"BABEŞ-BOLYAI" UNIVERSITY OF CLUJ-NAPOCA
DOCTORAL SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE

Variational properties of the solutions of nonlinear operator equations and systems

DOCTORAL THESIS SUMMARY

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Introduction

Variational methods represent an important chapter of nonlinear analysis, and of the theory of differential equations as well. The advantage of such methods in the theory of equations is that they provide variational properties of the solutions.

Numerous papers have been published on this approach based on critical point theory, which compared to the fixed point approach, gives an additional information about the solution, for example of it being a maxima, a minima, or a saddle point.

For results in this direction, the reader may consult, for example, any of the following references: A. Ambrosetti and P. H. Rabinowitz [4], O. Kavian [41], P. H. Rabinowitz [79], M. Struwe [94].

Fixed point theory gives us conditions which guarantee the existence of fixed points for classes of nonlinear operators. A natural question would be to ask about additional properties of the fixed points in case that the fixed point equation has a variational form given by an associated functional. For example, we may be interested that the fixed point is an extremum of that functional.

There is a strong motivation for addressing this problem, coming from physics, where in many cases the state of a process complies to the minimization principle of the energy.

Problems of this kind were considered, for instance, in the papers A. Budescu [17], A. Budescu and R. Precup [18, 19], R. Precup [78], D. Motreanu and C. Varga [55], and B. Ricceri [84, 86], in connection with Lipschitz properties of the nonlinear operators.

In this thesis, we are concerned with the study of the semilinear operator equation

$$Au = J'(u), \tag{0.0.1}$$

in a Hilbert space $(H, (.,.)_H)$, where A is a positively defined linear operator and the nonlinear term is the Fréchet derivative of a functional J.

We seek variational properties of the weak solutions $u \in H_A$, where H_A is the energetic space associated to the operator A.

Equation $(0.0.1)$ is equivalent to the fixed point equation

$$u = T(u), \tag{0.0.2}$$

where $T := A^{-1}J'$. The construction of the functional associated to equation $(0.0.1)$ is based on the theory of positive self-adjoint linear operators in Hilbert spaces (H. Brezis [15], H. Brezis and F. Browder [16], S. G. Michlin [53]). This functional is defined as follows

$$E : H_A \to \mathbb{R}, \quad E(u) = \frac{1}{2}\|u\|^2{}_{H_A} - J(u). \tag{0.0.3}$$

Similarly, we discuss the system

$$\begin{cases} A_1 u = J_{11}(u,v) \\ A_2 v = J_{22}(u,v), \end{cases} \tag{0.0.4}$$

associated to two positively defined linear operators A_1, A_2, and to two functionals J_1, J_2, where by $J_{11}(u,v), J_{22}(u,v)$ we mean the Fréchet derivatives of $J_1(.,v)$ and $J_2(u,.)$, respectively.

Next, we introduce the results given in the paper R. Precup [77], which we are going to apply to the equation $(0.0.1)$ and to the system $(0.0.4)$, respectively. In R. Precup [77], it was shown that the unique fixed point of a contraction T on a Hilbert space, which is identified to its dual, in case that T has the variational form

$$T(u) = u - E'(u), \tag{0.0.5}$$

minimizes the functional E.

Also, the unique fixed point (u^*, v^*) of a Perov contraction $(T_1(u,v), T_2(u,v))$, where T_1 and T_2 can be expressed as

$$T_1(u,v) = u - E_{11}(u,v) \tag{0.0.6}$$

$$T_2(u,v) = u - E_{22}(u,v), \tag{0.0.7}$$

is, under some conditions, a Nash-type equilibrium of the pair of functionals (E_1, E_2), that is:

$$\begin{aligned} E_1(u^*, v^*) &= \min_u E_1(u, v^*) \\ E_2(u^*, v^*) &= \min_v E_2(u^*, v). \end{aligned}$$

More informations about Nash equilibria, related problems, different approaches and applications, can be found in A. Kristály, V. Radulescu and C. Varga [44], J. F. Nash [60], and S. Park [68].

For the treatment of systems we will use a vectorial approach based on matrices instead of constants, and on vector-valued norms, as it was initiated in A. I. Perov and A. V. Kibenko [71] for a vector version of the contraction principle, and it was extended by R. Precup [74] in connection with other principles of nonlinear analysis.

Our aim is to develop a general theory based on the study of existence and variational characterizations of the weak solutions of equations type $(0.0.1)$ and systems type $(0.0.4)$, and then to apply the abstract results to specific problems: Dirichlet boundary value problems for elliptic equations and systems, elliptic problems under nonresonance conditions, singular second-order equations and systems.

Motivated by the large number of applications, we make a further step and we discuss the case of the fixed point equation $u = N(u)$, this time in the setting of topological fixed point theory. We assume that the operator N is connected to the energy functional E, by the relation

$$E'(u) = u - N(u)$$

in the case of Hilbert spaces, and correspondingly, by

$$E'(u) = J(u) - J(N(u))$$

in more general Banach spaces, where J is the duality map defined in Section 5.3.

By our approach, compared to the classic topological fixed point theorems of Schauder, Krasnoselskii, Darbo, and Sadovskii, part of the assumptions on the operator N is replaced by conditions asked to the functional E.

Structure of the thesis

The thesis is divided into 5 chapters, each chapter containing several sections, and is organized as follows.

Chapter 1: Preliminaries

In Section 1.1 we review some basic notions and results, more exactly: the Nemytskii superposition operator and its properties (P. Jebelean [39], D. Pascali and S. Sburlan [69], R. Precup [76]), the Ekeland variational principle (I. Ekeland [31, 32], D. G. de Figueiredo [33], M. Frigon [35, 36]), matrices with spectral radius less than one, or matrices that are convergent to zero, and their role for the treatment of systems (A. I. Perov [70], R. Precup [74]), some fixed point principles from the nonlinear analysis: Banach's contraction principle on complete metric spaces and Perov's fixed point theorem for contractive operators on generalized metric spaces (J. Dugunji and A. Granas [30], A. Granas and J. Dugunji [37], A. I. Perov [70], A. I. Perov and A. V. Kibenko [71], R. Precup [73], I. A. Rus [87, 88, 89]), abstract Fourier series in Hilbert spaces and their properties (L. V. Kantorovich and G. P. Akilov [40], R. Precup [76]).

In Section 1.2, the notions of compact, completely continuous operators, and some of their properties are presented (H. Brezis [14], K. Deimling [25], A. Granas and J. Dugunji [37], M. A. Krasnoselskii [43], R. Precup [73], E. Zeidler [97]); also presented are some topological fixed point theorems: Schauder's fixed point theorem, for compact operators, the Leray-Schauder principle for completely continuous operators, set-contractions, and condensing operators (J. Leray and J. Schauder [49], D. O' Regan and R. Precup [66], R. Precup [75], J. Schauder [93]), and the fixed point theorems of Krasnoselskii, Darbo, and Sadovskii (G. Darbo [23], A. Granas and J. Dugunji [37], M. A. Krasnoselskii [42, 43], B. N. Sadovskii [91, 92]).

Section 1.3 contains the variational characterization of the fixed points of contraction-type operators, given in R. Precup [77], which represents the starting point of this thesis.

Chapter 2: Semilinear operator equations and systems with potential-type nonlinearities

In this chapter, we develop a general, abstract theory of semilinear operator equations and systems with linear parts given by positively defined operators, and nonlinearities of potential-type. Also, we present applications to elliptic equations and systems.

Firstly, we are concerned with the semilinear operator equation $(0.0.1)$, and secondly, we discuss the semilinear operator system $(0.0.4)$. To this aim, we fully exploit S. G. Michlin's theory [53], on linear operator equations.

Our special interest in such kind of equations and systems is represented by the elliptic problem

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{0.0.8}$$

and correspondingly, by the following elliptic system

$$\begin{cases} -\Delta u = f(x,u,v) \text{ in } \Omega \\ -\Delta v = g(x,u,v) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ v = 0 \text{ on } \partial\Omega, \end{cases} \tag{0.0.9}$$

where Ω is a bounded, open subset of $\mathbb{R}^n$.

Elliptic boundary value problems were also the subject of study of various authors, such as G. Dinca, P. Jebelean and J. Mawhin [29], D. G. de Figueiredo [34], P. Jebelean [39], L. Nirenberg [61], and V. Radulescu [82].

In particular, we shall derive variational characterizations of the weak solutions of the Dirichlet problem $(0.0.8)$, and the system $(0.0.9)$. More exactly, we shall give conditions such that the solution of the Dirichlet problem $(0.0.8)$ is minimizing the associated energy functional, and that the solution of the corresponding problem for $(0.0.9)$ is a Nash-type equilibrium of the pair of energy functionals. Here, we adapt to systems the variational approach for one equation due to C. Bereanu, P. Jebelean and J. Mawhin [10].

*

The chapter is organized as follows. Section 2.1 is devoted to the overview and Section 2.2 contains additional preliminaries for this chapter. In Section 2.3, we discuss the case of the equation $(0.0.1)$, while in Section 2.4, we obtain theoretical results for the system $(0.0.4)$. Furthermore, in Section 2.5 we apply our first abstract result to the elliptic problem $(0.0.8)$, and in Section 2.6 we apply our second abstract result to the system $(0.0.9)$.

*

Our contributions in this chapter are: Theorems 2.3.1, 2.4.1, 2.5.1, 2.6.1. These results are contained in the papers A. Budescu [17], and A. Budescu and R. Precup [19].

Chapter 3: Semilinear equations under nonresonance conditions

In this chapter, we are concerned with the variational characterization of the weak solutions for the semilinear equation of the form

$$Au - cu = J'(u), \tag{0.0.10}$$

in a Hilbert space H under the *nonresonance* condition $c \neq \lambda_j$, $j = 1, 2, ...$, where λ_j are the eigenvalues of the operator A. Here A is a linear and positively defined operator, having all the properties required in Section 2.2.

Nonresonant problems were studied by several authors, such as H. Amann and E. Zehnder [2], R. P. Agarwal, D. O'Regan and R. Precup [7], D. G. de Figueiredo [34], J. Mawhin and J. R. Ward Jr. [50], D. Muzsi [56, 57], D. Muzsi and R. Precup [58, 59], D. O'Regan [63].

The main result of this chapter is that under some assumptions on the functional $J \in C^1(H)$, if E is the energy functional of the equation $(0.0.10)$ given by

$$E(u) = \frac{1}{2}\|u\|_{H_A}^2 - \frac{c}{2}\|u\|_H^2 - J(u),$$

then any solution u satisfies $E(u) \leq E(u + w)$, for every element w orthogonal on the first k eigenvectors of A.

The proof is based on the application of Ekeland's variational principle to a suitable modified functional, and differs essentially from the proof of the particular case when $c = 0$, which was discussed in Section 2.3.

In particular, we apply the general theory developed in Section 3.4 to the elliptic problem

$$\begin{cases} -\Delta u - cu = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.11}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$.

The result that we obtained is the following: if $\lambda_k < c < \lambda_{k+1}$, for some $k \in \{0, 1, ...\}$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the Carathéodory conditions and the Lipschitz condition

$$|f(x, u) - f(x, \bar{u})| \leq \alpha|u - \bar{u}|,$$

with $0 \leq \alpha \leq \dfrac{\lambda_1}{\lambda_{k+1}}(\lambda_{k+1} - c)$, then for any weak solution $u^* \in H_0^1(\Omega)$ of the problem $(0.0.11)$, the following variational property holds:

$$E(u^*) = \inf_{w \in H_k} E(u^* + w).$$

Here E is the associated energy functional and

$$H_k = \left\{ u \in H_0^1(\Omega) : (u, \phi_j)_{H_0^1} = 0 \text{ for } j = 1, 2, ..., k \right\}.$$

*

The chapter is organized as follows. Section 3.1 is devoted to the overview, while in Section 3.2 we present the spectral theory of linear, symmetric, and completely continuous operators (H. Brezis [14], L. V. Kantorovich and G. P. Akilov [40], R. Precup [76]). In Section 3.3, we present some auxiliary results concerning the Fourier series, namely Lemma 3.3.1 and Lemma 3.3.2, that we will use to state the main result of this chapter, Theorem 3.4.1. Furthermore, in Section 3.4, we discuss the case of the equation (0.0.10), developing a general theory for the nonresonance case, and in Section 3.6, we apply this theory to the elliptic problem (0.0.11).

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Our contributions in this chapter are: Theorems 3.4.1, 3.5.1, Lemmas 3.3.1, 3.3.2. These results are contained in the paper A. Budescu and R. Precup [18].

Chapter 4: Variational properties of the solutions of singular second-order differential equations and systems

This chapter contains results that are derived from the abstract theory worked out in Section 2.3 for one equation, and in Section 2.4 for systems. More exactly, we present applications to specific problems: second-order differential equations with singularities and systems of such equations. All the results from this chapter are based on the abstract theory developed in Chapter 2.

Singular boundary value problems were studied intensively in the recent years. We refer to the bibliographies of the papers R. P. Agarwal, R. P. Agarwal, D. O'Regan and R. Precup [7], K. Q. Lan [47], K. Q. Lan and J. R. L. Webb [48], D. O'Regan [62, 63, 64], D. O'Regan and R. P. Agarwal [65], I. Rachunkova [80], I. Rachunkova and J. Tomecek [81], for references.

First, in Section 4.2, we will approach the case of a single equation, and we will present results for the Dirichlet boundary value problem for a second-order equation type

$$\begin{cases} -(p(x)u'(x))' = f(x, u(x)) & \text{a.e. } x \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{0.0.12}$$

Here $p \in C[0,1] \cap C^1(0,1)$, $p > 0$ on $(0,1)$, and can vanish at $x = 0$ or $x = 1$, making the equation singular with respect to the independent variable x. We assume that $I = \int_0^1 \frac{1}{p(x)} dx$ is finite. In this specific case, $H = L^2(0,1)$, and $Au = -(pu')'$ with

$$\begin{aligned} D(A) &= \{u \in C[0,1] \cap C^1(0,1) : \ u(0) = u(1) = 0, \\ pu' &\in AC[0,1], \ (pu')' \in L^2(0,1)\}. \end{aligned}$$

Also, the energetic space H_A is the weighted Sobolev space

$$H_0^1(0,1;p) = \left\{u \in AC[0,1] : \ u(0) = u(1) = 0, \ \sqrt{p}u' \in L^2(0,1)\right\}.$$

For a basic presentation of Sobolev spaces, one can consult e.g. the book of R. A. Adams [1].

The result states that the unique solution of (0.0.12) is a minimium of the associated energy functional.

Next, we extend the theory towards systems of the form

$$\begin{cases} -\left(p(x)u'(x)\right)' = f\left(x, u\left(x\right), v\left(x\right)\right) \\ -\left(q(x)v'(x)\right)' = g\left(x, u\left(x\right), v\left(x\right)\right) \\ u(0) = u(1) = 0 \\ v(0) = v(1) = 0, \end{cases} \tag{0.0.13}$$

where both functions p and q may introduce singularities.

In the case of systems (0.0.13), our result is about the existence of a Nash-type equilibrium of the pair of the associated energy functionals.

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The chapter is organized as follows. Section 4.1 is devoted to the overview, while in Section 4.2 we present the main results, first in the case of a single equation, and then in the case of systems. In Section 4.3, two examples are given in order to illustrate the results obtained in the previous section.

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Our contributions in this chapter are: Theorems 4.2.1, 4.2.2 and Examples 4.3.1, 4.3.2. These results are contained in the paper A. Budescu and R. Precup [19].

Chapter 5: Fixed point theorems under combined topological and variational conditions

In this chapter, we obtain fixed points that minimize the associated functionals, this time in the setting of topological fixed point theory. We shall use the Ekeland variational principle and the Palais-Smale compactness condition guaranteed by the topological properties of the nonlinear operators.

More precisely, we consider the fixed point equation

$$u = N(u) \tag{0.0.14}$$

in a subset D of a Hilbert space H with inner product $(.,.)$ and norm $\|.\|$, which has a variational structure in the sense that there exists a functional $E \in C^1(H)$, such that

$$u - N(u) = E'(u) \tag{0.0.15}$$

for all $u \in D$.

The idea is to obtain variational characterizations for the solutions of the equation $(0.0.14)$. Compared to the classical approach (Schauder, Krasnoselskii, Darbo and Sadovskii), in our results, we shall replace part of the assumptions on the operator N with conditions asked to the energy functional E.

First, we obtain a result for the case where N is the sum of two operators, namely Theorem 5.2.1. Its consequence, Theorem 5.2.2, can be seen as a variational-topological version of Darbo's fixed point theorem.

Next, Theorem 5.2.2 is generalized for condensing operators. The result, Theorem 5.2.3, is a variational-topological version of Sadovskii's fixed point theorem.

Moreover, in Section 5.3, we obtain a variational-topological result in more general Banach spaces. Here, we consider the fixed point problem $(0.0.14)$ in a Banach space X with suitable geometric properties, which are expressed in terms of the properties of the duality map.

Duality mappings have become a most important tool in nonlinear functional analysis, in particular for questions involving nonlinear operators. For further reading, we recommend the following references for the study of geometric properties of Banach spaces and a lot more: V. Barbu [9], I. Cioranescu [22], G. Dinca [27], G. Dinca and P. Jebelean [28], G. Dinca, P. Jebelean and J. Mawhin [29].

The result we state in Section 5.3, namely Theorem 5.3.1, is a vectorial-topological analogue of Schauder's fixed point theorem.

Finally, an application is given to the two-point boundary-value problem

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = f(t,u(t)), & t \in [0,1] \\ u(0) = u(1) = 0, \end{cases} \tag{0.0.16}$$

using Theorem 5.3.1.

Here, the space $X := W_0^{1,p}(0,1)$ is endowed with the energetic norm

$$\|u\|_{1,p} = \left(\int_0^1 |u'(t)|^p dt\right)^{1/p},$$

and the energy functional is given by

$$E : W_0^{1,p}(0,1) \to \mathbb{R}, \quad E(u) = \int_0^1 \left(\frac{1}{p}|u'(t)|^p - F(t,u(t))\right) dt,$$

where $F(t,\tau) = \displaystyle\int_0^\tau f(t,s)ds$.

Also, the duality map of the space $W_0^{1,p}(0,1)$ is the mapping

$$J : W_0^{1,p}(0,1) \to W^{-1,q}(0,1), \quad (1/p + 1/q = 1)$$

given by

$$J(u) = -(|u'|^{p-2}u')'.$$

The result states that the problem $(0.0.16)$ has a solution $u \in W_0^{1,p}(0,1)$ with $\|u\|_{1,p} < R$, where $R > 0$ satisfies

$$h(R) := \frac{R^p}{p} - \frac{a}{q}R^q - bR > 0,$$

which minimizes the energy functional E, on the closed ball centered at origin and of radius R of $W_0^{1,p}(0,1)$.

*

The chapter is organized as follows. In Section 5.1 we present the overview, while Section 5.2 is devoted to the study of the fixed point problem $(0.0.14)$ in Hilbert spaces. In Section 5.3, we extend our theory to more general Banach spaces, where we shall assume that the duality mapping J is single-valued, invertible, and both J and J^{-1} are continuous. Section 5.4 contains an application of the general theory developed in Section 5.3 to the two-point boundary value problem $(0.0.16)$.

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Our contributions in this chapter are: Theorems 5.2.1, 5.2.2, 5.2.3, 5.3.1, 5.4.1. These results are contained in the paper A. Budescu and R. Precup [20].

*

Author's research activity

Most of the results in this doctoral thesis are part of the following papers:

- A. Budescu, *Semilinear operator equations and systems with potential-type nonlinearities*, Studia Univ. Babeş-Bolyai Math. **59** (2014), 199–212, MR3229442.
- A. Budescu and R. Precup, *Variational properties of the solutions of semilinear equations under nonresonance conditions*, J. Nonlinear Convex Anal., **17**(2016), Issue 8, 1517-1530.
- A. Budescu and R. Precup, *Variational properties of the solutions of singular second-order differential equations and systems*, J. Fixed Point Theory Appl. **18** (2016)3, 505-518.
- A. Budescu and R. Precup, *Fixed point theorems under combined topological and variational conditions*, Results Math., **70**(2016), 3, 487–797.

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Chapter 1

Preliminaries

1.1 Basic notions and results

The purpose of this introductory chapter is to present some definitions and basic results that will be required in what follows.

1.1.1 Ekeland's variational principle

In this section we present the Ekeland variational principle, which is one of the most frequently applied results of nonlinear analysis. See, for example, I. Ekeland [31], E. Bishop and R. R. Phelps [11], J. Danes [24] and M. Frigon [36], for more details.

We will apply the Ekeland variational principle in the proofs of several results obtained throughout the thesis.

We shall give the weak version of Ekeland's variational principle, which is enough for our considerations.

First, note that a real functional E defined on a metric space X is *lower semicontinuous* if the level set

$$(E \leq r) := \{u \in X : E(u) \leq r\}$$

is closed for every $r \in \mathbb{R}$.

Theorem 1.1.1 *Let (X, d) be a complete metric space and let $E : X \to \mathbb{R}$ be a lower semicontinuous function bounded from below. Then given $\epsilon > 0$ and $u_0 \in X$, there exists a point $u \in X$ such that*

$$E(v) - E(u) + \epsilon d(u, v) \geq 0 \quad \text{for all } v \in X, \tag{1.1.1}$$

$$E(u) \leq E(u_0) - \epsilon d(u, u_0). \tag{1.1.2}$$

Corollary 1.1.2 *Under the assumptions of Theorem 1.1.1, for each $\epsilon > 0$, there exists an element $u \in X$ such that* (1.1.1) *holds and*

$$E(u) \leq \inf_X E + \epsilon.$$

Corollary 1.1.3 *Under the assumptions of Theorem 1.1.1, if X is a Banach space with norm $\|\cdot\|$, and $E \in C^1(X)$, there exists a sequence (u_k) with*

$$E(u_k) \to \inf_X E \quad \text{and} \quad E'(u_k) \to 0. \tag{1.1.3}$$

Definition 1.1.4 *We say that functional E satisfies the* Palais-Smale (compacteness) condition *if any sequence satisfying*

$$E(u_k) \to c \quad \textit{and} \quad E'(u_k) \to 0$$

for every $c \in \mathbb{R}$, has a convergent subsequence.

Corollary 1.1.5 *Under the assumptions of Corollary 1.1.3, if in addition E satisfies the Palais-Smale condition, then there is a point $u \in X$ with*

$$E(u) = \inf_X E \quad \textit{and} \quad E'(u) = 0.$$

1.1.2 Matrices with spectral radius less than one

Next, we present the notion of matrices with spectral radius less than one , sometimes called convergent to zero matrices. We shall present their basic properties, particularly the relation between the concepts of convergent to zero matrix and inverse-positive matrix.

Definition 1.1.6 *A square matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is said to be* convergent to zero *if*

$$M^k \to 0 \quad \textit{as} \quad k \to \infty,$$

where by 0 we have denoted the zero matrix of order n.

Lemma 1.1.7 *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following statements are equivalent:*

(i) *M is a matrix that is convergent to zero;*

(ii) *$I - M$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \dots \tag{1.1.4}$$

(where I stands for the unit matrix of the same order as M);

(iii) *the eigenvalues of M are located inside the unit disc of the complex plane (the spectral radius $\rho(M)$ is less than 1);*

(iv) *$I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements (the matrix $I - M$ is inverse-positive).*

Note that, according to the equivalence of the statements (i) and (iv), a matrix M is convergent to zero if and only if the matrix $I - M$ is *inverse-positive*.

Also, using the well-known concept of *spectral radius* (A. I. Perov [70], R. Precup [74], O. Bolojan-Nica [12]), we can say that a matrix M is convergent to zero if and only if $\rho(M) < 1$, where the spectral radius $\rho(M)$ is defined by

$$\rho(M) = \max\left\{|\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue of } M\right\}.$$

Let us note that, for a square matrix of order 2

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}_+),$$

one has $\rho(M) < 1$ if and only if

$$a + d < \min\{2, 1 + ad - bc\} \tag{1.1.5}$$

(see O. Bolojan and R. Precup [13]). We refer to R. Precup [74], for a survey on the role of matrices with spectral radius less than one in the study of semilinear operator systems. In the paper, the author also points out the advantage of vector-valued norms, compared to scalar norms.

Some of our next results are based on the notion of *generalized contraction* in Perov's sense.

Let (X, d) be a generalized metric space with $d : X \times X \to \mathbb{R}^n$.

Definition 1.1.8 *An operator $T : X \to X$ is said to be a* generalized contraction *with respect to the vector-valued metric d, if there exists a convergent to zero matrix M of order n such that*

$$d(T(u), T(v)) \leq Md(u, v)$$

for all $u, v \in X$.

The matrix M is called a *Lipschitz matrix*.

This notion is used in the vector version of Banach's contraction principle, which is due to A. I. Perov [70].

1.2 Topological fixed point theorems

In this section we present some fundamental topological results from nonlinear analysis, which will be used along the thesis.

The fixed point theorem connected to the concept of completely continuous operator is *Schauder's fixed point theorem.* The next two theorems are known as the fixed point theorem of Schauder (see J. Schauder [93]). In applications the second one is most useful.

Theorem 1.2.1 ***(Schauder's fixed point theorem)*** *Let X be a Banach space, $K \subset X$ a nonempty, convex, compact set and let*

$$N : K \to K$$

be a continuous operator. Then N has at least one fixed point.

Theorem 1.2.2 ***(Schauder's fixed point theorem)*** *Let X be a Banach space, $D \subset X$ a nonempty, convex, bounded, closed set and let*

$$N : D \to D$$

be a completely continuous operator. Then N has at least one fixed point.

1.2.1 Krasnoselskii, Darbo and Sadovskii fixed point theorems

The topological fixed point theorems which follow, will be used in Chapter 5. We begin with some preliminary notions and results.

A result that unifies Banach's and Schauder's fixed point theorems is the following.

Theorem 1.2.3 ***(Krasnoselskii)*** *If $D \subset X$ is a nonempty, convex, bounded, closed set, $T : D \to X$ is a contraction, $S : D \to X$ is a compact operator, and*

$$T(D) + S(D) := \{T(u) + S(v) : \ u, \ v \in D\} \subset D,$$

then there exists at least one $u \in D$ such that $N(u) = u$, where $N := T + S$.

In applications, one of the drawbacks of the above fixed point theorems is the invariance condition $N(D) \subset D$. The Leray-Schauder principle (see, for e.g. J. Leray and J. Schauder [49]) makes it possible to avoid such a condition and requires instead that a boundary condition is satisfied.

Theorem 1.2.4 ***(The Leray-Schauder principle)*** *Let X be a Banach space. Also, let $K \subset X$ be a closed, convex set, $U \subset K$ a bounded set, open in K and $u_0 \in U$ a fixed element. We denote by ∂U the boundary of U with respect to K. Assume that the operator $N : \overline{U} \to K$ is compact and satisfies the boundary condition*

$$u \neq (1-\lambda)u_0 + \lambda N(u) \tag{1.2.1}$$

for all $u \in \partial U$ and $\lambda \in (0,1)$. Then N has at least one fixed point in $\overline{U}$.

For the proofs of Theorems 1.2.1, 1.2.2, 1.2.4 see, e.g. R. Precup [73].

Let $\mathcal{P}$ be the set of all bounded subsets of X. Denote by α the *Kuratowski measure of noncompactness* on X, i.e.,

$$\alpha(M) = \inf\{d > 0 : M \text{ admits a finite cover by sets of diameter } \leq d\}, \tag{1.2.2}$$

for any $M \in \mathcal{P}$.

Recall some of the basic properties of the Kuratowski measure of noncompactness (A. Ambrosetti [3], G. Darbo [23], K. Deimling [25], J. Dugunji and A. Granas [30], J. K. Hale and S. M. V. Lunel [38], K. Kuratowski [46]):

(a) $\alpha(M) = 0$ if and only if M is a relatively compact set;

(b) $\alpha(M_1 \cup M_2) = \max\{\alpha(M_1), \alpha(M_2)\}$;

(c) $\alpha(\text{conv } M) = \alpha\{M\} = \alpha\left(\overline{M}\right)$.

A continuous operator $N : D \subset X \to X$ is said to be

(a) of *Krasnoselskii type*, if $N = S + T$, where S is completely continuous and T is a contraction;

(b) a *k-set contraction* (of *Darbo type*) for some $k \in [0,1)$, if

$$\alpha(N(M)) \leq k\alpha(M), \tag{1.2.3}$$

for every $M \subset D$ bounded;

(c) a *condensing operator*, if

$$\alpha(N(M)) < \alpha(M), \tag{1.2.4}$$

for every $M \subset D$ bounded, with $\alpha(M) > 0$.

Obviously, each completely continuous map is of Krasnoselskii type and each Krasnoselskii type map is a set-contraction. Also, any set-contraction is a condensing operator.

Replacing in Theorem 1.2.4 the completely continuous operator T by a Krasnoselskii type map, a set-contraction, or a condensing operator, the Theorem 1.2.4 becomes, more generally, the Leray-Schauder principle for Krasnoselskii type maps, for set-contractions, and for condensing operators, respectively.

To the notions of Krasnoselskii type operator, set-contraction and condensing operator, one can associate the Theorem 1.2.3 and following generalizations of Schauder's fixed point theorem (see, e.g. M. A. Krasnoselkii [43], G. Darbo [23], B. N. Sadovskii [91, 92]).

Theorem 1.2.5 *(**Darbo's fixed point theorem**) Let X be a Banach space. If $D \subset X$ is a nonempty, convex, bounded, closed set and $N : D \to D$ is a k-set contraction, then there exists at least one $u \in D$ such that $N(u) = u$.*

Theorem 1.2.6 *(**Sadovskii's fixed point theorem**) Let X be a Banach space. If $D \subset X$ is a nonempty, convex, bounded, closed set and $N : D \to D$ is a condensing operator, then there exists at least one $u \in D$ such that $N(u) = u$.*

1.3 Variational properties for contraction-type operators

In this section, we summarize the abstract results from the paper R. Precup [77], concerning the variational characterization of the fixed points of contraction-type operators, which represent the starting point of this thesis.

The first result refers to usual contractions on a Hilbert space.

Theorem 1.3.1 *(**R. Precup** [77]) Let $(X, (.,.))$ be a Hilbert space identified to its dual and $T : X \to X$ be a contraction with the unique fixed point u^* (guaranteed by Banach's contraction theorem). If there exists a C^1-functional E bounded from below such that*

$$E'(u) = u - T(u) \tag{1.3.1}$$

for all $u \in X$, then u^* minimizes the functional E, i.e.

$$E(u^*) = \inf_X E.$$

The next result from R. Precup [77] is about systems of the form

$$\begin{cases} u = T_1(u, v) \\ v = T_2(u, v), \end{cases} \tag{1.3.2}$$

where $u \in X_1$, $v \in X_2$. In this case, instead of Lipschitz constants, we shall use matrices.

Refering to the system $(1.3.2)$, we assume that $(X_i, \|\cdot\|_i)$, $i = 1, 2$, are Hilbert spaces identified to their duals and we denote by $X := X_1 \times X_2$.

Also, assume that each equation of the system has a variational form, i.e., that there exist the continuous functionals

$$E_1, E_2 : X \to \mathbf{R},$$

such that $E_1(., v)$ is Fréchet differentiable for every $v \in X_2$, $E_2(u, .)$ is Fréchet differentiable for every $u \in X_1$, and

$$\begin{cases} E_{11}(u,v) = u - T_1(u,v) \\ E_{22}(u,v) = v - T_2(u,v). \end{cases} \tag{1.3.3}$$

Here $E_{11}(., v), E_{22}(u, .)$ are the Fréchet derivatives of $E_1(., v)$ and $E_2(u, .)$, respectively.

The next theorem gives us a variational characterization of the unique fixed point of a Perov contraction, being the vectorial version of Theorem 1.3.1.

Theorem 1.3.2 *(**R. Precup** [77]) Assume that the above conditions are satisfied. In addition assume that $E_1(., v)$ and $E_2(u, .)$ are bounded from below for every $u \in X_1$, $v \in X_2$, and that there are $R, c > 0$ such that one of the following conditions holds:*

$$\begin{aligned} E_1(u,v) &\geq \inf_{X_1} E_1(.,v) + c \ \text{ for } |u|_1 \geq R \text{ and } v \in X_2, \\ E_2(u,v) &\geq \inf_{X_2} E_2(u,.) + c \ \text{ for } |v|_2 \geq R \text{ and } u \in X_1. \end{aligned} \tag{1.3.4}$$

Then the unique fixed point (u^*, v^*) of $T_1, T_2)$ (guaranteed by Pe rov's fixed point theorem) is a Nash-type equilibrium of the pair of functionals (E_1, E_2), i.e.

$$\begin{aligned} E_1(u^*, v^*) &= \inf_{X_1} E_1(., v^*) \\ E_2(u^*, v^*) &= \inf_{X_2} E_2(u^*, .). \end{aligned}$$

Some applications of Theorems 1.3.1 and 1.3.2 to differential equations were given in R. Precup [77]. Extensions to nonsmooth functionals can be found in R. Precup [78].

Chapter 2

Semilinear operator equations and systems with potential-type nonlinearities

2.1 Overview

This chapter is based on the article A. Budescu [17], whose motivation was the article R. Precup [77].

The main results are concerning with variational properties of the solutions of semilinear equations having the form

$$Au = J'(u), \tag{2.1.1}$$

where A is a symmetric, linear, and positively defined operator, and the nonlinear term is the Fréchet derivative of a functional J.

2.2 Variational theory of linear operator equations

In this section, we present the variational theory of linear operator equations from S. G. Michlin [53, 54](see also D. Muzsi and R. Precup [58]) for linear equations associated to positively defined operators. The theory is used in the next sections.

Let H be a Hilbert space with the inner product denoted by $(.,.)_H$, and the norm $\|.\|_H$. Let

$$A : D(A) \to H$$

be a symmetric, linear, and densely defined operator.

We recall that the operator A is said to be *symmetric* if

$$(Au, v)_H = (u, Av)_H,$$

for every $u, v \in D(A)$.

A symmetric everywhere-defined operator is *self-adjoint*.

The symmetric operator A is said to be *positively defined* if

$$\inf_{\substack{u\in D(A)\\ u\neq 0}} \frac{(Au,u)_H}{\|u\|_H^2} > 0. \tag{2.2.1}$$

Note that the above definition for positively defined operators is equivalent to the following: the symmetric operator A is said to be *positively defined* if there exists a constant $\gamma > 0$ such that

$$(Au,u)_H \geq \gamma^2 \|u\|_H^2 \tag{2.2.2}$$

for every $u \in D(A)$.

For such a linear operator, we endow the linear subspace $D(A)$ of H with the bilinear functional:

$$(u,v)_{H_A} = (Au,v)_H$$

for every $u, v \in D(A)$.

One can verify that $(.,.)_{H_A}$ is an inner product. Consequently, $D(A)$ endowed with the inner product $(.,.)_{H_A}$ is a pre-hilbertian space. This space may not be complete. The completion H_A of $(D(A),(.,.)_{H_A})$ is called the *energetic space* of A.

By the construction, $D(A) \subset H_A \subset H$ with dense inclusions.

We use the same symbol $(.,.)_{H_A}$ to denote the inner product of H_A. The corresponding norm

$$\|u\|_{H_A} = \sqrt{(u,u)_{H_A}}$$

is called the *energetic norm* associated to A.

If $u \in D(A)$, then $\|u\|_{H_A} = \sqrt{(Au,u)_H}$, and in view of (2.2.2), one has the *Poincaré inequality*

$$\|u\|_H \leq \frac{1}{\gamma}\|u\|_{H_A}, \tag{2.2.3}$$

for every $u \in D(A)$. By density the above inequality extends to H_A. We use this inequality in order to identify the elements of H_A with elements from H.

Let H'_A be the dual space of H_A. If we identify H with its dual, then from $H_A \subset H$ we have $H \subset H'_A$.

We attach to the operator A the following problem

$$Au = f, \quad u \in H_A \tag{2.2.4}$$

where $f \in H'_A$.

By a *weak solution* of (2.2.4) we mean an element $u \in H_A$ with

$$(u,v)_{H_A} = (f,v) \tag{2.2.5}$$

for every $v \in H_A$, where the notation (f,v) stands for the value of the functional f on the element v.

In case that $f \in H$, then $(f,v) = (f,v)_H$.

Notice that, if $u \in D(A)$, then (2.2.5) becomes $(Au, v)_H = (f, v)$.

Theorem 2.2.1 *For every $f \in H'_A$ there exists a unique weak solution $u \in H_A$ of the problem* (2.2.4).

This result allows us to define the solution operator A^{-1} associated to operator A. Thus,

$$A^{-1} : H'_A \to H_A \,,$$

$$A^{-1} f = u, \tag{2.2.6}$$

where u is the unique weak solution of problem (2.2.4).

The operator A^{-1} is well defined by the above theorem, and one has

$$(A^{-1} f, v)_{H_A} = (f, v) \tag{2.2.7}$$

for all $v \in H_A$, and $f \in H'_A$.

Also, the operator A^{-1} is an isometry between H'_A and H_A, i.e.,

$$\|A^{-1} f\|_{H_A} = \|f\|_{H'_A} \tag{2.2.8}$$

for all $f \in H'_A$.

Indeed, in order to show that the inequality $\|A^{-1} f\|_{H_A} \leq \|f\|_{H'_A}$ holds, we replace v with $A^{-1} f$ in (2.2.7), to obtain

$$(A^{-1} f, A^{-1} f)_{H_A} = (f, A^{-1} f).$$

Therefore,

$$\|A^{-1} f\|^2_{H_A} = (f, A^{-1} f) \leq \|f\|_{H'_A} \|A^{-1} f\|_{H_A}.$$

Hence

$$\|A^{-1} f\|_{H_A} \leq \|f\|_{H'_A}.$$

On the other hand, we have that

$$\begin{aligned}
\|f\|_{H'_A} &= \sup_{\substack{v \in H_A \\ v \neq 0}} \frac{|(f, v)|}{\|v\|_{H_A}} = \sup_{\substack{v \in H_A \\ v \neq 0}} \frac{|(A^{-1} f, v)_{H_A}|}{\|v\|_{H_A}} \\
&\leq \sup_{\substack{v \in H_A \\ v \neq 0}} \frac{\|A^{-1} f\|_{H_A} \|v\|_{H_A}}{\|v\|_{H_A}} = \|A^{-1} f\|_{H_A}.
\end{aligned}$$

From the above inequalities, (2.2.8) follows.

We also mention Poincaré's inequality for the inclusion $H \subset H'_A$,

$$\|u\|_{H'_A} \leq \frac{1}{\gamma} \|u\|_H, \qquad u \in H. \tag{2.2.9}$$

This can be proved as follows:

$$\|u\|_{H'_A} = \sup_{\substack{v\in H_A\\ v\neq 0}} \frac{|(u,v)_H|}{\|v\|_{H_A}} \leq \sup_{\substack{v\in H_A\\ v\neq 0}} \frac{\|u\|_H\|v\|_H}{\|v\|_{H_A}}.$$

Now, using (2.2.3) we have

$$\sup_{\substack{v\in H_A\\ v\neq 0}} \frac{\|u\|_H\|v\|_H}{\|v\|_{H_A}} \leq \frac{1}{\gamma}\|u\|_H.$$

Therefore (2.2.9) holds.

Using (2.2.8) and (2.2.9), we see that, if $f \in H$, then

$$\|A^{-1}f\|_{H_A} = \|f\|_{H'_A} \leq \frac{1}{\gamma}\|f\|_H. \tag{2.2.10}$$

For a fixed $f \in H'_A$, one considers the functional

$$E : H_A \to \mathbb{R},$$
$$E(u) = \frac{1}{2}\|u\|^2{}_{H_A} - (f,u).$$

The functional E is Fréchet differentiable, and for any $u, v \in H_A$, we have:

$$(E'(u),v) = \lim_{t\to 0} \frac{E(u+tv)-E(u)}{t} = (u,v)_{H_A} - (f,v) = \left(u - A^{-1}f, v\right)_{H_A}. \tag{2.2.11}$$

Now, (2.2.11) shows that $u \in H_A$ is a weak solution of (2.2.4) if and only if u is a *critical point* of E, i.e.,

$$E'(u) = 0.$$

2.3 Minimizer for contraction-type operators

In this section, we present the variational formulation of the semilinear equation

$$Au = J'(u), \tag{2.3.1}$$

and we state the first result of this chapter concerning the variational property of the solution of equation (2.3.1).

Let H be a Hilbert space with the inner product denoted by $(.,.)_H$, and the norm $\|.\|_H$. Let A be a symmetric, linear, and densely defined operator as in the previous section, and let

$$J : H \to \mathbb{R}$$

be a C^1- functional.

We look for weak solutions $u \in H_A$ for the semilinear equation (2.3.1), where H_A is the energetic space defined in Section 2.2.

Equation (2.3.1) is equivalent to

$$u = A^{-1}J'(u),$$

this is, to the fixed point equation

$$u = T(u),$$

where $T := A^{-1}J'$.

We associate to the equation (2.3.1) the functional

$$E : H_A \to \mathbb{R}, \quad E(u) = \frac{1}{2}\|u\|^2{}_{H_A} - J(u). \tag{2.3.2}$$

We have the following result.

Theorem 2.3.1 *(**A. Budescu and R. Precup** [19]) Under the above conditions on A and J, assume that the following conditions are satisfied:*

(i) *there exists $\alpha < \gamma^2$ with*

$$\|J'(u) - J'(v)\|_H \leq \alpha\|u - v\|_H \tag{2.3.3}$$

for all $u, v \in H$;

(ii) *there exist $a < \frac{1}{2}$, and $b, c \in \mathbb{R}_+$ such that*

$$J(u) \leq a\|u\|^2{}_{H_A} + b\,\|u\|_{H_A} + c \tag{2.3.4}$$

for all $u \in H_A$.

Then there is a unique weak solution $u^* \in H_A$ of equation (2.3.1) such that

$$E(u^*) = \inf_{H_A} E.$$

2.4 Nash-type equilibrium for Perov contractions

This section is devoted to the study of systems type

$$\begin{cases} A_1u = J_{11}(u, v) \\ A_2v = J_{22}(u, v), \end{cases} \tag{2.4.1}$$

where A_1, A_2 are symmetric, linear, and positively defined operators on two Hilbert spaces H_1, H_2. We denote $H := H_1 \times H_2$.

Also, $J_1, J_2 : H \to \mathbb{R}$ are two C^1 - functionals, and by $J_{11}(u, v)$ we mean the partial derivative of J_1 with respect to u, and by $J_{22}(u, v)$ the partial derivative of J_2 with respect to v.

We express the above system as a fixed point equation of the form

$$w = T(w) \tag{2.4.2}$$

for the nonlinear operator $T = (T_1, T_2)$, where $w = (u, v)$. The operators T_1 and T_2 are defined as follows

$$T_1 : H_{A_1} \times H_{A_2} \to H_{A_1}, \quad T_1(u, v) = A_1^{-1} J_{11},$$

respectively

$$T_2 : H_{A_1} \times H_{A_2} \to H_{A_2}, \quad T_2(u, v) = A_2^{-1} J_{22}.$$

Hence (2.4.2) can be rewritten explicitly as follows

$$\begin{cases} u = T_1(u, v) \\ v = T_2(u, v). \end{cases} \tag{2.4.3}$$

This vectorial structure of (2.4.2) allows the two terms T_1 and T_2 to behave differently one from another, and also with respect to the two variables. Also, this requires the use of matrices instead of constants, when Lipschitz conditions are imposed to T_1 and T_2.

Next, we are going to describe the variational structure of each component equation of (2.4.3). We associate to the equations of (2.4.3) the functionals

$$E_1, E_2 : H_{A_1} \times H_{A_2} \to \mathbb{R}$$

defined by

$$\begin{aligned} E_1(u, v) &= \frac{1}{2}\|u\|_{H_{A_1}}^2 - J_1(u, v) \\ E_2(u, v) &= \frac{1}{2}\|v\|_{H_{A_2}}^2 - J_2(u, v). \end{aligned}$$

One has

$$\begin{aligned} E_{11}(u, v) &= u - T_1(u, v) \\ E_{22}(u, v) &= v - T_2(u, v), \end{aligned}$$

or equivalently

$$\begin{aligned} E_{11}(u, v) &= 0 \\ E_{22}(u, v) &= 0, \end{aligned}$$

where $E_{11}(., v)$, $E_{22}(u, .)$ are the Fréchet derivatives of $E_1(., v)$, and $E_2(u, .)$, respectively. The main result of this section is the following theorem.

Theorem 2.4.1 *(**A. Budescu and R. Precup** [19]) Let the above conditions on A_1, A_2, and J_1, J_2 hold. In addition, assume that the following conditions are satisfied:*

(i) *there exist $m_{ij} \in \mathbb{R}_+$ $(i, j = 1, 2)$ such that*

$$\begin{aligned} \|J_{11}(u, v) - J_{11}(\bar{u}, \bar{v})\|_{H_1} &\leq m_{11}\|u - \bar{u}\|_{H_1} + m_{12}\|v - \bar{v}\|_{H_2} \\ \|J_{22}(u, v) - J_{22}(\bar{u}, \bar{v})\|_{H_2} &\leq m_{21}\|u - \bar{u}\|_{H_1} + m_{22}\|v - \bar{v}\|_{H_2} \end{aligned}$$

for all $u, \bar{u} \in H_1$, and $v, \bar{v} \in H_2$, and the spectral radius of the matrix

$$M = \begin{bmatrix} \frac{m_{11}}{\gamma_1^2} & \frac{m_{12}}{\gamma_1^2} \\ \frac{m_{21}}{\gamma_2^2} & \frac{m_{22}}{\gamma_2^2} \end{bmatrix}$$

is strictly less than one;

(ii) *there exist $a_1, a_2 < 1/2$, and $b_1, b_2, c_1, c_2 \geq 0$ such that*

$$\begin{aligned} J_1(u,v) &\leq a_1\|u\|^2_{H_{A_1}} + b_1\|u\|_{H_{A_1}} + c_1 \\ J_2(u,v) &\leq a_2\|v\|^2_{H_{A_2}} + b_2\|u\|_{H_{A_2}} + c_2 \end{aligned}$$

for all $u \in H_{A_1}, v \in H_{A_2}$;

(iii) *there are R, $d > 0$ such that one of the following conditions holds:*

$$\begin{aligned} E_1(u,v) &\geq \inf_{H_{A_1}} E_1(.,v) + d \quad \textit{for} \quad \|u\|_{H_{A_1}} \geq R \text{ , and } v \in H_{A_2} \\ E_2(u,v) &\geq \inf_{H_{A_2}} E_2(u,.) + d \quad \textit{for} \quad \|v\|_{H_{A_2}} \geq R \text{ , and } u \in H_{A_1}. \end{aligned} \tag{2.4.4}$$

Then there is a unique solution $(u^*, v^*) \in H_{A_1} \times H_{A_2}$ of the system (2.4.1)*, which is a* Nash-type equilibrium *of the pair of functionals (E_1, E_2), i.e.,*

$$\begin{aligned} E_1(u^*,v^*) &= \inf_{H_{A_1}} E_1(.,v^*) \\ E_2(u^*,v^*) &= \inf_{H_{A_2}} E_2(u^*,.). \end{aligned}$$

2.5 Application to elliptic equations

In this section, we present an application of Theorem 2.3.1 to elliptic equations. More exactly, we deal with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega \\ \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.5.1}$$

Here Ω is a bounded, open subset of $\mathbb{R}^n$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, and Δ is the Laplacian.

In this specific case $H = L^2(\Omega)$, and $A = -\Delta$ with

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Also, $H_A = H_0^1(\Omega)$ with the inner product

$$(u,v)_{H_0^1} = \int_\Omega \nabla u \nabla v \mathrm{d}x,$$

and the norm

$$\|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{1/2}.$$

The functional $J : L^2(\Omega) \to \mathbb{R}$ is given by

$$J(u) = \int_\Omega F(x, u(x))dx,$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

Also $\gamma = \sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the Dirichlet problem for $-\Delta$ (see, e.g. H. Brezis [15], J. Mawhin and M. Willem [51], R. Precup [76]).

Hence the energy functional associated to (2.5.1) is the following one

$$E : H_0^1(\Omega) \to \mathbb{R}$$
$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(x, u(x)) \right) \mathrm{d}x.$$

Problem (2.5.1) is equivalent to the fixed point equation

$$u = (-\Delta)^{-1} N_f(u),$$

where N_f is the Nemytskii superposition operator assumed to act from $L^2(\Omega)$ to itself,

$$N_f(u)(x) = f(x, u(x))$$

(see more details in Section 1.1).

Notice that the functional J is C^1 on $L^2(\Omega)$,

$$J' = N_f,$$

and

$$E'(u) = u - (-\Delta)^{-1} N_f(u).$$

Theorem 2.5.1 *(**A. Budescu** [17]) Assume that the folllowing conditions are satisfied:*

(i) *f satisfies the Carathéodory conditions, i.e., $f(.,y) : \Omega \to \mathbb{R}$ is measurable for each $y \in \mathbb{R}$, and $f(x,.) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e $x \in \Omega$;*

(ii) *$f(.,0) \in L^2(\Omega)$;*

(iii) *exists $\alpha \in [0, \lambda_1)$ such that*

$$|f(x,u) - f(x,\bar{u})| \le \alpha |u - \bar{u}|$$

for all $u, \bar{u} \in \mathbb{R}$, and a.e. $x \in \Omega$.

Then (2.5.1) *has a unique weak solution* $u^* \in H_0^1(\Omega)$, *and*

$$E(u^*) = \inf_{H_0^1(\Omega)} E.$$

2.6 Application to elliptic systems

In this section, we present an application of Theorem 2.4.1 for the elliptic system

$$\begin{cases} -\Delta u = f(x,u,v) \text{ in } \Omega \\ -\Delta v = g(x,u,v) \text{ in } \Omega \\ \quad u = 0 \text{ on } \partial\Omega \\ \quad v = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.6.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where

$$f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}.$$

This problem is equivalent to the system

$$\begin{cases} u = (-\Delta)^{-1} f(.,u,v) \\ v = (-\Delta)^{-1} g(.,u,v). \end{cases}$$

A pair $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of (2.6.1) if and only if

$$\begin{cases} E_{11}(u,v) = 0 \\ E_{22}(u,v) = 0, \end{cases}$$

where $E_1, E_2 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ are defined by

$$\begin{aligned} E_1(u,v) &= \frac{1}{2}\|u\|_{H_0^1}^2 - \int_\Omega F(x,u(x),v(x))\mathrm{d}x, \\ E_2(u,v) &= \frac{1}{2}\|v\|_{H_0^1}^2 - \int_\Omega G(x,u(x),v(x))\mathrm{d}x, \end{aligned}$$

and

$$F(x,t,s) = \int_0^t f(x,\tau,s)\mathrm{d}\tau, \;\; G(x,t,s) = \int_0^s g(x,t,\tau)\mathrm{d}\tau.$$

The functionals $E_1(.,v)$ and $E_2(u,.)$ are C^1 for any fixed u and v, respectively, and

$$\begin{aligned} E_{11}(u,v) &= u - (-\Delta)^{-1} f(.,u,v) \\ E_{22}(u,v) &= v - (-\Delta)^{-1} g(.,u,v). \end{aligned}$$

Here again $E_{11}(.,v), E_{22}(u,.)$ are the Fréchet derivatives of $E_1(.,v)$, and $E_2(u,.)$, respectively.

We shall say that a function $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is of *coercive type* if the functional

$$\Phi : H_0^1(\Omega) \to \mathbb{R}$$
$$\Phi(v) = \frac{1}{2}\|v\|_{H_0^1}^2 - \int_\Omega H(x, v(x))\mathrm{d}x$$

is coercive, i.e.,

$$\Phi(v) \to \infty \quad \text{as} \quad \|v\|_{H_0^1} \to \infty.$$

We state the main result of this section.

Theorem 2.6.1 *(**A. Budescu** [17]) Let $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $f = f(x,y,z)$, $g = g(x,y,z)$ satisfy the Carathédory conditions. Assume that $f(.,0,0)$, $g(.,0,0) \in L^2(\Omega)$, and that the following conditions hold:*

(i) *there exist $m_{ij} \in \mathbb{R}_+$ $(i,j = 1,2)$ with:*

$$\begin{cases} |f(x,u,v) - f(x,\bar{u},\bar{v})| \le m_{11}|u - \bar{u}| + m_{12}|v - \bar{v}| \\ |g(x,u,v) - g(x,\bar{u},\bar{v})| \le m_{21}|u - \bar{u}| + m_{22}|v - \bar{v}| \end{cases} \tag{2.6.2}$$

for all $u, \bar{u}, v, \bar{v} \in \mathbb{R}$, and a.e. $x \in \Omega$;

(ii) *there exist $H, H_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ with*

$$H_1(x,v) \le G(x,u,v) \le H(x,v), \tag{2.6.3}$$

for all $u, v \in \mathbb{R}$, and a.e. $x \in \Omega$, where H and H_1 are of coercive type.

If the matrix

$$M = \frac{1}{\lambda_1}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

is convergent to zero, then (2.6.1) has a unique solution $(u^*, v^*) \in H_0^1(\Omega) \times H_0^1(\Omega)$ which is a Nash-type equilibrium of the pair of energy functionals (E_1, E_2) associated to the problem (2.6.1).

Chapter 3

Semilinear equations under nonresonance conditions

3.1 Overview

This chapter is based on the article A. Budescu and R. Precup [18], where we deal with weak solutions of the semilinear operator equation

$$Au - cu = J'(u) \tag{3.1.1}$$

in a Hilbert space. Here, as in the previous sections, A is a positively defined linear operator, and $J : H \to \mathbb{R}$ is a C^1-functional.

3.2 Spectral theory

We attach to the operator A from the previous section, the following problem

$$Au = f, \quad u \in H_A. \tag{3.2.1}$$

From the Riesz representation theorem, it follows that, for each $f \in H'_A$, there exists a unique $u_f \in H_A$, such that

$$(u_f, v)_{H_A} = (f, v) \quad \text{for every } \ v \in H_A, \tag{3.2.2}$$

where the notation (f, v) stands for the value of the functional f on the element v.

We denote u_f by $A^{-1}f$, and we call it the *weak solution* of the equation $(3.2.1)$. Thus,

$$A^{-1} : H'_A \to H_A$$
$$\left(A^{-1}f, v\right)_{H_A} = (f, v)$$

for $f \in H'_A, \quad v \in H_A$. It is easy to see that, the linear operator A^{-1} from H to H is positively defined.

From now on, in addition, we shall assume that, the embedding of H_A into H is compact. This guarantees that A^{-1} is a *compact operator* from H to itself.

Then, from the spectral theory of self-adjoint compact operators (see S. G. Mihlin [54], D. Muzsi and R. Precup [58]), we recall the following properties.

(i) The set of eigenvalues of the operator A^{-1} is nonempty and at most countable;

(ii) Zero is the only possible cluster point of the set of eigenvalues of the operator A^{-1};

(iii) To each eigenvalue corresponds a finite number of linearly independent eigenvectors;

(iv) The eigenvalues of A^{-1} are positive;

(v) There exists an orthonormal set $(\phi_k)_{k\geq 1}$ of eigenvectors of A^{-1}, with

$$\|\phi_k\|_H = 1,$$

which is at most countable and it is complete in the image of A^{-1}, i.e.,

$$A^{-1}u = \sum_{k\geq 1} \left(A^{-1}u, \phi_k\right)_H \phi_k$$

for all $u \in H$.

Assume that $D(A)$ is infinite dimensional. Then the image of A^{-1} is infinite dimensional and so, there exists a sequence $(\mu_j)_{j\geq 1}$ of eigenvalues of A^{-1}, and correspondingly, a sequence $(\phi_j)_{j\geq 1}$ of eigenvalues, orthonormal in H.

Let $\lambda_j := 1/\mu_j$. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots, \qquad \lambda_j \to \infty \quad \text{as} \quad j \to \infty,$$

and from

$$A^{-1}\phi_j = \mu_j \phi_j,$$

we have

$$(\phi_j, v)_{H_A} = \lambda_j (\phi_j, v)_H \quad \text{for all} \quad v \in H_A, \tag{3.2.3}$$

i.e.,

$$A\phi_j = \lambda_j \phi_j$$

in the weak sense. Hence λ_j and ϕ_j, $j \geq 1$, are the *eigenvalues* and *eigenvectors*, respectively of A, with $\|\phi_j\|_H = 1$.

Also, we shall use the following characterization for the eigenvalues (see S. G. Michlin [54]), namely

$$\lambda_j = \inf \left\{ \frac{\|u\|_{H_A}^2}{\|u\|_H^2} : \ u \in H_A \setminus \{0\},\ (u, \phi_i)_{H_A} = 0 \ \text{ for } i = 1, 2, ..., j-1 \right\}$$

for $j = 1, 2, \ldots$.

Note that, for $j = 1$, this shows that, the best constant γ^2 in (2.2.1) is $\gamma^2 = \lambda_1$.

If in H_A' we consider the inner product and norm

$$(u, v)_{H_A'} := \left(A^{-1}u, A^{-1}v\right)_{H_A}, \quad \|u\|_{H_A'} := \left\|A^{-1}u\right\|_{H_A},$$

then using (3.2.3) we obtain

$$\|\phi_j\|_H = 1, \quad \|\phi_j\|_{H_A} = \sqrt{\lambda_j}, \quad \|\phi_j\|_{H_A'} = \frac{1}{\sqrt{\lambda_j}}.$$

Also, for each $v \in H_A'$, one has

$$(v, \phi_j) = (A^{-1}v, \phi_j)_{H_A} = (A^{-1}v, A^{-1}(\lambda_j \phi_j))_{H_A} = \lambda_j (v, \phi_j)_{H_A'}.$$

This implies that, the systems

$$(\phi_j), \quad \left(\frac{1}{\sqrt{\lambda_j}}\phi_j\right), \quad \left(\sqrt{\lambda_j}\phi_j\right)$$

are orthonormal and complete (Hilbert bases) in H, H_A and H_A', respectively, and that for each $v \in H_A$, the Fourier series

$$\sum (v, \phi_j)_H \phi_j, \ \sum \left(v, \frac{1}{\sqrt{\lambda_j}}\phi_j\right)_{H_A} \frac{1}{\sqrt{\lambda_j}}\phi_j, \ \sum \left(v, \sqrt{\lambda_j}\phi_j\right)_{H_A'} \sqrt{\lambda_j}\phi_j$$

are identical and can be written as

$$\sum (v, \phi_j)\phi_j,$$

where by (v, ϕ_j) we mean the action of v as an element of H_A' over ϕ_j.

3.3 Auxiliary results concerning Fourier series

In this section, we give some auxiliary results concerning the properties of the Fourier series in Hilbert spaces, which are essential tools of this chapter.

The first lemma extends to the dual space H_A' of H_A, the corresponding result for H, used in D. Muzsi and R. Precup [58], and first proved for $A = -\Delta$ in R. Precup [72] (see also D. O'Regan and R. Precup[66, Lemma 6.1]).

Lemma 3.3.1 *(**A. Budescu and R. Precup [18]**) Let c be any constant with $c \neq \lambda_j$, for $j = 1, 2, \ldots$. For each $v \in H_A'$, there exists a unique weak solution $u \in H_A$ of the problem*

$$Lu := Au - cu = v, \quad u \in H_A$$

denoted by $L^{-1}v$, and the following eigenvector expansion holds

$$L^{-1}v = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - c}(v, \phi_j)\phi_j, \tag{3.3.1}$$

where the series converges in H_A. In addition,

$$\|L^{-1}v\|_{H_A} \leq \sigma_c \|v\|_{H_A'}, \tag{3.3.2}$$

where $\sigma_c = \max\left\{\left|\frac{\lambda_j}{\lambda_j - c}\right| : \ j = 1, 2, ...\right\}.$

Lemma 3.3.2 *(**A. Budescu and R. Precup [18]**) For every $w \in H_k$, the following inequalities hold:*

$$\|w\|_H \leq \frac{1}{\sqrt{\lambda_{k+1}}}\|w\|_{H_A}; \tag{3.3.3}$$

$$\left\|L^{-1}w\right\|_{H_A} \leq \frac{\sqrt{\lambda_{k+1}}}{\lambda_{k+1} - c}\|w\|_H; \tag{3.3.4}$$

$$\left\|L^{-1}w\right\|_{H_A} \leq \frac{1}{\lambda_{k+1} - c}\|w\|_{H_A}. \tag{3.3.5}$$

3.4 The main result

The main result of this chapter is concerning with a variational property of the solutions of semilinear equations of the form

$$Au - cu = J'(u), \tag{3.4.1}$$

where A is a linear operator having all the properties required in Section 2.2 and c is not an eigenvalue of A.

We shall work out a general theory of nonresonance, which in particular, for $c = 0$, contains the results from Section 2.3.

We look for weak solutions $u \in H_A$ of the semilinear equation (3.4.1), i.e., an element $u \in H_A$ with

$$(u, v)_{H_A} - c(u, v)_H = (J'(u), v)$$

for all $v \in H_A$. If we denote

$$Lu = Au - cu,$$

then (3.4.1) is equivalent to the fixed point equation

$$u = L^{-1}J'(u), \quad u \in H_A.$$

On the other hand, the equation (3.4.1) has the variational form

$$E'(u) = 0,$$

where $E : H_A \to \mathbb{R}$ is the energy functional given by

$$E(u) = \frac{1}{2}\|u\|_{H_A}^2 - \frac{c}{2}\|u\|_H^2 - J(u).$$

Note that

$$E'(u) = Lu - J'(u).$$

If we identify H'_A with H_A via A^{-1}, and we take into account that

$$A^{-1}(Lu - J'(u)) = A^{-1}(Au - cu - J'(u)) = u - A^{-1}[cu + J'(u)],$$

we obtain

$$E'(u) = u - A^{-1}[cu + J'(u)]. \tag{3.4.2}$$

We remark that the method we used in Section 2.3 can not longer be applied when $c \neq 0$, as we can see that

$$A^{-1}[cu + J'(u)] \neq L^{-1}J'(u).$$

Let H_k and $H_k^{\perp}$ be the subspaces of H_A defined by

$$H_k = \left\{u \in H_A : (u, \phi_j)_{H_A} = 0 \text{ for } j = 1, 2, ..., k\right\}; \tag{3.4.3}$$

$$H_k^{\perp} = \left\{u \in H_A : (u, \phi_j)_{H_A} = 0 \text{ for } j = k+1, k+2, ...\right\}.$$

In what follows, by P and $P^{\perp}$ we mean the projection operators on H_k and on its orthogonal complement $H_k^{\perp}$. So, any element $u \in H_A$ can be written as

$$u = P^{\perp}u + Pu.$$

Now we are ready to state the main result for the equation (3.4.1).

Theorem 3.4.1 *(**A. Budescu and R. Precup [18]**) Assume that all the above conditions on A, J and c hold. In addition, assume that there exist $\alpha < \lambda_{k+1} - c$, $p \leq \frac{1}{2} - \frac{c}{2\lambda_{k+1}}$ and $q, r \in \mathbb{R}_+$ such that*

$$\|J'(u) - J'(v)\|_H \leq \alpha \|u - v\|_H \tag{3.4.4}$$

for all $u, v \in H_A$ satisfying $P^{\perp}u = P^{\perp}v$, and

$$J(u) \leq p\|u\|_{H_A}^2 + q\|u\|_{H_A} + r \tag{3.4.5}$$

for all $u \in H_A$. Then for any weak solution $u^* \in H_A$ of the equation (3.4.1), the following variational property holds

$$E(u^*) = \inf_{w \in H_k} E(u^* + w). \tag{3.4.6}$$

As in Section 2.5, the abstract theory presented here can be applied to semilinear elliptic equations under nonresonance conditions, so , in the next section we give an application to the elliptic problem

$$\begin{cases} -\Delta u - cu = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $c \neq \lambda_j$, $j = 1, 2, ...,$.

A forthcoming direction of research is to extend the theory developed in this chapter to the nonresonance systems of the form

$$\begin{cases} A_1 u - c_1 u = J_{11}(u, v) \\ A_2 v - c_2 v = J_{22}(u, v), \end{cases}$$

where by $J_{11}(u,v), J_{22}(u,v)$ we mean the partial derivatives of two C^1 functionals $J_1, J_2 : H_1 \times H_2 \to \mathbb{R}$. We can anticipate that under suitable conditions, the solution (u,v) of the system is a Nash-type equilibrium of the pair of associated energy functionals on a suitable cross product subspace depending on indices k_1 and k_2, where $\lambda_{k_i} < c_i < \lambda_{k_i+1}$, $i = 1, 2$.

3.5 Application to elliptic problems under nonresonance conditions

In this section, we extend the results from Section 2.5. More exactly, we investigate the case of the problem

$$\begin{cases} -\Delta u - cu = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5.1}$$

under the *nonresonance* condition $c \neq \lambda_j$, $j = 1, 2, ...$, where λ_j are the eigenvalues of the Dirichlet problem for $-\Delta$.

The energy functional is

$$E(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - \frac{c}{2} u^2 - F(x, u(x)) \right) dx,$$

where $E : H_0^1(\Omega) \to \mathbb{R}$, and $F(x,t) = \int_0^t f(x,s)ds$.

However, if $c > \lambda_1$, E is not bounded from below and consequently the solution can not be a minimizer of E. Nevertheless, even in this case, a variational property holds for the solution.

Assume that

$$\lambda_k < c < \lambda_{k+1}$$

for some $k \in \{0, 1, ...\}$, where $\lambda_0 = -\infty$.

Let H_k and $H_k^\perp$ be the subspaces of H_0^1 defined by

$$H_k = \left\{ u \in H_0^1(\Omega) : (u, \phi_j)_{H_0^1} = 0 \text{ for } j = 1, 2, ..., k \right\};$$

$$H_k^\perp = \left\{ u \in H_0^1(\Omega) : (u, \phi_j)_{H_0^1} = 0 \text{ for } j = k+1, k+2, ... \right\}.$$

Furthermore, if we denote

$$Lu = -\Delta u - cu,$$

and we take into account that $c \neq \lambda_j$, $j = 1, 2, ...,$, then L is invertible and $(3.5.1)$ is equivalent to the fixed point equation

$$u = L^{-1}N_f(u), \quad u \in H_0^1(\Omega),$$

where N_f is the Nemytskii superposition operator, defined as follows

$$N_f : L^2(\Omega) \to L^2(\Omega), \quad N_f(u)(x) = f(x, u(x)),$$

where we assume that it is well-defined.

On the other hand, the equation $(3.5.1)$ has the variational form

$$E'(u) = 0,$$

and

$$E'(u) = Lu - N_f(u) = Lu - J'(u).$$

If we identify $H^{-1}(\Omega)$ with $H_0^1(\Omega)$ via $(-\Delta)^{-1}$, and we take into account that

$$(-\Delta)^{-1}[Lu - N_f(u)] = (-\Delta)^{-1}[-\Delta u - cu - N_f(u)] = u - (-\Delta)^{-1}[cu + N_f(u)],$$

we obtain

$$E'(u) = u - (-\Delta)^{-1}[cu + N_f(u)].$$

We remark that the method we used in Section 2.5 can not longer be applied when $c \neq 0$, as we can see that

$$(-\Delta)^{-1}[cu + N_f(u)] \neq L^{-1}N_f(u).$$

Using Theorem $3.4.1$, we obtain the following result.

Theorem 3.5.1 ***(A. Budescu and R. Precup [18])*** *Assume that the folllowing conditions are satisfied:*
(i) f satisfies the Carathéodory conditions;
(ii) $f(., 0) \in L^2(\Omega)$*;*
(iii) exists $0 \leq \alpha \leq \dfrac{\lambda_1}{\lambda_{k+1}}(\lambda_{k+1} - c)$ *such that*

$$|f(x, u) - f(x, \bar{u})| \leq \alpha |u - \bar{u}|$$

for all $u, \bar{u} \in \mathbb{R}$*, and a.e.* $x \in \Omega$*.*
Then for any weak solution $u^* \in H_0^1(\Omega)$ *of the problem* $(3.5.1)$ *the following variational property holds:*

$$E(u^*) = \inf_{w \in H_k} E(u^* + w).$$

Chapter 4

Variational properties of the solutions of singular second-order differential equations and systems

4.1 Overview

This chapter is based on the article A. Budescu and R. Precup [19], where we studied the existence and the variational characterization of the weak solutions of the Dirichlet boundary value problem for singular second-order ordinary differential equations and systems. The solution appears as a minimizer of the energy functional associated to the equation, and in the case of systems, as a Nash-type equilibrium of the set of energy functionals.

In this chapter, we consider the Dirichlet boundary value problem for a single second-order equation

$$\begin{cases} -\left(p(x)u'(x)\right)' = f\left(x, u\left(x\right)\right) & \text{a.e. } x \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{4.1.1}$$

and for a system of two equations

$$\begin{cases} -\left(p(x)u'(x)\right)' = f\left(x, u\left(x\right), v\left(x\right)\right) \\ -\left(q(x)v'(x)\right)' = g\left(x, u\left(x\right), v\left(x\right)\right) \\ u(0) = u(1) = 0 \\ v(0) = v(1) = 0. \end{cases} \tag{4.1.2}$$

Here $p, q \in C[0,1] \cap C^1(0,1)$, $p, q > 0$ on $(0,1)$ can vanish making the equations singular with respect to the independent variable x.

4.2 The main results

4.2.1 The case of a single equation

In this section, we present an application of Theorem 2.3.1 to the singular boundary value problem

$$\begin{cases} -\left(p(x)u'(x)\right)' = f\left(x, u\left(x\right)\right) \quad \text{a.e. } x \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{4.2.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $p \in C[0,1] \cap C^1(0,1)$, and we assume that

$$I = \int_0^1 \frac{1}{p\left(x\right)} dx$$

is finite. In particular, the last condition is satisfied if $p(x) > 0$ for all $x \in [0,1]$.

In this specific case, $H = L^2(0,1)$, and

$$Au = -\left(pu'\right)'$$

with

$$\begin{aligned} D(A) &= \{u \in C[0,1] \cap C^1(0,1) : \ u(0) = u(1) = 0, \\ pu' &\in AC[0,1], \ (pu')' \in L^2(0,1)\}. \end{aligned}$$

Furthermore, the inner product and the norm are given by

$$(u,v)_{H_A} = \int_0^1 p(x)u'(x)v'(x)\mathrm{d}x$$

and

$$\|u\|_{H_A} = \left(\int_0^1 p(x)u'(x)^2 \mathrm{d}x\right)^{1/2},$$

respectively. Notice that, here, the energetic space H_A is the weighted Sobolev space

$$H_0^1(0,1;p) = \left\{u \in AC[0,1] : \ u(0) = u(1) = 0, \ \sqrt{p}u' \in L^2(0,1)\right\},$$

or equivalently,

$$H_0^1(0,1;p) = \left\{u : \ u(x) = \int_0^x \frac{v(s)}{\sqrt{p(s)}}\mathrm{d}s, \ v \in L^2(0,1), \ \int_0^1 \frac{v(s)}{\sqrt{p(s)}}\mathrm{d}s = 0\right\}.$$

For simplicity, from now on, we shall use the symbol H_A instead of $H_0^1(0,1;p)$.

By a *(weak) solution* of (4.2.1) we mean a function $u \in H_A$, such that $f(.,u) \in L^2(0,1)$, and

$$(u,v)_{H_A} = (f(.,u),v)_{L^2}$$

for all $v \in H_A$.

Also, the functional $J : L^2(0,1) \to \mathbb{R}$ is given by

$$J(u) = \int_0^1 F(x, u(x))dx, \text{ where } F(x,t) = \int_0^t f(x,s)ds.$$

The operator A is positively defined and $\gamma = 1/\sqrt{I}$.

In what follows, we shall use Poincaré's inequality:

$$\|u\|_{L^2(0,1)} \leq \sqrt{I}\|u\|_{H_A}, \qquad u \in H_A. \tag{4.2.2}$$

The energy functional associated to (4.2.1) is given by

$$\begin{aligned} E &: H_A \to \mathbb{R}, \\ E(u) &= \frac{1}{2}\|u\|_{H_A}^2 - \int_0^1 F(x, u(x))\mathrm{d}x. \end{aligned}$$

Obviously, problem (4.2.1) is equivalent to the fixed point equation

$$u = A^{-1}N_f(u),$$

where N_f is the Nemytskii superposition operator $N_f(u)(x) = f(x, u(x))$, defined in Section 1.1.1.

Let us now establish the expression of $A^{-1}(h)$ for any $h \in L^2(0,1)$. We obtain

$$\left(A^{-1}h\right)(x) = \frac{1}{I}\int_0^1 \frac{1}{p(s)}\int_0^s h(\tau)\,\mathrm{d}\tau\mathrm{d}s \cdot \int_0^x \frac{1}{p(s)}\mathrm{d}s - \int_0^x \frac{1}{p(s)}\int_0^s h(\tau)\,\mathrm{d}\tau\mathrm{d}s.$$

With these preliminaries, we can state and prove our first result of this chapter.

Theorem 4.2.1 ***(A. Budescu and R. Precup [19])*** *Assume that the following conditions are satisfied:*

(i) *f satisfies the Carathéodory conditions;*

(ii) $f(.,0) \in L^2(0,1)$;

(iii) *there exists $\alpha \in [0, 1/I)$ such that*

$$|f(x,u) - f(x,\bar{u})| \leq \alpha|u - \bar{u}|$$

for all $u, \bar{u} \in \mathbb{R}$, and a.e. $x \in (0,1)$.

Then, (4.2.1) has a unique weak solution $u^* \in H_A$, and

$$E(u^*) = \inf_{H_A} E.$$

4.2.2 The case of systems

In this section, as an application of Theorem 2.4.1, we extend the theory from Section 4.2.1 towards systems type

$$\begin{cases} -\left(p(x)u'(x)\right)' = f\left(x, u\left(x\right), v\left(x\right)\right) \\ -\left(q(x)v'(x)\right)' = g\left(x, u\left(x\right), v\left(x\right)\right) \quad \text{a.e. } x \in (0,1) \\ u(0) = u(1) = 0 \\ v(0) = v(1) = 0, \end{cases} \tag{4.2.3}$$

where $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $p, q \in C^1[0,1]$, and

$$I_1 = \int_0^1 \frac{1}{p(x)} \mathrm{d}x, \quad I_2 = \int_0^1 \frac{1}{q(x)} \mathrm{d}x$$

are finite. Here, $H = H_1 \times H_2$, $H_1 = H_2 = L^2(0,1)$,

$$A_1 u = -\left(pu'\right)'$$

with

$$\begin{aligned} D(A_1) &= \{u \in C[0,1] \cap C^1(0,1) : \ u(0) = u(1) = 0, \\ pu' &\in AC[0,1], \ (pu')' \in L^2(0,1)\} \end{aligned}$$

and

$$A_2 v = -\left(qv'\right)'$$

with

$$\begin{aligned} D(A_2) &= \{v \in C[0,1] \cap C^1(0,1) : \ v(0) = v(1) = 0, \\ qv' &\in AC[0,1], \ (qv')' \in L^2(0,1)\}. \end{aligned}$$

Also, the corresponding energetic spaces H_{A_1}, H_{A_2} are

$$H_{A_1} = H_0^1\left(0,1;p\right) \quad \text{and} \quad H_{A_2} = H_0^1\left(0,1;q\right),$$

respectively, while the definition of a weak solution of the system (4.2.3) is similar to that for the equation (4.2.1).

Notice that, A_1, A_2 are two positively defined operators, and

$$\gamma_1 = \frac{1}{\sqrt{I_1}}, \ \gamma_2 = \frac{1}{\sqrt{I_2}}.$$

The functionals $J_1, J_2 : H \to \mathbb{R}$ are given by

$$J_1(u,v) = \int_0^1 F(x, u(x), v(x)) \mathrm{d}x, \quad J_2(u,v) = \int_0^1 G(x, u(x), v(x)) \mathrm{d}x,$$

with

$$F(x,t,s)=\int_0^t f(x,\tau,s)\mathrm{d}\tau,\quad G(x,t,s)=\int_0^s g(x,t,\tau)\mathrm{d}\tau.$$

The energy functionals $E_1, E_2 : H_{A_1}\times H_{A_2}\to\mathbb{R}$ are defined by

$$\begin{aligned} E_1(u,v) &= \frac{1}{2}\|u\|^2_{H_{A_1}} - \int_0^1 F(x,u(x),v(x))\mathrm{d}x \\ E_2(u,v) &= \frac{1}{2}\|v\|^2_{H_{A_2}} - \int_0^1 G(x,u(x),v(x))\mathrm{d}x. \end{aligned}$$

Following the same steps as for one equation we obtain that the system (4.2.3) is equivalent to the fixed point problem

$$\begin{cases} u = A_1^{-1}N_f(u,v) \\ v = A_2^{-1}N_g(u,v). \end{cases}$$

Here, $N_f,\ N_g$ are the Nemytskii superposition operators associated to f and g, respectively. Each equation of the system (4.2.3) has a variational form, so it can be rewritten as

$$\begin{aligned} E_{11}(u,v) &= u - A_1^{-1}N_f(u,v) \\ E_{22}(u,v) &= v - A_2^{-1}N_g(u,v), \end{aligned}$$

where $E_{11}(.,v), E_{22}(u,.)$ are the Fréchet derivatives of $E_1(.,v)$ and $E_2(u,.)$, respectively.

We have the following result.

Theorem 4.2.2 *Let $f,g : [0,1]\times\mathbb{R}^2\to\mathbb{R}$, $f=f(x,y,z)$, $g=g(x,y,z)$ satisfy the Carathéodory conditions. Assume that $f(.,0,0), g(.,0,0)\in L^2(0,1)$, and that the following conditions hold:*

(i) *there exist $m_{ij}\in\mathbb{R}_+$ $(i,j=1,2)$ with*

$$\begin{aligned} |f(x,u,v)-f(x,\bar u,\bar v)| &\le m_{11}|u-\bar u| + m_{12}|v-\bar v| \\ |g(x,u,v)-g(x,\bar u,\bar v)| &\le m_{21}|u-\bar u| + m_{22}|v-\bar v|, \end{aligned} \tag{4.2.4}$$

for all $u,\bar u, v,\bar v\in\mathbb{R}$, and a.e. $x\in[0,1]$;

(ii) *there exist $H,\ H_1 : [0,1]\times\mathbb{R}\to\mathbb{R}$ with*

$$H_1(x,v)\le G(x,u,v)\le H(x,v), \tag{4.2.5}$$

for all $u,v\in\mathbb{R}$, and a.e. $x\in[0,1]$, where H and H_1 are of coercive type, and the spectral radius of the matrix

$$M=\begin{bmatrix} m_{11}I_1 & m_{12}I_1 \\ m_{21}I_2 & m_{22}I_2 \end{bmatrix} \tag{4.2.6}$$

is strictly less than one.

Then, (4.2.3) *has a unique solution* $(u^*, v^*) \in H_{A_1} \times H_{A_2}$ *which is a Nash-type equilibrium of the pair of energy functionals* (E_1, E_2) *associated to the problem* (4.2.3)*, i.e.,*

$$\begin{aligned} E_1(u^*, v^*) &= \inf_{H_{A_1}} E_1(., v^*) \\ E_2(u^*, v^*) &= \inf_{H_{A_2}} E_2(u^*, .). \end{aligned}$$

4.3 Examples

In this section, two examples are given in order to illustrate the results from Sections 4.4.1 and 4.4.2, first one in the case of a single equation, and the second one, for systems also.

Example 4.3.1 *A typical example of function p as in (4.2.1) is:*

$$p(x) = x^\beta \ \ (x \in [0,1]), \ \textit{with } 0 < \beta < 1.$$

Here

$$I = \int_0^1 \frac{1}{p(x)} \mathrm{d}x = \frac{1}{1-\beta}.$$

The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *can be defined as*

$$f(x,u) = \alpha \sin u + g(x),$$

with $g \in L^2(0,1)$ *and* $\alpha < 1/I$*, that is,* $\alpha < 1 - \beta$*.*

In this case, the energy functional is

$$\begin{aligned} E &: H_A \to \mathbb{R}, \\ E(u) &= \frac{1}{2}\|u\|^2_{H_A} - \int_0^1 F(x, u(x))\mathrm{d}x, \end{aligned}$$

where

$$F(x,t) = \int_0^t f(x,s)\mathrm{d}s = \alpha(1 - \cos t) + g(x)t.$$

The example considered satisfies all the assumptions of Theorem 4.2.1.

Another example of function p is

$$p(x) = \frac{1}{\sqrt{2x - x^2}} \ \ (x \in [0,1]).$$

In this case,

$$I = \int_0^1 \frac{1}{\sqrt{2x - x^2}} \mathrm{d}x = \int_0^1 \frac{1}{\sqrt{1 - (1-x)^2}} \mathrm{d}x.$$

Substituting $1 - x := t$ *we obtain*

$$I = \int_0^1 \frac{1}{\sqrt{1 - t^2}} \mathrm{d}t = \frac{\pi}{2}.$$

Considering the function f from Example 4.3.1, we can easily show that all the assumptions of Theorem 4.2.1 are satisfied.

Example 4.3.2 *Referring to the system (4.2.3), let us consider*

$$\begin{aligned} p(x) &= x^{k_1}\ , \\ q(x) &= x^{k_2}\ \ (x \in [0,1]),\ \ \textit{with}\ 0 < k_1 < 1,\ \ 0 < k_2 < 1. \end{aligned}$$

Therefore,

$$I_1 = \int_0^1 \frac{1}{x^{k_1}}\mathrm{d}x = \frac{1}{1-k_1},\quad I_2 = \int_0^1 \frac{1}{x^{k_2}}\mathrm{d}x = \frac{1}{1-k_2}.$$

Also, consider

$$\begin{aligned} f(x,u,v) &= m_{11}\sin u + m_{12}v + h_1(x) \\ g(x,u,v) &= m_{21}\sin u + m_{22}v + h_2(x), \end{aligned}$$

where $h_1, h_2 \in L^2(0,1)$, and $m_{ij} \in \mathbb{R}_+$, $i,j = 1,2$. In this case, the matrix M given by $(4.2.6)$ is

$$M = \begin{bmatrix} \dfrac{m_{11}}{1-k_1} & \dfrac{m_{12}}{1-k_1} \\ \dfrac{m_{21}}{1-k_2} & \dfrac{m_{22}}{1-k_2} \end{bmatrix}.$$

The assumption $\rho(M) < 1$ implies by (1.1.5)

$$\textit{either}\ \frac{m_{11}}{1-k_1} < 1,\ \ \textit{or}\ \frac{m_{22}}{1-k_2} < 1.$$

We may assume that

$$\frac{m_{22}}{1-k_2} < 1, \tag{4.3.1}$$

otherwise interchange the system equations. Then, we show that, (4.3.1) guarantees (4.2.5) with two functions H, H_1 of coercive type. Indeed, in this case, we have

$$G(x,u,v) = m_{21}v\sin u + m_{22}\frac{v^2}{2} + h_2(x)v,$$

and

$$H_1(x,v) \le G(x,u,v) \le H(x,v),$$

where

$$\begin{aligned} H_1(x,v) &= -m_{21}|v| + m_{22}\frac{v^2}{2} + h_2(x)v, \\ H(x,v) &= m_{21}|v| + m_{22}\frac{v^2}{2} + h_2(x)v. \end{aligned}$$

It remains to show that H is of coercive type, i.e.,

$$\Phi(v) \to +\infty \quad \textit{as} \quad \|v\|_{H_{A_2}} \to \infty,$$

where

$$\Phi(v) = \frac{1}{2}\|v\|^2_{H_{A_2}} - \int_0^1 H(x, v(x))\mathrm{d}x.$$

One has

$$\Phi(v) = \frac{1}{2}\|v\|^2_{H_{A_2}} - m_{21}\|v\|_{L^1} - \frac{m_{22}}{2}\|v\|^2_{L^2} - \int_0^1 h_2(x)v(x)\mathrm{d}x.$$

Using Poincaré's inequality

$$\|v\|_{L^2} \leq \frac{1}{\sqrt{1-k_2}}\|v\|_{H_{A_2}},$$

Hölder's inequality, and the continuous embedding of H_{A_2} into $L^1(0,1)$, we obtain

$$\Phi(v) \geq \frac{1}{2}\left(1 - \frac{m_{22}}{1-k_2}\right)\|v\|^2_{H_{A_2}} - C\|v\|_{H_{A_2}}.$$

This clearly shows that

$$\Phi(v) \to +\infty \quad \textit{as} \quad \|v\|_{H_{A_2}} \to \infty.$$

Therefore, if the spectral radius of M is less than one, all the assumptions of Theorem 4.2.2 are satisfied.

We note that, the result obtained for a system of two equations can be extended to a general system of n equations with $n \geq 2$.

Chapter 5

Fixed point theorems under combined topological and variational conditions

5.1 Overview

This chapter is based on the article A. Budescu and R. Precup [20], where the main idea is to replace part of the conditions on the operator involved in the classical fixed point theorems of Schauder, Krasnoselskii, Darbo, and Sadovskii, by assumptions upon the associated functional, in case that the fixed point equation has a variational form.

We analyze the problem in the setting of topological fixed point theory. More precisely, we discuss the fixed point equation

$$u = N(u),$$

first, in a subset of a Hilbert space identified to its dual, where the operator N is connected to a C^1-functional E by the relation

$$N(u) = u - E'(u),$$

and then, in more general Banach spaces.

5.2 Variational-topological fixed point theorems in Hilbert spaces

In this section, we consider the fixed point equation

$$u = N(u) \tag{5.2.1}$$

in a subset D of a Hilbert space H with inner product $(.,.)$ and norm $\|.\|$, which has a variational structure, in the sense that, there exists a functional $E \in C^1(H)$, such that

$$u - N(u) = E'(u) \tag{5.2.2}$$

for all $u \in D$. Notice that, the representation formula (5.2.2) requires that the space H is identified to its dual, as guaranteed by the Riesz representation theorem.

Theorem 5.2.1 *Let $U \subset H$ be a nonempty, bounded open set, $D = \overline{U}$ and let*

$$T : H \to H \quad \textit{and} \quad S : D \to H$$

be two operators such that the following conditions are satisfied:

(A$_1$) *$I - T : H \to H$ is invertible and*

$$(I - T)^{-1} \;\; \textit{is } \gamma \textit{ - Lipschitz}, \tag{5.2.3}$$

with some $\gamma > 0$;

(A$_2$) *there exists $b \in [0, 1/\gamma)$, such that*

$$\alpha(S(M)) \leq b\alpha(M), \tag{5.2.4}$$

for all $M \subset D$.

In addition, assume that there exists a functional $E \in C^1(H)$, bounded from below on D, with

$$\inf_{\overline{U}} E < \inf_{\partial U} E \tag{5.2.5}$$

and such that

$$E'(u) = u - S(u) - T(u), \quad \textit{for all } \; u \in D. \tag{5.2.6}$$

Then, there exists $u^* \in U$, such that

$$E\left(u^*\right) = \inf_D E \quad \textit{and} \quad E'(u^*) = 0.$$

The following result is a direct consequence of Theorem 5.2.1, and can be seen as a variational-topological version of Darbo's fixed point theorem.

Theorem 5.2.2 *Let $U \subset H$ be a nonempty, bounded open set, $D = \overline{U}$, and let $N : D \to H$ be a b-set-contraction (of Darbo type), for some $b \in [0, 1)$. If there is a functional $E \in C^1(H)$, bounded from below on D, such that* (5.2.5) *holds and*

$$E'(u) = u - N(u), \;\; \textit{for all } \; u \in D, \tag{5.2.7}$$

then there exists $u^* \in U$, such that

$$E\left(u^*\right) = \inf_D E \quad \textit{and} \quad E'(u^*) = 0.$$

Compared to Darbo's theorem, we do not ask neither convexity for D, nor invariance for N. Instead, the domain D has nonempty interior, and the variational form (5.2.7) is required with the additional property (5.2.5).

Theorem 5.2.2 can be generalized for condensing operators by a slight modification in the proof of Theorem 5.2.1. The result is a variational-topological version of Sadovskii's fixed point theorem.

Theorem 5.2.3 *Let $U \subset H$ be a nonempty, bounded open set, $D = \overline{U}$, and let $N : D \to H$ be a condensing operator. If there exists a functional $E \in C^1(H)$, bounded from below on D, such that* (5.2.5) *and* (5.2.7) *hold, then, there exists $u^* \in U$, such that*

$$E(u^*) = \inf_D E \quad \text{and} \quad E'(u^*) = 0.$$

5.3 A variational-topological fixed point theorem in Banach spaces

In this section, we consider the fixed point problem (5.2.1), more generally in a Banach space with suitable geometric properties, which are expressed in terms of the properties of the duality map. Recall that, (see, e.g., C. Chidume [21], I. Cioranescu [22]), for a *gauge function,* i.e., a continuous and strictly increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$, the corresponding *duality map* is defined as

$$J : X \to 2^{X^*}, \tag{5.3.1}$$

by

$$J(u) := \{u^* \in X^* : u^*(u) = \|u\|\|u^*\|, \ \|u^*\| = \phi(\|u\|)\}.$$

In particular, if $\phi(t) = t$, then the duality map is called the *normalized duality map.*

In what follows, we assume that, for a given gauge function,

(C) the duality map J is single-valued, invertible, and both J and J^{-1} are continuous.

Sufficient conditions for J to have the above properties can be found in C. Chidume [21], I. Cioranescu [22].

We say that, the equation (5.2.1) has a variational structure in a subset $D \subset X$, if there exists a functional $E \in C^1(X)$, such that

$$E'(u) = J(u) - J(N(u))$$

for all $u \in D$.

The next result is a vectorial-topological analogue of Schauder's fixed point theorem.

Theorem 5.3.1 *Let X be a Banach space, such that the condition* (C) *is satisfied, let $U \subset X$ be a nonempty, bounded, open set and $D = \overline{U}$. Assume that $N : D \to X$ is a compact operator, and that there exists a functional $E \in C^1(X)$, bounded from below on D, such that* (5.2.5) *holds, and*

$$E'(u) = J(u) - J(N(u))$$

for all $u \in D$. Then, there exists $u^* \in U$, such that

$$E(u^*) = \inf_D E \quad \text{and} \quad E'(u^*) = 0.$$

5.4 Application

In this section, we present an application of Theorem 5.3.1 to the two-point boundary value problem

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = f(t, u(t)), & t \in [0,1] \\ u(0) = u(1) = 0, \end{cases} \tag{5.4.1}$$

where

(G) $p > 1$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, and

$$|f(t,u)| \leq a\,|u|^{q-1} + b \tag{5.4.2}$$

for a.e. $t \in [0,1]$, every $u \in \mathbb{R}$, and some constants $a, b \geq 0$, and $q \geq 1$.

Consider the Banach space $X := W_0^{1,p}(0,1)$, endowed with the energetic norm

$$\|u\|_{1,p} = \left(\int_0^1 |u'(t)|^p dt \right)^{1/p}$$

and the energy functional

$$E : W_0^{1,p}(0,1) \to \mathbb{R}, \quad E(u) = \int_0^1 \left(\frac{1}{p}|u'(t)|^p - F(t, u(t)) \right) dt,$$

where $F(t,\tau) = \int_0^\tau f(t,s)ds$. The functional E is C^1 in $W_0^{1,p}(0,1)$, and

$$E'(u) = -\left(|u'|^{p-2}u'\right)' - f(.,u).$$

Hence, the solutions of (5.4.1) are the critical points of E.

The duality map of the space $W_0^{1,p}(0,1)$ corresponding to the gauge function $\phi(t) = t^{p-1}$ is the mapping

$$J : W_0^{1,p}(0,1) \to W^{-1,q}(0,1), \quad (1/p + 1/q = 1)$$

given by

$$J(u) = -(|u'|^{p-2}u')',$$

which satisfies the required condition (C).

Also, the solutions of (5.4.1) are the fixed points of the operator $N : W_0^{1,p}(0,1) \to W_0^{1,p}(0,1)$, $N(u) = J^{-1}(f(.,u))$, which is completely continuous by standard arguments based on the Arzela-Ascoli theorem. Obviously,

$$E'(u) = J(u) - J(N(u)).$$

Take any $R > 0$ and $U = B_R(0)$, the open ball of $W_0^{1,p}(0,1)$ centered at origin and of radius R. In order to show that the functional E is bounded from below on $\overline{U}$, first, use the growth condition (5.4.2) to obtain the estimation

$$\begin{aligned} E(u) &= \frac{\|u\|_{1,p}^p}{p} - \int_0^1 F(t,u(t))dt \\ \geq & \frac{\|u\|_{1,p}^p}{p} - \int_0^1 \left(\frac{a}{q}|u(t)|^q + b|u(t)| \right) dt \\ = & \frac{\|u\|_{1,p}^p}{p} - \frac{a}{q} \|u\|_{L^q(0,1)}^q - b \|u\|_{L^1(0,1)} . \end{aligned}$$

Next, taking into consideration the inequalities

$$\|u\|_{L^q(0,1)} \leq \|u\|_{1,p} \quad \text{and} \quad \|u\|_{L^1(0,1)} \leq \|u\|_{1,p} \quad \left(u \in W_0^{1,p}(0,1)\right)$$

yields

$$E(u) \geq \frac{\|u\|_{1,p}^p}{p} - \frac{a}{q} \|u\|_{1,p}^q - b \|u\|_{1,p} . \tag{5.4.3}$$

Now, if $\|u\|_{1,p} \leq R$, then,

$$E(u) \geq -\frac{a}{q} R^q - bR,$$

which shows that E is bounded from below in $\overline{U}$.

Furthermore, we show that the condition (5.2.5) is satisfied provided that R is such that

$$h(R) := \frac{R^p}{p} - \frac{a}{q} R^q - bR > 0. \tag{5.4.4}$$

Indeed, in this case, if $u \in \partial U$, that is $\|u\|_{1,p} = R$, from (5.4.3), we have

$$E(u) \geq h(R) > 0 = E(0) \geq \inf_{\overline{U}} E,$$

whence

$$\inf_{\partial U} E \geq h(R) > \inf_{\overline{U}} E.$$

For example, if $q < p$, then $h(R) \to \infty$ as $R \to \infty$, and the condition (5.4.4) holds for all sufficiently large R.

Therefore, according to Theorem 5.3.1, we have the following result.

Theorem 5.4.1 *If the condition* (G) *holds, then for every number* $R > 0$ *satisfying* (5.4.4)*, the problem* (5.4.1) *has a solution* $u \in W_0^{1,p}(0,1)$*, with* $\|u\|_{1,p} < R$*, which minimizes the energy functional* E *on the closed ball centered at origin and of radius* R *of* $W_0^{1,p}(0,1)$.

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