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Contents

Introduction **5**

1 Preliminaries **11**
1.1 Brief Background Concerning Multicriteria Optimization Problems 11
1.2 Brief Background Concerning Lexicographic Optimization Problems 11
1.3 Brief Background Concerning Multilevel Optimization Problems 12

2 Lexicographic Multicriteria Bottleneck Problems and Optimal Points with Pipeline Property **14**
2.1 Lexicographic Multicriteria Bottleneck Problems 14
2.2 Optimal Solutions with Pipeline Property 15
2.3 Lexicographic Discrete Bottleneck Problems 17
2.4 A Method to Determine an Optimal Solution with Pipeline Property for (LDpBP) Problem 19

3 An Application Related to Firm's Costs Management **21**
3.1 The Milk Collection Problem 21
3.2 Generalization of the Mathematical Model for the Milk Collection Problem 22

4 Applications of Multilevel Optimization with Respect to Professional Training Programs **25**
4.1 Brief Background Concerning Assignment Problems 25
4.2 The Concrete Economic Problems 26
4.3 The Study of the Problem (AEP_1) 27
4.3.1 Mathematical Modeling of the Problem (AEP_1) 27
4.3.2 Necessary and Sufficient Optimality Conditions of (PS) 28
4.3.3 A Technique for Solving the Problem (PM) 31
4.4 The Study of the Problem (AEP_2) 31

5 Practical Applications Related to Portfolio Optimization **34**
5.1 A Relation between Portfolio Selection Problems and Bicriteria Optimization . . . 35
5.1.1 Basic Notions Related to Portfolio Theory 35
5.1.2 Portfolio Selection Models of Markowitz Type 35
5.1.3 Portfolio Selection Problem versus Bicriteria Optimization 35
5.1.4 A Kind of Boolean Portfolio Selection Problem 36
5.2 A Kind of Portfolio Selection Problem 36

5.2.1 Formulation of the Economic Problem 37
5.2.2 Modeling and Solving the Portfolio Selection Problem 37
5.2.3 Modeling and Solving the Extended Portfolio Selection Problem 39

6 Applications of Multilevel Optimization in the Technology Transfer Area 44
6.1 Brief Background Concerning Technology Transfer 45
6.2 Basic Framework of the Studied Economic Problem 45
6.3 Benchmark Case: No-licensing Case 46
6.3.1 Modeling and Solving the Economic Problem 46
6.3.2 Benchmark: No-licensing Case for One Differentiated Product 49
6.4 Per-unit royalty Licensing Case 50
6.4.1 Modeling and Solving the Economic Problem 50
6.4.2 Per-unit royalty Licensing Case for One Differentiated Product 52
6.5 Fixed-fee Licensing Case for One Differentiated Product 52
6.6 Two-part tariff Licensing Case for One Differentiated Product 53

References 55

Introduction

"Life is about decisions. Decisions, no matter if made by a group or an individual, usually involve several conflicting objectives. The observation that real world problems have to be solved optimally according to criteria, which prohibit an "ideal" solution - optimal for each decision-maker under each of the criteria considered - has led to the development of multicriteria optimization." (EHRGOTT M. [27])

Operations research, often considered to be a sub-field of mathematics, is a discipline that deals with the application of advanced analytical methods in helping to make better decisions. It leads at the optimal solutions or near optimal solutions to complex decision-making problems. It is often concerned with determining the maximum (of profit, performance or yield) or minimum (of loss, risk or cost) of some real-world objective.

The scientific results within the present thesis introduce some particular types of optimization problems generated by concrete economic problems. In this way we highlight the well-deserved high rank of optimization theory among mathematical areas due to its countless applications in practical areas. For each such a problem we give a method or an algorithm that can be used for solving it and some necessary and sufficient conditions for the optimal solutions of each problem.

As the title of the thesis suggests, we study different types of particular multicriteria and multilevel optimization problems, the bond being represented by the discrete point of view. All types of the studied problems are based on concrete applications that can be found in real life situations.

The opportunity of studying such an exciting research field represents a real privilege. The present thesis contains the author's own results, obtained alone or in joint works, addressing concrete economic problems from different economic fields, such as: costs management area, portfolio theory area, technology transfer area and assignment of the unemployed persons to professional training programs area.

In what follows, we give a description of how is the present thesis organized: the entire content is split into six chapters, followed by the Bibliography.

Chapter 1, entitled *Preliminaries*, contains a brief background concerning multicriteria, lexicographic and multilevel optimization problems. Also, we point out the case in which in a multilevel optimization problem the coefficients depend on one or more parameters. A problem of this kind, obtained by mathematically modeling a practical problem, is given in Chapter 6 of the present thesis.

Chapter 2, entitled *Lexicographic Multicriteria Bottleneck Problems and Optimal Points with Pipeline Property*, contains original results of scientific research belonging to the author of this thesis and can be found in papers [76], [118], [114] and [115]. We begin our exposure explaining what we understand by lexicographic multicriteria bottleneck problems with p bottleneck objective functions (LpBP). Furthermore, in Lemma 2.1.2 and Theorem 2.1.3 we study the structure of the set of optimal solutions of this type of problem. In Section 2.2 we introduce the notion of optimal solution with pipeline property for the lexicographic bottleneck problems and we discuss some aspects about the set of all optimal solutions with pipeline property. We introduce the notion of minimum point with pipeline property of a function on a set (Definition 2.2.1). We note that within this definition the function f does not appear, but Theorem 2.2.2 justifies the use of the term "minimum". An example of minimum points which does not have the pipeline property is given in Example 2.2.3. In Propositions 2.2.4 and 2.2.5 we give two different methods that can be used to verify that a point has the pipeline property. Some properties of the set of all minimum points with pipeline property are given in Theorem 2.2.2, Propositions 2.2.7, 2.2.8 and 2.2.9. In Section 2.3 we consider the lexicographic bottleneck problems in which the set of all feasible solutions is discrete. The structure of the set of optimal solutions (Theorems 2.3.6 and 2.3.7) and the structure of the set of optimal solutions with pipeline property for this particular type of problems are discussed (Proposition 2.3.8, Corollary 2.3.9, and Propositions 2.3.10 and 2.3.11). Section 2.4 contains a method (based on Theorem 2.4.2) to determine an optimal solution with pipeline property for lexicographic discrete bottleneck problems. This method is a type of weighted methods. The novelty is that the weighted type introduced by us allows the direct getting of the optimal points with pipeline property. It generalizes both the method used by Bandopadhyaya L. [6] and the method given by Zarepisheh M. and Khorram E. [135].

In **Chapter 3**, entitled *An Application Related to Firm's Costs Management*, based on a concrete problem concerning the planning of how to collect and transport the milk by a dairy products manufacturing company under the restrictions of minimizing the quantity of stored milk and the transport costs, and taking into account some other given requirements, we study a special type of bilevel optimization problem in which the set of feasible solutions is the set of subgraphs of a given graph. These subgraphs fulfill some given restrictions. We note that the studied problem mathematically models a concrete costs management problem and it can be considered a type of traveling salesman problem. The novelty of this problem is given, on one hand, by the mathematical model which we introduce and, on the other hand, by the fact that we use the splitting technique which allows us to reduce the solving of this problem (see Lemmas 3.2.1, 3.2.2, 3.2.3 and Theorems 3.2.4 and 3.2.5) to solving three problems: (PP), $(P_1(H_0))$, $(P_2(H_0))$, where H_0 is an optimal solution of the first problem. The problem $(P_1(H_0))$ is a classic problem of determining the minimum of a given function on a given set of subgraphs of a graph. The problem $(P_2(H_0))$ is a lexicographic bicriteria optimization problem. By introducing Theorem 3.2.6 and by using the method presented in Section 2.4 we reduce the solving of this last problem to solving again a classic problem of determining the minimum of a given function on a given set of subgraphs of a graph. The author's achievements within this chapter can be found in paper authored by Goina D. and Tuns (Bode) O.R. [43]. It completes the results obtained by Ruuska S., Miettinen K. and Wiecek M.M. [101].

Chapter 4, entitled *Applications of Multilevel Optimization with Respect to Professional Training Programs*, deals with the study, from the optimization point of view, of some concrete economic problems involving assigning unemployed persons to professional training programs. We begin by justifying the importance of the professional training programs for the unemployed persons. In Section 4.2 we formulate the two studied economic problems. We note that both problems represent new types of generalizations of classic assignment problems. Therefore, these problems complete the results obtained by PENTICO D.W. [92]. We continue by analysing in Section 4.3 the first economic problem, denoted by (AEP_1). We mathematically model it in Subsection 4.3.1 and then, in Subsection 4.3.2 we give some necessary and sufficient optimality conditions (Propositions 4.3.3, 4.3.5, 4.3.6 and 4.3.7 and Theorem 4.3.14). Based on Propositions 4.3.3 and 4.3.5 in Subsection 4.3.3 we give a polynomial technique for solving the problem (PM).

In Section 4.4 we study the second problem, denoted by (AEP_2). In Proposition 4.4.1 it is given a necessary and sufficient condition such that a feasible solution is optimal. Then, we present the way in which the above proposed technique can be used to solve this problem. The technique we introduce is more efficient than the one given by DELLA CROCE F., PASCHOS V.TH. and TSOUKIAS A. [24]. The author's achievements within this area of research can be found in papers [116] and [119].

In **Chapter 5**, entitled *Practical Applications Related to Portfolio Optimization*, we turn our attention to the economic-financial problems related to portfolio theory area.

In Section 5.1, after giving a brief background concerning modern portfolio theory in Subsection 5.1.1, we emphasize in Subsection 5.1.2 the most known portfolio selection models, i.e. the Markowitz's portfolio selection models. Furthermore, in Subsection 5.1.3 we introduce a relation between the portfolio selection models of Markowitz's type and the bicriteria optimization. By means of this relation, we give a new approach for the portfolio selection problem (Propositions 5.1.2 and 5.1.4). Using the results within this paragraph, in Subsection 5.1.4 we analyse a particular type of portfolio selection problem. The mathematical model attached to this type of problem is a fractional pseudo boolean optimization problem. The scientific results within Subsections 5.1.3 and 5.1.4 belong to the author and can be found in paper [110]. In Section 5.2 we begin by presenting two different problems which we want to study. We formulate in Subsection 5.2.1 the particular type of portfolio selection problem which we mathematically model and solve in Subsection 5.2.2. This mathematical model represents a bilevel assignment optimization problem of cost type. Based on the restrictions of this problem we use the splitting technique in order to solve it (see Theorems 5.2.1, 5.2.2 and Corollary 5.2.5). In subsection 5.2.3 we extend the economic problem. The mathematical model attached to this economic problem is a bilevel optimization problem for which the lower level function is bicriteria of cost-bottleneck type. As far as we know, this kind of bilevel optimization problem is not discussed in literature. For solving this problem we use both the splitting technique and the technique introduced in Section 2.4 (see Theorems 5.2.9, 5.2.10, 5.2.12, 5.2.13 and 5.2.14). An algorithm that can be used to solve this problem and an example to emphasize how this algorithm works are given.

Chapter 6, entitled *Applications of Multilevel Optimization in the Technology Transfer Area*, is devoted to the study of the economic problems related to technology transfer area. We begin our exposure by giving in Section 6.1 a brief background concerning technology transfer, then we continue with the formulation in Section 6.2 of our concrete economic problem: we consider a n differentiated Stackelberg model, when the leader firm engages in an research and development process that gives an endogenous cost-reducing innovation. Our goal is to study, in sections that follow, the no-licensing case and the licensing of the cost-reduction innovation, i.e. the patent licensing contracts cases (per-unit royalty licensing case, fixed-fee licensing case and two-part tariff licensing case) when the patentee is an insider.

In Section 6.3 we attach the mathematical model to the concrete economic problem in the benchmark case. The novelty consists in the fact that this mathematical model is a three-level parametric optimization problem with two parameters within the objective functions. This allows us to solve the mathematical problem by using the variables of the upper level problems as parameters in the lower level problems (Propositions 6.3.2, 6.3.4, 6.3.6, 6.3.8). We determine the feasibility domain of the parameter which represents the degree of the differentiation of goods in Proposition 6.3.6, Remarks 6.3.7 and 6.3.9. We note that for the particular case when $n = 1$ and both parameters belong to interval $]0, 1[$, the optimal solution of the mathematical problem coincides with the optimal solution of the economic problem that can be found in the next subsection. More, the result that the absolute value of the parameter which represents the degree of the differentiation of goods cannot exceed 1 has an important economic significance. The result justifies the condition that this parameter belongs to interval $]0, 1[$, which is frequently used in the economic literature. In Subsection 6.3.2 we recall, for the particular case when $n = 1$, the economic problem formulated in Section 6.2. Within this paragraph we determine the optimal value in this case of study for some other variables which have an important economic significance, such as: the profits of both firms (leader and follower), the consumer surplus and the social welfare. We remark that we denote all these variables by using the specific economic notations. We evaluate the effects of the degree of the differentiation of goods over all these variables and also over the optimal innovation size and optimal outputs of both firms (Theorem 6.3.11 and Remark 6.3.12). As the main novelty of these results we note the fact that the mathematical solutions for the values of the optimal innovation size and optimal outputs for both firms coincide with the values that we get strictly from the economic point of view when $n = 1$. In Section 6.4 we attach the mathematical model to the concrete economic problem in the per-unit royalty licensing case. The novelty consists in the fact that this mathematical model is a four-level parametric optimization problem with two parameters within the objective functions. Again we solve the mathematical problem by using the variables of the upper level problems as parameters in the lower level problems (Proposition 6.4.2). In Subsection 6.4.2 we recall, for the particular case when $n = 1$, the economic problem formulated in Section 6.2. As in Subsection 6.3.2, we determine the optimal value for this case of study for the innovation size, the output and the profit of both firms, the consumer surplus and the social welfare. Again, we evaluate the effects of the degree of the differentiation of goods over all these variables (Theorem 6.4.3 and Remark 6.4.4). Also, we note that for this case of study the mathematical solutions for the values of the optimal innovation size and optimal outputs for both firms coincide with the values that we get strictly from the economic point of view when $n = 1$. In Section 6.5, respectively 6.6, we solve the economic problem formulated in Section 6.2 for the particular case when $n = 1$ and when the technology license occurs by means of a fixed-fee, respectively by means of a two-part tariff. By means of Remark 6.5.1 and Theorem 6.5.2, respectively Theorem 6.6.2 and Theorem 6.6.3, we give the economic interpretation of the mathematical results that follows in this case of study. The personal contribution of the author in this area may be described by means of the following classification:

• the analysis and comparison of the pre-licensing case and of some possible cases of the licensing contract (licensing by means of a per-unit royalty, licensing by means of a fixed-fee and licensing by means of a two-part tariff) in a differentiated-good Stackelberg duopoly, where one of the

firms invests in research and development in order to get a cost-reducing innovation. Based on the identity of the patentee, we analyse the case when the leader firm is the innovator (i.e. only the leader firm engages in process innovation) [32], [33], [113] and [117];
• the analysis of the licensing by means of a per-unit royalty and the licensing by means of a fixed-fee, in the Cournot and Bertrand models; the comparison between these results and the ones obtained by Li C. and Ji X. [68], for Cournot and Bertrand duopolies [11].
• the mathematical modeling by using the multilevel parametric optimization problems of the benchmark case and per-unit royalty case in the n differentiated Stackelberg duopoly. In this way, we get some mathematical explanations for the economic restrictions.

We note that all the studied problems within the present thesis points out some new types of discrete optimization problems which have not been studied so far.

The results of this thesis are included in 19 papers, individual or in joint works (see [11], [32], [33], [43], [48], [49], [50], [75], [76], [110], [111], [112], [113], [118], [114], [115], [116], [117], [119]). 10 of these papers have been published, 6 of these papers have been accepted for publication and 3 of them are submitted for publication.

Keywords: mathematical modeling, multicriteria optimization, lexicographic optimization, multilevel optimization, bilevel optimization on a graph, multilevel parametric optimization, lexicographic multicriteria bottleneck problem, minimum lexicographic points with pipeline property, lexicographic discrete optimization problem with bottleneck objective functions, assignment problem, portfolio selection problem, licensing, differentiated Stackelberg model, benchmark, per-unit royalty licensing, fixed-fee licensing, two-part tariff licensing.

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Chapter 1

Preliminaries

For the mathematical modeling of the different types of economic problems studied in the present thesis we use different mathematical tools and notions. Therefore, in order to make this thesis as self-contained as possible, the present chapter contains some basic notions and results with respect to multicriteria optimization problems, lexicographic optimization problems and multilevel optimization problems.

1.1 Brief Background Concerning Multicriteria Optimization Problems

There exists many practical problems in which the aim is to realize concomitantly more objectives. These problems are named multi-objective or multicriteria optimization problems and are generated by many concrete problems which are based on real life situations. The solving of these problems can be approached in various ways, mentioned briefly in the thesis. We recall the notions: ideal point (global maximum point/global minimum point), max-efficient (respectively, min-efficient) point, non-dominated point of a function with respect to a preference relation.

Among the numerous papers in which the multicriteria optimization problems are studied we cite the followings: [27], [57],[72], [88], [100],[109]. Also, we recall some romanian papers very useful from the practical point of view, such as [3], [5], [90], [108] and [121].

1.2 Brief Background Concerning Lexicographic Optimization Problems

Lexicographic multiobjective optimization problems appear when conflicting objectives exist in a decision problem and, additional, those objectives have to be considered in a hierarchical order which is outside the control of the decision maker. Concrete examples which are mathematically modeled using this type of problems can be found in [10], [53] and [131].

In this section we recall the classical notions of lexicographic order relation, lexicographic maximum (respectively, minimum) point of a given function on a set, very used throughout this thesis.

Generally, we can write the lexicographic optimization problem as:

$$\text{(LP)} \quad \begin{cases} f(x) = (f_1(x), \ldots, f_m(x)) \ \rightarrow \ \text{lex} - \max \quad (\text{or lex} - \min), \\ x \in S. \end{cases}$$

There exist cases when not all criteria follow a maximization or all of them a minimization. Therefore, if this happens, we use a specific notation which specify what follows: maximization or minimization. We call optimal point of the problem (LP) any point which is a maximum (minimum) lexicographic point of the function f on S.

There exist many algorithms that can be used to solve the problem (LP). Zarepisheh M. and Khorram E. [135] notice that there exist two different methods: the sequential method and the weighted method. In the present thesis we use both methods.

1.3 Brief Background Concerning Multilevel Optimization Problems

Multilevel optimization and subsequently bilevel optimization have lately become important areas in optimization. A detailed bibliography of works in the field of bilevel and multilevel optimization problems is given in [123]. Multilevel optimization problems are used for modeling many types of concrete problems, such as: the network design problem [13], optimal pricing problem [71], the optimal signal setting problem [67], transportation problem [26], train set organization [40] and allocation problem [89].

Now, we present the bilevel optimization problem as it is given in [25].

Let us consider the sets $D \subseteq \mathbf{R}^n \times \mathbf{R}^m$, $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$, and let $F : D \to \mathbf{R}$, $f : D \to \mathbf{R}$, $G : D \to \mathbf{R}^p$ and $g : D \to \mathbf{R}^q$ be some given functions.

We introduce the set $S = \{x \in X,\ y \in Y \,|\, (x, y) \in D,\ G(x, y) \leqq 0_p,\ g(x, y) \leqq 0_q\}$. For each $x \in X$, we denote by $S_x = \{y \in Y \,|\, (x, y) \in D,\ G(x, y) \leqq 0_p\}$ and, under the hypothesis that $S_x \neq \emptyset$, we denote by $S_x^* = \{\arg\min(\arg\max)\,\{f(x, y) | y \in S_x\}\}$.

In mathematical terms, using the above notations, a bilevel optimization problem can be formulated as follows:

$$(BP) \quad \begin{cases} F(x, y) \ \rightarrow \ \min(\max), \\ \text{such that} \\ G(x, y) \leqq 0_p, \\ x \in X, \\ y \in S_x^*. \end{cases} \tag{1.1}$$

We give the terminology used in the literature for the case of bilevel optimization problems: relaxed feasible set, lower level feasible solutions set, follower's rational reaction set, set of feasible solutions, lower level optimal value, upper level objective function, lower level objective function, upper level problem, lower level problem.

We note that there exist some cases (see [35]) in which the solving of the bilevel optimization problem can be reduced to solving a multicriteria optimization problem. Then, the new problem can be solved by using the weighted method. In the present thesis we use an analogous way.

Another aspect, which is new and very used in practical applications, is the one which implies to consider the bi and multilevel optimization problem in which the coefficients of the objective functions or of the restrictions depend on one or more parameters. Such kind of problems are named bilevel/multilevel parametric optimization problems. In fact, to solve a parametric optimization problem means to specify, for each value of the parameter, which is the solution of the bilevel optimization problem obtained if the parameter is fixed to this value. It is well known that in case of the bilevel optimization we work under the hypothesis that both lower level objective functions and upper level objective function are bounded on the set of optimal solutions.

Chapter 2

Lexicographic Multicriteria Bottleneck Problems and Optimal Points with Pipeline Property

In the present chapter we augment the existent results regarding the lexicographic optimization problems with the special case of the lexicographic optimization problems with p objective functions of bottleneck type (or time type), generic denoted by us with LpBP. For this, we introduce a new type of optimal solutions named by us *optimal solution with pipeline property.* This type of points are introduced by using the idea of the papers authored by BANDOPADHYAYA L. [6]. The study of these points can be reached when solving bilevel optimization problems in which the lower level function is bicriteria, of bottleneck type. For example, in GOINA D. and TUNS (BODE) O.R. [43] or in TUNS (BODE) O.R. [111].

We note that the scientific results within this chapter belong to the author and can be found in the papers authored by TUNS (BODE) O.R. and LUPŞA L. [76], [118] or by TUNS (BODE) O.R. [114], [115].

2.1 Lexicographic Multicriteria Bottleneck Problems

In the present section we formulate the lexicographic optimization problem with p objective functions of bottleneck type and we study the structure of the set of its optimal solutions.

Let m, n, p be natural non-zero numbers such that $1 \leq p \leq m$.

Let $I := \{1, \ldots, m\}$, $J := \{1, \ldots, n\}$, $H := \{1, \ldots, p\}$.

Let $\Omega \subseteq \mathbb{R}^n_+$ and $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$. We say that the vector function $f = (f_1, \ldots, f_m)$ has p components of *bottleneck type* if there exist p natural numbers $i_1, \ldots, i_p$ and a real matrix $T = [t_{hj}]$ with p lines and n columns such that, for each $h \in H$,

$$f_{i_h}(x) = \max\{t_{hj} \cdot \mathrm{sgn}(x_j) \,|\, j \in J\}, \ \forall\, x \in \Omega \tag{2.1}$$

or

$$f_{i_h}(x) = \min\{t_{hj} \cdot \mathrm{sgn}(x_j) \,|\, j \in J\}, \ \forall\, x \in \Omega. \tag{2.2}$$

Let $I^c := I \setminus \{i_h | h \in H\}$.

In the present chapter we consider the case (2.1). In Chapter 4 of the present thesis we use the case (2.2). Generally, both cases can be used.

For each $h \in H$ we set

$$Z_h = \{t_{hj} | j \in J\},\ q_h := \text{card}(Z_h) \text{ and } K_h := \{1, \ldots, q_h\}.$$

We note that for each $h \in H$ the set Z_h is finite. Therefore, we renumber, for each $h \in H$, the elements of set Z_h by $z_{h1}, \ldots, z_{hq_h}$, such that to have

$$z_{h1} > z_{h2} > \cdots > z_{hq_h}. \tag{2.3}$$

For each $h \in H$ and $k \in K_h$ we set $L_{hk} := \{j \in J | t_{hj} = z_{hk}\}$.

Remark 2.1.1 (Tuns (Bode) O.R. [*114*]). *For each $h \in H$ we have $f_{i_h}(x) = z_{hr}$ if and only if*

$$\{j \in L_{hk} | x_j > 0\} = \emptyset,\ \forall\, k \in K_h,\ k < r,\ \textit{ and } \{j \in L_{hr} | x_j > 0\} \neq \emptyset.$$

Let S be a non-void subset of Ω. We consider the following optimization problem:

$$\text{(LpBP)} \quad \begin{cases} f(x) \to \text{lex} - \min & \text{(or lex} - \max), \\ x \in S, \end{cases}$$

named by us *lexicographic optimization problem with p bottleneck objective functions*.

A point x^0 of S is an optimal solution of the problem (LpBP) if x^0 is a minimum (maximum) point of function f on S. If we denote by $\hat{S}$ the set of all optimal solutions for (LpBP) and if, for each $i \in I$, by $\hat{S}_i$ we denote the set of all minimum points of f_i on set $\hat{S}_{i-1}$, with $\hat{S}_0 := S$, then $\hat{S} = \hat{S_m}$.

We study the structure of the set $\hat{S}$ with respect to the convexity property.

Lemma 2.1.2 (Tuns (Bode) O.R. [*114*]). *If $\Lambda \subseteq S$ is a convex set and $h \in H$, then the set $\hat{\Lambda}_{i_h}$ of all minimum points of f_{i_h} on Λ is convex.*

As a consequence of Lemma 2.1.2, we obtain a very relevant result for the practical applications.

Theorem 2.1.3 (Tuns (Bode) O.R. [*114*]). *If S is a non-void and convex set and functions f_k, $k \in I \setminus \{i_h | h \in H\}$, are convex, then set $\hat{S}$ is convex, too.*

2.2 Optimal Solutions with Pipeline Property

Let $\Lambda \subseteq S$, $h \in H$.

Definition 2.2.1 (Tuns (Bode) O.R. *and* Lupşa L. [*118*]). *A point $x^0 \in \Lambda$ is said to be a minimum point with pipeline property of f_{i_h} on Λ if, for all $x \in \Lambda$, we have*
a) $\sum\limits_{j \in L_{hk}} sgn(x_j^0) = \sum\limits_{j \in L_{hk}} sgn(x_j),\ \forall\, k \in K_h,$
or

b) there exists a natural number $r \in K_h$ such that $\sum\limits_{j \in L_{hr}} sgn(x_j^0) < \sum\limits_{j \in L_{hr}} sgn(x_j)$, and, if $r \geq 2$, then $\sum\limits_{j \in L_{hk}} sgn(x_j^0) = \sum\limits_{j \in L_{hk}} sgn(x_j)$, $\forall\, k \in \{1, ..., r-1\}$.

In what follows, for each $h \in H$, we denote by $\tilde{\Lambda}_{i_h}$ the set of all minimum points with pipeline property of f_{i_h} on Λ.

Theorem 2.2.2 (Tuns (Bode) O.R. [*114*]). *If $h \in H$ and $x^0 \in \tilde{\Lambda}_{i_h}$, then x^0 is a minimum point of f_{i_h} on Λ.*

Theorem 2.2.2 justify the use of the adjective "minimum" within Definition 2.2.1.

Recalling that we denoted by $\hat{\Lambda}_{i_h}$ the set of all minimum points of function f_{i_h} on Λ, from Theorem 2.2.2 we get that

$$\tilde{\Lambda}_{i_h} \subseteq \hat{\Lambda}_{i_h}. \tag{2.4}$$

But, based on the below example, there exist minimum points of f_{i_h} with respect to Λ which does not have the pipeline property.

Example 2.2.3 (Tuns (Bode) O.R. [*114*]). *Let us consider $\Lambda = [1, 2] \times [0, 2] \times [0, 1] \times [0, 1]$. Let $f = (f_1, f_2) : \Lambda \to \mathbb{R}^2$ be the function given by*
$f_1(x_1, x_2, x_3, x_4) = \max\{2sgn(x_1),\, 3sgn(x_2), 2sgn(x_3)\}$, for all $(x_1, x_2, x_3, x_4) \in \Lambda$,
$f_2(x_1, x_2, x_3, x_4) = \max\{3sgn(x_1),\, 2sgn(x_2), sgn(x_3)\}$, for all $(x_1, x_2, x_3, x_4) \in \Lambda$. We have

$$\hat{\Lambda}_1 = \{(x_1, 0, x_3, x_4) | x_1 \in [1,\, 2],\, x_3 \in [0,\, 1],\, x_4 \in [0,\, 1]\}$$

and $\tilde{\Lambda}_1 = \{(x_1, 0, 0, 0) | x_1 \in [1,\, 2]\} = [(1, 0, 0, 0),\, (2, 0, 0, 0)]$. Therefore $\hat{\Lambda}_1 \neq \tilde{\Lambda}_1$.

Under the hypothesis that $\Lambda \subseteq S \subseteq \mathbb{R}^n_+$ and $h \in H$, in the following two propositions we give necessary and sufficient conditions for a point to be minimum point with pipeline property, considering that such a point is already known. These propositions are very relevant from the theoretical point of view.

Proposition 2.2.4 (Tuns (Bode) O.R. [*114*]). *If $x \in \tilde{\Lambda}_{i_h}$, $f_{i_h}(x) = z_{hr}$ and $y \in \Lambda$, then $y \in \tilde{\Lambda}_{i_h}$ if and only if*

$$f_{i_h}(x) = f_{i_h}(y) \;\; and \sum_{j \in L_{hk}} sgn(x_j) = \sum_{j \in L_{hk}} sgn(y_j),\; \forall\, k \in \{r, \ldots, q_h\}. \tag{2.5}$$

Proposition 2.2.5 (Tuns (Bode) O.R. [*114*]). *If $x \in \tilde{\Lambda}_{i_h}$ and $y \in \Lambda$, then $y \in \tilde{\Lambda}_{i_h}$ if and only if*

$$\sum_{j \in L_{hk}} sgn(x_j) = \sum_{j \in L_{hk}} sgn(y_j),\; \forall\, k \in K_h. \tag{2.6}$$

Let $h \in H$. In what follows, we study the properties of the set $\tilde{\Lambda}_{i_h}$ with respect to convexity.

Let $\Lambda \subseteq S \subseteq \mathbb{R}^n_+$, $h \in H$ and $x, y \in \Lambda$.

For each $k \in K_h$, we set $L^x_{hk} := \{j \in L_{hk} | \mathrm{sgn}(x_j) = 1, \mathrm{sgn}(y_j) = 0\}$.

Proposition 2.2.7 (Tuns (Bode) O.R. [*114*]). *If set $\Lambda \subseteq S$ is convex, $h \in K_h$ and $y \in (\hat{\Lambda}_{i_h} \setminus \tilde{\Lambda}_{i_h})$, then there exists $x \in \tilde{\Lambda}_{i_h}$ such that $]x,\, y] \subseteq (\hat{\Lambda}_{i_h} \setminus \tilde{\Lambda}_{i_h})$.*

Proposition 2.2.8 (Tuns (Bode) O.R. [*114*]). *If set $\Lambda \subseteq S$ is convex, $h \in H$, $x, y \in \hat{\Lambda}_{i_h}$ and $]x,\, y[\cap\tilde{\Lambda}_{i_h} \neq \emptyset$, then $]x,\, y[\subseteq \tilde{\Lambda}_{i_h}$.*

Proposition 2.2.9 (Tuns (Bode) O.R. [*114*]). *If set $\Lambda \subseteq S$ is convex, $h \in H$, $x, y \in \tilde{\Lambda}_{i_h}$, $x \neq y$, then $[x,\, y] \subseteq \tilde{\Lambda}_{i_h}$ if and only if $L^x_{hk} = \emptyset$ (or equivalent $L^y_{hk} = \emptyset$), for all $k \in K_h$.*

We remark the fact that the condition $L^x_{hk} = \emptyset$, $\forall\, k \in K_h$ is essential, as can be seen also from the following example.

Example 2.2.10 (Tuns (Bode) O.R. [*114*]). *Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by $f_1(x_1, x_2) = \max\{2sgn(x_1), 2sgn(x_2)\}$ and $f_2(x_1, x_2) = x_1 + x_2$, $\forall\, x = (x_1, x_2) \in \mathbb{R}^2$. Let $\Lambda = [(2, 0),\, (0, 2)] = \{(2(1 - \lambda),\, 2\lambda) | \lambda \in [0,\, 1]\}$. One gets that $\hat{\Lambda}_1 = [(2, 0),\, (0, 2)]$, $Z_1 = \{2\}$, $q_1 = 1$ and $\tilde{\Lambda} = \{(2, 0),\, (0, 2)\}$. We note that, if we take $x = (2, 0)$ and $y = (0, 2)$, then $L^x_{11} = \{1\} \neq \emptyset$, which implies that the hypothesis of the Proposition 2.2.9 are not fulfilled.*

In what follows, we introduce the notion of optimal solution with pipeline property for (LpBP) problem.

Definition 2.2.11 (Tuns (Bode) O.R. *and* Lupşa L. [*118*]). *A point $x^0 \in S$ is called an optimal solution with pipeline property for (LpBP) or minimum lexicographic point with pipeline property of f on S if x^0 is a minimum point of function f_k on set S_{k-1} for each $k \in I$ and, additionally, if $k \in \{i_h | h \in H\}$ then it has the pipeline property, where $S_0 = S$ and S_k denotes:*
i) the set of all minimum points of function f_k with respect to the set S_{k-1}, i.e. the set $\hat{S}_{k-1}$, if $k \in I \setminus \{i_h | h \in H\}$; or
ii) the set of all minimum points with pipeline property of function f_k with respect to the set S_{k-1}, i.e. the set $\tilde{S}_{k-1}$, if $k \in \{i_h | h \in H\}$.

Let us denote by $\tilde{S}$ the set of all minimum lexicographic points with pipeline property of f on S. The below example illustrates how the set $\tilde{S}$ is determined.

Example 2.2.12 (Tuns (Bode) O.R. [*114*]). *Let us recall Example 2.2.3. If $S = \Lambda = [1,\, 2] \times [0,\, 2] \times [0,\, 1] \times [0, 1]$, then we have $\tilde{S}_1 = \{(x_1, 0, 0, 0) | x_1 \in [1,\, 2]\}$, $\tilde{S}_2 := \tilde{\Lambda} = \{(x_1, 0, 0, 0) | x_1 \in [1,\, 2]\}$. Therefore, $\tilde{\Lambda} = \{(x_1, 0, 0, 0) | x_1 \in [1,\, 2]\}$.*

2.3 Lexicographic Discrete Bottleneck Problems

In this section we consider the (LpBP) problem under the additional hypothesis that $S \subseteq \mathbb{N}^n$. We denote this problem by (LDpBP) and we call it lexicographic discrete optimization problem with p bottleneck objective functions. Although, exteriorly, such kind of problems are very restrictive, they appear in real life situations. Some concrete examples are presented in the following chapters of the present thesis.

Example 2.3.1 (Tuns (Bode) O.R. *and* Lupşa L. [*118*]).
Let $S = \{(-1, -1), (-1/2, -1)\}$ and $f : S \to \mathbb{R}$, $f(x_1, x_2) = \max\{2x_1, x_2\}$, for all $(x_1, x_2) \in S$. Both points $x^0 = (-1, -1)$ and $x = (-1/2, -1)$ are minimum points of f with respect to S, but only $(-1, -1)$ has the pipeline property.

Example 2.3.2 (TUNS (BODE) O.R. *and* LUPŞA L. [*118*]).
Let $S = \{0,1,2\} \times \{1,2,3\} \times \{0,1\}$ *and let* $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ *be the function given by:*
$f_1(x_1, x_2, x_3) = -5 + x_3^2 - x_3$,
$f_2(x_1, x_2, x_3) = \max\{2sgn(x_1), 2sgn(x_2), 3sgn(x_3)\}$,
$f_3(x_1, x_2, x_3) = -8 + x_1^2 + x_3^2$, *for all* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
We have $\tilde{S} = \{(0,1,0), (0,2,0), (0,3,0)\}$.

In what follows, some properties concerning the structure of the set of optimal solutions of (LDpBP) problem are studied. As this problem is discrete, in order to study the convexity properties of the set of all optimal solutions and of the set of all optimal solutions with pipeline property, we use the notions of strongly 2-convex set with respect to $\mathbb{N}^n$ and strongly 2-convex function with respect to $\mathbb{N}^n \times \mathbb{R}$ given by CRISTESCU G. and LUPŞA L. in [23].

Let $\Omega \subseteq \mathbb{R}^n_+$, $S \subseteq (\Omega \cap \mathbb{N}^n)$ and $f = (f_1, \dots, f_p) : \Omega \to \mathbb{R}^p$.

Theorem 2.3.6 (TUNS (BODE) O.R. *and* LUPŞA L. [*118*]). *If set* $\Lambda \subseteq S$ *is strongly 2-convex with respect to* $\mathbb{N}^n$ *and* $h \in H$, *then set* $\hat{\Lambda}_{i_h}$ *of all minimum points of* f_{i_h} *on* Λ *is strongly 2-convex with respect to* $\mathbb{N}^n$.

Corollary 2.3.7 (TUNS (BODE) O.R. *and* LUPŞA L. [*118*]). *If set* S *is non-void and strongly 2-convex with respect to* $\mathbb{N}^n$ *and functions* f_k, $k \in I \setminus \{i_h | h \in H\}$, *are strongly 2-convex with respect to* $\mathbb{N}^n$, *then set* $\hat{S}$ *is strongly 2-convex with respect to* $\mathbb{N}^n$, *too.*

Similar properties with the ones presented above can be obtained in case there are studied the properties of the set of optimal points with pipeline property.

Proposition 2.3.8 (LUPŞA L. *and* TUNS (BODE) O.R. [*76*]). *If set* $\Lambda \subseteq S$ *is strongly 2-convex with respect to* $\mathbb{N}^n$, $h \in H$, *y is a minimum point of* f_{i_h} *on* Λ, *but* $y \notin \tilde{\Lambda}_{i_h}$, *then there exists* $x \in \tilde{\Lambda}_{i_h}$ *such that if* $\lambda \in]0,\, 1[$ *and* $z := (1-\lambda)x + \lambda y \in \mathbb{N}^n$, *then* $z \notin \tilde{\Lambda}_{i_h}$.

Corollary 2.3.9 (LUPŞA L. *and* TUNS (BODE) O.R. [*76*]). *If set* $\Lambda \subseteq S$ *is strongly 2-convex with respect to* $\mathbb{N}^n$, $h \in H$, *x and y are minimum points of* f_{i_h} *on* Λ, *but* $x, y \notin \tilde{\Lambda}_{i_h}$, *and* $\lambda \in [0,\, 1]$ *is such that* $z := (1-\lambda)x + \lambda y \in \mathbb{N}^n$, *then* $z \notin \tilde{\Lambda}_{i_h}$.

Proposition 2.3.10 (LUPŞA L. *and* TUNS (BODE) O.R. [*76*]). *If set* $\Lambda \subseteq S$ *is strongly 2-convex with respect to* $\mathbb{N}^n$, $h \in H$, $x, y \in \hat{\Lambda}$ *and there exists* $z \in]x,\, y[\cap \mathbb{N}^n$ *such that* $z \in \tilde{\Lambda}_{i_h}$, *then* $]x,\, y[\cap \mathbb{N}^n \subseteq \tilde{\Lambda}_{i_h}$.

Proposition 2.3.11 (LUPŞA L. *and* TUNS (BODE) O.R. [*76*]). *If set* $\Lambda \subseteq S$ *is strongly 2-convex with respect to* $\mathbb{N}^n$, $h \in H$, $x, y \in \tilde{\Lambda}_{i_h}$ *and* $]x,\, y[\cap \mathbb{N}^n \neq \emptyset$, *then the set* $[x,\, y]$ *is strongly 2-convex with respect to* $\mathbb{N}^n$ *if and only if* $L^x_{hk} = \emptyset$ (*or equivalent* $L^y_{hk} = \emptyset$), *for all* $k \in K_h$.

We note that the condition $L^x_{hk} = \emptyset$, $\forall\, k \in K_h$ is essential, as can be seen also from the following example.

Example 2.3.12 (LUPŞA L. *and* TUNS (BODE) O.R. [*76*]).

Let $S = \{(2,0),\, (1,1),\, (0,2)\}$. *Let* $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ *be the function given by:* $f_1(x_1, x_2) = x_1 + x_2$ *and* $f_2(x_1, x_2) = \max\{2sgn(x_1), 2sgn(x_2)\}$, *for all* $x = (x_1, x_2) \in \mathbb{R}^2$. *One gets that* $\tilde{S} = \{(2,0),\, (0,2)\}$. *It is easy to see that* $(1,1) \in \hat{S}_2 \cap \mathbb{N}^2$, *but* $(1,1) \notin \tilde{S}$. *Therefore set* $\tilde{S}$ *is not strongly 2-convex with respect to* $\mathbb{N}^2$.

2.4 A Method to Determine an Optimal Solution with Pipeline Property for (LDpBP) Problem

In the present paragraph a method to determine an optimal solution with pipeline property for (LDpBP) problem is given. This method is a type of weighted methods. The novelty is that the weighted type introduced by us allows the direct getting of the optimal points with pipeline property. It generalizes both the method used by Bandopadhyaya L. [6], and the one described by Zarepisheh M. and Khorram E. [135].

Let m, n, p be natural non-zero numbers such that $1 \leq p \leq m$.

Let $I := \{1, \ldots, m\}$, $J := \{1, \ldots, n\}$, $H := \{1, \ldots, p\}$ and let i_h, $h \in H$, be p distinct natural numbers such that $i_1 < \cdots < i_p$.

Let $S \subseteq \mathbb{N}^n$. We consider $f = (f_1, \ldots, f_m) : S \to \mathbb{N}^m$ the vector function with p components of bottleneck type, given by (2.1), where the numbers t_{hj}, $h \in H$, $j \in J$, are natural. We consider the problem:

$$\text{(LDpBP)} \quad \begin{cases} f(x) = (f_1(x), \ldots, f_m(x)) \to \text{lex} - \min, \\ x \in S. \end{cases}$$

For each $h \in H$ we set $q_h + 1$ numbers, denoted by M_{hk}, $k \in \{q_h, q_{h-1}, \ldots, 0\}$, such that

$$M_{hq_h} = 1 \tag{2.7}$$

and

$$M_{hk} = 1 + \sum_{j=k+1}^{q_h} M_{hj} \cdot \operatorname{card}(L_{hj}), \ \forall\, k \in \{q_{h-1}, ..., 0\}. \tag{2.8}$$

It is easy to see that

$$M_{h0} = 1 + \sum_{j=1}^{q_h} M_{hj} \cdot \operatorname{card}(L_{hj}). \tag{2.9}$$

Let

$$M_0 := 1 + \max\{M_{h0} | h \in H\}. \tag{2.10}$$

Obviously, for each $r \in K_h$ we have

$$\sum_{k=r}^{q_h} M_{hk} \cdot \operatorname{card}(L_{hk}) = \begin{cases} M_{h,r-1} - 1 \leq M_{h0} - 1 \leq M_0 - 1 < M_0, \ \text{if } r > 1, \\ M_{h0} - 1 \leq M_0 - 1 < M_0, \ \text{if } r = 1. \end{cases} \tag{2.11}$$

For each $i \in I$ let us consider a real number $\bar{f}_i$, $\bar{f}_i \geq 2$, such that

$$f_i(x) \leq \bar{f}_i, \ \forall\, x \in S. \tag{2.12}$$

Let

$$\lambda = 1 + \max\{1 + M_0, \ \{\bar{f}_i | i \in I\}\}. \tag{2.13}$$

It is easy to see that

$$\lambda^{m+1-i} - 2\lambda^{m-i} > 0, \, \forall\, i \in I, \text{ and } \lambda^{m+2-i_{h_0}} - (M-0+1)\lambda^{m-i_{h_0}} > 0, \, \forall\, h \in H. \tag{2.14}$$

Remark 2.4.1 (Tuns (Bode) O.R. [*115*]). *Let $u, v \in S$, $u \neq v$, $i_0 \in I^c$, $h \in H$ and $k \in K_h$. The following inequalities fulfill:*

(i) *if* $f_{i_0}(u) > f_{i_0}(v)$, *then* $f_{i_0}(u) - f_{i_0}(v) \geq 1$;

(ii) $\sum\limits_{i \in I^c, i \geq i_0} \lambda^{m-i}(f_i(u) - f_i(v)) \geq -\sum\limits_{i \in I^c, i \geq i_0} \lambda^{m-i} f_i(v) > \sum\limits_{i \in I^c, i \geq i_0} \lambda^{m+1-i}$;

(iii) *if* $f_{i_h}(u) = z_{hs} \in Z_h$, *then* $\sum\limits_{j \in L_{hs}} sgn(u_j) \geq 1$;

(iv) $\sum\limits_{j \in L_{hk}} (sgn(u_j) - sgn(v_j)) \geq -\sum\limits_{j \in L_{hk}} sgn(v_j) \geq -card(L_{hk})$;

(v) $\sum\limits_{k \in K_h} M_{hk} \sum\limits_{j \in L_{hk}} (sgn(u_j) - sgn(v_j)) \geq -\sum\limits_{k \in K_h} M_{hk}\, card(L_{hk}) = 1 - M_{h0} > 1 - M_0$.

Now, let us consider the problem:

$$\text{(PUP)} \begin{cases} F(x) = M_0 \sum\limits_{i \in I^c} \lambda^{m-i} f_i(x) + \sum\limits_{h \in H} \lambda^{m+1-i_h} \sum\limits_{k \in K_h} (M_{hk} \sum\limits_{j \in L_{hk}} \operatorname{sgn}(x_j)) \to \min, \\ x \in S. \end{cases}$$

Theorem 2.4.2 (Tuns (Bode) O.R. [*115*]). *A point $x^0 \in S$ is an optimal solution with pipeline property of the problem (LDpBP) if and only if it is an optimal solution of the problem (PUP).*

Chapter 3

An Application Related to Firm's Costs Management

In the present chapter we present an application generated by a concrete costs management problem. We note that the problem studied in the present chapter can be seen, on one hand, as a traveling salesman problem, and, on the other hand, as a problem of generating new types of routes. We mathematically model this problem by using bilevel optimization on a graph. Therefore, we study a special type of bilevel optimization problem in which the set of feasible solutions is the set of subgraphs of a given graph. These subgraphs fulfill some given restrictions. The novelty of this problem is given, on one hand, by the mathematical model which we introduce and, on the other hand, by the fact that we use the splitting technique which allows us to reduce the solving of this problem to solving three problems: (PP), $(P_1(H_0))$, $(P_2(H_0))$, where H_0 is an optimal solution of the first problem. The problem $(P_1(H_0))$ is a classic problem of determining the minimum of a function on a given set of subgraphs of a graph. The problem $(P_2(H_0))$ is a lexicographic bicriteria optimization problem. We reduce the solving of this last problem to solving again a classic problem of determining the minimum of a function on a given set of subgraphs of a graph.

The author's achievements within this chapter can be found in the paper authored by Goina D. and Tuns (Bode) O.R. [43].

3.1 The Milk Collection Problem

A dairy products manufacturing company collects twice a day the milk from a certain area. Collection points are located only on roads linking villages in the area. The milk is brought to the collection points by the owners. The quantity of milk delivered depends on the time when the collection is scheduled. Some providers can bring the milk to the collection points only in the morning. Others only in the evening, and some of them both in the morning and in the evening. There exists a possibility for some providers who deliver milk in the morning to store it (in conditions that do not impair the milk quality) and to offer it for delivery only in the evening. The others do not have this possibility. The providers impose that either the entire quantity of milk offered is collected by the dairy products manufacturing company or nothing. The milk is collected by the dairy products manufacturing company in the morning and in the evening using a collector tank, which has a capacity denoted by $\overline{Q}$.

The problem that arises is that of planning the providers:
- those who bring milk to the collection points in the morning, and the milk is collected by the collector tank in the morning;
- those who bring milk to the collection points in the morning, but it is necessary to store it until evening, when it will be collected by the collector tank;
- those who bring milk to the collection points in the evening, and the milk is collected by the collector tank in the evening,
such that the total cost required for milk collection in a day to be minimum and a collection point to be visited by the collector tank at most once in the morning and at most once in the evening.

The providers planning must satisfy the following requirements:
a) the quantity of milk collected in the morning not exceed the capacity $\overline{Q}$ of the collector tank;
b) the quantity of milk collected in the evening (which may be from the evening milk or from the stored one) not exceed the capacity $\overline{Q}$ of the collector tank;
c) the quantity of milk collected in the morning and the quantity collected in the evening must be greater than a specified quantity, denoted by $\underline{Q}$, in order to ensure the continuity in the production process;
d) the quantity of stored milk to be minimum and in the same time fulfilling the conditions a)-c).

3.2 Generalization of the Mathematical Model for the Milk Collection Problem

In this paragraph, after we attach the mathematical model to the milk collection problem, we give a generalization of it.

Let n be a non-zero natural number, $N = \{1, \ldots, n\}$ be a finite set, $G = (N, E)$ be a weighted graph and let $I \subset N$.

Let Λ be the set of subgraphs $\Gamma = (N_\Gamma, E_\Gamma)$ of G with $N_\Gamma \neq \emptyset$ and $E_\Gamma \neq \emptyset$.

Let $\mathcal{C}_1$ be the set of those elements $G_1 = (N_1, E_1)$ of Λ which fulfill some given conditions imposed to be fulfilled by the set of nodes N_1. Also, let $\mathcal{C}_2$ be the set of those elements $G_2 = (N_2, E_2)$ of Λ which fulfill other given conditions imposed to be fulfilled by the set of nodes N_2.

Let $h : \Lambda \to \mathbb{R}$ and $g : \Lambda \to \mathbb{N}$ be some given functions. Let a and b be positive numbers. We consider the function $F : \Lambda \times \Lambda \to \mathbb{R}_+$ given by

$$F(G_1, G_2) \;=\; a \cdot h(G_1) + b \cdot h(G_2),\ \forall\, (G_1, G_2) \in \Lambda \times \Lambda. \tag{3.1}$$

The corresponding mathematical model for the milk collection problem is obtained, by the specified particularization, from the following bilevel optimization problem:

$$(PBG) \quad \begin{cases} F(G_1, G_2) \;\to\; \min, \\ G_1 \in \mathcal{C}_1, \\ G_2 \in S^*(G_1), \end{cases}$$

where $S^*(G_1)$ it is the set of optimal solutions of the problem

$$(P(G_1)) \quad \begin{cases} g(G_2) \ \rightarrow \ \min, \\ G_2 \in \mathcal{C}_2, \\ N_1 \cap N_2 \cap I = \emptyset. \end{cases}$$

We remark that the objective function of the lower level problem is bicriteria. As far as we know, problems of (PBG) type have not been studied in the literature.

In what follows, we give an exact method for solving the (PBG) problem, based on the splitting technique. For this, we use the particularities of the restrictions and of the objective function (more exactly, the restriction $N_1 \cap N_2 \cap I = \emptyset$).

Let us denote by $S = \{(G_1, G_2) \in \Lambda \times \Lambda \,|\, G_1 \in \mathcal{C}_1,\ G_2 \in \mathcal{C}_2,\ N_1 \cap N_2 \cap I = \emptyset\}$ and by $S_1 = \{G_1 \in \Lambda \mid \exists\, G_2 \in \Lambda \text{ s. t. } (G_1, G_2) \in S\}$, i.e. $S_1 = \{G_1 \in \mathcal{C}_1 \mid \exists\, G_2 \in \mathcal{C}_2 \text{ s. t. } N_1 \cap N_2 \cap I = \emptyset\}$. For each $G_1 \in S_1$ we consider the set $S(G_1) = \{G_2 \in \mathcal{C}_2 \mid (G_1, G_2) \in S\} = \{G_2 \in \mathcal{C}_2 \mid N_1 \cap N_2 \cap I = \emptyset\}$. It is easy to see that $S(G_1)$ it is the set of feasible solutions of the problem $(P(G_1))$.

Let $H \in 2^I$. We consider the problems:

$$(P_1(H)) \quad \begin{cases} h(G_1) \ \rightarrow \ \min, \\ G_1 \in \mathcal{C}_1, \\ N_1 \cap I = H \end{cases}$$

and

$$(P_2(H)) \quad \begin{cases} \begin{pmatrix} g(G_2) \\ h(G_2) \end{pmatrix} \ \rightarrow \ \text{lex} - \min, \\ G_2 \in \mathcal{C}_2, \\ N_2 \cap H = \emptyset. \end{cases}$$

Let us denote by h_1^H the optimal value of the problem $(P_1(H))$ and by (g_2^H, h_2^H) the optimal value of the problem $(P_2(H))$.

Let $\tilde{F} : 2^I \rightarrow \mathbb{R}$ be the function given by $\tilde{F}(H) = a \cdot h_1^H + b \cdot h_2^H,\ \forall\ H \in 2^I$. We consider the problem:

$$(PP) \quad \begin{cases} \tilde{F}(H) \rightarrow \min, \\ H \in 2^I. \end{cases}$$

Furthermore, we establish relations between the feasible solutions, respectively between the optimal solutions, of the problems (PBG) and (PP).

Lemma 3.2.1 (Goina D. *and* Tuns (Bode) O.R. [*43*]). *If $G_1^0 \in \mathcal{C}_1$, G_2^0 it is a feasible solution of the problem $(P(G_1^0))$ and $H^0 = N_1^0 \cap I$, then G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.*

Lemma 3.2.2 (Goina D. *and* Tuns (Bode) O.R. [*43*]). *If (G_1^0, G_2^0) it is a feasible solution of the problem (PBG) and $H^0 = N_1^0 \cap I$, then G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.*

Lemma 3.2.3 (Goina D. *and* Tuns (Bode) O.R. [*43*]). *If $H^0 \in 2^I$, G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and G_2^0 it is an optimal solution of the problem $(P_2(H^0))$, then (G_1^0, G_2^0) it is a feasible solution of the problem (PBG).*

Theorem 3.2.4 (Goina D. *and* Tuns (Bode) O.R. [*43*]). *If (G_1^0, G_2^0) it is an optimal solution of the problem (PBG), then taking $H^0 = N_1^0 \cap I$ the following sentences are true:*
i) G_1^0 it is an optimal solution of the problem $(P_1(H^0))$;
ii) G_2^0 it is an optimal solution of the problem $(P_2(H^0))$;
iii) H^0 it is an optimal solution of the problem (PP).

Theorem 3.2.5 (Goina D. *and* Tuns (Bode) O.R. [*43*]). *If H^0 it is an optimal solution of the problem (PP) and G_1^0, respectively G_2^0, it is an optimal solution of the problem $(P_1(H^0))$, respectively $(P_2(H^0))$, then (G_1^0, G_2^0) it is an optimal solution of the problem (PBG).*

We note that the $(\mathrm{P}_2(\mathrm{H}^0))$ problem is a lexicographic bicriteria optimization problem. We can solve this problem by using and particularizing the method given in Section 4.2.

Let $\lambda \geq 1 + \max\{F(G_1, G_2),\ \forall\,(G_1, G_2) \in \Lambda\}$.

Let $G_1 \in S_1$ and $H \in 2^I$, fulfilling the following condition:

$$N_1 \cap I = H. \tag{3.2}$$

Let us consider the problem:

$$(PL_2(H)) \quad \begin{cases} \lambda \cdot g(G_2) + F(G_1, G_2) \ \rightarrow \ \min, \\ G_2 \in \mathcal{C}_2, \\ H \cap N_2 = \emptyset. \end{cases}$$

Theorem 3.2.6 (Goina D. *and* Tuns (Bode) O.R. [*43*]). *If $G_1 \in S_1$ and $H \in 2^I$ such that the condition (3.2) is fulfilled, then an element G_2 it is an optimal solution of the problem $(PL_2(H))$ if and only if it is an optimal solution of the problem $(P_2(H))$.*

Remark 3.2.7 *Based on Theorem 3.2.6 we can reduce the solving of the problem (PBG) to solving a finite sequence of couples of problems $(P_1(H), P_2(H))$, where the parameter H belongs to the set 2^I.*

Chapter 4

Applications of Multilevel Optimization with Respect to Professional Training Programs

Education represents a life-long learning process. Thus imposes the need of professional training courses of the persons since well educated people with better qualification adapt more rapidly to technological changes and ensure growth of productivity on the long run. Different aspects concerning professional training courses can be found in the papers authored by Guţ C.M. and Bode O.R. [48], Guţ C.M., Vorzsak M., Chifu C.I. and Bode O.R. [49], Guţ C.M., Vorzsak M. and Bode (Tuns) O.R. [50].

Keeping in mind the importance of well-qualified persons, we should notice that one of the major problems faced by the national institutions from our country and from abroad is the lack of the financial resources allocated for the professional training programs of the unemployed persons. The problem that arises very often is that the budget allocated for these institutions is not enough to offer for free professional training programs from different areas to all unemployed persons. Therefore, different economic problems concerning the assignment of the persons to attend a professional training program or the restriction of the budget allocated for these type of courses can be found in real life situations. In the present chapter we study from the optimization point of view some of these economic problems that arise and which are solved in practice intuitively.

The scientific results within this chapter belong to the author and can be find in the papers authored by Tuns (Bode) O.R. and Neamţiu L. [119] or by Tuns (Bode) O.R. [116].

4.1 Brief Background Concerning Assignment Problems

In real life situations, we can find some concrete economic problems involving assigning unemployed persons to professional training programs. Since we use as a mathematical tool the assignment problems in the thesis we give a brief background concerning these type of problems. As useful papers in this area of research we recall [16], [18], [36], [38], [41], [44], [65], [92], [98], [125].

During the time, the assignment problems knew other different generalizations. A very useful overview regarding the variety of models of the assignment problems can be find in Pentico

D.W. [92]. This paper provides a comprehensive survey of the different variations on the assignment problem that have appeared in the literature such as: the lexicographic bottleneck problem, the assignment problem with side constraints and r-lexicographic multi-objective problem. The lexicographic bottleneck problem have been studied for example in Burkard R.E. and Rendl F. [16] or in Sokkalingam P.T. and Aneja Y.P. [105]. Another useful work for researchers and practitioners is the one authored by Burkard R., Dell'Amico M. and Martello S. [15]. It provides a comprehensive treatment of assignment problems from their conceptual beginnings through present day.

In [63] Khanmohammadi S., Hajiha A. and Jassbi J. introduce a qualification matrix used to classify and select the qualified individuals for different jobs to optimize the man power of the organization.

Based on the author's papers with a preponderant economic character (see [48], [49] and [50]), one of the critical problems that can be find in real life situations and must be solved is the problem of retraining the unemployed persons. Therefore, the mathematical models introduced by us in the present thesis in order to select suitable persons for different professional training programs are used to solve different types of optimization problems, which can be view as general assignment problems.

4.2 The Concrete Economic Problems

In the present section we formulate the two studied economic problems involving assigning unemployed persons to professional training programs.

Problem (AEP_1). Assume that in the same period of time there are organized different professional training programs for the unemployed persons. For each professional training program there is known its efficiency (defined from the point of view of finding a place to work by the unemployed persons after graduating it), the maximum number of the persons that can attend it and the score that each unemployed person has if attends it (this score was calculated based on historical data about each unemployed person taking into account his/her education or professional experience). The problem that arises is how to assign the registered unemployed persons to the professional training programs, based on each person's score, such that the following restrictions to be fulfilled:
i) all unemployed persons to attend the professional training programs (i.e. the case when the maximum number of the persons that can attend the professional training programs is bigger than the total number of the registered unemployed persons which need to attend the courses);
ii) each unemployed person to attend exactly one professional training program;
iii) the assignment of the unemployed persons to a professional training program to be done such that to maximize the minimum score of the assignments;
iv) the efficiency of the professional training program for which the minimum score is reached to be as small as possible and to be reached as few times as possible.

Problem (AEP_2). Now, we consider the above problem but under the hypothesis that restriction i) is not fulfilled, i.e. the case when the maximum number of the persons that can attend the professional training programs is smaller than the total number of the registered unemployed persons which need to attend the courses, while the above restrictions ii), iii) and iv) occur.

Furthermore, we give the mathematical model of each concrete economic problem, we study some properties of the optimal solutions of each problem and we propose an algorithm or a method for solving it, highlighted by different examples.

In each of the following sections, let us denote by:

-m the number of the total professional training programs identified by the variable i, $i \in \{1, \ldots, m\}$. Let $I = \{1, \ldots, m\}$;

-e_i, $i \in I$, the efficiency of the professional training program i;

-n the total number of the unemployed persons that need to attend the professional training programs. Let $J = \{1, \ldots, n\}$;

-a_i, $i \in I$, the maximum number of the persons that can participate to the professional training program i, $i \in I$;

-r_{ij}, $i \in I$, $j \in J$, the score corresponding to each unemployed person j if attends the professional training program i. Let $R \in \mathcal{M}_{m\times n}(\mathbb{R}_+^*)$ be the matrix which elements represent the scores r_{ij};

-y_{ij}, $i \in I$, $j \in J$, the binary variable having the significance $y_{ij} = 1$ if the unemployed person j will participate to the course i and $y_{ij} = 0$ otherwise;

4.3 The Study of the Problem (AEP_1)

The problem to be discussed in the present section represents a new kind of a generalized bottleneck assignment problem. It images the modeling of a concrete economic problem which involves assigning unemployed persons to professional training programs under the circumstances that, on one hand, there exists no restriction regarding the budget allocated for it and, on the other hand, the only restrictions are the ones concerning the number of the unemployed persons which must attend the professional training programs, the score of the assignments and the efficiency of each professional training program.

4.3.1 Mathematical Modeling of the Problem (AEP_1)

In the present section we mathematically model and solve the first economic problem based on some given restrictions. Within the restrictions of our practical problem the values of the efficiencies of the professional training programs does not interfere. It interferes just the arrangement of the efficiency of one professional training program in relation to the other professional training programs. Therefore, we assume that the arrangement of the professional training programs was done in a descending order of their efficiency, i.e. $e_i \geq e_{i+1}$, $\forall\, i \in I$.

Let $\mathcal{Y}$ be the set of the matrices $Y = [y_{ij}] \in M_{m\times n}(\mathbb{R})$ which fulfill the following conditions:

$$y_{ij} \in \{0, 1\}, \forall\, i \in I, \forall\, j \in J; \tag{4.1}$$

$$\sum_{i\in I} y_{ij} = 1, \forall\, j \in J; \tag{4.2}$$

$$\sum_{j\in J} y_{ij} \leq a_i, \forall\, i \in I. \tag{4.3}$$

Let $f = (f_1, f_2, f_3) : \mathcal{Y} \to \mathbb{R}^3$ be the function given by: $\forall\, Y \in \mathcal{Y}$,

$$f_1(Y) = \min\Big\{r_{ij} \,|\, i \in I,\, j \in J,\, y_{ij} = 1\Big\}, \tag{4.4}$$

$$f_2(Y) = \min\Big\{i \in I \,|\, \exists\, j \in J \quad \text{such that} \quad r_{ij}y_{ij} = f_1(Y)\Big\}, \tag{4.5}$$

$$f_3(Y) = \sum_{(i,j)\in I\times J;\, r_{ij}y_{ij}=f_1(Y);\, i\geq f_2(Y)} y_{ij}. \tag{4.6}$$

We work under the hypothesis that

$$\sum_{i\in I} a_i \geq n, \tag{4.7}$$

i.e. the total number of the persons that can attend the professional training programs is greater than the total number of the registered unemployed persons which need to attend it. Condition (4.7) assures that $\mathcal{Y} \neq \emptyset$. If condition (4.7) is not fulfilled, then $\mathcal{Y} = \emptyset$.

Our problem can be graphically given by the following problem:

$$(PS) \quad \begin{cases} f(Y) \to lex - \max - \max - \min, \\ Y \in \mathcal{Y}. \end{cases} \tag{4.8}$$

Definition 4.3.1 (Tuns (Bode) O.R. *and* Neamţiu L. [*119*]). *A point $Y^0 \in \mathcal{Y}$ is said to be an optimal solution of the problem (PS) if there is no other point $Y \in \mathcal{Y}$ such that neither one of the following restrictions to occur:*
i) $f_1(Y) > f_1(Y^0)$;
ii) $f_1(Y) = f_1(Y^0)$ and $f_2(Y) > f_2(Y^0)$;
iii) $f_1(Y) = f_1(Y^0)$, $f_2(Y) = f_2(Y^0)$ and $f_3(Y) < f_3(Y^0)$.

We remark that the mathematical model attached to the economic problem is a problem of lexicographic optimization type. Based on the restrictions (4.2) and (4.3), the problem (PS) can be view as a particular type of an unbalanced transportation problem of bottleneck type, having the property that all its variables have a boolean value. On the other hand, based on restriction (4.1), the problem (PS) can be view as a generalization of the bottleneck assignment problem. Also, the problem can be seen as a resources assignment problem [90]. Whatever we consider this problem, as far as we know a such type of problem have not been studied yet.

Based on the notions given in Chapter 2, (PS) is a lexicografic bottleneck threecriteria assignment problem. Its optimal solution is not a minimum point with pipeline property. But it fulfill a pipeline condition imposed by f_3. Therefore, we can consider the problem (PS) as a generalization of the problem studied in Chapter 2. The particular form of the scalar components of the objective function, as well as the set of feasible solutions, allows us to give a specific method for solving it.

4.3.2 Necessary and Sufficient Optimality Conditions of (PS)

Let

$$\lambda = \min\{r_{ij} \,|\, i \in I,\, j \in J\}, \tag{4.9}$$

$$h = \min\{i \in I \,|\, \exists\, k \in J \,\text{such that}\, r_{ik} = \lambda\}, \tag{4.10}$$

$$J_{h,\lambda} = \{j \in J \,|\, r_{hj} = \lambda\}, \tag{4.11}$$

and

$$q = card(J_{h,\lambda}). \tag{4.12}$$

Based on the sign of the inequality between $\sum\limits_{i\in I} a_i$ and $n+q$ two different cases can occur, highlighted by the following propositions.

Proposition 4.3.3 (Tuns (Bode) O.R. *and* Neamţiu L. [*119*]). *If*

$$\sum_{i\in I} a_i < n + q, \tag{4.13}$$

then for each $Y \in \mathcal{Y}$ the following conditions occur: $f_1(Y) = \lambda$ and $f_2(Y) = h$.

Remark 4.3.4 *If (4.13) holds, then for each $Y^0 \in \mathcal{Y}$ an optimal solution of the problem (PS), we have that $f_1(Y^0) = \lambda$ and $f_2(Y^0) = h$.*

Proposition 4.3.5 (Tuns (Bode) O.R. *and* Neamţiu L. [*119*]). *Let $Y^0 \in \mathcal{Y}$ be an optimal solution of the problem (PS). If $\sum_{i\in I} a_i \geq n + q$, then we have $y^0_{hj} = 0$, $\forall\, j \in J_{h,\lambda}$.*

From the above proposition, it results that if $q = n$, then to determine an optimal solution of the problem (PS) is equivalent to determine an optimal solution of a problem of the same type as the problem (PS), but in which in the scores matrix the line h does not appear. Therefore, in what follows, we suppose that $q < n$.

The method introduced for solving the problem (PS) is based on getting the optimal solution in two stages. In the first stage, we determine a minimum lexicographic point Y^0 of the bicriteria function for which the scalar components are f_1 and f_2 on the set $\mathcal{Y}$. In the second stage we verify if Y^0 is an optimal solution of the problem (PS). If not, we give a way of replacing Y^0 with another lexicographic point. This process continue until it is found the optimal solution of the problem (PS). Since the set $\mathcal{Y}$ is finite, obviously the method introduced by us is a method which leads, after a finite number of iterations, to an optimal solution.

In what follows we consider the problem:

$$(PM) \quad \begin{cases} \varphi(Y) = \begin{pmatrix} f_1(Y) \\ f_2(Y) \end{pmatrix} \to lex - \max - \max, \\ Y \in \mathcal{Y}. \end{cases} \tag{4.14}$$

Let $\lambda \in \mathbb{R}^n_+$ and $h \in I$. We set the matrix $C_{\lambda,h} = [c_{ij}]$ such that

$$c_{ij} = \begin{cases} 0, & \text{if} \quad r_{ij} > \lambda, \\ 1, & \text{if} \quad r_{ij} = \lambda \quad \text{and} \quad i \geq h, \\ n+1, & \text{if} \quad (r_{ij} < \lambda) \quad \text{or} \quad (r_{ij} = \lambda \quad \text{and} \quad i < h). \end{cases} \tag{4.15}$$

Now, let us consider the following optimization problem:

$$(AC_{\lambda,h}) \quad \begin{cases} \sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}y_{ij} \to \min, \\ Y \in \mathcal{Y}. \end{cases} \tag{4.16}$$

The problem $(AC_{\lambda,h})$ represents a transportation problem which can be solved using the potential plan algorithm.

The below results reflect the bond existing between the optimal solutions of the problems (PS), (PM) and (AC$_{\lambda,h}$).

Proposition 4.3.6 (Tuns (Bode) O.R. *and* Neamţiu L. [*119*]). *If $\tilde{Y} \in \mathcal{Y}$ is an optimal solution of the problem (PS), $\lambda = f_1(\tilde{Y})$ and $h = f_2(\tilde{Y})$, then $\tilde{Y}$ is an optimal solution of the problem (PM) and an optimal solution of the problem $(AC_{\lambda,h})$.*

Proposition 4.3.7 (Tuns (Bode) O.R. *and* Neamţiu L. [*119*]). *If $\tilde{Y} \in \mathcal{Y}$ is an optimal solution of the problem (PM) and an optimal solution of the problem $(AC_{\lambda,h})$, where $\lambda = f_1(\tilde{Y})$ and $h = f_2(\tilde{Y})$, then $\tilde{Y}$ is an optimal solution of the problem (PS).*

From Propositions 4.3.6 and 4.3.7 it results that the solving of the problem (PS) can be reduced by solving two problems: on one hand, we have to solve the problem (PM) and, on the other hand, we have to verify the optimality of the solution $\tilde{Y}$ of the problem (PM) for the problem $(AC_{\lambda,h})$, where $\lambda = f_1(\tilde{Y})$ and $h = f_2(\tilde{Y})$.

Since the problem $(AC_{\lambda,h})$ is an unbalanced transport problem, we transform it to a balanced transport problem.

Let $\overline{J} = J \cup \{n+1\}$, $b_j = 1$, $\forall\, j \in J$, $b_{n+1} = \sum\limits_{s\in I} a_s - n$,

$$c^*_{ij} = \begin{cases} c_{ij}, & \text{if} \quad i \in I,\ j \in J, \\ 0, & \text{if} \quad i \in I,\ j = n+1, \end{cases} \tag{4.17}$$

and $\mathcal{Z} = \{Z = [z_{ij}] \in M_{m\times\{n+1\}}(\{0,1\}) | \sum\limits_{j\in J} z_{ij} = a_i,\ \forall\, i \in I,\ \sum\limits_{i\in I} z_{ij} = b_j,\ \forall\, j \in J\}$.

Let us consider the balanced transport problem:

$$(AC^*_{\lambda,h}) \quad \begin{cases} f^*_c(Z) = \sum\limits_{i\in I}\sum_{j\in\overline{J}} c^*_{ij}z_{ij} \to \min, \\ Z \in \mathcal{Z}. \end{cases} \tag{4.18}$$

Applying the potential plan theorem to the (AC$^*_{\lambda,h}$) problem and based on the connection between the problems (AC$^*_{\lambda,h}$) and (AC$_{\lambda,h}$), it results the following necessary and sufficient optimality condition.

Theorem 4.3.8 (Tuns (Bode) O.R. [*116*]). *$Y \in \mathcal{Y}$ is an optimal solution of the problem $(AC_{\lambda,h})$ if and only if $\sum_{s\in I} c_{sj}y_{sj} \le c_{ij}$, $\forall\, i \in I$, $\forall\, j \in J$.*

From Proposition 4.3.6, Proposition 4.3.7 and Theorem 4.3.8 it results the following important result.

Theorem 4.3.14 (Tuns (Bode) O.R. [*116*]). *The matrix* $\tilde{Y} \in \mathcal{Y}$ *is an optimal solution of the problem (PS) if and only if* $\tilde{Y}$ *it is an optimal solution of the problem* (PM) *and* $\sum\limits_{s \in I} c_{sj} y_{sj} \leq c_{ij}$, $\forall\, i \in I$, $\forall\, j \in J$, *where*

$$c_{ij} = \quad \begin{cases} 0, & if \quad r_{ij} > f_1(\tilde{Y}), \\ 1, & if \quad r_{ij} = f_1(\tilde{Y}) \quad and \quad i \geq f_2(\tilde{Y}), \\ n+1, & if \quad (r_{ij} < f_1(\tilde{Y})) \quad or \quad (r_{ij} = f_1(\tilde{Y}) \;\; and \;\; i < f_2(\tilde{Y})). \end{cases} \tag{4.19}$$

4.3.3 A Technique for Solving the Problem (PM)

Based on Propositions 4.3.3 and 4.3.5 we give a polynomial technique for solving the problem (PM). The efficiency of this technique results from the fact that we pass through the scores matrix R from up to down, setting, at each iteration, at least the value of one of the variables y^0_{ij} from the optimal solution. We note that this technique can be used also for solving the bottleneck assignment problem type.

4.4 The Study of the Problem (AEP_2)

In the present section we mathematically model and solve the second economic problem under the circumstances that, on one hand, there exists no restriction regarding the budget allocated for it and, on the other hand, under the circumstances that the maximum number of the persons that can attend the professional training programs is smaller than the total number of the registered unemployed persons which need to attend the courses. The goal is to train as much as many unemployed persons, to maximize the minimum scores of the assignments and that the number of the unemployed persons for which the minimum score is reached to be as small as possible.

Under the above circumstances, furthermore we work under the hypothesis that

$$\sum_{i \in I} a_i < n. \tag{4.20}$$

Using the notations introduced in Subsection 4.2, we consider the following lexicographic optimization problem:

$$(PMR) \quad \begin{cases} \varphi_1(Y) = \begin{pmatrix} \sum\limits_{i \in I} \sum_{j \in J} y_{ij} \\ \min\left\{ r_{ij} \,|\, i \in I,\, j \in J,\, y_{ij} = 1 \right\} \end{pmatrix} \to lex - \max - \max, \\ \sum\limits_{j \in J} y_{ij} \leq a_i,\, \forall\, i \in I, \\ \sum\limits_{i \in I} y_{ij} \leq 1,\, \forall\, j \in J, \\ y_{ij} \in \{0, 1\},\, \forall\, i \in I,\, \forall\, j \in J. \end{cases} \tag{4.21}$$

Let us denote by Ω the set of the feasible solutions of the problem (PMR), i.e.

$$\Omega = \Big\{ Y = [y_{ij}] \in \mathcal{M}_{m\times n}(\{0,1\}) \,|\, \sum_{i\in I} y_{ij} \le 1,\, \forall\, j \in J;\, \sum_{j\in J} y_{ij} \le a_i,\, \forall\, i \in I \Big\}. \tag{4.22}$$

Since we work under the hypothesis (4.20), the maximum value of the sum $\sum_{i\in I}\sum_{j\in J} y_{ij}$ is $\sum_{i\in I} a_i$.

We remark that if we take

$$y^*_{ij} = \begin{cases} 1, \text{ if } i = 1,\, j \in \{1, \dots, a_1\}, \\ 0, \text{ if } i = 1,\, j \in \{a_1 + 1, \dots, \sum\limits_{i\in I} a_i\}, \\ 1, \text{ if } i \in I \setminus \{1\},\, j \in \Big\{ \sum\limits_{k=1}^{i-1}(a_k + 1), \dots, \sum\limits_{k=1}^{i} a_k \Big\}, \\ 0, \text{ if } i \in I \setminus \{1\},\, j \in \Big\{1, \dots, \sum_{k=1}^{i-1} a_k\Big\} \cup \Big\{ \sum_{k=1}^{i}(a_k+1), \dots, n\Big\}. \end{cases} \tag{4.23}$$

then we have that $Y^* = [y^*_{ij}] \in \Omega$. Moreover $\sum\limits_{i\in I}\sum\limits_{j\in J} y^0_{ij} = a_1 + \cdots + a_i + \cdots + a_m$. Based on the above, the solving of the problem (PMR) is reduced to solving the following problem:

$$(PMR_1) \quad \begin{cases} \min\Big\{ r_{ij} \,|\, i \in I,\, j \in J,\, \sum\limits_{j\in J} y_{ij} = a_i,\, \forall\, i \in I, \\ \sum\limits_{i\in I} y_{ij} \le 1,\, \forall\, j \in J, \\ y_{ij} \in \{0,1\},\, \forall\, i \in I,\, \forall\, j \in J. \end{cases} \tag{4.24}$$

Furthermore, we modify the problem (PMR_2) such that it can be solved by applying the method described for solving the problem (PM).

Let $r_{m+1,j} := 1 + \max\{r_{ij} \,|\, i \in I,\, j \in J\}$, and $a_{m+1} := n - \sum\limits_{i\in I} a_i$ and $\bar{I} := I \cup \{m+1\}$. Let us consider the problem:

$$(PMR_2) \quad \begin{cases} \min\Big\{ r_{ij}y_{ij} \,|\, i \in \bar{I},\, j \in J \Big\} \to \max, \\ \sum\limits_{j\in J} y_{ij} = a_i,\, \forall\, i \in \bar{I}, \\ \sum\limits_{i\in \bar{I}} y_{ij} = 1,\, \forall\, j \in J, \\ y_{ij} \in \{0,1\},\, \forall\, i \in \bar{I},\, \forall\, j \in J. \end{cases} \tag{4.25}$$

Proposition 4.4.1 (Tuns (Bode) O.R. [*116*]). *i) If $\bar{Y} = [\bar{y}_{ij}] \in \mathcal{M}_{(m+1)\times n}(\{0,1\})$ is an optimal solution of the problem (PMR_2), then $Y^* = [y^*_{ij}] \in \mathcal{M}_{m\times n}(\{0,1\})$ is an optimal solution of the problem (PMR_1).*

ii) If $Y^* = [y^*_{ij}] \in \mathcal{M}_{m\times n}(\{0,1\})$ is an optimal solution of the problem (PMR_1), then taking

$$\bar{y}_{m+1,j} = \begin{cases} 0, \ \textit{if } \sum\limits_{i\in I} y^*_{ij} = 1 \\ 1, \ \textit{if } \sum\limits_{i\in I} y^*_{ij} = 0 \end{cases}, \forall\, j \in J, \tag{4.26}$$

and $\bar{y}_{ij} = y^*_{ij}$, $\forall\, i \in I$, $\forall\, j \in J$, we have that the matrix $\bar{Y} = [\bar{y}_{ij}] \in \mathcal{M}_{(m+1)\times n}(\{0,1\})$ is an optimal *solution of the problem* (PMR_2).

The solving of the problem (PMR_2) can be done applying the technique described in the above paragraph. In the thesis an easiest example to point out how the technique works is given.

Chapter 5

Practical Applications Related to Portfolio Optimization

My professional experience acquired so far in the economic field proved that the mathematical tools can simplify very much the decision process of a firm's management, such that the results of its activity to be advantageous. One of the many difficult tasks which assume taking an important decision is investment, because it supposes to use an amount of money in order to increase that amount. But, if investment is not done taking into account some restrictions, the result can be an disadvantageous one for the investor. That is the reason why in the present chapter my goal is to identify economic-financial problems related to portfolio theory area wherein by using the optimization theory we are guided to optimal solutions viable from the practical point of view. So, within the present chapter we focus our attention on mathematical modeling and solving problems associated with portfolio optimization.

We note that the results within this chapter belong to the author and can be find in the papers authored by LUPŞA L. and TUNS (BODE) O.R. [75], respectively by TUNS (BODE) O.R. [110], [111] and [112]. In [110] we treat the portfolio selection problem as a bicriteria optimization problem. This allows us to obtain a new perspective concerning the investor's objective. In this way we obtain a new mathematical model for the portfolio selection problem, in which the new objective function is equal to risk/return ratio, while adding some specified restrictions. In [75], based on a concrete economic problem, we study the case in which both the objective function of the upper level problem and of the lower level problem are linear, while the problem restrictions coincide with the ones of a E-type problem. In [111] and [112] a kind of bilevel optimization problem in 0-1 variables, based on the mathematical model attached by us to a concrete portfolio optimization problem, is analyzed. The upper level function is to be maximized, while the lower level function (which is a bicriteria function) is to be maximized-minimized in the lexicographic sense. The core idea of these papers is to present a way for solving the proposed bilevel problem by reducing it to a finite number of couples of linear pseudo boolean optimization problems. One of these last types of problems is an assignment problem.

5.1 A Relation between Portfolio Selection Problems and Bicriteria Optimization

5.1.1 Basic Notions Related to Portfolio Theory

Modern portfolio theory represents the scientific approach to investment. It deals with the selection of portfolios for investors who wish to maximize the expected return for the level of risk each investor is willing to assume. We recall in the present subsection some basic notions related to portfolio theory. As useful papers in this area of research we recall [81], [82], [83], [54], [56], [79], [87], [130], [97].

5.1.2 Portfolio Selection Models of Markowitz Type

We recall the portfolio selection models of Markowitz type, i.e. the problem to maximize the expected return given a specific risk level, the problem to minimize the risk given a specific expected return level and the problem of balancing the return and the risk, emphasizing in mathematical terms the portfolio selection problem. Despite the fact that the portfolio selection models of Markowitz's type are considered to be simplified models because they take into account only the average and variance of a portfolio profit, they are the most commonly used nowadays, remaining the cornerstone of modern portfolio theory.

5.1.3 Portfolio Selection Problem versus Bicriteria Optimization

In the present paragraph we treat the portfolio selection problem as a bicriteria optimization problem. This allows us to obtain a new perspective concerning the investor's objective. In this way, we obtain a new mathematical model related to portfolio selection problem in which the new objective function is equal to risk/return ratio, while adding some specified restrictions.

Let $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$ be a vector function, where f_1 is the risk function and f_2 is the return function. We can view the general portfolio selection problem as a bicriteria minimization problem:

$$\text{(PV)} \qquad \begin{cases} (f(s)) = \begin{pmatrix} f_1(s) \\ -f_2(s) \end{pmatrix} \to \text{v} - \min, \\ s \in \Omega. \end{cases} \tag{5.1}$$

We remark that a point $s^0 \in \Omega$ is a global optimal solution of the problem (PV) if s^0 is both a minimum point of f_1 with respect to Ω and a maximum point of f_2 with respect to Ω (or a minimum point of $-f_2$ with respect to Ω). Generally, there are rare cases when global optimal points exist. Most of the times the minimum point of f_1 with respect to Ω is not the maximum point of f_2 with respect to Ω and vice versa. Therefore, for the problem (PV) the global optimal solutions usually do not exist (only in very special cases).

In real life situations, usually the risk can not exceed a specified value $\overline{m}$ and the expected return must be greater than a given value $\underline{m} > 0$.

Let $\tilde{\Omega} = \{s \in \Omega |\, f_1(s) \leq \overline{m},\ f_2(s) \geq \underline{m}\}$. Based on the idea given by Stancu-Minasian

I.M. [108] (§ 1.1), we introduce the function $F : \Omega \to \overline{\mathbb{R}}$ given by

$$F(s) = \begin{cases} \dfrac{f_1(s)}{f_2(s)}, \text{ if } f_2(s) \neq 0, \\ +\infty, \text{ if } f_2(s) = 0. \end{cases}$$

Using the function F we introduce the following preference relation.

Definition 5.1.1 (Tuns (Bode) O.R. [*110*]). *We say that a point $s^0 \in \Omega$ is strictly preferred to another point $s \in \Omega$ with respect to relation $\succ$, and we denote by $s^0 \succ s$, if*
a) $s^0 \in \tilde{\Omega}$ and $s \notin \tilde{\Omega}$; or b) $s^0 \in \tilde{\Omega}$, $s \in \tilde{\Omega}$ and $F(s) > F(s^0)$.
A point $s^0 \in \Omega$ is called non-dominated if $s^0 \in \tilde{\Omega}$ and there is no point $s \in \Omega$ strictly preferred to s^0.

Proposition 5.1.2 (Tuns (Bode) O.R. [*110*]). *If the (PV) problem admits an ideal point $s^0 \in \Omega$ and the values $\overline{m}$ and $\underline{m}$ fulfill the natural conditions $f_1(s^0) \leq \overline{m}$ and $f_2(s^0) \geq \underline{m}$, then s^0 is a non-dominated point with respect to the preference relation $\succ$.*

From Definition 5.1.1 it results that a point $x^0 \in \Omega$ is a non-dominated point with respect to the preference relation $\succ$ if and only if is the optimal solution of the following optimization problem:

$$(\mathrm{PO}) \qquad \begin{cases} F(s) \ \to \ \min, \\ \sum\limits_{j=1}^{n} s_j = M, \\ f_1(s) \leq \overline{m}, \\ f_2(s) \geq \underline{m}, \\ s \in \mathbb{R}_+^n. \end{cases}$$

Proposition 5.1.4 (Tuns (Bode) O.R. [*110*]). *If $s^0 \in \Omega$ is an optimal solution of the problem (PO) and $\underline{m} > 0$, then s^0 is a* min-*efficient point of the function $f = (f_1, -f_2)$ with respect to the set $\tilde{\Omega}$.*

Remark 5.1.5 *Based on the above and on the fact that, generally, the (PV) problem does not have a global optimal solution, we call optimal portfolio any point $s^0 \in \Omega$ which is non-dominated with respect to the preference relation $\succ$.*

5.1.4 A Kind of Boolean Portfolio Selection Problem

In the present section we emphasize the effect of the new type of objective function introduced by us by considering the particular case of the portfolio selection problem wherein we work with boolean values. A technique for solving this new problem is given.

5.2 A Kind of Portfolio Selection Problem

In the present section we study a kind of portfolio selection problem which represents the mathematical model attached to a concrete economic problem. By using the bilevel optimization

for mathematically modeling this problem, we consider two different cases which implies the study of two different problems:
(i) a boolean portfolio selection problem based on a single period of investment and on the case when a stock portfolio contains stocks which have the same quotation on the capital market;
(ii) a boolean portfolio selection problem based on a period of investment divided in several subperiods of time and on the case when there are different stock portfolios on the capital market and there exists more restrictions concerning the investment.

In each case we provide a method to solve the problem. For the second problem we also propose an algorithm to solve it.

5.2.1 Formulation of the Economic Problem

Let S be a firm which owns n subsidiaries denoted by S_j, $j \in J = \{1, \ldots, n\}$. The firm needs to invest in some stock portfolios available on the capital market.

Let P_i, $i \in I = \{1, \ldots, m\}$, $m > n$, be the stock portfolios in which the firm S will invest. For each stock portfolio P_i, $i \in I$, the firm S has historical data based on which it can predict the expected return for a certain level of risk undertaken for a period of time T.

The firm S can make transactions with the stock portfolios in two different ways:
i) directly, through its n branches;
ii) indirectly, through p companies denoted by C_k, $k \in K = \{1, \ldots, p\}$, within a group of companies C specialized in financial investment services.

Therefore, the firm's return is equal to the sum of direct return (achieved from the investment made through its branches) and indirect return (achieved from the investment made through the specialized investment companies). In the second case, the company engaged by the firm to invest on its behalf will get its own return from the transactions made and, based on an agreed share, will yield a part of the return to the firm.

We remark that both firms play a Stackelberg game: the leader (firm S) acts first and chooses those stock portfolios in which it will invest directly through its branches, after which the follower (company C) responds by its own transactions with the remained stock portfolios. We point out the game restrictions for both players:
- for the leader: each branch will transact with exactly one stock portfolio in such a way it maximizes its return;
-for the follower: the company (engaged by the firm to invest on its behalf) will get its own return from the transactions made and, based on an agreed share, will yield a part of the return to the firm. The investment company must transact with all stock portfolios that were not chosen by the leader to invest directly.

This economic problem leads us, thinking from the mathematical point of view, to a typical case of bilevel optimization problem.

5.2.2 Modeling and Solving the Portfolio Selection Problem

We first begin by considering the particular portfolio selection problem in the case there exists a single period of time of investment for each firm and a stock portfolio contains stocks which have the same quotation on the capital market.

Let $f_1 : \mathbb{R}^{m\times n} \to \mathbb{R}$ be the function given by $f_1(X) = \sum\limits_{i\in I}\sum\limits_{j\in J} a_{ij}x_{ij},\ \forall\, X = [x_{ij}] \in \mathbb{R}^{m\times n}$. It represents the total return of the firm S gained by its direct investment.

Let $g : \mathbb{R}^{m\times p} \to \mathbb{R}$ be the function given by $g(Y) = \sum\limits_{i\in I}\sum\limits_{k\in K} c_{ik}y_{ik},\ \forall\, Y = [y_{ik}] \in \mathbb{R}^{m\times p}$. It represents the total return of the companies C_k, $k \in K$.

Let $f_2 : \mathbb{R}^{m\times p} \to \mathbb{R}$ be the function given by $f_2(Y) = \sum\limits_{i\in I}\sum\limits_{k\in K} b_{ik}y_{ik},\ \forall\, Y = [y_{ik}] \in \mathbb{R}^{m\times p}$. It represents the total return of the firm S gained by its indirect investment.

The mathematical model for the portfolio selection problem is given by the following problem:

$$(EB)\quad \begin{cases} f_1(X) + f_2(Y) \to \max, \\ X \in \Lambda, \\ Y \in U^{*X}, \end{cases}$$

where $\Lambda = \{X = [x_{ij}] \in \{0,1\}^{m\times n} \mid \sum_{i\in I} x_{ij} = 1,\ \forall\, j \in J,\ \sum_{j\in J} x_{ij} \le 1, \forall i \in I\}$ and U^{*X} is the set of optimal solutions of the problem

$$(P2_X)\quad \begin{cases} g(Y) = \sum\limits_{i\in I}\sum\limits_{k\in K} c_{ik}y_{ik} \to \max, \\ Y \in U^X, \end{cases}$$

with $U^X = \{Y^X = [y^X_{ik}] \in \{0,1\}^{m\times p} | \sum\limits_{i\in I} y^X_{ik} = 1, \forall\, k \in K,\ \sum\limits_{k\in K} y^X_{ik} = 1 - \sum\limits_{j\in J} x_{ij},\ \forall\, i \in I\}$.

Furthermore, we analyse the way in which the (EB) problem can be solved by using the splitting technique. We introduce the set $V = \{v = (v_1, ..., v_m) \in \{0,1\}^m \mid v_1 + \cdots + v_m = n\}$.

For each $v = (v_1, ..., v_m) \in V$, we set $\Lambda^v = \{X = [x_{ij}] \in \Lambda \mid \sum\limits_{j\in J} x_{ij} = v_i,\ \forall\, i \in I\}$ and $U^v = \{Y = [y_{ik}] \in \{0,1\}^{m\times p} \mid \sum\limits_{i\in I} y_{ik} = 1, \forall\, k \in K,\ \sum\limits_{k\in K} y_{ik} = 1 - v_i, \forall\, i \in I\}$.

In what follows, for each $v \in V$, we consider the problems

$$(P_1^v)\quad \begin{cases} f_1(X) = \sum\limits_{i\in I}\sum\limits_{j\in J} a_{ij}x_{ij} \to \max, \\ X \in \Lambda^v \end{cases}$$

and

$$(P_3^v)\quad \begin{cases} g^v(Y) = \sum\limits_{i\in (I\setminus M_v)}\sum\limits_{k\in K} c_{ik}y_{ik} \to \max, \\ Y \in U^v, \end{cases}$$

where $M_v = \{i \in I \mid v_i = 1\}$.

Let F_1^v be the maximum value of f_1 on Λ^v, $\mathcal{X}^v$ the set of optimal solutions of the problem (P_1^v), G^v the maximum value of g^v on U^v, and $\mathcal{Y}^v$ the set of optimal solutions of the problem (P_3^v).

In what follows, we study the relation between the (EB) problem and the problems (P_1^v) and (P_3^v).

Theorem 5.2.1 (Tuns (Bode) O.R. [*111*]). *If (X^0, Y^0) is an optimal solution of the problem (EB), then there exists $v^0 = (v_1^0, \ldots, v_m^0) \in V$ such that X^0 is an optimal solution of the problem $(P_1^{v^0})$ and Y^0 is an optimal solution of the problem $(P_3^{v^0})$.*

Now, let $F : V \to \mathbb{R}$, $F(v) = F_1^v + \max\{f_2(Y) | Y \in U^v\}$ and let us consider the problem:

$$(EBV) \quad \begin{cases} F(v) \to \max, \\ v \in V. \end{cases}$$

Theorem 5.2.3 (Tuns (Bode) O.R. [*111*]). *If function g is injective and v^0 is an optimal solution of the problem (EBV), then (X^0, Y^0) is an optimal solution of the problem (EB), for each $X^0 \in \mathcal{X}^{v^0}$ and $Y^0 \in \mathcal{Y}^{v^0}$.*

Example 5.2.4 from the thesis present emphasize the way of solving the problem (EBV).

Corollary 5.2.6 (Lupşa L. *and* Tuns (Bode) O.R. [*75*]). *If v^0 is an optimal solution of the problem (EBV), then there exists $Y^0 \in \mathcal{Y}^{v^0}$ such that (X^0, Y^0) is an optimal solution of the problem (EB), for each $X^0 \in \mathcal{X}^{v^0}$.*

5.2.3 Modeling and Solving the Extended Portfolio Selection Problem

In the present section we consider the second problem formulated in Section 5.2.

The mathematical model attached to this economic problem is given by:

$$(EBCT) \begin{cases} f(X,Y) = \sum\limits_{i\in I}\sum\limits_{j\in J} a_{ij}x_{ij} + \sum\limits_{i\in I}\sum\limits_{k\in K} b_{ik}(\sum\limits_{h\in H} y_{ikh}) \to \max, \\ X = [x_{ij}] \in \Omega_1, \\ \max\{r_{ij}x_{ij} \,|\, i \in I,\, j \in J\} \le e, \\ Y = [y_{ikh}] \in \mathcal{Y}^*(X) \text{ with } Y \in \{0,1\}^{m\times p\times s}, \end{cases}$$

where $\mathcal{Y}^*(X)$ is the set of all points of $\mathcal{Y}(X)$ which are max-p min-max points (see Definition 5.2.9 from the thesis).

The (EBCT) problem is a bilevel optimization problem for which the lower level function is bicriteria of cost-bottleneck type. As far as we know, this kind of bilevel optimization problem have not been discussed in the literature. The bilevel optimization problem presented above it is new by its particular structure of the objective function of the lower level, on one side, being of discrete variables and, on the other side, allowing the elaboration of an easiest method to solve it.

The particularity of the constraints allows us to give a finite algorithm for solving the (EBCT) problem. If we consider X as a parameter, then there exists the possibility of splitting the set Ω_1 in a finite number of subsets, less or equal to C_m^n. Hence, we introduce the set

$$V = \{v = (v_1, ..., v_m) \in \{0,1\}^m \,|\, v_1 + \cdots + v_m = n\}.$$

For each $v = (v_1, ..., v_m) \in V$ we set

$$U^v = \{i \in I | v_i = 1\}, \quad \bar{U}^v = \{i \in I | v_i = 0\} = I \setminus U^v,$$

$$\Lambda^v = \{X = [x_{ij}] \in \{0,1\}^{m\times n} \,|\, \sum_{i\in I} x_{ij} = 1, \forall j \in J, \quad \sum_{j\in J} x_{ij} = v_i, \forall\, i \in I\}$$

and

$$\mathcal{X}^v = \{X = [x_{ij}] \in \Lambda^v \,|\, \max\{r_{ij}x_{ij} \,|\, i \in U^v,\, j \in J\} \le e\}.$$

Hence, if we set $\mathcal{V}_1 = \{v \in V \,|\, \mathcal{X}^v \neq \emptyset\}$, then we have $\bigcup\limits_{v\in\mathcal{V}_1} \Lambda^v = \mathcal{X}$. Also, for each $v \in V$, we set

$$\begin{aligned} W^v = &\Big\{ Y = [y_{ikh}] \in \{0,1\}^{m\times p\times s} \,| \\ &\operatorname{sgn}\Big(\sum_{i\in\bar{U}^v} \sum_{h\in H} y_{ikh} \Big) = 1, \, \forall\, k \in K, \\ &\operatorname{sgn}\Big(\sum_{k\in K} \sum_{h\in H} y_{ikh} \Big) = 1, \, \forall\, i \in \bar{U}^v, \\ &\sum_{k\in K} y_{ikh} \leq 1, \, \forall\, i \in \bar{U}^v, \, \forall\, h \in H, \\ &y_{ikh} = 0, \forall\, i \in U^v, \, \forall\, k \in K, \, \forall\, h \in H \Big\} \end{aligned}$$

and

$$\mathcal{Y}^v = \Big\{ Y \in W^v \,|\, \max\{d_{ikh} y_{ikh} \,|\, i \in \bar{U}^v, k \in K\} \leq e_h, \, \forall h \in H \Big\}.$$

Let $\mathcal{V}_2 = \{v \in V \,|\, \mathcal{Y}^v \neq \emptyset\}$ and $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$.

Remark 5.2.10 (Tuns (Bode) O.R. [*112*]). *For a given $v \in V$ and $X \in \Lambda^v$, it is obvious that:*

$$x_{ij} = 0, \quad \forall\, i \in \bar{U}^v, \quad \forall\, j \in J; \tag{5.2}$$

$$\sum_{j\in J} x_{ij} = 1, \forall\, i \in U^v, \quad \sum_{i\in U^v} x_{ij} = 1, \quad \forall\, j \in J; \tag{5.3}$$

$$\max\{r_{ij} x_{ij} \,|\, i \in I, \, j \in J\} = \max\{r_{ij} x_{ij} \,|\, i \in U^v, \, j \in J\}. \tag{5.4}$$

Remark 5.2.11 (Tuns (Bode) O.R. [*112*]). *Based on the above equalities, we deduce that:*
(i) If $X \in \mathcal{X}$, $Y \in \mathcal{Y}(X)$ and $v = (\sum_{j\in J} x_{1j}, \ldots, \sum_{j\in J} x_{mj})$, then $v \in \mathcal{V}$, $X \in \mathcal{X}^v$ and $Y \in \mathcal{Y}^v$.
(ii) If $v \in \mathcal{V}$, $X \in \mathcal{X}^v$ and $Y \in \mathcal{Y}^v$, then $X \in \mathcal{X}$ and $Y \in \mathcal{Y}(X)$.
(iii) If $v \in \mathcal{V}$ and X', $X'' \in \mathcal{X}^v$, then $\mathcal{Y}^*(X') = \mathcal{Y}^*(X'')$.

In what follows, for each $v \in \mathcal{V}$ we consider the problem

$$(\mathrm{P}_1^v) \qquad \begin{cases} f_1(X) = \sum\limits_{i\in I} \sum\limits_{j\in J} a_{ij} x_{ij} \to \max, \\ X \in \mathcal{X}^v. \end{cases}$$

Let us denote by $\mathcal{X}_0^v$ the set of optimal solutions of the problem (P_1^v).

We denote by (P_2^v) the problem of determining the $\max -p \min - \max$ points of the vector function φ with respect to the set $\mathcal{Y}^v$ and by $\mathcal{Y}_0^v$ the set of such kind of points.

Theorem 5.2.12 (Tuns (Bode) O.R. [*112*]). *If (X^0, Y^0) is an optimal solution of the (EBCT) problem and*

$$v^0 = \Big(\sum_{j\in J} x_{1j}^0, \ldots, \sum_{j\in J} x_{ij}^0, \ldots, \sum_{j\in J} x_{mj}^0 \Big), \tag{5.5}$$

then $v^0 \in \mathcal{V}$, $X^0 \in \mathcal{X}_0^{v^0}$ *and* $Y^0 \in \mathcal{Y}_0^{v^0}$.

Therefore, we can consider the function $F : \mathcal{V} \to \mathbb{R}$ given by

$$F(v) = \sum_{i \in U^v} \sum_{j \in J} a_{ij} x_{ij} + \sum_{i \in \bar{U}^v} \sum_{k \in K} b_{ik} \left(\sum_{h \in H} y_{ikh} \right), \quad \forall\, v \in \mathcal{V}, \tag{5.6}$$

where $X \in \mathcal{X}_0^v$ and $Y \in \mathcal{Y}_0^v$ are chosen arbitrary.

Now, let us consider the problem

$$(EBVT) \quad \begin{cases} \text{Find} \quad v^0 \in \mathcal{V} \quad \text{such} \quad \text{that} \\ F(v^0) \ = \ \max\{F(v) \,|\, v \in \mathcal{V}\}. \end{cases}$$

Theorem 5.2.13 (Tuns (Bode) O.R. [*112*]). *If* v^0 *is an optimal solution of the (EBVT) problem, then the couple* (X^0, Y^0) *is an optimal solution of the (EBCT) problem, for each* $X^0 \in \mathcal{X}_0^{v^0}$ *and* $Y^0 \in \mathcal{Y}_0^{v^0}$.

Theorems 5.2.12 and 5.2.13 allow us to reduce the solving of the (EBCT) problem by solving at most C_m^n couples of classical pseudo boolean optimization problems. In real cases, the number of such couples of problems is decreased based on the following restrictions:

$$\max\{r_{ij} x_{ij} \,|\, i \in U^v,\, j \in J\} \ \leq \ e,$$

$$\max\{d_{ikh} y_{ikh} \,|\, i \in \bar{U}^v,\, k \in K\} \leq e_h, \text{ for each } h \in H.$$

Furthermore, we analyze the way in which the (P_1^v) and (P_2^v) problems can be transformed in new problems such that, after applying the transformations, it become easier to solve. Therefore, we apply some transformations to the matrices A and C_h, $h \in H$.

Let us consider the number $\mu \ = \ 1 + \sum_{i \in I} \sum_{j \in J} a_{ij}$.

We set the matrices $\tilde{A} = [\tilde{a}_{ij}]$ and $\tilde{C}_h = [\tilde{c}_{ikh}]$ such that

$$\tilde{a}_{ij} \ = \begin{cases} \mu + a_{ij}, & \text{if} \quad r_{ij} \leq e \\ 0, & \text{if} \quad r_{ij} > e \end{cases}, \quad \text{for} \quad \text{each} \quad i \in I,\, j \in J \tag{5.7}$$

and

$$\tilde{c}_{ikh} \ = \begin{cases} c_{ikh}, & \text{if} \quad \mathrm{d_{ikh}} \leq \mathrm{e_h} \\ 0, & \text{if} \quad \mathrm{d_{ikh}} > \mathrm{e_h} \end{cases}, \quad \text{for} \quad \text{each} \quad \mathrm{i} \in \mathrm{I},\, \mathrm{k} \in \mathrm{K},\, \mathrm{h} \in \mathrm{H}. \tag{5.8}$$

Remark 5.2.14 (Tuns (Bode) O.R. [*112*]). *Let* $v \in V$. *The following sentences are true:*
(i) If there exists $i \in U^v$ *such that* $\sum_{j \in J} \tilde{a}_{ij} = 0$, *then the problem* (P_1^v) *is inconsistent. Indeed, in this case we have* $r_{ij} > e$, *for all* $j \in J$.
(ii) If there exists $j \in J$ *such that* $\sum_{i \in U^v} \tilde{a}_{ij} = 0$, *then the problem* (P_1^v) *is inconsistent. Indeed, in this case we have* $r_{ij} > e$, *for all* $i \in U^v$.
(iii) If there exists $i \in \bar{U}^v$ *such that* $\sum_{k \in K} \sum_{h \in H} \tilde{c}_{ikh} = 0$, *then the problem* (P_2^v) *is inconsistent.*
(iv) If there exists $k \in K$ *such that* $\sum_{i \in \bar{U}^v} \sum_{h \in H} \tilde{c}_{ikh} = 0$, *then the problem* (P_2^v) *is inconsistent.*

Let $\tilde{f}_1 : \{0,\,1\}^{m\times n} \to \mathbb{R}$ be the function given by $\tilde{f}_1(X) = \sum_{i\in U^v}\sum_{j\in J} \tilde{a}_{ij}x_{ij}, \forall\, X \in \{0,\,1\}^{m\times n}$.

Let $X \in \Lambda^v$. We set $\Delta(X) = \{(i,j) \in U^v \times J \,|\, x_{ij} = 1, \quad r_{ij} \le e\}$ and $\bar{\Delta}(X) = \{(i,j) \in U^v \times J \,|\, x_{ij} = 1, \quad r_{ij} > e\}$. Then $0 \le \mathrm{card}(\Delta(\mathrm{X})) \le \mathrm{n}$ and

$$\tilde{f}_1(X) = \sum_{(i,j)\in\Delta(X)} \tilde{a}_{ij}x_{ij} + \sum_{(i,j)\in\bar{\Delta}(X)} \tilde{a}_{ij}x_{ij} = \mu\cdot\mathrm{card}(\Delta(\mathrm{X})) + \sum_{(\mathrm{i,j})\in\Delta(\mathrm{X})} \mathrm{a_{ij}}. \tag{5.9}$$

For each $v \in V$ we consider the problem

$$(\mathrm{PM}_1^{\mathrm{v}}) \quad \begin{cases} \tilde{f}_1(X) \to \max, \\ x \in \Lambda^v. \end{cases} \tag{5.10}$$

Theorem 5.2.15 (Tuns (Bode) O.R. [112]). *Let $v \in V$.*
(i) The problem (P_1^v) has feasible solutions if and only if there exists $X^v \in \Lambda^v$ such that $\tilde{f}_1(X^v) > n\mu$.
ii) If $v \in \mathcal{V}_1$, then the problems (P_1^v) and (PM_1^v) have the same optimal solutions.

In view of Theorem 5.2.15 the solving of the (P_1^v) problem is reduced to solving a classical assignment problem. This problem can be solved by using the exact algorithms considered by Bertsekas D.P. [9] and by Kuhn H.W. [66], or genetic algorithms considered by Sahu A. and Tapadar R. [102].

In what follows, we consider the function $\tilde{\varphi} = (\tilde{\varphi}_1, \varphi_2, f_2)$ where

$$\tilde{\varphi}_1(Y) = \sum_{i\in\bar{U}^v}\sum_{k\in K}\sum_{h\in H} \tilde{c}_{ikh}y_{ikh}, \quad \forall\, Y \in \{0,1\}^{m\times p\times s}. \tag{5.11}$$

Let $v \in V$. By (PM_2^v) we denote the problem of determining the max $-p$ min $-$ max points of the vector function $\tilde{\varphi}$ with respect to the set $\tilde{W}^v$, where $\tilde{W}^v = \{Y \in W^v \,|\, \varphi(Y) - \tilde{\varphi}(Y) = 0\}$. Let $\tilde{\mathcal{Y}}_0^v$ be the set of all max $-p$ min $-$ max points of the vector function $\tilde{\varphi}$ with respect to the set $\tilde{W}^v$.

Let $v \in V$ and $Y \in \tilde{W}^v$. We set $\Gamma(Y) = \Big\{(i,k,h) \in \bar{U}^v \times K \times H \,|\, y_{ikh} = 1 \quad \text{and} \quad \mathrm{d_{ikh}} \le \mathrm{e_h}\Big\}$ and $\bar{\Gamma}(Y) = \Big\{(i,k,h) \in \bar{U}^v \times K \times H \,|\, y_{ikh} = 1 \quad \text{and} \quad \mathrm{d_{ikh}} > \mathrm{e_h}\Big\}$. Then,

$$\tilde{\varphi}_1(Y) = \sum_{(i,k,h)\in\Gamma(Y)} \tilde{c}_{ikh} + \sum_{(i,k,h)\in\bar{\Gamma}(X)} \tilde{c}_{ikh} = \sum_{(i,k,h)\in\Gamma(Y)} c_{ikh}. \tag{5.12}$$

Theorem 5.2.16 (Tuns (Bode) O.R. [112]). *Let $v \in V$. Then we have: $\mathcal{Y}^v = \tilde{W}^v$ and $\mathcal{Y}_0^v = \tilde{\mathcal{Y}}_0^v$.*

The problem (PM_2^v) is a kind of pseudo boolean three objective lexicographic optimization problem. We can reduce the solving of this problem to solving a linear pseudo boolean optimization problem. To do this, we define a sequence of numbers $M, M_0, M_1, \ldots, M_q$, where q is the cardinal of the set $\varphi_2(\{0,1\}^{m\times p\times s})$. Let $M = 1 + \sum_{i\in I}\sum_{k\in K} b_{ik} \ge 1 + \max\{f_2(Y)|Y \in \{0,\,1\}^{m\times p\times s}\}$.

By using the above notation, we set the numbers M_k, $k \in \{q, q-1, \ldots, 0\}$, such that $M_q = 1$ and $M_t = 1 + \sum_{\nu=t+1}^{q} M_\nu \cdot \mathrm{card}\,(\mathrm{L}_\nu), \quad \text{for} \quad \text{all} \quad \mathrm{t} \in \{\mathrm{q}-1, \ldots, 0\}$.

Now, let $FL : \{0,\, 1\}^{m \times p \times s} \to \mathbb{R}$ be the function given by

$$FL(Y) \,=\, -M \cdot M_0 \cdot \tilde{\varphi}_1(Y) \,+\, M \sum_{t=1}^{q} \Big(M_t \cdot \sum_{(i,k,h) \in L_t} y_{ikh} \Big) \,-\, f_2(Y), \tag{5.13}$$

for all $Y \in \{0,\, 1\}^{m \times p \times s}$.

Let us consider the problem:

$$(\mathrm{PL}^v) \quad \begin{cases} FL(Y) \to \min, \\ Y \in \tilde{W}^v. \end{cases}$$

Let us denote by $\tilde{W}_0^v$ the set of all optimal solutions of the (PL^v) problem.

Theorem 5.2.17 (Tuns (Bode) O.R. [*112*]). *Let $v \in \mathcal{V}$. An element $Y^0 \in \tilde{W}^v$ is an optimal solution of the problem (PL^v) if and only if is a* max −p min − max *point of the function $\tilde{\varphi}$ with respect to the set $\tilde{W}^v$.*

Theorem 5.2.17 can be used to determine a max $-p$ min − max point of φ with respect to $\mathcal{W}^v$.

We remark that the problem (PL^v) is a linear pseudo boolean optimization problem. Therefore, it can be solved by using the techniques presented by Berthold T., Heinz S. and Pfetsch M.E. [8], by Hammer P.L. and Rudeanu S. [51], and by Manquinho V. and Marques-Silva J. [77].

Furthermore, based on Theorems 5.2.12, 5.2.13, 5.2.15, 5.2.16, 5.2.17, and on the above Remarks, an algorithm for solving the (EBCT) problem is given. Example 5.2.18 highlights how the above algorithm works.

Chapter 6

Applications of Multilevel Optimization in the Technology Transfer Area

Technology transfer is defined in a lot of different ways, depending on one hand on the discipline of the research and on the other hand on the purpose of the research.

The present chapter studies the licensing, one of the most used methods for technology transfer between firms. We analyze different licensing contracts in a differentiated Stackelberg model, when one of the firms engages itself in an research and development ($R\&D$) process that gives an endogenous cost-reducing innovation. On the other hand, we study some types of particular multilevel optimization problems generated by the concrete economic problems related to the different types of licensing contracts.

The scientific research results within this chapter belong to the author and can be found in papers authored by Ferreira F. and Bode O.R. [32] and [33], or by Tuns (Bode) O.R. [113]. In [32] we consider a differentiated Stackelberg model, when the leader firm engages itself in a $R\&D$ process that gives an endogenous cost-reducing innovation. The aim is to study the licensing of the cost-reduction by a two-part tariff. By using comparative static analysis, we conclude that the degree of the differentiation of goods plays an important role in the results. We also do a direct comparison between the Stackelberg duopoly model and Cournot duopoly model. In [33] by considering the same differentiated Stackelberg duopoly model, when the leader firm engages itself in a $R\&D$ process that gives an endogenous cost-reducing innovation, we study the licensing of the cost-reduction by a per-unit royalty and a fixed-fee. We analyse the implications of these types of licensing contracts over the $R\&D$ effort, the profits of the firms, the consumer surplus and the social welfare. By using comparative static analysis, we also conclude that the degree of the differentiation of goods plays an important role in the results. In [113] we study the case when two firms compete on the market in a differentiated Stackelberg model and there is no technology transfer between the innovator firm and the follower firm. A mathematical model is attached to this particular economic problem and an optimal solution is found. For that, we consider a multilevel parametric optimization problem in which both the upper and the lower level functions are to be maximized under some given conditions.

6.1 Brief Background Concerning Technology Transfer

We begin our exposure with a brief background concerning technology transfer. Technology licensing has been the subject of much theoretical inquiry as can be found in papers [2], [19], [20], [31], [34], [37], [58], [59], [60], [61], [62], [68], [80], [93], [94], [99], [127], [128], [129], [138].

We note the fact that in the present thesis we work under the hypothesis that the market competition holds in a Stackelberg model since, from the mathematical point of view, this leads us to multilevel optimization problems. This aspect results from the papers [32], [33], [111], [112] and [113] which represent author's scientific results obtained by herself or as a joint work. But we remark that the author's research is also regarding the Bertrand and Cournot competition, as we can see in paper [11].

6.2 Basic Framework of the Studied Economic Problem

We introduce the concrete economic problem studied in the present chapter:

Let us consider a duopoly model where two firms, denoted by F^1 and F^0, produce n differentiated goods. The inverse demand functions are given by $p^h = 1 - q^h - < d, q^{1-h} >$, where:
• $p^h = (p^h_1, \ldots, p^h_n) \in \mathbb{R}^n$ represents the price of the firm F^h, $h = 0, 1$;
• $q^h = (q^h_1, \ldots, q^h_n) \in \mathbb{R}^n$ and $q^{1-h} = (q^{1-h}_1, \ldots, q^{1-h}_n) \in \mathbb{R}^n$ represent the outputs of firms F^h and F^{1-h}, respectively, where $h = 1$;
• d represents the degree of the differentiation of goods, $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$, with $d_j \in (0, 1)$, $\forall\, j \in \{1, \ldots, n\}$.

The duopoly market is modeled as a Stackelberg competition: the leader firm F^1 choose its output level and then the follower firm F^0 is free to choose its optimal output taking into account the leader's output. Initially, both firms have identical unit production cost $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, with $c_j \in (0, 1)$, $\forall\, j \in \{1, \ldots, n\}$. We consider that firm F^1, the leader firm, can engages itself in a R&D process in order to improve its technology. This allows a reduction of its production costs by an amount called *innovation size*. The cost-reducing innovation creates a new technology that reduces innovating firm's unit cost by the amount of k, while the amount invested in R&D is $k^2/2$.

In case there is a technology transfer between the two firms, we consider the following five stages game. In the first stage, the innovator firm decides the value of the innovation size (or, equivalently, the amount to invest in R&D). In the second stage, the innovator firm decides whether to license the technology or not, because licensing reduces the marginal cost of the licensee firm. If it decides to license the new technology, then it charges a payment from the licensee (either a per-unit royalty rate, a fixed-fee or a combination of both royalty and fixed-fee). In the third stage, the licensee firm decides whether to accept or reject the offer made by the licensor. Then, both firms represents the players of a Stackelberg game. Therefore, in the fourth stage the leader firm decides its output and in the last stage the follower firm, being aware of the leader's output, chooses the output to produce.

In the present chapter we consider both situations:
i) when there is no technology transfer between firms (benchmark case);
ii) when there is a technology transfer between firms based either on a per-unit royalty contract, a fixed-fee contract or a two-part tariff contract.

6.3 Benchmark Case: No-licensing Case

We begin this section by mathematically modeling and solving the concrete economic problem in the benchmark case. The mathematical model is a three-level parametric optimization problem in which both the upper and the lower level functions are to be maximized under some given conditions. Then, we complete the results with those obtained from the economic point of view.

6.3.1 Modeling and Solving the Economic Problem

We note that the results in this paragraph belong to the author and can be found in TUNS (BODE) O.R. [113].

Let $n \in \mathbb{N}^*$ be a natural number, $J = \{1, \ldots, n\}$, and let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) \in \mathbb{R}^n$. Let $T \subseteq \mathbb{R}^n$ be the set of variation of the parameter $d = (d_1, d_2, ..., d_n) \in \mathbb{R}^n$. Using d we can set the diagonal matrix $D \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

For each $d \in T$, let $f_d : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $F_d : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $g_d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the functions given, respectively, by

$$f_d(x, y, z) \ = <\gamma - x - Dy + z, x>, \forall\, (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n,$$

$$F_d(x, y, z) \ = <\gamma - x - Dy + z, x> - \frac{1}{2}\|z\|^2 \ = \ f_d(x, y, z) - \frac{1}{2}\|z\|^2, \forall\, (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$$

and

$$g_d(x, y) \ = <\gamma - y - Dx, y>, \forall\, (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Let us consider the three-level parametric optimization problem

$$(P;T) \qquad \begin{cases} F_d(x, y, z) \to \max \\ y \in S_d^*(x) \\ x \in S_d^*(z) \\ z \in \mathbb{R}^n \end{cases} , \ d \in T,$$

where $S_d^*(x) = \{y^x\} := argmax\{g_d(x, y) \,|\, y \in \mathbb{R}^n\}$, $x \in \mathbb{R}^n_+$, and $S_d^*(z) := argmax\{f_d(x, y^x, z) \,|\, x \in \mathbb{R}^n_+\}$, $z \in \mathbb{R}^n$. For each $d \in T$, by (P_d) we denote the three-level optimization problem obtained from $(P;T)$ if the parameter is fixed to d.

Remark 6.3.1 *If $T =]0, 1[^n$, then the problem $(P;T)$ is the mathematical model attached to the basic economic problem described above.*

The solving of the problem (P_d), $d \in T$, is reduced to solving three optimization problems.

Let $d \in T$ and $x \in \mathbb{R}^n_+$. We consider the problem:

$$(P^1_{d,x}) \quad \begin{cases} \varphi_{d,x}(y) \to \max, \\ y \in \mathbb{R}^n, \end{cases}$$

where $\varphi_{d,x}(y) = g_d(x,y) =< \gamma - y - Dx, y >, \forall\, y \in \mathbb{R}^n$.

Recalling the problem (P_d), $d \in T$, it results that for all $z \in \mathbb{R}^n$, if $S^*_d(z) \neq \emptyset$ and $x \in S^*_d(z)$, then $S^*_d(x) = \{y^x\} = \left\{\frac{1}{2}(\gamma - Dx)\right\}$.

Furthermore, we consider the following optimization problem:

$$(P^2_{d,z}) \quad \begin{cases} \phi_{d,z}(x) \to \max, \\ x \in \mathbb{R}^n_+, \end{cases}$$

where $\phi_{d,z}(x) = f_d(x, y^x, z) =< \gamma - \frac{1}{2}D\gamma + z, x > + < (D^2 - I_n)x, x >$, $\forall\, x \in \mathbb{R}^n_+$, I_n being the identity matrix in n dimensions.

We remark that $\phi_{d,z}(x) = \sum\limits_{j\in J} \Big(-(1 - \frac{d_j^2}{2})x_j^2 + (\gamma_j - \frac{d_j\gamma_j}{2} + z_j)x_j \Big)$, $\forall\, x \in \mathbb{R}^n$.

For $d \in T$ fixed, let us denote by $J^d_+ = \left\{j \in J \,|\, d_j^2 > 2\right\}$, $J^d_- = \left\{j \in J \,|\, d_j^2 < 2\right\}$, $J^d_0 = \left\{j \in J \,|\, d_j^2 = 2\right\}$.

For $d \in T$ and $z \in \mathbb{R}^n$, both fixed, we set $J^d_{0+}(z) := \left\{j \in J^d_0 \,|\, \gamma_j - \frac{d_j\gamma_j}{2} + z_j > 0\right\}$, $J^d_{00}(z) := \left\{j \in J^d_0 \,|\, \gamma_j - \frac{d_j\gamma_j}{2} + z_j = 0\right\}$, $J^d_{0-}(z) := \left\{j \in J^d_0 \,|\, \gamma_j - \frac{d_j\gamma_j}{2} + z_j < 0\right\}$.

Proposition 6.3.2 (Tuns (Bode) O.R. [113]). *Let $d \in T$.*
(i) If $J^d_+ \neq \emptyset$, then the function $\phi_{d,z}$ is upper unbounded on $\mathbb{R}^n_+$, for each $z \in \mathbb{R}^n$;
(ii) If $J^d_+ = \emptyset$ and $z \in \mathbb{R}^n$ such that $J^d_{0+}(z) \neq \emptyset$, then the function $\phi_{d,z}$ is upper unbounded on $\mathbb{R}^n_+$.

Let $d \in T$ and $z \in \mathbb{R}^n$ be such that $J^d_+ = \emptyset$ and $J^d_{0+}(z) = \emptyset$. Let us denote by $p = card(J^d_{00}(z))$ and by $q = card(J^d_{0-}(z))$. Let $m = n - p - q = card(J^d_-)$.

Remark 6.3.3 (Tuns (Bode) O.R. [113]). *It is not difficult to see that, if $m = 0$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_+$, then $x^z = (x^z_1, \ldots, x^z_n) \in \mathbb{R}^n_+$, with*

$$x^z_j = \begin{cases} 0, \text{ if } j \in J^d_{0-}(z), \\ \lambda_j, \text{ if } j \in J^d_{00}(z), \end{cases}$$

is a maximum point of $\phi_{d,z}$.

Under the hypothesis that $m > 0$, let $J^d_- = \{j_1, \ldots, j_m\}$, where $1 \le j_1 < \cdots < j_m \le n$. We consider the function $\tilde{\phi}_{d,z} : \mathbb{R}^m \to \mathbb{R}$,

$$\tilde{\phi}_{d,z}(x_{j_1}, \ldots, x_{j_m}) = -\sum_{h=1}^{m}(1 - \frac{d_{j_h}^2}{2})x_{j_h}^2 + \sum_{h=1}^{m}(\gamma_{j_h} - \frac{d_{j_h}\gamma_{j_h}}{2} + z_{j_h})x_{j_h},\ \forall\, (x_{j_1}, \ldots, x_{j_m}) \in \mathbb{R}^m.$$

Proposition 6.3.4 (Tuns (Bode) O.R. [113]). *The function $\tilde{\phi}_{d,z} : \mathbb{R}^m \to \mathbb{R}$ is strictly concave and its unique maximum point is $\tilde{x} = (\tilde{x}_{j_1}, ..., \tilde{x}_{j_m})$, where $\tilde{x}_{j_h} = \frac{2\gamma_{j_h} - d_{j_h}\gamma_{j_h} + 2z_{j_h}}{2(2 - d_{j_h}^2)}$, for each $h \in \{1, \ldots, m\}$.*

Let $d \in T$ and $z \in \mathbb{R}^n$.

We set $J^d_{--}(z) = \left\{ j \in J^d_- \mid \gamma_j - \frac{d_j\gamma_j}{2} + z_j < 0 \right\}$, $J^d_{-+}(z) = \left\{ j \in J^d_- \mid \gamma_j - \frac{d_j\gamma_j}{2} + z_j > 0 \right\}$ and $J^d_{-0}(z) = \left\{ j \in J^d_- \mid \gamma_j - \frac{d_j\gamma_j}{2} + z_j = 0 \right\}$. We note that $J^d_- = J^d_{--}(z) \cup J^d_{-+}(z) \cup J^d_{-0}(z)$.

From Remark 59 and Proposition 60 we obtain the following result.

Corollary 6.3.5 (Tuns (Bode) O.R. [*113*]). *If $d \in T$ and $z \in \mathbb{R}^n$ such that $J^d_+ = \emptyset$ and $J^d_{0+}(z) = \emptyset$, then, for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_+$, the point $x^z = (x^z_1, \ldots, x^z_n)$, where*

$$x^z_j = \begin{cases} \lambda_j, \ \textit{if}\, j \in J^d_{00}(z), \\ \\ 0, \ \textit{if}\, j \in (J^d_{0-}(z) \cup J^d_{--}(z) \cup J^d_{-0}(z)), \\ \\ \dfrac{2\gamma_j - d_j\gamma_j + 2z_j}{2(2-d_j^2)}, \ \textit{if}\, j \in J^d_{-+}(z), \end{cases} \tag{6.1}$$

is a maximum point of the function $\phi_{d,z}$.

Based on the fact that for a multilevel optimization problem the objective functions of the lower level must have maximum (respectively, minimum) points, Proposition 58 implies that the set T^* of feasibility of parameter d has the following property:

$$T^* \subseteq \{d \in T \mid J^d_+ = \emptyset \,\text{and}\, J^d_0 = \emptyset\} = \{d \in T \mid d_j^2 < 2, \, \forall\, j \in J\}. \tag{6.2}$$

In what follows, we consider that (6.2) holds. Under this hypothesis, for each $z \in \mathbb{R}$, the set $S^*_d(z)$ has exactly one element, i.e. we have $S^*_d(z) = \{x^z = (x^z_1, x^z_2, ..., x^z_n)\}$, where

$$x^z_j = \begin{cases} 0, \ \text{if}\ j \in J^d_{--}(z) \cup J^d_{-0}(z), \\ \\ \frac{2\gamma_j - d_j\gamma_j + 2z_j}{2(2-d_j^2)}, \ \text{if}\ j \in J^d_{-+}(z). \end{cases} \tag{6.3}$$

Under the hypothesis that (6.2) holds, we have $J = J^d_-$. Solving the initial problem (P_d) is equivalent to determining the set $argmax\{F_d(x^z, y^{x^z}, z) \mid z \in \mathbb{R}^n\}$. Therefore, now we solve the problem

$$(P^3) \quad \begin{cases} \theta_d(z) \to \max, \\ z \in \mathbb{R}^n, \end{cases}$$

where $\theta_d(z) = F_d(x^z, y^{x^z}, z) = < \gamma - x^z - Dy^{x^z} + z, x^z > - \frac{1}{2}\|z\|^2$.

Proposition 6.3.6 (Tuns (Bode) O.R. [*113*]). *If there exists $j \in J$ such that: (i) $2 > d_j^2 > 1$ or (ii) $d_j^2 = 1$ and $\gamma_j \neq 0$, then the function θ_d is upper unbounded on $\mathbb{R}^n$.*

Remark 6.3.7 *From Proposition 6.3.6 it follows that $T^* \subseteq \{d \in T \mid d_j^2 < 1, \, \forall\, j \in J\}$.*

Proposition 6.3.8 (Tuns (Bode) O.R. [*113*]). *If $d \in T$ and $d_j^2 < 1$, $\forall\, j \in J$, then the function θ_d has an unique maximum point $z^* = (z^*_1, z^*_2, ..., z^*_n)$, where $z^*_j = \frac{\gamma_j(2-d_j)}{2(1-d_j^2)}$, $\forall\, j \in J$.*

Remark 6.3.9 *From Proposition 6.3.8 one gets that* $\{d \in T \mid d_j^2 < 1,\ \forall j \in J\} \subseteq T^*$. *Then, in view of Remark 6.3.6 we obtain that* $T^* = \{d \in T \mid d_j^2 < 1,\ \forall j \in J\}$.

Remark 6.3.10 *The originality of the three-level optimization problem studied above is that it depends on the parameters* d *and* γ. *For the particular case when* $n = 1$, $d \in]0, 1[$ *and* $\gamma \in]0, 1[$, *the optimal solution of the problem coincide with the optimal solution obtained from the economic point of view by* FERREIRA F. *and* BODE O.R. *[32]. More, the result that the absolute value of parameter* d *cannot exceed* 1 *has an important economic significance. Since* d *denotes the degree of differentiation of goods, the result justifies the condition* $d \in]0, 1[$, *which is frequently used in the economic literature.*

6.3.2 Benchmark: No-licensing Case for One Differentiated Product

We begin our exposure by recalling the economic problem studied by FERREIRA F. and BODE O.R. [32] and then we present all the results obtained by studying the case when there is no technology licensing between firms. We note that these results can be obtained for the particular case when $n = 1$ of the parametric optimization problem introduced in Subsection 6.2. Additional, we analyse the consumer surplus CS and the social welfare W that are, respectively, defined by $CS = \frac{q_1^2 + 2dq_1q_2 + q_2^2}{2}$ and $W = \pi_1 + \pi_2 + CS$.

We study the case of pre-licensing in the differentiated Stackelberg duopoly model. Depending on the degree of the differentiation of the goods, we conclude that:
(i) if $0 < d < d_1$[1], then firm F_2 competes with firm F_1 using its old technology and gets positive profit (non-drastic innovation);
(ii) if $d_1 \le d < 1$, then firm F_2 finds unprofitable to produce any positive output (drastic innovation). In this case, firm F_1 gains the monopoly.

We compute explicitly the optimal outputs, the profits, the optimal innovation size, the consumer surplus and social welfare, in both non-drastic and drastic innovation cases.

We evaluate the effects of the degree d of the differentiation of goods over: the amount that reduces the leader's unit cost, the profits of both firms (leader and follower), the consumer surplus and the social welfare. Hence, we state the followings.

Theorem 6.3.11 (FERREIRA F. *and* BODE O.R. [*32*]). *If there exists no technology transfer, then:*
(i) For $d \in (d_2, d_1)$[2] (*respectively,* $d \in (0, d_2) \cup [d_1, 1)$), *the optimal innovation size decreases* (*respectively, increases*) *with the differentiation of the goods;*
(ii) For $d \in (0, 0.5) \cup (d_3, 1)$[3] (*respectively,* $d \in (0.5, d_3)$), *the profit of the innovator firm increases* (*respectively, decreases*) *with the differentiation of the goods;*
(iii) For $d \in (0, d_4) \cup [d_1, 1)$[4] (*respectively,* $d \in (d_4, d_1)$), *the consumer surplus increases* (*respectively, decreases*) *with the differentiation of the goods;*
(iv) For $d \in (0, d_5) \cup [d_1, 1)$[5] (*respectively,* $d \in (d_5, d_1)$), *the social welfare increases* (*respectively, decreases*) *with the differentiation of the goods.*

[1] d_1 is the solution belonging to interval $]0, 1[$ of the equation $d^2 + 2d - 2 = 0$, $d_1 \simeq 0.732$.
[2] d_2 is the solution belonging to interval $]0, 1[$ of the equation $d^2 - 4d + 1 = 0$, $d_2 \simeq 0.268$.
[3] d_3 is the solution belonging to interval $]0, 1[$ of the equation $d^4 + 2d^3 + 8d - 8 = 0$, $d_3 \simeq 0.268$.
[4] d_4 is the solution belonging to interval $]0, 1[$ of the equation $d^4 - 5d^3 - 3d^2 + 10d - 2 = 0$, $d_4 \simeq 0.219$.
[5] d_5 is the solution belonging to interval $]0, 1[$ of the equation $7d^4 - 13d^3 - 9d^2 + 28d - 10 = 0.$, $d_5 \simeq 0.458.$.

Remark 6.3.12 (Ferreira F. *and* Bode O.R. [32]). *We note that if there exists no technology transfer and the innovation is non-drastic (i.e.* $d \in (0, d_1)$*), the profit of the non-innovator firm increases with the differentiation of the goods.*

6.4 Per-unit royalty Licensing Case

In the present section we analyse the case when there exists a technology transfer from the leader firm to the follower firm based on a per-unit licensing contract. A mathematical model is attached to the concrete economic problem in this case. This mathematical model is a four-level parametric optimization problem in which both the upper and the lower level functions are to be maximized under some given conditions. We solve and determine the optimal solution of the problem from both mathematical and economic point of view.

6.4.1 Modeling and Solving the Economic Problem

Let $n \in \mathbb{N}^*$, $J = \{1, 2, ..., n\}$, $r = (r_1, r_2, ..., r_n) \in \mathbb{R}^n$ and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) \in \mathbb{R}^n$, with $\gamma_j \in]0, 1[$, $\forall\, j \in J$. Let $T \subseteq\,]0, 1[^n$ be the set of variation of the parameter $d = (d_1, d_2, ..., d_n) \in T$. Using d we can set the diagonal matrix $D \in \mathbb{R}^n \times \mathbb{R}^n$ given in Subsection 6.3.1.

For each $d \in T$, let $F_d : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f_d : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $g_d : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the functions given, respectively, by

$$F_d(x, y, z, r) = < \gamma - x - Dy + z, x > -\frac{1}{2}\|z\|^2 + < r, y >, \tag{6.4}$$

$$f_d(x, y, z, r) = < \gamma - x - Dy + z, x > + < r, y >, \tag{6.5}$$

$$g_d(x, y, z, r) = < \gamma - y - Dx + z - r, y >, \tag{6.6}$$

$\forall\, (x, y, z, r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$.

Let us consider the four-level parametric optimization problem

$$(P_{royalty}) \quad \begin{cases} F_d(x, y, z, r) \to \max \\ y \in S_d^*(x, z, r) \\ x \in S_d^*(z, r) \\ r \in S_d^*(z) \\ z \in \mathbb{R}^n \end{cases}, \ d \in T,$$

where $S_d^*(x, z, r) = \{y^{x,z,r}\} = argmax\{g_d(x, y, z, r) \,|\, y \in \mathbb{R}^n\}$, for each $z \in \mathbb{R}^n$ and $x, r \in \mathbb{R}^n_+$; $S_d^*(z, r) = \{x^{z,r}\} = argmax\{f_d(x, y^{x,z,r}, z, r) \,|\, x \in \mathbb{R}^n_+\}$, for each $z \in \mathbb{R}^n$, $y \in S_d^*(x, z, r)$ and $r \in \mathbb{R}^n_+$; and $S_d^*(z) = \{r^z\} = argmax\{F_d(x^{z,r}, y^{x,z,r}, z, r) \,|\, r \in \mathbb{R}^n_+\}$, for each $z \in \mathbb{R}^n$, $x \in S_d^*(z, r)$ and $y \in S_d^*(x, z, r)$.

For each $d \in T$, by $(P_{d,royalty})$ we denote the four-level optimization problem obtained from $(P_{royalty})$ if the parameter is fixed to d.

Remark 6.4.1 *If* $T =]0, 1[^n$*, then the problem* $(P_{royalty})$ *is the mathematical model attached to the basic economic problem described in Section 6.3.2 in case the technology transfer occurs by means of a per-unit royalty.*

We reduce the solving of the $(P_{royalty})$ problem to solving of four optimization problems.

Let $d \in T$, $x, r \in \mathbb{R}^n_+$ and $z \in \mathbb{R}^n$. We consider the problem

$$(P^1_{d,x,z,r}) \quad \begin{cases} \varphi_{d,x,z,r}(y) \to \max, \\ y \in \mathbb{R}^n, \end{cases}$$

where $\varphi_{d,x,z,r}(y) = g_d(x, y, z, r) = -\|y\|^2 + < \gamma - Dx + z - r, y >$, $\forall\, y \in \mathbb{R}^n$,

Recalling the problem $(P_{d,royalty})$, $d \in T$, it follows that for all $z \in \mathbb{R}^n$, if $r \in S^*_d(z)$ and $x \in S^*_d(z, r)$, then $S^*_d(x, z, r) = \{y^{x,z,r}\} = \left\{\frac{\gamma - Dx + z - r}{2}\right\}$.

Now, let $d \in T$, $z \in \mathbb{R}^n$ and $r \in \mathbb{R}^n_+$. We consider the following optimization problem

$$(P^2_{d,z,r}) \quad \begin{cases} \phi_{d,z,r}(x) \to \max, \\ x \in \mathbb{R}^n_+, \end{cases}$$

where $\phi_{d,z,r}(x) = f_d(x, y^{x,z,r}, z, r) =< \gamma - \frac{1}{2}D\gamma + z, x > + < (D^2 - I_n)x, x > + < r, y^{x,z,r} >$, for all $x \in \mathbb{R}^n_+$, I_n being the identity matrix in n dimensions.

Recalling the problem $(P_{d,royalty})$, $d \in T$, one gets that for all $z \in \mathbb{R}^n$, if $r \in S^*_d(z)$ then

$$S^*_d(z, r) = \{x^{z,r}\} = \begin{cases} \frac{(d-2)(\gamma+z)}{2(d^2-2)}, \text{ if } \gamma + z > 0, \\ \\ 0, \text{ if } \gamma + z \le 0. \end{cases}$$

Furthermore, let $d \in T$ and $z \in \mathbb{R}^n$. We consider the following optimization problem

$$(P^3_{d,z}) \quad \begin{cases} \rho_{d,z}(r) \to \max, \\ z \in \mathbb{R}^n, \end{cases}$$

where $\rho_{d,z}(r) = F_d(x^{z,r}, y^{x,z,r}, z, r)$.

Recalling the problem $(P_{d,royalty})$, $d \in T$, it results that for all $z \in \mathbb{R}^n$ we have

$$S^*_d(z) = \{r^z\} = \begin{cases} \frac{\gamma+z}{2}, \text{ if } \gamma + z > 0, \\ \\ 0, \text{ if } \gamma + z \le 0. \end{cases}$$

Under these circumstances, to solve the initial problem $(P_{d,royalty})$ is equivalent to determine the set $argmax\{F_d(x^{z,r}, y^{x,z,r}, z, r^z) \,|\, z \in \mathbb{R}^n\}$. Therefore, now we solve the problem

$$(P^4) \quad \begin{cases} \theta_d(z) \to \max, \\ z \in \mathbb{R}^n, \end{cases}$$

where $\theta_d(z) = F_d(x^{z,r}, y^{x,z,r}, z, r^z) \; =< \gamma - x^{z,r} - Dy^{x,z,r} + z, x^{z,r} > -\frac{1}{2}\|z\|^2 + < r^z, y^{x,z,r} >$.

Proposition 6.4.2 (Tuns (Bode) O.R. *and* Ferreira F. [*117*]). *If* $d \in T$, *then the function* θ_d *is strictly concave and it has an unique maximum point* $z^* = (z^*_1, z^*_2, ..., z^*_n)$ *with* $z^*_j = \frac{\gamma_j(2d_j-3)}{2d_j^2-2d_j-1}$, $\forall j \in J$.

6.4.2 Per-unit royalty Licensing Case for One Differentiated Product

This section deals with the case of licensing by means of a per-unit royalty and yields the main results. In the present paragraph, considering the particular case when $n = 1$ of the economic problem formulated in Subsection 6.2, we present all the results obtained by Ferreira F. and Tuns (Bode) O.R. in [33]. We note that these results can be obtained for the particular case when $n = 1$ of the four-level parametric optimization problem studied above.

We note that in the royalty licensing case the innovation in non-drastic for all $d \in (0, 1)$. We compute explicitly the optimal outputs, the profits, the optimal innovation size, the consumer surplus and social welfare, in both non-drastic and drastic innovation cases.

We evaluate the effects of the degree d of the differentiation of goods over: the amount that reduces the leader's unit cost, the optimal royalty rate, the consumer surplus, the social welfare and the profits of both firms (leader and follower). We can state the following.

Theorem 6.4.3 (Ferreira F. *and* Tuns (Bode) O.R. [*33*]). *If there exists a technology transfer based on a royalty licensing, then:*
(i) The optimal innovation size, the optimal royalty rate, the consumer surplus and the social welfare increase with the differentiation of the goods;
(ii) In the non-drastic innovation case ($d \in (0, d_1)$), *if the goods are sufficiently differentiated, then the interest of the innovator firm in licensing its technology increases with the differentiation of the goods;*
(iii) In the drastic innovation case ($d \in [d_1, 1)$), *if the goods are neither sufficiently differentiated nor sufficiently homogenous* ($d \in [d_1, d_6)$)[6] (*respectively, sufficiently homogenous* ($d \in (d_6, 1)$), *then the interest of the innovator firm in licensing its technology increases* (*respectively, decreases*) *with the differentiation of the goods.*

Remark 6.4.4 (Ferreira F. *and* Tuns (Bode) O.R. [*33*]). *In the non-drastic innovation case, the interest of the non-innovator firm in accepting the new technology by paying a per-unit royalty increases with the differentiation of the goods.*

6.5 Fixed-fee Licensing Case for One Differentiated Product

In this section we consider licensing by means of a fixed-fee only. We note that the results within this paragraph belong to the author and can be found in Ferreira F. and Tuns (Bode) O.R. [33].

In both non-drastic and drastic innovation cases we determine the maximum fixed-fee that the leader firm can charge, the corresponding cost reduction, the optimal output of each firm, the firms' profits, the consumer surplus and social welfare.

We remark that in this case, for the leader firm it is imposed one restrictive condition: it will license its technology if and only if its total profit (i.e. market profit + fixed-fee) will exceed the profit it makes with no-licensing.

[6] d_6 is the solution belonging to interval $]0.1[$ of the equation $4d^8 - 16d^6 + 28d^5 - 63d^4 + 50d^3 + 32d^2 - 24d - 8 = 0$, $d_6 \simeq 0.849$.

Remark 6.5.1 (Ferreira F. *and* Tuns (Bode) O.R. [*33*]).
(i) If the goods are sufficiently differentiated ($d \in (0, d_7)$)[7]*, then a fixed-fee licensing strictly dominates no-licensing;*
(ii) If the goods are sufficiently homogenous ($d \in [d_7, 1)$)*, then the innovator firm never license its technology by a fixed-fee only.*

We evaluate the effects of the degree d of the differentiation of goods over the main variables.

Theorem 6.5.2 (Ferreira F. *and* Tuns (Bode) O.R. [*33*]). *If there exists a technology transfer based on a fixed-fee licensing contract* (*i.e.* $d \in (0, d_7)$)*, then:*
(i) The optimal innovation size, the maximum fixed-fee that can be charged by the innovator firm, the consumer surplus and the social welfare increase with the differentiation of the goods;
(ii) The interest of the innovator firm in licensing its technology increases with the differentiation of the goods.

6.6 Two-part tariff Licensing Case for One Differentiated Product

In the following analyses we are going to consider the situation when there can exist a technology transfer from the innovator firm to the non-innovator firm based on a two-part tariff licensing contract, i.e. both fixed-fee and a royalty per-unit of output. We note that the results within this paragraph belong to the author and can be found in Ferreira F. and Bode O.R. [32].

We determine in both non-drastic and drastic innovation cases the maximum fixed-fee that the leader can charge, the optimal royalty, the optimal cost reduction, the optimal outputs and profits for the leader and follower firms, and the consumer surplus and social welfare. We note that the profit of the follower firm is equal to the profit that it gets by a fixed-fee contract. Standard computations yield that the following result.

Remark 6.6.1 (Ferreira F. *and* Bode O.R. [*32*]). *A two-part tariff licensing strictly dominates no-licensing.*

From the above results we remark that even the innovation is drastic, it is always better for the innovator firm to license its technology either by a per-unit royalty or by a two-part tariff. But this is not true if the innovation is licensed by a fixed-fee contract.

We evaluate the effects of the degree d of the differentiation of goods over the above main variables.

For the non-drastic innovation case (i.e. $d \in (0, d_1)$), we have the following result.

Theorem 6.6.2 (Ferreira F. *and* Bode O.R. [*32*]).
If the innovation is non-drastic ($d \in (0, d_1)$) *and the technology is licensed based on a two-part licensing contract, then:*

[7] d_7 is the solution belonging to interval $]0.1[$ of the equation $9d^7+24d^6-76d^5-160d^4+360d^3+160d^2-576d+256 = 0$, $d_7 \simeq 0.793$.

(i) The optimal innovation size, the maximum fixed-fee that the innovator firm can charge, the consumer surplus and the social welfare increase with the differentiation of the goods;
(ii) If the goods are sufficiently differentiated ($d \in (0, d_8)$)[8], (*respectively, neither sufficiently differentiated nor sufficiently homogenous* ($d \in (d_8, d_1)$)), *then the optimal royalty rate increases* (*respectively, decreases*) *with the differentiation of the goods.*

For the drastic innovation case (i.e. $d \in [d_1, 1)$), we have the following result.

Theorem 6.6.3 (Ferreira F. *and* Bode O.R. [*32*]). *If the innovation is drastic* ($d \in [d_1, 1)$) *and the technology is licensed based on a two-part tariff licensing contract, then:*
(i) If the goods are neither sufficiently differentiated nor sufficiently homogenous ($d \in [d_1, d_9)$)[9] (*respectively, sufficiently homogenous* ($d \in (d_9, 1)$)), *then the optimal innovation size increases* (*respectively, decreases*) *with the differentiation of the goods;*
(ii) The optimal royalty rate and the consumer surplus decrease with the differentiation of the goods;
(iii) The maximum fixed-fee that the innovator firm can charge increases with the differentiation of the goods;
(iv) If the goods are neither sufficiently differentiated nor sufficiently homogenous ($d \in [d_1, d_{10})$)[10] (*respectively, sufficiently homogenous* ($d \in (d_{10}, 1)$)), *then the social welfare increases* (*respectively, decreases*) *with the differentiation of the goods.*

[8] d_8 is the solution belonging to interval $]0.1[$ of the equation $6d^6 - 42d^5 + 125d^4 - 156d^3 + 14d^2 + 112d - 56 = 0$, $d_8 \simeq 0.721$..

[9] d_9 is the solution belonging to interval $]0.1[$ of the equation $6d^2 - 18d + 11 = 0$, $d_9 \simeq 0.855$.

[10] d_{10} is the solution belonging to interval $]0.1[$ of the equation $54d^{10} - 99d^9 - 621d^8 + 1866d^7 - 42d^6 - 4446d^5 + 3146d^4 + 3020d^3 - 3276d^2 - 344d + 736 = 0$, $d_{10} \simeq 0.863$..

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