) dP = \int_{\mathbb{R}} f dP_{Z(n,x)},$$

where P is a probability on the measurable space $(\Omega, \mathcal{C})$, $Z : \mathbb{N} \times I \to \mathcal{M}_2(\Omega)$, with I a subinterval of $\mathbb{R}$, $\mathcal{M}_2(\Omega)$ represents the space of all random variables whose square is integrable on Ω with respect to the probability P and $P_{Z(n,x)}$ denotes the distribution of the random variable $Z(n,x)$ with respect to P defined by $P_{Z(n,x)}(B) = P(Z^{-1}(n,x)(B))$, for al B-Borel measurable subset of $\mathbb{R}$. Then, denoting by $E(Z(n,x))$ and $Var(Z(n,x))$ the expectance and the variance of the random variable $Z(n,x)$, respectively, and supposing that $\lim_{n\to\infty} E(Z(n,x)) = x$, $\lim_{n\to\infty} Var(Z(n,x)) = 0$, uniformly on I, it is proved that for all f as above, $L_n(f)$ converges to f uniformly on each compact subinterval of I.

In addition, if for the random variable $Z(n,x)$, its probability density function $\lambda_{n,x}$ is known, then for any f we can write

$$\int_{\mathbb{R}} f dP_{Z(n,x)} = \int_{\mathbb{R}} f(t) \cdot \lambda_{n,x}(t) dP(t),$$

formula which is useful in the concrete construction of the approximation operators $L_n(f)(x)$.

In the very recent paper [33], the Feller's scheme was generalized to the case when the above classical integral is replaced with the nonlinear Choquet integral with respect to a monotone and subadditive set function.

By analogy with the above considerations, in the next subsection we consider a Feller kind scheme based on the possibilistic integral, for the construction of convergent sequences of nonlinear operators. In particular, in the discrete case, all the so-called max-product Bernstein type operators and their qualitative convergence are reobtained through this scheme. In Section 3, new discrete nonperiodic nonlinear possibilistic convergent operators of Picard type, Gauss-Weierstrass type and Poisson-Cauchy type suggested by Feller's scheme are considered. At the end, for future studies we consider discrete periodic(trigonometric) nonlinear possibilistic operators of de la Vallée-Poussin type, of Fejér type and of Jackson type.

2.2.2 Feller's scheme in terms of possibilistic integral

Firstly we summarize some known concepts for the discrete or non-discrete fuzzy variables in possibility theory, which will be useful in the next section. As it is easily seen, in fact they are the corresponding concepts for those in probability theory, like random variable, probability distribution, mean value, probability, so on. For details, see, e.g., [27] or [22].

Definition 2.2.1. Let Ω be a non-empty, discrete (i.e. at most countable) or non-discrete set.

(i) A fuzzy variable X is an application $X : \Omega \to \mathbb{R}$. If Ω is a discrete set, then X is called discrete fuzzy variable. If Ω is finite then X is called a finite fuzzy variable. If Ω is not discrete, then X is called non-discrete fuzzy variable.

(ii) A possibility distribution (on Ω), is a function $\lambda : \Omega \to [0, 1]$, such that $\sup\{\lambda(s); s \in \Omega\} = 1$.

(iii) The possibility expectation of a fuzzy variable X (on Ω), with the possibility distribution λ is defined by $M_{sup}(X) = \sup_{s\in\Omega} X(s)\lambda(s)$. The possibility variance of X is $V_{sup}(X) = \sup\{(X(s) - M_{sup}(X))^2\lambda(s); s \in \Omega\}$.

(iv) If Ω is a non-empty set, then a possibility measure is a mapping $P : \mathcal{P}(\Omega) \to [0, 1]$, satisfying the axioms $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(\bigcup_{i\in I} A_i) = \sup\{P(A_i); i \in I\}$ for all $A_i \in \Omega$, and any I, an at most countable family of indices (if Ω is finite then obviously I must be finite too). Note that if $A, B \subset \Omega$, satisfy $A \subset B$, then by the last property it easily follows that $P(A) \leq P(B)$ and that $P(A \bigcup B) \leq P(A) + P(B)$.

It is well-known (see, e.g., [27]) that any possibility distribution λ on Ω, induces a possibility measure $P_\lambda : \mathcal{P}(\Omega) \to [0, 1]$, given by the formula $P_\lambda(A) = \sup\{\lambda(s); s \in A\}$, for all $A \subset \Omega$.

For each fuzzy (possibilistic) variable $X : \Omega \to \mathbb{R}$, we can define its distribution measure with respect to a possibility measure P induced by a possibility distribution λ, by the formula

$$P_X : \mathcal{B} \to \mathbb{R}_+,\ P_X(B) = P(X^{-1}(B)) = P(\{\omega \in \Omega; X(\omega) \in B\}),\ B \in \mathcal{B},$$

where $\mathbb{R}_+ = [0, +\infty)$ and $\mathcal{B}$ is the class of all Borel measurable subsets in $\mathbb{R}$. It is clear that P_X is a possibility measure on $\mathcal{B}$, induced by the possibility distribution defined by

$$\lambda_X^* : \mathbb{R} \to [0,1],\ \lambda_X^*(t) = \sup\{\lambda(\omega); \omega \in X^{-1}(t)\}, \quad \text{if } X^{-1}(t) \neq \emptyset,$$

$$\lambda_X^*(t) = 0, \quad \text{if } X^{-1}(t) = \emptyset.$$

(v) (see, e.g., [22]) The possibilistic integral of $f : \Omega \to \mathbb{R}_+$ on $A \subset \Omega$, with respect to the possibilistic measure P_λ induced by the possibilistic distribution λ, is defined by

$$(Pos)\int_A f(t)dP_\lambda(t) = \sup\{f(t)\cdot\lambda(t); t \in A\}.$$

It is clear that this definition is a particular case of the possibilistic integral with respect to a semi-norm t, introduced in [22], by taking there $t(x, y) = x\cdot y$. Also, denoting $\Lambda_1 : \Omega \to [0,1]$, $\Lambda_1(x) = 1$, for all $x \in \Omega$, it is immediate that we can write

$$(Pos)\int_A f(t)dP_{\Lambda_1}(t) = \sup\{f(t); t \in A\},$$

$$(Pos)\int_A f(t)dP_\lambda(t) = (Pos)\int_A f(t)\cdot\lambda(t)dP_{\Lambda_1}$$

and $dP_\lambda(t) = \lambda(t)\cdot dP_{\Lambda_1}(t)$.

It is also worth noting that the above definition of the concept of possibilistic integral has a good sense only for positive-valued functions, because, for example, if we denote $\mathbb{R}_- = (-\infty, 0]$, then for any $f : \Omega \to \mathbb{R}_-$ with $f(\omega_0) = 0$ for a certain $\omega_0 \in A \subset \Omega$, we get $(Pos)\int_A f(t)dP_\lambda(t) = 0$.

In what follows, we also need in the frame of the possibility theory, a simple analogue of the Chebyshev's inequality in probability theory.

Theorem 2.2.2. (see [32]) *Let* Ω *be a discrete or non-discrete non-empty set,* $\lambda : \Omega \to [0, 1]$ *and consider* $X : \Omega \to \mathbb{R}$ *be with the possibility distribution* λ*. Then, for any* $r > 0$*, we have*

$$P_\lambda(\{s \in \Omega; |X(s) - M_{sup}(X)| \geq r\}) \leq \frac{V_{sup}(X)}{r^2},$$

where P_λ *is the possibilistic measure induced by* λ*.*

This result was proved by Theorem 2.2 in [32] for Ω discrete set, but analysing its proof it is obvious that it remains valid in the non-discrete case too.

In the particular case when $X : \Omega \to \mathbb{R}_+$, in terms of the possibility integral, the above Chebyshev inequality can be written as

$$P_\lambda(\{s \in \Omega; |X(s) - (Pos)\int_\Omega X(t)dP_\lambda(t)| \geq r\})$$

$$\leq \frac{(Pos)\int_\Omega (X - (Pos)\int_\Omega X(t)dP_\lambda(t))^2 dP_\lambda}{r^2}.$$

In what follows, by analogy with the Feller's random scheme in probability theory which produces nice linear and positive approximation operators, we will consider a similar approximation scheme, but which will produce nonlinear approximation operators constructed with the aid of the possibilistic integral.

For that purpose, let us denote by $Var^b(\Omega)$ the class of all bounded $X : \Omega \to \mathbb{R}$ and by $Var_+^b(\Omega)$ the class of all bounded $X : \Omega \to \mathbb{R}_+$. Also, for $I \subset \mathbb{R}$ a real interval (bounded or unbounded), let us consider the mapping Z defined on $\mathbb{N} \times I \to Y$ where $Y = Var^b(\Omega)$ or $Y = Var_+^b(\Omega)$, depending on the context.

Notice that if for any $(n, x) \in \mathbb{N} \times I$ we have $Z(n, x) \in Var_{+}^{b}(\Omega)$, then for the concepts of possibility expectation and possibility variance of $Z(n, x)$ (defined at the above Definition 2.1, (iii)) we can write the integral formulas

$$M_{sup}(Z(n,x)) = (Pos) \int_{\Omega} Z(n,x)(t) dP_{\lambda}(t) := \alpha_{n,x}, \tag{2.2}$$

$$V_{sup}(Z(n,x)) = (Pos) \int_{\Omega} (Z(n,x)(t) - \alpha_{n,x})^2 dP_{\lambda}(t) := \sigma_{n,x}^2. \tag{2.3}$$

Now, according to the Feller's scheme, to $f : \mathbb{R} \to \mathbb{R}_{+}$ let us attach a sequence of operators by the formula

$$L_n(f)(x) := (Pos) \int_{\mathbb{R}} f(t) dP_{Z(n,x)}(t), \ n \in \mathbb{N}, \ x \in I, \tag{2.4}$$

where $P_{Z(n,x)}$ is defined as in Definition 2.1, (iv), i.e. with respect to the possibility measure P_{λ} induced by the possibility distribution λ.

Firstly, for the operators given by (2.4) the following representation holds.

Lemma 2.2.3. (Coroianu-Gal-Opriş-Trifa [21]) *With the above notations, if $Z : \mathbb{N} \times I \to Var^{b}(\Omega)$ and, in addition, $f : \mathbb{R} \to \mathbb{R}_{+}$ is bounded on $\mathbb{R}$, then the formula*

$$L_n(f)(x) = (Pos) \int_{\mathbb{R}} f(t) dP_{Z(n,x)}(t) = (Pos) \int_{\Omega} f \circ Z(n,x) dP_{\lambda}, \ x \in I \tag{2.5}$$

holds and both integrals are finite.

If $f : I \to \mathbb{R}_{+}$ is bounded on I, where $I \subset \mathbb{R}$ is a subinterval and $P_{\lambda}(\{\omega \in \Omega; Z(n,x)(\omega) \notin I\}) = 0$, then we have

$$L_n(f)(x) = (Pos) \int_{\mathbb{I}} f(t) dP_{Z(n,x)}(t) = (Pos) \int_{\Omega} f \circ Z(n,x) dP_{\lambda}.$$

Remark. Explicitly, formula (2.5) can be written as

$$L_n(f)(x) = \sup\{f(t) \cdot \lambda_{Z(n,x)}^{*}(t); t \in \mathbb{R}\} = \sup\{f[Z(n,x)(t)] \cdot \lambda(t); t \in \Omega\},$$

where $\lambda^*_{Z(n,x)}(t)$is defined with respect to λ as in Definition 2.2.1, (iv).

Since the next main result will involve the quantity $\alpha_{n,x}$ given by formula (2.2), it will be necessarily to suppose that $Z(n,x) \in Var^b_+(\Omega)$.

The following Feller-type result holds.

Theorem 2.2.4. (Coroianu-Gal-Opriş-Trifa [21]) *Let $I \subset \mathbb{R}$ be a subinterval, $Z(n,x) \in Var^b_+(\Omega)$ for all $(n,x) \in \mathbb{N} \times I$ and let us suppose that $f : \mathbb{R} \to \mathbb{R}_+$ is uniformly continuous and bounded on $\mathbb{R}$. With the notations in the formulas (2.2), (2.3) and in the statement of Lemma 2.3, if $\lim_{n\to+\infty} \alpha_{n,x} = x$ and $\lim_{n\to+\infty} \sigma^2_{n,x} = 0$, uniformly with respect to $x \in I$, then $\lim_{n\to\infty} L_n(f)(x) = f(x)$, uniformly with respect to $x \in I$.*

Remarks. 1) Analyzing the proof of Theorem 2.2.4, it easily follows that without any change in its proof, the construction of the operators $L_n(f)(x)$ can be slightly generalized by considering that not just Z depends on n and x, but also that λ (and consequently P_λ too) may depend on n and x. More exactly, we can consider $L_n(f)(x)$ of the more general form

$$L_n(f)(x) := (Pos)\int_{\mathbb{R}} f(t)dP_{Z(n,x)}(t) = (Pos)\int_{\Omega} f \circ Z(n,x)dP_{\lambda_{n,x}},\ x \in I,$$

where $P_{\lambda_{n,x}} : \mathcal{P}(\Omega) \to [0,1]$, $(n,x) \in \mathbb{N}\times I$, is a family of possibility measures induced by the families of distributions $\lambda_{n,x}$, $(n,x) \in \mathbb{N} \times I$. This remark is useful in producing several concrete examples of such operators.

Also, let us note here that if we suppose that $P_\lambda(\{\omega \in \Omega; Z(n,x)(\omega) \notin I\} = 0$, then the operators L_n can be attached to continuous, bounded functions defined on a subinterval $I \subset \mathbb{R}$, $f : I \to \mathbb{R}_+$, by extending f to a function continuous and bounded, $f^* : \mathbb{R} \to \mathbb{R}_+$ and taking into account the obvious relationship

$$(Pos)\int_{\mathbb{R}} f^* dP_{Z(n,x)} = (Pos)\int_I f dP_{Z(n,x)}.$$

2) If $f : I \to \mathbb{R}$ is not necessarily positive, but bounded, then evidently that there exists a constant $c > 0$ such that $f(x) + c \geq 0$, for all $x \in I$ and in this case, for $n \in \mathbb{N}$, we can attach to f the approximation operators

$$L_n(f)(x)$$

$$= (Pos)\int_{\mathbb{I}}(f(t)+c)dP_{Z(n,x)}(t) - c = (Pos)\int_{\Omega}(f+c)\circ Z(n,x)dP_{\lambda_{n,x}} - c.$$

3) As particular cases of operators for which qualitative approximation properties can be derived by the Feller's scheme in Theorem 2.2.4, are all the so-called max-product Bernstein-type operators. Thus, for example, if we take $\Omega = \{0, 1, ..., n\}$, $I = [0, 1]$, $Z(n, x)(k) = \frac{k}{n}$, $f : [0, 1] \to \mathbb{R}_+$, $\lambda_{n,x}(k) = \frac{p_{n,k}(x)}{\bigvee_{j=0}^{n} p_{n,j}(x)}$, with $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ and $\bigvee_{j=0}^{n} p_{n,j}(x) = \max_{j=\{0,...,n\}}\{p_{n,j}(x)\}$, then by the formula in Lemma 2.2.3 and by the definition of the possibility integral, we get

$$L_n(f)(x) = (Pos)\int_{\Omega} f\circ Z(n,x)dP_{\lambda_{n,x}} = \frac{\bigvee_{k=0}^{n} p_{n,k}(x)f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)},$$

which are exactly the max-product Bernstein operators $B_n^{(M)}(f)(x)$. The qualitative approximation properties of $B_n^{(M)}(f)(x)$ can follow now from Theorem 2.2.4.

Analogously, if, for example, we take the countable $\Omega = \{0, 1, ..., k, ..., \}$ and $P_{\lambda_{n,x}}$ the possibility measure induced by the possibility distribution

$$\lambda_{n,x}(k) = \frac{s_{n,k}(x)}{\bigvee_{k=0}^{\infty} s_{n,k}(x)}, x \in [0, +\infty), k \in \mathbb{N}\bigcup\{0\},$$

with $s_{n,k}(x) = \frac{(nx)^k}{k!}$ and $\bigvee_{k=0}^{\infty} s_{n,k}(x) = \max_{k=\{0,1,...,k,...\}}\{s_{n,k}(x)\}$, then the formula in Lemma 2.3 gives the max-product Favard-Szász-Mirakjan operators.

In a similar way, from Theorem 2.2.4 can be obtained qualitative approximation properties for the other max-product operators, like those of Baskakov kind, of Bleimann-Butzer-Hahn kind and of Meyer-König-Zeller kind.

It is worth nothing that by using other (direct) methods, quantitative estimates in approximation by max-product type operators were obtained by the first two authors in a long series of papers (see, e.g., [8], [9], [17]-[20] and their References).

2.2.3 Approximation by convolution possibilistic operators

In this subsection, by using the above possibilistic Feller's scheme, we introduce and study possibilistic variants of the classical linear convolution operators of Picard, Gauss-Weierstrass and Poisson-Cauchy, formally given by the formulas

$$P_n(f)(x) = \frac{n}{2}\int_{\mathbb{R}} f(t)e^{-n|x-t|}dt, \quad W_n(f)(x) = \frac{\sqrt{n}}{\sqrt{\pi}}\int_{\mathbb{R}} f(t)e^{-n|t-x|^2}dt,$$

$$Q_n(f)(x) = \frac{n}{\pi}\int_{\mathbb{R}} \frac{f(t)}{n^2(t-x)^2+1},$$

respectively, where $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Denoting $\Omega = \{0, 1, ..., k, ..., \}$ and $Z(n, x)$ as in the previous Remark 3 and defining $\lambda_{n,x}(k) = \frac{e^{-n|x-k/n|}}{\bigvee_{k=-\infty}^{\infty} e^{-n|x-k/n|}}$, by the formula in Lemma 2.3

$$L_n(f)(x) = (Pos)\int_{\Omega} f \circ Z(n, x)dP_{\lambda_{n,x}},$$

we obtain the following discrete possibilistic (max-product !) Picard operators

$$P_n^{(M)}(f)(x) = \frac{\bigvee_{k=-\infty}^{+\infty} f(k/n)\cdot e^{-n|x-k/n|}}{\bigvee_{k=-\infty}^{+\infty} e^{-n|x-k/n|}}.$$

Similarly, for $\lambda_{n,x}(k) = \frac{e^{-n(x-k/n)^2}}{\bigvee_{k=-\infty}^{\infty} e^{-n(x-k/n)^2}}$ and $\lambda_{n,x}(k) = \frac{1/(n^2(x-k/n)^2+1)}{\bigvee_{k=0}^{\infty} 1/(n^2(x-k/n)^2+1)}$ we obtain the following discrete possibilistic (max-product !) operators,

$$W_n^{(M)}(f)(x) = \frac{\bigvee_{k=-\infty}^{+\infty} f(k/n) \cdot e^{-n(x-k/n)^2}}{\bigvee_{k=-\infty}^{+\infty} e^{-n(x-k/n)^2}}, \text{ - of Gauss-Weierstrass kind,}$$

$$Q_n^{(M)}(f)(x) = \frac{\bigvee_{k=-\infty}^{+\infty} f(k/n) \cdot \frac{1}{n^2(x-k/n)^2+1}}{\bigvee_{k=-\infty}^{+\infty} \frac{1}{n^2(x-k/n)^2+1}}, \text{ - of Poisson-Cauchy kind.}$$

Let us denote by $BUC_+(\mathbb{R})$, the space of all uniformly continuous, bounded and with positive values functions. The convergence of these operators can be proved by using Theorem 2.2.4. However, we can obtain quantitative estimates too, by direct proofs, as follows.

Theorem 2.2.5. (Coroianu-Gal-Opriş-Trifa [21]) *For all $f \in BUC_+(\mathbb{R})$ we have*

$$|P_n^{(M)}(f)(x) - f(x)| \leq 2 \cdot \omega_1(f; 1/n)_{\mathbb{R}}.$$

We also can consider truncations of the operator $P_n^{(M)}$. In this sense, we can state the following.

Corollary 2.2.6. (Coroianu-Gal-Opriş-Trifa [21]) *Let $(m(n))_{n\in\mathbb{N}}$ be a sequence of natural numbers with the property that $\lim_{n\to\infty} \frac{m(n)}{n} = +\infty$ and for $f \in BUC_+(\mathbb{R})$ let us define*

$$T_n^{(M)}(f)(x) = \frac{\bigvee_{k=-m(n)}^{+m(n)} f(k/n) \cdot e^{-n|x-k/n|}}{\bigvee_{k=-m(n)}^{+m(n)} e^{-n|x-k/n|}}.$$

Then, $T_n^{(M)}(f)$ converges uniformly (as $n \to \infty$) to f, on any compact subinterval of the form $[-A, A]$, $A > 0$.

In what follows, similar results we present for the other possibilistic operators, $W_n^{(M)}(f)(x)$, $Q_n^{(M)}(f)(x)$ and their corresponding truncated operators given by

$$S_n^{(M)}(f)(x) = \frac{\bigvee_{k=-m(n)}^{+m(n)} f(k/n) \cdot e^{-n(x-k/n)^2}}{\bigvee_{k=-m(n)}^{+m(n)} e^{-n(x-k/n)^2}}$$

and

$$U_n^{(M)}(f)(x) = \frac{\bigvee_{k=-m(n)}^{+m(n)} f(k/n) \cdot \frac{1}{n^2(x-k/n)^2+1}}{\bigvee_{k=-m(n)}^{+m(n)} \frac{1}{n^2(x-k/n)^2+1}}.$$

Theorem 2.2.7. (Coroianu-Gal-Opriş-Trifa [21]) *For all* $f \in BUC_+(\mathbb{R})$ *we have*

$$|W_n^{(M)}(f)(x) - f(x)| \leq 2 \cdot \omega_1(f; 1/\sqrt{n})_{\mathbb{R}}.$$

Corollary 2.2.8. (Coroianu-Gal-Opriş-Trifa [21]) *Let* $(m(n))_{n\in\mathbb{N}}$ *be a sequence of natural numbers with the property that* $\lim_{n\to\infty} \frac{m(n)}{n} = +\infty$. *Then, for any* $f \in BUC_+(\mathbb{R})$, $S_n^{(M)}(f)$ *converges uniformly (as* $n \to \infty$*) to* f*, on any compact subinterval of the form* $[-A, A]$, $A > 0$ *(*$S_n^{(M)}(f)$ *is defined just above the statement of Theorem 2.2.7).*

Theorem 2.2.9. (Coroianu-Gal-Opriş-Trifa [21]) *For all* $f \in BUC_+(\mathbb{R})$ *we have*

$$|Q_n^{(M)}(f)(x) - f(x)| \leq 2 \cdot \omega_1(f; 1/(2n))_{\mathbb{R}}.$$

Corollary 2.2.10. (Coroianu-Gal-Opriş-Trifa [21]) *Let* $(m(n))_{n\in\mathbb{N}}$ *be a sequence of natural numbers with the property that* $\lim_{n\to\infty} \frac{m(n)}{n} = +\infty$. *Then, for any* $f \in BUC_+(\mathbb{R})$, $U_n^{(M)}(f)$ *converges uniformly (as* $n \to \infty$*) to* f*, on any compact subinterval of the form* $[-A, A]$, $A > 0$ *(*$U_n^{(M)}(f)$ *is defined just above the statement of Theorem 2.2.7).*

Remarks. 1) We note that in [28] Favard introduced the discrete version of the above Gauss-Weierstrass singular integral by the formula

$$\mathcal{F}_n(f)(x) = \frac{1}{\sqrt{\pi n}} \cdot \sum_{k=-\infty}^{+\infty} f(k/n) \cdot e^{-n(x-k/n)^2}, n \in \mathbb{N}, x \in \mathbb{R}$$

and proved that if $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$, of the exponential growth $|f(t)| \leq Me^{At^2}$ for all $t \in \mathbb{R}$ (here $M, A > 0$), then $\mathcal{F}_n(f)(x)$ converges to $f(x)$ pointwise for each $x \in R$ and uniformly on any compact subinterval of $\mathbb{R}$. Other approximation properties of $\mathcal{F}_n(f)(x)$, especially in various weighted spaces, were studied in many papers, see, e.g., [1] and the References therein.

Exactly as it was proved for other max-product operators studied in previous papers (see, e.g., [17]-[20]), with respect to its linear counterpart $\mathcal{F}_n(f)(x)$, for the max-product operators $W_n^{(M)}(f)(x)$ can be proved that in some subclasses of functions f, have better global approximation properties and that present much stronger localization results. More precisely, they represent locally much better (probably best possible) the approximated function, in the sense that if f and g coincides on a strict subinterval $I \subset \mathbb{R}$, then for any subinterval I_0 strictly included in I, $W_n^{(M)}(f)$ and $W_n^{(M)}(g)$ coincide in I_0 for sufficiently large n.

2) By using the above possibilistic Feller's scheme, we can introduce for study possibilistic variants of the classical linear convolution trigonometric operators of de la Vallée-Poussin, Fejér and Jackson, formally defined by the formulas

$$V_n(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)k_n(x-t)dt, \ F_n(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)b_n(x-t)dt,$$

$$J_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)c_n(x-t)dt,$$

respectively, where f is 2π-periodic,

$$k_n(t) = \frac{(n!)^2}{(2n)!}\left(2\cos(t/2)\right)^{2n}, \ b_n(t) = \frac{1}{n}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2$$

and $c_n(t) = \frac{3}{2n(2n^2+1)}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^4$.

More precisely, denoting $\Omega = \{-n, ..., -1, 0, 1, ..., n\}$ and $Z_{n,x}(k) = \frac{k\pi}{n}$, for $f : [-\pi, \pi] \to \mathbb{R}$ and $\lambda_{n,x}(k) = \frac{k_n(x-k\pi/n)}{\bigvee_{k=-n}^{n} k_n(x-k\pi/n)}$, by the formula in Lemma 2.2.3 and by the definition of the possibility integral, we get the possibilistic de la Vallée-Poussin operators

$$V_n^{(M)}(f)(x) = (Pos) \int_{\Omega} f \circ Z(n,x) dP_{\lambda_{n,x}} = \frac{\bigvee_{k=-n}^{n} f(k\pi/n) k_n(x - k\pi/n)}{\bigvee_{k=-n}^{n} k_n(x - k\pi/n)}.$$

Similarly, we can obtain the possibilistic operators of Fejér type

$$F_n^{(M)}(f)(x) = \frac{\bigvee_{k=-n}^{n} f(k\pi/n) b_n(x - k\pi/n)}{\bigvee_{k=-n}^{n} b_n(x - k\pi/n)}$$

and of Jackson type

$$J_n^{(M)}(f)(x) = \frac{\bigvee_{k=-n}^{n} f(k\pi/n) c_n(x - k\pi/n)}{\bigvee_{k=-n}^{n} c_n(x - k\pi/n)}.$$

The study of the approximation properties of these operators will be made elsewhere.

Ch. 3

Approximation with an arbitrary order by Szász and Baskakov operators of real variable

Given an arbitrary sequence $\lambda_n > 0$, $n \in \mathbb{N}$, with the property that $\lim_{n\to\infty} \lambda_n = 0$ as fast we want, in this chapter we introduce modified/generalized Baskakov operators in such a way that on each compact subinterval in $[0, +\infty)$ the order of uniform approximation is $\omega_1(f; \sqrt{\lambda_n})$.

The idea of construction of these generalized operators is simple : in their classical formulas, we replace everywhere n with $\frac{1}{\lambda_n}$.

For example, starting from the classical formula for the Szász operators

$$S_n(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(k/n),$$

by replacing n by $\frac{1}{\lambda_n}$ we get the generalized Szász operator

$$S_n(f;\lambda_n)(x) = e^{-x/\lambda_n} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{\lambda_n}\right)^k \cdot f\left(k\lambda_n\right),$$

while starting from the classical formula for the Baskakov operator

$$V_n(f)(x) = (1+x)^{-n} \sum_{j=0}^{\infty} \binom{n+j-1}{j} \left(\frac{x}{1+x}\right)^j \cdot f\left(\frac{j}{n}\right)$$

$$= (1+x)^{-n} \sum_{j=0}^{\infty} \frac{(n+j-1)!}{(n-1)!j!} \left(\frac{x}{1+x}\right)^j \cdot f\left(\frac{j}{n}\right)$$

$$= (1+x)^{-n} \sum_{j=0}^{\infty} \frac{1}{j!} \cdot n(n+1) \cdot \ldots \cdot (n+j-1) \left(\frac{x}{1+x}\right)^j \cdot f\left(\frac{j}{n}\right),$$

by replacing n by $\frac{1}{\lambda_n}$ we get the modified/generalized Baskakov operator

$$V_n(f;\lambda_n)(x)$$

$$= (1+x)^{-1/\lambda_n} \sum_{j=0}^{\infty} \frac{1}{j!} \cdot \frac{1}{\lambda_n} \left(1+\frac{1}{\lambda_n}\right) \cdot \ldots \cdot \left(j-1+\frac{1}{\lambda_n}\right) \cdot \left(\frac{x}{1+x}\right)^j f\left(j\lambda_n\right).$$

3.1 Generalized Baskakov operators on $\mathbb{R}_+$

Given an arbitrary sequence $\lambda_n > 0$, $n \in \mathbb{N}$, with the property that $\lim_{n\to\infty} \lambda_n = 0$ as fast we want, in this section we introduce generalized Baskakov operators in such a way that on each compact subinterval in $[0,+\infty)$ the order of uniform approximation is $\omega_1(f;\sqrt{\lambda_n})$. These modified operators can uniformly approximate a Lipschitz 1 function, on each compact subinterval of $[0,\infty)$ with the arbitrary good order of approximation $\sqrt{\lambda_n}$. Also, similar considerations are made for modified/generalized q_n-Baskakov operators, with $0 < q_n < 1$, $\lim_{n\to\infty} q_n = 1$.

3.1.1 Introduction

Let $(\lambda_n)_n$ be a sequence of real positive numbers with the properties that $\lim_{n\to\infty} \lambda_n = 0$.

In [15] Cetin and Ispir introduced a remarkable generalization of the Szász-Mirakjan type operators attached to analytic functions f of exponential growth in a compact disk,

$$S_n(f;\lambda_n)(z) = e^{-z/\lambda_n} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{z}{\lambda_n}\right)^k \cdot f(k\lambda_n),$$

which approximate f in any compact disk $|z| \leq r$, $r < R$, with the approximation order λ_n.

Involving in their construction the Faber polynomials too, these operators and their order of approximation were extended in Gal [36] in order to approximate analytic functions in compact subsets (continuums) of the complex plane. The great advantage of all these constructions is that the sequence λ_n, $n \in \mathbb{N}$, can evidently be chosen to converge to zero with an arbitrary small order. Note that in fact, all the above mentioned results were obtained for λ_n written in the unnecessary more complicated form, $\lambda_n = \frac{\beta_n}{\alpha_n}$.

The first main aim of this section is to introduce and study the linear and positive modified/generalized Baskakov-type operators, defined by

$$L_n(f;\lambda_n)(x) = \sum_{j=0}^{\infty} \frac{(-1)^j \varphi^{(j)}(\lambda_n;x) x^j}{j!} f(j\lambda_n), \tag{3.1}$$

for functions $f : [0,b) \to \mathbb{R}$ (here b can be $+\infty$ too) such that the above series converges (e.g. if f is bounded or uniformly continuous on $[0,b)$), where the sequence of analytic functions $\varphi_n : [0,b) \to \mathbb{R}$, $n \in \mathbb{N}$, satisfy the hypothesis : (i) $\varphi(\lambda_n;0) = 1$; (ii) $(-1)^j \varphi^{(j)}(\lambda_n;x) \geq 0$, for all $n, j \in \mathbb{N}$, $x \in [0,b]$.

It is worth noting that for the particular case $\lambda_n = \frac{1}{n}$ and under the additional hypothesis

(iii) there exists a sequence $m(n)$, $n \in \mathbb{N}$ with $\lim_{n\to\infty} \frac{n}{m(n)} = 1$ such that $\varphi_n^{(k)}(\lambda_n; x) = -n\varphi_n^{(k-1)}(\lambda_n; x)$, for all $x \in [0, b)$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, the operators in (3.1) were introduced and investigated in Baskakov [7].

Choosing $\varphi(\lambda_n; x) = (1+x)^{-1/\lambda_n}$ in (3.1), because of the formula

$$\varphi^{(j)}(\lambda_n; x) = (-1)^j \frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\cdot \ldots \cdot \left(j-1+\frac{1}{\lambda_n}\right)\cdot (1+x)^{-j-1/\lambda_n}, \quad (3.2)$$

we immediately get the modified/generalized Baskakov-type operators defined by

$$V_n(f;\lambda_n)(x)$$

$$= (1+x)^{-1/\lambda_n}\sum_{j=0}^{\infty}\frac{1}{j!}\cdot\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\cdot \ldots \cdot \left(j-1+\frac{1}{\lambda_n}\right)\cdot\left(\frac{x}{1+x}\right)^j f\left(j\lambda_n\right), \quad (3.3)$$

$x \geq 0$, where by convention $\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\cdot \ldots \cdot \left(j-1+\frac{1}{\lambda_n}\right) = 1$ for $j = 0$.

For these operators $V_n(f;\lambda_n)(x)$ in (3.3), in the next Subsection we prove that on each compact subinterval in $[0, +\infty)$, the order of uniform approximation obtained is $\omega_1(f; \sqrt{\lambda_n})$, and consequently uniformly approximate a Lipschitz 1 function, on each compact subinterval of $[0, \infty)$ with an arbitrary good order of approximation $\sqrt{\lambda_n}$. In other words, from the point of view of approximation theory, between all kinds of Baskakov-type operators in literature, these modified/generalized Baskakov operators represent the best possible construction. In the same time, the results obtained have also a strong unifying character, in the sense that one can recapture from them all the results previously obtained by other authors, for various choices of the nodes λ_n. It is also remarked that by modifying a Baskakov-type operator introduced in Lopez-Moreno [53], similar considerations can be made for the operator defined by

$$L_{n,r}(f;\lambda_n)(x)=\sum_{j=0}^{\infty}(-1)^r f\left(j\lambda_n\right)\cdot\frac{\varphi^{(j+r)}(\lambda_n;x)\cdot(-x)^j}{j!}\cdot\left(\lambda_n\right)^r, r,n\in\mathbb{N}. \tag{3.4}$$

Then, in the next Subsection we make similar considerations for modified/generalized q-Baskakov-type operators, $0<q<1$.

3.1.2 Main results

Firstly, we need the following two auxiliary results.

Lemma 3.1.1. (Gal-Opriş [43]) *Let* $\lambda_n>0$, $n\in\mathbb{N}$, *be with* $\lim_{n\to\infty}\lambda_n=0$.

(i) If $L_n(f;\lambda_n)(x)$ *given by (3.1) is well-defined, then we can write*

$$L_n(f;\lambda_n)(x)=\sum_{j=0}^{\infty}\left(\lambda_n\right)^j\cdot(-1)^j\cdot\varphi^{(j)}(\lambda_n;0)\cdot[0,\lambda_n,...,j\lambda_n;f]\cdot x^j, x\in[0,b],$$

where $[0,\lambda_n,...,j\lambda_n;f]$ *is the divided difference of* f *on the knots* $0,\lambda_n,...,j\lambda_n$.

(ii) Denoting $e_k(x)=x^k$, *we have* $L_n(e_0;\lambda_n)(x)=1$, $L_n(e_1;\lambda_n)(x)=-x\lambda_n\varphi'(\lambda_n;0)$,

$$L_n(e_2;\lambda_n)(x)=\left(\lambda_n\right)^2\cdot[x^2\varphi''(\lambda_n;0)-x\varphi'(\lambda_n;0)].$$

Remark. In the case when $\lambda_n=\frac{1}{n}$, the formula in Lemma 3.1.1, (i) was obtained by Lupas [54].

Corollary 3.1.2. (Gal-Opriş [43]) *(i) If* $(1+\lambda_n)\ldots(1+(j-1)\lambda_n)=1$ *for* $j=0$ *(by convention), then for* $V_n(f;\lambda_n)(x)$ *given by (3.3), we have*

$$V_n(f;\lambda_n)(x)=\sum_{j=0}^{\infty}(1+\lambda_n)\ldots(1+(j-1)\lambda_n)\cdot[0,\lambda_n,...,j\lambda_n;f]x^j, x\geq 0.$$

(ii) $V_n(e_0;\lambda_n)(x) = 1$, $V_n(e_1;\lambda_n)(x) = x$, $V_n(e_2;\lambda_n)(x) = x^2+\lambda_n\cdot x(1+x)$;

$$V_n((\cdot - x)^2;\lambda_n)(x) = \lambda_n x(1+x).$$

Since $V_n(f;\lambda_n)$, $n \in \mathbb{N}$, are positive and linear operators, we can state the following result.

Theorem 3.1.3. (Gal-Opriş [43]) *Let* $f : [0,\infty) \to \mathbb{R}$ *be uniformly continuous on* $[0,\infty)$*. Denote* $\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; |x - y| \le \delta,\ x, y \in [0,\infty)\}$*. For all* $x \in [0,\infty)$*,* $n \in \mathbb{N}$ *we have*

$$|V_n(f;\lambda_n)(x) - f(x)| \le 2\cdot\omega_1\left(f;\sqrt{\lambda_n}\cdot\sqrt{x(1+x)}\right).$$

As an immediate consequence of Theorem 3.1.3 we get the following.

Corollary 3.1.4. (Gal-Opriş [43]) *If there exists* $L > 0$ *such that* $|f(x) - f(y)| \le L|x - y|$*, for all* $x, y \in [0,\infty)$*, then*

$$|V_n(f;\lambda_n)(x) - f(x)| \le 2L\sqrt{x(1+x)}\cdot\sqrt{\lambda_n}, n \in \mathbb{N}, x \ge 0.$$

Remarks. 1) If x belong to a compact subinterval of $[0,+\infty)$, then evidently that we get uniform convergence in that subinterval.

2) The optimality of the estimates in Theorem 3.1.3 and Corollary 3.1.4 consists in the fact that given an arbitrary sequence of strictly positive numbers $(\gamma_n)_n$, with $\lim_{n\to\infty}\gamma_n = 0$ and a compact subinterval $[0,b]$, we can find a sequence λ_n, satisfying $2\omega_1(f;\sqrt{\lambda_n}\cdot\sqrt{x(1+x)}) \le \gamma_n$ for all $n \in \mathbb{N}$, $x \in [0,b]$ in the case of Theorem 3.1.3 and $2L\sqrt{\lambda_n}\cdot\sqrt{x(1+x)} \le \gamma_n$ for all $n \in \mathbb{N}$, $x \in [0,b]$ in the case of Corollary 3.1.4.

3) If f is uniformly continuous on $[0,+\infty)$ then it is well known that its growth on $[0,+\infty)$ is linear, i.e. there exist $\alpha,\beta > 0$ such that $|f(x)| \le \alpha x + \beta$, for all $x \in [0,+\infty)$ (see e.g. [25], p. 48, Problème 4, or [26]). This implies that in this case $V_n(f;\lambda_n)(x)$ is well-defined for all $x \in [0,\infty)$.

4) In the paper [53], the Baskakov-type approximation operators of the form

$$L_{n,r}(f)(x) = \sum_{j=0}^{\infty}(-1)^r f\left(\frac{j}{n}\right) \cdot \frac{\varphi_n^{(j+r)}(x) \cdot (-x)^j}{j!} \cdot \left(\frac{1}{n}\right)^r, r, n \in \mathbb{N}$$

were studied, obtaining for example if $\varphi_n(x) = (1+x)^{-n}$, quantitative estimates of the order $\Omega(f; n^{-1/2}) + \frac{C}{n}$, where $\Omega(f;\delta)$ is a suitable weighted modulus of continuity. By following the lines of proofs in [53], choosing $\varphi(\lambda_n; x) = (1+x)^{-1/\lambda_n}$ in the modified/generalized Baskakov-type operator $L_{n,r}(f;\lambda_n)(x)$ given by formula (3.4), the order of approximation $\Omega\left(f; \sqrt{\lambda_n}\right) + C\lambda_n$ is obtained, where λ_n can be chosen to converge to 0 as fast we want.

3.1.3 The case of q-Baskakov operators, $0 < q < 1$

Firstly, we need the following concepts in quantum calculus (see e.g. [50], pp. 7-13).

For $0 < q$, $q \neq 1$, and $a \in \mathbb{R}$, de q analogue of a is defined by $[a]_q = \frac{1-q^a}{1-q}$. For $n \in \mathbb{N} \cup \{0\}$, we get $[n]_q = 1 + q + ... + q^{n-1}$, $n \in \mathbb{N}$, $[0]_q = 1$. The q-factorial is defined by $[n]_q! = [1]_q \cdot [2]_q \cdot ... \cdot [n]_q$ and the q-binomial coefficient is given by $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}$, $k = 0, 1, ...n$.

Note that for $q = 1$ we get $[n]_q = n$ and as a consequence, $[n]_q! = n!$ and $\binom{n}{k}_q = \binom{n}{k}$.

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by $D_q(f)(x) = \frac{f(x)-f(qx)}{x(1-q)}$, $x \neq 0$, $D_q(f)(0) = \lim_{x\to 0} D_q(f)(x)$, and the q-derivatives of higher order are given recursively by $D_q^0(f) = f$, $D_q^n(f) = D_q(D_q^{n-1}(f))$, $n \in \mathbb{N}$.

Everywhere in what follows, we consider $0 < q < 1$.

Various kinds of q-Baskakov operators were studied in the e.g. the papers [2], [60], [4]–[6], [49], [30].

Following the previous ideas and suggested by the q-Baskakov operators introduced and studied in [60] and [2], we introduce here a modified q-Baskakov operator, as follows.

Let $\lambda_n > 0$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \frac{1}{\lambda_n} = +\infty$. It is clear that without any lost of generality, we may suppose that $\frac{1}{\lambda_n} \geq 1$, $n \in \mathbb{N}$. For $\varphi(\lambda_n; \cdot) : [0, \infty) \to \mathbb{R}$, $n \in \mathbb{N}$, a sequence of analytic functions satisfying the hypothesis (i) $\varphi(\lambda_n; 0) = 1$; (ii) $(-1)^j \varphi^{(j)}(\lambda_n; x) \geq 0$, for all $n, j \in \mathbb{N}$, $x \in [0, \infty)$, let us introduce the q-Baskakov operator given by

$$T_{n,q}(f; \lambda_n)(x) = \sum_{j=0}^{\infty} \frac{(-x)^j}{[j]_q!} \cdot q^{(k(k-1)/2} D_q^k \varphi(\lambda_n; x) f\left(\frac{[j]_q}{q^{k-1}} \cdot \frac{1}{[1/\lambda_n]_q}\right), \quad (3.5)$$

attached to functions for which $T_{n,q}(f; \lambda_n)(x)$ is well-defined.

Note that for $1/\lambda_n = n$ we recapture the q-Baskakov operators in [60], [2].

Following exactly the lines in the proof of Lemma 1 in [60] and also using relationships (21) and(22) in [2], we immediately get the following.

Lemma 3.1.5. (Gal-Opriş [43]) *Let $\lambda_n > 0$, $\frac{1}{\lambda_n} \geq 1$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \frac{1}{\lambda_n} = +\infty$. For all $n \in \mathbb{N}, x \geq 0$ and $0 < q < 1$, we have :*

(i) $T_{n,q}(e_0; \lambda_n)(x) = 1$; $T_{n,q}(e_1; \lambda_n)(x) = -x \cdot D_q(\varphi(\lambda_n; \cdot))(0) \cdot \frac{1}{[1/\lambda_n]_q}$;

(ii) $T_{n,q}(e_2; \lambda_n)(x) = x^2 \cdot D_q^2(\varphi(\lambda_n; \cdot))(0) \cdot \frac{1}{q \cdot [1/\lambda_n]_q^2} - x \cdot D_q(\varphi(\lambda_n; \cdot))(0) \cdot \frac{1}{[1/\lambda_n]_q^2}$;

(iii) $T_{n,q}((\cdot - x)^2; \lambda_n)(x) = A_{n,q}x^2 + B_{n,q}x$, *where*

$$A_{n,q} = 1 + D_q^2(\varphi(\lambda_n; \cdot))(0) \cdot \frac{1}{q \cdot [1/\lambda_n]_q^2} + 2 \cdot D_q(\varphi(\lambda_n; \cdot))(0) \cdot \frac{1}{[1/\lambda_n]_q}$$

and

$$B_{n,q} = -\frac{D_q \varphi(\lambda_n; 0)}{[1/\lambda_n]_q^2}.$$

Denoting by $C_B(\mathbb{R}_+)$ the space of all bounded continuous real-valued functions on $[0,\infty)$ and following exactly the lines in the proof of Theorem 2 in [2], we can state the following.

Theorem 3.1.6. (Gal-Opriş [43]) *Let $\lambda_n > 0$, $\frac{1}{\lambda_n} \geq 1$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \frac{1}{\lambda_n} = +\infty$ and let $(q_n)_{n\in\mathbb{N}}$ be a sequence such that $0 < q_n < 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} q_n = 1$. Then, for $f \in C_B(\mathbb{R}_+)$ uniformly continuous, the q_n-operators given by (3.5) satisfy*

$$|T_{n,q_n}(f;\lambda_n)(x) - f(x)| \leq (1 + \sqrt{\max\{x, x^2\}}) \cdot \omega_1(f; \sqrt{C_{n,q_n}}), n \in \mathbb{N}, x \geq 0,$$

where $C_{n,q_n} = |A_{n,q_n}| + B_{n,q_n}$, $(A_{n,q_n})_n, (B_{n,q_n})_n$ are given in Lemma 3.1.5, (iii) and $\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,\infty), |x - y| \leq \delta\}$.

As consequences of Theorem 3.1.6, we get the following two corollaries.

Corollary 3.1.7. (Gal-Opriş [43]) *Let $\lambda_n > 0$, $\frac{1}{\lambda_n} \geq 1$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \frac{1}{\lambda_n} = +\infty$ and $(q_n)_{n\in\mathbb{N}}$ be a sequence such that $0 < q_n < 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} q_n = 1$. Then, for $f \in C_B(\mathbb{R}_+)$ uniformly continuous, the q_n-operators given by*

$$T_{n,q_n}(f;\lambda_n)(x) = \frac{1}{(1+x)^{1/\lambda_n}} \cdot \sum_{j=0}^{\infty} \frac{[1/\lambda_n]_{q_n} \cdot [1/\lambda_n + 1]_{q_n} \cdot \ldots \cdot [1/\lambda_n + j - 1]_{q_n}}{[j]_{q_n}!}$$
$$\cdot q^{j(j-1)/2} \cdot \frac{x^j}{(1+x)^j} \cdot f\left(\frac{[j]_{q_n}}{q_n^{j-1}} \cdot \frac{1}{[1/\lambda_n]_{q_n}}\right), \tag{3.6}$$

for all $n \in \mathbb{N}, x \geq 0$, satisfy the estimate

$$|T_{n,q_n}(f;\lambda_n)(x) - f(x)| \leq (1+\sqrt{\max\{x,x^2\}}) \cdot \omega_1\left(f; \sqrt{\frac{1+q_n}{q_n}} \cdot \frac{1}{\sqrt{[1/\lambda_n]_{q_n}}}\right).$$

Corollary 3.1.8. (Gal-Opriş [43]) *Let $\lambda_n > 0$, $\frac{1}{\lambda_n} \geq 1$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \frac{1}{\lambda_n} = +\infty$ and $(q_n)_{n\in\mathbb{N}}$ be a sequence such that $0 < q_n < 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} q_n = 1$. Then, for $f \in C_B(\mathbb{R}_+)$ uniformly continuous, the q_n-operators given by*

$$S_{n,q_n}(f;\lambda_n)(x)$$

$$= \sum_{j=0}^{\infty} \frac{([1/\lambda_n]_{q_n} x)^j}{[j]_{q_n}!} \cdot q_n^{j(j-1)} \cdot E_{q_n}(-[1/\lambda_n]_{q_n} q_n^j x) \cdot f\left(\frac{[j]_{q_n}}{q_n^{j-1}} \cdot \frac{1}{[1/\lambda_n]_{q_n}}\right), \quad (3.7)$$

for all $n \in \mathbb{N}, x \geq 0$, *satisfy the estimate*

$$|S_{n,q_n}(f;\lambda_n)(x) - f(x)| \leq (1 + \sqrt{\max\{x, x^2\}}) \cdot \omega_1\left(f; \frac{1}{\sqrt{[1/\lambda_n]_{q_n}}}\right).$$

Remark. The order of approximation for the q_n-Baskakov-type operators in Corollary 3.1.7 and for the q_n-Szász-Mirakjan operators in Corollary 3.4 is $O(1/\sqrt{[1/\lambda_n]_{q_n}})$. On the other hand, for $q_n = 1$, for all $n \in \mathbb{N}$, the order of approximation is $O(1/\sqrt{1/\lambda_n}) = O(\sqrt{\lambda_n})$ (see Theorem 3.1.3 in the case of Baskakov-type operators).

However, for $0 < q_n < 1$ for all $n \in \mathbb{N}$, it is easy to see that $\sqrt{\lambda_n} \leq \frac{\sqrt{2}}{\sqrt{[1/\lambda_n]_{q_n}}}$, because $[1/\lambda_n]_{q_n} \leq 2/\lambda_n$.

Indeed, denoting with $[a]_*$ the integer part of a, we have $1/\lambda_n \leq [1/\lambda_n]_* + 1$, which by $0 < q_n < 1$ implies $q_n^{[1/\lambda_n]_*+1} \leq q_n^{1/\lambda_n}$, leading to $[1/\lambda_n]_{q_n} \leq [[1/\lambda_n]_* + 1]_{q_n} \leq [1/\lambda_n]_* + 1 \leq 2/\lambda_n$.

On the other hand, by [24], Lemma 3.4, $n \leq C'[n]_{q_n}$, for all $n \in \mathbb{N}$ (with $C' > 0$ independent of n), if and only if there exists a constant $c > 0$ and $n_0 \in \mathbb{N}$ (independent of n) such that $q_n^n \geq c$, for all $n \geq n_0$. Therefore, in this case, we obtain

$$1/\lambda_n \leq [1/\lambda_n]_* + 1 \leq C'[[1/\lambda_n]_* + 1]_{q_n}$$

$$\leq C'[2[1/\lambda_n]_*]_{q_n} \leq 2C'[[1/\lambda_n]_*]_{q_n} \leq 2C'[1/\lambda_n]_{q_n}.$$

In conclusion, if in Corollaries 3.1.7 and 3.1.8 q_n is chosen to satisfy $q_n^n \geq c$, for all $n \geq n_0$, $0 < q_n < 1$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} q_n = 1$, then the approximation orders for the corresponding q_n-Baskakov and q_n-Szász-Mirakjan operators are $\omega_1\left(f; \sqrt{\lambda_n}\right)$, which can be chosen to converge to 0 as fast we want.

3.2 Generalized Szász-Stancu operators on $[0, +\infty)$

Let $0 \le \alpha, \beta$ and $\lambda_n > 0$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \lambda_n = 0$.

In this section we obtain estimates in approximation by the generalized Szász-Stancu by the formula

$$L_n^{(\alpha,\beta)}(f;\lambda_n)(x) = e^{-x/\lambda_n} \sum_{k=0}^{\infty} \frac{x^k}{\lambda_n^k k!} \cdot f\left(\frac{\lambda_n(j+\alpha)}{1+\beta\lambda_n}\right), x \ge 0.$$

It is clear that $L_n^{(\alpha,\beta)}(f;\lambda_n)$ is a positive linear operator on $[0, +\infty)$, for any $n \in \mathbb{N}$.

Firstly, we need the following auxiliary result.

Lemma 3.2.1. (Opriş [59]) *Let $0 \le \alpha, \beta$ and $\lambda_n > 0$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \lambda_n = 0$. Denote $e_k(x) = x^k$, $k = 0, 1, 2, ...,$. For all $n \in \mathbb{N}$ and $x \ge 0$ we have :*

(i) $L_n^{(\alpha,\beta)}(e_0;\lambda_n)(x) = 1$; $L_n^{(\alpha,\beta)}(e_1;\lambda_n)(x) = \frac{x+\lambda_n\alpha}{1+\lambda_n\beta}$;

(ii) $L_n^{(\alpha,\beta)}(e_2;\lambda_n)(x) = \frac{(x+\lambda_n\alpha)^2+\lambda_n x}{(1+\lambda_n\beta)^2} = \left[\frac{x+\lambda_n\alpha}{1+\lambda_n\beta}\right]^2 + \frac{\lambda_n x}{(1+\lambda_n\beta)^2}$;

(iii) $L_n^{(\alpha,\beta)}((\cdot - x)^2;\lambda_n)(x) = \lambda_n \cdot \frac{\lambda_n(\alpha - x\beta)^2+x}{(1+\lambda_n\beta)^2}$.

Denote by $C_B(\mathbb{R}_+)$ the space of all bounded continuous real-valued functions on $[0, \infty)$. We can state the following.

Theorem 3.2.2. (Opriş [59]) *Let $0 \le \alpha, \beta$ and $\lambda_n > 0$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \lambda_n = 0$, as fast we want. Then, for $f \in C_B(\mathbb{R}_+)$ uniformly continuous, the following estimate holds*

$$|L_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \le 2\omega_1\left(f; \sqrt{\lambda_n} \cdot \sqrt{\lambda_n(\alpha - x\beta)^2 + x}\right), x \ge 0,$$

where $\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,\infty), |x - y| \le \delta\}$ denotes the modulus of continuity of f.

As an immediate consequence of Theorem 3.2.2, we get the following corollary.

Corollary 3.2.3. (Opriş [59]) *Under the hypothesis of Theorem 3.2.2, if, in addition, there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$, for all $x, y \in [0, \infty)$ (f is Lipschitz function), then*

$$|L_n^{(\alpha,\beta)}(f; \lambda_n)(x) - f(x)| \leq 2L\sqrt{\lambda_n} \cdot \sqrt{\lambda_n(\alpha - x\beta)^2 + x}, n \in \mathbb{N}, x \geq 0.$$

Remarks. 1) If x belong to a compact subinterval of $[0, +\infty)$, then evidently that we get uniform convergence in that subinterval.

2) Since the sequence λ_n can be chosen to converge to zero as fast as we want, the results in Theorem 3.2.2 Corollary 3.2.3 are of definitive type, that is are the best possible (cannot be improved).

3.3 Generalized Baskakov-Stancu operators on $[0, +\infty)$

Let $0 \leq \alpha, \beta$ and $\lambda_n > 0$, $n \in \mathbb{N}$ be with $\lim_{n\to\infty} \lambda_n = 0$.

In this section we obtain estimates for the generalized Baskakov-Stancu operators given by the formula

$$K_n^{(\alpha,\beta)}(f; \lambda_n)(x)$$

$$= (1+x)^{-1/\lambda_n} \sum_{j=0}^{\infty} \frac{1}{j!} \cdot \frac{1}{\lambda_n} \cdot \left(1 + \frac{1}{\lambda_n}\right) \cdot ... \cdot \left(j - 1 + \frac{1}{\lambda_n}\right) \cdot \frac{x^j}{(1+x)^j} f\left(\frac{\lambda_n(j+\alpha)}{1+\beta\lambda_n}\right)$$

$$= \sum_{j=0}^{\infty} (1 + \lambda_n) \cdot ... \cdot (1 + (j-1)\lambda_n) \left[\frac{\lambda_n \alpha}{1 + \lambda_n \beta}, ..., \frac{\lambda_n(\alpha + j)}{1 + \lambda_n \beta}; f\right] x^j, x \geq 0.$$

Evidently, $K_n^{(\alpha,\beta)}$ is positive linear operator on $[0, +\infty)$, for any $n \in \mathbb{N}$.

Firstly, we need the following auxiliary result.

Lemma 3.3.1. (Opriş [59]) *Let* $0 \leq \alpha, \beta$ *and* $\lambda_n > 0$, $n \in \mathbb{N}$ *be with* $\lim_{n\to\infty} \frac{1}{\lambda_n} = +\infty$. *For all* $n \in \mathbb{N}$ *and* $x \geq 0$ *we have :*

(i) $K_n^{(\alpha,\beta)}(e_0;\lambda_n)(x) = 1$; $K_n^{(\alpha,\beta)}(e_1;\lambda_n)(x) = \frac{x+\lambda_n\alpha}{1+\lambda_n\beta}$;

(ii) $K_n^{(\alpha,\beta)}(e_2;\lambda_n)(x) = \frac{(x+\lambda_n\alpha)^2+\lambda_n x(x+1)}{(1+\lambda_n\beta)^2} = \left[\frac{x+\lambda_n\alpha}{1+\lambda_n\beta}\right]^2 + \frac{\lambda_n x(x+1)}{(1+\lambda_n\beta)^2}$;

(iii) $K_n^{(\alpha,\beta)}((\cdot - x)^2;\lambda_n)(x) = \lambda_n \cdot \frac{\lambda_n(\alpha-x\beta)^2+x(1+x)}{(1+\lambda_n\beta)^2}$.

Denote by $C_B(\mathbb{R}_+)$ the space of all bounded continuous real-valued functions on $[0,\infty)$. We can state the following.

Theorem 3.3.2. (Opriş [59]) *Let* $0 \leq \alpha, \beta$ *and* $\lambda_n > 0$, $n \in \mathbb{N}$ *be with* $\lim_{n\to\infty} \lambda_n = 0$, *as fast we want. Then, for* $f \in C_B(\mathbb{R}_+)$ *uniformly continuous, the following estimate holds*

$$|K_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \leq 2\omega_1\left(f; \sqrt{\lambda_n} \cdot \sqrt{\lambda_n(\alpha - x\beta)^2 + x(1+x)}\right), x \geq 0,$$

where $\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,\infty), |x-y| \leq \delta\}$ *denotes the modulus of continuity of* f.

Remark. For $\alpha = \beta = 0$ we recapture the estimate for $V_n(f;\lambda_n)$ obtained in Corollary 2.1, (ii) in [43] (see also the previous Section 3.1).

As an immediate consequence of Theorem 3.3.2, we get the following corollary.

Corollary 3.3.3. (Opriş [59]) *Under the hypothesis of Theorem 3.3.2, if, in addition, there exists* $L > 0$ *such that* $|f(x) - f(y)| \leq L|x-y|$, *for all* $x, y \in [0,\infty)$ *(*f *is Lipschitz function), then*

$$|K_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \leq 2L\sqrt{\lambda_n} \cdot \sqrt{\lambda_n(\alpha - x\beta)^2 + x(1+x)}, n \in \mathbb{N}, x \geq 0.$$

Remarks. 1) If x belong to a compact subinterval of $[0,+\infty)$, then evidently that we get uniform convergence in that subinterval.

2) Since the sequence λ_n can be chosen to converge to zero as fast as we want, the results in Theorem 3.3.2 Corollary 3.3.3 are of definitive type, that is are the best possible (cannot be improved).

Ch. 4

Approximation with an arbitrary order by Szász and Baskakov kind operators of complex variable

In this chapter we consider the ideas in the previous chapter, but applied now to the case of approximation of analytic functions by complex Szász and Baskakov type operators, in compact sets in $\mathbb{C}$. Two cases are studied : (i) approximation in compact disks with center at origin ; (ii) approximation in arbitrary compacts by using the Faber polynomials attached to these compact sets.

4.1 Arbitrary order in compact disks

By using a sequence $\lambda_n > 0$, $n \in \mathbb{N}$ with the property that $\lambda_n \to 0$ as fast we want, in this section we obtain the approximation order $O(\lambda_n)$ for some generalized/modified Szász, Szász-Kantorovich, and Baskakov complex operators attached to entire functions or to analytic functions of exponential growth in compact disks and without to involve the values on $[0, +\infty)$.

4.1.1 Introduction

In [15], with the notations there for two sequences a_n and b_n, $n \in \mathbb{N}$, and denoting here $\lambda_n = \frac{b_n}{a_n}$, the authors introduced the generalized complex Szász operator by

$$S_n(f;\lambda_n)(z) = e^{-z/\lambda_n} \sum_{j=0}^{\infty} \frac{(z/\lambda_n)^j}{j!} \cdot f(j\lambda_n), \tag{4.1}$$

where $\lambda_n > 0$, $\lambda_n \to 0$.

For this operator, attached to functions $f : \overline{\mathbb{D}_R} \bigcup [R, +\infty) \to \mathbb{C}$ of exponential growth in $\overline{\mathbb{D}_R} \bigcup [R, +\infty)$, analytic in the disk $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$, $R > 1$ and continuous on $[0, +\infty)$, the exact order of approximation $O(\lambda_n)$ is obtained in [15]. Also, in the same paper a Voronovskaja-type result with an upper estimate of order $O(\lambda_n^2)$ is proved.

The first goal of the present section is to extend the results in [15] to the case of entire functions and then, to a kind od Szász operator which does not involve the values of f on $[0, +\infty)$. Also, a complex operator of Szász-Kantorovich type is introduced, for which similar results are proved, essentially improving the order of approximation $O(1/n)$ obtained in [58].

The second goal is to introduce generalized/modified complex Baskakov type operators, for which similar results with those obtained for the Szász operators are proved.

4.1.2 Szász and Szász-Kantorovich operators

In the case of complex Szász operator, we can prove the following result.

Theorem 4.1.1. (Gal-Opriş [46]) *Let* $\lambda_n > 0$, $n \in \mathbb{N}$ *be with* $\lambda_n \to 0$ *as fast we want. Let* $f : \mathbb{D}_R \to \mathbb{C}$, $1 < R \leq +\infty$, *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Suppose that there exist* $M > 0$ *and* $A \in (1/R, 1)$, *with the property* $|c_k| \leq M\frac{A^k}{k!}$, *for all* $k = 0, 1, ...$, *(which implies* $|f(z)| \leq Me^{A|z|}$ *for all* $z \in \mathbb{D}_R$*). Consider* $1 \leq r < \frac{1}{A}$.

(i) If $R = +\infty$, $(1/R = 0)$, *i.e.* f *is an entire function, then* $S_n(f;\lambda_n)(z)$ *is entire function, we have* $S_n(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k S_n(e_k;\lambda_n)(z)$ *for all* $z \in \mathbb{C}$, $n \in \mathbb{N}$ *and for all* $|z| \leq r$ *the following estimates hold :*

$$|S_n(f;\lambda_n)(z) - f(z)| \leq C_{r,M,A} \cdot \lambda_n,$$

$$|S_n^{(p)}(f;\lambda_n)(z) - f^{(p)}(z)| \leq \frac{p! r_1 \cdot C_{r_1,M,A}}{(r_1 - r} \cdot \lambda_n,$$

$$\left|S_n(f;\lambda_n)(z) - f(z) - \frac{\lambda_n}{2} z f''(z)\right| \leq M_r(f)(z) \cdot \lambda_n^2 \leq C_r(f) \cdot \lambda_n^2,$$

$$\|S_n^{(p)}(f;\lambda_n) - f^{(p)}\|_r \sim \lambda_n,$$

the last equivalence holding if f *is not a polynomial of degree* $\leq p \in \mathbb{N}$ *and the constants in the equivalence depend on* f, r, p.

Above, $C_{r,M,A} = \frac{M}{2r} \sum_{k=2}^{\infty} (k+1)(rA)^k < \infty$, $p \in \mathbb{N}$, $1 \leq r < r_1 < \frac{1}{A}$, $M_r(f)(z) = \frac{3MA|z|}{r^2} \cdot \sum_{k=2}^{\infty} (k+1)(rA)^{k-1} < \infty$, $C_r(f) = \frac{3MA}{r} \cdot \sum_{k=2}^{\infty} (k+1)(rA)^{k-1}$ *and* $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$.

(ii) If $R < +\infty$, *then the complex approximation operator*

$$S_n^*(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k \cdot S_n(e_k;\lambda_n)(z), z \in \overline{\mathbb{D}_r},$$

is well-defined and $S_n^*(f;\lambda_n)(z)$ *satisfies all the estimates from the point (i), for all* $1 \leq r < \frac{1}{A} < R$.

In what follows, we can define the generalized/modified complex Szász-Kantorovich type operator by the formula

$$K_n(f;\lambda_n)(z) = e^{-z/\lambda_n}\sum_{j=0}^{\infty}\frac{(z/\lambda_n)^j}{j!}\cdot\frac{1}{\lambda_n}\cdot\int_{j\lambda_n}^{(j+1)\lambda_n} f(v)dv$$

$$= e^{-z/\lambda_n}\sum_{j=0}^{\infty}\frac{(z/\lambda_n)^j}{j!}\cdot\int_0^1 f\left((t+j)\lambda_n\right)dt.$$

Denoting $F(z)=\int_0^z f(t)dt$, simple calculation leads to the formula (under the hypothesis that the series $S_n(F;\lambda_n)(z)$ is uniformly convergent)

$$K_n(f;\lambda_n)(z) = S_n'(F;\lambda_n)(z). \tag{4.2}$$

We can prove the following results.

Theorem 4.1.2. (Gal-Opriş [46]) *Let $\lambda_n > 0$, $n\in\mathbb{N}$ be with $\lambda_n \to 0$ as fast we want. Let $f:\mathbb{D}_R\to\mathbb{C}$, $1<R\le+\infty$, i.e. $f(z)=\sum_{k=0}^{\infty}c_k z^k$, for all $z\in\mathbb{D}_R$. Suppose that there exist $M>0$ and $A\in(1/R,1)$, with the property $|c_k|\le M\frac{A^k}{k!}$, for all $k=0,1,...,$ (which implies $|f(z)|\le Me^{A|z|}$ for all $z\in\mathbb{D}_R$). Also, consider $1\le r<1/A$.*

(i) If $R=+\infty$, ($1/R=0$), i.e. f is an entire function, then, $K_n(f;\lambda_n)(z)$ is entire function, we have $K_n(f;\lambda_n)(z)=\sum_{k=0}^{\infty}c_k K_n(e_k;\lambda_n)(z)$ for all $z\in\mathbb{C}$, $n\in\mathbb{N}$ and for all $|z|\le r$ the following estimates hold :

$$\left|K_n(f;\lambda_n)(z)-f(z)-\frac{\lambda_n}{2}[f'(z)+zf''(z)]\right|\le C_r'(f)\cdot\lambda_n^2,$$

$$\|K_n^{(p)}(f;\lambda_n)-f^{(p)}\|_r \sim \lambda_n,$$

the last equivalence holding if f is not a polynomial of degree $\le p$ and the constants in the equivalence depend on f, r, p.

Above $p\in\mathbb{N}\bigcup\{0\}$, $C_r'(f)<\infty$ is a constant independent of n and z and $\|f\|_r=\max\{|f(z)|;|z|\le r\}$.

(ii) If $R < +\infty$, then the complex approximation operator

$$K_n^*(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k \cdot K_n(e_k;\lambda_n)(z), z \in \overline{\mathbb{D}_r},$$

is well-defined and $K_n^*(f;\lambda_n)(z)$ satisfies all the estimates from the point (i), for all $1 \le r < \frac{1}{A} < R$.

Remarks. 1) In conclusion, the results in the complex case in Theorems 4.1.1 and 4.1.2, are of definitive type, in the sense that they exhibit operators which can approximate the functions with an arbitrary chosen order.

2) The first estimate in the statement of Theorem 4.1.1, (i), was extended (with a different constant, of course) in [36] to the approximation by generalized Szász-Faber type operators in compact sets in $\mathbb{C}$.

4.1.3 Generalized Baskakov operators

For x real and ≥ 0, the original formula of the classical now Baskakov operator is given by (see [7])

$$Z_n(f)(x) = (1+x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f(k/n)$$

and many approximation results of this operators were published.

According to [54], Theorem 2, under the same hypothesis on f that $Z_n(f)(x)$ is well defined and denoting by $[0, 1/n, ..., j/n; f]$ the divided difference of f on the knots 0, ..., j/n, for $x \ge 0$ we can write $Z_n(f)(x) = W_n(f)(x)$, $x \ge 0$, where

$$W_n(f)(x) := \sum_{j=0}^{\infty} \left(1 + \frac{1}{n}\right) \cdot ... \cdot \left(1 + \frac{j-1}{n}\right) \cdot [0, 1/n, ..., j/n; f] x^j, x \ge 0, \tag{4.3}$$

(here for $j = 0$ and $j = 1$ we take $(1 + 1/n) \cdot ... \cdot (1 + (j-1)/n) = 1$).

For $\lambda_n \searrow 0$, arbitrary, by formula (1) in the paper [43] (particularizing there $\varphi_n(\lambda_n; x) = (1+x)^{-1/\lambda_n}$), $Z_n(f)(x)$ can be generalized to

$$Z_n(f; \lambda_n)(x)$$

$$= (1+x)^{-1/\lambda_n} \cdot \sum_{j=0}^{\infty} \frac{1}{j!} \cdot \frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right) \cdot ... \cdot \left(j-1+\frac{1}{\lambda_n}\right) \cdot \left(\frac{x}{1+x}\right)^j f\left(j\lambda_n\right),$$

$x \geq 0$, where by convention $\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right) \cdot ... \cdot \left(j-1+\frac{1}{\lambda_n}\right) = 1$ for $j=0$.

For this generalization, in [43] the order of approximation $\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{x(1+x)})$ was obtained.

Accordingly, $W_n(f)(x)$ given by (4.3), can be generalized to

$$W_n(f; \lambda_n)(x) = \sum_{j=0}^{\infty} (1+\lambda_n) ... (1+(j-1)\lambda_n) \cdot [0, \lambda_n, ..., j\lambda_n; f]x^j, x \geq 0,$$

where by convention, $(1+\lambda_n) ... (1+(j-1)\lambda_n) = 1$ for $j=0$.

It is clear that $Z_n(f; \lambda_n)(x) = W_n(f; \lambda_n)(x)$ for all $x \geq 0$, but as it was remarked in [34], p. 124, in the particular case $\lambda_n = \frac{1}{n}$, if $|x| < 1$ is not positive then $W_n(f; \lambda_n)(x)$ and $Z_n(f; \lambda_n)(x)$ do not necessarily coincide and because of this reason in Section 1.8 of the book [34], pp. 124-138, they were studied separately, under different hypothesis on f and $z \in \mathbb{C}$.

In what follows we study the approximation properties of the complex generalized Baskakov type operators $W_n(f; \lambda_n)(z)$ attached to analytic functions satisfying some exponential-type growth condition.

In this sense, we can state the following.

Theorem 4.1.3. (Gal-Opriş [46]) *Let* $0 < \lambda_n \leq \frac{1}{2}$, $n \in \mathbb{N}$ *be with* $\lambda_n \to 0$ *as fast we want. Let* $f : \mathbb{D}_R \to \mathbb{C}$, $1 < R \leq +\infty$, *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Suppose that there exist* $M > 0$ *and* $A \in (1/R, 1)$, *with the property* $|c_k| \leq M\frac{A^k}{k!}$, *for all* $k = 0, 1, ...$, *(which implies* $|f(z)| \leq Me^{A|z|}$ *for all* $z \in \mathbb{D}_R$*). Consider* $1 \leq r < \frac{1}{A}$.

(i) If $R=+\infty$, $(1/R=0)$, i.e. f is an entire function, then for $|z|\le r$ $W_n(f;\lambda_n)(z)$ is analytic, we have $W_n(f;\lambda_n)(z)=\sum_{k=0}^{\infty}c_kW_n(e_k;\lambda_n)(z)$ and the following estimates hold :

$$|W_n(f;\lambda_n)(z)-f(z)|\le C_{r,M,A}\cdot\lambda_n,$$

$$|W_n^{(p)}(f;\lambda_n)(z)-f^{(p)}(z)|\le\frac{p!r_1\cdot C_{r_1,M,A}}{(r_1-r}\cdot\lambda_n,$$

$$\left|W_n(f;\lambda_n)(z)-f(z)-\frac{\lambda_n}{2}zf''(z)\right|\le M_r(f)\cdot\lambda_n^2,$$

$$\|W_n^{(p)}(f;\lambda_n)-f^{(p)}\|_r\sim\lambda_n,$$

the last equivalence holding if f is not a polynomial of degree $\le p\in\mathbb{N}$ and the constants in the equivalence depend on f, r, p.

Above, $C_{r,M,A}=6M\sum_{k=2}^{\infty}(k+1)(k-1)(rA)^k<\infty$, $p\in\mathbb{N}$, $1\le r<r_1<\frac{1}{A}$, $M_r(f)=16M\cdot\sum_{k=3}^{\infty}(k-1)(k-2)(rA)^k<\infty$ and $\|f\|_r=\max\{|f(z)|;|z|\le r\}$.

(ii) If $R<+\infty$, then the complex approximation operator

$$W_n^*(f;\lambda_n)(z)=\sum_{k=0}^{\infty}c_k\cdot W_n(e_k;\lambda_n)(z),z\in\overline{\mathbb{D}_r},$$

is well-defined and $W_n^*(f;\lambda_n)(z)$ satisfies all the estimates from the point (i), for all $1\le r<\frac{1}{A}<R$.

Remark. Due to the results in the real case in [43] and to those in the complex case in Theorem 4.1.3, we can say that they seem to be of definitive type, in the sense that exhibit Baskakov type operators which can approximate the functions with an arbitrary chosen order.

4.2 Arbitrary order by Baskakov-Faber operators

By using a sequence $\lambda_n > 0$, $n \in \mathbb{N}$ with the property that $\lambda_n \to 0$ as fast we want, in this paper we obtain the approximation order $O(\lambda_n)$ for a generalized Baskakov-Faber type operator attached to analytic functions of exponential growth in a continuum $G \subset \mathbb{C}$. Several concrete examples of continuums G are given for which this operator can explicitly be constructed.

In this way, the results obtained in the previous section for compact disks, are generalized to the case when the disk is replaced by a compact set in $\mathbb{C}$.

4.2.1 Introduction

According to the considerations in Subsection 4.1.1, denoting

$$W_n(f)(z) = \sum_{j=0}^{\infty} \left(1 + \frac{1}{n}\right) \cdot ... \cdot \left(1 + \frac{j-1}{n}\right) \cdot [0, 1/n, ..., j/n; f] z^j,$$

for analytic functions satisfying some exponential-type growth condition, quantitative estimates of order $O\left(\frac{1}{n}\right)$ in approximation by $W_n(f)(z)$ in compact disks with center at origin were obtained in [34], Section 1.9, pp. 124-138. For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, all the quantitative results are based on the formula $W_n(f)(z) = \sum_{k=0}^{\infty} a_k \cdot W_n(e_k)(z)$, with $e_k(z) = z^k$, i.e. by using (4.3) too,

$$W_n(f)(z) = \sum_{k=0}^{\infty} a_k \cdot \sum_{j=0}^{k} \left(1 + \frac{1}{n}\right) \cdot ... \cdot \left(1 + \frac{j-1}{n}\right) \cdot [0, 1/n, ..., j/n; e_k] z^j. \tag{4.4}$$

Also, it is worth noting that similar quantitative estimates in approximation by other complex operators can be found in, e.g., the books [34], [35], [48] and in the papers [15], [37], [38]-[47], [55]-[57].

By using a sequence of real positive numbers, $(\lambda_n)_{n\in\mathbb{N}}$, with the properties that $\lambda_n \to 0$ as fast we want, suggested by the formula (4.4) too, the aim of this note is to generalize the approximation by the operators $W_n(f)(z)$, to the approximation by the so-called by us generalized Baskakov-Faber type operators attached to analytic functions of some exponential growth in a continuum in $\mathbb{C}$, obtaining the approximation order $O(\lambda_n)$.

Since $\lambda_n \to 0$, obviously that without to loose the generality, everywhere in the paper we may suppose that $0 < \lambda_n \leq \frac{1}{2}$, for all $n \in \mathbb{N}$.

4.2.2 Preliminaries

Firstly, we briefly recall some basic concepts on Faber polynomials and Faber expansions.

For $G \subset \mathbb{C}$ a compact set such that $\tilde{\mathbb{C}} \setminus G$ is connected, let $A(G)$ be the Banach space of all functions that are continuous on G and analytic in the interior of G, endowed with the norm $\|f\|_G = \sup\{|f(z)|; z \in G\}$. Denoting $\mathbb{D}_r = \{z \in \mathbb{C}; |z| < r\}$, according to the Riemann Mapping Theorem, there exists a unique conformal mapping Ψ of $\tilde{\mathbb{C}} \setminus \overline{\mathbb{D}}_1$ onto $\tilde{\mathbb{C}} \setminus G$ such that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. Then, to G one may attach the polynomial of exact degree n, $F_n(z)$, called *Faber polynomial*, defined by $\frac{\Psi'(w)}{\Psi(w)-z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}$, $z \in G, |w| > 1$.

If $f \in A(G)$ then

$$a_n(f) = \frac{1}{2\pi i}\int_{|u|=1} \frac{f(\Psi(u))}{u^{n+1}}du = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\Psi(e^{it}))e^{-int}dt, n \in \mathbb{N} \cup \{0\}$$

are called the Faber coefficients of f and $\sum_{n=0}^{\infty} a_n(f)F_n(z)$ is called the

Faber expansion (series) attached to f on G. It is worth noting that the Faber series represent a natural generalization of the Taylor series, when the unit disk is replaced by an arbitrary simply connected domain bounded by a "nice" curve.

Detailed properties of Faber polynomials and Faber expansions can be found in e.g. [31], [62].

Let G be a connected compact subset in $\mathbb{C}$ (that is a continuum) and suppose that f is analytic on G, that is there exists $R > 1$ such that f is analytic in G_R, given by $f(z) = \sum_{k=0}^{\infty} a_k(f)F_k(z)$, $z \in G_R$. Recall here that G_R denotes the interior of the closed level curve Γ_R given by $\Gamma_R = \{\Psi(w); |w| = R\}$ (and that $G \subset \overline{G_r}$ for all $1 < r < R$).

Suggested by the formula (4.4), we can introduce the following.

Definition 4.2.1. (Gal-Opriş [45]) The generalized Baskakov-Faber type operators attached to G and f is defined by

$$W_n(f;\lambda_n, G; z) = \sum_{k=0}^{\infty} a_k(f) \cdot W_n(e_k; \lambda_n, G; z),$$

i.e.,

$$W_n(f;\lambda_n, G; z)$$

$$= \sum_{k=0}^{\infty} a_k(f) \cdot \sum_{j=0}^{k} (1+\lambda_n) \cdot ... \cdot (1+(j-1)\lambda_n) \cdot [0, \lambda_n, ..., j\lambda_n; e_k] \cdot F_j(z), \quad (4.5)$$

where for $j = 0$ and $j = 1$, by convention $(1+\lambda_n) \cdot ... \cdot (1+(j-1)\lambda_n) = 1$.

Remark. For $\lambda_n = 1/n$, $n \in \mathbb{N}$ and $G = \overline{\mathbb{D}}_1$, since $F_j(z) = z^j$, the above generalized Baskakov-Faber type operators reduce to the classical complex Baskakov operators, introduced and studied in [34], Section 1.9.

4.2.3 Main results

For the proof of the main result, we need two lemmas, as follows.

Lemma 4.2.2. (Gal-Opriş [45]) *Let* $0 < \lambda_n \le \frac{1}{2} < 1$, $n \in \mathbb{N}$, *be with* $\lambda_n \to 0$. *For all* $k, n \in \mathbb{N}$ *with* $k \le [1/\lambda_n]$ *(here* $[a]$ *denotes the integer part of* a*) we have the inequality*

$$E_{k,n} := \sum_{j=0}^{k-1} (1+\lambda_n) \cdot ... \cdot (1+(j-1)\lambda_n) \cdot [0, \lambda_n, ..., j\lambda_n; e_k] \le \lambda_n \cdot (k+3)!.$$

Here, by convention, for $j = 0$ *and* $j = 1$ *we take* $(1+\lambda_n) \cdot ... \cdot (1+(j-1)\lambda_n) = 1$.

Also, we can prove the following.

Lemma 4.2.3. (Gal-Opriş [45]) *Let* $0 < \lambda_n \le \frac{1}{2}$, $n \in \mathbb{N}$, *be with* $\lambda_n \to 0$. *For all* $k \ge 0$ *and* $n \in \mathbb{N}$, *we have*

$$G_{k,n} := \sum_{j=0}^{k} (1+\lambda_n) \cdot ... \cdot (1+(j-1)\lambda_n) \cdot [0, \lambda_n, ..., j\lambda_n; e_k] \le (k+1)!.$$

The main result is the following.

Theorem 4.2.4. (Gal-Opriş [45]) *Let* f *be analytic on the continuum* G, *that is there exists* $R > 1$ *such that* f *is analytic in* G_R, *given by* $f(z) = \sum_{k=0}^{\infty} a_k(f) F_k(z)$, $z \in G_R$. *Also, suppose that there exist* $M > 0$ *and* $A \in \left(\frac{1}{R}, 1\right)$, *with* $|a_k(f)| \le M \frac{A^k}{k!}$, *for all* $k = 0, 1, ...$, *(which implies* $|f(z)| \le C(r) M e^{Ar}$ *for all* $z \in G_r$, $1 < r < R$*).*

Let $1 < r < \frac{1}{A}$ *be arbitrary fixed. Then, there exist an index* $n_0 \in \mathbb{N}$ *and a constant* $C(r, f) > 0$ *depending on* r *and* f *only, such that for all* $z \in \overline{G_r}$ *and* $n \ge n_0$ *we have*

$$|W_n(f; \lambda_n, G; z) - f(z)| \le C(r, f) \cdot \lambda_n.$$

Remarks. 1) Theorem 4.2.4 generalizes Theorem 1.9.1, p. 126 in [34], in two senses : firstly, it is extended from compact disks with center at origin to compact sets and secondly, the order of approximation $O\left(\frac{1}{n}\right)$ is essentially improved to the order $O(\lambda_n)$, with $\lambda_n \to 0$ as fast we want.

2) It is clear that Theorem 4.2.4 holds under the more general hypothesis $|a_k(f)| \le P_m(k) \cdot \frac{A^k}{k!}$, for all $k \ge 0$, where P_m is an algebraic polynomial of degree m with $P_m(k) > 0$ for all $k \ge 0$.

3) There are many concrete examples for G when the conformal mapping Ψ and the Faber polynomials associated to G, and consequently when the Baskakov-Faber type operators too, can explicitly be written (see, e.g., [35], pp. 81-83, or [36]), as follows : $G = [-1, 1]$, G is the continuum bounded by the m-cusped hypocycloid, G is the regular m-star ($m = 2, 3, ...,$), G is the m-leafed symmetric lemniscate, $m = 2, 3, ...,$, G is a semidisk, or G is a circular lune.

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