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Continuous-Time Dynamical Systems for Solving Constraint Satisfaction Problems

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Abstract

In the present Thesis two major topics are investigated. In the first part we will introduce an analog computer model which is capable of solving Boolean satisfiability problems, while in the second part we present a continuous-time dynamical system which can predict the global optimum energy of an NP-hard problem.

After the introductory notions in Chapter 1, which discusses the history of analog computing and introduces the concept of the Cellular Neural Network (CNN) computer model[9,8] we present a continuous-time dynamical system for solving Boolean satisfiability problems[17] (k-SAT) on which our novel model is based.

In Chapter 2 our Asymmetric Cellular Neural Network model is presented which in contrast to the original dynamical system[17] is implementation friendly (through analog circuits) preserving, nevertheless, the most important properties of the original SAT-solver dynamics. In this Chapter we provide also proof of three fundamental theorems which underpin our model: 1) the variables remain bounded; 2) every SAT solution has a corresponding stable fixed point; 3) a stable fixed point always corresponds to a solution. Our numerical results are presented in Chapter 3 where we discuss the chaotic behavior of the system and we demonstrate that there is an optimal range of interpretation for the main two parameters of our system, which is fairly independent of the size and complexity of the problem. We will also show that, however limit cycles are possible the proper choice for the parameters help to avoid these type of attractors. We have also developed a real-time limit cycle detection algorithm, which is described in this Chapter as well.

If talking about physical implementation of a device it is fundamental to discuss the effect of different noise types on the system. In Chapter 4 we bring proof that our system tolerates very well different kind of noises that can appear in electronic devices up to a magnitude around three times higher than the highest possible noise intensity allowed in this type of devices. We studied three types of noise: white noise, colored noise and connection weight errors, the latter simulating the imperfections of electronic connections and/or circuit elements.

Finally, in the last Chapter we introduce a continuous-time dynamical system, and we show that using its chaotic properties we can predict the global optimum of an NP-hard problem long before reaching it. We have tested our system on the *Maximum Satisfiability Problem* (*max*-SAT), which is the NP-hard version of k-SAT. The main advantage of this algorithm lies in its capability of properly approximating the global optimum of the *max*-SAT problem in a very short period of time and giving an estimation how trustworthy this prediction is, providing at the same time a rough estimation on how much time would it need to reach that state. From these results and simulations one can see that the information hidden within the chaotic behavior can predict useful information.

Contents

Introduction

MANY SYSTEMS CAN BE DESCRIBED USING LOGICAL MODELS, LIKE CONSTRAINT SATISFIABILITY PROBLEMS (CSP). In this study two SAT problems were considered: 1) Boolean Satisfiability Problem (k-SAT) and 2) Maximum Satisfiability Problem (*max*-SAT). Boolean Satisfiability Problems (such as k-SAT) constitute one of the hardest classes of constraint satisfaction problems lying at the basis of many decision, scheduling, error-correction and bio-computational applications. Being classified as NP-complete means that every NP problem can be transformed into this form in polynomial time[11]. If one can solve k-SAT problems optimally, in relative short time than it means that the whole family of these problems can be solved. NP-complete problems are efficiently (in polynomial time) checkable, but the worst case complexity of finding a solution on Turing-machines is exponential[20]. Because these problems are predominating fields such as science and technology it is more pressing to develop efficient solvers for NP-complete problems.

The development of CMOS technology presents a exponential growth and trends towards saturation[33,12] due to the approach of the atomic level. As of this trend of CMOS technology more and more interest is given to the analog computers and analog dynamical systems, which are designed in such way that the attractors of these dynamical systems represent the output of the computation.[6,44,32,5]. As lately the engineers designed analog devices imitating the nervous system, such as the Cellular Neural/Nonlinear Networks (CNN)[9,40], or analog VLSI devices[28] to solve technological problems, including in robotics and sensory computing (vision, hearing). The question of whether technologies based on continuous-time dynamical systems can be used in solving NP-hard problems was raised.

This Thesis is organized in 5 chapters as follows: in Chapter 1 the basics of analog computing is discussed and an introduction to CNN computational systems in general is given. In Chapter 2 the basics of Boolean Satisfiability problems are discussed followed by the presentation of the proposed CNN model for solving *k*-SAT problems. Simulation results are presented in Chapter 3. In Chapter 4 the effects of different kind of noise types on the functioning and behavior of the CNN model are presented. Finally in Chapter 5 the *max*-SAT problem is discussed, together with their solvability using the proposed novel algorithm. Finally at the end of this Thesis some of the visions about future research in this field and how the findings presented in this present Thesis can be considered as basis for application purposes are discussed.

Music is the one incorporeal entrance into the higher world of knowledge which comprehends mankind but which mankind cannot comprehend.

Ludwig van Beethoven

1

Basics of analog computing CNN computers in general

The analog computer is a form of computation which in contrary to digital computers does not operate on discreet values like 0 and 1, but it operates on continuous values of a physical phenomena or a particular signal such as mechanical, electrical or hydraulic quantities. Due to this nature of analog computers measurements or calculations cannot be exactly reproduced in a later time, unlike on Turing machines. In this Chapter a brief introduction in the world of analog computers in general is given, followed by a detailed presentation of the theory and main principals of CNN computers.

As digital computers appeared they gained huge popularity due their capabilities of being easily programmable, however analog computational devices still remained in use for specific tasks. This thread was picked up by Leon O. Chua and Lin Yang[9,8] in 1988 in Berkeley when they introduced a revolutionary analog computing device called CNN computer, which is a cellular wave computer[39]. The core of the CNN computer is a Cellular Neural/Nonlinear Network (CNN), an array of analog dynamic processors, so called cells. The revolutionary nature of the CNN computers consists in the high resemblance to the real-life neural networks: it is capable of processing multiple signals in parallel and it features a continuous-time dynamics allowing real-time signal processing. The roots of these kind of analog computers are in the bio-inspired information technology. The host processors of the CNN computer are generating and obtaining as input analog signals on which they operate in continuous time. In general they mimic the anatomy and behavior of some sensory and processing organ, like the retina in the eyes. The computer implementation of the Cellular Neural Networks is the CNN Universal Machine (CNN-UM)[40,7] which is available commercially in various forms of implementation. If the CNN-UM is implemented on a CMOS chip it is a fully programmable stored-program dynamic array computer. The CNN Universal Machine is not only universal in the Turing sense, but also on analog array signals. Since 2003, the International Technology Roadmap for Semiconductors (ITRS, published biannually) considers CNN technology as one of the major emerging architectures.[16] The implementation of CNN computers can be easily differentiated based on the tasks for which they are built: mixed-mode CMOS, emulated digital CMOS, FPGA, optical solutions, image processing, cellular automata models or to solve partial differential equations[49]. The most promising out of the previously given implementation forms is the image processing and other optical solutions in robotics or in sensory computing[13]. This defined the main path in the practical development which had as principal aim the development of a visual microprocessor[10]. As one of the latest achievements we would like to highlight is the *Bi-i V301HD* chip manufactured by Analogic Computers, which is capable of processing optical data up to the FullHD resolution scale. An other area of image processing is the high speed cameras: some of the CNN computer specially built for this purpose can capture *10000fps* up to *100000fps*. The basics of these systems is the following: the optical sensors of the CNN chip can record motion picture up to *100000fps*, which is passed over to the digital computers and played back with a normal *29-30fps* so very detailed investigations can be conducted about such high speed events, which are otherwise impossible to investigate. Many control devices use CNN computers attached to digital computers such as registration plate identification systems. The operating mechanism of these devices is the following: the optical sensors of CNN computer capture the image of a passing car, which is processed instantly by the chip. The single dynamic process on the chip identifies the letters from the obtained image and the result is instantly passed over to the digital computer.

Inspiration is a guest that does not willingly visit the lazy.

Pyotr Ilyich Tchaikovsky

2

CNN model for solving *k*-SAT

In order to face the computational challenges of our modern world there was a need to rethink the structure and purpose of computers. On can already see that it is not enough just to use the common digital computers in our everyday life, nor in scientific research. The fast saturation of the CMOS technology[33,12] forced scientists to search for alternative solutions.

In this chapter the preliminary research results are presented on which our novel CNN model is based[31,29]. After a brief introduction of the Constraint Satisfaction Problem (CSP) in general and of the Boolean Satisfiability Problem (k-SAT) in particular, we are going to discuss a continuous-time dynamical approach to constraint satisfaction problem, which was introduced by Ercsey-Ravasz and Toroczkai in a recent paper[17]. Finally Cellular Neural/Nonlinear Network model is introduced and its main properties and principles presented[31,29]. In this Chapter three Theorems and their proofs are discussed, which lay the basis of our novel model[31,29]:

- Variables remain bounded
- Every k-SAT solution has a corresponding stable fixed point
- A stable fixed point always corresponds to a solution.

In this Chapter a novel Asymmetric Cellular Neural Network (ACNN) was presented with the possibility of applications in analog computation. This presented model is based on the continuous-time dynamical system introduced by Ercsey-Ravasz and Toroczkai[17] and is designed to solve Boolean satisfiability problems, which is one of the most fundamental constraint satisfaction problem. Realized on an analog device it would take only a single operation to find the optimal solution. The connection weights (template) are based on the matrix elements of the given k-SAT instance. The system is started from any initial condition and it converges into a solution, without the need of further intervention. In contrast with the original model we are not able to exclude limit cycles. We have shown that the fixed points of the system are only the solutions, but this does not guarantee that there no limit cycles in the system. In the next Chapter we will show that limit cycles are in fact possible. This is a drawback compared to the original system but a major region of the possible values of the main parameters eliminate the limit cycles, as we will see later, and it will not constitute a problem. As expected from the similarity with the original dynamical system[17] this model also exhibits chaotic behavior especially in the *hard*-SAT phase (see numerical evidence in the next Chapter) reinforcing the equivalence between optimization hardness and chaotic behavior present in analog search algorithms.

Beware of missing chances; otherwise it may be altogether too late some day.

Franz Liszt

3
Numerical results

The novel Asymmetric cellular neural network model presented in the previous Chapter together with the three Theorems laying at the basis of this model, they guarantee that all stable fixed points of the dynamical system correspond to k-SAT solutions. Although the theory presented in the previous Chapter indicates that we have a robust system, there is no guarantee that there are no other attractors in our system. These other types of attractors can either be limit cycles or other chaotic attractors. However, the existence or non-existence of such kind of attractors is very hard to show with analytical methods, but the computer simulation results indicate that these kind of attractors do exist in our system. This property is another major difference between the original dynamical system[17] and our novel SAT solver[31,29]. But also the simulation results indicate that there is a robust optimal region of parameters (A, B) fairly independent of the properties of the problem, where the dynamics avoids getting trapped in limit cycles and converges into a *k*-SAT solution.

We realized that it is hardly possible to specify a fixed value for the main parameters, rather we have to search for and define an optimal region for these parameters. To answer this question we tested the behavior of our system on the whole interpretation interval of the main parameters A and B. Color maps were realized for different system sizes (changing the constraint density α) and also varying the *k*. Changing the constraint density α implies a change in the complexity and hardness of the problem. In the first row of Figure 3.1 the maps for a series of 3-SAT problems are drawn having a system size fixed at $N = 40$ and the constraint density varying $\alpha = 3.5, 4, 4.25$ (from left to right). The last figure in the first row shows the frames of the optimal regions of the three maps placed on the top of each other. This figure shows an excellent match meaning that this region is also independent of the typical hardnesses of the problems. This investigation was conducted in case of 4-SAT, second row of Figure 3.1 and in case of 5-SAT problems, third row of Figure 3.1. In both of the higher *k*-order cases problems with $N = 20$ variables were considered. In case of the 4-SAT the α values were set to the following values: $8.5, 9, 9.25$ (from left to right), respectively the 5-SAT instances had the $\alpha = 15, 18, 20.55$ constraint densities. One can see that in all three cases we have a fairly consistent match in considering the optimal regions for A, B, the optimal parameter region stays in the same section for all different values of *k* and different system sizes (N) or the typical hardnesses (α) of the problems.

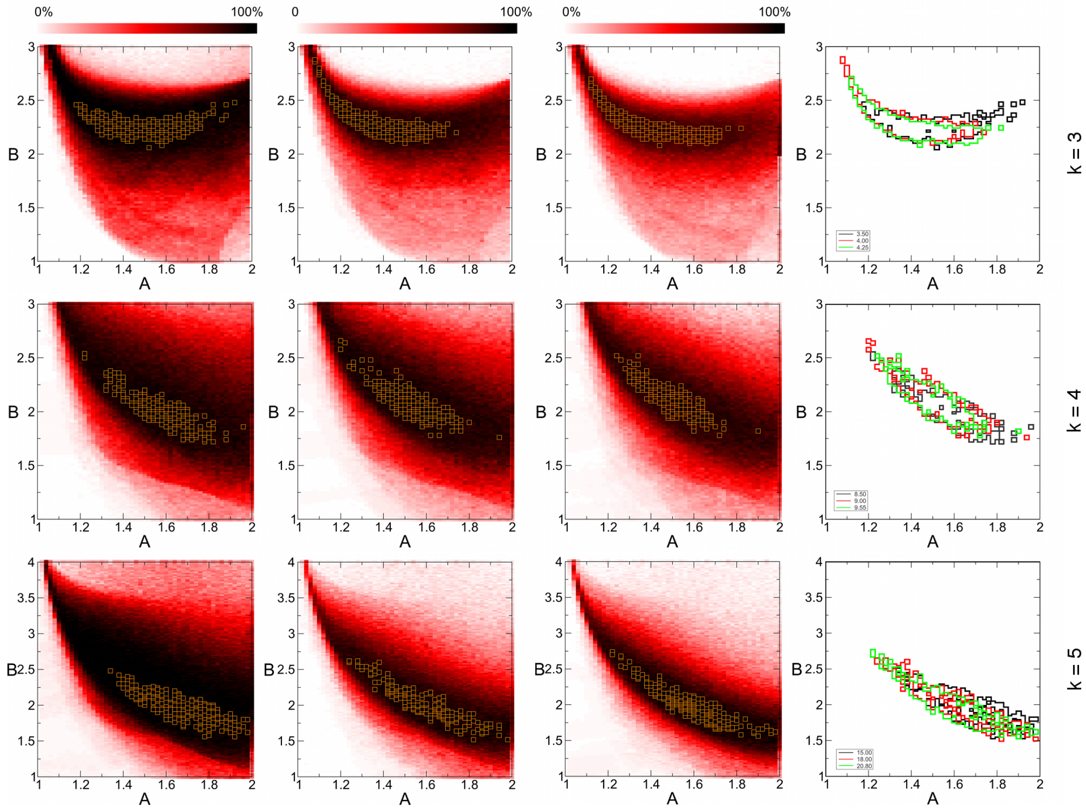

Figure 3.1: Parameter dependence of dynamics in 3-SAT, 4-SAT and 5-SAT problems with fixed size and varying constraint density. For each (A, B) on the map 100 randomly chosen satisfiable instances are solved. The color indicates the fraction of solved problems (see color bar). Simulations were performed on 3-SAT problems (first row) with $N = 40$ and constraint densities $\alpha = 3.5, 4.0, 4.25$ (left to right), 4-SAT (second row) with $N = 20$ and $\alpha = 8.5, 9.0, 9.55$, and 5-SAT (third row) with $N = 20$ and $\alpha = 15, 18, 20.80$. The optimal parameter regions are shown with orange squares on the color maps. In the last column the optimal regions of the three maps are compared in each particular row (black, red, green from left to right), by drawing the frames of these regions (see legends)

Those who have achieved all their aims probably set them too low.

Herbert von Karajan

4

Noise on CNN

To PHYSICALLY IMPLEMENT A SYSTEM it is essential that a deep study of the effects of different kind of noises to be conducted. In the previous Chapters we have presented the Cellular Neural Network model for solving the Boolean satisfiability (k-SAT) problem. We saw that when using ACNN for solving these hard problems the system exhibits a transiently chaotic behavior in its dynamics. This raises the question of viability of this novel analog system in presence of noise, which is unavoidable during the implementation (e.g. electric circuits) and use of analog devices. In the current Chapter the robustness of our system against white and colored ($1/f^2$) noises was tested. We have also tested the system for potential errors in connection weights would influence its operation [46,30].

In real-life systems there can be several types of noises present, like shot, thermal, burst, flicker, avalanche noise, etc. [21]. To conduct an extensive study of these effects we modelled the most commonly occurring noises in electronic circuits with the following three types [46,30]:

- white noise: uncorrelated in time
- colored noise: correlated in time, $1/f^2$ type noise
- small random errors in connection weights: constant in time, modelling imperfections of electronic junctions or circuit elements.

Noise is a fundamental characteristic of all electronic circuits. It is caused by small fluctuations in the current or the voltage, imperfections of the circuit elements, etc. The presence of noise until the last couple of years was considered as a bad, unwanted effect which needs to be eliminated. Fortunately recently more and more systems were built which actually benefit from the presence of noise. We show in the current Chapter how the efficiency of our CNN system improves from the presence of noise. During the circuit implementations of CNN models another high concern is the precision of the connection weights. When producing circuit elements, like resistors, capacitors, etc. the parameters of these elements will show small variations compared to the theoretically proposed values. These fluctuations can also be called a type of noise in the system. The connection weight errors were introduced in equations and therefore in the simulations as a small random value, which is constant in time, and it is randomly added or subtracted from the connection weights in this way simulating the implementation errors.

It is interesting to notice that the presence of noise can improve the performance of our dynamical system, especially when the values of the two main parameters A, B are not from the optimal parameter region. This is shown on Figure 4.1 in case of a small 3-SAT instance with $N = 20$ in the *hard*-SAT phase, having a constraint density of $\alpha = 4.25$. On the left-hand side of Figure 4.1 the fraction of

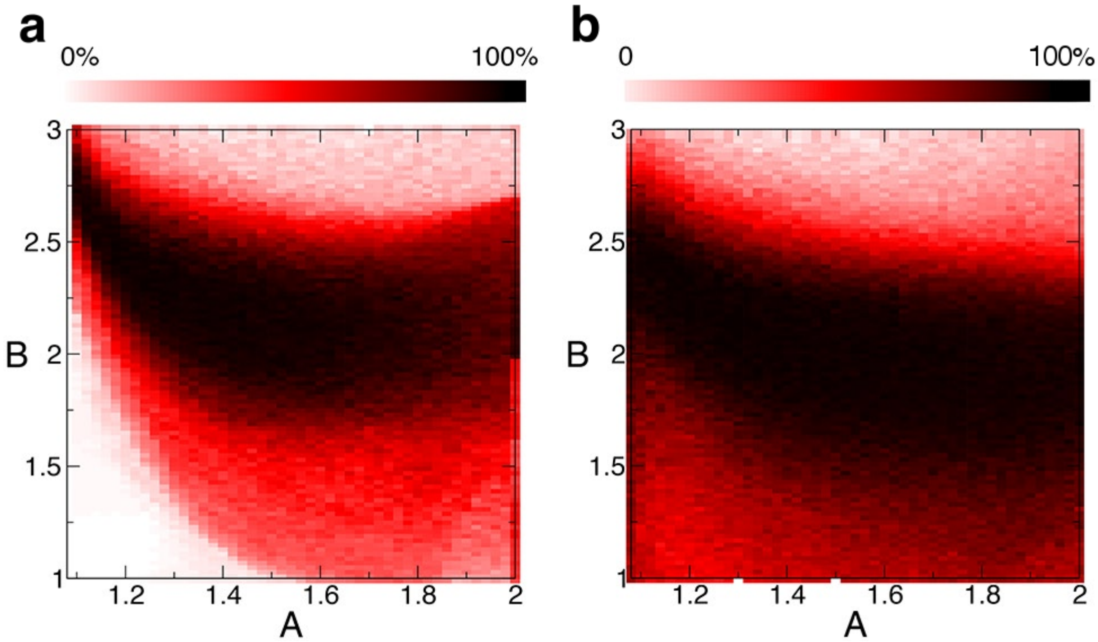

Figure 4.1: Color map on a small system to map the A, B parameter region in presence of colored noise. For each $A \in [1,2), B \in [1,3)$ on the map there are 100 randomly chosen satisfiable 3-SAT instances with $N = 20, \alpha = 4.25, t_{max} = 5000$ solved. The color represents the fraction of solved problem (see color bars): a) without noise, b) with colored noise ($\tau = 1, I = 0.01$)

solved problems is shown in the original case, when no noise is present in the system. The same problem, from the same initial conditions was studied and plotted on the right-hand side of Figure 4.1 but in the presence of correlated colored noise with an intensity of $I = 0.01$ and correlation time $\tau = 1$. One can see that the presence of noise enlarges the optimal region for the two optimal parameters (A, B). This underlies our initial assumptions, that these parameters do not need careful tuning in order for the system to function properly. The tolerated large noise intensity levels promise the possibility for highly robust and efficient physical implementations of the system.

Melodic invention is one of the surest signs of Divine gift.

Gustav Mahler

5

Continuous-time dynamics for predicting global optimum of NP-hard problems

Many real-life problems are in fact falling in the category of NP-hard problems, while the k-SAT system we have presented and investigated earlier in this Thesis is NP-complete. NP-hard problems lie at the basis of many optimization, decision making, error correction, etc. problems.

There are two major classes of algorithm for solving NP-hard problems: exact solvers are extremely inefficient and slow, while heuristic methods can be efficient in finding good approximations, but they are unable to provide information about the correctness of the calculated optimum. This means that they could find the global minimum of the system, but they could also easily be trapped in local minima and the user is not provided with a feedback about the true nature of the found optimum. Many real-life problems can be easily translated into Constraint Satisfaction Problems such as *max*-SAT. From spin-glasses, through protein folding to Sudoku-puzzles[18] and various industrial applications all can be written mathematically in the form of *max*-SAT. It is a fundamental problem which lies at the basis of real-life problems also, like the ground-state problem of Ising spin-glasses[3] from statistical physics, the travelling salesman problem[15], protein folding in bioinformatics[45], industry applications such as scheduling[48], design debugging[41], FPGA routing[50], probabilistic reasoning[35]. The *max*-SAT designation comes from the *maximum satisfiability* and it is the generalized form of the Boolean Satisfiability (k-SAT) problem.

Our novel *max*-SAT solver algorithm is based on the original continuous-time dynamical system presented by Ercsey-Ravasz and Toroczkai[17]. The greatest advantage of this model comprises in the one-to-one correspondence between the stable attractors of the system and the SAT solutions. Starting the dynamics from any initial condition the algorithm will converge into a solution without any further need of an input from the user. In the *hard*-SAT phase the dynamics becomes transiently chaotic leading to interesting conclusions about the relation between chaos and optimization hardness. However, in case of *max*-SAT problems when there is no solution satisfying all constraints at the same time it means that the global attractor of the system is not a stable attractor anymore.

The algorithm is illustrated on a very hard benchmark problem taken from a set of benchmark instances listed on a SAT problem solving competition website[SAT]. In order to test our algorithm we solved a set of benchmark problems, which were very useful because the real minimum energy level was known, so we could compare our results with the available data. We have chosen this particular instance for illustration because it seems to be an extremely hard problem having $N = 250$ variables and a typical hardness $\alpha = 4.0$. We used the complete algorithm named *maxsatz*,[26,27] which won the 2006, 2007 and 2013 *max*-SAT solving competitions for testing the correctness of our *max*-SAT solver. The *maxsatz* algorithm has been running for 5.5 weeks (!) working on this extremely hard benchmark problem and the smallest energy found was $E = 9$. Our algorithm finds an energy level $E = 5$ at $P = 189562$ running around only 20 hours, but even better: it convincingly predicts this global optimum even starting from $P = 7000$. On Figure 5.1 we presented the performance of the

a

P	E_s	$n(E_s)$	E_0	E^{pred}_{min}	$P^{pred}(E_s\text{-}1)$
119	10	4	2.9	3	213
...	...	...	...		...
373	10	8	6.6	7	598
398	9	1	7.0	8	4452
...	...	...	...		...
1385	**9**	**10**	**6.1**	**7**	**2196**
1472	8	1	6.3	7	33096
...	...	...	...		...
5821	**8**	**11**	**5.4**	**6**	**10793**
5909	7	1	5.3	6	240872
6694	7	2	5.0	6	86523
7370	7	3	4.8	5	57276
7379	7	4	4.6	5	39759
7745	7	5	4.5	5	33454
9215	7	6	4.4	5	31589
13434	**7**	**7**	**4.5**	**5**	**40357**
15777	6	1	4.4	5	2320334
86072	6	2	4.8	5	79708643
93334	6	3	4.6	5	9680801
176968	6	4	4.7	5	26144392
177629	**6**	**5**	**4.6**	**5**	**10482130**
189562	**5**	**1**	**4.1**	**5**	

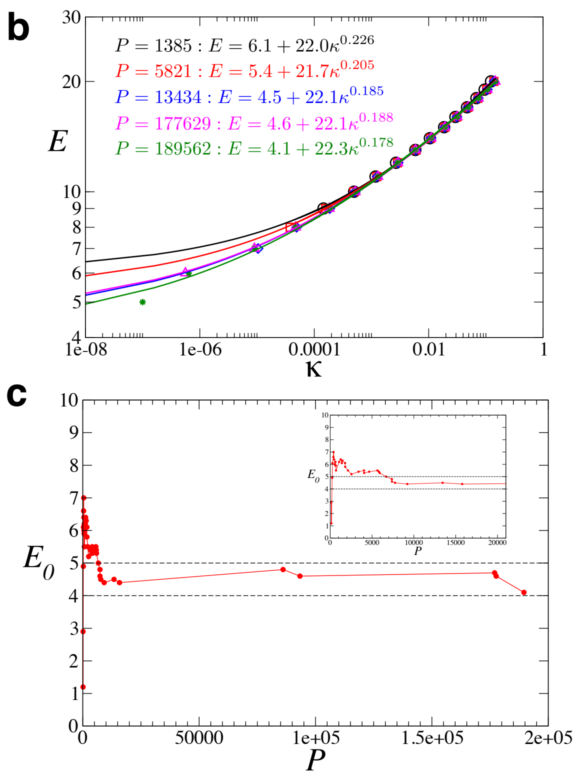

Figure 5.1: Performance of the algorithm illustrated on an extremely hard benchmark problem having $N = 250$ variables and a typical hardness $\alpha = 4.0$, the maximum time set for $t_{max} = 50$ and $b = 0.002375$. a) P - number of trajectories, E_s - the lowest energy found until that point, $n(E_s)$ - the number of times this minimum was found, E_0 - the parameter obtained by fitting and predicting E^{pred}_{min} and estimating $P^{pred}(E_s - 1)$ - the number of trajectories needed to find a lower energy. The algorithm estimates the escape rate and performs a prediction at each P shown in the table, for the lines outlined with bold and colored we show the fit in b). c) The relevant parameter E_0 is shown as function of P. It heavily fluctuates at the beginning when the statistics is small, but as the statistics increases it stabilizes in the $E_0 \in [4, 5)$ interval, convincingly predicting $E^{pred}_{min} = 5$ already after $P = 7000$ up until the point when it finds this energy at $P = 189562$. At this point we do not have a precise estimation for $\kappa(5)$ because it has been found only once ($n(5) = 1$), but the estimation E^{pred}_{min} remains the same, convincing the algorithm to accept $E^{dec}_{min} = 5$ and stop searching further.

algorithm.

Based on the presented chaotic dynamics we designed a novel algorithm, which is capable of efficiently predicting the global minimum of NP-hard problems, using many short dynamical trajectories started from different initial conditions, and more importantly it can provide an estimate for the time needed to find the next lower energy level. This feature of providing feedback on how close one is to find the global optimum of the system is unique. None of the existing algorithm have even a comparable feature. This can become very useful especially in solving extremely hard problems. In this Chapter our main goal was to introduce this novel algorithm, but its details can be further improved. One can redefine the rules that decide when the prediction should be performed, as well as redefine the stopping conditions. We would like to emphasize that the stopping conditions can be further improved. In this Chapter we have also shown that prediction errors mainly occur only in easy problems, but as these easy problems are almost easy enough to be solved with exact solver the main goal is not to apply this method to these problem, but to the hard problems. The power of the algorithm lies in the prediction of global optimum in very hard problems, where it can also provide information about how close it is to finding the global optimum, and it can become more time efficient than exact solvers (as we saw in case of extremely hard problems).

The energy based escape rate is the generalized form of the escape rate used in transient chaos theory[24,47]. We have also shown that the scaling of this new measure reveals crucial information about the structure of state space: the lowest energy level, probabilities of finding lower energy levels, etc. Despite one can never predict the route of a single chaotic trajectory, the statistical properties of the system are robust and can be used to obtain useful information about the system. This could become a novel approach in studying other chaotic dynamical systems and certainly provides an intriguing aspect of the predicting power of chaos. Nevertheless, opens new doors of studying NP-hard problems.

6

Conclusion

The world of analog computing has not reached the end of the road, but rather every day there are emerging new technologies and ideas which prove the viability of this kind of calculating machines. Our results, presented in this Thesis support this idea by bringing examples and showing new ways how these systems can be improved and on how they can be applied in solving complex problems.

In the current Thesis we have introduced a novel analog computer design based on the CNN model introduced by Leon O. Chua and Lin Yang[9,8], as well as on the continuous-time dynamical system introduced by Mária Ercsey-Ravasz and Zoltán Toroczkai to solve Boolean satisfiability problems (*k*-

SAT)[17]. Our revolutionary asymmetric cellular neural network model is implementation-friendly, while preserving the main advantages of the original SAT-solver dynamics[17]. The biggest advantage of the CNN architecture lies in its high processing capability this justifying the choice of the implementation model. By design it is very complicated to implement the dynamical system introduced by Ercsey-Ravasz and Toroczkai in physical form due to the use of the auxiliary variables associated to each constraint in the k-SAT problem. These auxiliary variables are unbounded so their value can increase to infinity in order to prevent the system getting trapped in local minima. If realized on an analog device it would only take a single operation to find the optimal solution. The main advantage of the system is that starting from any initial condition it converges into a solution without further interventions. In contrast to the original dynamical system one cannot exclude limit cycles, but we have shown that the only fixed points of the problem are the solutions, although this does not guarantee that there are no limit cycles in the system, as we also demonstrated their existence. This is a drawback compared to the original system but as one can see a major region of the two main parameters (A and B) govern the system on such trajectories which omit the limit cycles and therefore this will not constitute a significant problem. In order to simulate the system more efficiently we needed a method to detect whether the system will be trapped in a limit cycle so we can stop the simulation and restart it from different initial conditions. Since there is no known algorithm which can detect in real-time (without using the whole data-series already generated ahead) if a trajectory follows a path leading to a limit cycle, we have developed a new algorithm for this task. Measured the efficiency of the algorithm we found the prediction to be correct in $\sim 96\%$ of the cases.

Since physical implementations are usually different from the idealistic environment of models, the presence of different types of noise is inevitable and it is a fundamental question to study their effects on the system. How do they affect the behavior and efficiency of the system? We have presented a detailed study on the effects of typical noise types that are present in analog circuits: white noise (uncorrelated in time), colored noise (correlated in time) and small errors in connection weights (constant in time, modelling imperfections of electronic junctions and/or circuit elements). A series of extensive simulations were run for different system sizes and constraint densities using different noise types. We have shown that the presence of noise in the system does not have a negative influence on the operation of the system, in fact it helps to avoid more easily the limit cycles by widening the optimal range of the main parameters (A, B). We have also shown that our system can tolerate noise intensities up to $I = 10^{-2}$, which is three magnitude higher than the allowed maximum noise intensities in a physical device ($I < 1.6 \times 10^{-6}$).

The concept of the asymmetric cellular neural network model was welcomed by the engineers as well. This model can be considered one of the first attempts to generalize the applications of the CNN computers, which nowadays are mainly used in visual processing applications. An engineering group from the Faculty of Information Technology and Bionics of the Pázmány Péter Catholic University (Budapest, HU) were the first who started to work on a physical realization of our model. The work was started by Prof. Tamás Roska and now continued by his students. After several test circuit implementations they have also realized a $10 + 20$ cell system (s-type, respectively a-type cells) on printed circuit boards (PCB). One of these PCBs is shown on Figure 6.1 (work in progress). Another research group from the Department of Electrical Engineering and Computer Science from the University of Tennessee (Knoxville, TN, USA) lead by David Basford independently started to work on the CNN implementation of our model[4]. A research group lead by Prof. X. Sharon Hu from the University of Notre Dame (Notre Dame, IN, USA) started to work on the implementation of the original dynamical system with small modifications that apply boundaries on the a-type variables[51].

Finally we have introduced a modified version of the original dynamical system[17] in order to be used for predicting the global optimum of NP-hard problems. Many real-life problems fall in this category, which cannot be solved efficiently by any algorithm. In many cases NP-hard problems lack a complete (null-energy) solution but the question is still valid: what is the lowest possible energy level in this system? We studied the NP-hard problems through the *maximum satisfiability* problem (*max*-SAT). The novel algorithm introduced in this Thesis can predict the number of unsatisfied clauses in the global optimum long before reaching these states and provides information on how much time would

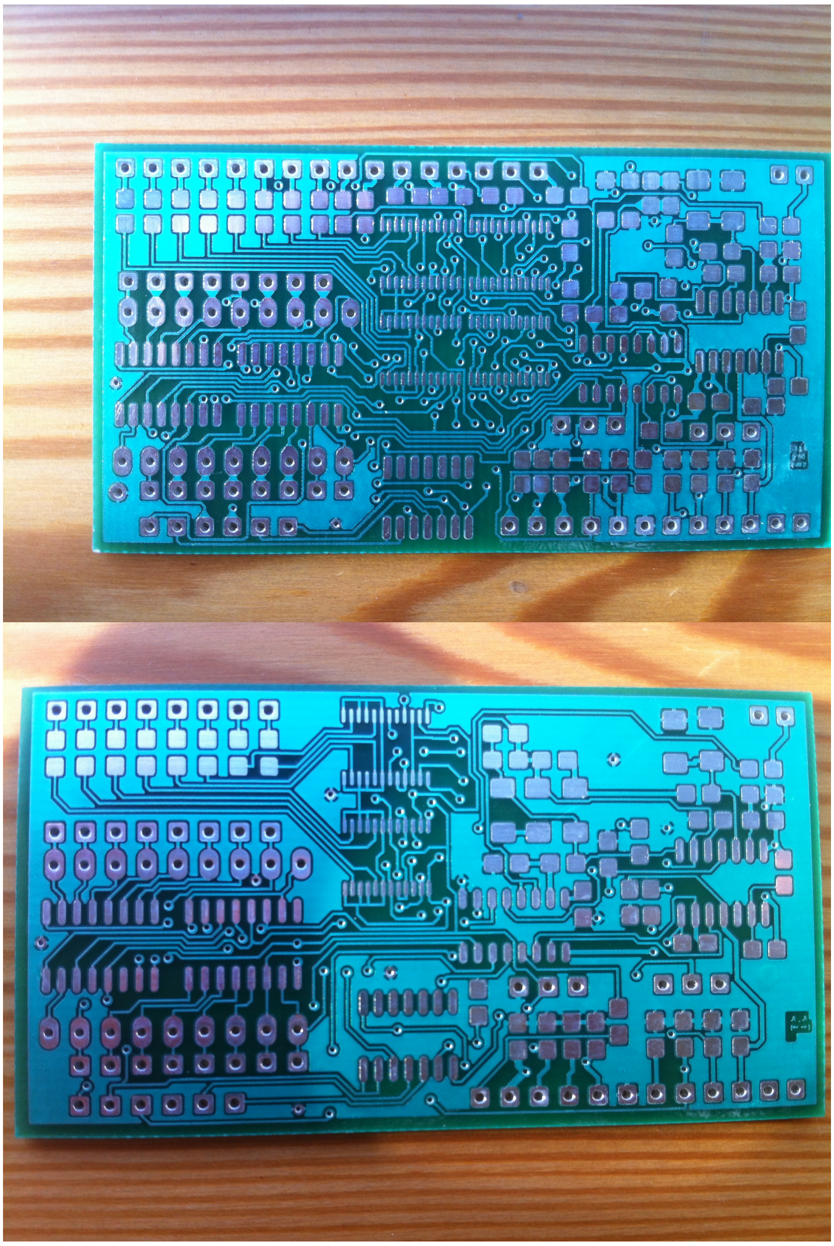

Figure 6.1: Asymmetric Cellular Neural Network on printed circuit board. Printed circuit board realization of our Asymmetric Cellular Neural Network model, created by András Horváth and Dóra Babicz from the Faculty of Information Technology and Bionics of the Pázmány Péter Catholic University (Budapest, HU)

it take or how many different trajectories should be run in order to reach that state. We preserved all the advantages of the original dynamical algorithm and we have added a few enhancements so it can efficiently operate in the new environment of NP-hard problems. We have also shown that this novel algorithm can predict the global optimum of a *max*-SAT problems and can provide information about the trustworthiness of the prediction it made.

One can see that these studies opened the possibilities to study optimization problems in new ways. Furthermore based on our model a completely new direction of the analog computer development is emerging. The algorithm presented at the end of this Thesis gives a trustworthy prediction of global minima in the majority of NP-hard problems in relatively short time which is an unmatched advantage compared to any known algorithm. We are currently using it to study the Ramsey problem and to predict further values of Ramsey-numbers[38,19]. Even more, we are now working on an update to the algorithm which can predict not only the lowest energy level but a solution (an assignment of Boolean variables if considering a *max*-SAT problem) which can correspond to that level. There are endless possibilities of application so for the asymmetric cellular neural networks as for the global optimum predictor in various arias where optimization problems occur.

All these continuous-time dynamical systems show that hardness of optimization problems corresponds to chaotic dynamics in physical systems. It is fascinating that in spite of the unpredictability of individual chaotic trajectories, physical properties of chaotic dynamics can predict properties of the global energy landscape.

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Publications

Scientific papers (ISI indexed)

- Molnár, B., Varga, M., Toroczkai, Z. and Ercsey-Ravasz, M. (to be submitted) *Predicting global optima of NP-hard problems through analog chaos*
- Sumi, R., Molnár, B. and Ercsey-Ravasz, M. (2014) *Robust optimization with transiently chaotic dynamical systems*, European Physics Letters, 106(4).
- Molnár, B. and Ercsey-Ravasz, M. (2013) *Asymmetric Continuous-Time Neural Networks with-*

out Local Traps for Solving Constraint Satisfaction Problems, PLoS ONE, 8(9).

- Molnár, B. and Ercsey-Ravasz, M. (2014), *Analog dynamics for solving max-SAT problems* In Niemier, M and Porod, W (Ed.), 14th Int. Workshop on Cellular Nanoscale Networks and their Applications (CNNA), University of Notre Dame, Notre Dame, IN, JUL 29-31, 2014.

- Molnár, B., Sumi, R. and Ercsey-Ravasz, M. (2014), *A CNN SAT-solver robust to noise*, In Niemier, M and Porod, W (Ed.), 14th Int. Workshop on Cellular Nanoscale Networks and their Applications (CNNA), University of Notre Dame, Notre Dame, IN, JUL 29-31, 2014.

- Molnár, B., Toroczkai, Z. and Ercsey-Ravasz, M. (2012), *Continuous-time Neural Networks Without Local Traps for Solving Boolean Satisfiability*, 13th Int. Workshop on Cellular Nanoscale Networks and their Applications (CNNA), Turin, IT, Aug 29-31, 2012.

Conference contributed talks

- *Analog dynamics for solving max-SAT problems*, 14th Int. Workshop on Cellular Nanoscale Networks and their Applications (CNNA), University of Notre Dame, Notre Dame, IN, Jul 29-31, 2014.

- *A CNN SAT-solver robust to noise*, 14th Int. Workshop on Cellular Nanoscale Networks and their Applications (CNNA), University of Notre Dame, Notre Dame, IN, Jul 29-31, 2014.

- *Continuous-time Neural Networks Without Local Traps for Solving Boolean Satisfiability*, 13th Int. Workshop on Cellular Nanoscale Networks and their Applications (CNNA), Turin, IT, Aug 29-31, 2012.

Posters

- *CNN Computers for Solving SAT Problems*, IV Summer School on Statistical Physics of Complex and Small Systems (GEFENOL), IFISC, Palma de Mallorca, Spain, Sept 8-19, 2014.

Previous publications (ISI indexed)

- Néda, Z., Sumi, R., Ercsey-Ravasz, M., Varga, M., Molnár, B. and Cseh, Gy. *Correlation clustering on networks*, Journal of Physics A: Math. Theor. 42 (2009) 345003

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Keywords

Constraint Satisfaction Problem, analog computing, NP-hard problems, continuous-time dynamical systems, Cellular Neural/Nonliniar Networks (CNN), transient chaos, noisy dynamical systems, global optimization problems

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