UNIUNEA EUROPEANĂ
GUVERNUL ROMÂNIEI MINISTERUL MUNCII, FAMILIEI ŞI PROTECŢIEI SOCIALE AMPOSDRU
Fondul Social European POSDRU 2007-2013
Instrumente Structurale 2007-2013
MINISTERUL EDUCAŢIEI CERCETĂRII TINERETULUI ŞI SPORTULUI
OIPOSDRU
UNIVERSITATEA BABEŞ-BOLYAI CLUJ-NAPOCA

Babeş-Bolyai University, Cluj Napoca
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Contributions to the theory of variational inequalities, equilibrium and optimization problems via duality

Ph.D. Thesis Summary

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FONDUL SOCIAL EUROPEAN
Investeşte în OAMENI

Titlul proiectului
„Studii doctorale inovative într-o societate bazată pe cunoaştere"
POSDRU/88/1.5/S/60185
Proiect cofinanţat din Fondul Social European
prin Programul Operaţional Sectorial Dezvoltarea Resurselor Umane
2007-2013

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Introduction

The power and importance of science has been demonstrated once again in the Second World War when the victory was imposed by those who have used the discoveries of physics, chemistry, mathematics and computer science. In 1939, Leonid Kantorovich developed what today is known as linear programming, to plan expenditures and returns of the army in order to minimize costs and maximize enemy losses. Later, in 1947, George B. Dantzig published the simplex method in order to solve the linear programming while John von Neumann developed duality theory as a linear optimization solution along with its applications in the field of game theory. Many of the concepts from optimization theory, such as duality and the importance of convexity and its generalizations were inspired by ideas coming from linear programming which has greatly expanded due to a widespread use in microeconomics, business, engineering, company management, such as transportation, planning, production, and others. In practice, most of the times we deal with an optimization problem which implies convex functions in order to be minimized, but it does not have to be necessarily linear. Because of that and due to the interest in the calculus of variations, the study of convex sets and convex functions recorded an increased interest. The first works in this field are due to Fenchel, Moreau, Brondsted, and Rockafellar, works that include notions of the theory of convex functions, conjugate, and duality. Necessary conditions for a solution to be optimal in nonlinear programming, were given by Karush-Kuhn-Tucker, known also as Kuhn-Tucker conditions, that allow inequality constraints, generalizing the Lagrange multiplier method which allows only equality constraints. In order to fulfill the Karush-Kuhn-Tucker conditions, a minimum point must satisfy certain regularity conditions.

The study of the classical optimization problem is well presented in the literature. In most of the papers, the optimal solutions of this problem can be attained by attaching to the optimization problem its dual one. Duality is an extensively studied method, let us mention here the Lagrange and Fenchel duality. Optimization problems are used as tools in solving other classes of problems, namely variational inequalities and equilibrium problems.

The main purpose of this thesis is to study different types of variational inequalities, equilibrium problems and optimization problems and characterize the solutions of those problems via duality.

The thesis consists of four chapters, which are briefly presented in the following lines highlighting the most important results.

In **Chapter 1** we begin with some preliminary notions and results from functional analysis and convex analysis presented in Section 1.1. Then we present in Section 1.2 some regularity conditions regarding the general scalar primal optimization problem and its conjugate dual one, which are expressed by means of a perturbation function. The regularity conditions are used in order to ensure strong duality between the two problems. We also present regularity conditions for some particular cases.

In **Chapter 2** we deal with the variational inequalities. In the first section of this chapter we introduce the general ε-variational inequality, and its dual one formulated by the help of a perturbation function and a set valued operator.

The main results of the Subsection 2.1 ensures that if the function involved is proper, convex and a regularity condition is fulfilled, if the primal ε-variational inequality is solvable then also its dual one is solvable. Conversely, when the dual ε-variational inequality is solvable and the function involved is proper, convex and lower semicontinuous, then also the primal one is solvable. We underline the fact that we do not need regularity conditions for the validity of this theorem. Further, we give similar results for some particular cases when we specialize the perturbation function. Among our special instances we rediscover dual scheme consider by Kum, Kim and Lee given in finite dimensional settings. We improve the results given in [94] by showing that the theorem concerning the implication primal ε-variational inequality is solvable than its dual one is solvable, holds under weaker hypotheses (the lower semicontinuity of the function f is not needed and instead of the regularity condition considered in [94] we use a weaker one) and that another result (which addresses the implication the dual is solvable then the primal is solvable) is valid also in the absence of any regularity condition. An example justifies the use of weaker regularity conditions than the one considered in [94]. Duality scheme proposed by Mosco in [113] concerning variational inequalities can be seen as another particular instance of our main results. In the end of this section, we give duality results concerning dual pair of ε-variational inequalities in which the composition case is involved.

In Section 2.2 we introduce the general (Stampacchia type) variational inequality problem. In Subsection 2.2.1 we reformulate this general variational inequality into an optimization problem depending on a fixed variable. We attach to this optimization problem a dual one and we define a function by means of the optimal value of the dual problem. Regularity conditions guaranteeing strong duality for the primal-dual pair of optimization problems play a significant role when proving that the introduced function is gap function for the general variational inequality. The convexity of the gap function is ensured by convexity and monotonicity hypothesis. In Subsection 2.2.2 we present several gap functions for particular cases of the general variational inequalities and we rediscover several gap functions introduced in the literature by Altangerel, Boţ and Wanka. We give an improved result as the one given by Altangerel et al. by using weaker regularity conditions. An example comes to justify the use of such regularity conditions. In Subsection 2.2.3 we introduce a dual gap function for the general variational inequality by considering the (Minty type) general variational inequality. We show that under monotonicity and mild continuity properties the Stampacchia type variational inequality and the Minty type variational inequality are equivalent. We construct the dual gap function by reformulating this general (Minty type) variational inequality into an optimization problem depending on a fixed variable. We attach to this optimization problem a dual one and we define this function by means of the optimal value of the dual problem. It is proved, under monotonicity assumptions, that the gap function and the dual gap function can be related. The main result of this subsection gives conditions in order to ensure that the function introduced is gap function for the general Stampacchia type variational inequality. We point out that this function is a convex function. In the last part of this subsection we particularize the dual gap function for particular cases, rediscovering a dual gap function introduced in [3, Section 4]. We also provide an example which underlines the fact that in general the gap function and the dual gap function do not coincide, even if all the hypotheses of theorems which ensures that this functions are gap functions are fulfilled.

In the last section of this chapter, Section 2.3, we deliver optimality conditions for variational inequalities based on subdifferential calculus for the general variational inequality and for some particular cases of it. In Subsection 2.3.2 we give sequential characterizations of the solutions of general variational inequality and for some of its particular cases. This type of optimality conditions are applicable even if no regularity condition is fulfilled. We use as tool the sequential optimality conditions given by Boţ, Csetnek and Wanka in [28, 29]. Some examples are given in order to justify the usefulness of such characterizations.

Chapter 3 is dedicated to equilibrium problems. This chapter is split in three sections.

In Section 3.1 we generalize equilibrium problem considered by Bigi, Castellani and Kassay [19] and we consider an equilibrium problem with sum of two functions, one being composed with a linear mapping. In subsection 3.1.1 we give a regularity condition for an optimization problem formed by a sum of three functions, one being composed with a linear operator, regularity condition needed in the next subsection. In the Subsection 3.1.2 we attach to the composed equilibrium problem an optimization problem. We give results which establish the connection between the composed equilibrium problem, the associated optimization problem and a set of which elements are in subdifferential of the functions involved in the composed equilibrium problem. Using the characterization of subdifferential of a given function and its conjugate, we attach a dual problem to the composed equilibrium problem. Then duality results are given for the dual pair of equilibrium problem proposed. Furthermore, we attach a Lagrangian function to the composed equilibrium problem and we prove that the solutions of the composed equilibrium problem respectively of its dual are the saddle points of the Lagrangian function. We consider also the Fenchel dual optimization problem of the optimization problem attach to the composed equilibrium problem. This dual optimization problem is not the optimization form of the dual composed equilibrium problem, but we deliver a result which ensure a relation between them. We also give results which guarantees that all the solutions of the composed equilibrium problem and its dual one can be found using the attach optimization problem respectively, its Fenchel dual one. This results are given in hypothesis of which the regularity condition presented in Subsection 3.1.1 is used. In the last part of this section we present some particular cases of our results, rediscovering results introduced in literature by Bigi et al. [19] for equilibrium problems and also results given in Subsection 2.1.2 for variational inequalities if we consider a special instance of them.

In Section 3.2 we deal with gap functions introduced for a general equilibrium problem (in sense of Stampacchia) formulated by the help of a perturbation function. Using the same techniques presented in Section 2.2 for variational inequalities, we construct gap functions for the general equilibrium problem. Under convexity hypotheses and the fulfillment of a regularity condition we ensure that the introduced function becomes a gap function for the general equilibrium problem. By particularizing the perturbation function we rediscover some gap functions introduced for equilibrium problems in literature by Boţ and Capătă in [26] and by Altangerel et al. in [2]. We mention also that by specializing the generalized equilibrium problem to variational inequalities we rediscover the same framework as in Section 2.2. In Subsection 3.2.3 we introduce a dual gap function for the general equilibrium problem by using the so-called dual equilibrium problems (Minty type) which is closely related to the Stampacchia type general equilibrium problem. In order to formulate the dual gap function for the general equilibrium problem using the dual equilibrium problem we use some generalized monotonicity and convexity notions. Using hypothesis like monotonicity, convexity, hemicontinuity we establish inclusions between the solution set of the Stampacchia type equilibrium problem and the solution set of the corresponding Minty type equilibrium problem. The main result of this subsection gives conditions in order to ensure that the function introduced by the help of Minty type equilibrium problem becomes gap function for the general equilibrium problem in sense of Stampacchia. Also, some particular instances of our results are considered and among them we rediscover the settings considered for variational inequalities in Subsection 2.2.3 and we can rediscover the dual gap function considered by Altangerel et al. in [2].

In the last section of this chapter, Section 3.3, we characterize the solutions of the general equilibrium problem by means of the (convex) subdifferential. Similar characterizations are given for some particular cases. The particularization of those results to variational inequalities brings to us the results presented in Subsection 2.3.1. In Subsection 3.3.2 are given regularity free conditions in order to characterize the solutions of the general equilibrium problem and its particular cases. We show that the particularizations of the sequential optimality conditions given for the equilibrium problem to variational inequalities are exactly the sequential characterizations of the solutions of the general variational inequality. There are also presented sequential optimality conditions for the equilibrium problems with the sum of two functions. We rediscover also similar theorem given for variational inequality when we specialize the bifunction used in formulation of the equilibrium problem to a given operator.

In **Chapter 4** we turn our attention to an optimization problem to which we attach an approximate optimization problem which is constructed by a second order η-approximation of the constraint functions at an arbitrary but fixed feasible point and which is called the $(0,2)$ η-approximated optimization problem. In order to prove the equivalence between the original optimization problem and its $(0,2)$ η-approximated optimization problem we use second order invexity. Some examples illustrating theoretical notions are presented. We establish connections between the feasible solutions of (0,2) η-approximated optimization problem and the feasible solutions of the original problem. Then we study the connections between the optimal solutions of the (0,2) η-approximated optimization problem and the optimal solutions of original optimization problem via the saddle points of associated Lagrangian functions of the two problems. In the last section of this chapter we attach to the original optimization problem its dual. Some theorems ensures, under appropriate hypothesis, that if the dual optimization problem of the original problem is solvable, then the $(0,2)$ η-approximated optimization problem is also solvable, and vice versa.

The author's original contributions are: In Chapter 2: Theorems: 2.1.3, 2.1.5, 2.1.9, 2.1.12, 2.1.17, 2.1.19, 2.1.27, 2.1.28, 2.1.41, 2.1.45, 2.2.1, 2.2.4, 2.2.5, 2.2.7, 2.2.9, 2.2.10, 2.2.14, 2.2.16, 2.2.20, 2.2.22, 2.3.1, 2.3.2, 2.3.4, 2.3.5, 2.3.6, 2.3.7, 2.3.8, 2.3.9, 2.3.10, 2.3.11, 2.3.12, 2.3.17, 2.3.18; Propositions: 2.2.3, 2.2.19; Remarks: 2.1.2, 2.1.4, 2.1.6, 2.1.7, 2.1.8, 2.1.10, 2.1.11, 2.1.13, 2.1.14, 2.1.15, 2.1.16, 2.1.18, 2.1.20, 2.1.21, 2.1.22, 2.1.23, 2.1.24, 2.1.25, 2.1.26, 2.1.29, 2.1.30, 2.1.31, 2.1.32, 2.1.33, 2.1.34, 2.1.35, 2.1.37, 2.1.39, 2.1.40, 2.1.42, 2.1.43, 2.1.44, 2.1.46, 2.1.47, 2.1.47, 2.1.48, 2.1.49, 2.2.2, 2.2.6, 2.2.8, 2.2.11, 2.2.12, 2.2.15, 2.2.17, 2.2.21, 2.2.24, 2.2.25; 2.3.3; Examples:

2.1.36, 2.2.13, 2.2.23, 2.3.13, 2.3.14, 2.3.15, 2.3.16;

In Chapter 3: Theorems: 3.1.1, 3.1.3, 3.1.5, 3.1.7, 3.1.9, 3.1.10, 3.1.11, 3.1.14, 3.1.15, 3.1.16, 3.2.1, 3.2.13, 3.3.1, 3.3.3, 3.3.5, 3.3.8, 3.3.11, 3.3.13, 3.3.15, 3.3.16, 3.3.18; Corollaries: 3.1.12, 3.1.13, 3.2.11; Propositions: 3.2.8, 3.2.9, 3.2.12; Remarks: 3.1.2, 3.1.4, 3.1.6, 3.1.8, 3.2.2, 3.2.3, 3.2.4, 3.2.10, 3.2.14, 3.2.15, 3.2.16; 3.3.2, 3.3.4, 3.3.6, 3.3.6, 3.3.7, 3.3.9, 3.3.10, 3.3.12, 3.3.14, 3.3.17; Example 3.3.19.

In Chapter 4: Theorems: 4.2.3, 4.2.4, 4.3.2, 4.3.3, 4.3.5, 4.3.6, 4.3.7, 4.4.3, 4.4.4; Definitions: 4.1.3, 4.1.8; Examples: 4.1.6, 4.1.7, 4.2.1, 4.2.2, 4.3.4; Remarks: 4.1.5, 4.1.9, 4.1.11.

The results of this thesis are included in the following papers: L. Cioban [45], L. Cioban [46], L. Cioban [47], L. Cioban [48], L. Cioban [49], L. Cioban and E.R. Csetnek [50], L. Cioban and E.R. Csetnek [51], L. Cioban and D. Duca [52].

Keywords: convex analysis, conjugate functions, (convex) subdifferential, optimal solution, perturbation theory, Fenchel and Lagrange duality, ε-variational inequalities, ε-subdifferential, ε-optimality conditions, variational inequalities, equilibrium problems, gap functions, dual gap functions, sequential optimality conditions, Lagrangian function, saddle point, optimization problem, $(0,2)-\eta-$ approximated optimization problem.

Acknowledgements

My first debt of gratitude must go to my advisor, Prof. Dr. Dorel Duca. He patiently encouraged and advised me to proceed through the doctoral program.

I want to express my deeply-felt thanks to Dr. Robert Ernö Csetnek for his invaluable help during my eight-month scholarship at the research department of Faculty of Mathematics, Chemnitz University of Technology, who introduced me in new research areas, guided me and worked together in achieving meaningful results. I want to express my gratitude to Prof. Dr. Gert Wanka for providing me a great research environment during my research internship at Chemnitz University of Technology, and I want to thank also to Dr. Radu Ioan Boţ and Dr. Sorin-Mihai Grad for the kindly receiving in their research group and for their friendship.

It has been a great privilege to spend the last three years in the Department of Analysis and Optimization of Faculty of Mathematics and Computer Science at Babeş Bolyai University. Its members will always remain dear to me.

Special thanks to my guidance committee, Prof. Dr. Liana Lupşa, Prof. Dr. Kassay Gábor, Dr. Tiberiu Trif for their support and helpful suggestions.

I am also grateful to the Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca and to the Institute for Doctoral Studies for financial support through the Doctoral studies: through science towards society project (POSDRU/88/1.5/S/60185).

Last but not least, I wish to thank my family. My mother's love provided me the inspiration and it was my driving force. I could never thank enough to my father for his unconditional support. I owe them everything and wish I could show them just how much I love and appreciate them. My brother's priceless family has offered me spiritual support, warmth and cheer through the smiles and innocence of my beloved nephew and niece, David and Tania.

Finally, I would like to dedicate this work to my parents who encouraged me and believed in me all this time. I hope that this work makes you proud.

Chapter 1

Preliminaries

In this chapter we give the necessary notions and results in order to make this thesis as self-contained as possible.

1.1 Notions and results

Throughout this thesis we will use the classical notations and notions from functional analysis and convex analysis, see [14, 22, 23, 24, 34, 57, 58, 59, 79, 80, 81, 119, 122, 133].

1.2 Regularity conditions for Optimization Problems

It is known that the optimal objective value of the dual optimization problem does not surpass the value of the primal optimization problem. In literature it is called weak duality. The regularity conditions are used in order to ensure strong duality, that is the situation when the optimal values of both problems are equal and the dual has an optimal solution.

1.2.1 The general scalar optimization problem

The aim of this section is to present some regularity conditions regarding the general scalar primal optimization problem and its conjugate dual one, which are expressed by means of the perturbation function.

1.2.2 Particular cases

In this section we present regularity conditions for some particular cases that are used at times in the author own results and proofs. The cases presented here are the composition case, the case $f + g \circ A$, the case $f + \delta_{-K} \circ A$, the case $g \circ A$, the case $f + g$, the case $f + \delta_K$, the case with geometric and cone constraints.

Chapter 2

Variational inequalities

An inequality involving a functional and whose solution must be found for all values of a variable that belongs usually to a convex set, is called the variational inequality. Mathematical theory on variational inequalities was developed based on Signorini's problem which consists in finding the elastic equilibrium configuration of an anisotropic non-homogeneous elastic body that lies in a subset of the three-dimensional euclidean space resting on a rigid frictionless surface and subject only to its mass forces [63]. The functional involved in Signorini's problem was obtained from a variational problem. Applicability of this theory has been extended and includes problems in physics, optimization, economics, finance, game theory.

2.1 Duality for ε-variational inequalities via the subdifferential calculus

The research from this section is motivated by the following dual scheme concerning ε-variational inequalities proposed by Kum, Kim and Lee in [94]. For $\varepsilon \geq 0$ we consider the ε-variational inequalities:

$$
\begin{array}{rl}
(\mathrm{VI})_{\varepsilon}^{f,A} & \text{Find } \overline{x} \in \mathbb{R}^n \text{ for which there exists } v \in F(\overline{x}), \\
 & \text{s.t. } \langle v, x - \overline{x} \rangle \geq f(A\overline{x}) - f(Ax) - \varepsilon \ \forall x \in \mathbb{R}^n, \\
(\mathrm{DVI})_{\varepsilon}^{f,A} & \text{Find } \overline{y} \in \mathbb{R}^m \text{ for which there exists } w \in A(F^{-1}(-A^*\overline{y})), \\
 & \text{s.t. } \langle w, y - \overline{y} \rangle \leq f^*(y) - f^*(\overline{y}) + \varepsilon \ \forall y \in \mathbb{R}^m,
\end{array}
$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set valued operator, $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping fulfilling $A^{-1}(\operatorname{dom} f) \neq \emptyset$, f^* is the (Fenchel) conjugate of f and A^* is the adjoint operator of A.

In the following we extend the dual scheme of Kum, Kim and Lee to the infinite dimensional setting. Instead of the function $f \circ A$ in the formulation of $(\mathrm{VI})_{\varepsilon}^{f,A}$, we consider a general perturbation function. We attach to this primal ε-variational inequality a dual one, in which the conjugate of the perturbation function is used. By using the powerful techniques of the (convex)

ε-subdifferential calculus (which is well developed in the literature, see [30, 32, 78, 133]) we show that if the function involved is proper, convex and a regularity condition is fulfilled, if the primal ε-variational inequality is solvable then also its dual one is solvable. Conversely, when the dual ε-variational inequality is solvable and the function involved is proper, convex and lower semicontinuous, then also the primal one is solvable (notice that for this implication no regularity condition is needed). We consider several particular cases of our general results. We show that the dual scheme of Kum, Kim and Lee follows as a particular instance of the main results of this paper. Finally, let us mention that the duality scheme proposed by Mosco in [113] concerning variational inequalities can be seen as another particular instance of our main results.

2.1.1 A perturbation approach of ε-variational inequalities

In this section we introduce the general ε-variational inequality and its dual one. We show that under appropriate hypotheses the primal one is solvable if and only if its dual one is solvable. The techniques are based on ε-subdifferential calculus, in the context of which the regularity conditions play a significant role.

Let us consider now the announced ε-variational inequalities. Let $F : X \rightrightarrows X^*$ be a set-valued map. Notice that $G(F) = \{(x, x^*) \in X \times X^* : x^* \in F(x)\}$ denotes the *graph* of F and $F^{-1} : X^* \rightrightarrows X$ is the multivalued operator having as graph the set $G(F^{-1}) = \{(x^*, x) \in X^* \times X : (x, x^*) \in G(F)\}$.

For $\varepsilon \geq 0$ consider the following ε-variational inequality:

$$
\begin{aligned}
(\mathrm{VI})_\varepsilon^\Phi \quad & \text{Find } \overline{x} \in X \text{ for which there exists } v \in F(\overline{x}), \\
& \text{s.t. } \langle v, x - \overline{x} \rangle \geq \Phi(\overline{x}, 0) - \Phi(x, 0) - \varepsilon \ \forall x \in X.
\end{aligned} \tag{2.1}
$$

To $(\mathrm{VI})_\varepsilon^\Phi$ we attach the following dual ε-variational inequality:

$$
\begin{aligned}
(\mathrm{DVI})_\varepsilon^\Phi \quad & \text{Find } (\overline{x}^*, \overline{y}^*) \in X^* \times Y^* \text{ for which there exists } w \in F^{-1}(-\overline{x}^*), \\
& \text{s.t. } \langle w, x^* - \overline{x}^* \rangle \leq \Phi^*(x^*, y^*) - \Phi^*(\overline{x}^*, \overline{y}^*) + \varepsilon \ \forall (x^*, y^*) \in X^* \times Y^*.
\end{aligned} \tag{2.2}
$$

In what follows we show that, if one of the regularity conditions above is fulfilled and Φ is proper, convex and lower semicontinuous, then $(\mathrm{VI})_\varepsilon^\Phi$ is solvable if and only if $(\mathrm{DVI})_\varepsilon^\Phi$ is solvable.

Theorem 2.1.3 (**L. Cioban**, E.R. Csetnek, [50])*Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper and convex function such that $0 \in \mathrm{pr}_Y(\mathrm{dom}\,\Phi)$ and assume that one of the regularity conditions (RC_i^Φ), $i \in \{1, 2, 3, 4, 5, 6, 7\}$, is fulfilled. Suppose that for a fixed $\varepsilon \geq 0$ $(VI)_\varepsilon^\Phi$ is solvable, that is $\overline{x} \in X$ is a solution of $(VI)_\varepsilon^\Phi$ and $v \in F(\overline{x})$ satisfies (2.1). Then also $(DVI)_\varepsilon^\Phi$ is solvable and a solution of it can be constructed as follows: take any $\overline{y}^*$ such that $(-v, \overline{y}^*) \in \partial_\varepsilon \Phi(\overline{x}, 0)$. Then $(-v, \overline{y}^*)$ is a solution of $(DVI)_\varepsilon^\Phi$ and $\overline{x} \in F^{-1}(v)$ satisfies (2.2).*

Theorem 2.1.5 (**L. Cioban**, E.R. Csetnek, [50])*Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function such that $0 \in \mathrm{pr}_Y(\mathrm{dom}\,\Phi)$. Suppose that for a fixed $\varepsilon \geq 0$ $(DVI)_\varepsilon^\Phi$*

is solvable, that is $(\overline{x}^*, \overline{y}^*) \in X^* \times Y^*$ is a solution of $(DVI)_\varepsilon^\Phi$ and $w \in F^{-1}(-\overline{x}^*)$ satisfies (2.2). Then also $(VI)_\varepsilon^\Phi$ is solvable, w is a solution of $(VI)_\varepsilon^\Phi$ and $-\overline{x}^* \in F(w)$ satisfies (2.1).

Remark 2.1.6 *Let us underline the fact that for the validity of the above theorem we need no regularity condition.*

Remark 2.1.7 *It is obvious from the formulation of $(VI)_\varepsilon^\Phi$ that $\overline{x}$ must belong to* $\operatorname{dom}\Phi(\cdot,0)$. *Moreover, the element $(\overline{x}^*, \overline{y}^*)$ from $(DVI)_\varepsilon^\Phi$ belongs to* $\operatorname{dom}\Phi^*$ *and* $(w,0) \in \operatorname{dom}\Phi^{**}$.

2.1.2 Particular cases

In this subsection we consider several particular cases of the general results obtained above. Among others, we rediscover and improve the duality schemes concerning variational inequalities considered by Kum, Kim & Lee [94] and Mosco [113]. In this section (unless otherwise specified) X and Y are real separated locally convex spaces.

The case $f + g \circ A$

Let $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$ be proper, convex functions and $A : X \to Y$ a linear continuous operator, fulfilling $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \neq \emptyset$.

For $\varepsilon, \varepsilon_1, \varepsilon_2 \geq 0$ we consider the following ε-variational inequalities:

$$(\mathrm{VI})_\varepsilon^{f,g,A} \quad \begin{array}{l} \text{Find } \overline{x} \in X \text{ for which there exists } v \in F(\overline{x}), \\ \text{s.t. } \langle v, x - \overline{x}\rangle \geq (f + g\circ A)(\overline{x}) - (f + g \circ A)(x) - \varepsilon \ \forall x \in X \end{array} \tag{2.3}$$

$$(\mathrm{DVI})_{\varepsilon_1,\varepsilon_2}^{f,g,A} \quad \begin{array}{l} \text{Find } (\overline{x}^*, \overline{y}^*) \in X^* \times Y^* \text{ for which there exists} \\ (w_1, w_2) \in \big(F^{-1}(-\overline{x}^* - A^*\overline{y}^*) \times A(F^{-1}(-\overline{x}^* - A^*\overline{y}^*))\big) \cap \Delta_X^A, \\ \text{s.t. } \left\{ \begin{array}{l} \langle w_1, x^* - \overline{x}^*\rangle \leq f^*(x^*) - f^*(\overline{x}^*) + \varepsilon_1 \ \forall x^* \in X^*, \\ \langle w_2, y^* - \overline{y}^*\rangle \leq g^*(y^*) - g^*(\overline{y}^*) + \varepsilon_2 \ \forall y^* \in Y^*, \end{array}\right. \end{array} \tag{2.4}$$

where $\Delta_X^A = \{(x, Ax) : x \in X\}$.

The following two theorems summarize the duality results obtained for $(\mathrm{VI})_\varepsilon^{f,g,A}$ and $(\mathrm{DVI})_{\varepsilon_1,\varepsilon_2}^{f,g,A}$.

Theorem 2.1.9 (**L. Cioban**, E.R. Csetnek, [50]) *Let $g : Y \to \overline{\mathbb{R}}$ and $f : X \to \overline{\mathbb{R}}$ be proper, convex functions and $A : X \to Y$ a linear continuous operator fulfilling* $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \neq \emptyset$ *and assume that one of the regularity conditions $(RC_i^{f,g,A})$, $i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Suppose that for a fixed $\varepsilon \geq 0$ $(VI)_\varepsilon^{f,g,A}$ is solvable, that is $\overline{x} \in X$ is a solution of $(VI)_\varepsilon^{f,g,A}$ and $v \in F(\overline{x})$ satisfies (2.3). Then there exist $\varepsilon_1, \varepsilon_2 \geq 0$, $\varepsilon_1 + \varepsilon_2 = \varepsilon$, such that $(DVI)_{\varepsilon_1,\varepsilon_2}^{f,g,A}$ is solvable and a solution of it can be constructed as follows: take any $\alpha_1 \in \partial_{\varepsilon_1} f(\overline{x})$ and $\alpha_2 \in \partial_{\varepsilon_2} g(A\overline{x})$ with $v = -\alpha_1 - A^*\alpha_2$. Then $(\overline{x}^*, \overline{y}^*) := (\alpha_1, \alpha_2) \in X^* \times Y^*$ is a solution of $(DVI)_{\varepsilon_1,\varepsilon_2}^{f,g,A}$ and $(w_1, w_2) := (\overline{x}, A\overline{x}) \in \big(F^{-1}(-\alpha_1 - A^*\alpha_2) \times A(F^{-1}(-\alpha_1 - A^*\alpha_2))\big) \cap \Delta_X^A$ satisfies (2.4).*

Remark 2.1.10 *The approach of constructing the solution of $(DVI)_{\varepsilon_1,\varepsilon_2}^{f,g,A}$ is well defined.*

Remark 2.1.11 *Let us notice that instead of* $(DVI)^{f,g,A}_{\varepsilon_1,\varepsilon_2}$ *one can consider the following dual* ε-*variational inequality*

$$\begin{array}{ll}(\overline{DVI})^{f,g,A}_{\varepsilon} & \text{Find } (\overline{x}^*,\overline{y}^*) \in X^* \times Y^* \text{ for which there exists } (w_1,w_2) \in \\ & \big(F^{-1}(-\overline{x}^* - A^*\overline{y}^*) \times A(F^{-1}(-\overline{x}^* - A^*\overline{y}^*))\big) \cap \Delta^A_X \text{ s.t. } \forall (x^*,y^*) \in X^* \times Y^* \\ & \langle w_1, x^* - \overline{x}^* \rangle + \langle w_2, y^* - \overline{y}^* \rangle \leq f^*(x^*) + g^*(y^*) - f^*(\overline{x}^*) - g^*(\overline{y}^*) + \varepsilon. \end{array} \tag{2.5}$$

One can prove that if for a given $\varepsilon \geq 0$ $(VI)^{f,g,A}_{\varepsilon}$ *is solvable, then also* $(\overline{DVI})^{f,g,A}_{\varepsilon}$ *is solvable and a solution of it can be constructed as in Theorem 2.1.9.*

Theorem 2.1.12 (**L. Cioban**, E.R. Csetnek, [50])*Let* $g : Y \to \overline{\mathbb{R}}$ *and* $f : X \to \overline{\mathbb{R}}$ *be proper, convex, lower semicontinuous functions and* $A : X \to Y$ *a linear continuous operator fulfilling* $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \neq \emptyset$. *Suppose that for fixed* $\varepsilon_1, \varepsilon_2 \geq 0$ $(DVI)^{f,g,A}_{\varepsilon_1,\varepsilon_2}$ *is solvable, that is* $(\overline{x}^*,\overline{y}^*) \in X^* \times Y^*$ *is a solution of* $(DVI)^{f,g,A}_{\varepsilon_1,\varepsilon_2}$ *and* $(w_1,w_2) \in \big(F^{-1}(-\overline{x}^* - A^*\overline{y}^*) \times A(F^{-1}(-\overline{x}^* - A^*\overline{y}^*))\big) \cap \Delta^A_X$ *satisfies (2.4). Then, for* $\varepsilon = \varepsilon_1 + \varepsilon_2$ $(VI)^{f,g,A}_{\varepsilon}$ *is solvable,* w_1 *is a solution of* $(VI)^{f,g,A}_{\varepsilon}$ *and* $-\overline{x}^* - A^*\overline{y}^* \in F(w_1)$ *satisfies (2.3).*

The case $f + \delta_{-K} \circ A$

Let us particularize the duality statements for $(\mathrm{VI})^{f,g,A}_{\varepsilon}$ and $(\mathrm{DVI})^{f,g,A}_{\varepsilon_1,\varepsilon_2}$ to the case $g = \delta_{-K}$, where $K \subseteq Y$ is a non-empty closed convex cone. As in the previous subsection, $f : X \to \overline{\mathbb{R}}$ is a proper and convex function and $A : X \to Y$ is a linear continuous operator, fulfilling $\operatorname{dom} f \cap A^{-1}(-K) \neq \emptyset$. One can easily prove that the conjugate function of g is $g^* = \delta_{K^*}$.

For $\varepsilon, \varepsilon_1, \varepsilon_2 \geq 0$ we consider the following ε-variational inequalities:

$$\begin{array}{ll}(\mathrm{VI})^{f,K,A}_{\varepsilon} & \text{Find } \overline{x} \in A^{-1}(-K), \text{ for which there exists } v \in F(\overline{x}), \\ & \text{s.t. } \langle v, x - \overline{x} \rangle \geq f(\overline{x}) - f(x) - \varepsilon \ \forall x \in A^{-1}(-K). \end{array} \tag{2.6}$$

$$\begin{array}{ll}(\mathrm{DVI})^{f,K,A}_{\varepsilon_1,\varepsilon_2} & \text{Find } (\overline{x}^*,\overline{y}^*) \in X^* \times K^* \text{ for which there exists} \\ & (w_1,w_2) \in \big(F^{-1}(-\overline{x}^* - A^*\overline{y}^*) \times A(F^{-1}(-\overline{x}^* - A^*\overline{y}^*))\big) \cap \Delta^A_X \\ & \text{s.t. } \begin{cases} \langle w_1, x^* - \overline{x}^* \rangle \leq f^*(x^*) - f^*(\overline{x}^*) + \varepsilon_1 \ \forall x^* \in X^* \\ \langle w_2, y^* - \overline{y}^* \rangle \leq \varepsilon_2 \ \forall y^* \in K^*. \end{cases} \end{array} \tag{2.7}$$

The following results are particular cases of Theorem 2.1.9 and Theorem 2.1.12.

Theorem 2.1.17 (**L. Cioban**, E.R. Csetnek, [50])*Let* $f : X \to \overline{\mathbb{R}}$ *be a proper and convex function,* $K \subseteq Y$ *a closed convex cone and* $A : X \to Y$ *a linear continuous operator fulfilling* $\operatorname{dom} f \cap A^{-1}(-K) \neq \emptyset$ *and assume that one of the regularity conditions* $(RC^{f,K,A}_i)$, $i \in \{1,2,3,4,5,6,7\}$ *is fulfilled. Suppose that for a fixed* $\varepsilon \geq 0$ $(VI)^{f,K,A}_{\varepsilon}$ *is solvable, that is* $\overline{x} \in A^{-1}(-K)$ *is a solution of* $(VI)^{f,K,A}_{\varepsilon}$ *and* $v \in F(\overline{x})$ *satisfies (2.6). Then there exist* $\varepsilon_1, \varepsilon_2 \geq 0$, $\varepsilon_1 + \varepsilon_2 = \varepsilon$, *such that* $(DVI)^{f,K,A}_{\varepsilon_1,\varepsilon_2}$ *is solvable and a solution of it can be constructed as follows: take any* $\alpha_1 \in \partial_{\varepsilon_1} f(\overline{x})$ *and* $\alpha_2 \in N^{\varepsilon_2}_{-K}(A\overline{x})$ *with* $v = -\alpha_1 - A^*\alpha_2$. *Then* $(\alpha_1,\alpha_2) \in X^* \times K^*$ *is a solution of* $(DVI)^{f,K,A}_{\varepsilon_1,\varepsilon_2}$ *and* $(\overline{x}, A\overline{x}) \in \big(F^{-1}(-\alpha_1 - A^*\alpha_2) \times A(F^{-1}(-\alpha_1 - A^*\alpha_2))\big) \cap \Delta^A_X$ *satisfies (2.7).*

Theorem 2.1.19 (**L. Cioban**, E.R. Csetnek, [50])*Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex, lower semicontinuous function, $K \subseteq Y$ a closed convex cone and $A : X \to Y$ a linear continuous operator fulfilling $\operatorname{dom} f \cap A^{-1}(-K) \neq \emptyset$. Suppose that for fixed $\varepsilon_1, \varepsilon_2 \geq 0$ $(DVI)^{f,K,A}_{\varepsilon_1,\varepsilon_2}$ is solvable, that is $(\overline{x}^*, \overline{y}^*) \in X^* \times K^*$ is a solution of $(DVI)^{f,K,A}_{\varepsilon_1,\varepsilon_2}$ and $(w_1, w_2) \in \big(F^{-1}(-\overline{x}^* - A^*\overline{y}^*) \times A(F^{-1}(-\overline{x}^* - A^*\overline{y}^*))\big) \cap \Delta^A_X$ satisfies (2.7). Then, for $\varepsilon = \varepsilon_1 + \varepsilon_2$ $(VI)^{f,K,A}_{\varepsilon}$ is solvable, w_1 is a solution of $(VI)^{f,K,A}_{\varepsilon}$ and $-\overline{x}^* - A^*\overline{y}^* \in F(w_1)$ satisfies (2.6).*

Remark 2.1.23 *Let us notice that Theorem 2.1.17 and Theorem 2.1.19 can be obtained by applying Theorem 2.1.3 and Theorem 2.1.5 to the perturbation function $\Phi_{f,K,A} : X \times Y \to \overline{\mathbb{R}}$,*

$$\Phi_{f,K,A}(x, y) = \begin{cases} f(x), & \text{if } Ax + y \in -K \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 2.1.24 *(i) The duality statements obtained in this section can be particularized to the case $X = Y$ and $A : X \to X$ is the identity operator.*

(ii) Finally, another particular instance of the above results can be provided for $\varepsilon = \varepsilon_1 = \varepsilon_2 = 0$.

The case $g \circ A$

This subsection is devoted to the particularization of the duality statements for $(\mathrm{VI})^{f,g,A}_{\varepsilon}$ and $(\mathrm{DVI})^{f,g,A}_{\varepsilon_1,\varepsilon_2}$ to the case $f : X \to \mathbb{R}$, $f(x) = 0$ for all $x \in X$.

For $\varepsilon \geq 0$ we consider the following two ε-variational inequalities:

$$\begin{array}{ll} (\mathrm{VI})^{g,A}_{\varepsilon} & \text{Find } \overline{x} \in X \text{ for which there exists } v \in F(\overline{x}), \\ & \text{s.t. } \langle v, x - \overline{x}\rangle \geq g(A\overline{x}) - g(Ax) - \varepsilon \ \forall x \in X. \end{array} \tag{2.8}$$

$$\begin{array}{ll} (\mathrm{DVI})^{g,A}_{\varepsilon} & \text{Find } \overline{y}^* \in Y^* \text{ for which there exists } w \in A(F^{-1}(-A^*\overline{y}^*)), \\ & \text{s.t. } \langle w, y^* - \overline{y}^*\rangle \leq g^*(y^*) - g^*(\overline{y}^*) + \varepsilon \ \forall y^* \in Y^*. \end{array} \tag{2.9}$$

We call $(\mathrm{DVI})^{g,A}_{\varepsilon}$ the dual variational inequality of $(\mathrm{VI})^{g,A}_{\varepsilon}$.

Remark 2.1.25 *Let us mention that this pair of ε-variational inequalities was considered in [94] in a finite dimensional setting, namely for $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, where $m, n \in \mathbb{N}$.*

Theorem 2.1.27 (**L. Cioban**, E.R. Csetnek, [50])*Let $g : Y \to \overline{\mathbb{R}}$ be a proper and convex function, $A : X \to Y$ a linear continuous operator fulfilling $A^{-1}(\operatorname{dom} g) \neq \emptyset$ and assume that one of the regularity conditions $(RC^{g,A}_i)$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$ is fulfilled. Suppose that for a fixed $\varepsilon \geq 0$ $(VI)^{g,A}_{\varepsilon}$ is solvable, that is $\overline{x} \in X$ is a solution of $(VI)^{g,A}_{\varepsilon}$ and $v \in F(\overline{x})$ satisfies (2.8). Then $(DVI)^{g,A}_{\varepsilon}$ is solvable and a solution of it can be constructed as follows: take any $\alpha \in \partial_{\varepsilon} g(A\overline{x}) \cap (A^*)^{-1}(-v)$. Then $\alpha \in Y^*$ is a solution of $(DVI)^{g,A}_{\varepsilon}$ and $A\overline{x} \in A(F^{-1}(-A^*\alpha))$ satisfies (2.9).*

Theorem 2.1.28 (**L. Cioban**, E.R. Csetnek, [50])*Let $g : Y \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function and $A : X \to Y$ a linear continuous operator fulfilling $A^{-1}(\operatorname{dom} g) \neq \emptyset$. Suppose that for fixed $\varepsilon \geq 0$ $(DVI)^{g,A}_{\varepsilon}$ is solvable, that is $\overline{y}^* \in Y^*$ is a solution of $(DVI)^{g,A}_{\varepsilon}$ and*

$w \in A(F^{-1}(-A^*\overline{y}^*))$ satisfies (2.9). Then $(VI)_\varepsilon^{g,A}$ is solvable and a solution of it can be constructed as follows: take any $\beta \in F^{-1}(-A^*\overline{y}^*) \cap A^{-1}(w)$. Then β is a solution of $(VI)_\varepsilon^{g,A}$ and $-A^*\overline{y}^* \in F(\beta)$ satisfies (2.8).

Remark 2.1.32 *Let us notice that the duality statements concerning $(VI)_\varepsilon^{g,A}$ and $(DVI)_\varepsilon^{g,A}$ can be obtained from Theorem 2.1.3 and Theorem 2.1.5 by taking $\Phi_{g,A} : X \times Y \to \overline{\mathbb{R}}$, $\Phi_{g,A}(x,y) = g(Ax+y)$ for all $(x,y) \in X \times Y$.*

Remark 2.1.35 *In Theorem 2.1.27 and Theorem 2.1.28 we extended to the infinite dimensional setting similar results given in [94] in finite dimensional spaces. Notice that in [94] the duality statements are given under the regularity condition $\operatorname{Im} A \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$, which is actually $(RC_7^{g,A})$. We have considered more general regularity conditions which can be applied also in the infinite dimensional framework. Let us underline other improvements of [94, Theorem 2.1]. By Theorem 2.1.27 we have that [94, Theorem 2.1(i)] holds even if the function g is not lower semicontinuous. Moreover, [94, Theorem 2.1(ii)] holds even in the absence of any regularity condition.*

In the following example justifies the use of weaker regularity conditions.

Example 2.1.36 (**L. Cioban**, E.R. Csetnek, [50])*Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $A : \mathbb{R} \to \mathbb{R}^2$, $A(x) = (x, 0)$,*

$$g : \mathbb{R}^2 \to \overline{\mathbb{R}},\ g(x,y) = \begin{cases} \frac{1}{2}x^2, & \text{if } x \in \mathbb{R},\ y \geq 0, \\ +\infty, & \text{otherwise} \end{cases}$$

and $F : \mathbb{R} \rightrightarrows \mathbb{R}$ an arbitrary set valued map. It is proved that for this function the regularity condition used in [94, Theorem 2.1(i)] fails, but the closedness-type regularity condition $(RC_6^{g,A})$ holds and we can apply Theorem 2.1.27.

Rediscovering the dual variational inequality of Mosco

If we consider X a separated locally convex space, $g : X \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function and $F : \operatorname{dom} F \to X^*$ an operator, $\operatorname{dom} F$ being the domain of F. For $\varepsilon = 0$, $X = Y$ and $A : X \to X$ the identity operator ($Ax = x$ for all $x \in X$) and if we suppose that F is injective the problems $(\mathrm{VI})_\varepsilon^{g,A}$ and $(\mathrm{DVI})_\varepsilon^{g,A}$ are the ones considered by Mosco in [113] and the results given by Mosco follows from the Theorem 2.1.27 and Theorem 2.1.28.

2.1.3 The composition case

Let X, Y be separated locally convex spaces, $K \subseteq Y$ a nonempty closed convex cone, $g : Y \to \overline{\mathbb{R}}$ be a proper, convex, K-increasing function with $g(\infty_K) = +\infty$, $f : X \to \overline{\mathbb{R}}$ a proper, convex function and $h : X \to Y^\bullet$ a proper, K-convex function such that $h(\operatorname{dom} f \cap \operatorname{dom} h) \cap \operatorname{dom} g \neq \emptyset$.

Let us consider now for $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ the following ε-variational inequalities:

$$(\mathrm{VI})^{CC}_{\varepsilon} \quad \text{Find } \overline{x} \in X \text{ for which there exists } v \in F(\overline{x}),$$
$$\text{s.t. } \langle v, x - \overline{x} \rangle \geq (f + g \circ h)(\overline{x}) - (f + g \circ h)(x) - \varepsilon \ \forall x \in X. \tag{2.10}$$

$$(\mathrm{DVI})^{CC}_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \quad \text{Find } (v_1, v_2, \lambda) \in X^* \times X^* \times K^* \text{ for which there exists } (w_1, w_2, w_3) \in$$
$$\big(F^{-1}(-v_1 - v_2) \times F^{-1}(-v_1 - v_2) \times h(F^{-1}(-v_1 - v_2))\big) \cap \Delta^h_X,$$
$$\text{s.t. } \begin{cases} \langle w_1, x^* - v_1 \rangle \leq f^*(x^*) - f^*(v_1) + \varepsilon_1 \ \forall x^* \in X^* \\ \langle w_2, \widetilde{x}^* - v_2 \rangle \leq (\lambda h)^*(\widetilde{x}^*) - (\lambda h)^*(v_2) + \varepsilon_2 \ \forall \widetilde{x}^* \in X^* \\ \langle w_3, y^* - \lambda \rangle \leq g^*(y^*) - g^*(\lambda) + \varepsilon_3 \ \forall y^* \in Y^* \end{cases} \tag{2.11}$$

where $\Delta^h_X = \{(x, x, h(x)) : x \in \operatorname{dom} h\}$.

We are now ready to state the duality results concerning $(\mathrm{VI})^{CC}_{\varepsilon}$ and $(\mathrm{DVI})^{CC}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$.

Theorem 2.1.41 (**L. Cioban**, E.R. Csetnek, [50])*Let $g : Y \to \overline{\mathbb{R}}$ be a proper, convex and K-increasing function, $f : X \to \overline{\mathbb{R}}$ proper, convex, $h : X \to Y^\bullet$ a proper and K-convex such that $h(\operatorname{dom} f \cap \operatorname{dom} h) \cap \operatorname{dom} g \neq \emptyset$ and assume that one of the regularity conditions $(RC^{CC_2}_i)$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$ is fulfilled. Suppose that for a fixed $\varepsilon \geq 0$ $(VI)^{CC}_{\varepsilon}$ is solvable, that is $\overline{x} \in X$ is a solution of $(VI)^{CC}_{\varepsilon}$ and $v \in F(\overline{x})$ satisfies (2.10). Then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$, such that $(DVI)^{CC}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is solvable and a solution of it can be constructed as follows: take any $v_1 \in \partial_{\varepsilon_1} f(\overline{x})$, $\lambda \in K^* \cap \partial_{\varepsilon_3} g(h(\overline{x}))$ and $v_2 \in \partial_{\varepsilon_2}(\lambda h)(\overline{x})$ with $v = -v_1 - v_2$. Then $(v_1, v_2, \lambda) \in X^* \times X^* \times K^*$ is a solution of $(DVI)^{CC}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ and $(w_1, w_2, w_3) := (\overline{x}, \overline{x}, h(\overline{x})) \in \big(F^{-1}(-v_1 - v_2) \times F^{-1}(-v_1 - v_2) \times h(F^{-1}(-v_1 - v_2))\big) \cap \Delta^h_X$ satisfies (2.11).*

Theorem 2.1.45 (**L. Cioban**, E.R. Csetnek, [50])*Let $g : Y \to \overline{\mathbb{R}}$ be a proper, convex, K-increasing and lower semicontinuous function, $f : X \to \overline{\mathbb{R}}$ proper, convex and lower semicontinuous and $h : X \to Y^\bullet$ a proper, K-convex and star K-lower semicontinuous function such that $h(\operatorname{dom} f \cap \operatorname{dom} h) \cap \operatorname{dom} g \neq \emptyset$. Suppose that for fixed $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ $(DVI)^{CC}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is solvable, that is $(v_1, v_2, \lambda) \in X^* \times X^* \times K^*$ is a solution of $(DVI)^{CC}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ and $(w_1, w_2, w_3) \in \big(F^{-1}(-v_1 - v_2) \times F^{-1}(-v_1 - v_2) \times h(F^{-1}(-v_1 - v_2))\big) \cap \Delta^h_X$ satisfies (2.11). Then, for $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ $(VI)^{CC}_{\varepsilon}$ is solvable, w_1 is a solution of $(VI)^{CC}_{\varepsilon}$ and $v = -v_1 - v_2 \in F(w_1)$ satisfies (2.10).*

2.2 Gap functions for variational inequalities via conjugate duality

In [3] the authors considered the following mixed variational inequality which consists in finding an element $x \in K$ such that

$$(MVI) \quad F(x)^T (y - x) + f(y) - f(x) \geq 0 \ \forall y \in K,$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper, convex function, $K \subseteq \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function. The authors associated to (MVI) a (convex) optimization problem. By using Fenchel duality and under a regularity condition they constructed a gap function for (MVI) (see [3]).

The aim of this section is to introduce gap functions for more general variational inequalities. We extend (MVI) to the infinite dimensional setting and instead of the function f in the formulation of (MVI) we consider a general perturbation function (see [25, 31, 57, 133]). We use the techniques from [1, 2, 3]: we reformulate this general variational inequality into an optimization problem depending on a fixed variable. We attach to this optimization problem a dual one and the gap function is defined by means of the optimal value of the dual problem. Regularity conditions guaranteeing strong duality for the primal-dual pair of optimization problems play a significant role when proving that the functions introduced are indeed gap functions for the variational inequalities. We use weaker regularity conditions and by examples we justify the use of them. By particularizing the perturbation function we rediscover several gap functions introduced in literature.

2.2.1 A gap function for the general variational inequality

We consider the generalized (Stampacchia type) variational inequality problem which consists in finding an element $\overline{x} \in X$ such that

$$(VI)^{\Phi} \qquad \langle F(\overline{x}), x - \overline{x} \rangle + \Phi(x, 0) - \Phi(\overline{x}, 0) \geq 0 \ \forall x \in X,$$

where X and Y are real separated locally convex spaces, $\Phi : X \times Y \to \overline{\mathbb{R}}$ is a proper function fulfilling $0 \in \mathrm{pr}_Y(\mathrm{dom}\,\Phi)$ and $F : X \to X^*$ is a given operator. Let us mention that we are looking actually for an element $\overline{x} \in \mathrm{dom}\,\Phi(\cdot, 0)$ and it is enough to require that the inequality $(VI)^{\Phi}$ holds for all $x \in \mathrm{dom}\,\Phi(\cdot, 0)$.

A common approach in order to solve the problem $(VI)^{\Phi}$ is to formulate a gap function for problem $(VI)^{\Phi}$. A function $\gamma : X \to \overline{\mathbb{R}}$ is said to be a gap function for the problem $(VI)^{\Phi}$ if it satisfies the following properties:

(i) $\gamma(x) \geq 0 \ \forall x \in X$;

(ii) $\gamma(x) = 0$ if and only if x solves the problem $(VI)^{\Phi}$.

Let $\overline{x} \in \mathrm{dom}\,\Phi(\cdot, 0)$ be fixed. To the problem $(VI)^{\Phi}$ one can associate the following optimization problem

$$(P^{\Phi}, \overline{x}) \quad \inf_{x \in X} \{\langle F(\overline{x}), x \rangle + \Phi(x, 0)\} - \langle F(\overline{x}), \overline{x} \rangle - \Phi(\overline{x}, 0).$$

It is immediate that $\overline{x}$ is a solution of the variational inequality $(VI)^{\Phi}$ if and only if $v(P^{\Phi}, \overline{x}) = 0$.

The dual problem to $(P^{\Phi}, \overline{x})$ is (see [31, page 110] or [133, page 117])

$$(D^{\Phi}, \overline{x}) \quad \sup_{y^* \in Y^*} \{-\Phi^*(-F(\overline{x}), y^*)\} - \langle F(\overline{x}), \overline{x} \rangle - \Phi(\overline{x}, 0).$$

Following the idea from [3], we introduce the function $\gamma^{\Phi} : X \to \overline{\mathbb{R}}$ defined for all $\overline{x} \in \mathrm{dom}\,\Phi(\cdot, 0)$ by

$$\gamma^{\Phi}(\overline{x}) := -v(D^{\Phi}, \overline{x}) = \inf_{y^* \in Y^*} \{\Phi^*(-F(\overline{x}), y^*)\} + \langle F(\overline{x}), \overline{x} \rangle + \Phi(\overline{x}, 0)$$

and $\gamma^{\Phi}(\overline{x}) = +\infty$ for $\overline{x} \notin \mathrm{dom}\,\Phi(\cdot, 0)$.

Let us show that under appropriate conditions γ^{Φ} becomes a gap function for $(VI)^{\Phi}$.

Theorem 2.2.1 (**L. Cioban**, E.R. Csetnek, [51]) *Let X and Y be real separated locally convex spaces, $\Phi : X \times Y \to \overline{\mathbb{R}}$ a proper and convex function such that $0 \in \mathrm{pr}_Y(\mathrm{dom}\,\Phi)$ and assume that one of the regularity conditions $(RC_i^{\Phi}), i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Then γ^{Φ} is a gap function for the problem $(VI)^{\Phi}$.*

Remark 2.2.2 *Let us mention that we do not need the existence of optimal solutions of the dual problem $(D^{\Phi}, \overline{x})$, the proof uses only the equality $v(P^{\Phi}, \overline{x}) = v(D^{\Phi}, \overline{x})$, which is called in the literature "zero-duality gap". This means that instead of the regularity conditions considered above one can use weaker conditions, see for example [30, 84] and the references therein.*

Proposition 2.2.3 (**L. Cioban**, E.R. Csetnek, [51]) *(Convexity of γ^{Φ}) Assume that $F : X \to X^*$ is affine and monotone and the function $\Phi(\cdot, 0)$ is convex. Then γ^{Φ} is a convex function.*

2.2.2 Particular cases

In this subsection we consider several particular cases of the general results obtained above and show that several gap functions from the literature can be rediscovered.

The first composition case

Let $\Phi^{CC_1} : X \times Y \to \overline{\mathbb{R}}$ be the perturbation function defined by $\Phi^{CC_1}(x,y) = f(x) + g(h(x)+y)$ for all $(x,y) \in X \times Y$, where X and Y are real separated locally convex spaces, $K \subseteq Y$ is a nonempty cone, $g : Y^{\bullet} \to \overline{\mathbb{R}}$ is a proper function with $g(\infty_K) = +\infty$, $f : X \to \overline{\mathbb{R}}$ is proper and $h : X \to Y^{\bullet}$ is a proper function such that $h(\mathrm{dom}\, f \cap \mathrm{dom}\, h) \cap \mathrm{dom}\, g \neq \emptyset$.

The variational inequality $(VI)^{\Phi^{CC_1}}$ is nothing else than: find an element $\overline{x} \in \mathrm{dom}\, f \cap \mathrm{dom}\, h \cap h^{-1}(\mathrm{dom}\, g)$ such that

$$(VI)^{CC} \quad \langle F(\overline{x}), x - \overline{x} \rangle + f(x) + g(h(x)) - f(\overline{x}) - g(h(\overline{x})) \geq 0 \ \forall x \in X.$$

The function $\gamma^{\Phi^{CC_1}}$ becomes for $\overline{x} \in \mathrm{dom}\, f \cap \mathrm{dom}\, h \cap h^{-1}(\mathrm{dom}\, g)$

$$\gamma^{CC_1}(\overline{x}) = \inf_{y^* \in K^*} \{g^*(y^*) + (f + y^*h)^*(-F(\overline{x}))\} + \langle F(\overline{x}), \overline{x} \rangle + f(\overline{x}) + g(h(\overline{x}))$$

and $\gamma^{CC_1}(\overline{x}) = +\infty$ for $\overline{x} \notin \mathrm{dom}\, f \cap \mathrm{dom}\, h \cap h^{-1}(\mathrm{dom}\, g)$.

Theorem 2.2.4 (**L. Cioban**, E.R. Csetnek, [51]) *Let X and Y be real separated locally convex spaces, $K \subseteq Y$ a nonempty convex cone, $g : Y \to \overline{\mathbb{R}}$ a proper, convex, K-increasing function with $g(\infty_K) = +\infty$, $f : X \to \overline{\mathbb{R}}$ a proper, convex function and $h : X \to Y^{\bullet}$ a proper, K-convex function such that $h(\mathrm{dom}\, f \cap \mathrm{dom}\, h) \cap \mathrm{dom}\, g \neq \emptyset$ and assume that one of the regularity conditions $(RC_i^{CC_1}), i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Then γ^{CC_1} is a gap function for the problem $(VI)^{CC}$.*

The second composition case

Let $\Phi^{CC_2} : X \times X \times Y \to \overline{\mathbb{R}}$ be the perturbation function defined by $\Phi^{CC_2}(x,z,y) = f(x+z) + g(h(x)+y)$ for all $(x,z,y) \in X \times X \times Y$, where X and Y are real separated locally convex spaces, $K \subseteq Y$ is a nonempty cone, $g : Y^{\bullet} \to \overline{\mathbb{R}}$ is a proper function with $g(\infty_K) = +\infty$, $f : X \to \overline{\mathbb{R}}$ is proper and $h : X \to Y^{\bullet}$ is a proper function such that $h(\mathrm{dom}\, f \cap \mathrm{dom}\, h) \cap \mathrm{dom}\, g \neq \emptyset$.

The variational inequality $(VI)^{\Phi^{CC_2}}$ is nothing else than $(VI)^{CC}$: find an element $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} h \cap h^{-1}(\operatorname{dom} g)$ such that

$$(VI)^{CC} \quad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) + g(h(x)) - f(\overline{x}) - g(h(\overline{x})) \geq 0 \ \forall x \in X.$$

The function $\gamma^{\Phi^{CC_2}}$ becomes for $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} h \cap h^{-1}(\operatorname{dom} g)$

$$\gamma^{CC_2}(\overline{x}) = \inf_{z^* \in X^*, y^* \in K^*} \{g^*(y^*) + f^*(z^*) + (y^*h)^*(-F(\overline{x}) - z^*)\} + \langle F(\overline{x}), \overline{x}\rangle + f(\overline{x}) + (g \circ h)(\overline{x})$$

and $\gamma^{CC_2}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom} f \cap \operatorname{dom} h \cap h^{-1}(\operatorname{dom} g)$.

Theorem 2.2.5 (**L. Cioban**, E.R. Csetnek, [51]) *Let X and Y be real separated locally convex spaces, $K \subseteq Y$ a nonempty convex cone, $g : Y \to \overline{\mathbb{R}}$ a proper, convex, K-increasing function with $g(\infty_K) = +\infty$, $f : X \to \overline{\mathbb{R}}$ a proper, convex function and $h : X \to Y^\bullet$ a proper, K-convex function such that $h(\operatorname{dom} f \cap \operatorname{dom} h) \cap \operatorname{dom} g \neq \emptyset$ and assume that one of the regularity conditions $(RC_i^{CC_2}), i \in \{1, 2, 3, 4, 5, 6, 7\}$ is fulfilled. Then γ^{CC_2} is a gap function for the problem $(VI)^{CC}$.*

The case $f + g \circ A$

We specialize the first composition case to the situation $K = \{0\}$ and $h : X \to Y$, $h(x) = Ax$ for all $x \in X$, where A is a linear and continuous operator. The problem $(VI)^{CC}$ is nothing else than finding an element $\overline{x} \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$ such that

$$(VI)^{f,g,A} \quad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) + g(Ax) - f(\overline{x}) - g(A\overline{x}) \geq 0 \ \forall x \in X.$$

The function γ^{CC_1} becomes for $\overline{x} \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$

$$\gamma^{f,g,A}(\overline{x}) = \inf_{y^* \in Y^*} \{g^*(y^*) + f^*(-F(\overline{x}) - A^*y^*)\} + \langle F(\overline{x}), \overline{x}\rangle + f(\overline{x}) + g(A\overline{x})$$

and $\gamma^{f,g,A}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$.

Theorem 2.2.7 (**L. Cioban**, E.R. Csetnek, [51]) *Let X and Y be real separated locally convex spaces, $A : X \to Y$ a linear continuous operator, $g : Y \to \overline{\mathbb{R}}$ and $f : X \to \overline{\mathbb{R}}$ proper and convex functions fulfilling $\operatorname{dom} g \cap A^{-1}(\operatorname{dom} f) \neq \emptyset$ and assume that one of the regularity conditions $(RC_i^{f,g,A}), i \in \{1, 2, 3, 4, 5, 6, 7\}$ is fulfilled. Then $\gamma^{f,g,A}$ is a gap function for the problem $(VI)^{f,g,A}$.*

The case $f + g$

This is a particular case of Section 2.2.2 when we take $X = Y$ and $A = \mathrm{id}_X$. In this case the variational inequality reduces to finding an element $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$ such that

$$(VI)^{f,g} \quad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) + g(x) - f(\overline{x}) - g(\overline{x}) \geq 0 \ \forall x \in X.$$

The function $\gamma^{f,g,A}$ becomes for $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$

$$\gamma^{f,g}(\overline{x}) = \inf_{y^* \in X^*} \{f^*(-F(\overline{x}) - y^*) + g^*(y^*)\} + \langle F(\overline{x}), \overline{x}\rangle + f(\overline{x}) + g(\overline{x})$$

and $\gamma^{f,g}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom} f \cap \operatorname{dom} g$.

Notice that one can use the following reformulation based on the infimal convolution:

$$\gamma^{f,g}(\overline{x}) = (f^* \Box g^*)(-F(\overline{x})) + \langle F(\overline{x}), \overline{x}\rangle + f(\overline{x}) + g(\overline{x}).$$

Theorem 2.2.9 (**L. Cioban**, E.R. Csetnek, [51]) *Let X be a real separated locally convex space, $f, g : X \to \overline{\mathbb{R}}$ be proper and convex functions fulfilling* $\operatorname{dom} g \cap \operatorname{dom} f \neq \emptyset$ *and assume that one of the regularity conditions $(RC_i^{f,g}), i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Then $\gamma^{f,g}$ is a gap function for the problem $(VI)^{f,g}$.*

The case $f + \delta_K$

We particularize the results from Section 2.2.2 to the case $g = \delta_K$, where $K \subseteq X$ is a nonempty set. In this case the variational inequality $(VI)^{f,g}$ becomes: find an element $\overline{x} \in \operatorname{dom} f \cap K$ such that

$$(VI)^{f,K} \quad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) - f(\overline{x}) \geq 0 \ \forall x \in K.$$

The function $\gamma^{f,g}$ becomes for $\overline{x} \in \operatorname{dom} f \cap K$

$$\gamma^{f,K}(\overline{x}) = \inf_{y^* \in X^*} \{f^*(y^* - F(\overline{x})) + \sigma_K^*(-y^*)\} + \langle F(\overline{x}), \overline{x}\rangle + f(\overline{x}) + \delta_K(\overline{x})$$

and $\gamma^{f,K}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom} f \cap K$.

Theorem 2.2.10 (**L. Cioban**, E.R. Csetnek, [51]) *Let X be a real separated locally convex space, $K \subseteq X$ a nonempty convex set, $f : X \to \overline{\mathbb{R}}$ a proper and convex function fulfilling* $\operatorname{dom} f \cap K \neq \emptyset$ *and assume that one of the regularity conditions $(RC_i^{f,K}), i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Then $\gamma^{f,K}$ is a gap function for the problem $(VI)^{f,K}$.*

Remark 2.2.11 *Notice that in case $X = \mathbb{R}^n$, we rediscover the function introduced in [3].*

Remark 2.2.12 *In case $f \equiv 0$, γ_F^{MVI} becomes the Auslender's gap functions [12] (see also [3]).*

Example 2.2.13 (**L. Cioban**, E.R. Csetnek, [51]) *Let $X = \mathbb{R}$, $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined for $x \in \mathbb{R}$ by*

$$f(x) = \begin{cases} \frac{1}{2}x^2, & \text{if } x \geq 0 \\ +\infty, & \text{otherwise,} \end{cases}$$

$K = (-\infty, 0]$ and $F : \mathbb{R} \to \mathbb{R}$, $F(x) = x$ for all $x \in \mathbb{R}$.

We proved that in this case Theorem 2.2.10 can be applied, while [3, Theorem 3.1] fails because $(RC_7^{f,K})$ is not fulfilled, justifying in this way the use of closedness-type regularity conditions.

The case with geometric and cone constraints

The results from Section 2.2.1 are particularized now to the function $\Phi^{C_L} : X \times Z \to \overline{\mathbb{R}}$,

$$\Phi^{C_L}(x, z) = \begin{cases} f(x), & \text{if } x \in S, g(x) \in z - C, \\ +\infty, & \text{otherwise,} \end{cases}$$

where X and Z are real separated locally convex spaces, Z partially ordered by a nonempty cone $C \subseteq Z$, $S \subseteq X$ is a nonempty set, $f : X \to \overline{\mathbb{R}}$ is a proper function and $g : X \to Z^\bullet$ is a proper function fulfilling $\operatorname{dom} f \cap S \cap g^{-1}(-C) \neq \emptyset$.

The variational inequality $(VI)^{\Phi^{C_L}}$ becomes: find an element $\overline{x} \in \operatorname{dom} f \cap \mathcal{A}$ such that

$$(VI)^C \qquad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) - f(\overline{x}) \geq 0 \ \forall x \in \mathcal{A},$$

where $\mathcal{A} = S \cap g^{-1}(C)$.

The function $\gamma^{\Phi^{C_L}}$ becomes for $\overline{x} \in \operatorname{dom} f \cap \mathcal{A}$

$$\gamma^{C_L}(\overline{x}) = \inf_{z^* \in C^*} \{(f + (z^*g))^*_S(-F(\overline{x}))\} + \langle F(\overline{x}), \overline{x}\rangle + f(\overline{x}) + \delta_{\mathcal{A}}(\overline{x})$$

and $\gamma^{C_L}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom} f \cap \mathcal{A}$.

Theorem 2.2.14 (**L. Cioban**, E.R. Csetnek, [51]) *Let X and Z be real separated locally convex spaces, Z partially ordered by the convex cone $C \subseteq Z$, $S \subseteq X$ be a nonempty convex set, $f : X \to \overline{\mathbb{R}}$ a proper convex function and $g : X \to Z^\bullet$ a proper C-convex function fulfilling* $\operatorname{dom} f \cap S \cap g^{-1}(-C) \neq \emptyset$ *and assume that one of the regularity conditions $(RC_i^{C_L}), i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Then γ^{C_L} is a gap function for the problem $(VI)^C$.*

Remark 2.2.15 *In the case $X = S = \mathbb{R}^n$, $Z = \mathbb{R}^m$, $C = \mathbb{R}^m_+$ and $g(x) = (g_1(x), ..., g_m(x))^T$, where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, ..., m\}$, we obtain the gap function introduced in [3, Section 3]. Moreover, in case $f \equiv 0$, we obtain Giannessi's gap function [67] (see also [3]).*

We consider another perturbation function i.e. $\Phi^{C_{FL}} : X \times X \times Z \to \overline{\mathbb{R}}$,

$$\Phi^{C_{FL}}(x, y, z) = \begin{cases} f(x + y), & \text{if } x \in S, g(x) \in z - C, \\ +\infty, & \text{otherwise.} \end{cases}$$

The function $\gamma^{\Phi^{C_{FL}}}$ becomes for $\overline{x} \in \operatorname{dom} f \cap \mathcal{A}$

$$\gamma^{C_{FL}}(\overline{x}) = \inf_{z^* \in C^*, y^* \in X^*} \{f^*(y^*) + (z^*g)^*_S(-F(\overline{x}) - y^*)\} + \langle F(\overline{x}), \overline{x}\rangle + f(\overline{x}) + \delta_{\mathcal{A}}(\overline{x})$$

and $\gamma^{C_{FL}}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom} f \cap \mathcal{A}$.

Theorem 2.2.16 (**L. Cioban**, E.R. Csetnek, [51]) *Let X and Z be real separated locally convex spaces, Z partially ordered by the convex cone $C \subseteq Z$, $S \subseteq X$ be a nonempty convex set, $f : X \to \overline{\mathbb{R}}$ a proper convex function and $g : X \to Z^\bullet$ a proper C-convex function fulfilling* $\operatorname{dom} f \cap S \cap g^{-1}(-C) \neq \emptyset$ *and assume that one of the regularity conditions $(RC_i^{C_{FL}}), i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Then $\gamma^{C_{FL}}$ is gap function for the problem $(VI)^C$.*

Remark 2.2.17 *(a) Following the idea from [3] one can prove the inequality: $\gamma^{C_L} \leq \gamma^{C_{FL}}$.*

(b) In the hypotheses of Remark 2.2.15 we obtain the gap functions introduced in [3, Section 3].

2.2.3 Dual gap functions for the general variational inequality

In the following we consider the (Minty type) general variational inequality: find an element $\overline{x} \in X$ such that

$$(VI)'^{\Phi} \qquad \langle F(x), x - \overline{x} \rangle + \Phi(x, 0) - \Phi(\overline{x}, 0) \geq 0 \ \forall x \in X.$$

Let us introduce now dual gap functions for $(VI)^{\Phi}$. For $\overline{x} \in \operatorname{dom} \Phi(\cdot, 0)$ fixed, to the variational inequality $(VI)'^{\Phi}$ we attach the optimization problem

$$(P'^{\Phi}, \overline{x}) \quad \inf_{x \in X} \{ \langle F(x), x - \overline{x} \rangle + \Phi(x, 0) \} - \Phi(\overline{x}, 0)$$

and its Fenchel dual one:

$$(D'^{\Phi}, \overline{x}) \quad \sup_{x^* \in X^*} \left\{ - \sup_{x \in X} \{ \langle x^*, x \rangle - \langle F(x), x - \overline{x} \rangle \} - (\Phi(\cdot, 0))^*(-x^*) \right\} - \Phi(\overline{x}, 0).$$

Following [3], let us introduce now the function γ'^{Φ} defined for all $\overline{x} \in \operatorname{dom} \Phi(\cdot, 0)$ by

$$\gamma'^{\Phi}(\overline{x}) = \inf_{x^* \in X^*, y^* \in Y^*} \left\{ \sup_{x \in X} \{ \langle x^*, x \rangle - \langle F(x), x - \overline{x} \rangle \} + \Phi^*(-x^*, y^*) \right\} + \Phi(\overline{x}, 0)$$

and $\gamma'^{\Phi}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom} \Phi(\cdot, 0)$.

In the following we compare the function γ^{Φ} with the new one introduced, namely γ'^{Φ}.

Proposition 2.2.19 (**L. Cioban**, E.R. Csetnek, [51]) *If $F : X \to X^*$ is monotone then it holds*

$$\gamma'^{\Phi}(\overline{x}) \leq \gamma^{\Phi}(\overline{x}) \ \forall \overline{x} \in X.$$

Theorem 2.2.20 (**L. Cioban**, E.R. Csetnek, [51]) *Let X and Y be real separated locally convex spaces, $F : X \to X^*$ a monotone and upper hemicontinuous operator, $\Phi : X \times Y \to \overline{\mathbb{R}}$ a proper convex function such that $0 \in \operatorname{pr}_Y(\operatorname{dom} \Phi)$ and assume that one of the regularity conditions $(RC_i^{\Phi}), i \in \{1, 2, 3, 4, 5, 6, 7\}$ is fulfilled. Then γ'^{Φ} is a gap function for $(VI)^{\Phi}$.*

Remark 2.2.21 *Let us notice that in contrast to the function γ^{Φ}, the function γ'^{Φ} is always a convex function if we suppose that $\Phi(\cdot, 0)$ is convex (see also Proposition 2.2.3).*

In what follows we particularize the perturbation function Φ and we work in the same settings as in Section 2.2.2. In this particular case one has the variational inequality: find an element $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$ such that

$$(VI)'^{f,g} \qquad \langle F(x), x - \overline{x} \rangle + f(x) + g(x) - f(\overline{x}) - g(\overline{x}) \geq 0 \ \forall x \in X.$$

The function γ'^{Φ} becomes for $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$

$$\gamma'^{f,g}(\overline{x}) = \inf_{x^* \in X^*, y^* \in Y^*} \left\{ \sup_{x \in X} \{ \langle x^*, x \rangle - \langle F(x), x - \overline{x} \rangle \} + f^*(-x^* - y^*) + g^*(y^*) \right\} + f(\overline{x}) + g(\overline{x})$$

and $\gamma'^{f,g}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom} f \cap \operatorname{dom} g$. We have the following result.

Theorem 2.2.22 (**L. Cioban**, E.R. Csetnek, [51]) *Let X be a real separated locally convex space, $F : X \to X^*$ a monotone and upper hemicontinuous operator, $f, g : X \to \overline{\mathbb{R}}$ proper and convex functions fulfilling* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ *and assume that one of the regularity conditions $(RC_i^{f,g})$, $i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Then $\gamma'^{f,g}$ is a gap function for $(VI)^{f,g}$.*

Notice that we introduced two functions, $\gamma^{f,g}$ and a dual one $\gamma'^{f,g}$, which under appropriate hypotheses are gap functions for the variational inequality $(VI)^{f,g}$. Let us give now an example to show that in general these functions do not coincide, even if all the hypotheses of Theorem 2.2.9 and Theorem 2.2.22 are fulfilled.

Example 2.2.23 (**L. Cioban**, E.R. Csetnek, [51]) *Let us consider the case $X = \mathbb{R}$ and $f, g, F : \mathbb{R} \to \mathbb{R}$, $f(x) = g(x) = F(x) = x$ for all $x \in \mathbb{R}$. One can show that $f^* = g^* = \delta_{\{1\}}$, hence for all $\overline{x} \in \mathbb{R}$ we have*

$$\begin{aligned}\gamma'^{f,g}(\overline{x}) &= \inf_{x^* \in \mathbb{R}} \left\{ \sup_{x \in \mathbb{R}} \{x^* x - x^2 + x\overline{x}\} + \inf_{y^* \in \mathbb{R}} \{\delta_{\{1\}}(-x^* - y^*) + \delta_{\{1\}}(y^*)\} \right\} + 2\overline{x} \\ &= \sup_{x \in \mathbb{R}} \{-2x - x^2 + x\overline{x}\} + 2\overline{x} = (\overline{x} - 2)^2/4 + 2\overline{x} = (\overline{x} + 2)^2/4\end{aligned}$$

and

$$\gamma^{f,g}(\overline{x}) = \inf_{y^* \in \mathbb{R}} \{\delta_{\{1\}}(-\overline{x} - y^*) + \delta_{\{1\}}(y^*)\} + \overline{x}^2 + 2\overline{x} = \delta_{\{-2\}}(\overline{x}).$$

Remark 2.2.24 *Let us consider now $f \equiv 0$ and $g = \delta_K$ where K is a nonempty set. In this case we rediscover exactly the function $\gamma_F^{VI'}$ introduced in [3].*

Remark 2.2.25 *Let us take the perturbation function considered in the first part of Section 2.2.2 in the case $X = S = \mathbb{R}^n$, $Z = \mathbb{R}^m$, $C = \mathbb{R}^m_+$, $f \equiv 0$ and $g(x) = (g_1(x), ..., g_m(x))^T$, where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, ..., m\}$. We rediscover in this case the gap function introduced in [3, Section 4].*

2.3 Optimality conditions for variational inequalities

In this section we deliver optimality conditions for variational inequalities based on subdifferential calculus. In this context the regularity conditions are again involved. Further, we show that even in the absence of any regularity condition, one can obtain sequential characterizations of the solutions of variational inequalities by applying the results given in [28, 29] for optimization problems. Examples justifying the usefulness of having such characterizations are also provided.

2.3.1 Optimality conditions for variational inequalities based on subdifferential calculus

The aim of this section is to characterize the solution of the variational inequalities considered in this thesis by means of the (convex) subdifferential.

Theorem 2.3.1 (**L. Cioban**, E.R. Csetnek, [51]) *Suppose that the hypotheses of Theorem 2.2.1 are fulfilled. Then $\overline{x}$ is a solution of the variational inequality $(VI)^{\Phi}$ if and only if*

$$-F(\overline{x}) \in \operatorname{Pr}_{X^*}(\partial\Phi(\overline{x}, 0)).$$

Theorem 2.3.2 (**L. Cioban**, E.R. Csetnek, [51]) *Suppose that the hypotheses of Theorem 2.2.7 are fulfilled. Then $\overline{x}$ is a solution of the variational inequality $(VI)^{f,g,A}$ if and only if*

$$-F(\overline{x}) \in \partial f(\overline{x}) + A^*\partial g(A\overline{x}).$$

Theorem 2.3.4 (**L. Cioban**, E.R. Csetnek, [51]) *Suppose that the hypotheses of Theorem 2.2.9 are fulfilled. Then $\overline{x}$ is a solution of the variational inequality $(VI)^{f,g}$ if and only if*

$$-F(\overline{x}) \in \partial f(\overline{x}) + \partial g(\overline{x}).$$

Theorem 2.3.5 (**L. Cioban**, E.R. Csetnek, [51]) *Suppose that the hypotheses of Theorem 2.2.10 are fulfilled. Then $\overline{x}$ is a solution of the variational inequality $(VI)^{f,K}$ if and only if*

$$-F(\overline{x}) \in \partial f(\overline{x}) + N_K(\overline{x}),$$

which is equivalent to: there exists $x^* \in \partial f(\overline{x})$ such that the following variational inequality holds

$$\langle F(\overline{x}) + x^*, x - \overline{x}\rangle \geq 0 \ \forall x \in K.$$

2.3.2 Sequential optimality conditions for variational inequalities

Notice that in the above characterizations of the solutions of the variational inequalities (via gap functions or by means of the subdifferential, see the above sections), the fulfillment of a regularity condition was of great importance. We show in this section that even in the absence of a regularity condition, we can still characterize these solutions. We use as tool the sequential optimality conditions given in [28, 29].

Sequential optimality condition for the general variational inequality

In what follows we deliver sequential conditions in order to characterize the solution of the general variational inequality $(VI)^{\Phi}$. In what is follows we consider X a reflexive Banach space and Y a Banach space.

Theorem 2.3.6 (**L. Cioban**, [48]) *Let $\Phi : X\times Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous and $0 \in \operatorname{Pr}_Y(\operatorname{dom}\Phi)$. Then $\overline{x} \in \operatorname{dom}\Phi(\cdot, 0)$ solves $(VI)^{\Phi}$ if and only if there exists $(x_n, y_n) \in \operatorname{dom}\Phi$ and $(x_n^*, y_n^*) \in \partial\Phi(x_n, y_n)$ such that*

$$x_n^* \to -F(\overline{x}),\ x_n \to \overline{x},\ y_n \to 0\,(n \to +\infty)\,\textit{and}$$

$$\Phi(x_n, y_n) - \langle y_n^*, y_n\rangle - \Phi(\overline{x}, 0) \to 0\,(n \to +\infty).$$

Sequential optimality condition for the case $g + f \circ h$

For this case we work in the following settings: X is a reflexive Banach space and Y is a Banach space partially ordered by the non-empty convex cone $K \subseteq X$, $f : X \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous, $h : X \to Y^\bullet$ is proper and $K-$*convex* and $g : Y^\bullet \to \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous with $g(+\infty_K) = +\infty$. We also suppose that $\operatorname{dom} f \cap \operatorname{dom} h \cap h^{-1}(\operatorname{dom} g) \neq \emptyset$.

We consider two special cases of this case. The first one, when we assume that h is K-epi closed and the second one when we assume that h is continuous.

1. *The case h is K-epi closed*

For this particular instance we assume in additionally that Y is reflexive, h is K-epi-closed and g is K-increasing on $h(\operatorname{dom} h) + K$.

Theorem 2.3.7 (**L. Cioban**, [48]) *The element $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} h \cap h^{-1}(\operatorname{dom} g)$ solves $(VI)^{CC}$ if and only if*

$$\begin{cases} \exists (x_n, p_n, q_n, q'_n) \in X \times \operatorname{dom} f \times \operatorname{dom} g \times Y,\ h(x_n) \leq_K q'_n \\ \exists (u_n^*, e_n^*, u_n'^*, q_n^*), q_n^* \in K^*, u_n^* \in \partial f(p_n), q_n^* + e_n^* \in \partial g(q_n), \\ u_n'^* \in \partial (q_n^* h)(x_n), \langle q_n^*, q'_n - h(x_n) \rangle = 0\, \forall n \in \mathbb{N}, \\ u_n^* + u_n'^* \to -F(\overline{x}), e_n^* \to 0, p_n \to \overline{x}, q_n \to h(\overline{x}), q'_n \to h(\overline{x})\ (n \to +\infty), \\ f(p_n) - \langle u_n^*, p_n - x_n \rangle + \langle F(\overline{x}), x_n - \overline{x} \rangle + \langle q_n^*, h(x_n) - h(\overline{x}) \rangle - f(\overline{x}) \to 0\ (n \to +\infty)\ \textit{and} \\ g(q_n) - \langle q_n^*, q_n - h(\overline{x}) \rangle - g(h(\overline{x})) \to 0\ (n \to +\infty). \end{cases}$$

2. *The case h is continuous*

For this case we consider in addition that $h : X \to Y$ is continuous and $g : Y \to \overline{\mathbb{R}}$ is K-increasing on Y.

Theorem 2.3.8 (**L. Cioban**, [48]) *The element $\overline{x} \in \operatorname{dom} f \cap h^{-1}(\operatorname{dom} g)$ solves $(VI)^{CC}$ if and only if*

$$\begin{cases} \exists (x_n, y_n) \in \operatorname{dom} f \times \operatorname{dom} g, \exists (u_n^*, v_n^*, y_n^*) \in X^* \times X^* \times K^*, \\ u_n^* - F(\overline{x}) \in \partial f(x_n), v_n^* \in \partial (y_n^* h)(x_n), y_n^* \in \partial g(y_n)\, \forall n \in \mathbb{N}, \\ u_n^* + v_n^* \to 0, x_n \to \overline{x}, y_n \to h(\overline{x})\ (n \to +\infty), \\ f(x_n) + \langle y_n^*, h(x_n) - h(\overline{x}) \rangle + \langle F(\overline{x}), x_n - \overline{x} \rangle - f(\overline{x}) \to 0\ (n \to +\infty),\ \textit{and} \\ g(y_n) - \langle y_n^*, y_n - h(\overline{x}) \rangle - g(h(\overline{x})) \to 0\ (n \to +\infty). \end{cases}$$

Sequential optimality condition for the case $f + g \circ A$

The next theorems are particular cases of Theorem 3.3 and Theorem 3.4 in [54] for $(VI)^{f,g,A}$.

Theorem 2.3.9 (**L. Cioban**, [48]) *Let $A : X \to Y$ be a continuous linear mapping, $f, g : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $A(\operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$. Then $\overline{x} \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$ solves $(VI)^{f,g,A}$ if and only if*

$$\exists \{\varepsilon_n\} \downarrow 0, \exists x_n^* \in \partial_{\varepsilon_n} f(\overline{x}), \exists y_n^* \in \partial_{\varepsilon_n} g(A\overline{x})\ \textit{such that}\ x_n^* + A^* y_n^* \to -F(\overline{x}),\ (n \to +\infty).$$

Theorem 2.3.10 (**L. Cioban**, [48]) *Let $A : X \to Y$ be a continuous linear mapping, $f, g : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $A(\operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$. Then*

$\overline{x} \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$ *solves* $(VI)^{f,g,A}$ *if and only if*

$$\begin{cases} \exists (x_n, y_n) \in \operatorname{dom} f \times \operatorname{dom} g,\ \exists x_n^* \in \partial f(x_n),\ \exists y_n^* \in \partial g(y_n) \textit{ such that} \\ x_n^* + A^* y_n^* \to -F(\overline{x}),\ x_n \to \overline{x},\ y_n \to A\overline{x}\, (n \to +\infty), \\ f(x_n) - \langle x_n^*, x_n - \overline{x} \rangle - f(\overline{x}) \to 0\, (n \to +\infty), \\ g(y_n) - \langle y_n^*, y_n - A\overline{x} \rangle - g(A\overline{x}) \to 0\, (n \to +\infty). \end{cases}$$

Sequential optimality condition for the case $f + g$

Theorem 2.3.11 (**L. Cioban**, E.R. Csetnek, [51]) *Let X be a reflexive Banach space and $f, g : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then* $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$ *is a solution of the variational inequality* $(VI)^{f,g}$ *if and only if*

$$\begin{cases} \exists (x_n, y_n) \in \operatorname{dom} f \times \operatorname{dom} g,\ \exists x_n^* \in \partial f(x_n),\ \exists y_n^* \in \partial g(y_n) \textit{ such that} \\ x_n^* + y_n^* \to -F(\overline{x}),\ x_n \to \overline{x},\ y_n \to \overline{x}\, (n \to +\infty), \\ f(x_n) - \langle x_n^*, x_n - \overline{x} \rangle - f(\overline{x}) \to 0\, (n \to +\infty), \\ g(y_n) - \langle y_n^*, y_n - \overline{x} \rangle - g(\overline{x}) \to 0\, (n \to +\infty). \end{cases}$$

Sequential optimality condition for the case $f + \delta_K$

In case $g = \delta_K$, where $K \subseteq X$ is a nonempty we get the following result.

Theorem 2.3.12 (**L. Cioban**, E.R. Csetnek, [51]) *Let X be a reflexive Banach space, $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function and $K \subseteq X$ a closed and convex set such that* $\operatorname{dom} f \cap K \neq \emptyset$. *Then* $\overline{x} \in \operatorname{dom} f \cap K$ *is a solution of the variational inequality* $(VI)^{f,K}$ *if and only if*

$$\begin{cases} \exists (x_n, y_n) \in \operatorname{dom} f \times K,\ \exists x_n^* \in \partial f(x_n),\ \exists y_n^* \in N_K(y_n) \textit{ such that} \\ x_n^* + y_n^* \to -F(\overline{x}),\ x_n \to \overline{x},\ y_n \to \overline{x}\, (n \to +\infty), \\ f(x_n) - \langle x_n^*, x_n - \overline{x} \rangle - f(\overline{x}) \to 0\, (n \to +\infty), \\ \langle y_n^*, y_n - \overline{x} \rangle \to 0\, (n \to +\infty). \end{cases}$$

In the following example we underline the advantage of having such sequential characterizations.

Example 2.3.13 (**L. Cioban**, E.R. Csetnek, [51]) *Let* $X = \mathbb{R}^2$, $K = \{0\} \times \mathbb{R}$, $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $f(x, y) = -\sqrt{xy} + \delta_{\mathbb{R}^2_+}(x, y)$ *and define* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $F(x, y) = (x, y)$ *for all* $x, y \in \mathbb{R}$. *In this case we proved that neither Theorem 2.2.10 nor Theorem 2.3.5 can be applied, however Theorem 2.3.12 works.*

Chapter 3

Equilibrium problems

Equilibrium problems provide a unified framework to the study of different problems in optimization, saddle and fixed point theory, variational inequalities, etc. see the seminal paper of Blum-Oettli [20].

In what follows we characterize the solutions of some equilibrium problems by making use of the saddle points of an associated Lagrangian function, via duality, by means of gap functions, through the properties of the convex subdifferential and we deliver also necessary and sufficient sequential characterizations for these solutions.

3.1 Duality for an extended equilibrium problem

Bigi, Castellani and Kassay [19] introduced the so-called "generalized equilibrium problem" and consists in finding a point $\overline{x} \in \mathbb{R}^n$ such that

$$\text{(GEP)} \quad \varphi(\overline{x}, y) + f(y) \geq f(\overline{x}), \ \forall y \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a function satisfying the conditions $\varphi(x, \cdot)$ is convex for all $x \in \operatorname{dom} f$ and $\varphi(x, x) = 0$ for all $x \in \operatorname{dom} f$, and it is been proved that the solution of (GEP) and its dual are strictly related to the saddle points of an associated Lagrangian function.

In what follows we study an extended form of (GEP), a generalized equilibrium problem with sum of two functions, one being composed with a linear continuous mapping and we introduce and study a dual problem associated to it.

3.1.1 Optimality conditions for an optimization problem

In this section we give optimality conditions for an optimization problem which is formed of a sum of three functions, one being composed with a linear operator.

We consider the optimization problem

$$(P_3) \quad \inf_{x \in \mathbb{R}^n} \Big\{ f_1(x) + f_2(x) + f_3(Bx) \Big\}.$$

The dual problem to (P_3) is

$$(D_3) \quad \sup_{\substack{x_1^*, x_2^* \in \mathbb{R}^n, x_3^* \in \mathbb{R}^m, \\ x_1^* + x_2^* + B^* x_3^* = 0}} \Big\{ - f_1^*(x_1^*) - f_2^*(x_2^*) - f_3^*(x_3^*) \Big\}.$$

Using as tool the regularity conditions given for problems having the composition with a linear continuous mapping in the objective function (see [31, section 3.2.2]) in finite dimensional case we give the following result.

Theorem 3.1.1 (**L. Cioban**, [45]) *Let $B : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping, $f_1, f_2 : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f_3 : \mathbb{R}^m \to \overline{\mathbb{R}}$ are proper functions fulfilling $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap B^{-1}(\operatorname{dom} f_3) \neq \emptyset$. Let $(\overline{x}_2^*, \overline{x}_3^*) \in \mathbb{R}^n \times \mathbb{R}^m$ be an optimal solution to (D_3) and assume that $\operatorname{ri}\operatorname{dom} f_1^* \cap (-\operatorname{ri}\operatorname{dom} f_2^* - B^* \operatorname{ri}\operatorname{dom} f_3^*) \neq \emptyset$. Then there exists $\overline{x} \in \mathbb{R}^n$, an optimal solution to the dual optimization problem of (D_3), such that*

1. *$\overline{x} \in \partial f_1^*(-\overline{x}_2^* - B^* \overline{x}_3^*)$;*

2. *$\overline{x} \in \partial f_2^*(\overline{x}_2^*)$;*

3. *$B\overline{x} \in \partial f_3^*(\overline{x}_3^*)$.*

3.1.2 Duality for the equilibrium problem (CEP)

Let us consider now the following equilibrium problem which consists in finding a point $\overline{x} \in \mathbb{R}^n$ such that

$$\text{(CEP)} \quad \varphi(\overline{x}, y) + f(y) + g(Ay) \geq f(\overline{x}) + g(A\overline{x}), \ \forall y \in \mathbb{R}^n,$$

where $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ are proper and convex functions fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $\varphi(x, \cdot)$ is a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0 \ \forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$.

For $\overline{x} \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ we can rewrite the equilibrium problem (CEP) as an optimization problem

$$(P_{\overline{x}}) \quad \inf_{y \in \mathbb{R}^n} \Big\{ \varphi(\overline{x}, y) + f(y) + g(Ay) \Big\}.$$

The following theorem establishes the connection between (CEP) and $(P_{\overline{x}})$.

Theorem 3.1.3 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper and convex functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0 \ \forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ and assume that $A(\operatorname{ri}\operatorname{dom} f) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$. The following statements are equivalent:*

(i) $\overline{x}$ is a solution of (CEP);

(ii) $\overline{x}$ is a solution of $(P_{\overline{x}})$;

(iii) $\mathcal{D}(\overline{x}) \neq \emptyset$;

where $\mathcal{D}(x) := \Big\{ (u^*, v^*) : u^* \in \partial f(x), v^* \in \partial g(Ax), -u^* - A^* v^* \in \partial \varphi(x, \cdot)(x) \Big\}$.

When the regularity condition $A(\operatorname{ri}\operatorname{dom} f) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$ is fulfilled, problem (CEP) can be formulated as finding $\overline{x} \in \mathbb{R}^n$ such that there exists $(\overline{u}^*, \overline{v}^*) \in \mathbb{R}^n \times \mathbb{R}^m$ with

$$\begin{array}{ll} (p1) & -\overline{u}^* - A^*\overline{v}^* \in \partial\varphi(\overline{x}, \cdot)(\overline{x}); \\ (p2) & \overline{u}^* \in \partial f(\overline{x}); \\ & \overline{v}^* \in \partial g(A\overline{x}). \end{array}$$

We can attach the following dual problem to (CEP) which consists in finding a point $(\overline{u}^*, \overline{v}^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that there exists $\overline{x} \in \mathbb{R}^n$ with

$$\begin{array}{lll} (DCEP) & (d1) & \overline{x} \in \partial\varphi^*(\overline{x}, \cdot)(-\overline{u}^* - A^*\overline{v}^*); \\ & (d2) & \overline{x} \in \partial f^*(\overline{u}^*); \\ & & A\overline{x} \in \partial g^*(\overline{v}^*); \end{array}$$

where $\varphi^*(x, \cdot)(x^*)$ is the conjugate of φ on its second variable, $\varphi^*(x, \cdot)(x^*) = (\varphi(x, \cdot))^*(x^*)$.

Theorem 3.1.5 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper and convex functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1}\operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g$ and assume that $A(\operatorname{ri}\operatorname{dom} f) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$. If $\overline{x} \in \mathbb{R}^n$ solves (CEP) then any element of $\mathcal{D}(\overline{x})$ is a solution of $(DCEP)$.*

We consider the set:

$$\mathcal{P}(u^*, v^*) = \Big\{x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g : x \in \partial\varphi^*(x, \cdot)(-u^* - A^*v^*) \cap \partial f^*(u^*) \cap A^{-1}\partial g^*(v^*)\Big\}.$$

The next theorem characterizes the solutions of $(DCEP)$ by the set $\mathcal{P}(u^*, v^*)$.

Theorem 3.1.7 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1}\operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g$. If $(\overline{u}^*, \overline{v}^*)$ is a solution of $(DCEP)$ then any element of $\mathcal{P}(\overline{u}^*, \overline{v}^*)$ is a solution of (CEP).*

Remark 3.1.8 *Theorem 3.1.7 tell us that any element $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$ generate a solution of (CEP). If we suppose in addition that the regularity condition $A(\operatorname{ri}\operatorname{dom} f) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$ is fulfilled then the set $\mathcal{P}(\overline{u}^*, \overline{v}^*)$ is nothing else but the set of solutions of (CEP) associated to $\overline{u}^* + A^*\overline{v}^*$.*

Theorem 3.1.9 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper and convex functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1}\operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g$. If $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$ then $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$.*

Theorem 3.1.10 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1}\operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1}\operatorname{dom} g$. If $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$ then $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$.*

The solutions of (CEP) respectively $(DCEP)$ are strictly related to the saddle points of the Lagrangian function:

$$\mathcal{L}_{\overline{x}}(x, y, u^*, v^*) = f(x) - \langle u^*, x\rangle + g(y) - \langle v^*, y\rangle - \varphi^*(\overline{x}, \cdot)(-u^* - A^*v^*)$$

as the following theorem shows.

Theorem 3.1.11 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. The following statements are equivalent:*

(i) $(\overline{u}^*, \overline{v}^*) \in \mathcal{D}(\overline{x})$;

(ii) $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$;

(iii) $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ *is a saddle point of* $\mathcal{L}_{\overline{x}}$.

Next results are consequences of Theorem 3.1.11.

Corollary 3.1.12 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper and convex functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ and assume that $A(\operatorname{ri} \operatorname{dom} f) \cap \operatorname{ri} \operatorname{dom} g \neq \emptyset$. $\overline{x} \in \mathbb{R}^n$ is a solution of (CEP) if and only if there exists $(\overline{u}^*, \overline{v}^*)$ such that $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ is a saddle point of $\mathcal{L}_{\overline{x}}$.*

Corollary 3.1.13 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. $(\overline{u}^*, \overline{v}^*) \in \mathbb{R}^n \times \mathbb{R}^m$ is a solution of $(DCEP)$ if and only if there exists $\overline{x} \in \mathbb{R}^n$ such that $(\overline{x}, A\overline{x}, \overline{u}^*, \overline{v}^*)$ is a saddle point of $\mathcal{L}_{\overline{x}}$.*

If we consider the optimization problem $(P_{\overline{x}})$ we can formulate its Fenchel dual problem as

$$(D_{\overline{x}}) \quad \sup_{\substack{x^* \in \mathbb{R}^n \\ y^* \in \mathbb{R}^m}} \Big\{ -\varphi^*(\overline{x}, \cdot)(-x^*) - g^*(y^*) - f^*(x^* - A^*y^*) \Big\}.$$

We can observe that problem $(D_{\overline{x}})$ is not the optimization form of the problem $(DCEP)$, but we can prove the following relation between them.

Theorem 3.1.14 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper and convex functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. If $(\overline{u}^*, \overline{v}^*)$ is a solution of $(DCEP)$ then there exists $\overline{x} \in \mathcal{P}(\overline{u}^*, \overline{v}^*)$ such that $(\overline{u}^* + A^*\overline{v}^*, \overline{v}^*)$ is a solution of $(D_{\overline{x}})$.*

In what follows we give results which guarantees that all the solutions of (CEP) or $(DCEP)$ can be found using the problem $(P_{\overline{x}})$ respectively $(D_{\overline{x}})$ if the following property of function φ is fulfilled:

$$\varphi(x, y) \leq \varphi(x, z) + \varphi(z, y),\ \forall x, y, z \in \mathbb{R}^n, \tag{3.1}$$

property used for the same reason in [17, 19, 20].

Theorem 3.1.15 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper and convex functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. If the function φ satisfies property (3.1) then $\overline{x}$ is a solution of the problem (CEP) if and only if there exists $z \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ such that $\overline{x}$ is a solution of (P_z).*

Theorem 3.1.16 (**L. Cioban**, [45]) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper and convex functions and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ fulfilling $\operatorname{dom} f \cap A^{-1} \operatorname{dom} g \neq \emptyset$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\varphi(x, \cdot)$ a convex function $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$, $\varphi(x, x) = 0$, $\forall x \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$. Assume that $\operatorname{ri} \operatorname{dom} \varphi^*(\overline{x}, \cdot) \cap (-\operatorname{ri} \operatorname{dom} f^* - A^* \operatorname{ri} \operatorname{dom} g^*) \neq \emptyset$ for all $x \in \mathbb{R}^n$ and function φ satisfies (3.1). Then, $(\overline{u}^*, \overline{v}^*)$ is a solution of $(DCEP)$ if and only if there exists $z \in \operatorname{dom} f \cap A^{-1} \operatorname{dom} g$ such that $(\overline{u}^* + A^*\overline{v}^*, \overline{v}^*)$ is a solution of (D_z).*

3.1.3 Particular cases

Equilibrium problems

Let us particularize the duality statements for the problems (CEP) and $(DCEP)$ to the case when $m = n$, $g(x) = 0, \forall x \in \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator. We show that the results given in [19, Section 3] are particular instances of results presented in Section 3.1.2.

Variational inequalities

If we consider $X = R^n$ and $Y = R^m$, $\varepsilon = \varepsilon_1 = \varepsilon_2 = 0$, in the formulation of $(\mathrm{VI})_{\varepsilon}^{g,f,A}$ and $(\mathrm{DVI})_{\varepsilon_1,\varepsilon_2}^{g,f,A}$ from Section 2.1.2, we obtain the same problems as in the case if we consider $\varphi(x, y) = \langle F(x), y - x\rangle$ where $F : \mathbb{R}^n \to \mathbb{R}^n$ for the problem (CEP). Furthermore, if we consider $n = m$, $g \equiv 0$, A is the identical operator and if F is an injective mapping, we rediscover the dual-pair of variational inequality introduced by Mosco in [113].

3.2 Gap functions for equilibrium problems

It is the aim of this section to apply the same techniques as in Section 2.2 to a broader class of problems, namely equilibrium problems. We construct gap functions for a general equilibrium problem and consider several particular cases rediscovering some of the gap functions introduced in literature.

3.2.1 A gap function for the general equilibrium problem

Let us consider the equilibrium problem which consists in finding a point $\overline{x} \in X$ such that

$$(PEP) \qquad F(\overline{x}, x) + \Phi(x, 0) \geq \Phi(\overline{x}, 0) \ \forall x \in X,$$

where X and Y be real separated locally convex spaces, $\Phi : X \times Y \to \overline{\mathbb{R}}$ is a proper function fulfilling $0 \in \operatorname{pr}_Y(\operatorname{dom} \Phi)$ and $F : X \times X \to \overline{\mathbb{R}}$ is a bifunction satisfying the relation $F(x, x) = 0$ for all $x \in \operatorname{dom} \Phi(\cdot, 0)$.

Let $\overline{x} \in \operatorname{dom}\Phi(\cdot,0)$ be fixed. To the problem (PEP) one can associate the following optimization problem

$$(P^{PEP},\overline{x}) \quad \inf_{x\in X}\{F(\overline{x},x)+\Phi(x,0)\}-\Phi(\overline{x},0).$$

One can see that $\overline{x}$ is a solution of the equilibrium problem (PEP) if and only if $v(P^{PEP},\overline{x})=0$. Let us consider now the Fenchel dual problem to $(P^{PEP},\overline{x})$:

$$(D^{PEP},\overline{x}) \quad \sup_{x^*\in X^*}\big\{-F^*_x(\overline{x},x^*)-(\Phi(\cdot,0))^*(-x^*)\big\}-\Phi(\overline{x},0),$$

where $F^*_x(\overline{x},x^*)=(F(\overline{x},\cdot))^*(x^*)$.

Let us introduce now the function $\gamma^{PEP}:X\to\overline{\mathbb{R}}$ defined for all $\overline{x}\in\operatorname{dom}\Phi(\cdot,0)$ by

$$\gamma^{PEP}(\overline{x})=\inf_{x^*\in X^*,y^*\in Y^*}\big\{F^*_x(\overline{x},x^*)+\Phi^*(-x^*,y^*)\big\}+\Phi(\overline{x},0)$$

and $\gamma^{PEP}(\overline{x})=+\infty$ for $\overline{x}\notin\operatorname{dom}\Phi(\cdot,0)$.

Theorem 3.2.1 (**L. Cioban**, E.R. Csetnek, [51]) *Let X and Y be real separated locally convex spaces, $\Phi:X\times Y\to\overline{\mathbb{R}}$ a proper and convex function, $F:X\times X\to\overline{\mathbb{R}}$ a proper bifunction such that for all $\overline{x}\in\operatorname{dom}\Phi(\cdot,0)$, $F(\overline{x},\overline{x})=0$, $\operatorname{dom}\Phi(\cdot,0)\cap\operatorname{dom}F(\overline{x},\cdot)\neq\emptyset$ and $F(\overline{x},\cdot)$ is convex and continuous at a point in $\operatorname{dom}\Phi(\cdot,0)\cap\operatorname{dom}F(\overline{x},\cdot)$. Assume that one of the regularity conditions (RC^{Φ}_i), $i\in\{1,2,3,4,5,6,7\}$ is fulfilled. Then γ^{PEP} is a gap function for the problem (PEP).*

Remark 3.2.2 (**L. Cioban**, E.R. Csetnek, [51]) *The continuity of the function $F(\overline{x},\cdot)$ has been used in order to guarantee the equality $v(P^{PEP},\overline{x})=v(D^{PEP},\overline{x})$. Alternatively, one can replace the continuity with an interiority type regularity condition, or a closedness type regularity condition.*

3.2.2 Particular cases

In what follows we particularize the perturbation function Φ and we show that we rediscover some gap functions for equilibrium problems considered in the literature in [2, 26].

The case $g\circ h$

Let $\Phi^{CC_1}:X\times Y\to\overline{\mathbb{R}}$ be the perturbation function defined by $\Phi^{CC_1}(x,y)=\delta_{\mathcal{A}}(x)+g(h(x)+y)$ for all $(x,y)\in X\times Y$, where X and Y are real separated locally convex spaces, $\mathcal{A}\subseteq X$ is a nonempty set, $K\subseteq Y$ is a nonempty cone, $g:Y\to\overline{\mathbb{R}}$ is a proper function with $g(\infty_K)=+\infty$ and $h:X\to Y^\bullet$ is a proper function such that $h(\mathcal{A}\cap\operatorname{dom}h)\cap\operatorname{dom}g\neq\emptyset$. In this case we rediscover the function introduced in [26, Section 4] in case g has full domain.

Remark 3.2.3 (**L. Cioban**, E.R. Csetnek, [51]) *If we consider the perturbation function $\Phi^{f,g}:X\times X\to\overline{\mathbb{R}}$ and if we further specialize this case to $f=\delta_K$, where $K\subseteq X$ is a nonempty set and $g\equiv 0$, we rediscover the function introduced in [2].*

Remark 3.2.4 (**L. Cioban**, E.R. Csetnek, [51]) *If we particularize the results given for the equilibrium problems to variational inequalities we rediscover the problems and the results considered in Section 2.2.1.*

3.2.3 Dual gap function for the general equilibrium problem

In this section we introduce another class of gap functions for the problem (PEP). In what follows we deal with the so-called dual equilibrium problems (Minty type) which is closely related to (PEP) and consists in finding $\overline{x} \in X$ such that

$$(DPEP) \qquad F(x,\overline{x}) + \Phi(\overline{x},0) \leq \Phi(x,0), \forall x \in X.$$

We denote by S^{PEP} and S^{DPEP} the solution set of the problems (PEP) and $(DPEP)$ respectively.

In order to formulate another gap function for (PEP) using the dual equilibrium problem $(DPEP)$ we recall some definitions (see [18, 20, 87, 89, 109]): monotonicity and pseudomonotonicity of a bifunction, quasiconvexity, explicitly quasiconvexity, (explicitly) quasiconcavity, u-hemicontinuity and l-hemicontinuity of a function.

Proposition 3.2.8 (**L. Cioban**, [49]) *If F is a monotone bifunction, then $S^{PEP} \subseteq S^{DPEP}$.*

Proposition 3.2.9 (**L. Cioban**, [49]) *Let $F(x,\cdot)$ convex $\forall x \in X$, $F(\cdot,x)$ u-hemicontinuous $\forall x \in X$ and $\Phi(\cdot,0)$ be proper, convex and l-hemicontinuous. Then $S^{DPEP} \subseteq S^{PEP}$.*

Remark 3.2.10 (**L. Cioban**, [49]) *We can replace the convexity of the functions $F(x,\cdot)$ and $\Phi(\cdot,0)$ with explicitly quasiconvexity of the function $F(x,\cdot) + \Phi(\cdot,0)$ in Proposition 3.2.9.*

To the dual equilibrium problem $(DPEP)$ one can attach the following optimization problem:

$$(P^{DPEP},\overline{x}) \quad \inf_{x\in X}\{-F(x,\overline{x}) + \Phi(x,0)\} - \Phi(\overline{x},0)$$

where $\overline{x}$ is fixed. The Fenchel dual problem to $(P^{DPEP},\overline{x})$ is (see [57, 133]):

$$(D^{DPEP},\overline{x}) \quad \sup_{x^*\in X^*}\left\{-\sup_{x\in X}[\langle x^*,x\rangle + F(x,\overline{x})] - (\Phi(\cdot,0))^*(-x^*)\right\} - \Phi(\overline{x},0).$$

Following [2, 3], we introduce now the function γ^{DPEP} defined for all $\overline{x} \in \operatorname{dom}\Phi(\cdot,0)$ by

$$\gamma^{DPEP}(\overline{x}) = \inf_{x^*\in X^*, y^*\in Y^*}\left\{\sup_{x\in X}[\langle x^*,x\rangle + F(x,\overline{x})] + \Phi^*(-x^*,y^*)\right\} + \Phi(\overline{x},0),$$

and $\gamma^{DPEP}(\overline{x}) = +\infty$ for $\overline{x} \notin \operatorname{dom}\Phi(\cdot,0)$.

Corollary 3.2.11 (**L. Cioban**, [49]) *Let X and Y be real separated locally convex spaces, $\Phi : X \times Y \to \overline{\mathbb{R}}$ a proper and convex function, $F : X \times X \to \overline{\mathbb{R}}$ a proper bifunction such that for all $\overline{x} \in \operatorname{dom}\Phi(\cdot,0)$, $F(\overline{x},\overline{x}) = 0$, $\operatorname{dom}\Phi(\cdot,0) \cap \operatorname{dom}F(\overline{x},\cdot)$ and $-F(\cdot,\overline{x})$ is convex and continuous at a point x in $\operatorname{dom}\Phi(\cdot,0) \cap \operatorname{dom}F(\overline{x},\cdot)$. Assume that one of the regularity conditions (RC_i^{Φ}), $i \in \{1,2,3,4,5,6,7\}$ is fulfilled. Then γ^{DPEP} is gap function for the problem $(DPEP)$.*

We can compare now the function γ^{PEP} with the new one introduced, namely γ^{DPEP}.

Proposition 3.2.12 (**L. Cioban**, [49]) *Assume that F is monotone bifunction. Then it holds*

$$\gamma^{DPEP}(x) \leq \gamma^{PEP}(x), \forall x \in X.$$

In what follows we give conditions for which the function γ^{DPEP} becomes gap function for the general equilibrium problem in sense of Stampacchia, (PEP).

Theorem 3.2.13 (**L. Cioban**, [49]) *Let X and Y be real separated locally convex spaces, $\Phi : X \times Y \to \overline{\mathbb{R}}$ a proper and convex function, $\Phi(\cdot, 0)$ l-hemicontinuous, $F : X \times X \to \overline{\mathbb{R}}$ a proper and monotone bifunction such that for all $\overline{x} \in \operatorname{dom} \Phi(\cdot, 0)$, $F(\overline{x}, \overline{x}) = 0$, $F(x, \cdot)$ convex $\forall x \in X$, $\operatorname{dom} \Phi(\cdot, 0) \cap \operatorname{dom} F(\overline{x}, \cdot)$ and $F(\cdot, \overline{x})$ is convex and continuous at a point $x \in \operatorname{dom} \Phi(\cdot, 0) \cap \operatorname{dom} F(\overline{x}, \cdot)$. Assume that one of the regularity conditions (RC_i^Φ), $i \in \{1, 2, 3, 4, 5, 6, 7\}$ is fulfilled. Then γ^{DPEP} is gap function for (PEP).*

Remark 3.2.15 *Notice that the function γ^{PEP} and the function γ^{DPEP} considered in this section, are gap functions for the equilibrium problem (PEP) under appropriate assumptions. In general these functions do not coincide, even if all the conditions in Theorem 3.2.1 and Theorem 3.2.13 are fulfilled.*

Particular cases

In this subsection we particularize problem $(DPEP)$ and we show that we rediscover some gap function for equilibrium problems considered in the literature in [2] and for variational inequalities which are presented in Section 2.2.3.

3.3 Optimality conditions for equilibrium problems

In what follows we characterize the solutions of the general equilibrium problem but also for some particular cases of it by means of the properties of the convex subdifferential in case we are working in the convex setting and if a regularity condition is fulfilled. In case no regularity conditions is fulfilled we give also necessary and sufficient sequential optimality conditions for these solutions.

3.3.1 Optimality conditions for equilibrium problems based on subdifferential calculus

In this section we characterize the solution of the general equilibrium problem by means of the (convex) subdifferential. We state below the announced characterization for (PEP).

Theorem 3.3.1 (**L. Cioban**, [46]) *Let X and Y be real separated locally convex spaces, $\Phi : X \times Y \to \overline{\mathbb{R}}$ a proper and convex function, $F : X \times X \to \overline{\mathbb{R}}$ a proper bifunction such that for all $\overline{x} \in \operatorname{dom} \Phi(\cdot, 0)$, $F(\overline{x}, \overline{x}) = 0$, $\operatorname{dom} \Phi(\cdot, 0) \cap \operatorname{dom} F(\overline{x}, \cdot) \neq \emptyset$ and $F(\overline{x}, \cdot)$ is convex and continuous at a point in $\operatorname{dom} \Phi(\cdot, 0) \cap \operatorname{dom} F(\overline{x}, \cdot)$. Assume that one of the regularity conditions (RC_i^Φ), $i \in \{1, 2, 3, 4, 5, 6, 7\}$ is fulfilled. Then $\overline{x}$ is a solution of the equilibrium problem (PEP) if and only if*

$$0 \in \partial(F(\overline{x}, \cdot))(\overline{x}) + \operatorname{Pr}_{X^*}(\partial\Phi(\overline{x}, 0)).$$

Particular cases

1. *The case* $f + g \circ A$

Theorem 3.3.3 (**L. Cioban**, [46]) *Let* X *and* Y *be real separated locally convex spaces,* $g : Y \to \overline{\mathbb{R}}$, $f : X \to \overline{\mathbb{R}}$ *be proper, convex functions and* $A : X \to Y$ *a linear continuous operator, fulfilling* $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \neq \emptyset$, $F : X \times X \to \overline{\mathbb{R}}$ *a proper bifunction such that for all* $\overline{x} \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$, $F(\overline{x}, \overline{x}) = 0$, $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \cap \operatorname{dom} F(\overline{x}, \cdot) \neq \emptyset$ *and* $F(\overline{x}, \cdot)$ *is convex and continuous at a point in* $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \cap \operatorname{dom} F(\overline{x}, \cdot)$. *Assume that one of the regularity conditions* $(RC_i^{f,g,A})$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$ *is fulfilled. Then* $\overline{x}$ *is a solution of the equilibrium problem* $(EP)^{f,g,A}$ *if and only if*

$$0 \in \partial(F(\overline{x}, \cdot))(\overline{x}) + \partial f(\overline{x}) + A^*\partial g(A\overline{x}).$$

2. *The case* $f + g$

Theorem 3.3.5 (**L. Cioban**, [46]) *Let* X *be real separated locally convex space,* $f, g : X \to \overline{\mathbb{R}}$, *be proper, convex functions fulfilling* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, $F : X \times X \to \overline{\mathbb{R}}$ *a proper bifunction such that for all* $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$, $F(\overline{x}, \overline{x}) = 0$, $\operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} F(\overline{x}, \cdot) \neq \emptyset$ *and* $F(\overline{x}, \cdot)$ *is convex and continuous at a point in* $\operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} F(\overline{x}, \cdot)$. *Assume that one of the regularity conditions* $(RC_i^{f,g})$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$ *is fulfilled. Then* $\overline{x}$ *is a solution of the equilibrium problem* $(EP)^{f,g}$ *if and only if*

$$0 \in \partial(F(\overline{x}, \cdot))(\overline{x}) + \partial f(\overline{x}) + \partial g(\overline{x}).$$

3. *The case* $f + \delta_K$

Theorem 3.3.8 (**L. Cioban**, [46]) *Let* X *be a real separated locally convex space,* $K \subseteq X$ *a nonempty convex set,* $f : X \to \overline{\mathbb{R}}$ *a proper and convex function fulfilling* $\operatorname{dom} f \cap K \neq \emptyset$, $F : X \times X \to \overline{\mathbb{R}}$ *a proper bifunction such that for all* $\overline{x} \in \operatorname{dom} f \cap K$, $F(\overline{x}, \overline{x}) = 0$, $\operatorname{dom} f \cap K \cap \operatorname{dom} F(\overline{x}, \cdot) \neq \emptyset$ *and* $F(\overline{x}, \cdot)$ *is convex and continuous at a point in* $\operatorname{dom} f \cap K \cap \operatorname{dom} F(\overline{x}, \cdot)$. *Assume that one of the regularity conditions* $(RC_i^{f,K})$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$ *is fulfilled. Then* $\overline{x}$ *is a solution of the equilibrium problem* $(EP)^{f,K}$ *if and only if*

$$0 \in \partial(F(\overline{x}, \cdot))(\overline{x}) + \partial f(\overline{x}) + N_K(\overline{x}).$$

Remark 3.3.9 *The above theorem can be obtained by applying Theorem 3.3.1 to the perturbation function* $\Phi_{f,K} : X \times X \to \overline{\mathbb{R}}$ *defined by*

$$\Phi_{f,K}(x, y) = \begin{cases} f(x), & \text{if } x + y \in K \\ +\infty, & \text{otherwise.} \end{cases}$$

3.3.2 Sequential optimality conditions for equilibrium problems

In this section we give characterizations for these solutions with no regularity conditions. We use as tool the sequential optimality conditions given in [28, 29].

Sequential optimality conditions for the general equilibrium problem

The next result uses sequential conditions in order to characterize the solution of the general equilibrium problem (PEP).

Theorem 3.3.11 (**L. Cioban**, [46]) *Let X a reflexive Banach space and Y a Banach space, $\Phi : X\times Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous and $0 \in \mathrm{Pr}_Y(\mathrm{dom}\,\Phi)$, $F : X\times X \to \overline{\mathbb{R}}$ a proper bifunction such that for all $\overline{x} \in \mathrm{dom}\,\Phi(\cdot,0)$, $F(\overline{x},\overline{x}) = 0$, $\mathrm{dom}\,\Phi(\cdot,0)\cap \mathrm{dom}\,F(\overline{x},\cdot) \neq \emptyset$ and $F(\overline{x},\cdot)$ is convex and continuous at a point in $\mathrm{dom}\,\Phi(\cdot,0)\cap\mathrm{dom}\,F(\overline{x},\cdot)$. Then $\overline{x} \in \mathrm{dom}\,\Phi(\cdot,0)\cap \mathrm{dom}\,F(\overline{x},\cdot)$ solves (PEP) if and only if there exists $(x_n, y_n) \in (\mathrm{dom}\,\Phi(\cdot,y)\cap \mathrm{dom}\,F(\overline{x},\cdot))\times \mathrm{dom}\,\Phi(x,\cdot)$, $(x_n^*, y_n^*) \in \partial\Phi(x_n,y_n)$, $z_n^* \in \partial(F(\overline{x},\cdot))(x_n)$, such that*

$$x_n^* + z_n^* \to 0,\ x_n \to \overline{x},\ y_n \to 0, (n\to+\infty),$$

$$\Phi(x_n,y_n) - \langle y_n^*, y_n\rangle - \Phi(\overline{x},0) \to 0\,(n\to+\infty).$$

In Theorem 3.3.11 are given sequential optimality conditions in order to characterize the solution of the general equilibrium problem (PEP) by using continuity of the function $F(\overline{x},\cdot)$. In what follows we give sequential optimality conditions for characterization of the solutions of (PEP) without asking the continuity of F in its second variable.

Theorem 3.3.13 (**L. Cioban**, [47]) *Let X a reflexive Banach space and Y a Banach space, $\Phi : X\times Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous and $0 \in \mathrm{Pr}_Y(\mathrm{dom}\,\Phi)$, $F : X\times X \to \overline{\mathbb{R}}$ a proper bifunction such that for all $\overline{x} \in \mathrm{dom}\,\Phi(\cdot,0)$, $F(\overline{x},\overline{x}) = 0$, $F(\overline{x},\cdot)$ is convex and lower semicontinuous and $\mathrm{dom}\,\Phi(\cdot,0)\cap\mathrm{dom}\,F(\overline{x},\cdot) \neq \emptyset$. Then $\overline{x} \in \mathrm{dom}\,\Phi(\cdot,0)\cap\mathrm{dom}\,F(\overline{x},\cdot)$ solves (PEP) if and only if there exists $(x_n,y_n) \in \mathrm{dom}\,\Phi$, $z_n \in \mathrm{dom}\,F(\overline{x},\cdot)$, $(x_n^*,y_n^*) \in \partial\Phi(x_n,y_n)$, $z_n^* \in \partial(F(\overline{x},\cdot))(z_n)$, such that*

$$\begin{cases} x_n^* + z_n^* \to 0,\ x_n \to \overline{x},\ y_n \to 0, z_n \to \overline{x}\,(n\to+\infty),\\ F(\overline{x},z_n) - \langle z_n^*, z_n - \overline{x}\rangle \to 0\,(n\to+\infty),\\ \Phi(x_n,y_n) - \langle x_n^*, x_n-\overline{x}\rangle - \langle y_n^*, y_n\rangle - \Phi(\overline{x},0) \to 0\,(n\to+\infty).\end{cases}$$

Sequential optimality conditions for $\sum f_i$

Let us consider the following optimization problem:

$$(P^{\Sigma})\quad \inf_{x\in X}\left\{\sum_{i=1}^m f_i(x)\right\}.$$

In what follows we derive from [54, Theorem 3.4] a sequential optimality condition for the problem (P^{Σ}).

Theorem 3.3.15 (**L. Cioban**, [47]) *Let X a reflexive Banach space, $f_i : X \to \overline{\mathbb{R}}$, $i = 1,...,m$, are proper, convex and lower semicontinuous such that $\bigcap_{i=1}^m \mathrm{dom}\,f_i \neq \emptyset$. Then $\overline{x} \in \bigcap_{i=1}^m \mathrm{dom}\,f_i$ solves (P^{Σ}) if and only if $\exists(x_n^1,...,x_n^m) \in \mathrm{dom}\,f_1\times...\times\mathrm{dom}\,f_m$, $\exists(x_n^{1*},...,x_n^{m*}) \in \partial f_1(x_n^1)\times...\times\partial f_m(x_n^m)$, such that*

$$\begin{cases} x_n^{1*} + ... + x_n^{m*} \to 0,\ x_n^i \to \overline{x}, i = 1,...,m\,(n\to+\infty),\\ f_i(x_n^i) - \langle x_n^{i*}, x_n^i - \overline{x}\rangle - f_i(\overline{x}) \to 0\,(n\to+\infty), i = 1,...,m.\end{cases} \tag{3.2}$$

In case $m = 3$ and $f_1(x) = F(\overline{x}, x)$, $f_2 = f(x)$, $f_3 = g(x)$, where $f, g : X \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions, $F : X \times X \to \overline{\mathbb{R}}$ is a proper bifunction such that for all $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$, $F(\overline{x}, \overline{x}) = 0$, $\operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} F(\overline{x}, \cdot) \neq \emptyset$ and $F(\overline{x}, \cdot)$ is convex and lower semicontinuous, we have the following result for the equilibrium problem $(EP)^{f,g}$.

Theorem 3.3.16 (**L. Cioban**, [47]) *Let X a reflexive Banach space, $f, g : X \to \overline{\mathbb{R}}$ proper, convex and lower semicontinuous functions, $F : X \times X \to \overline{\mathbb{R}}$ a proper bifunction such that for all $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$, $F(\overline{x}, \overline{x}) = 0$, $\operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} F(\overline{x}, \cdot) \neq \emptyset$ and $F(\overline{x}, \cdot)$ is convex and lower semicontinuous. Then $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} F(\overline{x}, \cdot)$ solves $(EP)^{f,g}$ if and only if there exists $(x_n, y_n, z_n) \in \operatorname{dom} F(\overline{x}, \cdot) \times \operatorname{dom} f \times \operatorname{dom} g$, $(x_n^*, y_n^*, z_n^*) \in \partial(F(\overline{x}, \cdot))(x_n) \times \partial f(y_n) \times \partial g(z_n)$, such that*

$$\begin{cases} x_n^* + y_n^* + z_n^* \to 0,\ x_n \to \overline{x},\ y_n \to \overline{x},\ z_n \to \overline{x},\ (n \to +\infty), \\ F(\overline{x}, x_n) - \langle x_n^*, x_n - \overline{x} \rangle \to 0\,(n \to +\infty), \\ f(y_n) - \langle y_n^*, y_n - \overline{x} \rangle - f(\overline{x}) \to 0\,(n \to +\infty), \\ g(z_n) - \langle z_n^*, z_n - \overline{x} \rangle - g(\overline{x}) \to 0\,(n \to +\infty). \end{cases}$$

Theorem 3.3.18 (**L. Cioban**, [47]) *Let X a reflexive Banach space, $f : X \to \overline{\mathbb{R}}$ proper, convex and lower semicontinuous function, K a closed and convex subset of X, $F : X \times X \to \overline{\mathbb{R}}$ a proper bifunction such that for all $\overline{x} \in \operatorname{dom} f \cap K$, $F(\overline{x}, \overline{x}) = 0$, $\operatorname{dom} f \cap K \cap \operatorname{dom} F(\overline{x}, \cdot) \neq \emptyset$ and $F(\overline{x}, \cdot)$ is convex and lower semicontinuous, $\overline{x} \in \operatorname{dom} f \cap K \cap \operatorname{dom} F(\overline{x}, \cdot)$ solves $(EP)^{f,K}$ if and only if there exists $(x_n, y_n, z_n) \in \operatorname{dom} F(\overline{x}, \cdot) \times \operatorname{dom} f \times K$, $(x_n^*, y_n^*, z_n^*) \in \partial(F(\overline{x}, \cdot))(x_n) \times \partial f(y_n) \times N_K(z_n)$, such that*

$$\begin{cases} x_n^* + y_n^* + z_n^* \to 0,\ x_n \to \overline{x},\ y_n \to \overline{x},\ z_n \to \overline{x}\,(n \to +\infty), \\ F(\overline{x}, x_n) - \langle x_n^*, x_n - \overline{x} \rangle - F(\overline{x}, \overline{x}) \to 0\,(n \to +\infty), \\ f(y_n) - \langle y_n^*, y_n - \overline{x} \rangle - f(\overline{x}) \to 0\,(n \to +\infty), \\ \langle z_n^*, z_n - \overline{x} \rangle \to 0\,(n \to +\infty). \end{cases}$$

In the following we give an example in order to show the applicability of sequential characterization.

Example 3.3.19 (**L. Cioban**, [47]) *Let $X = \mathbb{R}^2$, $K = -\mathbb{R}^2_+$, $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$,*
$f(x) = \begin{cases} x^2 - \sqrt{y}, & y \geq 0 \\ +\infty, & otherwise, \end{cases}$ $F : \mathbb{R}^2 \times R^2 \to \mathbb{R}^2$, $F((\overline{x}, \overline{y}), (x, y)) = \langle (\overline{x}, \overline{y}), (x, y) - (\overline{x}, \overline{y}) \rangle$. *In this case we prove that all the condition in Theorem 3.3.18 are fulfilled.*

If we particularize the function F, i.e. $F(\overline{x}, x) = \langle G(\overline{x}), x - \overline{x} \rangle$ for all the results presented in this section, we rediscover all the results considered for variational inequalities in Section 2.3.

Chapter 4

Optimization problems and $(0, 2)$ η-approximated optimization problems

In this chapter, we attach to the initial optimization problem an approximate optimization problem which is constructed by a second order η-approximation of the constraint functions at an arbitrary but fixed feasible point x and which is called the $(0, 2)$ η-approximated optimization problem. In order to prove the equivalence between the original optimization problem and its associated optimization problem we use second order invexity. In general, the approximated optimization problem is less complicated than the original problem. We will study the connections between the feasible solutions of the η-approximated problem and the feasible solutions of the original problem. Then we will study the connections between the optimal solutions of the approximated optimization problem and the optimal solutions of original optimization problem via the saddle points of associated Lagrangian functions of the two problems. Then, we attach to the original optimization problem its dual, and we prove that, in appropriate hypothesis, if the dual optimization problem is solvable, then the $(0, 2)$ η-approximated optimization problem is also solvable, and vice versa.

4.1 Introduction and preliminaries

We consider the optimization problem

$$\begin{array}{lll} (P) & \min & f(x) \\ & \text{s.t.} & x \in X \\ & & g(x) \leqq 0 \\ & & h(x) = 0, \end{array}$$

where X is a subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ and $g = (g_1, ..., g_m) : X \to \mathbb{R}^m$ and $h = (h_1, ..., h_q) : X \to \mathbb{R}^q$ are three functions, $m, n, q \in \mathbb{N}$.

Let

$$\mathcal{F}(P) := \{x \in X : g(x) \leqq 0,\ h(x) = 0\}$$

denote the set of all feasible solutions of Problem (P), and let $L_P : X \times (\mathbb{R}^m \times \mathbb{R}^q) \to \mathbb{R}$ defined by

$$L_P(x,(v,w)) = f(x) + \langle v, g(x)\rangle + \langle w, h(x)\rangle,$$

for all $(x,(v,w)) \in X \times (\mathbb{R}^m \times \mathbb{R}^q)$, denote the lagrangian of Problem (P).

A point $(x^0,(v^0,w^0)) \in X \times (\mathbb{R}^m_+ \times \mathbb{R}^q)$ is a saddle point of langrangian L_P if

$$L_P\left(x^0,(v,w)\right) \leq L_P\left(x^0,(v^0,w^0)\right) \leq L_P\left(x,(v^0,w^0)\right),$$

for all $(x,(v,w)) \in X \times (\mathbb{R}^m_+ \times \mathbb{R}^q)$.

We define the functions $G : X \to \mathbb{R}^m$, $H : X \to \mathbb{R}^q$ by

$$\begin{aligned} G(x) &:= g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\eta\left(x,x^0\right) + \frac{1}{2}\left[\eta\left(x,x^0\right)\right]^T\left[\nabla^2 g\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right), \qquad (4.1)\\ H(x) &:= h\left(x^0\right) + \left[\nabla h\left(x^0\right)\right]\eta\left(x,x^0\right) + \frac{1}{2}\left[\eta\left(x,x^0\right)\right]^T\left[\nabla^2 h\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right), \end{aligned}$$

for all $x \in X$.

Let $\eta : X \times X \to \mathbb{R}^n$ be a function, x^0 be an interior point of X. Assume that g and h are twice differentiable at x^0.

In what follows, we attach to Problem (P), the problem

$$\begin{array}{lll} (AP) & \min & f(x) \\ & s.t. & x \in X \\ & & G(x) \leqq 0 \\ & & H(x) = 0, \end{array}$$

called the $(0,2)$ $\eta-$approximated optimization problem and we study the connections beetween optimal solutions of this Problem and the optimal solutions of Problem (P). This problem depends not only on X, f, and g, but also on x^0 and η.

Let

$$\mathcal{F}(AP) := \{x \in X : \ G(x) \leqq 0, \ H(x) = 0\}$$

denote the set of all feasible solutions of Problem (AP).

We recall some definitions and notions used below which are very well synthesized in [110] like invexity, incavity, second order invexity, second order incavity, second order quasiinvexity at a point with respect to a function.

Definition 4.1.3 (**L. Cioban**, D. Duca, [52]) *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $f : X \to \mathbb{R}$ be a differentiable function at x^0, and $\eta : X \times X \to \mathbb{R}^n$ be a function. We say that the function f is avex at x^0 w.r.t. η if*

$$f(x) - f\left(x^0\right) = \left\langle \nabla f\left(x^0\right), \eta\left(x,x^0\right)\right\rangle, \ \textit{for all } x \in X. \qquad (4.2)$$

In what follows we give some examples to illustrate the above notions.

Example 4.1.6 (**L. Cioban**, D. Duca, [52]) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$f(x_1,x_2) = x_1^2 + \sin\frac{\pi x_2}{2}, \ \textit{for all } (x_1,x_2) \in \mathbb{R}^2.$$

The function f is second order invex at $x^0 = (0,0)$ w.r.t. $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\eta\left(\left(x_1, x_2\right), \left(u_1, u_2\right)\right) = \left(x_1, \frac{2}{\pi} \sin \frac{\pi x_2}{2} - x_1^2 - x_2^2\right),$$

for all $\left(\left(x_1, x_2\right), \left(u_1, u_2\right)\right) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Definition 4.1.8 (**L. Cioban**, D. Duca, [52]) *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $f : X \to \mathbb{R}$ be a twice differentiable function at x^0, and $\eta : X \times X \to \mathbb{R}^n$ be a function. We say that the function f is second order avex at x^0 w.r.t. η if*

$$f(x) - f\left(x^0\right) = \left\langle \nabla f\left(x^0\right), \eta\left(x, x^0\right)\right\rangle + \frac{1}{2}\left[\eta\left(x, x^0\right)\right]^T \left[\nabla^2 f\left(x^0\right)\right](y), \tag{4.3}$$

for all $x \in X$ and $y \in \mathbb{R}^n$.

4.2 Connections between the feasible solutions of (0,2) η-approximated optimization problem and the feasible solutions of the original problem

We can notice that the sets of feasible solutions of the original problem and its approximated problem are not related as some examples show. But if the functions g and h fulfil certain conditions, then we can establish connections between the feasible solutions of (0,2) $\eta-$approximated optimization problem and the feasible solutions of the original problem.

Theorem 4.2.3 (**L. Cioban**, D. Duca, [52]) *Let X be a subset of $\mathbb{R}^q$, x^0 be an interior point of X, $\eta : X \times X \to \mathbb{R}^n$, $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^q$ are three functions. Assume that:*

(i) g is twice differentiable at x^0 and second order incave at x^0 w.r.t. η;

(ii) h is twice differentiable at x^0 and second order avex at x^0 w.r.t. η.

Then any feasible solution for Problem (AP) is also feasible for Problem (P), that is

$$\mathcal{F}(AP) \subseteq \mathcal{F}(P).$$

Theorem 4.2.4 (**L. Cioban**, D. Duca, [52]) *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to \mathbb{R}^n$, $g : X \to \mathbb{R}^m$, $h : X \to \mathbb{R}^q$ are three functions. Assume that:*

(i) g is twice differentiable at x^0 and second order invex at x^0 w.r.t. η;

(ii) h is twice differentiable at x^0 and second order avex at x^0 w.r.t. η.

Then any feasible solution for Problem (P) is also feasible for Problem (AP), that is

$$\mathcal{F}(P) \subseteq \mathcal{F}(AP).$$

4.3 Connections between optimal solutions of (0,2) η-approximated problem and optimal solutions of the original problem

The lagrangian of Problem (AP) will be denoted by L_{AP}, i.e. $L_{AP} : X \times \mathbb{R}^m_+ \times \mathbb{R}^q \to \mathbb{R}$ is defined by $L_{AP}(x,(v,w)) := f(x) + \langle v, g(x^0)\rangle + \left\langle \eta(x,x^0), [\nabla g(x^0)]^T(v)\right\rangle + \frac{1}{2}\left\langle v, [\eta(x,x^0)]^T [\nabla^2 g(x^0)](\eta(x,x^0))\right\rangle + \langle w, h(x^0)\rangle + \left\langle \eta(x,x^0), [\nabla h(x^0)]^T(w)\right\rangle + \frac{1}{2}\left\langle w, [\eta(x,x^0)]^T [\nabla^2 h(x^0)](\eta(x,x^0))\right\rangle$, for all $(x,(v,w)) \in X \times \left(\mathbb{R}^m_+ \times \mathbb{R}^q\right)$.

Theorem 4.3.3 (**L. Cioban**, D. Duca, [52]) *Let X be a subset of $\mathbb{R}^n$ and x^0 an interior point of X, $\eta : X \times X \to \mathbb{R}^n$, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$, $h : X \to \mathbb{R}^q$ are four functions. Assume that:*

(i) $\eta(x^0, x^0) = 0$;

(ii) *g is twice differentiable at x^0 and g_i, $i \in I(x^0)$ are second order quasiinvex at x^0;*

(iii) *h is twice differentiable at x^0 and h is second order avex at x^0 w.r.t. η.*

If $(x^0,(v^0,w^0)) \in X \times \left(\mathbb{R}^m_+ \times \mathbb{R}^q\right)$ is a saddle point of the lagrangian L_{AP} of Problem (AP), then x^0 is an optimal solution of Problem (P).

In what follows we consider an example in order to justify the above theorem. We consider an optimization problem involving second order invex functions with respect to the same function η, which is not linear with respect to the first component. A similar example was given by T. Antczak, in [7].

Example 4.3.4 (**L. Cioban**, D. Duca, [52]) *Consider the following nonlinear mathematical programming problem*

$$(P_1) \qquad \min \quad \begin{aligned} & f(x) = \tfrac{1}{2}\arctan^2 x + \arctan x \\ & g(x) = (1+x^4)(\arctan^2 x - \arctan x) \le 0 \\ & h(x) = 0, \end{aligned}$$

where $f, g, h : \mathbb{R} \to \mathbb{R}$. Note that $\mathcal{F}(P_1) = \{x \in \mathbb{R}, 0 \le x \le \frac{\pi}{4}\}$, and $\overline{x} = 0$ is optimal solution for (P_1). We consider $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\eta(x,\overline{x}) = \frac{1}{2}(\arctan x - \arctan \overline{x})$. We show that we can apply the above theorem.

Theorem 4.3.5 (**L. Cioban**, D. Duca, [52]) *Let X be a subset of $\mathbb{R}^n$ and x^0 an interior point of X, $\eta : X \times X \to \mathbb{R}^n$, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$, $h : X \to \mathbb{R}^q$ are four functions. Assume that:*

(i) $\eta(x^0, x^0) = 0$;

(ii) *f is invex at x^0 w.r.t. η;*

(iii) *g is twice differentiable at x^0 and g_i, $i \in I(x^0)$ are second order quasiinvex at x^0 w.r.t. η;*

(iv) *h is twice differentiable at x^0 and h is second order avex at x^0 w.r.t. η;*

(v) *a suitable constraint qualification for Problem (P) satisfied at x^0;*

(*vi*) $[\eta(x,x^0)]^T [\nabla^2 g(x^0)] (\eta(x,x^0)) + [\eta(x,x^0)]^T [\nabla^2 h(x^0)] (\eta(x,x^0)) \geqq 0$, *for all* $x \in X$.

If $x^0 \in X$ *is an optimal solution of Problem* (P) *then there exists a point* $(v^0, w^0) \in \mathbb{R}^m_+ \times \mathbb{R}^q$ *such that* $(x^0, (v^0, w^0)) \in X \times (\mathbb{R}^m_+ \times \mathbb{R}^q)$ *is a saddle point of the lagrangian* L_{AP} *of Problem* (AP).

Theorem 4.3.6 (**L. Cioban**, D. Duca, [52]) *Let* X *be a subset of* $\mathbb{R}^n$ *and* x^0 *be an interior point of* X, $\eta : X \times X \to \mathbb{R}^n$, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$, $h : X \to \mathbb{R}^q$ *are four functions. Assume that:*

(*i*) $\eta(x^0, x^0) = 0$;

(*ii*) f *differentiable at* x^0 *and invex at* x^0 *w.r.t.* η;

(*iii*) g *is twice differentiable at* x^0 *and* g_i, $i \in I(x^0)$ *are second order quasiinvex at* x^0 *w.r.t.* η;

(*iv*) h *is twice differentiable at* x^0 *and* h *is second order avex at* x^0 *w.r.t.* η;

(*v*) *a suitable constraint qualification for Problem* (P) *satisfied at* x^0;

(*vi*) $[\eta(x,x^0)]^T [\nabla^2 g(x^0)] (\eta(x,x^0)) + [\eta(x,x^0)]^T [\nabla^2 h(x^0)] (\eta(x,x^0)) \geqq 0$, *for all* $x \in X$.

If $x^0 \in X$ *is an optimal solution of Problem* (P) *then* x^0 *is an optimal solution for Problem* (AP).

Theorem 4.3.7 (**L. Cioban**, D. Duca, [52]) *Let* X *be a subset of* $\mathbb{R}^n$ *and* x^0 *be an interior point of* X, η, $\mu : X \times X \to \mathbb{R}^n$, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$, $h : X \to \mathbb{R}^q$, *are five functions,* f *is differentiable at* x^0, g, h *are twice differentiable at* x^0, $G : X \to \mathbb{R}^m$, $H : X \to \mathbb{R}^q$ *deffined by* (4.1). *Assume that:*

(*i*) $\eta(\cdot, x^0) : X \to \mathbb{R}^n$ *is differentiable at* x^0 *and* $\eta(x^0, x^0) = 0$;

(*ii*) g_i, $i \in I(x^0)$ *are second order quasiinvex at* x^0 *w.r.t.* η;

(*iii*) h *is second order avex at* x^0 *w.r.t.* μ;

(*iv*) f, G *are invex at* x^0 *w.r.t.* μ;

(*v*) H *is avex at* x^0 *w.r.t.* μ;

(*vi*) *a suitable constraint qualification for Problem* (AP) *is satisfied at* x^0.

If $x^0 \in X$ *is an optimal solution for Problem* (AP) *then* x^0 *is an optimal solution of Problem* (P).

4.4 Duality

We attach to problem (P) the dual optimization problem

$$
\begin{array}{lll}
(D) & \max & f(x) + \langle v, g(x) \rangle + \langle w, h(x) \rangle \\
 & \text{s. t.} & (x, v, w) \in X \times \mathbb{R}^m \times \mathbb{R}^q \\
 & & -v \leqq 0 \\
 & & \nabla f(x) + [\nabla g(x)]^T (v) + [\nabla h(x)]^T (w) = 0,
\end{array}
$$

where X is an open subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$, $g = (g_1, ..., g_m) : X \to \mathbb{R}^m$ and $h = (h_1, ..., h_q) : X \to \mathbb{R}^q$ are three differentiable functions.

We denote by $\mathcal{F}(D) := \{(x, v, w) \in X \times \mathbb{R}^m \times \mathbb{R}^q : -v \leqq 0, \nabla f(x^0) + [\nabla g(x)]^T (v) + [\nabla h(x)]^T (w) = 0\}$, the set of all feasible solutions of Problems (D).

Theorem 4.4.3 *Let $X \subseteq \mathbb{R}^n$ be a nonempty, open and convex set, x^0 an interior point of X, $\eta : X \times X \to \mathbb{R}^n$, $\mu : X \times X \to \mathbb{R}^q$, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m, h : X \to \mathbb{R}^q$ are functions. Assume that:*

(i) $\eta(x^0, x^0) = 0$;

(ii) *g is twice differentiable at x^0 and g_i, $i \in I(x^0)$ are second order quasiinvex function at x^0 w.r.t. η;*

(iii) *h is twice differentiable at x^0 and h_j are second order avex function at x^0 w.r.t. η;*

(iv) *a suitable constraint qualification for problem (P) satisfied at x^0;*

(v) $[\eta(x, x^0)]^T [\nabla^2 g(x^0)] (\eta(x, x^0)) + [\eta(x, x^0)]^T [\nabla^2 h(x^0)] (\eta(x, x^0)) \geqq 0$, *for all $x \in X$.*

If (x^0, v^0, w^0) is an optimal solution for Problem (D) and there exists a neighborhood $V \times W$ of (v^0, w^0) and a function $\gamma : V \times W \to X$, differentiable on (v^0, w^0) such that $\gamma(v^0, w^0) = x^0$ and $\nabla f(\gamma(v, w)) + [\nabla g((v, w))](v^0) + [\nabla h((v, w))](w^0) = 0$, for all $(v, w) \in \mathcal{F}(D)$, then x^0 is an optimal solution of Problem (AP).

Theorem 4.4.4 *Let $X \subseteq \mathbb{R}^n$ be a nonempty and open set, x^0 an interior point of X, $\eta : X \times X \to \mathbb{R}^n$, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$, $h : X \to \mathbb{R}^q$ three functions. Assume that:*

(i) *η is differentiable at x^0 and $\eta(x^0, x^0) = 0$;*

(ii) *g is twice differentiable at x^0 and g_i, $i \in I(x^0)$ are second order quasiinvex function at x^0 w.r.t. μ;*

(iii) *h is twice differentiable at x^0 and h_j are second order avex function at x^0 w.r.t. μ;*

(iv) *f is invex at x^0 w.r.t. μ;*

(v) *a suitable constraint qualification for (AP) is satisfied at x^0.*

If x^0 is an optimal solution of Problem (AP) and Problem (P) satisfies Kukn-Tucker conditions at x^0, then there exists a point (v^0, w^0) such that (x^0, v^0, w^0) is an optimal solution of Problem (D).

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