"Babeş-Bolyai" University Cluj-Napoca
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HARNACK TYPE INEQUALITIES AND MULTIPLE POSITIVE SOLUTIONS OF NONLINEAR PROBLEMS

Ph.D. Thesis Summary

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Introduction

The purpose of the present thesis is to emphasize the role of Harnack type inequalities for the existence, localization and multiplicity of positive solutions for some classes of nonlinear equations and systems.

Krasnosel'skiĭ's fixed point theorem in cones

In the center of this thesis is Krasnosel'skiĭ's compression-expansion theorem (i.e. Krasnosel'skiĭ's fixed point theorem in cones), which helps us to obtain existence, localization and multiplicity results of solutions in a conical shell of a Banach space (see M. A. Krasnosel'skiĭ [51, 52]).

The idea is to find solutions of an operator equation of the form $u = N(u)$, in a cone K of a normed linear space $(X, \|\cdot\|)$, with $r \leq \|u\| \leq R$, where r and R are two positive numbers $0 < r < R$. If such an existence result can be established, then we immediately obtain multiple solutions in K provided that the assumptions of the existence theorem are satisfied for several pairs of numbers (r, R). Thus we can obtain several solutions $u_1, u_2, ..., u_k$ in K, localized as $r_i \leq \|u_i\| \leq R_i$, $i = 1, 2, ..., k$. The solutions are distinct if $r_1 < R_1 < r_2 < R_2 < ... < r_k < R_k$. Similarly we may obtain infinite sequences of solutions.

The fundamental existence result which allows to apply the above strategy is Krasnosel'skiĭ's compression-expansion theorem.

Theorem (**Krasnosel'skiĭ**) Let $(X, \|.\|)$ be a normed linear space; $K \subset X$ a cone; $r, R \in \mathbb{R}_+$, $0 < r < R$; $K_{r,R} = \{u \in K : r \leq \|u\| \leq R\}$, and let $N : K_{r,R} \to K$ be a compact map. Assume that one of the following conditions is satisfied:

(a) $N(u) \not< u$ if $\|u\| = r$, and $N(u) \not> u$ if $\|u\| = R$;

(b) $N(u) \not> u$ if $\|u\| = r$, and $N(u) \not< u$ if $\|u\| = R$.

Then N has a fixed point u in K with $r \leq \|u\| \leq R$.

Note that the condition (a) expresses a property of the operator N, of compressing the conical shell $K_{r,R}$, while the condition (b) expresses the expansion property.

The previous strategy described for the case of an equation, can be extended to systems in a component-wise manner. Thus, for a system of two equations

$$\begin{cases} u_1 = N_1(u_1, u_2) \\ u_2 = N_2(u_1, u_2) \end{cases}$$

we could be interested to find solutions (u_1, u_2), where u_1 belongs to a cone K_1, u_2 belongs to a cone K_2 and each one of them is localized as follows

$$r_1 \leq \|u_1\| \leq R_1, \quad r_2 \leq \|u_2\| \leq R_2.$$

Hence, in this case, two conical shells appear. The idea is to allow N_1 and N_2 to satisfy either the compression condition, or the expansion condition, individually. Thus, three cases are possible:
(1) Both operators N_1, N_2 are compressive;
(2) Both operators N_1, N_2 are expansive;
(3) One of the operators N_1, N_2 is compressive, while the other one is expansive.

The fundamental existence result which makes possible the above strategy for systems is the following vector version of Krasnosel'skiĭ's theorem that is presented for a general system of n equations.

Theorem(**[78]**) Let $(X, \|.\|)$ be a normed linear space; $K_1, K_2, ..., K_n \subset X$ cones; $K := K_1 \times K_2 \times ... \times K_n$; $r, R \in \mathbb{R}_+^n$, $r = (r_1, r_2, ..., r_n)$, $R = (R_1, R_2, ..., R_n)$ with $0 < r_i < R_i$ for all i, $K_{r,R} = \{u \in K : r_i \leq \|u_i\| \leq R_i, \ i = 1, 2, ..., n\}$, and let $N : K_{r,R} \to K$, $N = (N_1, N_2, ..., N_n)$ be a compact map. Assume that for each $i = 1, 2, ..., n$, one of the following conditions is satisfied in $K_{r,R}$:

(a) $N_i(u) \not< u_i$ if $\|u_i\| = r_i$, and $N_i(u) \not> u_i$ if $\|u_i\| = R_i$;

(b) $N_i(u) \not> u_i$ if $\|u_i\| = r_i$, and $N_i(u) \not< u_i$ if $\|u_i\| = R_i$.

Then N has a fixed point $u = (u_1, u_2, ..., u_n)$ in K with $r_i \leq \|u_i\| \leq R_i$ for $i = 1, 2, ..., n$.

For some applications of compression-expansion principles to integral and differential equations and systems we refer to R. P. Agarwal, M. Meehan, D. O'Regan and R. Precup [2], R. P. Agarwal, D. O'Regan and P. J. Y. Wong [4], S. Budisan [15], A. Cabada and J. A. Cid [17], L. H. Erbe, S. Hu and H. Wang [24], S. Li [55], W.-C. Lian, F.-H. Wong and C.-C. Yeh [58], B. Liu and J. Zhang [60], M. Meehan and D. O'Regan [64], R. Precup [79, 80, 83], Y. N. Raffoul [87], W. Sun, S. Chen, Q. Zhang and C. Wang [89], P. J. Torres [92], F. Wang and F. Zhang [93], J. R. L. Webb [97]. In [54] R. W. Leggett and L. R. Williams obtained a remarkable generalization of Krasnosel'skiĭ's original result and applied their fixed point theorem to the nonlinear equation modelling certain infectious diseases.

In applications, the technique based on the Krasnosel'skiĭ's theorem requires the construction of a suitable cone of functions for which the compression and expansion conditions can be satisfied. To this end, in the case of most boundary value problems, the corresponding Green's functions and their properties play an important role (see for example A. Boucherif [13], F. Haddouchi and S. Benaicha [31], J. R. L. Webb [96]).

Green's functions are named after the British mathematician George Green, who first developed the concept in the 1830s. A Green's function is the impulse response of an ordinary differential equation defined on a domain, with specified initial conditions or boundary conditions. According to D. G. Duffy [23], the application of Green's functions to ordinary differential equations involving boundary value problems began with the work of Burkhardt(1861-1914). Later on, Bôcher (1867-1918) extended these results to nth order boundary value problems.

In the paper R. Precup [85] it was noticed that in the case of many problems for which Green's functions are not known, or their properties are not good enough, one can use instead, weak Harnack type inequalities associated to the differential operators and the boundary conditions (see also R. Precup [79, 83]). This type of inequalities helps to obtain lower estimations that are useful in order to achieve the compression-expansion condition. In some cases, such inequalities arise as a consequence of the concavity of the positive solutions.

The paper M. Kassmann [50] presents an introduction to certain inequalities named after Carl Gustav Axel von Harnack. These inequalities were originally defined for harmonic functions in the plane. Much later J. Serrin [88] and J. Moser [65] generalized Harnack's inequality to solutions of elliptic or parabolic partial differential equations. Many other authors have proved such type of inequalities for different problems (see W. Hebisch and L. Saloff-Coste [37], T. Kuusi [53], R. Precup [85], R. Zacher [102]).

The role of Harnack type inequalities

Harnack type inequalities are stated in connection with a given ordered Banach space $(X, \leq)$ with monotone norm, and a given operator $L : D(L) \subset X \to X$.

We say that a Harnack type inequality holds for L if there is some nonzero element φ in the positive cone $K \subset X$ such that

$$u \geq \|u\|\varphi,$$

for every positive supersolution of the equation $Lu = 0$, i.e. $u \in D(L)$ with $u \geq 0$ and $Lu \geq 0$.

This inequality is accompanied by a reverse one, namely

$$u \leq \|u\|\psi,$$

where $\psi \geq 0$ and $\psi \neq 0$, which in applications to function spaces is trivially satisfied, for example with $\psi \equiv 1$.

We shortly explain the use of Harnack type inequalities in guaranteeing the condition

$$N(u) \not< u \quad if\ u \in K \text{ and } \|u\| = r$$

required by Krasnosel'skiĭ's theorem, in case of an equation of the form $Lu = F(u)$, where $N = L^{-1}F$.

Assuming that the operator N is positive and increasing with respect to the ordering $\leq$, the proof goes as follows:
We assume the contrary, that is

$$N(u) < u \quad \text{for some } u \in K \text{ with } \|u\| = r.$$

From the Harnack inequality

$$u \geq \|u\|\varphi = r\varphi,$$

using the fact that N is increasing, we obtain

$$N(u) \geq N(r\varphi).$$

On the other hand, from $u \le \|u\|\psi$ and $N(u) < u$, we have

$$r\psi \ge u > N(u).$$

Then $r\psi > N(r\varphi)$, and since the norm is monotone, we deduce that $r\|\psi\| \ge \|N(r\varphi)\|$ and a contradiction arises if we ask as a hypothesis that

$$r\|\psi\| < \|N(r\varphi)\|.$$

More details about the use of Harnack inequalities in connection with Krasnosel'skiĭ's theorem, in an abstract setting, are given in the last chapter of the thesis.

Structure of the thesis

The thesis is divided into four chapters, each chapter being organized in several sections and subsections, an **Introduction** and a list of **References**.

Chapter 1 is entirely dedicated in presenting some preliminary notions, results and notations that we use throughout this work. Here, in Section 1.1 we introduce the concepts of an ordered Banach space, of a compact and completely continuous operator and we recall an important tool for our investigation, namely the well known Krasnosel'skiĭ's fixed point theorem in cones. In Section 1.2 we present a comparison result for Dirichlet boundary value problems, while Section 1.3 deals with an auxiliary existence and uniqueness result.

In **Chapter 2** we discuss four classes of nonlinear differential equations with different boundary conditions, motivated by some nonlinear problems that arise from mathematical modeling of real processes from engineering, mechanics, physics, economics and so on.

Section 2.1 contains a short overview of the chapter, where we explain the contents of the next sections and we present the main tools and methods that are used.

In Section 2.2 we present new existence, localization and multiplicity results for positive solutions of nonlocal boundary value problems for first order differential equations of the form

$$\begin{cases} u' = f(t,u) \\ u(0) - au(1) = g[u]. \end{cases}$$

Here g is a bounded linear functional on $C[0,1]$. Two cases are included: the discrete one, when

$$g[u] = \sum_{k=1}^{m} a_k u(t_k),$$

and the continuous case when g is given by a Stieltjes integral,

$$g[u] = \int_0^1 u(s)\, d\gamma(s).$$

Notice that, in particular, when $a = 1$ and $g[u] = 0$ we have the periodicity condition $u(0) = u(1)$.

Nonlocal problems for different classes of differential equations and systems have been intensively studied in the literature (see, for example, A. Boucherif [13], A. Boucherif and R. Precup [14], L. Byszewski [16], G. Infante [45], O. Nica [67, 68], O. Nica and R. Precup [69] for multi-point nonlocal conditions; R. Precup and D. Trif [86], J. R. L. Webb and G. Infante [99] for nonlocal conditions given by Stieltjes integrals).

We also mention some other papers on nonlocal problems for several classes of differential equations and systems: O.-M. Bolojan [12], X. Hao, L. Liu and Y. Wu [35], G. Infante [46], J. R. L. Webb and G. Infante [98].

The main results in this section are: Theorem 2.2.1, Theorem 2.2.2 and Theorem 2.2.3. These results are part of the work D.-R. Herlea [40].

In Section 2.3 we study the existence, localization and multiplicity of positive solutions of the Dirichlet-Neumann boundary value problem

$$\begin{cases} (\phi(u'))' + f(t,u) = 0, & 0 < t < 1 \\ u'(0) = u(1) = 0, \end{cases}$$

where $\phi : (-a,a) \to (-b,b)$, $0 < a,b \leq \infty$, is a homeomorphism such that $\phi(0) = 0$.

The study of the ϕ-Laplace equations and systems is a classical topic that has attracted the attention of many experts because of its interest in applications (see for example R. P. Agarwal, D. O'Regan and S. Stanek [3]). These problems, with different boundary conditions have been studied in a large number of papers using fixed point methods, degree theory, upper and lower solution techniques and variational methods. We refer to the papers C. Bereanu and J. Mawhin [8], C. Bereanu, P. Jebelean and J. Mawhin [10, 11], A. Cabada and R. L. Pouso [18], H. Dang and S. F. Oppenheimer [21], P. Drábek and J. Hernández [22], M. García-Huidobro and P. Ubilla [26], M. García-Huidobro, R. Manásevich and J. R. Ward [27], D. D. Hai and K. Schmitt [32], D. D. Hai and R. Shivaji [33], D. D. Hai and H. Wang [34], J. Henderson and H. Wang [39], P. Jebelean and C. Popa [47], P. Jebelean, C. Popa and C. Şerban [48], J. Marcos do Ó and P. Ubilla [61], J. Mawhin [62], D. O'Regan [70]-[72], D. O'Regan and R. Precup [73], I. Peral [75], V. Polášek and I. Rachůnková [76, 77], W. Sun and W. Ge [90], C. Şerban [91], J. Y. Wang [94], Z. Wang and J. Zhang [95], Z. Yang [100], Z. Yang and D. O'Regan [101], to the survey work J. Mawhin [63], and the bibliographies therein.

Contrary to the above papers, our aproach is based on a weak Harnack inequality associated to the problem, namely the following result:

Lemma 2.3.1 *For each* $c \in (0,1)$, *and any* $u \in C^1[0,1] \cap C([0,1];\mathbb{R}_+)$ *with* $u'(0) = u(1) = 0$, $u'(t) \in (-a,a)$ *for every* $t \in [0,1]$, $\phi \circ u' \in W^{1,1}(0,1)$ *and* $(\phi(u'))' \leq 0$ *on* $[0,1]$, *one has*

$$u(t) \geq (1-c)\|u\|_\infty, \quad \textit{for all} \ \ t \in [0,c].$$

The main results in this section are: Lemma 2.3.1, Theorem 2.3.2, Theorem 2.3.3 and Theorem 2.3.4; Example 2.3.5, Example 2.3.6, Example 2.3.7 and Example 2.3.8 that present some numerical applications. Most part of these results can be found in the paper D.-R. Herlea and R. Precup [43].

Section 2.4 is devoted to the study of ordinary differential equations of the same form as in the previous section, but this time with Dirichlet boundary conditions

$$\begin{cases} (\phi(u'))' + f(t,u) = 0, & 0 < t < 1 \\ u(0) = u(1) = 0. \end{cases}$$

Here ϕ is a homeomorphism from $(-a, a)$ to $\mathbb{R}$, $0 < a \leq \infty$ and the useful weak Harnack inequality is given by the following Lemma:

Lemma 2.4.2 *For each $t_0, t_1 \in (0,1)$ with $t_0 < t_1$, and any $u \in C^1[0,1] \cap C([0,1]; \mathbb{R}_+)$ with $u(0) = u(1) = 0$, $u'(t) \in (-a, a)$ for every $t \in [0,1]$, $\phi \circ u' \in W^{1,1}(0,1)$ and $(\phi(u'))' \leq 0$ a.e. on $[0,1]$, one has*

$$u(t) \geq \gamma(t)\|u\|_\infty, \quad \text{for all } t \in [0,1],$$

where

$$\gamma(t) = \begin{cases} \min\{t_0, 1-t_1\}, & \text{for all } t \in [t_0, t_1] \\ 0, & \text{otherwise.} \end{cases}$$

The most relevant results in this section are: Lemma 2.4.1, Theorem 2.4.2, Theorem 2.4.3 and Theorem 2.4.4. These contributions can be found in the paper D.-R. Herlea [41].

In Section 2.5 we discuss ϕ-Laplace equations with Neumann-Robin boundary conditions

$$\begin{cases} (\phi(u'))' + f(t,u) = 0, & 0 < t < 1 \\ u(0) - au'(0) = 0, \\ u'(1) = 0, \end{cases}$$

where $a > 0$ and ϕ is a homeomorphism from $\mathbb{R}$ to $(-b, b)$, $0 < b \leq \infty$. Problems with general Robin conditions

$$\alpha_1 u(0) - \beta_1 u'(0) = 0 = \alpha_2 u(1) + \beta_2 u'(1),$$

were studied by many authors in order to obtain the existence of positive solutions (see L. H. Erbe and H. Wang [25], W. G. Ge and J. Ren [28]). Some other authors worked with special cases. For example A. Benmezaï, S. Djebali and T. Moussaoui [5], W. G. Ge and J. Ren [29] and D.-R. Herlea [41] studied the case $\beta_1 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = 1$, while D.-R. Herlea and R. Precup [43] discussed the case $\alpha_1 = \beta_2 = 0$, $\alpha_2 = 1$ and $\beta_1 = -1$.

In order to apply Krasnosel'skiĭ's technique to our problem we first establish a weak Harnack inequality:

Lemma 2.5.1 *For each $d \in (0,1)$, and any $u \in C^1[0,1] \cap C([0,1]; \mathbb{R}_+)$ with $u(0) - au'(0) = u'(1) = 0$, $\phi \circ u' \in W^{1,1}(0,1)$ and $(\phi(u'))' \leq 0$ a.e. on $[0,1]$, one has*

$$u(t) \geq \gamma(t)\|u\|_\infty, \quad \text{for all } t \in [0,1],$$

where

$$\gamma(t) = \begin{cases} \dfrac{a+d}{a+1}, & \text{for } t \in [d,1] \\ 0, & \text{for } t \in [0,d). \end{cases}$$

The main results in this section are: Lemma 2.5.1, Theorem 2.5.2, Theorem 2.5.3 and Theorem 2.5.4; Example 2.5.5, Example 2.5.6, Example 2.5.8 and Example 2.5.7, numerical applications of the theoretical results. The results from this section have been published in the paper D.-R. Herlea [42].

Chapter 3 extends to the general case of systems the results from Chapter 2, using this time the vector version of Krasnosel'skiĭ's fixed point theorem. After an overview on the problems and the contents of the chapter given in Section 3.1, in Section 3.2 we present some existence, localization and multiplicity results for a system of two first order differential equations with nonlocal conditions

$$\begin{cases} u_1' = f_1(t, u_1, u_2) \\ u_2' = f_2(t, u_1, u_2) \\ u_1(0) - a_1 u_1(1) = g_1[u_1] \\ u_2(0) - a_2 u_2(1) = g_2[u_2]. \end{cases}$$

where g_1, g_2 are bounded linear functionals on $C[0,1]$. The theoretical results are then illustrated by some relevant examples.

The aim of Section 3.3 is to illustrate the applicability of the vector version of Krasnosel'skiĭ's theorem to the Dirichlet-Neumann boundary value problem for the ϕ-Laplace system

$$\begin{cases} \left(\phi_i\left(u_i'\right)\right)' + f_i\left(t, u_1, u_2, ..., u_n\right) = 0, \quad 0 < t < 1 \\ u_i'\left(0\right) = u_i\left(1\right) = 0 \quad \left(i = 1, 2, ..., n\right). \end{cases}$$

where ϕ_i are different homeomorphisms from $(-a_i, a_i)$ to $(-b_i, b_i)$, $0 < a_i, b_i \leq \infty$.

Section 3.4 is devoted to the study of the ϕ-Laplace system with Dirichlet conditions

$$\begin{cases} \left(\phi_i\left(u_i'\right)\right)' + f_i\left(t, u_1, u_2, ..., u_n\right) = 0, \quad 0 < t < 1 \\ u_i\left(0\right) = u_i\left(1\right) = 0 \quad \left(i = 1, 2, ..., n\right). \end{cases}$$

where ϕ_i are different homeomorphisms from $(-a_i, a_i)$ to $\mathbb{R}$, $0 < a_i \leq \infty$.

In Section 3.5 we present some existence and localization results for positive solutions of the Neumann-Robin boundary value problem

$$\begin{cases} \left(\phi_i\left(u_i'\right)\right)' + f_i\left(t, u_1, u_2, ..., u_n\right) = 0, \quad 0 < t < 1 \\ u_i\left(0\right) - a_i u_i'\left(0\right) = 0 \\ u_i'\left(1\right) = 0 \qquad \left(i = 1, 2, ..., n\right), \end{cases}$$

where $a_i > 0$, ϕ_i are different homeomorphisms from $\mathbb{R}$ to $(-b_i, b_i)$, $0 < b_i \leq \infty$.

The most relevant results in this chapter are: Theorem 3.2.1, Theorem 3.2.2, Theorem 3.3.1, Theorem 3.3.2, Theorem 3.4.1 and Theorem 3.5.1; Example 3.2.3 and Example 3.2.5 that present two numerical applications. These results appear in the papers D.-R. Herlea [40]-[42], D.-R. Herlea and R. Precup [43].

The purpose of **Chapter 4** is to give an abstract theory. After a short overview given by

Section 4.1, in Section 4.2 we shall concentrate on the abstract problem for a single equation

$$\begin{cases} Lu = F(u) \\ u \in B, \end{cases}$$

in a Banach space $(X, \|\cdot\|)$, where $L : D(L) \subset X \to X$ and $F : X \to X$ are two given operators and $B \subset X$.

Then, in Section 4.3, we shall extend the results to the case of systems

$$\begin{cases} L_i u_i = F_i(u_1, u_2, ..., u_n) \\ u_i \in B_i \quad (i = 1, 2, ..., n). \end{cases}$$

The theory uses Krasnosel'skiĭ's technique and is based on an abstract Harnack type inequality which is now assumed as hypothesis.

The main contributions here are as follows: Theorem 4.2.2, Theorem 4.2.3, Theorem 4.2.4 and Theorem 4.3.1. The results from this chapter will appear in the paper D.-R. Herlea and R. Precup [44].

As above mentioned in the summary of each chapter, most of the results presented in this thesis are part of the following publications:

- D.-R. Herlea, *Existence and localization of positive solutions to first order differential systems with nonlocal conditions*, Studia Univ. Babeş-Bolyai Math., **59**(2014), 221-231.
- D.-R. Herlea, *Positive solutions for second-order boundary-value problems with ϕ-Laplacian*, Electron. J. Differential Equations, **2016**(2016), 1-8.
- D.-R. Herlea and R. Precup, *Existence, localization and multiplicity of positive solutions to ϕ-Laplace equations and systems*, Taiwanese J. Math., **20**(2016), 77-89.
- D.-R. Herlea, *Existence, localization and multiplicity of positive solutions for the Dirichlet BVP with ϕ-Laplacian*, Fixed Point Theory, to appear.
- D.-R. Herlea and R. Precup, *Abstract weak Harnack type inequalities and multiple positive solutions of nonlinear problems*, submitted.

Some ideas for further work

The method that we have used throughout the thesis can be applied to other classes of problems, for instance, to equations and systems of higher order with different boundary conditions, to functional-differential equations and partial differential equations. Some advances in this direction are due to A. Cabada, R. Precup, L. Saavedra and S. Tersian [19], Y. Li [56], H. Lian, J. Zhao and R. P. Agarwal [59], M. Naceri, R. P. Agarwal, E. Çetin and A. El-Haffaf [66].

Another idea is to study positive radial solutions for some classes of boundary value problems which introduce singularities in equations (for problems on radial solutions we refer to the papers

C. Bereanu, P. Jebelean and J. Mawhin [11], D. D. Hai and K. Schmitt [32], X. He [36], D. Jiang and H. Liu [49]).

Another direction is to use Harnack type inequalities together with some principles from critical point theory as already suggested in R. Precup [81].

Keywords

Weak Harnack type inequalities, positive solutions, Krasnosel'skiĭ's fixed point theorem, cone, nonlinear equations and systems, ϕ-Laplacian.

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Chapter 1

Preliminaries

In this chapter we list some notions and results that we use throughout the Ph.D. thesis. Ordered Banach spaces, compactness and completely continuous operators and Krasnosel'skiĭ's fixed point theorems in cones are the main tools in our work.

1.1 Basic notions and results

1.1.1 Ordered Banach spaces

Definition 1.1.1 Let X be a real linear space. By a *cone* K of X we understand a closed convex subset of X such that $\lambda K \subset K$ for all $\lambda \in \mathbb{R}_+$ and $K \cap (-K) = \{0\}$.

Proposition 1.1.2 *Let X be a linear space and $K \subset X$ be a cone. The relation $\leq_K$ on X defined by*

$$u \leq_K v \quad \textit{if and only if } v - u \in K,$$

is an order (reflexive, antisymmetric and transitive) relation on X (called the order relation induced by K), compatible with the linear structure of X, i.e., whenever $u_i, v_i \in X$, $u_i \leq_K v_i$, $i = 1, 2$, and $\lambda \in \mathbb{R}_+$, we have

$$u_1 + u_2 \leq_K v_1 + v_2, \quad \lambda u_1 \leq_K \lambda v_1.$$

Conversely, if $\leq$ is an order relation on X compatible with the linear structure of X, then the set

$$K_+ = \{u \in X : 0 \leq u\}$$

is a cone (called the positive cone) and $\leq = \leq_{K_+}$.

A Banach space endowed with a cone, equivalently with an order relation compatible with linear structure is called an *ordered Banach space*.

1.1.2 Compactness and completely continuous operators

Definition 1.1.3 A metric space (X, d) is said to be *compact* if every sequence of elements of X has a convergent subsequence in X.

Let (X, d) be a compact metric space and $C(X; \mathbb{R})$ be the Banach space of all continuous functions from X to $\mathbb{R}$ under the norm $\|\cdot\|_\infty$.

Theorem 1.1.4 (Ascoli-Arzela) *A subset Y of $C(X; \mathbb{R})$ is relatively compact if and only if the following conditions are satisfied:*

(i) Y is bounded, i.e., there exists a constant $C > 0$ such that

$$\|u(x)\| \leq C,$$

for all $x \in X$ and $u \in Y$.

(ii) Y is equicontinuous, i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $u \in Y$,

$$\|u(x_1) - u(x_2)\| < \epsilon,$$

whenever $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta$.

Definition 1.1.5 Let X, Y be Banach spaces and $T : D \subset X \to Y$.

(a) The operator T is said to be *bounded* if it maps any bounded subset of D into a bounded subset of Y.

(b) The operator T is said to be *completely continuous* if it is continuous and maps any bounded subset of D into a relatively compact subset of Y.

(c) The operator T is said to be *compact* if it is continuous and $T(D)$ is relatively compact.

1.1.3 Krasnosel'skiĭ's fixed point theorem in cones

Theorem 1.1.6 (Krasnosel'skiĭ) *Let $(X, \|.\|)$ be a normed linear space; $K \subset X$ a cone; $r, R \in \mathbb{R}_+$, $0 < r < R$; $K_{r,R} = \{u \in K : r \leq \|u\| \leq R\}$, and let $N : K_{r,R} \to K$ be a compact map. Assume that one of the following conditions is satisfied:*

(a) *$N(u) \not< u$ if $\|u\| = r$, and $N(u) \not> u$ if $\|u\| = R$;*

(b) *$N(u) \not> u$ if $\|u\| = r$, and $N(u) \not< u$ if $\|u\| = R$.*

Then N has a fixed point u in K with $r \leq \|u\| \leq R$.

Theorem 1.1.7 ([78]) *Let $(X, \|.\|)$ be a normed linear space; $K_1, K_2, ..., K_n \subset X$ cones; $K := K_1 \times K_2 \times ... \times K_n$; $r, R \in \mathbb{R}_+^n$, $r = (r_1, r_2, ..., r_n)$, $R = (R_1, R_2, ..., R_n)$ with $0 < r_i < R_i$ for all i, $K_{r,R} = \{u \in K : r_i \leq \|u_i\| \leq R_i,\ i = 1, 2, ..., n\}$, and let $N : K_{r,R} \to K$, $N = (N_1, N_2, ..., N_n)$ be a compact map. Assume that for each $i = 1, 2, ..., n$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) *$N_i(u) \not< u_i$ if $\|u_i\| = r_i$, and $N_i(u) \not> u_i$ if $\|u_i\| = R_i$;*

(b) *$N_i(u) \not> u_i$ if $\|u_i\| = r_i$, and $N_i(u) \not< u_i$ if $\|u_i\| = R_i$.*

Then N has a fixed point $u = (u_1, u_2, ..., u_n)$ in K with $r_i \leq \|u_i\| \leq R_i$ for $i = 1, 2, ..., n$.

1.2 A comparison result

Let the intervals I_0 and $J = [t_0, t_1]$, the functions $\phi : J \times I_0 \to \mathbb{R}$ and $q : J \times \mathbb{R}^2 \to \mathbb{R}$ and the differential operator A be defined by

$$\begin{aligned} &Au(t) := -\frac{d}{dt}\phi\,(t, u'(t)) - q(t, u(t), u'(t)), \quad t \in J, \\ &u \in Y := \{u \in C^1(J)|\ u'[J] \subseteq I_0 \text{ and } \phi(\cdot, u'(\cdot)) \in AC(J)\}. \end{aligned} \tag{1.2.1}$$

Theorem 1.2.1 *([38]) Let the functions $q : J \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi : J \times I_0 \to \mathbb{R}$ have the following properties*

(ϕ_0) $\phi(t, z) < \phi(t, y)$ whenever $t \in J$, $y, z \in I_0$ and $z < y$;

(q_1) $q(t, x, z) \leq q(t, y, z)$ for a.a. $t \in J$ and for all $x, y, z \in \mathbb{R}$, $x \geq y$;

(q_2) $q(t, x, z) - q(t, x, y) \leq h(t, \phi(t, y) - \phi(t, z))$ for a.a. $t \in J$ and for all $x \in \mathbb{R}$, $y, z \in I_0$, $y > z$, $0 < \phi(t, y) - \phi(t, z) \leq r$, where $r > 0$, $h : J \times [0, r] \to \mathbb{R}_+$, and $x(t) \equiv 0$ is the only function in $AC(J)$ which satisfies

$$x'(t) \leq h(t, x(t)) \text{ a.e. in } J, \quad x(t_0) = 0.$$

Assume that $u, w \in Y$ satisfy

$$Au(t) \leq Aw(t) \text{ a.e. in } J, \quad u(t_0) \leq w(t_0), \quad u(t_1) \leq w(t_1).$$

Then $u(t) \leq w(t)$ for each $t \in J$. In particular, under the conditions on q and ϕ, the Dirichlet problem

$$\begin{cases} -\dfrac{d}{dt}\phi\,(t, u'(t)) = q(t, u(t), u'(t)) \quad \text{a.e. in } J \\ u\,(t_0) = c_0, \quad u\,(t_1) = c_1 \end{cases}$$

has at most one solution.

1.3 An auxiliary existence and uniqueness result

Let us assume that

(H_Φ) $\phi : (-a, a) \to \mathbb{R}$, $0 < a \leq \infty$ is a homeomorphism such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi : (-a, a) \to (-\infty, 0]$ of class C^1, and strictly convex.

So, ϕ is strictly monotone on $(-a, a)$.

If $\Phi^* : \mathbb{R} \to \mathbb{R}$ is the Legendre-Fenchel transform of Φ defined by

$$\Phi^*(v) = \langle \phi^{-1}(v), v\rangle - \Phi[\phi^{-1}(v)] = \sup_{u \in (-a,a)} \{\langle u, v\rangle - \Phi(u)\},$$

then Φ^* is also strictly convex and

$$\Phi^*(v) \leq a|v| - \inf_{|v|<a} \Phi \circ \phi^{-1} =: a|v| + d. \tag{1.3.2}$$

Now, using the nonnegativity of Φ,

$$\Phi^*(v) \geq \sup_{u\in(-a,a)} \langle v, u\rangle = a|v|, \tag{1.3.3}$$

we have that Φ^* is coercive on $\mathbb{R}$. Also Φ^* is of class C^1. Hence

$$\phi^{-1} = \nabla\Phi^*,$$

so that

$$v = \nabla\Phi(u) = \phi(u), u \in (-a, a) \Leftrightarrow u = \phi^{-1}(v) = \nabla\Phi^*(v), v \in \mathbb{R}.$$

Given $h, H \in C[0,1]$, $H := \int_0^t h(s)\, ds$ and $b \in \mathbb{R}$, we define

$$\begin{aligned} G(b; H) &= \int_0^1 \phi^{-1}[b - H(t)]\, dt = \int_0^1 \nabla_b\Phi^*[b - H(t)]\, dt \\ &= \nabla_b \int_0^1 \Phi^*[b - H(t)]\, dt = \nabla_b g(b; H), \end{aligned}$$

where

$$g(b; H) = \int_0^1 \Phi^*[b - H(t)]\, dt.$$

Lemma 1.3.1 *([7]) If $\phi = \nabla\Phi$, with Φ satisfying assumption (H_Φ), then, for each $H \in C[0,1]$, the equation*

$$\int_0^1 \phi^{-1}[b - H(t)]\, dt = 0 \tag{1.3.4}$$

has a unique solution $b := Q_\phi(H)$. Moreover, $Q_\phi : C[0,1] \to \mathbb{R}$ is continuous and takes bounded subsets of $C[0,1]$ into bounded subsets of $\mathbb{R}$.

Chapter 2

Positive solutions for some classes of nonlinear equations

2.1 Overview

This chapter is devoted to existence, localization and multiplicity of positive solutions for some boundary value problems.

2.2 First order differential equations with nonlocal conditions

We present existence, localization and multiplicity results for positive solutions of the problem

$$\begin{cases} u' = f(t,u) \\ u(0) - au(1) = g[u] \end{cases} \tag{2.2.1}$$

where $f \in C([0,1] \times \mathbb{R}_+; \mathbb{R}_+)$; $g : C[0,1] \to \mathbb{R}$ is a linear functional given by

$$g[u] = \int_0^1 u(s)\, d\gamma(s), \tag{2.2.2}$$

with $g[1] < 1$; $\gamma \in C^1[0,1]$ increasing and $0 < a < 1 - g[1]$.

We seek nonnegative solutions u on $[0,1]$. The problem (2.2.1) is equivalent to the following integral equation

$$u(t) = \int_0^t [c(\gamma(1) - \gamma(s) + a) + 1]h(s)\, ds + \int_t^1 c(\gamma(1) - \gamma(s) + a)h(s)\, ds, \tag{2.2.3}$$

where $c := 1/(1 - g[1] - a)$, $c > 0$. If now, to the nonlocal condition $u(0) - au(1) = g[u]$, we associate the Green's function

$$G(t,s)=\begin{cases} c[\gamma(1)-\gamma(s)+a]+1 & \text{for } \ 0\le s\le t\le 1\\ c[\gamma(1)-\gamma(s)+a] & \text{for } \ 0\le t< s\le 1, \end{cases} \tag{2.2.4}$$

then (2.2.3) can be written as

$$u(t)=\int_0^1 G(t,s)h(s)\,ds. \tag{2.2.5}$$

Thus we have obtained the inverse of the operator L, $L^{-1}: C[0,1]\to C[0,1]$,

$$(L^{-1}h)(t)=\int_0^1 G(t,s)h(s)\,ds.$$

The following properties are essential for the applicability of Krasnosel'skiĭ's technique:
1) $G(t,s)\le H(s)$, for all $t,s\in[0,1]$, where

$$H(s)=c[\gamma(1)-\gamma(s)+a]+1$$

2) $\delta H(s)\le G(t,s)$ for all $t,s\in[0,1]$, where

$$\delta=\min_{s\in[0,1]}\frac{c[\gamma(1)-\gamma(s)+a]}{c[\gamma(1)-\gamma(s)+a]+1}.$$

Notice that $\delta>0$ since $a,c>0$ and $\gamma(1)\ge\gamma(s)$, for all $s\in[0,1]$. Also, it is clear that $\delta<1$.
Let $N: C([0,1];\mathbb{R}_+)\to C([0,1];\mathbb{R}_+)$ be defined by

$$N(u)(t)=\int_0^1 G(t,s)f(s,u(s))\,ds.$$

The above properties of the Green's functions imply that for any $t,t^*\in[0,1]$, one has:

$$N(u)(t)\ge\delta N(u)(t^*). \tag{2.2.6}$$

If t^* is such that $N(u)(t^*)=\|N(u)\|_\infty$, then (2.2.6) shows that

$$N(u)(t)\ge\delta\|N(u)\|_\infty \quad \text{for all} \ \ t\in[0,1] \tag{2.2.7}$$

and any nonnegative function $u\in C[0,1]$.

Based on these estimations we define the cone

$$K=\{u\in C[0,1]: u(t)\ge\delta\|u\|_\infty, \ \text{for all } t\in[0,1]\}. \tag{2.2.8}$$

Due to (2.2.7) we have the invariance property $N(K)\subset K$. Therefore, the problem of nonnegative solutions of (2.2.1) is equivalent to the fixed point problem $u=Nu$, $u\in K$, for the self-mapping N of K. Note that the continuity of f implies the complete continuity of N by standard arguments based on Ascoli-Arzela's theorem.

Notice that (2.2.7) represents a weak Harnack type inequality for the nonnegative super solutions of the problem (2.2.1).

2.2.1 Existence and localization results

Theorem 2.2.1 *Assume that there exist α, $\beta > 0$ with $\alpha \neq \beta$, such that*

$$A\lambda > \alpha, \quad B\Lambda < \beta, \tag{2.2.9}$$

where

$$\begin{aligned} A &= \int_0^1 G(t^*, s)\, ds, \ \text{for a chosen point} \ \ t^* \in [0,1], \\ B &= \max_{0 \leq t \leq 1} \int_0^1 G(t,s)\, ds, \\ \lambda &= \min\{f(t,u) : 0 \leq t \leq 1, \ \delta\alpha \leq u \leq \alpha\}, \\ \Lambda &= \max\{f(t,u) : 0 \leq t \leq 1, \ \delta\beta \leq u \leq \beta\}, \end{aligned}$$

Then (2.2.1) *has at least one positive solution u with $r \leq \|u\|_\infty \leq R$, where $r = \min\{\alpha, \beta\}$, $R = \max\{\alpha, \beta\}$.*

The next theorem is about the existence of at least one pair α, β satisfying the conditions from (2.2.9).

Theorem 2.2.2 *Let f be a nondecreasing function with $f = f(u)$. Assume that one of the following conditions is satisfied:*

(i) $\lim_{x\to\infty} \dfrac{f(x)}{x} = \infty$ *and* $\lim_{x\to 0} \dfrac{f(x)}{x} = 0$;

(ii) $\lim_{x\to 0} \dfrac{f(x)}{x} = \infty$ *and* $\lim_{x\to\infty} \dfrac{f(x)}{x} = 0$.

Then (2.2.1) *has at least one positive solution.*

2.2.2 A multiplicity result

Theorem 2.2.1 guarantees the existence of solutions in an annular set. Clearly, if the assumptions of Theorem 2.2.1 are satisfied for several disjoint annular sets, then multiple solutions are obtained.

Theorem 2.2.3 *Let f be a nondecreasing function with $f = f(u)$. If the condition*

(iii) $\limsup_{x\to\infty} \dfrac{f(x)}{x} > \dfrac{1}{\delta A}$ *and* $\liminf_{x\to\infty} \dfrac{f(x)}{x} < \dfrac{1}{B}$

holds, then (2.2.1) *has a sequence of positive solutions $(u_n)_{n\geq 1}$ such that $\|u_n\|_\infty \to \infty$ as $n \to \infty$.*

If the condition

(iv) $\limsup_{x\to 0} \dfrac{f(x)}{x} > \dfrac{1}{\delta A}$ *and* $\liminf_{x\to 0} \dfrac{f(x)}{x} < \dfrac{1}{B}$

holds, then (2.2.1) *has a sequence of positive solutions $(u_n)_{n\geq 1}$ such that $\|u_n\|_\infty \to 0$ as $n \to \infty$.*

2.3 The Dirichlet-Neumann BVP for ϕ-Laplace equations

We present existence, localization and multiplicity results for positive solutions of the two-point boundary value problem

$$\begin{cases} (\phi(u'))' + f(t,u) = 0, & 0 < t < 1 \\ u'(0) = u(1) = 0. \end{cases} \tag{2.3.10}$$

Important motivations for this study are the cases of the equations with p-Laplacian and curvature operators in Euclidian and Minkowski spaces, for which problem (2.3.10) respectively becomes

$$\begin{cases} \left(|u'|^{p-2} u'\right)' + f(t,u) = 0, & 0 < t < 1 \\ u'(0) = u(1) = 0, \end{cases} \tag{2.3.11}$$

$$\begin{cases} \left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' + f(t,u) = 0, & 0 < t < 1 \\ u'(0) = u(1) = 0, \end{cases} \tag{2.3.12}$$

$$\begin{cases} \left(\dfrac{u'}{\sqrt{1-u'^2}}\right)' + f(t,u) = 0, & 0 < t < 1 \\ u'(0) = u(1) = 0. \end{cases} \tag{2.3.13}$$

Inspired by these three typical examples, in the literature, the cases of homeomorphisms of $\mathbb{R}$, $\phi : \mathbb{R} \to \mathbb{R}$; of homeomorphisms with bounded range, $\phi : \mathbb{R} \to (-b,b)$; and of homeomorphisms with bounded domain, $\phi : (-a,a) \to \mathbb{R}$, have been discussed separately. In this section, these three cases will be treated unitarily by assuming that ϕ is a homeomorphism from $(-a,a)$ to $(-b,b)$, and $0 < a, b \leq \infty$.

2.3.1 Existence and localization results

This section deals with positive solutions for the problem (2.3.10). We make the following assumptions: $\phi : (-a,a) \to (-b,b)$, $0 < a, b \leq \infty$ is an increasing homeomorphism such that $\phi(0) = 0$ and $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $f(t,x) < b$.

First we obtain the equivalent integral equation to the problem (2.3.10)

$$u(t) = -\int_t^1 \phi^{-1}\left(-\int_0^\tau f(s,u(s))\,ds\right) d\tau. \tag{2.3.14}$$

Conversely, if a function $u \in C([0,1];\mathbb{R}_+)$ satisfies (2.3.14), which implicitly means that

$$\int_0^\tau f(s,u(s))\,ds < b$$

for all $\tau \in [0,1]$, then u is a positive solution of the problem (2.3.10).

Next, assuming in addition that $f(t,x) < b$ for all $t \in [0,1]$ and $x \in \mathbb{R}_+$, we may define the integral operator $N : C([0,1];\mathbb{R}_+) \to C([0,1];\mathbb{R}_+)$ by

$$N(u)(t) = -\int_t^1 \phi^{-1}\left(-\int_0^\tau f(s,u(s))\,ds\right)d\tau, \tag{2.3.15}$$

and thus, finding positive solutions to (2.3.10) is equivalent to the fixed point problem for the operator N on $C([0,1];\mathbb{R}_+)$. Note that by standard arguments based on Ascoli-Arzela's theorem, N is completely continuous. Let $\|.\|_\infty$ denote the max norm on $C\,[0,1]$.

In order to apply Krasnosel'skiĭ's fixed point theorem in cones we need a weak Harnack type inequality for the differential operator $Lu := -(\phi(u'))'$ and the boundary conditions $u'(0) = u(1) = 0$.

Lemma 2.3.1 *For each $c \in (0,1)$, and any $u \in C^1[0,1] \cap C([0,1];\mathbb{R}_+)$ with $u'(0) = u(1) = 0$, $u'(t) \in (-a,a)$ for every $t \in [0,1]$, $\phi \circ u' \in W^{1,1}(0,1)$ and $(\phi(u'))' \leq 0$ on $[0,1]$, one has*

$$u(t) \geq (1-c)\|u\|_\infty, \quad \textit{for all} \;\; t \in [0,c]. \tag{2.3.16}$$

For our first result we make the following assumptions:

(A1) $\phi : (-a,a) \to (-b,b)$, $0 < a,b \leq \infty$ is an increasing homeomorphism such that $\phi(0) = 0$;

(A2) $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $f(t,.)$ is nondecreasing on $\mathbb{R}_+$ for each $t \in [0,1]$, and $f(t,x) < b$ for all $t \in [0,1]$ and $x \in \mathbb{R}_+$.

Theorem 2.3.2 *Let the conditions* (A1) *and* (A2) *hold and assume that there exist $c, \alpha, \beta > 0$ with $c < 1$ and $\alpha \neq \beta$ such that*

$$\Phi(\alpha) := -\int_0^c \phi^{-1}\left(-\int_0^\tau f(s,(1-c)\alpha)\,ds\right)d\tau > \alpha, \tag{2.3.17}$$

$$\Psi(\beta) := -\int_0^1 \phi^{-1}\left(-\int_0^\tau f(s,\beta)\,ds\right)d\tau < \beta. \tag{2.3.18}$$

Then (2.3.10) *has at least one positive solution u with $r \leq \|u\|_\infty \leq R$, where $r = \min\{\alpha,\beta\}$, $R = \max\{\alpha,\beta\}$.*

The next result is about the existence of at least one pair of numbers (α,β).

Theorem 2.3.3 *Let* (A1) *and* (A2) *hold and assume that one of the following conditions is satisfied:*

(i) $\limsup_{\lambda\to\infty} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\Psi(\lambda)}{\lambda} < 1$;

(ii) $\limsup_{\lambda\to 0} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\Psi(\lambda)}{\lambda} < 1$.

Then (2.3.10) *has at least one positive solution.*

2.3.2 A multiplicity result

The next result is about a sequence of positive solutions of the problem 2.3.10, whose existence is guaranteed by the oscillations of f towards infinity or zero.

Theorem 2.3.4 *Let* (A1) *and* (A2) *hold. If the condition*

(iii) $\limsup_{\lambda\to\infty} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\Psi(\lambda)}{\lambda} < 1$

holds, then (2.3.10) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $\|u_n\|_\infty \to \infty$ *as* $n \to \infty$.

If the condition

(iv) $\limsup_{\lambda\to 0} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\Psi(\lambda)}{\lambda} < 1$

holds, then (2.3.10) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $\|u_n\|_\infty \to 0$ *as* $n \to \infty$.

2.3.3 Examples

Example 2.3.5 Consider the problem (2.3.12) where

$$f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+, \quad f(t,x) = \frac{\gamma x}{x+\delta} \tag{2.3.19}$$

and $\gamma, \delta > 0$. In this case $a = \infty$, $b = 1$ and one can easily check that the condition (A2), particularly, the inequality $f(t,x) < 1$, holds if and only if $\gamma \leq 1$. Direct computation shows that

$$\Phi(\alpha) = \frac{1-\sqrt{1-A^2c^2}}{A}, \quad \Psi(\beta) = \frac{1-\sqrt{1-B^2}}{B},$$

where

$$A = \frac{\gamma(1-c)\alpha}{(1-c)\alpha+\delta}, \quad B = \frac{\gamma\beta}{\beta+\delta}. \tag{2.3.20}$$

Now it is easy to see that

$$\lim_{\lambda\to 0} \frac{\Phi(\lambda)}{\lambda} = \frac{\gamma(1-c)c^2}{2\delta} \quad \text{and} \quad \lim_{\lambda\to\infty} \frac{\Psi(\lambda)}{\lambda} = 0. \tag{2.3.21}$$

Hence the condition (ii) from Theorem 2.3.3 is satisfied if $[\gamma(1-c)c^2]/2\delta > 1$. Thus, if

$$\delta < \gamma\frac{(1-c)c^2}{2} \quad \text{and} \quad \gamma \leq 1,$$

then the problem (2.3.12) has at least one positive solution.

Example 2.3.6 Consider the problem (2.3.13) for the same function (2.3.19). In this case $a = 1$, $b = \infty$ and the condition (A2) holds for any $\gamma, \delta > 0$. We have

$$\Phi(\alpha) = \frac{\sqrt{1+A^2c^2}-1}{A}, \quad \Psi(\beta) = \frac{\sqrt{1+B^2}-1}{B},$$

where A, B are given by (2.3.20), and the limits (2.3.21) also hold. Thus, if

$$\delta < \gamma \frac{(1-c)c^2}{2} \quad \text{and} \quad \gamma > 0,$$

then the problem (2.3.13) has at least one positive solution.

Example 2.3.7 Consider the problem (2.3.12) where, this time

$$f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+, \quad f(t,x) = \frac{\gamma\sqrt{x}}{\delta + \sqrt{x}} \tag{2.3.22}$$

and $\gamma, \delta > 0$. In this case $a = \infty$, $b = 1$ and one can easily check that the condition (A2), particularly, the inequality $f(t,x) < 1$, holds if and only if $\gamma \leq 1$. Direct computation shows that

$$\Phi(\alpha) = \frac{1 - \sqrt{1 - A^2c^2}}{A}, \quad \Psi(\beta) = \frac{1 - \sqrt{1 - B^2c^2}}{B},$$

where

$$A = \frac{\gamma\sqrt{(1-c)\,\alpha}}{\sqrt{(1-c)\,\alpha} + \delta}, \quad B = \frac{\gamma\sqrt{\beta}}{\sqrt{\beta} + \delta}. \tag{2.3.23}$$

Now it is easy to see that

$$\lim_{\lambda \to 0} \frac{\Phi(\lambda)}{\lambda} = \infty \quad \text{and} \quad \lim_{\lambda \to \infty} \frac{\Psi(\lambda)}{\lambda} = 0. \tag{2.3.24}$$

Hence the condition (ii) from Theorem 2.3.3 is satisfied. Thus the problem (2.3.12) has at least one positive solution.

Example 2.3.8 Consider the problem (2.3.13) for the same function (2.3.22). In this case $a = 1$, $b = \infty$ and the condition (A2) holds for any $\gamma, \delta > 0$. We have

$$\Phi(\alpha) = \frac{\sqrt{1 + A^2c^2} - 1}{A}, \quad \Psi(\beta) = \frac{\sqrt{1 + B^2c^2} - 1}{B},$$

where A, B are given by (2.3.23), and the limits (2.3.24) also hold. Thus the problem (2.3.13) has at least one positive solution.

2.4 The Dirichlet BVP for ϕ-Laplace equations

In this section, we focus on the existence, localization and multiplicity of positive solutions for the following Dirichlet boundary value problem

$$\begin{cases} (\phi(u'))' + f(t,u) = 0, \quad 0 < t < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{2.4.25}$$

where ϕ is a homeomorphism from $(-a, a)$ to $\mathbb{R}$, $0 < a \leq \infty$.

Under these assumptions there are two basic models (see C. Bereanu, P. Jebelean and J. Mawhin [11], S.-S. Chen and Z.-H. Ma [20]):

(1) For $a = \infty$ we have the p-Laplacian operator,

$$\phi(u) = |u|^{p-2}u, \text{ with } p > 1.$$

(2) For $a = 1$ we have the curvature operator in Minkowski space,

$$\phi(u) = \frac{u}{\sqrt{1-u^2}}.$$

2.4.1 Existence and localization results

In this subsection, we prove existence of positive solutions for the problem (2.4.25). We make the following assumptions: $\phi : (-a, a) \to \mathbb{R}$, $0 < a \leq \infty$ is an increasing homeomorphism such that $\phi(0) = 0$ and $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

In order to obtain the equivalent integral equation to the problem (2.4.25), let us first consider the problem:

$$\begin{cases} (\phi(u'))' + h(t) = 0, & 0 < t < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{2.4.26}$$

where $h \in C[0, 1]$. The integral equation equivalent to the problem (2.4.26) is

$$u(t) = \int_0^t \phi^{-1}\left(b - \int_0^\tau h(s)\, ds\right) d\tau, \tag{2.4.27}$$

where $b = \phi(u'(0))$. According to Lemma 1.3.1 (given by C. Bereanu and J. Mawhin [7]), there exists a unique $b = b(h)$. In addition, the mapping $b : C[0, 1] \to \mathbb{R}$ is continuous and takes bounded sets into bounded sets.

Taking this into account, for all $t \in [0, 1]$ we may define the integral operator $S : L^1[0, 1] \to C^1[0, 1]$ by

$$(Sh)(t) = \int_0^t \phi^{-1}\left(b(h) - \int_0^\tau h(s)\, ds\right) d\tau, \tag{2.4.28}$$

which has the following properties:
(a) For each $h \geq 0$, $Sh \geq 0$;
(b) If $h_1 \geq h_2 \geq 0$ then $Sh_1 \geq Sh_2$.

Now, returning to our problem (2.4.25), we have its equivalence to the integral equation

$$u = S \circ F(u), \tag{2.4.29}$$

where $F(u) = f(\cdot, u)$.

Next, we may define the integral operator $N : C([0, 1]; \mathbb{R}_+) \to C([0, 1]; \mathbb{R}_+)$ by

$$N(u)(t) = \int_0^t \phi^{-1}\left(b - \int_0^\tau f(s, u(s))\, ds\right) d\tau, \tag{2.4.30}$$

where $b = b(f(\cdot, u(\cdot)))$. Thus, finding positive solutions to (2.4.25) is equivalent to the fixed point problem for the operator N on $C([0, 1]; \mathbb{R}_+)$. Note that standard arguments based on Ascoli-Arzela's theorem, guarantee that N is completely continuous.

In order to apply Krasnosel'skiĭ's fixed point theorem in cones we need a weak Harnack type inequality for the differential operator $Lu := -(\phi(u'))'$.

Lemma 2.4.1 *For each* $t_0, t_1 \in (0,1)$ *with* $t_0 < t_1$, *and any* $u \in C^1[0,1] \cap C([0,1];\mathbb{R}_+)$ *with* $u(0) = u(1) = 0$, $u'(t) \in (-a, a)$ *for every* $t \in [0,1]$, $\phi \circ u' \in W^{1,1}(0,1)$ *and* $(\phi(u'))' \leq 0$ *a.e. on* $[0,1]$, *one has*

$$u(t) \geq \gamma(t)\|u\|_\infty, \quad \textit{for all } t \in [0,1], \tag{2.4.31}$$

where

$$\gamma(t) = \begin{cases} \min\{t_0, 1-t_1\}, & \textit{for all } t \in [t_0, t_1] \\ 0, & \textit{otherwise.} \end{cases}$$

For our following results we make the assumptions:

(B1) $\phi : (-a, a) \to \mathbb{R}$, $0 < a \leq \infty$ is an increasing homeomorphism such that $\phi(0) = 0$;

(B2) $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $f(t, .)$ is nondecreasing on $\mathbb{R}_+$ for each $t \in [0,1]$.

Theorem 2.4.2 *Let* (B1) *and* (B2) *hold and assume that there exist* $\alpha, \beta > 0$ *with* $\alpha \neq \beta$ *such that*

$$\|Sf(\cdot, \gamma(\cdot)\alpha)\|_\infty > \alpha, \tag{2.4.32}$$

$$\|Sf(\cdot, \beta)\|_\infty < \beta. \tag{2.4.33}$$

Then (2.4.25) *has at least one positive solution* u *with* $r \leq \|u\|_\infty \leq R$, *where* $r = \min\{\alpha, \beta\}$, $R = \max\{\alpha, \beta\}$.

The next theorem is about the existence of at least one pair α, β satisfying the conditions (2.4.32), (2.4.33).

Theorem 2.4.3 *Let* (B1) *and* (B2) *hold and assume that one of the following conditions is satisfied:*

(i) $\limsup_{\lambda\to\infty} \dfrac{\|Sf(\cdot, \gamma(\cdot)\lambda)\|_\infty}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\|Sf(\cdot, \lambda)\|_\infty}{\lambda} < 1$;

(ii) $\limsup_{\lambda\to 0} \dfrac{\|Sf(\cdot, \gamma(\cdot)\lambda)\|_\infty}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\|Sf(\cdot, \lambda)\|_\infty}{\lambda} < 1$.

Then (2.4.25) *has at least one positive solution.*

2.4.2 A multiplicity result

Theorem 2.4.4 *Let* (B1) *and* (B2) *hold. If the condition*

(iii) $\limsup_{\lambda\to\infty} \dfrac{\|Sf(\cdot, \gamma(\cdot)\lambda)\|_\infty}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\|Sf(\cdot, \lambda)\|_\infty}{\lambda} < 1$

holds, then (2.4.25) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $\|u_n\|_\infty \to \infty$ *as* $n \to \infty$.

If the condition

(iv) $\limsup_{\lambda\to 0} \dfrac{\|Sf(\cdot, \gamma(\cdot)\lambda)\|_\infty}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\|Sf(\cdot, \lambda)\|_\infty}{\lambda} < 1$

holds, then (2.4.25) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $\|u_n\|_\infty \to 0$ *as* $n \to \infty$.

2.5 The Neumann-Robin BVP for ϕ-Laplace equations

The aim of this section is to present some new results regarding the existence, localization and multiplicity of positive solutions to the following problem

$$\begin{cases} (\phi(u'))' + f(t,u) = 0, \quad 0 < t < 1 \\ u(0) - au'(0) = 0 \\ u'(1) = 0, \end{cases} \tag{2.5.34}$$

where $a > 0$, ϕ is a homeomorphism from $\mathbb{R}$ to $(-b, b)$ and $0 < b \leq \infty$.

According to the related literature C. Bereanu and J. Mawhin [9], C. Bereanu, P. Jebelean and J. Mawhin [11], A. Cabada and R. L. Pouso [18], S.-S. Chen and Z.-H. Ma [20], there are two remarkable models in this context:

(1) The p-Laplacian operator, where $b = \infty$ and

$$\phi(u) = |u|^{p-2}u, \quad \text{with} \quad p > 1. \tag{2.5.35}$$

(2) The curvature operator, where $b < \infty$ and

$$\phi(u) = \frac{u}{\sqrt{1+u^2}}. \tag{2.5.36}$$

2.5.1 Existence and localization results

We make the following assumptions: $\phi : \mathbb{R} \to (-b, b)$, $0 < b \leq \infty$ is an increasing homeomorphism such that $\phi(0) = 0$ and $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $f(t,x) < b$.

First we obtain the equivalent integral equation to the problem (2.5.34)

$$u(t) = a\phi^{-1}\left(\int_0^1 f(s,u(s))\,ds\right) + \int_0^t \phi^{-1}\left(\int_\tau^1 f(s,u(s))\,ds\right)d\tau. \tag{2.5.37}$$

Next, assuming in addition that $f(t,x) < b$ for all $t \in [0,1]$ and $x \in \mathbb{R}_+$, we may define the integral operator $N : C([0,1];\mathbb{R}_+) \to C([0,1];\mathbb{R}_+)$ by

$$N(u)(t) = a\phi^{-1}\left(\int_0^1 f(s,u(s))\,ds\right) + \int_0^t \phi^{-1}\left(\int_\tau^1 f(s,u(s))\,ds\right)d\tau, \tag{2.5.38}$$

and thus, finding positive solutions to (2.5.34) is equivalent to the fixed point problem for the operator N on $C([0,1];\mathbb{R}_+)$. Note that by standard arguments, N is completely continuous. Let $\|.\|_\infty$ denotes the max norm on $C[0,1]$.

In order to apply Krasnosel'skiĭ's fixed point theorem in cones we need a weak Harnack type inequality for the differential operator $Lu := -(\phi(u'))'$ subjected to the boundary conditions.

Lemma 2.5.1 *For each $d \in (0,1)$, and any $u \in C^1[0,1] \cap C([0,1];\mathbb{R}_+)$ with $u(0) - au'(0) = u'(1) = 0$, $\phi \circ u' \in W^{1,1}(0,1)$ and $(\phi(u'))' \leq 0$ a.e. on $[0,1]$, one has*

$$u(t) \geq \gamma(t)\|u\|_\infty, \quad \text{for all } t \in [0,1], \tag{2.5.39}$$

where

$$\gamma(t) = \begin{cases} \dfrac{a+d}{a+1}, & \text{for } t \in [d,1] \\ 0, & \text{for } t \in [0,d). \end{cases}$$

For our first result we make the following assumptions:

(C1) $\phi : \mathbb{R} \to (-b,b)$, $0 < b \le \infty$ is an increasing homeomorphism such that $\phi(0) = 0$;

(C2) $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $f(t,.)$ is nondecreasing on $\mathbb{R}_+$ for each $t \in [0,1]$ and $f(t,x) < b$ for all $t \in [0,1]$ and $x \in \mathbb{R}_+$.

Theorem 2.5.2 *Let* (C1) *and* (C2) *hold and assume that there exist $\alpha, \beta > 0$ with $\alpha \neq \beta$ such that*

$$\Phi(\alpha) := a\phi^{-1}\left(\int_0^1 f(s,\gamma(s)\alpha)\,ds\right) + \int_0^1 \phi^{-1}\left(\int_\tau^1 f(s,\gamma(s)\alpha)\,ds\right)d\tau > \alpha, \tag{2.5.40}$$

$$\Psi(\beta) := a\phi^{-1}\left(\int_0^1 f(s,\beta)\,ds\right) + \int_0^1 \phi^{-1}\left(\int_\tau^1 f(s,\beta)\,ds\right)d\tau < \beta. \tag{2.5.41}$$

Then (2.5.34) *has at least one positive solution u with $r \le \|u\|_\infty \le R$, where $r = \min\{\alpha,\beta\}$, $R = \max\{\alpha,\beta\}$.*

Theorem 2.5.3 *Let* (C1) *and* (C2) *hold and assume that one of the following conditions is satisfied:*

(i) $\limsup_{\lambda\to\infty} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\Psi(\lambda)}{\lambda} < 1$;

(ii) $\limsup_{\lambda\to 0} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\Psi(\lambda)}{\lambda} < 1$.

Then (2.5.34) *has at least one positive solution.*

2.5.2 A multiplicity result

The next theorem guarantees the existence of a sequence of positive solutions of the problem (2.5.34).

Theorem 2.5.4 *Let* (C1) *and* (C2) *hold. If the condition*

(iii) $\limsup_{\lambda\to\infty} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\Psi(\lambda)}{\lambda} < 1$

holds, then (2.5.34) *has a sequence of positive solutions $(u_n)_{n\ge1}$ such that $\|u_n\|_\infty \to \infty$ as $n \to \infty$.*

If the condition

(iv) $\limsup_{\lambda\to 0} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\Psi(\lambda)}{\lambda} < 1$

holds, then (2.5.34) has a sequence of positive solutions $(u_n)_{n\ge1}$ such that $\|u_n\|_\infty \to 0$ as $n \to \infty$.

2.5.3 Some particular cases

In this subsection, we shall take into consideration some particular cases of the problem (2.5.34).

First, we consider the remarkable model with the p-Laplacian operator (2.5.35). Then problem (2.5.34) becomes

$$\begin{cases} \left(|u'|^{p-2}\,u'\right)' + f\,(t,u) = 0, \quad 0 < t < 1 \\ u(0) - au'\,(0) = 0 \\ u'\,(1) = 0. \end{cases} \tag{2.5.42}$$

Now, we shall take into consideration the model with the curvature operator (2.5.36). Then problem (2.5.34) becomes

$$\begin{cases} \left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' + f\,(t,u) = 0, \quad 0 < t < 1 \\ u(0) - au'\,(0) = 0 \\ u'\,(1) = 0. \end{cases} \tag{2.5.43}$$

2.5.4 Examples

Example 2.5.5 *(on (ii), case $b < \infty$)* We consider the problem (2.5.43) where

$$f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+, \quad f(t,x) = f(x) = \frac{x}{x+1}. \tag{2.5.44}$$

In this case $b = 1$ and one can easily check that the condition (C2), particularly, the inequality $f\,(t,x) < 1$ holds. Direct computation shows that

$$\Phi(\lambda) = A\left(\frac{a+d}{\sqrt{1-A^2}} + \frac{1-d}{1+\sqrt{1-A^2}}\right), \quad \Psi\,(\lambda) = B\left(\frac{a}{\sqrt{1-B^2}} + \frac{1}{1+\sqrt{1-B^2}}\right),$$

where

$$A = \frac{\lambda(a+d)(1-d)}{\lambda(a+d)+(a+1)}, \quad \text{and} \quad B = \frac{\lambda}{\lambda+1}. \tag{2.5.45}$$

Now it is easy to see that

$$\lim_{\lambda\to 0} \frac{\Phi(\lambda)}{\lambda} = \frac{(a+d)(1-d)(2a+d+1)}{2(a+1)} \quad \text{and} \quad \lim_{\lambda\to\infty} \frac{\Psi(\lambda)}{\lambda} = 0. \tag{2.5.46}$$

Hence the condition (ii) from Theorem 2.5.3 is satisfied if

$$C := \frac{2(a+1)}{(a+d)(1-d)(2a+d+1)} < 1,$$

which holds for sufficiently large a. For example we can choose $a = 7$ and $d = 0.5$.

Example 2.5.6 *(on (iii), case $b = \infty$)* If in (2.5.34) we let $\phi(u) = u$ then the conditions (2.5.40) and (2.5.41) become

$$\Phi(\lambda) = f\left(\frac{a+d}{a+1}\lambda\right)\left(\frac{(1-c)(2a+d+1)}{2}\right), \quad \Psi(\lambda) = f(\lambda)\left(\frac{2a+1}{2}\right).$$

Consider

$$f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+, \quad f(t,x) = f(x) = mx + nx\sin(p\ln(x+1)).$$

In this case $b = \infty$ and the condition (C2) holds if

$$m \geq n(p+1). \tag{2.5.47}$$

Now it is easy to see that

$$\limsup_{\lambda\to\infty} \frac{\Phi(\lambda)}{\lambda} = (m+n)\frac{(a+d)(1-d)(2a+d+1)}{2(a+1)}$$

and

$$\liminf_{\lambda\to\infty} \frac{\Psi(\lambda)}{\lambda} = (m-n)\frac{2a+1}{2}.$$

Hence the condition (iii) from Theorem 2.5.4 is satisfied if

$$m+n > A \quad \text{and} \quad m-n < B, \tag{2.5.48}$$

where

$$A = \frac{2(a+1)}{(a+d)(1-d)(2a+d+1)} \quad \text{and} \quad B = \frac{2}{2a+1}.$$

For example, conditions (2.5.47) and (2.5.48) hold for

$$a = 2.5, \quad d = 0.3 \quad m = 0.46, \quad n = 0.15, \quad p = 2.$$

Example 2.5.7 *(on (iv), case $b < \infty$)* Consider

$$\phi(u) = \frac{u}{\sqrt{1+u^2}}$$

and the function $f(t,x) = f(x)$ which is defined on a small interval $(0,\epsilon)$ by

$$f(x) = mx + nx\sin\left(p\ln\frac{1}{x}\right).$$

Here $\epsilon > 0$ is chosen such that $f(x) < 1$ on $(0,\epsilon)$. The function is increasing on $(0,\epsilon)$ if

$$m \geq n(p+1). \tag{2.5.49}$$

Here

$$\Phi(\lambda) = (a+d)\phi^{-1}\left((1-d)f\left(\frac{a+d}{a+1}\lambda\right)\right) + \int_d^1 \phi^{-1}\left((1-\tau)f\left(\frac{a+d}{a+1}\lambda\right)\right)d\tau.$$

Since

$$\int_d^1 \phi^{-1}\left((1-\tau)f\left(\frac{a+d}{a+1}\lambda\right)\right)d\tau \geq 0,$$

a sufficient condition for $\Phi(\lambda) > \lambda$ to hold is

$$\phi^{-1}\left((1-d)f\left(\frac{a+d}{a+1}\lambda\right)\right) > \frac{\lambda}{a+d},$$

or equivalently

$$(1-d)f\left(\frac{a+d}{a+1}\lambda\right) > \phi\left(\frac{\lambda}{a+d}\right).$$

This gives the condition

$$\frac{f\left(\frac{a+d}{a+1}\lambda\right)}{\frac{a+d}{a+1}\lambda} > \frac{a+1}{(a+d)^2(1-d)\sqrt{1+\left(\frac{\lambda}{a+d}\right)^2}}.$$

Letting $\lambda \to 0$ yields

$$m+n > \frac{a+1}{(a+d)^2(1-d)}.$$

Also

$$\Psi(\lambda) = a\phi^{-1}\left(f(\lambda)\right) + \int_0^1 \phi^{-1}\left((1-\tau)f\left(\lambda\right)\right)\,d\tau,$$

and since

$$\int_0^1 \phi^{-1}\left((1-\tau)f\left(\lambda\right)\right)\,d\tau \le \phi^{-1}\left(f\left(\lambda\right)\right),$$

a sufficient condition for $\Psi(\lambda) < \lambda$ to hold is

$$\phi^{-1}\left(f\left(\lambda\right)\right) < \frac{\lambda}{a+1},$$

or equivalently

$$f\left(\lambda\right) < \phi\left(\frac{\lambda}{a+1}\right).$$

This gives the condition

$$\frac{f\left(\lambda\right)}{\lambda} < \frac{1}{(a+1)\sqrt{1+\left(\frac{\lambda}{a+1}\right)^2}},$$

which letting $\lambda \to 0$ yields

$$m-n < \frac{1}{a+1}.$$

Hence the condition (iv) from Theorem 2.5.4 is satisfied if

$$m+n > A \quad\text{and}\quad m-n < B, \tag{2.5.50}$$

where

$$A = \frac{a+1}{(a+d)^2(1-d)} \quad\text{and}\quad B = \frac{1}{a+1},$$

For example conditions (2.5.49) and (2.5.50) hold for

$$a = 2.5,\quad d = 0.1,\quad m = 0.43,\quad n = 0.17,\quad p = 1.5.$$

Example 2.5.8 *(on (iv), case $b = \infty$)* We consider $\phi(u) = u$ and the function

$$f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+, \quad f(t,x) = f(x) = mx + nx \sin\left(p \ln \frac{1}{x}\right),$$

for $x > 0$ and $f(0) = 0$. In this case $b = \infty$ and the condition (C2) holds if

$$m \geq n(p+1). \tag{2.5.51}$$

Now it is easy to see that

$$\limsup_{\lambda \to 0} \frac{\Phi(\lambda)}{\lambda} = (m+n) \frac{(a+c)(1-c)(2a+c+1)}{2(a+1)}$$

and

$$\liminf_{\lambda \to 0} \frac{\Psi(\lambda)}{\lambda} = (m-n) \frac{2a+1}{2}.$$

Hence the condition (iv) from Theorem 2.5.4 is satisfied if

$$m+n > A \quad \text{and} \quad m-n < B, \tag{2.5.52}$$

where

$$A = \frac{2(a+1)}{(a+c)(1-c)(2a+c+1)} \quad \text{and} \quad B = \frac{2}{2a+1}.$$

For example, conditions (2.5.51) and (2.5.52) hold for

$$a = 2, \quad d = 0.2, \quad m = 0.54, \quad n = 0.16, \quad p = 2.$$

Chapter 3

Positive solutions for some classes of nonlinear systems

3.1 Overview

Having in mind the problems and techniques that have been considered in Chapter 2, in this chapter we extend the results from the equations to the general case of systems. The approach is based on the vector version of Krasnosel'skiĭ's theorem given in R. Precup [78].

3.2 First order differential systems with nonlocal conditions

In this section we consider the following first order differential system with nonlocal boundary conditions given by linear functionals:

$$\begin{cases} u_1' = f_1(t, u_1, u_2) \\ u_2' = f_2(t, u_1, u_2) \\ u_1(0) - a_1 u_1(1) = g_1[u_1] \\ u_2(0) - a_2 u_2(1) = g_2[u_2] \end{cases} \tag{3.2.1}$$

where $f_1, f_2 \in C([0,1] \times \mathbb{R}_+^2; \mathbb{R}_+)$; $g_1, g_2 : C[0,1] \to \mathbb{R}$ are two linear functionals given by

$$g_i[u] = \int_0^1 u(s)\, d\gamma_i(s), \tag{3.2.2}$$

with $g_i[1] < 1$; $\gamma_i \in C^1[0,1]$ increasing and $0 < a_i < 1 - g_i[1]$ $(i = 1, 2)$.

We seek nonnegative solutions (u_1, u_2), $u_1 \geq 0$, $u_2 \geq 0$ on $[0,1]$. Based on (2.2.5), the problem of nonnegative solutions of (3.2.1) is equivalent to the integral system:

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 G_1(t,s) f_1(s, u_1(s), u_2(s))\, ds \\ u_2(t) = \displaystyle\int_0^1 G_2(t,s) f_2(s, u_1(s), u_2(s))\, ds, \end{cases} \tag{3.2.3}$$

where $G_1(t,s)$ and $G_2(t,s)$ are the Green's functions corresponding to the two nonlocal conditions,

$$G_i(t,s) = \begin{cases} c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1 & \text{for } \ 0 \le s \le t \le 1 \\ c_i[\gamma_i(1) - \gamma_i(s) + a_i] & \text{for } \ 0 \le t < s \le 1, \end{cases}$$

where

$$c_i = \frac{1}{1 - g_i[1] - a_i} \quad (i = 1, 2).$$

The following properties are essential for the applicability of Krasnosel'skiĭ's technique:
1) $G_i(t,s) \le H_i(s)$, for all $t, s \in [0,1]$, where

$$H_i(s) = c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1 \ (i = 1, 2)$$

2) $\delta_i H_i(s) \le G_i(t,s)$ for all $t, s \in [0,1]$, where

$$\delta_i = \min_{s \in [0,1]} \frac{c_i[\gamma_i(1) - \gamma_i(s) + a_i]}{c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1} \ (i = 1, 2).$$

Notice that $\delta_i > 0$ and $\delta_i < 1$. Now let $N : C([0,1]; \mathbb{R}^2_+) \to C([0,1]; \mathbb{R}^2_+)$, $N = (N_1, N_2)$ be defined by

$$N_i(u_1, u_2)(t) = \int_0^1 G_i(t,s) f_i(s, u_1(s), u_2(s))\, ds \ (i = 1, 2).$$

The above properties of the Green's functions imply that for any $t, t^* \in [0,1]$, one has:

$$N_i(u_1, u_2)(t) \ge \delta_i N_i(u_1, u_2)(t^*).$$

This yields the estimation from below

$$N_i(u_1, u_2)(t) \ge \delta_i \|N_i(u_1, u_2)\|_\infty \quad \text{for all} \quad t \in [0,1] \quad (i = 1, 2) \tag{3.2.4}$$

and any nonnegative functions $u_1, u_2 \in C[0,1]$.

Based on these estimations we define the cones

$$K_i = \{u_i \in C[0,1] : u_i(t) \ge \delta_i \|u_i\|_\infty, \text{for all } t \in [0,1]\} \ (i = 1, 2), \tag{3.2.5}$$

and the product cone $K := K_1 \times K_2$ in $C([0,1]; \mathbb{R}^2)$. Due to (3.2.4) we have the invariance property $N(K) \subset K$.
Therefore, the problem of nonnegative solutions of (3.2.1) is equivalent to the fixed point problem $u = Nu$, $u \in K$, for the self-mapping N of K. Note that the continuity of f_1, f_2 implies the complete continuity of N by standard arguments based on Ascoli-Arzela's theorem.

Notice that (3.2.4) represents a weak Harnack type inequality for the nonnegative super solutions of the problem (3.2.1).

3.2.1 Existence and localization results

Theorem 3.2.1 *Assume that there exist α_i, $\beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, such that*

$$\begin{aligned} A_1\lambda_1 > \alpha_1, \quad B_1\Lambda_1 < \beta_1, \\ A_2\lambda_2 > \alpha_2, \quad B_2\Lambda_2 < \beta_2, \end{aligned} \tag{3.2.6}$$

where

$$\begin{aligned} A_i &= \int_0^1 G_i(t^*, s)\, ds, \ \textit{for a chosen point} \ \ t^* \in [0,1], \\ B_i &= \max_{0\le t\le 1} \int_0^1 G_i(t,s)\, ds, \\ \lambda_1 &= \min\{f_1(t,u_1,u_2) : 0 \le t \le 1, \ \delta_1\alpha_1 \le u_1 \le \alpha_1, \ \delta_2 r_2 \le u_2 \le R_2\}, \\ \lambda_2 &= \min\{f_2(t,u_1,u_2) : 0 \le t \le 1, \ \delta_1 r_1 \le u_1 \le R_1, \ \delta_2\alpha_2 \le u_2 \le \alpha_2\}, \\ \Lambda_1 &= \max\{f_1(t,u_1,u_2) : 0 \le t \le 1, \ \delta_1\beta_1 \le u_1 \le \beta_1, \ \delta_2 r_2 \le u_2 \le R_2\}, \\ \Lambda_2 &= \max\{f_2(t,u_1,u_2) : 0 \le t \le 1, \ \delta_1 r_1 \le u_1 \le R_1, \ \delta_2\beta_2 \le u_2 \le \beta_2\}, \end{aligned}$$

and $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ $(i = 1, 2)$. Then (3.2.1) has at least one positive solution $u = (u_1, u_2)$ with $r_i \le \|u_i\|_\infty \le R_i$ $(i = 1, 2)$.

In particular, if f_1 and f_2 do not depend on t, i.e., $f_1 = f_1(u_1, u_2)$ and $f_2 = f_2(u_1, u_2)$, and f_1, f_2 have some monotonicity properties in u_1 and u_2, then we can specify the numbers λ_1, λ_2, Λ_1, Λ_2 and the conditions (3.2.6) are expressed by values of f_1, f_2 on only four points. There are sixteen possible cases.

3.2.2 A multiplicity result

Theorem 3.2.1 guarantees the existence of solutions in an annular set. Clearly, if the assumptions of Theorem 3.2.1 are satisfied for several disjoint annular sets, then finitely or infinitely many solutions are obtained (see R. Precup [79]).

Theorem 3.2.2 (A) *Let $(r^j)_{1\le j\le k}$, $(R^j)_{1\le j\le k}$ $(k \le \infty)$ be increasing finite or infinite sequence in $\mathbb{R}^2_+$, with $0 \le r^j < R^j$ and $R^j < r^{j+1}$ for all j. If the assumptions of Theorem 3.2.1 are satisfied for each couple (r^j, R^j), then (3.2.1) has k (respectively, when $k = \infty$, an infinite sequence of) distinct positive solutions.*

(B) *Let $(r^j)_{j\ge 1}$, $(R^j)_{j\ge 1}$ be decreasing infinite sequence with $0 < r^j < R^j$ and $R^j < r^{j+1}$ for all j. If the assumptions of Theorem 3.2.1 are satisfied for each couple (r^j, R^j), then (3.2.1) has an infinite sequence of distinct positive solutions.*

3.2.3 Examples

Example 3.2.3 Let

$$f_i(u_1, u_2) = \frac{1}{15}\sqrt{u_1 + u_2 + 1}, \quad (i = 1, 2),$$

and

$$\gamma_1(t) = \frac{1}{2}t, \quad \gamma_2(t) = \frac{3}{4}t, \quad a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{8}.$$

Then (3.2.1) becomes

$$\begin{cases} u_1' = \dfrac{1}{15}\sqrt{u_1 + u_2 + 1} \\ u_2' = \dfrac{1}{15}\sqrt{u_1 + u_2 + 1} \\ u_1(0) - \dfrac{1}{4}u_1(1) = \dfrac{1}{2}\displaystyle\int_0^1 u_1(t)\,dt \\ u_2(0) - \dfrac{1}{8}u_2(1) = \dfrac{3}{4}\displaystyle\int_0^1 u_2(t)\,dt, \end{cases} \tag{3.2.7}$$

or equivalently

$$\begin{cases} u_1(t) = \dfrac{1}{15}\displaystyle\int_0^1 G_1(t,s)\sqrt{u_1(s) + u_2(s) + 1}\,ds \\ u_2(t) = \dfrac{1}{15}\displaystyle\int_0^1 G_2(t,s)\sqrt{u_1(s) + u_2(s) + 1}\,ds, \end{cases} \tag{3.2.8}$$

where $G_1(t,s)$ and $G_2(t,s)$ are the Green's functions

$$G_1(t,s) = \begin{cases} 6 - 4s & \text{for} \ \ 0 \le s \le t \le 1 \\ 5 - 4s & \text{for} \ \ 0 \le t < s \le 1 \end{cases} \quad , \quad G_2(t,s) = \begin{cases} 10 - 8s & \text{for} \ \ 0 \le s \le t \le 1 \\ 9 - 8s & \text{for} \ \ 0 \le t < s \le 1. \end{cases}$$

In this case, the constants $\delta_1, \delta_2 > 0$ are $\delta_1 = \delta_2 = \dfrac{1}{2} =: \delta$. Now we have to determine A_i and B_i for $i \in \{1, 2\}$. We have

$$A_1 = \int_0^1 G_1(t^*, s)\,ds = \int_0^{t^*} (6 - 4s)\,ds + \int_{t^*}^1 (5 - 4s)\,ds = t^* + 3.$$

If we choose $t^* = 0$, then $A_1 = 3$. Also

$$A_2 = \int_0^1 G_2(t^*, s)\,ds = \int_0^{t^*} (10 - 8s)\,ds + \int_{t^*}^1 (9 - 8s)\,ds = t^* + 5,$$

and if we choose $t^* = 0$, then $A_2 = 5$. In addition

$$B_1 = \max_{0 \le t \le 1} \int_0^1 G_1(t,s)\,ds = 4 \ \ \text{and} \ \ B_2 = \max_{0 \le t \le 1} \int_0^1 G_2(t,s)\,ds = 6.$$

In this case $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ are both nondecreasing in u_1 and u_2 for $u_1, u_2 \in \mathbb{R}_+$. We choose $\alpha_1 = \alpha_2 =: \alpha$, $\beta_1 = \beta_2 =: \beta$, with $\alpha < \beta$, then $r_1 = r_2 = \alpha$, $R_1 = R_2 = \beta$ and $\lambda_1 = f_1(\delta\alpha, \delta\alpha)$, $\Lambda_1 = f_1(\beta, \beta)$, $\lambda_2 = f_2(\delta\alpha, \delta\alpha)$, $\Lambda_2 = f_2(\beta, \beta)$. The values of α and β will be precised in what follows. Since

$$\lim_{x \to \infty} \frac{f_i(x,x)}{x} = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{f_i(x,x)}{x} = \infty,$$

we may find α small enough and β large enough such that the conditions

$$\frac{f_i(\delta\alpha, \delta\alpha)}{\delta\alpha} > \frac{1}{\delta A_i}, \quad \frac{f_i(\beta, \beta)}{\beta} < \frac{1}{B_i} \quad (i = 1, 2)$$

are satisfied. For instance, we can choose $\alpha = 0,2$ and $\beta = 0,7$.

Hence the following result holds.

Theorem 3.2.4 *Under the above assumptions, the system (3.2.7) has at least one positive solution* $u = (u_1, u_2)$ *with* $0,2 < \|u_i\|_\infty < 0,7 \quad (i = 1, 2)$.

Example 3.2.5 Let

$$f_1(u_1, u_2) = \frac{1}{15}\sqrt{u_1 + u_2 + 1}, \quad f_2(u_1, u_2) = \frac{1}{(2 + u_1^2)(4 + u_2^2)},$$

and

$$\gamma_1(t) = \frac{1}{2}t, \quad \gamma_2(t) = \frac{3}{4}t, \quad a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{8}.$$

Then (3.2.1) becomes

$$\begin{cases} u_1' = \dfrac{1}{15}\sqrt{u_1 + u_2 + 1} \\ u_2' = \dfrac{1}{(2 + u_1^2)(4 + u_2^2)} \\ u_1(0) - \dfrac{1}{4}u_1(1) = \dfrac{1}{2}\displaystyle\int_0^1 u_1(t)\,dt \\ u_2(0) - \dfrac{1}{8}u_2(1) = \dfrac{3}{4}\displaystyle\int_0^1 u_2(t)\,dt, \end{cases} \tag{3.2.9}$$

or equivalently

$$\begin{cases} u_1(t) = \dfrac{1}{15}\displaystyle\int_0^1 G_1(t,s)\sqrt{u_1(s) + u_2(s) + 1}\,ds \\ u_2(t) = \displaystyle\int_0^1 G_2(t,s)\frac{1}{(2 + u_1(s)^2)(4 + u_2(s)^2)}\,ds. \end{cases} \tag{3.2.10}$$

The Green's functions $G_i(t,s)$ and the values of δ_i, A_i, B_i $(i = 1, 2)$ are the same from the Example 3.2.3. In this case $f_1(u_1, u_2)$ is nondecreasing in u_1 and u_2, while $f_2(u_1, u_2)$ is nonincreasing in u_1 and u_2, for $u_1, u_2 \in \mathbb{R}_+$. We choose $\alpha_1 = \alpha_2 =: \alpha$, $\beta_1 = \beta_2 =: \beta$, with $\alpha < \beta$. Then $r_1 = r_2 = \alpha$, $R_1 = R_2 = \beta$ and $\lambda_1 = f_1(\delta\alpha, \delta\alpha)$, $\Lambda_1 = f_1(\beta, \beta)$, $\lambda_2 = f_2(\beta, \alpha)$, $\Lambda_2 = f_2(\delta\alpha, \delta\beta)$, where α and β will be precised in what follows. Since

$$\lim_{y\to\infty} \frac{f_1(y,y)}{y} = 0 \quad \text{and} \quad \lim_{y\to\infty} \frac{f_2(x,y)}{y} = 0,$$

uniformly in $x \geq 0$, we may find $\beta > 0$ large enough such that

$$\frac{f_1(\beta, \beta)}{\beta} < \frac{1}{B_1}, \quad \frac{f_2(\delta\alpha, \delta\beta)}{\delta\beta} < \frac{1}{\delta B_2}.$$

And since

$$\lim_{x\to 0}\frac{f_1(x,x)}{x}=\infty \quad \text{and} \quad \lim_{x\to 0}\frac{f_2(y,x)}{x}=0,$$

with β fixed as above, we choose α small enough such that

$$\frac{f_1(\delta\alpha,\delta\alpha)}{\delta\alpha}>\frac{1}{\delta A_1},\quad \frac{f_2(\beta,\alpha)}{\alpha}>\frac{1}{A_2}.$$

For example, we can choose $\beta = 0,9$ and $\alpha = 0,2$.

Hence the following result holds.

Theorem 3.2.6 *Under the above assumptions, the system (3.2.9) has at least one positive solution* $u=(u_1,u_2)$ *with* $0,2<\|u_i\|_\infty<0,9 \quad (i=1,2)$.

3.3 The Dirichlet-Neumann BVP for ϕ-Laplace systems

The problem (2.3.10) can be considered as a particular case, as $n=1$, of the corresponding problem for an n-dimensional system,

$$\begin{cases} (\phi_i\left(u_i'\right))'+f_i\left(t,u_1,u_2,...,u_n\right)=0, & 0<t<1 \\ u_i'\left(0\right)=u_i\left(1\right)=0 \quad \left(i=1,2,...,n\right). \end{cases} \tag{3.3.11}$$

For any index $i\in\{1,2,...,n\}$, we shall say that the homeomorphism $\phi_i:(-a_i,a_i)\to(-b_i,b_i)$ satisfies (A1) if ϕ_i is increasing and $\phi_i(0)=0$, and that the continuous function $f_i:[0,1]\times\mathbb{R}_+^n\to\mathbb{R}_+$ satisfies (A2) if for each $t\in[0,1]$, $f_i(t,x_1,...,x_n)$ is nondecreasing on $\mathbb{R}_+$ with respect to any variable x_j, $j=1,2,...,n$, and $f_i\left(t,x\right)<b_i$ for all $t\in[0,1]$ and $x\in\mathbb{R}_+^n$.

Under these assumptions problem (3.3.11) is equivalent to the integral system

$$u_i(t)=-\int_t^1\phi_i^{-1}\left[-\int_0^\tau f_i(s,u(s))\,ds\right]d\tau \qquad \left(i=1,2,...,n\right),$$

where $u=\left(u_1,u_2,...,u_n\right)$.

According to Lemma 2.3.1, for each i and any constant $c_i\in(0,1)$, a weak Harnack type inequality holds for the differential operator $L_iv:=-(\phi_i(v'))'$ and the boundary conditions $v'\left(0\right)=v\left(1\right)=0$. Based on this we define the cones

$$K_i=\{u_i\in C([0,1];\mathbb{R}_+):u_i(t)\geq(1-c_i)\|u_i\|_\infty,\quad \text{for all } t\in[0,c_i]\}, \tag{3.3.12}$$

for $i=1,2,...,n$, and take the product cone

$$K:=K_1\times K_2\times...\times K_n$$

in $C([0,1];\mathbb{R}^n)$.

Let $N : C([0,1];\mathbb{R}^n_+) \to C([0,1];\mathbb{R}^n_+)$, $N = (N_1, N_2, ..., N_n)$ be defined by

$$N_i(u)(t) = -\int_t^1 \phi_i^{-1}\left[-\int_0^\tau f_i(s,u(s))\,ds\right] d\tau \qquad (i = 1,2,...,n).$$

If $u_j \in K_j$ for each j, then $f_i(s,u(s)) \geq 0$ and from Lemma 2.3.1, one has $N_i(u) \in K_i$. Thus the cone K is invariant by N. Moreover, the operator N is completely continuous since, by standard arguments, the components N_i are completely continuous.

3.3.1 Existence and localization results

The following result is a generalization of Theorem 2.3.2.

Theorem 3.3.1 *Let ϕ_i, f_i satisfy* (A1) *and* (A2) *for $i = 1,2,...,n$. Assume that there exist c_i, α_i, $\beta_i > 0$ with $c_i < 1$ and $\alpha_i \neq \beta_i$ such that*

$$\Phi_i(\alpha) := -\int_0^{c_i} \phi_i^{-1}\left(-\int_0^\tau f_i(s,(1-c_1)\alpha_1,...,(1-c_n)\,\alpha_n)\,ds\right) d\tau > \alpha_i, \tag{3.3.13}$$

$$\Psi_i(\beta) := -\int_0^1 \phi_i^{-1}\left(-\int_0^\tau f_i(s,\beta)\,ds\right) d\tau < \beta_i, \tag{3.3.14}$$

for $i = 1,2,...,n$, where $\alpha = (\alpha_1,\alpha_2,...,\alpha_n)$ and $\beta = (\beta_1,\beta_2,...,\beta_n)$. Then (3.3.11) *has at least one positive solution $u = (u_1,u_2,...,u_n)$ with $r_i \leq \|u_i\|_\infty \leq R_i$, where $r_i = \min\{\alpha_i,\beta_i\}$, $R_i = \max\{\alpha_i,\beta_i\}$, $i = 1,2,...,n$.*

We shall say that for a given index i, the condition (i) from Theorem 2.3.3 holds if for every $\lambda_1, \lambda_2, ..., \lambda_{i-1} > 0$,

$$\limsup_{\lambda_i\to\infty} \frac{\Phi_i(\lambda)}{\lambda_i} > 1 \quad \text{and} \quad \liminf_{\lambda_i\to 0} \frac{\Psi_i(\lambda)}{\lambda_i} < 1,$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_n \in (0,\infty)$. Here by λ we mean $(\lambda_1,\lambda_2,...,\lambda_n)$. We shall understand the condition (ii) in a similar manner. Analogously, we say that condition (iii) from Theorem 2.3.4 holds for some index i, if for every $\lambda_1, \lambda_2, ..., \lambda_{i-1} > 0$,

$$\limsup_{\lambda_i\to\infty} \frac{\Phi_i(\lambda)}{\lambda_i} > 1 \quad \text{and} \quad \liminf_{\lambda_i\to\infty} \frac{\Psi_i(\lambda)}{\lambda_i} < 1,$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_n \in (0,\infty)$. The condition (iv) is understood in a similar manner.

3.3.2 A multiplicity result

In this subsection we give the following theorem regarding the existence of a sequence of positive solutions to the problem (3.3.11).

Theorem 3.3.2 *Let ϕ_i, f_i satisfy* (A1) *and* (A2) *for every $i = 1,2,...,n$. Assume that the set of indices $I = \{1,2,...,n\}$ admits the partition $I = I_1 \cup I_2 \cup I_3 \cup I_4$, $I_j \cap I_k = \emptyset$ for $j \neq k$, such* *that condition (i) holds for every $i \in I_1$, condition (ii) holds for every $i \in I_2$, condition (iii) holds for every $i \in I_3$, and condition (iv) holds for every $i \in I_4$. If $I_3 \neq \emptyset$ or $I_4 \neq \emptyset$, then the problem* (3.3.11) *has a sequence of positive solutions.*

3.4 The Dirichlet BVP for ϕ-Laplace systems

In this section we study the following problem for an n-dimensional system,

$$\begin{cases} (\phi_i (u_i'))' + f_i (t, u_1, u_2, ..., u_n) = 0, \quad 0 < t < 1 \\ u_i (0) = u_i (1) = 0 \quad (i = 1, 2, ..., n) . \end{cases} \tag{3.4.15}$$

We shall allow the homeomorphisms ϕ_i have different domains, namely $\phi_i : (-a_i, a_i) \to \mathbb{R}$, $0 < a_i \leq \infty$ and we shall say that (B1) holds if ϕ_i is increasing and $\phi_i(0) = 0$. The continuous function $f_i : [0,1] \times \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies (B2) if for each $t \in [0,1]$, $f_i(t, x_1, ..., x_n)$ is nondecreasing on $\mathbb{R}_+$ with respect to any variable x_j, $j = 1, 2, ..., n$.

Under these assumptions, problem (3.4.15) is equivalent to the integral system

$$u_i(t) = \int_0^t \phi_i^{-1} \left(b_i - \int_0^\tau f_i(s, u(s))\, ds \right) d\tau \qquad (i = 1, 2, ..., n) ,$$

where $u = (u_1, u_2, ..., u_n)$ and $b_i = b_i(f_i(\cdot, u(\cdot)))$.

According to Lemma 2.4.1, for each i a weak Harnack type inequality holds for the differential operator $L_i v := -(\phi_i(v'))'$ and the boundary conditions $v(0) = v(1) = 0$. Based on this we define the cones

$$\begin{aligned} K_i = \{ & u_i \in C([0,1]; \mathbb{R}_+) : u_i(0) = u_i(1) = 0 \text{ and } u_i(t) \geq \gamma_i(t) \|u_i\|_\infty, \\ & \text{for all } t \in [0,1] \}, \end{aligned} \tag{3.4.16}$$

for $i = 1, 2, ..., n$. We note that the functions γ_i are given by Lemma 2.4.1 for possibly different subintervals $[t_0, t_1]$. Now we consider the product cone

$$K := K_1 \times K_2 \times ... \times K_n$$

in $C([0,1]; \mathbb{R}^n)$.

Let $N : C([0,1]; \mathbb{R}_+^n) \to C([0,1]; \mathbb{R}_+^n)$, $N = (N_1, N_2, ..., N_n)$ be defined by

$$N_i(u)(t) = \int_0^t \phi_i^{-1} \left(b_i - \int_0^\tau f_i(s, u(s))\, ds \right) d\tau \qquad (i = 1, 2, ..., n) .$$

If $u_j \in K_j$ for each j, then $f_i(s, u(s)) \geq 0$ and from Lemma 2.4.1, one has $N_i(u) \in K_i$. Thus the cone K is invariant by N.

Existence and localization results

The following result guarantees the existence of positive solutions to the problem (3.4.15) and their component-wise localization.

Theorem 3.4.1 *Let ϕ_i, f_i satisfy* (B1) *and* (B2) *for $i = 1, 2, ..., n$. Assume that there exist α_i, $\beta_i > 0$ with $\alpha_i \neq \beta_i$ such that*

$$\|Sf_i(\cdot, \gamma_1(\cdot)\alpha_1, ..., \gamma_n(\cdot)\alpha_n)\|_\infty > \alpha_i, \tag{3.4.17}$$

$$\|Sf_i(\cdot, \beta)\|_\infty < \beta_i, \tag{3.4.18}$$

for $i = 1, 2, ..., n$, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\beta = (\beta_1, \beta_2, ..., \beta_n)$ and S is given by (2.4.28). Then (3.4.15) *has at least one positive solution $u = (u_1, u_2, ..., u_n)$ with $r_i \leq \|u_i\|_\infty \leq R_i$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$, $i = 1, 2, ..., n$.*

3.5 The Neumann-Robin BVP for ϕ-Laplace systems

This section deals with the following problem for an n-dimensional system

$$\begin{cases} (\phi_i(u_i'))' + f_i(t, u_1, u_2, ..., u_n) = 0, & 0 < t < 1 \\ u_i(0) - a_i u_i'(0) = 0 & \\ u_i'(1) = 0 & (i = 1, 2, ..., n), \end{cases} \tag{3.5.19}$$

where $a_i > 0$. For any index $i \in \{1, 2, ..., n\}$, we shall say that the homeomorphism $\phi_i : \mathbb{R} \to (-b_i, b_i)$ satisfies (C1) if ϕ_i is increasing and $\phi_i(0) = 0$, and that the continuous function $f_i : [0, 1] \times \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies (C2) if for each $t \in [0, 1]$, $f_i(t, x_1, ..., x_n)$ is nondecreasing on $\mathbb{R}_+$ with respect to any variable x_j, $j = 1, 2, ..., n$, and $f_i(t, x) < b_i$ for all $t \in [0, 1]$ and $x \in \mathbb{R}_+^n$.

Under these assumptions problem (3.5.19) is equivalent to the integral system

$$u_i(t) = a_i \phi_i^{-1}\left(\int_0^1 f_i(s, u(s))\, ds\right) + \int_0^t \phi_i^{-1}\left(\int_\tau^1 f_i(s, u(s))\, ds\right) d\tau,$$

for $i = 1, 2, ..., n$ and $u = (u_1, u_2, ..., u_n)$.

According to Lemma 2.5.1, for each i and any constant $d_i \in (0, 1)$, a weak Harnack type inequality holds for the differential operator $L_i v := -(\phi_i(v'))'$ and the boundary conditions $v(0) - av'(0) = v'(1) = 0$. Based on this result we define the cones

$$\begin{aligned} K_i = &\{u_i \in C([0, 1]; \mathbb{R}_+) : u_i(0) - a_i u_i'(0) = u_i'(1) = 0 \text{ and } u_i(t) \geq \gamma_i(t)\|u_i\|_\infty, \\ &\text{for all } t \in [0, 1]\}, \end{aligned}$$

for $i = 1, 2, ..., n$. We note that the functions γ_i are given by Lemma 2.5.1 for possibly different d_i and a_i. Now we consider the product cone $K := K_1 \times K_2 \times ... \times K_n$ in $C([0, 1]; \mathbb{R}^n)$.

Let $N : C([0,1];\mathbb{R}^n_+) \to C([0,1];\mathbb{R}^n_+)$, $N = (N_1, N_2, ..., N_n)$ be defined by

$$N_i(u)(t) = a_i \phi_i^{-1}\left(\int_0^1 f_i(s,u(s))\,ds\right) + \int_0^t \phi_i^{-1}\left(\int_\tau^1 f_i(s,u(s))\,ds\right) d\tau,$$

for $i = 1, 2, ..., n$.

If $u_j \in K_j$ for each j, then $f_i(s,u(s)) \geq 0$ and from Lemma 2.5.1, one has $N_i(u) \in K_i$. Thus the cone K is invariant by N. Moreover, the operator N is completely continuous since, by standard arguments, the components N_i are completely continuous.

Existence and localization results

Theorem 3.5.1 *Let ϕ_i, f_i satisfy* (C1) *and* (C2) *for $i = 1, 2, ..., n$. Assume that there exist a_i, α_i, $\beta_i > 0$ and $\alpha_i \neq \beta_i$ such that*

$$\begin{aligned}\Phi_i(\alpha) := \ & a_i \phi_i^{-1}\left(\int_0^1 f_i(s, \gamma_1(s)\alpha_1, ..., \gamma_n(s)\alpha_n)\,ds\right)\\ &+ \int_0^1 \phi_i^{-1}\left(\int_\tau^1 f_i(s, \gamma_1(s)\alpha_1, ..., \gamma_n(s)\alpha_n)\,ds\right) d\tau > \alpha_i,\end{aligned}$$

$$\Psi_i(\beta) := \ a_i \phi_i^{-1}\left(\int_0^1 f_i(s,\beta)\,ds\right) + \int_0^1 \phi_i^{-1}\left(\int_\tau^1 f_i(s,\beta)\,ds\right) d\tau < \beta_i,$$

for $i = 1, 2, ..., n$, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$. Then (3.5.19) *has at least one positive solution $u = (u_1, u_2, ..., u_n)$ with $r_i \leq \|u_i\|_\infty \leq R_i$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$, $i = 1, 2, ..., n$.*

Chapter 4

Abstract theory

4.1 Overview

In this chapter we present an abstract theory regarding equations and systems. We study existence, localization and multiplicity of positive solutions using Krasnosel'skiĭ's fixed point theorem in cones.

4.2 The case of equations

We consider the abstract problem

$$\begin{cases} Lu = F(u) \\ u \in B, \end{cases} \tag{4.2.1}$$

where $(X, \|\cdot\|)$ is a Banach space, $L : D(L) \subset X \to X$; $F : X \to X$ and $B \subset X$. By a solution of (4.2.1) we mean an element $u \in D(L) \cap B$ for which $Lu = F(u)$. Next we assume that L is invertible, i.e., for every $v \in X$ there is a unique $u \in D(L) \cap B$ with $Lu = v$. Then we write the equivalent equation to the problem (4.2.1),

$$u = L^{-1}F(u), \ u \in X. \tag{4.2.2}$$

We look for solutions u in a cone $K_0 \subset X$. In what follows we shall call such solutions *positive solutions*. To this aim we shall require some additional conditions:

(D1) F is positive and increasing with respect to the ordering induced by K_0, i.e.

$$0 \leq u \leq v \ \text{ implies } \ 0 \leq F(u) \leq F(v); \tag{4.2.3}$$

(D2) L is invertible and

$$0 \leq u \leq v \ \text{ implies } \ 0 \leq L^{-1}u \leq L^{-1}v; \tag{4.2.4}$$

(D3) There exists $\psi \in K_0 - \{0\}$, such that for each $u \in K_0$ we have

$$u \leq \|u\|\psi; \tag{4.2.5}$$

(D4) There exists $\varphi \in K_0 - \{0\}$ with $\|\varphi\| \leq 1$, such that for each $u \geq 0$ satisfying $Lu \geq 0$, we have

$$u \geq \|u\|\varphi. \tag{4.2.6}$$

Note that the symbol $\leq$ from (4.2.3), (4.2.4), (4.2.5) and (4.2.6) is used to denote the ordering induced by the cone K_0, i.e., $u \leq v$ if $v - u \in K_0$.

Based on the estimates from (D3), (D4) we define a smaller cone:

$$K = \{u \in K_0 : u \geq \|u\|\varphi\}. \tag{4.2.7}$$

Let $N : X \to X$ be defined by

$$N(u) = L^{-1}F(u), \tag{4.2.8}$$

and thus finding positive solutions in K_0 to (4.2.1) is equivalent to the fixed point problem in K_0 for the operator N. In what follows the operator N is assumed to be completely continuous.

The following lemma gives the invariance property of N that we need.

Lemma 4.2.1 *Assume that the conditions (D1)-(D4) hold. Then $N(K) \subset K$.*

4.2.1 Existence and localization results

Theorem 4.2.2 *Let the conditions (D1)-(D4) hold and assume that the norm $\|\cdot\|$ is monotone with respect to K_0, i.e, from $0 \leq u \leq v$ one has $\|u\| \leq \|v\|$. Assume, in addition, that there exist α, $\beta > 0$ with $\alpha \neq \beta$, such that*

$$\|N(\alpha\varphi)\| > \alpha, \tag{4.2.9}$$

$$\|N(\beta\psi)\| < \beta. \tag{4.2.10}$$

Then (4.2.1) has at least one positive solution with $r \leq \|u\| \leq R$, where $r = \min\{\alpha, \beta\}$, $R = \max\{\alpha, \beta\}$.

Theorem 4.2.3 *Let* (D1)-(D4) *hold and assume that one of the following conditions is satisfied:*

(i) $\limsup_{\lambda\to\infty} \dfrac{\|N(\lambda\varphi)\|}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\|N(\lambda\psi)\|}{\lambda} < 1$;

(ii) $\limsup_{\lambda\to 0} \dfrac{\|N(\lambda\varphi)\|}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\|N(\lambda\psi)\|}{\lambda} < 1$.

Then (4.2.1) *has at least one positive solution.*

4.2.2 A multiplicity result

Theorem 4.2.4 *Let* (D1)-(D4) *hold. If the condition*

(iii) $\limsup_{\lambda\to\infty} \dfrac{\|N(\lambda\varphi)\|}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\|N(\lambda\psi)\|}{\lambda} < 1$;

holds, then (4.2.1) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $\|u_n\| \to \infty$ *as* $n \to \infty$.

If the condition

(iv) $\limsup_{\lambda\to 0} \dfrac{\|N(\lambda\varphi)\|}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\|N(\lambda\psi)\|}{\lambda} < 1$;

holds, then (4.2.1) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $\|u_n\| \to 0$ *as* $n \to \infty$.

4.3 The case of systems

In this section we study the problem for the n-dimensional system

$$\begin{cases} L_i u_i = F_i(u_1, u_2, ..., u_n) \\ u_i \in B_i \quad (i = 1, 2, ..., n). \end{cases} \tag{4.3.11}$$

We shall assume that for any index $i \in \{1, 2, ..., n\}$, $(X_i, \|\cdot\|_i)$ are Banach spaces, $L_i : D(L_i) \subset X_i \to X_i$, $F_i : X \to X_i$, $B_i \subset X_i$, where $X = X_1 \times X_2 \times ... \times X_n$. Next we assume that L_i are invertible $(i = 1, 2, ..., n)$. Then we write the equivalent system to the problem (4.3.11)

$$u_i = L_i^{-1} F_i(u), \quad u_i \in X_i, \tag{4.3.12}$$

for $i = 1, 2, ..., n$, where $u = (u_1, u_2, ..., u_n)$.

We shall look for solutions $u = (u_1, u_2, ..., u_n)$ with u_i in a cone $K_0^i \subset X_i$. To this aim we shall require some additional conditions:

(D1') F_i are positive and increasing, i.e.

$$0 \leq u_i \leq v_i,\ i = 1, 2, ..., n \ \text{ implies } \ 0 \leq F_j(u) \leq F_j(v),$$

for $j = 1, 2, ..., n$.

(D2') L_i are invertible and

$$0 \leq u_i \leq v_i,\ i = 1, 2, ..., n \ \text{ implies } \ 0 \leq L_i^{-1} u_i \leq L_i^{-1} v_i,$$

for $i = 1, 2, ..., n$.

(D3') There exists $\psi_i \in K_0^i - \{0\}$, such that for each $u_i \in K_0^i$ we have

$$u_i \leq \|u_i\|_i \psi_i,$$

for $i = 1, 2, ..., n$.

(D4') There exists $\varphi_i \in K_0^i - \{0\}$, with $\|\varphi_i\| \leq 1$, such that for each $u_i \geq 0$ satisfying $L_i u_i \geq 0$, we have

$$u_i \geq \|u_i\|_i \varphi_i,$$

for $i = 1, 2, ..., n$.

Based on these conditions we define the cones

$$K_i = \{v \in K_0^i : v \geq \|v\|_i \varphi_i\}. \tag{4.3.13}$$

for $i = 1, 2, ..., n$, and we consider the product cone $K := K_1 \times K_2 \times ... \times K_n$.

Let $N_i : X \to X_i$ be defined by

$$N_i(u) = L_i^{-1} F_i(u), \tag{4.3.14}$$

for $i = 1, 2, ..., n$ and $u = (u_1, u_2, ..., u_n)$. Similarly to Lemma 4.2.1, one can show that the cone K is invariant by N. In what follows, the operators N_i are assumed to be completely continuous, which guarantees that $N = (N_1, N_2, ..., N_n)$ is completely continuous.

Existence and localization results

Theorem 4.3.1 *Let the conditions (D1')-(D4') hold and assume that the norms $\|\cdot\|_i$ are monotone with respect to K_0^i $(i = 1, 2, ..., n)$. Assume, in addition, that there exist $\alpha_i,\ \beta_i > 0$ with $\alpha_i \neq \beta_i$, such that*

$$\|N_i(\alpha_1\varphi_1, \alpha_2\varphi_2, ..., \alpha_n\varphi_n)\|_i > \alpha_i \tag{4.3.15}$$

$$\|N_i(\beta_1\psi_1, \beta_2\psi_2, ..., \beta_n\psi_n)\|_i < \beta_i, \tag{4.3.16}$$

for $i = 1, 2, ..., n$. Then (4.3.11) has at least one solution $u = (u_1, u_2, ..., u_n)$ with $u_i \in K_0^i$ and $r_i \leq \|u_i\|_i \leq R_i$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$, $i = 1, 2, ..., n$.

We shall say that for a given index i, the condition (i) from Theorem 4.2.3 holds if for every $\lambda_1, \lambda_2, ..., \lambda_{i-1} > 0$,

$$\limsup_{\lambda_i \to \infty} \frac{\|N_i(\alpha_1\varphi_1, \alpha_2\varphi_2, ..., \alpha_n\varphi_n)\|_i}{\lambda_i} > 1$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_n \in (0, 1)$, and

$$\liminf_{\lambda_i \to 0} \frac{\|N_i(\beta_1\psi_1, \beta_2\psi_2, ..., \beta_n\psi_n)\|_i}{\lambda_i} < 1,$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_n \in (0, \infty)$.

We shall understand the condition (ii) in a similar manner. Analogously, we say that (iii) from Theorem 4.2.4 holds for some index i, if for every $\lambda_1, \lambda_2, ..., \lambda_{i-1} > 0$,

$$\limsup_{\lambda_i \to \infty} \frac{\|N_i(\alpha_1\varphi_1, \alpha_2\varphi_2, ..., \alpha_n\varphi_n)\|_i}{\lambda_i} > 1$$

and

$$\liminf_{\lambda_i \to \infty} \frac{\|N_i(\beta_1\psi_1, \beta_2\psi_2, ..., \beta_n\psi_n)\|_i}{\lambda_i} < 1,$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_n \in (0, \infty)$. The condition (iv) is understood in a similar manner. Under such type of conditions we may obtain analogous results to Theorems 4.2.3 and 4.2.4, and as consequences, existence and multiplicity results for the system (4.3.11).

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