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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

OPERATORIAL INCLUSIONS BY FIXED POINT TECHNIQUE IN VECTOR METRIC SPACES

PH.D. THESIS SUMMARY

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Contents

Introduction

Foreword

In recent decades, if we consult the electronic databases, we observe that the fixed point theory have recorded significant growth as the number of papers written by authors from various areas of mathematics. Also, it represents a powerful method of solving several problems arising in these areas of mathematics, especially in pure and applied mathematics.

The fixed point idea came to determine the solutions of the equation $x = f(x)$ and became famous in the study of existence and uniqueness problems of differential equations and inclusions.

Its development continues in research teams with the theme fixed point theory.

Among authors who have made major contributions in fixed point theory are found: S. Banach, B. Knaster, K. Kuratowski, S. Mazurkiewicz, R. Caccioppoli, V.V. Niemytzki, L. Kantorovich, M.A. Krasnoselskii, T. Wazewski, M. Edelstein, A.I. Perov, F.E. Browder, W.A. Kirk, L.B. Ćirić, I.A. Rus, K. Goebel, J. Caristi, B. Fisher, S. Heikkila, B.E. Rhoades, S. Seikkalä, M. Kwapisz, J. Matkowski, S. Reich, J. Dugundji, A. Granas, T.A. Makarevich, P.P. Zabrejko, E. De Pascale, T. Burton, R.P. Agarwal, D. O'Regan, C. Avramescu.

Motivation research

The Contraction Principle of Banach–Caccioppoli has a central role in the metrical theory of fixed point. Nowadays there are many generalizations of this result which were given in different types of metric spaces, including the generalized metric space, b-metric and E-metric space.

Through mathematical modeling, many problems from physics, biology, chemistry, engineering, etc. give rise to a two-dimensional or multi-dimensional equations system, respectively differential or integral inclusions.

Such systems of equations, respectively inclusions can be studied as an operatorial equation, respectively operatorial inclusion also in the context of a vector metric space. To determine if there is a solution, respectively a set of solutions for an unknown parameter of the system we need the fixed point technique.

Thus, appears the need to obtain fixed point results in an operatorial way used in many current works. After obtaining the solution existence of a system using a metrical fixed point theorem is sometimes wish some of its properties, called theories.

In this way, the problem may be complicated. For example, to consider as a model a system of equations, respectively inclusions each having in components a sum of two integral operators. Is required to find necessary conditions such that the system admits solution, respectively a set of solutions. So on the metric can be replaced by a b-metric or the system can be abstracted and to require the imposition of necessary conditions such that the system admits solution in a set of function spaces.

In this respect, the aim of this dissertation is to continue the scientific research on the following topics:

- fixed point theorems, which ensure the existence of solution in a generalized metric space for operators which satisfies a multivalued A-contraction condition in Nadler's sense, an open problem from [19];

- new fixed point theorems for singlevalued, respectively multivalued operators in E-metric spaces starting from the idea of [25] and other properties given in [5], [45], [99], [24];

- development of a theory for an E-metrical fixed point theorem for singlevalued, respectively multivalued operators using the classical model of [89], respectively [75];

- various extensions of the Contraction Principle for the case of φ-contractions in the context of E-metric spaces using the model from [43];

- Krasnoselskii type theorems and other connected results for a sum of two singlevalued and multivalued operators in generalized Banach spaces starting from classical results obtained in [48], [72], [20];

- existence results of the solution, respectively of the set of the solutions for the abstract case of Fredholm–Volterra type integral equations and inclusions systems in generalized Banach spaces starting from results obtained in [71], [72], [76], [77];

- Krasnoselskii type theorems for the sum of two singlevalued and multivalued operators in E-Banach spaces using ideas from [29], [31], [42], [96], [97];

- existence results of the solution, respectively of the set of the solutions for the case of Fredholm–Volterra type integral equations and inclusions systems in E-Banach spaces with the benchmark [71], [72], [29], [31], [42], [96], [97];

- fixed and strict fixed point theorems in generalized b-metric spaces for singlevalued and multivalued operators using the Picard and weak Picard operators technique from the ideas of [14], [16], [12];

- fixed point theorems in E-b-metric spaces for singlevalued and multivalued operators using the Picard and weak Picard operators technique in the cone of strict order unit elements, concept introduced in [57].

Thesis contents. Original results

The Ph.D. Thesis entitled „*Operatorial inclusions by fixed point technique in vector metric spaces*" is divided in four chapters, each chapter containing several sections.

Chapter 1: Vector metric spaces

In the first chapter we define the notions of generalized metric space and vector metric space, in particular when the metric takes values in a Riesz space. Also, we discuss about Archimedean and order completeness property (Dedekind), that can enjoy a Riesz space. In these metric spaces endowed with a finite and infinite dimensional vector metric, we define the notions of convergent sequence, Cauchy sequence, complete sequence, closed subset, the diameter and bounds of a subset of a vector metric space, some properties and topological elements. Also, we enounce some metrical and topological fixed point results preliminary to Krasnoselskii's theorem in a generalized Banach space, like Perov's theorem in a generalized metric space and Schauder's theorem in a generalized Banach space. For the case of multivalued operators we obtain new extended results.

Personal contributions: lemmas 1.3.16, 1.3.17, 1.3.18 and theorem 1.3.19, which represents Perov's theorem for an operator which satisfies a multivalued A-contraction condition in Nadler's sense, being given in response to an open problem enounced in A. Bucur, L. Guran, A. Petruşel [19]. In lemma 1.3.20 we establish a data dependence result for the excess between the fixed points sets of two multivalued operators which satisfies a multivalued A-contraction condition in Nadler's sense, and in theorem 1.3.21 we extend a preliminary result to Krasnoselskii's theorem given in L. Rybinski [92], which ensure the existence of a continuous selection for a multivalued operator, in the context of generalized Banach spaces.

The scientific paper which contain the original results of this chapter is:

I.-R. Petre, A. Petruşel, *Krasnoselskii's Theorem in generalized Banach spaces and applications*, Electron. J. Qual. Theory Differ. Equ., No. 85, 2012, 1-20.

Chapter 2: The theory of an E-metrical fixed point theorem

In the second chapter, starting from ideas of [29], [5] and [25], the new results include extensions of some metrical fixed point theorems for singlevalued and multivalued operators, by the classical theory to E-metrical spaces situation (theorems 2.1.3, 2.1.4, 2.1.6, 2.1.8 and 2.1.11). Also, we discuss a theory of the Contraction Principle for singlevalued and multivalued operators in E-metric spaces, which contains the extended Cauchy lemma 2.2.1, respectively the theorems 2.2.3, 2.2.12, 2.2.13 and 2.2.14 using the concept of theory of a metrical fixed point theorem introduced and studied by prof. I.A. Rus in the classical metric case. This consist in the study of some fixed point properties as: the existence and uniqueness of fixed points and strict fixed points, the data dependence of fixed points, the convergence of fixed points set, the Ulam–Hyers stability of the fixed point problem, well-posedness of fixed point problem property, the limit shadowing property and so on (see

[89] and [75]). In a similar way, the theory can be extended on the others studied metrical fixed point theorems which satisfies generalized contraction conditions in E-metric spaces. It provides examples for extensions made (examples 2.1.2 and 2.1.9) and applications for the Contraction Principle for singlevalued operators (Gronwall's lemma 2.2.5 and the comparison theorem 2.2.6).

In the context of E-metric spaces are presented various global and local extensions of the Contraction Principle for singlevalued operators (theorems 2.3.3, 2.3.4, 2.3.5, lemmas 2.3.6, 2.3.7 and theorem 2.3.9), respectively for multivalued operators (theorem 2.3.10 and problem 2.3.14) which satisfies a nonlinear φ-contraction condition. These results generalize well-known fixed point principles in the literature (historically, see [84], [47], [46], [98], [29], [96], [97] and [25]). Also, we present other metrical and topological fixed point results preliminary to Krasnoselskii's theorem in order complete E-metric spaces (lemmas 2.3.15, 2.3.16, theorem 2.3.17 and problem 2.3.19, but also an extended version of Cantor's theorem and of Cesaro's lemma, which found in lemma 2.3.20, respectively lemma 2.3.21).

The scientific papers which contains the original results of this chapter are:

I.-R. Petre, *Fixed points for φ-contractions in E-Banach spaces*, Fixed Point Theory, Vol. 13 (2), 2012, 623-640.

I.-R. Petre, *Fixed point theorems in vector metric spaces for single-valued operators*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity, Vol. 9, 2011, 59-80.

I.-R. Petre, *Fixed point theorems in vector metric spaces for multivalued operators*, Topol. Methods Nonlinear Anal. (submitted for publication).

Chapter 3: Topological fixed point theorems and applications in vector Banach spaces

In the third chapter we prove Krasnoselskii's theorem and other possible existence results of the fixed point for a sum of two operators in a generalized Banach space, respectively in an E-Banach space. The study of the fixed point occurs for singlevalued and multivalued operators which satisfies an A-contraction condition, respectively φ-contraction and a compactness condition, theorems 3.1.1, 3.1.2, 3.1.3, 3.1.4, 3.3.1 and problem 3.3.2. We present some open problems and also, the applying mode of theorems in the study of abstract systems of Fredholm–Volterra type integral equations and inclusions in a generalized Banach space, respectively in an E-Banach space, theorems 3.2.1, 3.2.3, 3.4.1 and problem 3.4.2.

The scientific papers which contains the original results of this chapter are:

I.-R. Petre, A. Petruşel, *Krasnoselskii's Theorem in generalized Banach spaces and applications*, Electron. J. Qual. Theory Differ. Equ., No. 85, 2012, 1-20.

I.-R. Petre, *Fixed points for φ-contractions in E-Banach spaces*, Fixed Point Theory, Vol. 13 (2), 2012, 623-640.

I.-R. Petre, *A multivalued version of Krasnoselskii's theorem in generalized Banach spaces*, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. (submitted for publication).

Chapter 4: Fixed point theorems in vector b-metric spaces

In the last chapter, we start from the classical notion of b-metric space appeared in [30] and several reference works, as [12], [38], [14] and [16]. Thus, we introduce the notions of generalized b-metric space, respectively E-b-metric space and a relevant concept of strict positivity in a Riesz space (see [57]). An advantage of this concept follows immediately by renunciation to the hypothesis $\varphi(t) < t$ for $t \in E_+$ on the o-comparison operator φ and using a kind of "ε–δ" formalism to prove our results in sections 4.3 and 4.4.

The personal contributions are found in examples 4.1.2, 4.1.3, 4.1.4, lemmas 4.1.5, 4.1.6, 4.1.7, 4.1.8, 4.1.9, 4.1.10 and in a nontrivial mode in propositions 4.3.3, 4.3.4, 4.3.5, lemma 4.3.6, corollary 4.3.7 and lemma 4.3.8, which works with the concept of strict positivity. Also, the fixed point theorems 4.2.1, 4.2.3, 4.2.4, 4.2.6 and strict fixed point 4.2.7, 4.2.8 in generalized b-metric spaces, respectively in E-b-metric spaces (the fixed point theorems 4.4.2 and 4.4.4) using the Picard and weak Picard operators technique, represents new results.

The scientific papers which contains the original results of this chapter are:

Zs. Páles, I.-R. Petre, *Iterative fixed point theorems in E-metric spaces*, Acta Math. Hung., DOI: 10.1007/s10474-012-0274-8.

I.-R. Petre, M. Bota, *Fixed point theorems on generalized b-metric spaces*, Publ. Math. Debrecen (accepted for publication).

I.-R. Petre, *Fixed point theorems in E-b-metric spaces*, Arabian J. Math. (submitted for publication).

Scientific papers. Conferences

The original contributions of the author, integral or partial included in this thesis, makes part by the following scientific papers:

1. I.-R. Petre, A. Petruşel, *Krasnoselskii's Theorem in generalized Banach spaces and applications*, Electron. J. Qual. Theory Differ. Equ., No. 85, 2012, 1-20, ISI indexed, Web of Science, ISSN: 1417-3875.

2. Zs. Páles, I.-R. Petre, *Iterative fixed point theorems in E-metric spaces*, Acta Math. Hung., DOI: 10.1007/s10474-012-0274-8, ISI indexed, Web of Science, ISSN: 0236-5294, ISSN: 1588-2632.

3. I.-R. Petre, *Fixed points for φ-contractions in E-Banach spaces*, Fixed Point Theory, Vol. 13 (2), 2012, 623-640, ISI indexed, CNCS A, ISSN: 1583-5022.

4. I.-R. Petre, M. Bota, *Fixed point theorems on generalized b-metric spaces*, Publ. Math. Debrecen, ISI indexed, Web of Science, ISSN: 0033-3883 (accepted for publication).

5. I.-R. Petre, *Fixed point theorems in vector metric spaces for single-valued operators*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity, Vol. 9, 2011, 59-80, BDI indexed, CNCS B+, ISSN: 1584-4536.

6. I.-R. Petre, *On the solution operator of a differential inclusion*, JP J. of Fixed Point Theory and Appl., Vol. 6 (2), 2011, 107-117, BDI indexed, ISSN: 0973-4228.

7. I.-R. Petre, *Fixed point theorems in vector metric spaces for multivalued operators*, Topol. Methods Nonlinear Anal., ISI indexed, Web of Science, ISSN: 1230-3429 (submitted for publication).

8. I.-R. Petre, *Fixed point theorems in E-b-metric spaces*, Arabian J. Math., ISI indexed, Web of Science, ISSN: 2193-5343 (submitted for publication).

9. I.-R. Petre, *A multivalued version of Krasnoselskii's theorem in generalized Banach spaces*, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. (submitted for publication).

A part of the original results which are founded in the above mentioned papers were annually presented in an international conference:

1. The 7th Internationale Conference of Applied Mathematics (ICAM7), 1-4 September, 2010, Baia Mare, Romania;

2. The International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA), 5-8 July, 2011, Cluj-Napoca, Romania;

3. The 10th International Conference of Fixed Point Theory and its Applications (IC-FPTAC), 9-15 July, 2012, Cluj-Napoca, Romania.

Keywords: A-contraction, applications of Krasnoselskii's theorem, convex cone, generalized contraction, multivalued contraction, Reich contraction, φ-contraction, Hausdorff convergence, vector convergence, data dependence, Fredholm–Volterra integral equation, order unit element, Contraction Principle extensions, linear lattice, Gronwall lemma, Fredholm–Volterra integral inclusion, matrix convergent to zero, iterative method, vector metric, Bielecki norm, vector norm, compact operator, contractive operator, multivalued operator, vector Picard operator, vector weak Picard operator, comparison principle, well-posedness of fixed point problem property, limit shadowing property, fixed point, strict fixed point, vector Banach space, vector metric space, vector b-metric space, Riesz space, Ulam–Hyers stability, sum of two operators, Krasnoselskii's theorem, Perov's theorem, the theory of a fixed point theorem.

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Chapter 1

Vector metric spaces

1.1 Generalized metric space. E-metric space

It is well known that historically, A.I. Perov [58], respectively A.I. Perov and A.V. Kibenko [59]) extended the classical Banach contraction principle to contraction mappings endowed with vector-valued metrics. In this sense, we recall in section 1.3 some basically results from A.-D. Filip, A. Petruşel [33], R. Precup [79] and some new ones. Also, in R. Precup [79], are pointed out some advantages of a vector-valued norm with respect to the usual scalar norm.

In the late of XX-th century and the beginning of XXI century appear works which treat results where the vector metric takes values in an infinite dimensional space, in particular, the Riesz space (see C. Çevik, I. Altun [25], W.A.J. Luxemburg, A.C. Zaanen [45], A.C. Zaanen [99]). These metric spaces as he called P.P. Zabrejko in [100] extend the generalized metric spaces. But in many results to obtain applications in these spaces is often difficult due to the abstracted work mode. Next we define the notion of generalized metric space and E-metric space.

Definition 1.1.1. ([58]) Let X be a nonempty set and consider the space $\mathbb{R}^m_+$ endowed with the usual component-wise partial order. The mapping $d : X \times X \to \mathbb{R}^m_+$ which satisfies all the usual axioms of the metric is called a *generalized metric* in Perov's sense and (X, d) is called a *generalized metric space*.

Let (X, d) be a generalized metric space in Perov's sense. Thus, if $v, r \in \mathbb{R}^m$, $v := (v_1, v_2, \ldots, v_m)$ and $r := (r_1, r_2, \ldots, r_m)$, then by $v \le r$ we mean

$v_i \leq r_i$, for each $i \in \{1, 2, \ldots, m\}$ and by $v < r$ we mean $v_i < r_i$, for each $i \in \{1, 2, \ldots, m\}$. Also, $|v| := (|v_1|, |v_2|, \ldots, |v_m|)$.

If $u, v \in \mathbb{R}^m$, with $u := (u_1, u_2, \ldots, u_m)$ and $v := (v_1, v_2, \ldots, v_m)$, then $\max(u, v) := (\max(u_1, v_1), \ldots, \max(u_m, v_m))$. If $c \in \mathbb{R}$, then $v \leq c$ means $v_i \leq c$, for each $i \in \{1, 2, \ldots, m\}$.

Definition 1.1.2. ([5]) A set E equipped with a partial order „$\leq$” is called a *partially ordered set*. In a partially ordered set $(E, \leq)$ the notation $x < y$ means $x \leq y$ and $x \neq y$. An order interval $[x, y]$ is the set $\{z \in E : x \leq z \leq y\}$. Notice that if $x \nleq y$, then $[x, y] = \emptyset$.

Definition 1.1.3. ([5]) An element z is the *supremum* of a pair of elements $x, y \in E$ if:

(i) z is an upper bound of the set $\{x, y\}$, i.e. $x \leq z$ and $y \leq z$;

(ii) z is the least such bound, i.e. $x \leq u$ and $y \leq u$ imply $z \leq u$.

The *infimum* of two elements $x, y \in E$ is defined similarly and we denote the supremum of such elements by $x \vee y = \sup\{x, y\}$, respectively the infimum by $x \wedge y = \inf\{x, y\}$.

Definition 1.1.4. ([5]) A partially ordered set $(E, \leq)$ is a *latice* if each pair of elements $x, y \in E$ has a supremum and an infimum.

The functions $(x, y) \mapsto x \vee y$ and $(x, y) \mapsto x \wedge y$ are the lattice operations on E. In a lattice, every finite nonempty set has a supremum and an infimum. If $\{x_1, \ldots, x_n\}$ is a finite subset of a lattice, then we write $\bigvee\limits_{i=1}^{n} x_i = \sup\{x_1, \ldots, x_n\}$, respectively $\bigwedge\limits_{i=1}^{n} x_i = \inf\{x_1, \ldots, x_n\}$.

Definition 1.1.5. Let E be a real linear space. We say that $K \subset E$ is a *convex cone* if:

(i) K is a cone, i.e. $tK \subset K$, for all $t > 0$ (equivalent, $t > 0$ and $x \in K$ imply $tx \in K$);

(ii) K is convex, i.e. $K + K \in K$ (equivalent, $x, y \in K$ imply $x + y \in K$).

Definition 1.1.6. ([5]) A real linear space E with an order relation „$\leq$” that is compatible with the algebraic structure of E in the sense that for any $x, y \in E$ are satisfied two properties:

(1) $x \leq y$ imply $x + z \leq y + z$, for any $z \in E$;

(2) $x \leq y$ imply $tx \leq ty$, for any $t > 0$;

is called an *ordered linear space.*

Definition 1.1.7. ([5]) An ordered linear space that is also a lattice is called a *Riesz space* or *linear lattice.*

The geometric interpretation of the lattice structure on a Riesz space is shown in Figure 1.

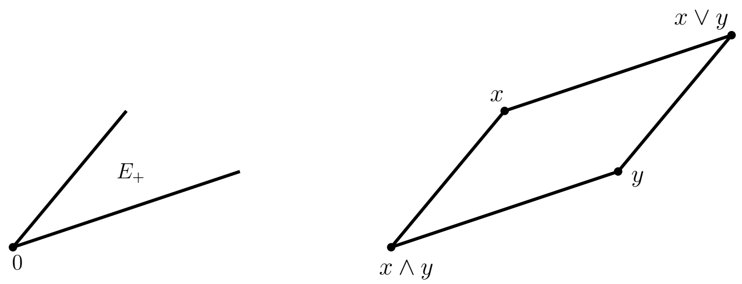

Figure 1: The geometry of sup and inf.

Definition 1.1.8. ([5]) For a vector x in a Riesz space, we define

$$|x| = x \vee (-x) \text{ the absolute value of } x.$$

Many familiar spaces are Riesz spaces, as the following examples show.

Example 1.1.9. ([5]) The Euclidean space $\mathbb{R}^n$ with norm defined by

$$||x|| = \left(\sum_{i=1}^{n} x_i \right)^{\frac{1}{2}}$$

and with the usual ordering relation, where $x = (x_1, \ldots, x_n) \leq y = (y_1, \ldots, y_n)$ whenever $x_i \leq y_i$, for each $i = 1, \ldots, n$ is a Riesz space. The infimum and supremum of two vectors x and y are given by

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\}) \text{ and}$$
$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}).$$

Example 1.1.10. ([5]) Both the vector space $C(X)$ of all continuous real functions (with X a compact set) and the vector space $C_b(X)$ of all bounded continuous real functions on the topological space X, with norms defined by

$$||f||_\infty = \sup\{|f(x)| : x \in X\}$$

and with the ordering relation defined pointwise, i.e. $f \leq g$ whenever $f(x) \leq g(x)$, for each $x \in X$ are Riesz spaces. The lattice operations of the real functions f and g are given by

$$(f \vee g)(x) = \max(f(x), g(x)) \text{ and}$$
$$(f \wedge g)(x) = \min(f(x), g(x)).$$

Example 1.1.11. ([5]) The vector space $L_p(\mu), 0 \leq p \leq \infty$, with norm defined by

$$||f||_p = \begin{cases} \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}, 0 \leq p < \infty \\ ess\sup |f|, p = \infty \end{cases}$$

and with the almost everywhere pointwise ordering relation, i.e. $f \leq g$ in $L_p(\mu)$ whenever $f(x) \leq g(x)$, for μ-almost every x is a Riesz space. The lattice operations are given by

$$(f \vee g)(x) = \max(f(x), g(x)) \text{ and}$$
$$(f \wedge g)(x) = \min(f(x), g(x)).$$

Definition 1.1.12. (L. Kantorovich, [25], [100]) Let X be a nonempty set and let E be a Riesz space. The function $d : X \times X \to E$ is said to be a *vector metric* or *E-metric* if it satisfies the following properties:

(a) $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Also, the triple (X, d, E) is said to be a *vector metric space* or an *E-metric space*.

It is obvious that E-metric spaces generalize the notion of metric spaces and for arbitrary elements x, y, z, w of an E-metric space, the following properties hold:

(i) $0 \leq d(x, y)$;

(ii) $d(x, y) = d(y, x)$;

(iii) $|d(x, z) - d(y, z)| \leq d(x, y)$;

(iv) $|d(x, z) - d(y, w)| \leq |d(x, y) - d(z, w)|$.

Example 1.1.13. (L. Kantorovich, [25], [100]) A Riesz space E is an E-metric space with $d : E \times E \to E$ defined by

$$d(x, y) = |x - y|.$$

This E-metric is called to be the *absolute valued metric* on E. For more examples of E-metric spaces, see C. Çevik, I. Altun [25].

If X is a nonempty set and $f : X \to X$ is a singlevalued operator, we denote by $\mathrm{Fix}(f) := \{x \in X \mid x = f(x)\}$, and if $F : X \to P(X)$ is a multivalued operator, we denote by

$$\begin{aligned}
\mathrm{Fix}(F) &:= \{x \in X \mid x \in F(x)\}; \\
\mathrm{SFix}(F) &:= \{x \in X \mid \{x\} = F(x)\}; \\
s(X) &:= \left\{(x_n)_{n\in\mathbb{N}^*} \mid x_n \in X,\, n \in \mathbb{N}^*\right\}.
\end{aligned}$$

In the context of a metric space (X, d), we will denote by

$$\begin{aligned}
\mathcal{P}(X) &:= \{Y \mid Y \subseteq X\}; \\
P(X) &:= \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}; \\
P_{cl}(X) &:= \{Y \in P(X) \mid Y \text{ is closed}\}; \\
P_{b,cl} &:= \{Y \in P(X) \mid Y \text{ is bounded and closed}\} \\
P_{b,cl,cv}(X) &:= \{Y \in P(X) \mid Y \text{ is bounded, closed and convex}\}; \\
P_{cp}(X) &:= \{Y \in P(X) \mid Y \text{ is compact}\}; \\
P_{cp,cv} &:= \{Y \in P(X) \mid Y \text{ is compact and convex}\}; \\
\mathrm{Graph}(F) &:= \{(x, y) \in X \times X \mid y \in F(x)\}
\end{aligned}$$

and we will use the following functionals:

$D_d : P(X) \times P(X) \to \mathbb{R}_+$, $D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ - the gap functional;

$\delta_d : P(X) \times P(X) \to \mathbb{R}_+$, $\delta_d(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$ - the diameter functional;

$\rho_d : P(X) \times P(X) \to \mathbb{R}_+$, $\rho_d(A, B) = \sup\{D_d(a, B) : a \in A\}$ - the excess functional;

$H_d : P(X) \times P(X) \to \mathbb{R}_+$, $H_d(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}$ - the Pompeiu–Hausdorff functional.

If A and B are two nonempty sets of a generalized metric space (X, d) with $d(x,y) := \begin{bmatrix} d_1(x,y) \\ \vdots \\ d_m(x,y) \end{bmatrix}$, then we denote by

$$D(A,B) = \begin{bmatrix} D_{d_1}(A,B) \\ \vdots \\ D_{d_m}(A,B) \end{bmatrix}, \ \delta(A,B) = \begin{bmatrix} \delta_{d_1}(A,B) \\ \vdots \\ \delta_{d_m}(A,B) \end{bmatrix},$$

$$\rho(A,B) = \begin{bmatrix} \rho_{d_1}(A,B) \\ \vdots \\ \rho_{d_m}(A,B) \end{bmatrix}, \ H(A,B) = \begin{bmatrix} H_{d_1}(A,B) \\ \vdots \\ H_{d_m}(A,B) \end{bmatrix}.$$

Notice that if A and B are two nonempty sets of an E-metric space (X, d, E), then these functionals can be defined as in the context of metric spaces, in the particular case when A and B are two E-bounded sets and the Riesz space E is order complete. These restrictive conditions, in view of Definition 1.2.9, ensure that there exists a supremum, respectively infimum in E.

The multivalued operator $F : X \to P(X)$ is called E-closed if Graph (F) is E-closed in $X \times X$. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$, recurrently defined by

$$\begin{cases} x_0 = x, x_1 = y; \\ x_{n+1} \in F(x_n), \text{ for all } n \in \mathbb{N} \end{cases}$$

is called the sequence of successive approximations of F starting from $(x, y) \in X \times X$.

For definitions of E-boundedness, E-closedness of a set and for order completeness property of a Riesz space, see next section.

1.2 Properties and topological elements

In the case of generalized metric spaces in the sense of Perov, the notions of convergent sequence, Cauchy sequence, completeness, open and closed subset are similar to those for usual metric spaces. Also, in what follows we present some elements of topology (see, for example, A. Granas and J. Dugundji [35], P.P. Zabrejko [99], E. Zeidler [101]).

Definition 1.2.1. ([25]) Let (X, d) be a generalized metric space. A subset $A \subset X$ is called *open* if, for any $x \in A$, there exists $r \in \mathbb{R}_+^m$ with $r > 0$ such that $B(x_0, r) \subset A$, where $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ denotes the open ball centered in x_0 with radius r. Any open ball is an open set and the collection of all open balls of X generates the *generalized metric topology* on X.

Definition 1.2.2. ([99]) Let (X, d) be a generalized metric space. A subset C of X is called *compact* if, every open cover of C has a finite subcover. A subset C of X is *sequentially compact* if, every sequence in C contains a convergent subsequence with limit in C.

A subset C of X is *totally bounded* if, for each $\varepsilon \in \mathbb{R}_+^m$ with $\varepsilon > 0$, there exists a finite number of elements $x_1, x_2, \dots, x_n$ in X such that $C \subset \bigcup\limits_{i=1}^{n} B(x_i, \varepsilon)$. The set $\{x_1, x_2, \dots, x_n\}$ is called a *finite ε-net.*

A set C of a topological space is *relatively compact* if its closure is compact, i.e., $\bar{C}$ is compact. The set C is *sequentially relatively compact* if, every sequence in C contains a convergent subsequence (the limit need not be an element of C), i.e., $\bar{C}$ is sequentially compact.

Proposition 1.2.3. *([99]) If C is a subset of X, then the following affirmations hold:*

(i) C is compact $\Leftrightarrow$ C is sequentially compact $\Leftrightarrow$ C is closed and totally bounded;

(ii) C relatively compact $\Leftrightarrow$ C sequentially relatively compact $\Leftrightarrow$ C totally bounded.

Definition 1.2.4. ([86], [100]) Let X be a nonempty set and let $|\cdot| : X \to \mathbb{R}^m$ be a norm on X. Then, the pair $(X, |\cdot|)$ is called a *generalized normed space.* If, moreover, $(X, |\cdot|)$ has the property that any Cauchy sequence from X is convergent in norm, then we say that $(X, |\cdot|)$ is a *generalized Banach space.*

Definition 1.2.5. ([101]) Let X, Y be two generalized normed spaces, $K \subset X$ and let $f : K \to Y$ be an operator. Then f is called:

(i) *compact*, if for any bounded subset $A \subset K$ we have $f(A)$ is relatively compact or $\overline{f(A)}$ is compact;

(ii) *complete continuous*, if f is continuous and compact;

(iii) *with relatively compact range*, if f is continuous and $f(K)$ is relatively compact or $\overline{f(K)}$ is compact.

In the following we present some order and lattice properties that can enjoy Riesz spaces and in the context of E-metric spaces, we will define the notions of E-convergent sequence, E-Cauchy sequence, E-complete sequence, E-open and E-closed subset, the E-diameter and the E-boundedness of a subset of an E-metric space as well as other properties, which works in a different manner that the usual ones.

Definition 1.2.6. ([5]) Let E be a Riesz space. A subset $A \subset E$ is *order bounded from above* (*from below*) if there is a vector u called an upper bound (lower bound) of A that dominates (is dominated by) each element of A, that is, $a \leq u$ $(a \geq u)$, for each $a \in A$. Therefore, A is *order bounded* if A is both order bounded from above and from below.

Definition 1.2.7. ([5]) Let E be a Riesz space and let (x_n) be a sequence in E. We say that (x_n) is *decreasing* (*increasing*), we denote $x_n \downarrow$ $(x_n \uparrow)$, if $n \geq m$ imply $x_n \leq x_m$ $(x_n \geq x_m)$ and the symbol $x_n \downarrow \geq x$ $(x_n \uparrow \leq x)$ denotes a decreasing sequence (increasing) order bounded from below (above) by x.

The notation $x_n \downarrow x$ means that x_n is a decreasing sequence and $\inf\{x_n\} = x$. If $(x_n), (y_n) \subset E$, then some basic properties of decreasing sequences are:

$x_n \downarrow x$ and $y_m \downarrow y$ imply $x_n + y_m \downarrow x + y$;

$x_n \downarrow x$ imply $\lambda x_n \downarrow \lambda x$, for $\lambda > 0$ and $\lambda x_n \uparrow \lambda x$, for $\lambda < 0$;

$x_n \downarrow x$ and $y_m \downarrow y$ imply $x_n \vee y_m \downarrow x \vee y$ and $x_n \wedge y_m \downarrow x \wedge y$.

The meaning of the notation $x_n \uparrow x$ and some basic properties of increasing sequences are similar.

Definition 1.2.8. ([5]) We say that a Riesz space E is *Archimedean* if, $\frac{1}{n}x \downarrow 0$ for each $x \in E_+$, where

$$E_+ := \{x \in E : x \geq 0\} \text{ is the positive cone of } E.$$

Definition 1.2.9. ([5]) We say that a Riesz space E is *order complete* or *Dedekind complete* if every nonempty subset of E which is bounded from above has a supremum (equivalently, every nonempty subset of E which is bounded from below has an infimum).

Lemma 1.2.10. *([5]) Every order complete Riesz space is Archimedean.*

The converse of Lemma 1.2.10 is false, an Archimedean Riesz space, but not order complete is $C[0,1]$.

Example 1.2.11. ([5]) Consider the sequence of piecewise linear functions $(f_n)_{n\geq 2}$ and $(g_n)_{n\geq 2}$ in $C[0,1]$ defined by

$$f_n(x) = \begin{cases} 1, \text{ if } 0 \leq x \leq \frac{1}{2} - \frac{1}{n}, \\ -n\left(x - \frac{1}{2}\right), \text{ if } \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2}, \\ 0, \text{ if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

$$g_n(x) = \begin{cases} 1, \text{ if } 0 \leq x \leq 1 - \frac{1}{n}, \\ n(1-x), \text{ if } 1 - \frac{1}{n} \leq x \leq 1. \end{cases}$$

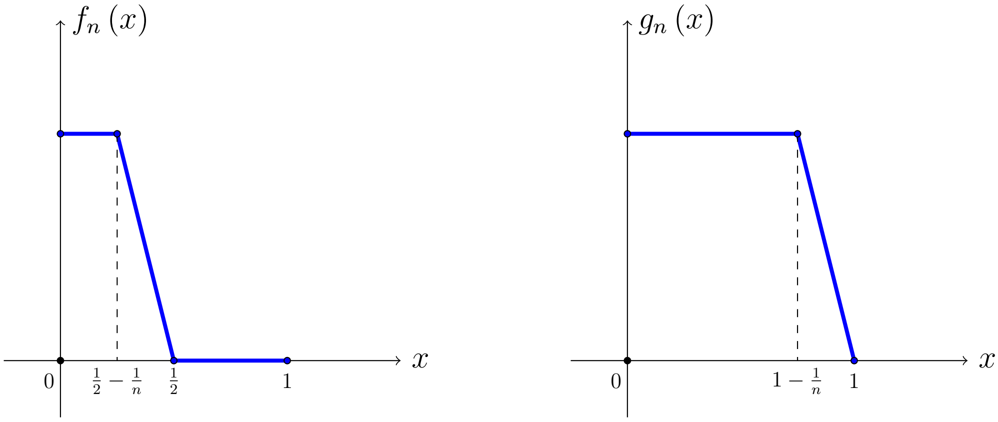

Figure 2: Graphics of functions f_n and g_n.

We have $0 \leq f_n \uparrow\leq \mathbf{1}$ in $C[0,1]$, where $\mathbf{1}$ is the constant function one, but $\{f_n\}$ does not have a supremum in $C[0,1]$ (see Figure 2), thus $C[0,1]$ is not order complete. Also, notice that the implication

$$f_n(x) \uparrow f(x), \text{ for each } x \in [0,1] \text{ imply } f_n \uparrow f$$

is true in the lattice sense. On the other hand, $f_n \uparrow f$ in the lattice sense does not imply that $f_n(x) \uparrow f(x)$, for each $x \in [0,1]$. A such example is function g_n (see Figure 2), where $g_n \uparrow \mathbf{1}$ in the lattice sense, while $g_n(1) = 0$, for all $n \in \mathbb{N}^*$.

Definition 1.2.12. ([25], [29], [100]) Let E be a Riesz space. A sequence (b_n) in E *o-converges* to some $b \in E$, written $b_n \xrightarrow{o} b$, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b| \leq a_n$, for all $n \in \mathbb{N}$.

Definition 1.2.13. ([25], [29], [100]) Let E, F be two Riesz spaces and let $f : E \to F$ be a function. We say that f is *o-continuous* if, $b_n \xrightarrow{o} b$ in E imply $f(b_n) \xrightarrow{o} f(b)$ in F.

Definition 1.2.14. ([25], [29], [100]) Let E be a Riesz space. A sequence (b_n) in E is called *o-Cauchy*, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ şi $|b_n - b_{n+p}| \le a_n$, for all $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$.

Definition 1.2.15. ([25], [29], [100]) We say that a Riesz space E is *o-complete* if each o-Cauchy sequence is o-convergent.

For many aspects of order and lattice properties, order convergence and order continuity in a Riesz space, the interested reader may consult the book of C.D. Aliprantis, K.C. Border [5].

Definition 1.2.16. ([25], [29], [100]) Let (X, d, E) be an E-metric space. A sequence (x_n) in X *E-converges* to some $x \in E$, written $x_n \xrightarrow{d,E} x$, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \le a_n$, for all $n \in \mathbb{N}$.

Definition 1.2.17. ([25], [29], [100]) Let (X, d, E) be an E-metric space. A sequence (x_n) in X is called *E-Cauchy*, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \le a_n$, for all $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$.

Definition 1.2.18. ([25], [29], [100]) An E-metric space (X, d, E) is called *E-complete* if each E-Cauchy sequence in X E-converges to a limit in X.

Definition 1.2.19. ([25], [29], [100]) LetX, Y be two E-metric spaces and let $f : X \to Y$ be a function. We say that f is *E-continuous* if, $x_n \xrightarrow{d,E} x$ in X imply $f(x_n) \xrightarrow{d,E} f(x)$ in Y.

Lemma 1.2.20. *([25]) If $x_n \xrightarrow{d,E} x$, then the following properties hold:*

1. *The limit x is unique;*
2. *Any subsequence of (x_n) E-converges to x;*
3. *If $y_n \xrightarrow{d,E} y$ then $d(x_n, y_n) \xrightarrow{o} d(x, y)$.*

If (X, d, E) is an E-metric space, then a subset $A \subset X$ is called *E-open* if for any $x \in A$, there exists $r > 0$ in E such that $B(x, r) \subset A$, where $B(x, r) = \{y \in X \mid d(x, y) < r\}$. Any E-open ball is an E-open set and the collection of all E-open subsets of X represents the E-metric topology on X denoted by $\tau_{d,E}$.

In an E-metric space, the notions of compact set, relatively compact set, totally bounded set, compact operator, complete continuous operator and operator with relatively compact range, respectively the notions of sequentially compactness (characterized by sequences) are defined as in the context of generalized metric spaces by replacing the space $\mathbb{R}^m$ with the Riesz space E. Proposition 1.2.3 also holds in the case of E-metric spaces (see A.C. Zaanen [99], pp. 500). Similarly, the concepts of E-closed and E-bounded set are defined in the E-metric sense.

Definition 1.2.21. ([25]) Let (X, d, E) be an E-metric space. We say that a subset $Y \subset X$ is *E-closed* if, $(x_n) \subset Y$ and $x_n \xrightarrow{d,E} x$ imply $x \in Y$.

Definition 1.2.22. ([25]) Let (X, d, E) be an E-metric space. If $A \subset X$ is a nonempty set, then the symbol

$$\delta(A) = \sup\{d(x, y) : x, y \in A\}$$

is called the *E-diameter* of A if $\sup\{d(x, y) : x, y \in A\}$ exists in E. Furthermore, if there exists an $a > 0$ in E such that $d(x, y) \leq a$, for any $x, y \in A$, then A is called an *E-bounded set.*

Corollary 1.2.23. *([25]) If E is an order complete Riesz space, then any E-bounded set of X has an E-diameter.*

Theorem 1.2.24. *([25]) Let (X, d, E) be an E-metric space. Then the following affirmations hold:*

(i) Any E-convergent sequence is E-Cauchy;

(ii) Any E-Cauchy sequence is E-bounded;

(iii) If an E-Cauchy sequence (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{d,E} x$, then $x_n \xrightarrow{d,E} x$;

(iv) If (x_n) and (y_n) are two E-Cauchy sequences, then $(d(x_n, y_n))$ is an o-Cauchy sequence.

Remark 1.2.25. ([25]) If $E = \mathbb{R}$, the concepts of E-convergence and metric convergence are the same, respectively the concepts of E-Cauchy sequence and Cauchy sequence are the same. If $X = E$ and d is the absolute valued vector metric on X, then the concepts of E-convergence and o-convergence are the same.

Definition 1.2.26. ([29]) Let X be a linear space and let E be a Riesz space. A function $||\cdot|| : X \to E$ is called an *E-norm* on X if it satisfies the following properties:

(a) $||x|| \geq 0$, for all $x \in X$;

(b) $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in X$.

Moreover, the triple $(X, ||\cdot||, E)$ is called an *E-normed space*.

Remark 1.2.27. ([29]) If $||\cdot||$ is an E-norm on X, then the function $d : X \times X \to E$, $d(x, y) = ||x - y||$ is an E-metric on X and d is called the E-metric generated by the E-norm $||\cdot||$.

Definition 1.2.28. ([29]) An E-normed space $(X, ||\cdot||, E)$ is called a *vector Banach space* (or *E-Banach space*) if any E-Cauchy sequence in X is E-convergent with respect to $||\cdot||$.

If $|\cdot|$ represents the absolute value of the Riesz space E, then $(E, |\cdot|, E)$ is an E-Banach space. Any E-normed Riesz space is Archimedean and thus, an E-Banach space is obviously Archimedean (see A.C. Zaanen [99]).

1.3 Fixed point results in generalized metric spaces

Already knowing what means a vector metric, which takes values in the finite dimensional space $\mathbb{R}^m$ or the Riesz space E, we propose in this section to pay attention to notions and to metrical and topological fixed point results, which are based on generalized metric in the sense of Perov. Next chapter will be dedicated exclusively to E-metric.

Now, we recall how to define the contraction and other known helpful results for the proof of Krasnoselskii's theorem for singlevalued operators in generalized Banach spaces.

Definition 1.3.1. ([94]) A square matrix of real numbers is said to be *convergent to zero* if and only if, $A^n \longrightarrow 0$ as $n \to \infty$.

Definition 1.3.2. ([58]) Let (X, d) be a generalized metric space and let $f : X \to X$ be a singlevalued operator. Then, f is called a *singlevalued A-contraction* if and only if, $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and

$$d(f(x), f(y)) \leq Ad(x, y), \text{ for any } x, y \in X.$$

Definition 1.3.3. ([86]) Let (X, d) be a generalized metric space. Then $f : X \to X$ is a *Picard operator* (briefly *PO*), if:

(i) $\text{Fix}(f) = \{x^*\};$

(ii) for any $x_0 \in X$, the sequence $x_n = f^n(x_0)$ converges to the fixed point of f.

Definition 1.3.4. ([86]) Let (X, d) be a generalized metric space and let $f : X \to X$ be a Picard operator. Then f is a *M-Picard operator* (briefly *M-PO*) if and only if, $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and there exists the operator $f^\infty : X \to X$, $f^\infty(x) = \lim\limits_{n\to\infty} f^n(x_0)$ such that

$$d[x_0, f^\infty(x_0)] \leq Md[x_0, f(x_0)], \text{ for any } x_0 \in X.$$

Theorem 1.3.5. *(Perov [58]). Let (X, d) be a complete generalized metric space and let $f : X \to X$ be a singlevalued A-contraction, then:*

(i) there exists a unique fixed point x^* for f and the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x_0)$ is convergent to x^*, for all $x_0 \in X$ and each $n \in \mathbb{N}^*$;

(ii) $d(x_n, x^*) \leq A^n (I - A)^{-1} d(x_0, x_1)$, for all $n \in \mathbb{N}^*$.

Lemma 1.3.6. *([79], [87]) Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the following statements are equivalent:*

(i) A is a matrix convergent to zero;

(ii) The eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;

(iii) The matrix $I - A$ is non-singular and $(I - A)^{-1} = I + A + \cdots + A^n + \cdots$;

(iv) The matrix $I - A$ *is non-singular and* $(I-A)^{-1}$ *has nonnegative elements;*

(v) $A^n q \longrightarrow 0$ *and* $qA^n \longrightarrow 0$ *as* $n \to \infty$*, for any* $q \in \mathbb{R}^m$*.*

Example 1.3.7. ([79]) Some examples of matrices convergent to zero are:

1) $A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

2) $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

3) $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

Theorem 1.3.8. *(Schauder [29]). Let* $(X, |\cdot|)$ *be a generalized Banach space, let* $Y \subset X$ *be a closed and convex set and let* $f : Y \to Y$ *be an operator with relatively compact range. Then* f *has at least one fixed point in* Y*.*

For the case of multivalued operators we have the following notions.

Definition 1.3.9. ([19]) Let (X, d) be a generalized metric space, $Y \subset X$ and let $F : Y \to P(X)$ be a multivalued operator. Then, F is called a *multivalued A-contraction* if and only if, $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and for any $x, y \in Y$ and for each $u \in F(x)$, there exists $v \in F(y)$ such that

$$d(u, v) \leq Ad(x, y).$$

We recall that a multivalued operator $F : X \to P(Y)$ is called *lower semi-continuous* (briefly l.s.c.) in $x_0 \in X$ if and only if, for any open set $U \subset X$ such that $F(x_0) \cap U \neq \emptyset$, there exists a neighborhood V for x_0 such that for any $x \in V$, we have that $F(x) \cap U \neq \emptyset$.

The multivalued operator F is called *Hausdorff lower semi-continuous* (briefly H-l.s.c.) in $x_0 \in X$ if and only if, for any $\varepsilon \in \mathbb{R}^m_+$ with $\varepsilon > 0$, there exists $\eta_\varepsilon \in \mathbb{R}^m_+$ with $\eta_\varepsilon > 0$ such that for any $x \in B(x_0, \eta_\varepsilon)$, we have $F(x_0) \subset V(F(x), \varepsilon)$, where

$$V(F(x), \varepsilon) = \{x \in X : D(x, F(x)) \leq \varepsilon\}.$$

Notice now that a generalized Pompeiu–Hausdorff functional

$$H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbb{R}^m_+$$

can be introduced on a generalized metric space in the sense of Perov and thus, the concept of multivalued contraction mapping introduced by S.B. Nadler Jr. in [50] can be extended to generalized metric spaces in the sense of Perov.

Definition 1.3.10. ([19]) Let $Y \subset X$ and let $F : Y \to P_{b,cl}(X)$ be a multivalued operator. Then, F is called a *multivalued A-contraction in Nadler's sense* if and only if, $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and

$$H[F(x), F(y)] \leq Ad(x, y), \text{ for any } x, y \in Y.$$

To obtain dual type results, in a similar way we can consider the condition

$$H^t[F(x), F(y)] \leq d^t(x, y)A, \text{ for any } x, y \in Y,$$

where A^t denotes transposed of matrix A.

Also, by properties of H, if F is a multivalued A-contraction in Nadler's sense it follows that F is a multivalued A-contraction.

Definition 1.3.11. ([19]) Let (X, d) be a generalized metric space. Then $F : X \to P(X)$ is a *multivalued weak Picard operator* (briefly MWP operator), if for any $x \in X$ and $y \in F(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in F(x_n)$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point of F.

For examples of MWP operators, see A. Petruşel [73] and I.A. Rus, A. Petruşel, A. Sântămărian [90].

Definition 1.3.12. ([73]) Let (X, d) be a generalized metric space and let $F : X \to P(X)$ be a MWP operator. Then we define the multivalued operator

$$F^\infty : \text{Graph}(F) \to P(\text{Fix}(F))$$

by the formula $\{F^\infty(x, y) = z \in \text{Fix}(F) :$ there exists a sequence of successive approximations of F starting from (x, y) which converges to $z\}$.

Definition 1.3.13. ([82]) Let $X, Y \neq \emptyset$ and let $F : X \to P(Y)$ be a multivalued operator. Then the singlevalued operator $f : X \to Y$ is a *selection* for F if and only if, $f(x) \in F(x)$ for any $x \in X$.

Definition 1.3.14. ([73]) Let (X, d) be a generalized metric space and let $F : X \to P(X)$ be a MWP operator. Then F is called a *multivalued M-weak Picard operator* (briefly M-MWP operator) if and only if, $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and there exists a selection f^∞ for F^∞ such that $d[x, f^\infty(x, y)] \leq Md(x, y)$, for any $(x, y) \in \mathrm{Graph}(F)$.

Lemma 1.3.15. *([86]) Let $(X, |\cdot|)$ be a generalized Banach space. Then:*

$$H(Y + Z, Y + W) \leq H(Z, W), \text{ for each } Y, Z, W \in P_b(X).$$

We recall that a measurable multivalued operator $F : [a, b] \to P_{cp}(\mathbb{R}^n)$ is said to be *integrable bounded* if and only if, there exists a Lebesgue integrable function $m : [a, b] \to \mathbb{R}^n$ such that for all $v \in F(t)$, we have $|v| \leq m(t)$, a.e. on $[a, b]$. For a measurable and integrable bounded multivalued operator F, the set S^1_F of all Lebesgue integrable selections for F is closed and nonempty (see H. Covitz, S.B. Nadler Jr. [27]).

To prove Krasnoselskii's theorem in our section 3.1 and the other connected results for a sum of two multivalued operators in generalized Banach spaces, we need to present now some new auxiliary results:

- we prove Perov's Theorem for an operator which satisfies a multivalued A-contraction condition in Nadler's sense, an answer to an open problem enounced in A. Bucur, L. Guran, A. Petruşel [19];

- we establish a data dependence result for the excess between the fixed points sets of two operators which satisfies a multivalued A-contraction condition in Nadler's sense;

- we extend a preliminary result to Krasnoselskii's theorem given in L. Rybinski [92].

Lemma 1.3.16. *Let (X, d) be a generalized metric space and let $A, B \subset P_{cl}(X)$, $q > 1$. Then, for any $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq qH(A, B).$$

Lemma 1.3.17. *Let (X, d) be a generalized metric space, $A \in P(X)$ and $x \in X$. Then $D(x, A) = 0$ if and only if $x \in \bar{A}$.*

Lemma 1.3.18. *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ be a matrix convergent to zero. Then, there exists $Q > 1$ such that for any $q \in (1, Q)$ we have that qA is convergent to 0.*

Theorem 1.3.19. *Let (X, d) be a complete generalized metric space and let $F : X \to P_{cl}(X)$ be a multivalued A-contraction in Nadler's sense. Then, F is a $(I - A)^{-1}$-MWP operator.*

Lemma 1.3.20. *Let (X, d) be a complete generalized metric space and $F_1, F_2 : X \to P_{b,cl}(X)$ be two multivalued A-contractions in Nadler' sense. Then:*

$$\rho[\mathrm{Fix}(F_1), \mathrm{Fix}(F_2)] \leq (I - A)^{-1} \begin{bmatrix} \sup\limits_{x\in X}\rho_{d_1}[F_1(x), F_2(x)] \\ \vdots \\ \sup\limits_{x\in X}\rho_{d_m}[F_1(x), F_2(x)] \end{bmatrix}.$$

Theorem 1.3.21. *Let (X, d) be a generalized metric space and Y be a closed subset of a generalized Banach space $(Z, \|\cdot\|)$. Assume that the multivalued operator $F : X \times Y \to P_{cl,cv}(Y)$ satisfies the following conditions:*

(i) *A is a matrix convergent to zero and* $H(F(x, y_1), F(x, y_2)) \leq A\|y_1 - y_2\|$, *for each* $(x, y_1), (x, y_2) \in X \times Y$;

(ii) *for every $y \in Y$, $F(\cdot, y)$ is H-l.s.c. on X.*

Then there exists a continuous mapping $f : X \times Y \to Y$ such that:

$$f(x, y) \in F(x, f(x, y)), \text{ for each } (x, y) \in X \times Y.$$

Chapter 2

The theory of an E-metrical fixed point theorem

2.1 E-metrical fixed point theorems

A reference result to study the existence and uniqueness of fixed points in an E-metric space is given in a theorem by C. Çevik and I. Altun in [25].

Theorem 2.1.1. *([25]) Let (X, d, E) be an E-complete vector metric space with E is Archimedean and let $f : X \to X$ be an operator, which satisfies a k-contraction condition, i.e. there exists a constant $k \in [0, 1)$ such that*

$$d\left[f(x), f(y)\right] \leq kd(x, y), \text{ for any } x, y \in X.$$

Then f has a unique fixed point $x^* \in X$ and for any $x_0 \in X$, the iterative sequence (x_n) defined by $x_n = f(x_{n-1})$ for any $n \in \mathbb{N}^*$, E-converges to the fixed point of f.

Example 2.1.2. ([39]) Let $E = \mathbb{R}^2$ with coordinatwise ordering (thus E is Archimedean) and we consider

$$X = \left\{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\right\} \cup \left\{(0, x) \in \mathbb{R}^2 : 0 \leq x \leq 1\right\}.$$

The mapping $d : X \times X \to E$ is defined by

$$d\left[(x, 0), (y, 0)\right] = \left(\frac{4}{3}|x - y|, |x - y|\right),$$
$$d\left[(0, x), (0, y)\right] = \left(|x - y|, \frac{2}{3}|x - y|\right),$$
$$d\left[(x, 0), (0, y)\right] = \left(\frac{4}{3}x + y, x + \frac{2}{3}y\right).$$

Then X is an E-complete vector metric space. Let $f : X \to X$ with $f((x,0)) = (0,x)$ and $f((0,x)) = \left(\frac{x}{2}, 0\right)$, then f satisfies the singlevalued k-contraction condition for $k = \frac{3}{4}$, which follows by the inequalities

$$\begin{aligned}
d\left[f(x,0), f(y,0)\right] &\leq \frac{3}{4} d\left[(x,0),(y,0)\right], \\
d\left[f(x,0), f(0,y)\right] &\leq \frac{3}{4} d\left[(x,0),(0,y)\right], \\
d\left[f(0,x), f(y,0)\right] &\leq \frac{3}{4} d\left[(0,x),(y,0)\right], \\
d\left[f(0,x), f(0,y)\right] &\leq \frac{3}{4} d\left[(0,x),(0,y)\right]
\end{aligned}$$

and by Theorem 2.1.1, f has a unique fixed point in X, but f is not a contraction mapping (for example, using the Cebisev metric) on $X \subseteq \mathbb{R}^2$, thus we can not apply the classical Banach's fixed point theorem.

Next, we extend another fixed point results for singlevalued operators, which satisfies generalized contraction conditions in the context of E-metric spaces.

Theorem 2.1.3. *Let X be an E-complete vector metric space with E is Archimedean. Suppose that $f : X \to X$ is an (a,b,c)-contraction, i.e. there exists $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ such that*

$$d\left[f(x), f(y)\right] \leq a d(x,y) + b d\left[x, f(x)\right] + c d\left[y, f(y)\right], \text{ for all } x,y \in X.$$

Then f has a unique fixed point in X and for any $x_0 \in X$, the iterative sequence (x_n) defined by $x_n = f(x_{n-1})$ for any $n \in \mathbb{N}^*$, E-converges to the fixed point of f.

Theorem 2.1.4. *Let X be an E-complete vector metric space with E is Archimedean. Suppose that $g : X \to X$ is an (a,b,c,e,f)-contracion, i.e. there exists $a,b,c,e,f \in \mathbb{R}_+$ with $a+b+c+2f<1$ such that for any $x,y \in X$, we have*

$$\begin{aligned}
&d\left[g(x), g(y)\right] \\
&\qquad \leq a d(x,y) + b d\left[x, g(x)\right] + c d\left[y, g(y)\right] + e d\left[y, g(x)\right] + f d\left[x, g(y)\right].
\end{aligned}$$

Then g has a unique fixed point in X and for any $x_0 \in X$, the iterative sequence (x_n) defined by $x_n = g(x_{n-1})$, for any $n \in \mathbb{N}^*$, E-converges to the fixed point of g.

Remark 2.1.5. Theorem 2.1.4 generalize Theorem 2.1.3 by choosing the constants $e = f = 0$ and Theorem 2.1.3 generalize Theorem 2.1.1 by choosing the constants $b = c = 0$.

Another metrical fixed point theorem for singlevalued operators which can be extended to E-metric spaces is the first form of Ćirić's Theorem. Notice that this result is another particular case of Theorem 2.1.4, the proof being similar.

Theorem 2.1.6. *Let X be an E-complete vector metric space with E is Archimedean. Suppose that $g : X \to X$ satisfies a Ćirić type α-contraction condition, i.e. there exists $\alpha \in [0, 1)$ such that for any $x, y \in X$, we have*

$$d\left[g\left(x\right), g\left(y\right)\right] \leq \alpha \max\left\{d\left(x, y\right), d\left[x, g\left(x\right)\right], d\left[y, g\left(y\right)\right], \frac{1}{2}\left[d\left(x, g\left(y\right)\right) + d\left(y, g\left(x\right)\right)\right]\right\}.$$

Then g has a unique fixed point in X and for any $x_0 \in X$, the iterative sequence (x_n) defined by $x_n = g(x_{n-1})$, for any $n \in \mathbb{N}^*$, E-converges to the fixed point of g.

In the following we present several fixed point theorems for multivalued operators in E-metric spaces.

Definition 2.1.7. Let (X, d, E) be an E-metric space. The operator $F : X \to P_{cl}(X)$ is a multivalued k-contraction, if and only if $k \in [0, 1)$ and for any $x, y \in X$ and any $u \in F(x)$, there exists $v \in F(y)$ such that

$$d\left(u, v\right) \leq kd\left(x, y\right).$$

Theorem 2.1.8. *Let (X, d, E) be an E-complete vector metric space with E is Archimedean and $F : X \to P_{cl}(X)$ be a multivalued k-contraction. Then F has a fixed point in X and for any $x \in X$, there exists a sequence of successive approximations of F starting from $(x, y) \in Graph(F)$, which E-converges in (X, d, E) to the fixed point of F.*

Example 2.1.9. Let $E = \mathbb{R}^2$ with coordinatwise ordering (thus E is Archimedean) and we consider

$$X = \left\{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\right\} \cup \left\{(0, x) \in \mathbb{R}^2 : 0 \leq x \leq 1\right\}.$$

The mapping $d : X \times X \to E$ is defined by

$$d[(x,0),(y,0)] = \left(\frac{4}{3}|x-y|, |x-y|\right),$$
$$d[(0,x),(0,y)] = \left(|x-y|, \frac{2}{3}|x-y|\right),$$
$$d[(x,0),(0,y)] = \left(\frac{4}{3}x+y, x+\frac{2}{3}y\right).$$

Then X is an E-complete vector metric space. Let $F : X \to P_{cl}(X)$ with $F(x_1,x_2) = \{u(x_1,x_2), v(x_1,x_2)\}$, where $u, v : X \to X$ are defined by

$$u((x,0)) = (0,x) \text{ şi } u((0,x)) = \left(\frac{x}{2},0\right);$$
$$v((x,0)) = (0,x) \text{ şi } v((0,x)) = \left(\frac{x}{3},0\right).$$

We have the following possibilities:

Case 1: for any $(x,0),(y,0) \in X$ and any $(0,x) \in F(x,0)$, there exists $(0,y) \in F(y,0)$;

Case 2: for any $(x,0),(0,y) \in X$ and any $(0,x) \in F(x,0)$, there exists $\left(\frac{y}{2},0\right) \in F(0,y)$ or $\left(\frac{y}{3},0\right) \in F(0,y)$;

Case 3: for any $(0,x),(y,0) \in X$ and any $\left(\frac{x}{2},0\right) \in F(0,x)$ or for any $\left(\frac{x}{3},0\right) \in F(0,x)$, there exists $(0,y) \in F(y,0)$;

Case 4: for any $(0,x),(0,y) \in X$ and any $\left(\frac{x}{2},0\right) \in F(0,x)$, there exists $\left(\frac{y}{2},0\right) \in F(0,y)$, respectively for any $(0,x),(0,y) \in X$ and any $\left(\frac{x}{3},0\right) \in F(0,x)$, there exists $\left(\frac{y}{3},0\right) \in F(0,y)$.

For all of these cases the multivalued k-contraction condition holds for $k = \frac{3}{4}$. By Theorem 2.1.8, it follows that F has a fixed point in X, but F is not a contraction mapping (for example, using the Cebisev metric) on $X \subseteq \mathbb{R}^2$, thus we can not apply the classical Nadler's fixed point theorem.

Definition 2.1.10. Let (X,d,E) be an E-metric space. The operator $F : X \to P_{cl}(X)$ is a multivalued (a,b,c)-contraction, if and only if $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ and for any $x,y \in X$ and any $u \in F(x)$, there exists $v \in F(y)$ such that

$$d(u,v) \le ad(x,y) + bd(x,u) + cd(y,v).$$

Theorem 2.1.11. *Let (X,d,E) be an E-complete vector metric space with E is Archimedean and $F : X \to P_{cl}(X)$ be a multivalued (a,b,c)-contraction.*

Then F has a fixed point in X and for any $x \in X$, there exists a sequence of successive approximations of F starting from $(x, y) \in Graph\,(F)$, which E-converges in (X, d, E) to the fixed point of F.

Remark 2.1.12. Theorems 2.1.8 and 2.1.11 generalizes several known results in the theory of fixed points (see Nadler [85], Covitz-Nadler [75], S. Reich [80], A. Bucur, L. Guran and A. Petruşel [19]), etc. Notice also that here we do not need a closed graph condition on F, as in [19], for example.

2.2 The theory of an E-metrical fixed point theorem

In this section, we discuss the theory of Banach–Caccioppoli's fixed point theorem in E-metric spaces (see Theorem 2.1.1) from the above section. The notion of theory of a metrical fixed point theorem was introduced by I.A. Rus in [89] for the classical metric spaces. Also, we discuss the multivalued case using the model given by A. Petruşel and I.A. Rus in [75].

Similarly, the theory can be extended for the rest of the metrical fixed point theorems from the above section, which satisfies generalized contraction conditions in E-metric spaces.

We start our aim with a Cauchy lemma given by I.A. Rus, M.-A. Şerban in [91] and we extend it to the case of Riesz spaces.

Lemma 2.2.1 (Extended Cauchy Lemma)**.** *Let E be an order complete Riesz space. Let $a_n \in \mathbb{R}_+$, $b_n \in E_+$, $n \in \mathbb{N}^*$ such that $\sum_{i=0}^{\infty} |a_i| < +\infty$ and $b_n \xrightarrow{o} 0$ as $n \to \infty$. Then*

$$\sum_{i=0}^{n} a_{n-i} b_i \xrightarrow{o} 0 \text{ as } n \to \infty.$$

Remark 2.2.2. If (X, d, E) is an E-metric space and $f : X \to X$ is a singlevalued k-contraction, it is easy to observe that f is E-continuous.

Theorem 2.2.3. *Let (X, d, E) be an E-complete vector metric space with E is Archimedean and let $f : X \to X$ be a singlevalued k-contraction. Then the following affirmations hold:*

i) $\mathrm{Fix}\,(f) = \{x_f^*\}$ *and* $f^n\,(x) \xrightarrow{d,E} x_f^*$ *as $n \to \infty$, for any $x \in X$, i.e., f is a vector Picard operator (briefly E-PO);*

ii) $\text{Fix}(f) = \text{Fix}(f^n) = \{x_f^*\}$, *for any* $n \in \mathbb{N}^*$, *i.e.,* f *is a vector Bessaga operator;*

iii) $d\left[f^n(x), x_f^*\right] \leq \frac{k^n}{1-k} d[x, f(x)]$, *for any* $x \in X$ *and* $n \in \mathbb{N}^*$*;*

iv) $d\left(x, x_f^*\right) \leq \frac{1}{1-k} d[x, f(x)]$, *for any* $x \in X$, *i.e.,* f *is* E-$\frac{1}{1-k}$-*PO;*

v) $\sum_{n \in \mathbf{N}} d\left[f^n(x), f^{n+1}(x)\right] \leq \frac{1}{1-k} d[x, f(x)]$, *for any* $x \in X$, *i.e.,* f *is a good* E-*PO;*

vi) $\sum_{n \in \mathbf{N}} d\left[f^n(x), x_f^*\right] \leq \frac{1}{1-k} d\left(x, x_f^*\right)$, *for any* $x \in X$, *i.e.,* f *is a special* E-*PO;*

vii) If $x_n \in X$, $n \in \mathbb{N}$ *are such that*

$$d[x_n, f(x_n)] \xrightarrow{d,E} 0 \text{ as } n \to \infty,$$

then

$$x_n \xrightarrow{d,E} x_f^* \text{ as } n \to \infty,$$

i.e., the fixed point problem for the operator f *is well posed;*

viii) If $x_n \in X, n \in \mathbb{N}$ *and*

$$(d(x_{n+1}, f(x_n))) \xrightarrow{o} 0 \text{ as } n \to \infty,$$

imply that there exists $x \in X$ *such that*

$$d[x_n, f^n(x)] \xrightarrow{o} 0 \text{ as } n \to \infty,$$

i.e., the operator f *has the limit shadowing property;*

ix) If $(x_n)_{n \in \mathbf{N}}$ *is an* E-*bounded sequence in* X, *then*

$$f^n(x_n) \xrightarrow{d,E} x_f^* \text{ as } n \to \infty;$$

x) If $g : X \to X$ *is such that there exists* $\eta \in E_+$ *with*

$$d[f(x), g(x)] \leq \eta, \text{ for any } x \in X,$$

then:

$$x_g^* \in \text{Fix}(g) \text{ imply } d\left(x_f^*, x_g^*\right) \leq \frac{1}{1-k}\eta;$$

xi) *If* $f_n : X \to X$, $f_n \overset{unif.}{\longrightarrow} f$, $x_n^* \in \mathrm{Fix}(f_n)$, $n \in \mathbb{N}$, *then*

$$x_n^* \overset{d,E}{\longrightarrow} x_f^* \text{ as } n \to \infty;$$

xii) *If* (X, d, E) *is an* E*-normed metric space, with* $d(x, y) = |x - y|$, *where* $|\cdot| : X \to E_+$, *then* $1_X - f : X \to X$ *is an* E*-topological isomorphism;*

xiii) *If* (X, d, E) *is an* E*-bounded vector metric space with* E *order complete, then*

$$\underset{n \in \mathbb{N}}{\cap} f^n(X) = \{x_f^*\},$$

i.e., f *is a Janos operator.*

Now we present two applications of the studied E-metrical fixed point theorem for singlevalued operators. In this scope we need the following request.

Definition 2.2.4. By definition, $(X, d, E, \leq)$ is an *ordered* E*-metric space* if (X, d, E) is an E-metric space and „$\leq$" is a partial ordering relation on X, such that the following implication holds: if $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in X are such that

i) $x_n \leq y_n$, for all $n \in \mathbb{N}$;

ii) $x_n \overset{d,E}{\longrightarrow} x$, $y_n \overset{d,E}{\longrightarrow} y$ as $n \to \infty$,

then

$$x \leq y.$$

Lemma 2.2.5 (Gronwall Lemma for k-contractions)**.** *Let* $(X, d, E, \leq)$ *be an ordered* E*-complete vector metric space such that* E *is Archimedean and let* $f : X \to X$ *be an operator. Supposing that* f *is a singlevalued* k*-contraction and* f *is increasing, then:*

i) $x \leq f(x)$ *imply* $x \leq x_f^*$;

ii) $x \geq f(x)$ *imply* $x \geq x_f^*$.

Theorem 2.2.6 (Comparison Theorem for k-contractions)**.** *Let* $(X, d, E \leq)$ *be an ordered* E*-complete vector metric space such that* E *is Archimedean and let* $f, g, h : X \to X$ *be three operators. We suppose that:*

i) $f \leq g \leq h$;

ii) f, g, h *are singlevalued* k*-contractions;*

iii) *the operator* g *is increasing.*

Then:

$$x_f^* \leq x_g^* \leq x_h^*.$$

For the multivalued case (see Theorem 2.1.8), also, we study other fixed point properties, as well as:

- the existence of fixed and strict fixed points;
- the data dependence of fixed points;
- the convergence of fixed points set for a sequence of multivalued operators;
- the Ulam–Hyers stability of the inclusion $x \in F(x)$;
- the well-posedness property of the fixed point problem;
- the limit shadowing property of the multivalued operator;
- others.

Definition 2.2.7. Let (X, d, E) be an E-metric space. Then the multivalued operator $F : X \to P(X)$ is called *vector weak Picard* (briefly E-MWP operator), if and only if for each $x \in X$ and each $y \in F(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is E-convergent and its limit is a fixed point of F.

Definition 2.2.8. Let (X, d, E) be an E-metric space and let $F : X \to P(X)$ be an E-MWP operator. Then we define the multivalued operator $F^\infty : \text{Graph}(F) \to P(\text{Fix}(F))$ by the formula $\{F^\infty(x, y) = x^* \in \text{Fix}(F) :$ there exists a sequence of successive approximations of F starting from (x, y) that E-converges to $x^*\}$.

Definition 2.2.9. Let (X, d, E) be an E-metric space and let $F : X \to P(X)$ be an E-MWP operator. Then the multivalued operator F is called *vector* *c-weak Picard* (briefly E-c-MWP operator) if and only if $c \in \mathbb{R}_+^*$ and there exists a selection f^∞ of F^∞ such that $d[x, f^\infty(x,y)] \leq cd(x,y)$, for all $(x,y) \in \text{Graph}(F)$.

Definition 2.2.10. Let (X,d,E) be an E-metric space and let $(F_n)_{n\in\mathbb{N}}$ be a sequence of ordered sets in $P_{cl}(X)$. Then F_n is called to be *Hausdorff E-convergent* to an ordered and closed set F of X, denoted by $F_n \overset{H,d,E}{\longrightarrow} F$ as $n \to \infty$ if and only if there exists a sequence $(a_n) \subset E$ such that $a_n \downarrow 0$ as $n \to \infty$ and for any $u_n \in F_n(x)$, there exists $v \in F(x)$ (respectively for any $v \in F(x)$, there exists $u_n \in F_n(x)$) such that

$$d(u_n, v) \leq a_n, \text{ for any } n \in \mathbb{N}.$$

Definition 2.2.11. Let (X,d,E) be an E-complete vector metric space and let $F : X \to P(X)$ be a multivalued operator. Then the multivalued operator F is called *vector Picard operator* (briefly E-MPO) if and only if:

(i) $\text{SFix}(F) = \text{Fix}(F) = \{x^*\}$;

(ii) $F^n(x) \overset{H,d,E}{\longrightarrow} \{x^*\}$ as $n \to \infty$, for any $x \in X$.

A first result for multivalued k-contractions in E-metric spaces is the following.

Theorem 2.2.12. *Let (X,d,E) be an E-complete vector metric space with E is Archimedean and let $F : X \to P_{cl}(X)$ be a multivalued k-contraction. Then the following statements hold:*

i) $\text{Fix}(F) \neq \emptyset$;

ii) *F is an E-$\frac{1}{1-k}$-MWP operator;*

iii) *Let $G : X \to P_{cl}(X)$ be a multivalued k-contraction and $\eta \in E_+$ such that for any $u \in G(x)$, there exists $v \in F(x)$ such that $d(u,v) \leq \eta$ (respectively for any $u \in F(x)$, there exists $v \in G(x)$ such that $d(u,v) \leq \eta$). Then for any $p \in \text{Fix}(G)$, there exists $q \in \text{Fix}(F)$ such that $d(p,q) \leq \frac{1}{1-k}\eta$ (respectively for any $p \in \text{Fix}(F)$, there exists $q \in \text{Fix}(G)$ such that $d(p,q) \leq \frac{1}{1-k}\eta$).*

iv) *Let $F_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ be a sequence of multivalued k-contractions such that $F_n(x) \overset{H,d,E}{\longrightarrow} F(x)$ as $n \to \infty$, uniformly with respect to $x \in X$. Then, $\text{Fix}(F_n) \overset{H,d,E}{\longrightarrow} \text{Fix}(F)$ as $n \to \infty$;*

v) *(Ulam–Hyers stability of the inclusion $x \in F(x)$) Let $\epsilon \in E_+$ be such that there exists $y \in F(x) : d(x,y) \leq \epsilon$. Then, there exists $x^* \in \operatorname{Fix}(F)$ such that $d(x,x^*) \leq \frac{1}{1-k}\epsilon$.*

A second result for multivalued k-contractions in E-metric spaces is as follows.

Theorem 2.2.13. *Let (X,d,E) be an E-complete vector metric space with E is Archimedean and let $F : X \to P_{cl}(X)$ be a multivalued k-contraction with $\operatorname{SFix}(F) \neq \emptyset$. Then the following statements hold:*

i) $\operatorname{Fix}(F) = \operatorname{SFix}(F) = \{x^*\}$;

ii) $\operatorname{Fix}(F^n) = \operatorname{SFix}(F^n) = \{x^*\}$, *for* $n \in \mathbb{N}^*$;

iii) $F^n(x) \xrightarrow{H,d,E} \{x^*\}$ *as* $n \to \infty$, *for any* $x \in X$;

iv) *Let $G : X \to P_{cl}(X)$ be a multivalued operator and $\eta \in E_+$ such that $\operatorname{Fix}(G) \neq \emptyset$ and for any $u \in G(x)$, there exists $v \in F(x)$ such that $d(u,v) \leq \eta$ (respectively for any $u \in F(x)$, there exists $v \in G(x)$ such that $d(u,v) \leq \eta$). Then for any $p \in \operatorname{Fix}(G)$, there exists $q \in \operatorname{Fix}(F)$ such that $d(p,q) \leq \frac{1}{1-k}\eta$ (respectively for any $p \in \operatorname{Fix}(F)$, there exists $q \in \operatorname{Fix}(G)$ such that $d(p,q) \leq \frac{1}{1-k}\eta$);*

v) *Let $F_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ be a sequence of multivalued operators such that $\operatorname{Fix}(F_n) \neq \emptyset$ for each $n \in \mathbb{N}$ and $F_n(x) \xrightarrow{H,d,E} F(x)$ as $n \to \infty$, uniformly with respect to $x \in X$. Then, $\operatorname{Fix}(F_n) \xrightarrow{H,d,E} \operatorname{Fix}(F)$ as $n \to \infty$.*

vi) *(Well-posedness property of the fixed point problem) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that there exists $y_n \in F(x_n)$, $n \in \mathbb{N}$ with the property*

$$d(x_n,y_n) \xrightarrow{o} 0 \text{ as } n \to \infty,$$

then $x_n \xrightarrow{d,E} x^*$ *as* $n \to \infty$.

vii) *(Limit shadowing property of the multivalued operator) If $(y_n)_{n\in\mathbb{N}}$ is a sequence in X such that there exists $u_n \in F(y_n)$, $n \in \mathbb{N}$ with the property*

$$d(y_{n+1},u_n) \xrightarrow{o} 0 \text{ as } n \to \infty,$$

then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ *of successive approximations for* F, *such that*

$$d(x_n, y_n) \xrightarrow{o} 0 \text{ as } n \to \infty.$$

A third result for multivalued k-contractions in E-metric spaces is the following.

Theorem 2.2.14. *Let* (X, d, E) *be an* E*-complete vector metric space with* E *is Archimedean and let* $F : X \to P_{cp}(X)$ *be a multivalued* k*-contraction such that* $F(\mathrm{Fix}(F)) = \mathrm{Fix}(F)$. *Then the following statements hold:*

i) $F(x) = \mathrm{Fix}(F)$, *for each* $x \in \mathrm{Fix}(F)$;

ii) If $(x_n)_{n\in\mathbb{N}} \subset X$ *is a sequence such that* $x_n \xrightarrow{d,E} x^* \in \mathrm{Fix}(F)$ *as* $n \to \infty$, *then* $F(x_n) \xrightarrow{H,d,E} \mathrm{Fix}(F)$ *as* $n \to \infty$.

2.3 Nonlinear fixed point results in E-metric spaces

The purpose of this section is to give some extensions of the Contraction Principle to the case of E-metric spaces. More precisely we will realize the study of the fixed point theory for (local and global) nonlinear contractions with an o-comparison function in E-metric spaces. Our results generalize some theorems given in J. Matkowski [47], M. Kwapisz [43], R. Cristescu [29], F. Voicu [96], [97], P.P. Zabrejko [100], C. Çevik, I. Altun [25]. Also, are presented new auxiliary fixed point results for Krasnoselskii's theorem in E-metric spaces and are pointed out some open problems.

Definition 2.3.1. An increasing operator $\varphi : E_+ \to E_+$ with $\varphi(t) \leq t$ and $\varphi^n(t) \xrightarrow{o} 0$ for any $t > 0$ is called an *order comparison operator* (briefly o-comparison operator).

Definition 2.3.2. Let (X, d, E) be an E-metric space and let $\varphi : E_+ \to E_+$ be an o-comparison operator. We say that the operator $f : X \to X$ is a *singlevalued nonlinear* φ*-contraction*, if and only if

$$d[f(x), f(y)] \leq \varphi[d(x, y)], \text{ for any } x, y \in X.$$

Theorem 2.3.3. *Let* (X, d, E) *be an* E*-complete metric space with* E*-Archimedean and let* $f : X \to X$ *be a singlevalued nonlinear* φ*-contraction. Then:*

i) *there exists a unique fixed point* x^* *for* f *in* X *and for any* $x \in X$*, the sequence* $f^n(x) \xrightarrow{d,E} x^*$*;*

ii) $d[x^*, f^n(x)] \leq \varphi^n[d(x^*, x)]$*, for any* $n \in \mathbb{N}$*.*

A local version of Theorem 2.3.3 can be obtained in the closed ball

$$\bar{B}(x_0, r) := \{x \mid x \in X, d(x_0, x) \leq r\}.$$

Theorem 2.3.4. *Let* (X, d, E) *be an* E*-complete metric space with* E*-Archimedean,* $x_0 \in X$*,* $r \in E_+$*, let* $f : \bar{B}(x_0, r) \to X$ *be an operator and there exists an increasing operator* $\varphi : [0, r] \to [0, r] \subset E_+$ *such that* $\varphi^n(t) \xrightarrow{o} 0$*, for any* $t \in (0, r]$ *with the property* $d[f(x), f(y)] \leq \varphi[d(x, y)]$ *and* $d(x, y) \leq r$*, for any* $x, y \in \bar{B}(x_0, r)$*. We assume that* $d[x_0, f(x_0)] \leq r - \varphi(r)$*. Then:*

i) *there exists a unique fixed point* x^* *for* f *în* $\bar{B}(x_0, r)$ *and for any* $x \in \bar{B}(x_0, r)$*, the sequence* $f^n(x) \xrightarrow{d,E} x^*$*;*

ii) $d[x^*, f^n(x)] \leq \varphi^n(b)$*, for any* $n \in \mathbb{N}$*.*

Following the ideas from M. Kwapisz [43], other results with equivalent conclusions with Theorems 2.3.3 and 2.3.4 can be obtained in the space

$$X(x_0, r) := \bigcup_{\lambda \in E_+} \bar{B}(x_0, \lambda r) = \bigcup_{\lambda \in E_+} \{x \mid x \in X, d(x, x_0) \leq \lambda r\}.$$

Theorem 2.3.5. *Let* (X, d, E) *be an* E*-complete metric space with* E*-Archimedean,* $x_0 \in X$*,* $r \in E_+$*, let* $f : X(x_0, r) \to X$ *be an operator and there exists an increasing operator* $\varphi : E_+ \to E_+$ *such that* $\varphi^n(t) \xrightarrow{o} 0$*, for any* $t > 0$*, with properties:*

1) $\varphi(\lambda r) \leq \varphi(\lambda) r$*, for* $\lambda \in E_+$*;*

2) $d[f(x), f(y)] \leq \varphi[d(x, y)]$ *şi* $d(x, y) \leq \lambda r$*, for any* $x, y \in X(x_0, r)$ *and for* $\lambda \in E_+$*;*

3) $d[x_0, f(x_0)] \leq \lambda_0 r$*, for* $\lambda_0 \in E_+$*.*

Then:

i) *there exists a unique fixed point* x^* *for* f *în* $X(x_0, r)$ *and for any* $x \in X(x_0, r)$*, the sequence* $f^n(x) \xrightarrow{d,E} x^*$*;*

ii) $d[x^*, f^n(x)] \leq \varphi^n d(x^*, x)$*, for any* $n \in \mathbb{N}$*.*

Lemma 2.3.6. *If $(y_n) \subset X(x_0, r)$ and $y_n \xrightarrow{d,E} y$, then $y \in X(x_0, r)$, i.e., $X(x_0, r)$ is E-closed in X with respect to the convergence $\xrightarrow{d,E}$.*

If we endow the space $X(x_0, r) \subset X$ with an E-metric $\rho : X(x_0, r) \times X(x_0, r) \to E_+$, given by

$$\rho(x, y) = \inf_{\lambda \in E_+} \{d(x, y) \leq \lambda r\},$$

we have the following:

Lemma 2.3.7. *Let (X, d, E) be an E-complete metric space with E-Archimedean. Then, the space $X(x_0, r)$ is E-complete with respect to ρ.*

If we use the same conditions as in Theorem 2.3.5, we can obtain another existence and uniqueness result in the E-metric space $X(x_0, b)$. Notice that, this time, the proof is based on the relationship between the E-metrics ρ and d. Thus, we will not apply Theorem 2.3.5 to show that the sequence of successive approximations of f converges with respect to $\xrightarrow{d,E}$ to the unique fixed point of f in $X(x_0, b)$.

Theorem 2.3.8. *If all the assumptions of Theorem 2.3.5 hold, then:*

i) f is a singlevalued nonlinear φ-contraction in $X(x_0, r)$ with respect to ρ;

ii) there exists a unique fixed point x^* for f in $X(x_0, r)$ and for any $y_0 \in X(x_0, r)$, we have that $f^n(y_0) \xrightarrow{d,E} x^*$.

In the following we present how to define a multivalued nonlinear φ-contraction, a nonlinear version of Theorem 2.1.1 for multivalued operators and open problems on this direction.

Definition 2.3.9. Let (X, d, E) be an E-metric space and let $\varphi : E_+ \to E_+$ be an o-comparison operator. We say that the operator $F : X \to P_{cl}(X)$ is a *multivalued nonlinear φ-contraction* if and only if, for any $x, y \in X$ and any $u \in F(x)$, there exists $v \in F(y)$ such that

$$d(u, v) \leq \varphi[d(x, y)].$$

Theorem 2.3.10. *Let (X, d, E) be an E-complete vector metric space. Assume that the operator $F : X \to P_{cl}(X)$ is a multivalued nonlinear φ-contraction, then F has a fixed point in X and for any $x \in X$, there exists a sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$ for F starting from $(x, y) \in \mathrm{Graph}(F)$, which E-converges in (X, d, E) to the fixed point F.*

If we choose the particular case when $\varphi(t) = kt$, $k \in [0,1)$ we obtain Theorem 2.1.8.

Remark 2.3.11. Theorems 2.3.3, 2.3.4 and 2.3.10 represents the extensions to the case of E-metric spaces of some classical fixed point metrical principles from the nonlinear analysis. If we choose $X = E$ and d the absolute valued metric on E, then we obtain fixed point theorems in the Riesz space E (see R. Cristescu [29]).

Problem 2.3.12. To prove other fixed point theorems in E-metric spaces for singlevalued operators which satisfies nonlinear generalized φ-contraction conditions and also, to study the theory of such a theorem (see I.-R. Petre [63], A. Petruşel, I.A. Rus [75] and I.A. Rus [89]).

We consider the following definition:

Definition 2.3.13. Let (X, d, E) be an E-metric space with E order complete and let $\varphi : E_+ \to E_+$ be an o-comparison operator. We say that the operator $F : X \to P_{b,cl}(X)$ is a *multivalued nonlinear φ-contraction in Nadler's sense*, if and only if

$$H[F(x), F(y)] \leq \varphi[d(x, y)], \text{ for any } x, y \in X. \tag{2.3.1}$$

Problem 2.3.14. To define a lucrative functional H for which the condition (2.3.1) occurs when E is a linear lattice, to prove Theorem 2.3.10 in terms of Definition 2.3.13 and to establish a data dependence lemma for the excess of fixed points sets of two operators, which satisfies a multivalued nonlinear φ-contraction condition in Nadler's sense.

The difficulty arises because usually we can not deny the inequality $u \leq v$ for $u, v \in E_+$. For example, for $E_+ = \mathbb{R}^2_+$, we have the following figure:

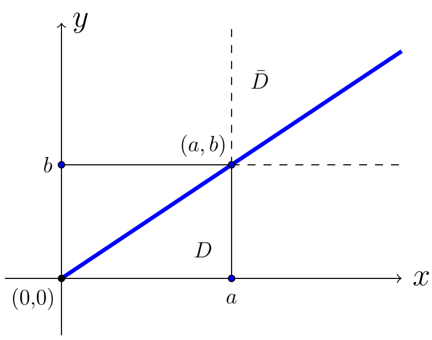

Figura 3: Difficulties of denial in a linear latice.

From here you see that the comparable elements are on the bold diagonal and the elements outside the rectangle $D = [0, a] \times [0, b]$ are not just those in the portion $\bar{D}$. Thus, we can not establish a Lemma of type 1.3.16 in the context of E-metric spaces. To avoid such issues for demonstration such a lemma, in section 4.3 we introduce a notion of strict positivity in a Riesz space and thus, we have a new way to approach for Problem 2.3.14.

Lemma 2.3.15. *Let (X, d, E) be an E-metric space with E order complete and let $A \subset P_b(X)$. Then $D(x, A) = 0$ if and only if $x \in \bar{A}$.*

Notice that Ky Fan's Lemma and Schauder's Theorem can be proved, in an E-Banach space X, by a similar method to the classical case, where we assume that E is order complete and $Y \subset X$ is an E-bounded set (thus, the order completeness guarantees that $\inf_{x\in Y} ||x - f(y_0)||$ exists in E). More precisely, we have the following results.

Lemma 2.3.16. *Let X be an order complete E-normed space, let $Y \subset X$ be an E-compact and E-convex set and let $f : Y \to X$ be an E-continuous operator. Then $||y_0 - f(y_0)|| = \inf_{x\in Y} ||x - f(y_0)||$.*

Theorem 2.3.17. *Let $(X, ||\cdot||, E)$ be an E-Banach space with E order complete, let $Y \subset X$ be an E-bounded, E-closed and E-convex set and let $f : Y \to Y$ be an operator with E-relatively compact range. Then f has at least one fixed point in Y.*

Remark 2.3.18. For another Schauder type theorem in Hausdorff Archimedean vector lattice, see T. Kawasaki, M. Toyoda, T. Watanabe [42].

Problem 2.3.19. To prove a Rybinski type theorem before the model of Theorem 1.3.21 in the context of E-metric spaces, which ensure that there exists an E-continuous selection for a given multivalued operator, which satisfies a multivalued nonlinear φ-contraction condition in Nadler's sense with respect to the second argument.

We need an extended version of Cantor's intersection theorem and of Cesaro's lemma.

Lemma 2.3.20. *Let (X, d, E) be an E-complete metric space with the property that for every descending sequence $\{F_n\}_{n\geq 1}$ of nonempty E-closed subsets of X we have that $\delta(F_n) \xrightarrow{o} 0$ as $n \to \infty$. Then the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one element.*

Lemma 2.3.21. *Let (X, d, E) be an E-complete metric space such that E is Archimedean and let (x_n) be an E-bounded sequence in X. Then, there exists an E-convergent subsequence (x_{n_k}) in X.*

Chapter 3

Topological fixed point theorems and applications in vector Banach spaces

3.1 Krasnoselskii's theorem in generalized Banach spaces

In this section we prove Krasnoselskii's fixed point theorem in generalized Banach spaces for singlevalued and multivalued operators. Also, we discuss other possible fixed point existence results for a sum of two multivalued operators, where one of them satisfies a multivalued A-contraction condition and the other one satisfies a compactness condition.

Theorem 3.1.1. *Let $(X, |\cdot|)$ be a generalized Banach space and let $Y \in P_{cl,cv}(X)$. Assume that the operators $f, g : Y \to X$ satisfies the properties:*

i) f is a singlevalued A-contraction;

ii) g is continuous;

iii) $g(Y)$ is relatively compact and $f(x) + g(y) \in Y$ for any $x, y \in Y$.

Then $f + g$ has a fixed point in Y.

For a matrix $A := (a_{ij})_{i,j\in\{1,\cdots,m\}} \in \mathcal{M}_{m,m}(\mathbb{R})$ we denote by

$$|A| := (|a_{ij}|)_{i,j\in\{1,\cdots,m\}} \in \mathcal{M}_{m,m}(\mathbb{R}_+).$$

In this context, we say that a non-singular matrix A has the *absolute value property* if $A^{-1}|A| \leq I$. Some examples of matrices convergent to zero $A \in \mathcal{M}_{2,2}(\mathbb{R})$, which also satisfies the property

$$(I - A)^{-1}|I - A| \leq I$$

are:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$;

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1, c < 1$;

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a > 1, b > 0$ şi $|a - b| < 1$.

For the study of existence of fixed point for a sum of two multivalued operators, the basic form of Krasnoselskii's theorem in a generalized Banach space and other connected results are presented in the following.

Theorem 3.1.2. *Let $(X, |\cdot|)$ be a generalized Banach space and let $Y \in P_{cp,cv}(X)$. Assume that the operators $F : Y \to P_{b,cl,cv}(X)$, $G : Y \to P_{cp,cv}(X)$ satisfies the properties:*

i) $F(y_1) + G(y_2) \subset Y$ *for each* $y_1, y_2 \in Y$;

ii) F *is a multivalued* A*-contraction in Nadler's sense;*

iii) G *is l.s.c and* $G(Y)$ *is relatively compact;*

iv) the matrix $I - A$ *has the absolute value property.*

Then $F + G$ *has a fixed point in* Y.

Extending an idea of T.A. Burton (see [20]), let us observe that the condition i) in our basic result (Theorem 3.1.2) can be relaxed.

Theorem 3.1.3. *Let $(X, |\cdot|)$ be a generalized Banach space and let $Y \in P_{cp,cv}(X)$. Assume that the operators $F : Y \to P_{b,cl,cv}(X)$, $G : Y \to P_{cp,cv}(X)$ satisfies the properties:*

i) $y \in F(y) + G(x), x \in Y$ *then* $y \in Y$;

ii) F *is a multivalued* A*-contraction in Nadler's sense;*

iii) G *is l.s.c and* $G(Y)$ *is relatively compact;*

iv) the matrix $I - A$ *has the absolute value property.*

Then $F + G$ *has a fixed point in* Y.

Theorem 3.1.4. *Let us suppose that the conditions ii), iii) and iv) of Theorem 3.1.3 hold. If there exists $r \in \mathbb{R}_+^m$ such that for $Y = \{x \in X : |x| \leq r\}$, we have $G(Y) \subset Y$ and $|y| \leq D[y, F(y)]$, $y \in Y$, then the conclusion of Theorem 3.1.3 holds.*

Notice that in the terms of an abstract measure of noncompactness on Y, we can say that the multivalued operator $F : Y \to P_{b,cl}(X)$ is called a multivalued (α, A)-contraction if and only if, $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and $\alpha(F(B)) \leq A\alpha(B)$, for each $B \in P_b(Y)$. In this sense, we can rise another direction in the study of Krasnoselskii's theorem using the classical results given by A. Petruşel in [70], I.A. Rus in [88] and A. Petruşel in [72].

Problem 3.1.5. Let $(X, |\cdot|)$ be a generalized Banach space and let $F_1, F_2 : X \to P_{b,cl}(X)$ be two multivalued operators, such that F_1 is a multivalued A-contraction and F_2 is compact. Then $F_1 + F_2$ is a multivalued (α, A)-contraction.

Problem 3.1.6. Let $(X, |\cdot|)$ be a generalized Banach space and let $Y \in P_{b,cl,cv}(X)$. If $F : Y \to P_{cl,cv}(Y)$ is an upper semicontinuous (briefly u.s.c.), (α, A)-contraction multivalued operator, then F has a fixed point in Y.

Problem 3.1.7. Let $(X, |\cdot|)$ be a generalized Banach space and let $Y \in P_{b,cl,cv}(X)$. Assume that the operators $F, G : Y \to P_{cp,cv}(X)$ satisfies the properties:

i) $F(y) + G(y) \subset Y$ for each $y \in Y$;
ii) F is a multivalued A-contraction in Nadler's sense;
iii) G is u.s.c. and compact.

Then $F + G$ has a fixed point in Y.

3.2 Applications

It is known that the classical form of Theorems 3.1.1 and 3.1.2 have a lot of interesting applications. For example, L. Collatz [26] established the existence of a solution for the integral equation

$$x(t) = \frac{1}{3}\left[x^2(t) + t\right] + \frac{1}{3}\int_0^1 |x(s) - t|^{\frac{1}{2}}\, ds,\; t \in [0,1].$$

Another application of Krasnoselskii's theorem in Hilbert spaces can be found in M. Zuluaga [102] or for the multivalued case, see, for example A. Petruşel [72].

Our purpose is to extend this applications for our results obtained in the above section, imposing an A-contraction condition on one of the integral operators and other conditions to get the existence of the solution for an integral equation and inclusion system in a generalized Banach space. Using Theorems 3.1.1 and 3.1.2 we can obtain existence results for systems of differential equations and inclusions. In the following we consider the abstract case of systems of two Fredholm–Volterra type integral equations.

Theorem 3.2.1. *Let $I = [0, a]$ (with $a > 0$) be an interval of the real axis and let us consider the following equations system in $C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$:*

$$\begin{cases} x_1(t) = \lambda_{11} \int_0^t k_1(t, s, x_1(s), x_2(s))\, ds + \lambda_{12} \int_0^a l_1(t, s, x_1(s), x_2(s))\, ds \\ x_2(t) = \lambda_{21} \int_0^t k_2(t, s, x_1(s), x_2(s))\, ds + \lambda_{22} \int_0^a l_2(t, s, x_1(s), x_2(s))\, ds \end{cases}$$

for $t \in I$, where $\lambda_{ij} \in \mathbb{R}$ for $i, j \in \{1, 2\}$. We assume that:

i) $k_1, l_1 \in C(I^2 \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n)$ and $k_2, l_2 \in C(I^2 \times \mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^p)$;

ii) there exists the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2,2}(\mathbb{R}_+)$ such that

$$|k_i(t, s, u_1, u_2) - k_i(t, s, v_1, v_2)| \le a_{i1}|u_1 - v_1| + a_{i2}|u_2 - v_2|,$$

for each $(t, s, u_1, u_2), (t, s, v_1, v_2) \in I^2 \times \mathbb{R}^n \times \mathbb{R}^p$, $i \in \{1, 2\}$;

iii) $\begin{pmatrix} |\lambda_{11}| \\ |\lambda_{21}| \end{pmatrix} \le \begin{pmatrix} \frac{r_1}{2a(a_{11}r_1 + a_{12}r_2)} \\ \frac{r_2}{2a(a_{21}r_1 + a_{22}r_2)} \end{pmatrix}$ and $\begin{pmatrix} |\lambda_{12}| \\ |\lambda_{22}| \end{pmatrix} \le \begin{pmatrix} \frac{r_1}{2M_{l_1}} \\ \frac{r_2}{2M_{l_2}} \end{pmatrix}$, where

$r := \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$, cu $r_1, r_2 > 0$ and $M_{l_i} = \max_{t \in [0,a]} \int_0^a |l_i(t, s, x_1(s), x_2(s))|\, ds$, for $i \in \{1, 2\}$.

Then, there exists $x_0 := (x_1^0, x_2^0) \in C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$ such that our equations system has at least one solution $x^* := (x_1^*, x_2^*) \in \bar{B}(x_1^0, r_1) \times \bar{B}(x_2^0, r_2) \subset C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$.

Remark 3.2.2. Theorems 3.2.1 and 3.2.3 can be improved supposing instead of the existence of a real positive number square matrix A, another square matrix $A = (a_{ij})_{i,j=1,2}$, where $a_{ij} \in L^p([0,a], \mathbb{R}_+), i,j \in \{1,2\}$ and using the Hölder's inequality which guarantees that

$$\int_0^t a_{ij}(s) e^{\tau s} ds \leq |a_{ij}|_{L^p} \left(\int_0^t a_{ij}(s) e^{q\tau s} ds \right)^{\frac{1}{q}}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1,$$

we can obtain too, another similar results.

Using Theorem 3.1.2 we will prove a nice existence result for an abstract system of Fredholm–Volterra integral inclusions in a generalized Banach space. Notice that in the following result the existence of the fixed point holds for a set of multivalued operators $F_i : Y_i \to P_{cp,cv}(X_i)$, where $(X_i, |\cdot|_{X_i})$ is a Banach space and $Y_i \in P_{b,cl,cv}(X_i)$ for $i \in \{1,2\}$.

To show that F_i is a multivalued contraction in Nadler's sense means to show that for $i \in \{1,2\}$, there exists a matrix convergent to zero $A = (a_{ij})_{i,j=1,2} \in \mathcal{M}_{2,2}(\mathbb{R}_+)$ with the property that for any $x := (x_1, x_2)$, $y := (y_1, y_2) \in Y_1 \times Y_2$ and for any $u_i \in F_i(x_1, x_2)$, there exists $v_i \in F_i(y_1, y_2)$, respectively for any $v_i \in F_i(y_1, y_2)$, there exists $u_i \in F_i(x_1, x_2)$ such that

$$|u_i - v_i|_{X_i} \leq a_{i1} |x_1 - y_1|_{X_1} + a_{i2} |x_2 - y_2|_{X_2}.$$

We consider the multivalued operator $F : Y_1 \times Y_2 \to P_{cp,cv}(X_1 \times X_2)$ defined by $F := (F_1, F_2)$ and denote by $Y := Y_1 \times Y_2$, $X := X_1 \times X_2$, $|u - v|_X := \begin{pmatrix} |u_1 - v_1|_{X_1} \\ |u_2 - v_2|_{X_2} \end{pmatrix}$. Then the above inequalities can be represented in the matrix form: for any $x, y \in Y$ and for any $u \in F(x)$, there exists $v \in F(y)$, respectively for any $v \in F(y)$, there exists $u \in F(x)$ such that

$$|u - v|_X \leq A |x - y|_X, \text{ where } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}_+).$$

Theorem 3.2.3. *Let $I = [0,a]$ (with $a > 0$) be an interval of the real axis and let us consider the following inclusions system in $C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$:*

$$\begin{cases} x_1(t) \in \lambda_{11} \int_0^t K_1(t, s, x_1(s), x_2(s)) ds + \lambda_{12} \int_0^a L_1(t, s, x_1(s), x_2(s)) ds \\ x_2(t) \in \lambda_{21} \int_0^t K_2(t, s, x_1(s), x_2(s)) ds + \lambda_{22} \int_0^a L_2(t, s, x_1(s), x_2(s)) ds \end{cases}$$

for $t \in I$, where $\lambda_{ij} \in \mathbb{R}$, $i,j \in \{1,2\}$. We assume that:

i) $K_1 : I^2 \times \mathbb{R}^n \times \mathbb{R}^p \to P_{cl,cv}(\mathbb{R}^n)$, $K_2 : I^2 \times \mathbb{R}^n \times \mathbb{R}^p \to P_{cl,cv}(\mathbb{R}^p)$ *are two l.s.c., measurable and integrable bounded multivalued operators;*

ii) $L_1 : I^2 \times \mathbb{R}^n \times \mathbb{R}^p \to P_{cp,cv}(\mathbb{R}^n)$, $L_2 : I^2 \times \mathbb{R}^n \times \mathbb{R}^p \to P_{cp,cv}(\mathbb{R}^p)$ *are two l.s.c., measurable and integrable bounded (by two integrable functions* m_{L_1}, m_{L_2}*) multivalued operators;*

iii) there exists the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2,2}(\mathbb{R}_+)$ *such that for each* (t, s, u_1, u_2), $(t, s, v_1, v_2) \in I^2 \times \mathbb{R}^n \times \mathbb{R}^p$ *and for* $i \in \{1, 2\}$, *we have:*

$$H(K_i(t, s, u_1, u_2), K_i(t, s, v_1, v_2)) \le a_{i1}|u_1 - v_1| + a_{i2}|u_2 - v_2|;$$

iv) $\begin{pmatrix} |\lambda_{11}| \\ |\lambda_{21}| \end{pmatrix} \le \begin{pmatrix} \frac{R_1}{2a(a_{11}R_1 + a_{12}R_2)} \\ \frac{R_2}{2a(a_{21}R_1 + a_{22}R_2)} \end{pmatrix}$ *and* $\begin{pmatrix} |\lambda_{12}| \\ |\lambda_{22}| \end{pmatrix} \le \begin{pmatrix} \frac{R_1}{2M_{L_1}a} \\ \frac{R_2}{2M_{L_2}a} \end{pmatrix}$, *where*

$$M_{L_1} = \max_{t\in[0,a]} |m_{L_1}|_{\mathbb{R}^n}, \; M_{L_2} = \max_{t\in[0,a]} |m_{L_2}|_{\mathbb{R}^p}$$

and m_{L_i} *represents the set of continuous selections for the multivalued operator* $t \longrightarrow \lambda_{i2}\int_0^a L_i(t, s, x_1(s), x_2(s))\,ds$, *for* $i \in \{1, 2\}$.

v) the matrix $I - M$ *has the absolute value property, where*

$$M = \left(\frac{|\lambda_{i1}|\, a_{ij}}{\tau}\right)_{i,j=1,2}, \; \tau > 0.$$

Then, there exists $(x_1^0, x_2^0) \in C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$ *such that our inclusions system has at least one solution* $x^* := (x_1^*, x_2^*) \in \bar{B}(x_1^0, R_1) \times \bar{B}(x_2^0, R_2) \subset C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$.

3.3 Krasnoselskii's theorem in E-Banach spaces

In this section we prove a nonlinear version of Krasnoselskii's fixed point theorem in E-Banach spaces for singlevalued operators. The multivalued case is an open problem, which rises from Problem 2.3.14.

Theorem 3.3.1. *Let* $(X, ||\cdot||, E)$ *be an* E*-Banach space with* E *order complete and let* Y *be a nonempty,* E*-bounded,* E*-convex and* E*-closed subset of* X. *Assume that the operators* $f, g : Y \to X$ *satisfy the properties:*

i) f *is a nonlinear* φ*-contraction and the operator* $\psi : E_+ \to E_+$ *defined by* $\psi(t) = t - \varphi(t)$ *satisfies the following relation:*

$$\text{if } (\psi(t_n)) \downarrow 0 \text{ as } n \to +\infty, \text{ then } (t_n) \downarrow 0 \text{ as } n \to +\infty.$$

ii) g *is* E*-continuous;*

iii) $g(Y)$ *is* E*-relatively compact and* $f(x) + g(y) \in Y$ *for any* $x, y \in Y$.

Then $f + g$ *has a fixed point in* Y.

Problem 3.3.2. To prove Krasnoselskii's Theorem in an E-Banach space for a sum of two multivalued operators, where one of the operators satisfies a multivalued nonlinear φ-contraction condition in Nadler's sense and the second operator satisfies a compactness condition.

3.4 Applications

In this last section, we give an existence result for the solution of a Fredholm–Volterra type integral equation in an E-Banach space in which we need to apply Theorem 3.3.1. To obtain an existence result for the solution of a Fredholm–Volterra type integral inclusion in an E-Banach space represents an open problem, consequence of Problem 3.3.2.

Theorem 3.4.1. *Let* E *be an order complete Riesz space,* $r \in E_+$ *with* $r > 0$ *and let* $I := [0, a]$ *(where* $a > 0$*) be an interval of real axis. We consider the following Fredholm–Volterra type integral equation in* $C(I, E)$*:*

$$x(t) = \int_I k(t, s, x(s))\, ds + \int_0^t l(t, s, x(s))\, ds,\ t \in I. \tag{3.4.1}$$

We assume that:

i) $k \in C(I^2 \times E, E)$ *and* $l \in C(I^2 \times E, E)$ *are two o-continuous operators;*

ii) there exists $\omega \in C(I^2, E_+)$ *with* $\sup_{t\in I} \int_I \omega(t, s)\, ds \le 1$*, such that*

$$|k(t, s, x) - k(t, s, y)| \le \omega(t, s)\, \varphi(|x - y|)\,, \text{ for any } t, s \in I,\ x, y \in E,$$

where $\varphi : E_+ \to E_+$ *is an o-comparison operator and the operator* $\psi : E_+ \to E_+$ *defined by* $\psi(t) = t - \varphi(t)$ *satisfies the following relation:*

$$\text{if } (\psi(t_n)) \downarrow 0 \text{ as } n \to +\infty, \text{ then } (t_n) \downarrow 0 \text{ as } n \to +\infty.$$

iii) we have that $M_l := \sup\limits_{t\in I} \int_0^t l\,(t,s,x\,(s))\,ds \leq \frac{1}{2}r$ *and* $\psi(r) \geq \delta$*, where*

$$\delta := \sup_{x\in\bar{B}(0,r)} \left| \sup_{t\in I} \int_I k\,(t,s,x\,(s))\,ds \right| \in E_+.$$

Then, the equation (3.4.1) has a solution x^* *in* $\bar{B}\,(0,r) \subset C\,(I,E)$*.*

Problem 3.4.2. Let E be an order complete Riesz space, $r \in E_+$ with $r > 0$ and let $I = [0,a]$ (where $a > 0$) be an interval of real axis. We consider the following Fredholm–Volterra type integral inclusion in $C\,(I,E)$:

$$x\,(t) \in \int_I K\,(t,s,x\,(s))\,ds + \int_0^t L\,(t,s,x\,(s))\,ds,\ t \in I. \tag{3.4.2}$$

Impose conditions on the multivalued operators $K, L : I^2 \times E \to P(E)$ such that the inclusion (3.4.2) to admit a solution x^* in $\bar{B}\,(0,r) \subset C\,(I,E)$.

Chapter 4

Fixed point theorems in vector b-metric spaces

4.1 Generalized b-metric space

In this section we introduce a larger notion of generalized b-metric space, which extends the classical notion of b-metric space used by other authors: V. Berinde in [12], S. Czerwik in [30], J. Heinonen in [38], M. Boriceanu, A. Petruşel, I.A. Rus in [14], M. Bota in [16]. Using this context, we present some useful properties and auxiliary results to prove our fixed point theorems in the following section.

Definition 4.1.1. Let X be a set and let $S \geq I$ be a square $m \times m$ matrix of nonnegative real numbers, where I denotes the identity matrix. A functional $d : X \times X \to \mathbb{R}^m_+$ is said to be a generalized b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq S[d(x, y) + d(y, z)]$.

The pair (X, d) is called a *generalized b-metric space*.

The class of generalized b-metric spaces is larger then the class of generalized metric spaces, since a generalized b-metric space is a generalized metric space when $S = I$ in the third assumption of the above definition. Some examples of b-metric spaces are given by V. Berinde [12], S. Czerwik [30], J. Heinonen [38]. Here we give some examples of generalized b-metric spaces. Notice that if $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $A = [a_{ij}]$, $B = [b_{ij}]$, for $i, j \in \{1, 2, \ldots, m\}$ then by $A \leq B$ we mean $a_{ij} \leq b_{ij}$, for $i, j \in \{1, 2, \ldots, m\}$.

Example 4.1.2. Let X be a set with the cardinal $card(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geq 2$. Let $S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \geq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be a matrix of real numbers. Then, the functional $d : X \times X \to \mathbb{R}^2_+$ defined by:

$$d(x,y) := \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & x = y \\ 2\begin{bmatrix} s_{11} \\ s_{22} \end{bmatrix}, & x, y \in X_1 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & \text{otherwise} \end{cases}$$

is a generalized b-metric on X.

Example 4.1.3. The set $\ell^p(\mathbb{R})$ (with $0 < p < 1$), where $\ell^p(\mathbb{R}) := \{(x_n) \subset \mathbb{R} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the functional $d : (\ell^p(\mathbb{R}) \times \ell^q(\mathbb{R}))^2 \to \mathbb{R}^2_+$,

$$d(x,y) := \begin{bmatrix} (\sum_{n=1}^{\infty} |x_{1n} - y_{1n}|^p)^{1/p} \\ (\sum_{n=1}^{\infty} |x_{2n} - y_{2n}|^q)^{1/q} \end{bmatrix}$$

is a generalized b-metric space with $S = \begin{bmatrix} 2^{1/p} & s_{12} \\ s_{12} & 2^{1/q} \end{bmatrix} > \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that the above example holds for the general case $l^p(X)$ with $0 < p < 1$, where X is a generalized Banach space.

Example 4.1.4. The space $L^p[0,1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional

$$d(x,y) := \begin{bmatrix} (\int_0^1 |x_1(t) - y_1(t)|^p dt)^{1/p} \\ (\int_0^1 |x_2(t) - y_2(t)|^q dt)^{1/q} \end{bmatrix}, \text{ for each } (x_1, y_1), (x_2, y_2) \in L^p[0,1] \times L^q[0,1]$$

is a generalized b-metric space with $S = \begin{bmatrix} 2^{1/p} & 0 \\ 0 & 2^{1/q} \end{bmatrix}$.

Notice that in a generalized b-metric space (X, d) the notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces, but the generalized b-metric is not continuous in general and does not induces a topology on X.

Lemma 4.1.5. *Let (X, d) be a generalized b-metric space and let $A, B \in P(X)$. We suppose that there exists $\eta \in \mathbb{R}^m_+$, $\eta > 0$ such that:*

(i) for each $a \in A$ there is $b \in B$ such that $d(a, b) \leq \eta$;

(ii) for each $b \in B$ there is $a \in A$ such that $d(a, b) \leq \eta$.

Then, $H(A, B) \leq \eta$.

Lemma 4.1.6. *Let (X, d) be a generalized b-metric space, $A \in P(X)$ and $x \in X$. Then $D(x, A) = 0$ if and only if $x \in \overline{A}$.*

Lemma 4.1.7. *Let (X, d) be a generalized b-metric space and let $\{x_k\}_{k=0}^n \subset X$. Let $S \in M_{m,m}(\mathbb{R})$, with $S \geq I$. Then:*

$$d(x_0, x_n) \leq Sd(x_0, x_1) + \cdots + S^{n-1}d(x_{n-2}, x_{n-1}) + S^{n-1}d(x_{n-1}, x_n).$$

Lemma 4.1.8. *Let (X, d) be a generalized b-metric space and let $S \in M_{m,m}(\mathbb{R})$, with $S \geq I$. Then for all $A, B, C \in P(X)$, we have:*

$$H(A, C) \leq S[H(A, B) + H(B, C)].$$

Lemma 4.1.9. *Let (X, d) be a generalized b-metric space and let $A, B \in P_{cl}(X)$. Then for each $\alpha \in \mathbb{R}^m_+$, $\alpha > 0$ and for each $b \in B$, there exists $a \in A$ such that*

$$d(a, b) \leq H(A, B) + \alpha.$$

If, moreover, $A, B \in P_{cp}(X)$ and $S \in M_{m,m}(\mathbb{R})$ with $S \geq I$. Then for each $b \in B$, there exists $a \in A$ such that

$$d(a, b) \leq SH(A, B).$$

Lemma 4.1.10. *Let (X, d) be a generalized b-metric space and let $A, B \in P_b(X)$, $q > 1$. Then, for all $a \in A$, there exists $b \in B$ such that:*

$$\delta(A, B) \leq qd(a, b).$$

4.2 Fixed point theorems in generalized b-metric spaces

In this section we present some fixed and strict fixed point theorems in E-b-metric spaces using the Picard and weak Picard operators technique. The definition mode of Picard, M-Picard, multivalued weak Picard and multivalued M-weak Picard operators in a generalized b-metric space is similar to those used in a generalized metric space (see section 1.3).

We start to present some fixed point theorems in generalized b-metric spaces for singlevalued operators.

Theorem 4.2.1. *Let (X, d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $S \geq I$ and let $f : X \to X$ be a singlevalued A-contraction such that $AS = SA$ and $SA < I$. Then f is a $(I - SA)^{-1} S$-Picard operator.*

Definition 4.2.2. Let (X, d) be a generalized b-metric space and let $f : X \to X$ be a singlevalued operator. Then, f is called a *singlevalued* (A, B, C)-*contraction* if and only if there exists the matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where A is convergent to zero with $A + B + C < I$ such that

$$d[f(x), f(y)] \leq Ad(x, y) + Bd[x, f(x)] + Cd[y, f(y)], \text{ for any } x, y \in X.$$

Theorem 4.2.3. *Let (X, d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $S \geq I$ and let $f : X \to X$ be a singlevalued (A, B, C)-contraction such that $KS = SK$, where $K := (I - C)^{-1}(A + B)$ and $SA < I$. Then f is a $(I - SA)^{-1} S (I - B)$-Picard operator.*

It is easy to observe (see S. Czerwik [30]) that if (X, d) is a generalized b-metric space, then the functional $H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbb{R}^m_+$ is a generalized b-metric in $P_{b,cl}(X)$. Also, if (X, d) is a complete generalized b-metric space, we have that $(P_{b,cl}(X), H)$ is a complete generalized b-metric space.

In the following we present some fixed and strict fixed point theorems in generalized b-metric spaces for multivalued operators.

Theorem 4.2.4. *Let (X, d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $S \geq I$ and let $F : X \to P_{cl}(X)$ be a multivalued A-contraction in Nadler's sense such that $AS = SA$ and $SA < I$. Then F is a multivalued $(I - SA)^{-1} S$-weak Picard operator.*

Definition 4.2.5. Let $Y \subset X$ be a nonempty set and let $F : Y \to P_{cl}(X)$ be a multivalued operator. Then, F is called a *multivalued* (A, B, C)*-contraction* if and only if there exists the matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where A is convergent to zero with $A + B + C < I$ such that

$$H[F(x), F(y)] \leq Ad(x, y) + BD[x, F(x)] + CD[y, F(y)], \text{ for any } x, y \in Y.$$

Theorem 4.2.6. *Let (X, d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $S \geq I$ and let $F : X \to P_{cl}(X)$ be a multivalued (A, B, C)-contraction such that $KS = SK$, where $K := (I - qC)^{-1}(A + B)$, $q \in \left(1, \frac{1}{\rho(A+B+C)}\right)$ and $SA < I$. Then F is a multivalued $(I - SA)^{-1} S (I - B)$-weak Picard operator.*

We have two additional results for the strict fixed point set of F. The first one in the terms of functional H, and the second one in the terms of functional δ.

Theorem 4.2.7. *If all the assumption of Theorem 4.2.6 holds and* $\text{SFix}(F)$ *is nonempty, then:*

$$\text{Fix}(F) = \text{SFix}(F) = \{x^*\}.$$

Theorem 4.2.8. *Let (X, d) be a complete generalized b-metric space with $S \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $S \geq I$ and let $F : X \to P_b(X)$ be such that $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where A is convergent to zero with $A + B + C < I$, $KS = SK$, where $K := (I - C)^{-1}(A + B)$, $SA < I$ and*

$$\delta[F(x), F(y)] \leq Ad(x, y) + B\delta[x, F(x)] + C\delta[y, F(y)], \textit{ for any } x, y \in X.$$

Then $\text{SFix}(F) = \{x^*\}$.

Remark 4.2.9. If we choose $B = C = 0$ in Theorem 4.2.8 implies that $\delta[F(x), F(x)] = 0$, for any $x \in X$ which yields that F is a singlevalued operator. Therefore the statement of Theorem 4.2.8 is nontrivial if $B+C > 0$.

4.3 E-b-metric space

In this section we introduce the notion of E-b-metric space and a relevant concept of strict positivity in a Riesz space. Also, we present some auxiliary results, which works with the concept of strict positivity and are used to prove the fixed point theorems in the following section.

Definition 4.3.1. Let X be a nonempty set and let $s \geq 1$ be a real number. A functional $d : X \times X \to E_+$ is called an *E-b-metric* if and only if, for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d, E) is called an *E-b-metric space.*

We recall from C.D. Aliprantis, K.C. Border [5] that an element $e \in E_+$ in a Riesz space E is called an *order unit element* if, for any $x \in E$, there exists $\lambda \in \mathbb{R}_+$ such that $|x| \leq \lambda e$. However, this notion of strict positiveness is insufficient for our purposes. Therefore, we introduce the following concept.

Definition 4.3.2. We say that $e \in E_+$ is a *strict order unit element*, written $e \gg 0$ if, for any subset $H \subset E_+$ with $\inf H = 0$, there exist $h_1, \ldots, h_n \in H$ such that $\min(h_1, \ldots, h_n) \leq e$.

For example, if $E = \mathbb{R}^2$, $E_+ = \mathbb{R}^2_+$, then $e = (e_1, e_2) \gg 0$ if and only if $e_1 > 0$ and $e_2 > 0$. Thus, in this case, we can see that order unit elements are strict order unit elements as well.

Proposition 4.3.3. *If E is Archimedean and e is a strict order unit element, then e is an order unit element.*

The reversed implication in the above proposition is not true in general as is shown by the next proposition.

Proposition 4.3.4. *In the space $E = \ell^\infty$ with the positive cone*

$$E_+ = \{(e_1, e_2, \ldots) : e_i \geq 0\} \subset \ell^\infty,$$

$e \in E_+$ is an order unit element if and only if $\inf\{e_1, e_2, \ldots\} > 0$. However, there is no strict order unit element in E.

Let us denote by E_{++} the set of strict order unit elements in E.

Proposition 4.3.5. *E_{++} is a convex cone.*

In the following results we characterize the convergence of sequences in terms of strict order unit elements.

Lemma 4.3.6. *Let E be order complete and assume that E_{++} is nonempty. Then $h_n \xrightarrow{o} 0$ if and only if, for all $e \in E_{++}$, there exists $n_0 \in \mathbb{N}$ such that*

$$|h_n| \leq e, \text{ for all } n \geq n_0.$$

Corollary 4.3.7. *Let (X, d, E) be an E-complete metric space, where E is order complete such that E_{++} is nonempty. Then $x_n \xrightarrow{d,E} x^*$ if and only if, for any $e \in E_{++}$, there exists $n_0 \in \mathbb{N}$ such that*

$$d(x_n, x^*) \leq e, \text{ for all } n \geq n_0.$$

Lemma 4.3.8. *Let (X, d, E) be an E-complete metric space, where E is order complete such that E_{++} is nonempty. Then $x_n \xrightarrow{d,E} x^*$ is an E-Cauchy sequence if and only if, for any $e \in E_{++}$, there exists $n_0 \in \mathbb{N}$ such that*

$$d(x_n, x_m) \leq e, \text{ for all } m > n \geq n_0.$$

4.4 Fixed point theorems in E-b-metric spaces

In this section we present some fixed point theorems in E-b-metric spaces using the Picard and weak Picard operators technique. The study of fixed point is realized in the strict order unit elements cone E_{++}. The notion was introduced in the above section, see also Zs. Páles, I.-R. Petre [57] and I.-R. Petre [61]. Moreover, the following theorems does not need to impose the condition $\varphi(t) < t$ on the o-comparison operator φ.

Definition 4.4.1. Let (X, d, E) be an E-metric space and let $f : X \to X$ be a vector Picard operator. Then, the operator f is called a *vector ψ-Picard operator* iff, the operator $\psi : E_+ \to E_+$ have the properties: for any decreasing sequence $(t_n) \subset E_+$ with $t_n \downarrow t$, we have $\psi(t_n) \downarrow \psi(t)$, for any $t \in E_+$ with $t > 0$ and $d(x, x^*) \leq \psi[d(x, f(x))]$, for any $x \in X$.

Theorem 4.4.2. *Let (X, d, E) be a complete E-b-metric space with E order complete and let $s \geq 1$. We assume that E_{++} is nonempty and let $f : X \to X$ be a nonlinear φ-contraction. If for any decreasing sequence $(t_n) \subset E_+$ with $t_n \downarrow t$, we have $\varphi(t_n) \downarrow \varphi(t)$, and the operator $\psi : E_+ \to E_+$ defined by $\psi(t) = \frac{1}{s}t - \varphi(t)$ is inversable, then f is a vector ψ^{-1}-Picard operator.*

Definition 4.4.3. Let (X, d, E) be an E-metric space and let $F : X \to P(X)$ be a multivalued vector weak Picard operator. Then, the operator F is called a *vector ψ-weak Picard operator* iff, the operator $\psi : E_+ \to E_+$ have the properties: for any decreasing sequence $(t_n) \subset E_+$ with $t_n \downarrow t$, we have $\psi(t_n) \downarrow \psi(t)$, for any $t \in E_+$ with $t > 0$ and there exists a selection f^∞ for F^∞ such that $d[x, f^\infty(x, y)] \leq \psi[d(x, y)]$, for any $(x, y) \in \text{Graph}(F)$.

Theorem 4.4.4. *Let (X, d, E) be a complete E-b-metric with E order complete and let $s \geq 1$. We assume that E_{++} is nonempty and let $F : X \to P_{cl}(X)$ be a multivalued nonlinear φ-contraction. If for any decreasing sequence $(t_n) \subset E_+$ with $t_n \downarrow t$, we have $\varphi(t_n) \downarrow \varphi(t)$ and $\varphi(st) \leq s\varphi(t)$, for any $t \in E_+$ with $t > 0$, and the operator $\psi : E_+ \to E_+$ defined by $\psi(t) = \frac{1}{s}t - s^2\varphi(t)$ is inversable, then F is a vector ψ^{-1}-weak Picard operator.*

Bibliography

[1] M. Abbas, A.R. Khan, S.Z. Németh, *Complemetarity problems via common fixed points in vector lattices*, Fixed Point Theory Appl., Vol. 2012:60, 2012, 26 pp.

[2] R.P. Agarwal, *Contraction and approximate contraction with an application to multi-point boundary value problems*, J. Comput. Applied Math., Vol. 9, 1983, 315-325.

[3] R.P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2004.

[4] R.P. Agarwal, D. O'Regan, D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, Dordrecht, 2009.

[5] C.D. Aliprantis, K.C. Border, *Infinite Dimensional Analysis*, Springer, Berlin, 1999.

[6] C.D. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, Dordrecht, 2006.

[7] G. Allaire, S.M. Kaber, *Numerical Linear Algebra*, Texts in Applied Mathematics, Vol. 55, Springer, New York, 2008.

[8] A. Amini-Harandi, H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal., Vol. 72, 2010, 2238-2242.

[9] J.-P. Aubin, A. Celina, *Differential Inclusions*, Springer, Berlin, 1984.

[10] I.A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Funct. Anal., Unianowsk Gos. Ped. Inst., Vol. 30, 1989, 26-37.

[11] D. Barbu, G. Bocşan, *Contractive mappings in spaces with vector norm*, Bull. Belg. Math. Soc. - Simon Stevin, Vol. 15, No. 4, 2008, 577-587.

[12] V. Berinde, *Generalized contractions in quasimetric spaces*, Semin. Fixed Point Theory, Preprint No. 3, 1993, 3-9.

[13] L.M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford, 1953.

[14] M. Boriceanu, A. Petruşel, I.A. Rus, *Fixed point theorems for some multivalued generalized contractions in b-metric spaces*, Int. J. Math. Stat., Vol. 6, 2010, 65-76.

[15] M. Boriceanu, *Fixed point theory for multivalued generalized contraction on a set with two b-metrics*, Studia Univ. Babeş-Bolyai, Math., Vol. 3, 2009, 3-14.

[16] M. Bota, *Dynamical Aspects in the Theory of Multivalued Operators*, Cluj University Press, 2010.

[17] M. Bota, A. Molnár, Cs. Varga, *On Ekeland's Variational Principle in b-metric spaces*, Fixed Point Theory, Vol. 12, No. 2, 2011, 21-28.

[18] N. Bourbaki, *Topologie Générale*, Herman, Paris, 1974.

[19] A. Bucur, L. Guran, A. Petruşel, *Fixed points for multivalued operators on a set endowed with vector-valued metrics and applications*, Fixed Point Theory, Vol. 10, No. 1, 2009, 19-34.

[20] T.A. Burton, *A fixed-point theorem of Krasnoselskii*, Appl. Math. Lett., Vol. 11, 1998, 85-88.

[21] T.A. Burton, *Integral equations, implicit functions, and fixed points*, Proc. Am. Math. Soc., Vol. 124, No. 8, 1996, 2383-2390.

[22] T.A. Burton, *Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem*, Nonlinear Stud., Vol. 9, No. 2, 2002, 181-190.

[23] T.A. Burton, T. Furumochi, *Krasnoselskii's fixed point theorem and stability*, Nonlinear Anal., Theory Methods Appl., Vol. 49, 2002, 445-454.

[24] S. Carl, S. Heikkila, *Fixed Point Theory in Ordered Sets and Applications*, Springer, New York, 2011.

[25] C. Çevik, I. Altun, *Vector metric spaces and some properties*, Topol. Methods Nonlinear Anal., Vol. 34, 2009, 375-382.

[26] L. Collatz, *Some applications of functional analysis to analysis, particularly to nonlinear integral equations*, Nonlinear Functional Anal. and Appl. (Proc. Advanced Sem., Math. Res. Center, Univ. of Wisconsin, Madison, Wis, 1970), Academic Press, New York, 1-43.

[27] H. Covitz, S.B. Nadler Jr., *Multivalued contraction mappings in generalized metric spaces*, Isr. J. Math., Vol. 8, 1970, 5-11.

[28] R. Cristescu, *Ordered Linear Spaces*, Editura Academiei, 1959 (in Romanian).

[29] R. Cristescu, *Order Structures in Normed Vector Spaces*, Editura Ştiinţifică şi Enciclopedică, Bucureşti, 1983 (in Romanian).

[30] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Univ. Modena, Vol. 46, 1998, 263-276.

[31] E. De Pascale, G. Marino, P. Pietramala, *The use of the E-metric spaces in the search for fixed points*, Matematiche, Vol. 48, No. 2, 1993, 367-376.

[32] K. Deimling, *Multivalued Differential Equations*, W. de Gruyter, 1992.

[33] A.-D. Filip, A. Petruşel, *Fixed point theorems on spaces endowed with vector-valued metrics*, Fixed Point Theory Appl., Vol. 2010, 2010, Article ID 281381, 15 pp.

[34] M. Fréchet, *Les Espaces Abstraits*, Gauthier-Villars, Paris, 1928.

[35] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, New York, 2003.

[36] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publisher, Dordrecht, 1996.

[37] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal., Vol. 71, 2009, 3403-3410.

[38] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer, Berlin, 2001.

[39] L.-G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., Vol. 332, 2007, 1468-1476.

[40] J. Jachymski, J. Matkowski, T. Swiatkowski, *Nonlinear contractions on semimetric spaces*, J. Appl. Anal., Vol. 1, 1995, 125-134.

[41] L. Kantorovich, *The method of successive approximations for functional equations*, Acta. Math., Vol. 71, 1939, 63-97.

[42] T. Kawasaki, M. Toyoda, T. Watanabe, *Schauder-Tychonoff's fixed point theorem in a vector lattice*, Fixed Point Theory, Vol. 11, No. 1, 2010, 37-44.

[43] M. Kwapisz, *Some remarks on abstract form of iterative methods in functional equation theory*, Proc. 16 Internationales Symposium űber Funktionalgleichungen, Graz, 1978.

[44] T. Lazăr, G. Moţ, G. Petruşel, S. Szentesi, *The theory of Reich's fixed point theorem for multivalued operators*, Fixed Point Theory Appl., Vol. 2010, 2010, Article ID 178421, 10 pp.

[45] W.A.J. Luxemburg, A.C. Zaanen, *Riesz Spaces*, North-Holland Publishing Company, Amsterdam, Vol. 1, 1971.

[46] J. Matkowski, *Fixed point theorems for contractive mappings in metric spaces*, Cas. Pěst. Mat., Vol. 105, 1980, 341-344.

[47] J. Matkowski, *Integrable solutions of functional equations*, Dissertationes Math. (Rozprawy Mat.), Vol. 127, 1975, 68 pp.

[48] I. Muntean, *Capitole speciale de analiză funcţională*, Cluj-Napoca, 1990 (in Romanian).

[49] I. Muntean, *În legătură cu o teoremă de punct fix în spaţii local convexe*, Rev. Roum. Math. Pures Appl., Vol. 19, 1974, 1105-1109 (in Russian).

[50] S.B. Nadler Jr., *Multi-valued contraction mappings*, Pac. J. Math., Vol. 30, 1969, 475-487.

[51] Juan J. Nieto, Rosana Rodriguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, Vol. 22, 2005, 223-239.

[52] Juan J. Nieto, Rodrigo L. Pouso, Rosana Rodriguez-López, *Fixed point theorems in ordered abstract spaces*, Amer. Math. Soc., Vol. 153, No. 8, 2007, 2505-2517.

[53] D. O'Regan, R. Precup, *Continuation theory for contractions on spaces with two vector-valued metrics*, Appl. Anal., Vol. 82, 2003, 131-144.

[54] D. O'Regan, A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl., Vol. 341, No. 2, 2008, 1241-1252.

[55] D. O'Regan, N. Shahzad, R.P. Agarwal, *Fixed point theory for generalized contractive maps on spaces with vector-valued metrics*, Fixed Point Theory Appl., Vol. 6, Nova Sci. Publ., New York, 2007, 143-149.

[56] D. O'Regan, *Fixed-point theory for the sum of two operators*, Appl. Math. Let., Vol. 9, 1996, 1-8.

[57] Zs. Páles, **I.-R. Petre**, *Iterative fixed point theorems in E-metric spaces*, Acta Math. Hung., DOI: 10.1007/s10474-012-0274-8.

[58] A.I. Perov, *On the Cauchy problem for a system of ordinary differential equations*, Pviblizhen. Met. Reshen. Differ. Uvavn., Vol. 2, 1964, 115-134 (in Russian).

[59] A.I. Perov, A.V. Kibenko, *On a certain general method for investigation of boundary value problems*, Izv. Akad. Nauk SSSR, Ser. Mat., Vol. 30, 1966, 249-264 (in Russian).

[60] **I.-R. Petre**, *A multivalued version of Krasnoselskii's theorem in generalized Banach spaces*, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. (submitted for publication).

[61] **I.-R. Petre**, *Fixed point theorems in E-b-metric spaces*, Arabian J. Math. (submitted for publication).

[62] **I.-R. Petre**, *Fixed point theorems in vector metric spaces for multivalued operators*, Topol. Methods Nonlinear Anal. (submitted for publication).

[63] **I.-R. Petre**, *Fixed point theorems in vector metric spaces for single-valued operators*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity, Vol. 9, 2011, 59-80.

[64] **I.-R. Petre**, M. Bota, *Fixed point theorems on generalized b-metric spaces*, Publ. Math. Debrecen (accepted for publication).

[65] **I.-R. Petre**, *Fixed points for φ-contractions in E-Banach spaces*, Fixed Point Theory, Vol. 13, No. 2, 2012, 623-640.

[66] **I.-R. Petre**, A. Petruşel, *Krasnoselskii's theorem in generalized Banach spaces and applications*, Electron. J. Qual. Theory Differ. Equ., No. 85, 2012, 1-20.

[67] **I.-R. Petre**, *On the solution operator of a differential inclusion*, JP J. of Fixed Point Theory and Appl., Vol. 6, No. 2, 2011, 107-117.

[68] A. Petruşel, *A generalization of Krasnoselskii's fixed point theorem*, Proc. Sci. Comm. Meeting of "Aurel Vlaicu" Univ. Arad, Vol. 14A, 1996, 109-112.

[69] A. Petruşel, I.A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., Vol. 134, 2006, 411-418.

[70] A. Petruşel, *Fixed points and integral inclusions*, Rev. Anal. Numér. Théor. Approx., Vol. 35, No. 2, 2006, 183-188.

[71] A. Petruşel, *Integral inclusions. Fixed point approaches*, Annales Soc. Math. Pol., Ser. I, Commentat. Math. XL, 2000, 147-158.

[72] A. Petruşel, *Multivalued operators and fixed points*, Pure Math. Appl., Vol. 11, No. 2, 2000, 361-368.

[73] A. Petruşel, *Multivalued weakly Picard operators and applications*, Sci. Math. Jap., Vol. 59, No. 1, 2004, 169-202.

[74] A. Petruşel, *Operatorial Inclusions*, House of the Book of Science, Cluj-Napoca, 2002.

[75] A. Petruşel, I.A. Rus, *The theory of a metric fixed point theorem for multivalued operators*, Fixed Point Theory and its Applications, Yokohama Publ., 2010, 167-176.

[76] R. Precup, A. Viorel, *Existence results for systems of nonlinear evolution equations*, Int. J. Pure Appl. Math., Vol. 47, No. 2, 2008, 199-206.

[77] R. Precup, A. Viorel, *Existence results for systems of nonlinear evolution inclusions*, Fixed Point Theory, Vol. 11, No. 2, 2010, 337-346.

[78] R. Precup, *Methods in Nonlinear Integral Equations*, Kluwer, Dordrecht, 2002.

[79] R. Precup, *The role of matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comput. Modelling, Vol. 49, No. 3-4, 2009, 703-708.

[80] S. Reich, *Fixed points of contractive functions*, Boll. Unione Mat. Ital., Vol. 5, No. 4, 1972, 26-42.

[81] S. Reich, *Kannan's fixed point theorem*, Boll. Unione Mat. Ital., Vol. 4, No. 4, 1971, 1-11.

[82] D. Repovš, P.V. Semenov, *Continuous Selections of Multivalued Mappings*, Kluwer, Dordrecht, 1998.

[83] Sh. Rezapour, R.H. Haghi, *Fixed point of multifunctions on cone metric spaces*, Numer. Funct. Anal. Optim., Vol. 30, 2009, 825-832.

[84] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Am. Math. Soc., Vol. 226, 1977, 257-290.

[85] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008.

[86] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[87] I.A. Rus, *Principles and Applications of the Fixed Point Theory*, Dacia, Cluj-Napoca, 1979 (in Romanian).

[88] I.A. Rus, *Technique of the fixed point structures for multivalued mappings*, Math. Jap., Vol. 38, 1993, 289-296.

[89] I.A. Rus, *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, Vol. 9, No. 2, 2008, 541-559.

[90] I.A. Rus, A. Petruşel, A. Sântămărian, *Data dependence of the fixed point set of multivalued weakly Picard operators*, Nonlinear Anal., Vol. 52, 2003, 1947-1959.

[91] I.A. Rus, M.-A. Şerban, *Some generalizations of a Cauchy lemma and applications*, Topics in Mathematics, Computer Science and Philosophy, A Festschrift for Wolfgang W. Breckner, 173-181.

[92] L. Rybinski, *An application of the continuous selection theorem to the study of the fixed points of multivalued mappings*, J. Math. Anal. Appl., Vol. 153, 1990, 391-396.

[93] M. Turinici, *Nonlinear contractions and applications to Volterra functional equations*, An. Şt. Univ. Iaşi, S.I., Vol. 23, 1977, 43-50.

[94] R.S. Varga, *Matrix Iterative Analysis*, Vol. 27 of Springer Series in Computational Mathematics, Springer, Berlin, 2000.

[95] A. Viorel, *Contributions to the study of nonlinear evolution equations*, Ph.D. Thesis, Babeş-Bolyai University Cluj-Napoca, 2011.

[96] F. Voicu, *Contractive operators in partially ordered spaces*, Seminar on Diff. Eq., Vol. 3, 1989, 181-214.

[97] F. Voicu, *Fixed point theorems in spaces with vectorial metric*, Stud. Univ. Babeş-Bolyai, Math., Vol. 36, No. 4, 1991, 53-56.

[98] R. Węgrzyk, *Fixed point theorems for multifunctions and their applications to functional equations*, Dissertationes Math., Vol. 201, 1982, 1-28.

[99] A.C. Zaanen, *Riesz Spaces*, North-Holland Publishing Company, Amsterdam, Vol. 2, 1983.

[100] P.P. Zabrejko, *K-metric and K-normed linear spaces: survey*, Collect. Math., Vol. 48, No. 4-6, 1997, 825-859.

[101] E. Zeidler, *Nonlinear Functional Analysis, Vol. I, Fixed Point Theorems*, Springer, Berlin, 1993.

[102] M. Zuluaga, *On a fixed point theorem and application to a two-point boundary value problem*, Comment. Math. Univ. Carolinae, Vol. 27, 1986, 731-735.