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Fuzzy Multiple Criteria Decision Making Methods

PhD Thesis Summary

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2. O.I. Ban, A.I. Ban, **D.A. Tuş**, *Importance-performance analysis by fuzzy c-means algorithm*, Expert Systems with Applications, 50 (2016), pp. 9-16, DOI: 10.1016/j.eswa.2015.12.023, (<http://www.sciencedirect.com/science/article/pii/S0957417415008258>), IF 2014: 2.240.

3. O.I. Ban, I.Gh. Tara, V. Bogdan, **D.A. Tuş**, G. Bologa, *Evaluation of hotel quality attribute importance through fuzzy correlation coefficient*, acceptat la Technological and Economic Development of Economy (2016), DOI: 10.3846/20294913.2016.1144657, (<http://www.tandfonline.com/doi/abs/10.3846/20294913.2016.1144657>), IF 2014: 1.563.

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5. **D.A. Tuş**, *An interval fuzzy multicriteria decision making method based on the expected value*, Studia Univ. Babeş-Bolyai, Informatica, LIX, 1 (2014), pp. 216-227, (<http://www.cs.ubbcluj.ro/~studia-i/2014-macs/17Tuse.pdf>), MaCS 2014: 10th Joint International Conference on Mathematics and Computer Science, 21-25 May 2014, Cluj-Napoca, Romania.

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7. A.I. Ban, **D.A. Tuş**, *Trapezoidal/triangular intuitionistic fuzzy numbers versus interval-valued trapezoidal/triangular fuzzy numbers and applications to multicriteria decision making methods*, Notes on Intuitionistic Fuzzy Sets, 20, 2 (2014), pp. 43-51, (http://www.ifigenia.org/wiki/Issue:Trapezoidal/triangular_intuitionistic_fuzzy_numbers_versus_interval-valued_trapezoidal/triangular_fuzzy_numbers_and_applications_to_multicriteria_decision_making_methods), 18th International Conference on Intuitionistic Fuzzy Sets, 10-11 May 2014, Sofia, Bulgaria.

List of keywords

Fuzzy sets, triangular norm, fuzzy number, trapezoidal fuzzy number, triangular fuzzy number, interval of trapezoidal fuzzy numbers, trapezoidal intuitionistic fuzzy number, interval-valued fuzzy number, ambiguity, value, cardinal, core, support, expected interval, expected value, index, score, accuracy, defuzzification, aggregation operator, ranking method, linguistic variables, decision, fuzzy MCDM, multicriteria analysis, criterion, alternative, decision matrix, weight, fuzzy weight, performance, correlation coefficient, importance-performance analysis, IPA, fuzzy group, fuzzy classification, fuzzy clustering, fuzzy partition, prototype, category, fuzzy c-means algorithm.

Chapter 1

Introduction

1.1 Research problem

The mathematical models and the computer's usage became in the last period instruments increasingly used in the decision making, for solving complex problems which involves the factors of uncertainty and risk. Most MCDM (Multiple Criteria Decision Making) problems from real life should be considered, normally, as fuzzy MCDM problems (see [58]).

1.2 Research motivation

Direct methods for the determination of the weight of the criteria have numerous disadvantages. Over the past few decades, many authors consider that the weight of the criteria can be inferred through mathematical methods (see, for example, [27]). The most appropriate method to obtain derived weight of the criteria is still subject to continuous debates (see [26]).

The generalization of the existing MCDM methods is necessary either when there are real situations that are not dealt with in literature, therefore they can not be treated by existing MCDM methods, or some existing MCDM methods can be improved.

The classical partitioning of a set of criteria in the traditional IPA (Importance-Performance Analysis) has disadvantages. A more objective approach is the fuzzy clustering which determines a membership degree of every criterion to each category.

1.3 Research objectives

The overall objective of this thesis is to study, design and implement methods and algorithms for solving multicriteria decision making problems using fuzzy numbers.

This thesis contains the results of the study on indirect methods for computing the fuzzy weight of the criteria, fuzzy MCDM methods for ranking of the alternatives, as well as fuzzy clustering methods on a set of criteria.

The first objective is to obtain indirect methods for calculating the fuzzy weights of the criteria following the idea in the crisp case, namely that the weight of a criterion is given by the correlation coefficient between the performance of the alternatives related to criteria and the overall customer satisfaction, all in the opinion of the decision makers and represented by fuzzy numbers and getting the fuzzy weights of the considered criteria.

The second objective consists in the generalization of the existing MCDM methods for the evaluation of the alternatives, when the performance are expressed by fuzzy numbers.

The third objective is to obtain a fuzzy classification of a set of criteria based on an adapted form of the fuzzy c -means algorithm.

1.4 Thesis structure

The thesis is structured as follows.

In Chapter 2 we recall notions related to fuzzy mathematics and in Chapter 3 basic elements of decision theory. Chapters 4-6 contain original contributions. Chapter 4 shows two proposed methods for the indirect determination of the fuzzy weights of criteria, both through correlation method between performances related to considered criteria and the overall level of satisfaction, all in the opinion of the customers and represented by fuzzy numbers. Due to the difficulties of the first proposed method derived from the use of the fuzzy arithmetic based on T_M norm, in the second proposed method we use the fuzzy arithmetic based on T_W norm, resulting an analytical solution with low computational resources. For an immediate interpretation of the results, the obtained fuzzy values must be defuzzified using the expected value. They are also presented the corresponding algorithms, illustrative examples and a case study based

on the results of a recent survey regarding the quality of hotel services in Oradea, Romania. The proposed indirect methods are compared between them and both with the direct method for calculation of the fuzzy weights of the criteria. Each section ends with the immediate conclusions. In Chapter 5 we propose two fuzzy MCDM methods that generalizes some methods described in the recent literature. The first original method uses the intervals of trapezoidal fuzzy numbers, modeling the situations when are allowed two choices or even an intermediate response in a survey. For the ranking of the intervals of trapezoidal fuzzy numbers is used the expected value. They are also given the corresponding algorithm, theoretical examples, the comparing of the results obtained by the proposed method with results obtained by other methods and the immediate conclusions. The second proposed method is based on the trapezoidal intuitionistic fuzzy numbers, on the two aggregation operators and on the four ranking methods of trapezoidal intuitionistic fuzzy numbers. They are also presented the corresponding algorithm, numerical examples, the comparing of the obtained results with the results obtained by other methods and the immediate conclusions. In Chapter 6 it is approached a refinement of classical IPA analysis using fuzzy numbers. The first original method for fuzzy classification of the criteria in the four categories corresponding to classical IPA analysis is based on the fuzzy c -means algorithm in an adapted form. They are also presented the corresponding algorithm, case studies and comparisons of the results and immediate conclusions. The second proposed method for fuzzy partitioning use s categories of criteria. They are presented the proposed method, the corresponding algorithm, case studies and immediate conclusions. Chapter 7 presents the general conclusions of the thesis, as well as the fulfilling the proposed objectives and the possible further interest of the ideas launched in this thesis and in Appendices 1 - 6 there are described the applications that implement the proposed methods in this thesis.

At the end of this introduction, we mention that this thesis contains original contributions published in the papers [7, 8, 9, 11, 12, 51, 52]. Original contributions are listed at the end of each section, in the conclusions.

Chapter 2

Fuzzy mathematics preliminaries

2.1 Fuzzy sets

Definition 1. ([13]) Let X be a non-empty set. A fuzzy set A (fuzzy subset of X) is defined as a mapping $A : X \rightarrow [0, 1]$, where $A(x)$ is the membership degree of x to the fuzzy set A .

2.2 Fuzzy numbers

Definition 2. (see [13] or [22]) A fuzzy number A is a fuzzy subset of the real line, $A : \mathbb{R} \rightarrow [0, 1]$, satisfying the following properties:

- (i) A is normal (i. e. there exists $x_0 \in \mathbb{R}$ such that $A(x_0) = 1$);
- (ii) A is fuzzy convex (i. e. $A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(A(x_1), A(x_2))$, for every $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$);
- (iii) A is upper semicontinuous on \mathbb{R} (i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that $A(x) - A(x_0) < \varepsilon$, whenever $|x - x_0| < \delta$);
- (iv) A is compactly supported, i.e. $cl\{x \in \mathbb{R}; A(x) > 0\}$ is compact, where $cl(M)$ denotes the closure of a set M .

2.3 Classes of fuzzy numbers

2.3.1 Trapezoidal fuzzy numbers

Definition 3. ([13]) A trapezoidal fuzzy number $A^T = (a, b, c, d)$, $a \leq b \leq c \leq d \in \mathbb{R}$ is the fuzzy set

$$A^T(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c < x \leq d \\ 0 & \text{otherwise} \end{cases}.$$

The r -level set of a trapezoidal fuzzy number $A^T = (a, b, c, d)$ is defined as:

$$A_r^T = [a + (b - a)r, d - (d - c)r], \quad r \in [0, 1].$$

For two trapezoidal fuzzy numbers $A_1^T = (a_1, b_1, c_1, d_1)$, $A_2^T = (a_2, b_2, c_2, d_2)$ and $\lambda \in \mathbb{R}$, the sum $A_1^T + A_2^T$ and the scalar multiplication λA_1^T are given by:

$$\begin{aligned} A_1^T + A_2^T &= (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2), \\ \lambda A_1^T &= \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1), & \text{if } \lambda \geq 0 \\ (\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1), & \text{if } \lambda < 0. \end{cases} \end{aligned}$$

Remark 4. *The product and the division of the two trapezoidal fuzzy numbers are not trapezoidal fuzzy numbers; instead, the cross product of two trapezoidal fuzzy numbers is a trapezoidal fuzzy number.*

A trapezoidal fuzzy number $A^T = (a, b, c, d)$ is positive if $a \geq 0$ and respectively negative if $d \leq 0$.

If the trapezoidal fuzzy numbers $A_1^T = (a_1, b_1, c_1, d_1)$ and $A_2^T = (a_2, b_2, c_2, d_2)$ are both positive or both negative, then the cross product $A_1^T \odot A_2^T$ is given by:

$$A_1^T \odot A_2^T = (a_1 b_2 + b_1 a_2 - b_1 b_2, \quad b_1 b_2, \quad c_1 c_2, \quad c_1 d_2 + d_1 c_2 - c_1 c_2).$$

If one of the trapezoidal fuzzy numbers $A_1^T = (a_1, b_1, c_1, d_1)$ and $A_2^T = (a_2, b_2, c_2, d_2)$ is positive and the other is negative, then the cross product $A_1^T \odot A_2^T$ is given by:

$$A_1^T \odot A_2^T = (-c_1 d_2 - d_1 c_2 + c_1 c_2, \quad -c_1 c_2, \quad -b_1 b_2, \quad -a_1 b_2 - b_1 a_2 + b_1 b_2).$$

2.3.2 Triangular fuzzy numbers

The triangular fuzzy numbers are particular trapezoidal fuzzy numbers, obtained when $b = c$ in Definition 3. The particular case $a = b = c = d = e$ leads to real number e . A triangular fuzzy number is denoted by $A^\Delta = (a, b, c)$, where $a \leq b \leq c$.

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The r -level set of a triangular fuzzy number $A^\Delta = (a, b, c)$ is given by:

$$A_r^\Delta = [a + (b - a)r, c - (c - b)r], \quad r \in [0, 1].$$

If $A_1^\Delta = (a_1, b_1, c_1)$ and $A_2^\Delta = (a_2, b_2, c_2)$ are two triangular fuzzy numbers and $\lambda \in \mathbb{R}$, the sum $A_1^\Delta + A_2^\Delta$ and the scalar multiplication λA_1^Δ are given by:

$$\begin{aligned} A_1^\Delta + A_2^\Delta &= (a_1 + a_2, b_1 + b_2, c_1 + c_2), \\ \lambda A_1^\Delta &= \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1), & \text{if } \lambda \geq 0 \\ (\lambda c_1, \lambda b_1, \lambda a_1), & \text{if } \lambda < 0. \end{cases} \end{aligned}$$

In the literature, for a triangular fuzzy number $A^\Delta = (a, b, c)$ is also used the notation $A^\Delta = (a, \alpha, \beta)$ where $\alpha = b - a$ and $\beta = c - b$. In this case, the triangular fuzzy number (see [31]) $A^\Delta = (a, \alpha, \beta)$ has the following membership function:

$$A^\Delta(x) = \begin{cases} 1 - \frac{a}{\alpha} + \frac{1}{\alpha}x, & \text{if } a - \alpha \leq x \leq a \\ 1 + \frac{a}{\beta} - \frac{1}{\beta}x, & \text{if } a \leq x \leq a + \beta \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

We summarize the usual T_W -based arithmetic operations based on Zadeh's extension principle (see (2.24)), on triangular fuzzy numbers.

Let $A = (a, \alpha, \beta)$ and $B = (b, \gamma, \delta)$ be two triangular fuzzy numbers and $\lambda \in \mathbb{R}$, $\lambda > 0$. We have (see [16], [29], [30], [31], [32], [37], [41]):

$$(a, \alpha, \beta) + (b, \gamma, \delta) = (a + b, \max(\alpha, \gamma), \max(\beta, \delta)), \quad (2.2)$$

$$(a, \alpha, \beta) - (b, \gamma, \delta) = (a - b, \max(\alpha, \delta), \max(\beta, \gamma)), \quad (2.3)$$

$$\lambda \cdot (a, \alpha, \beta) = (\lambda a, \alpha, \beta), \quad (2.4)$$

$$(a, \alpha, \beta) \cdot (b, \gamma, \delta) = \begin{cases} (ab, \max(\alpha b, \gamma a), \max(\beta b, \delta a)), & \text{if } a, b \geq 0 \\ (ab, -\max(\alpha b, \gamma a), -\max(\beta b, \delta a)), & \text{if } a, b \leq 0 \\ (ab, \max(\alpha b, -\delta a), \max(\beta b, -\gamma a)), & \text{if } a \leq 0, b \geq 0 \\ (ab, \max(\gamma a, -\beta b), \max(\delta a, -\alpha b)), & \text{if } a \geq 0, b \leq 0, \end{cases} \quad (2.5)$$

$$\sqrt{(a, \alpha, \beta)} = \left(\sqrt{a}, \frac{\alpha}{\sqrt{a}}, \frac{\beta}{\sqrt{a}} \right), \quad \text{for every } a > 0, \quad (2.6)$$

$$\begin{aligned}
 \frac{(a, \alpha, \beta)}{(b, \gamma, \delta)} &= \\
 &= \begin{cases} \left(\frac{a}{b}, \max\left(\frac{\alpha}{b}, \frac{a\delta}{b(b+\delta)}\right), \max\left(\frac{\beta}{b}, \frac{a\gamma}{b(b-\gamma)}\right) \right), & \text{if } a > 0, b > 0, b - \gamma > 0 \\ \left(\frac{a}{b}, \max\left(-\frac{\beta}{b}, -\frac{a\gamma}{b(b-\gamma)}\right), \max\left(-\frac{\alpha}{b}, -\frac{a\delta}{b(b+\delta)}\right) \right), & \text{if } a < 0, b < 0, b + \delta < 0 \\ \left(0, \frac{\alpha}{b}, \frac{\beta}{b} \right), & \text{if } a = 0, b > 0, b - \gamma > 0 \\ \left(0, -\frac{\beta}{b}, -\frac{\alpha}{b} \right), & \text{if } a = 0, b < 0, b + \delta < 0 \\ \left(\frac{a}{b}, \max\left(-\frac{\beta}{b}, \frac{a\delta}{b(b+\delta)}\right), \max\left(-\frac{\alpha}{b}, \frac{a\gamma}{b(b-\gamma)}\right) \right), & \text{if } a > 0, b < 0, b + \delta < 0 \\ \left(\frac{a}{b}, \max\left(\frac{\alpha}{b}, -\frac{a\gamma}{b(b-\gamma)}\right), \max\left(\frac{\beta}{b}, -\frac{a\delta}{b(b+\delta)}\right) \right), & \text{if } a < 0, b > 0, b - \gamma > 0. \end{cases} \quad (2.7)
 \end{aligned}$$

2.3.3 Other classes of fuzzy numbers

Another two interesting classes of fuzzy numbers are gaussian fuzzy numbers and exponential fuzzy numbers.

2.4 Generalizations of fuzzy numbers

2.4.1 Intervals of fuzzy numbers

Definition 5. ([51]) An interval of fuzzy numbers is a pair $\tilde{A} = [A^L, A^U]$, where A^L and A^U are fuzzy numbers such that $(A^L)_r^- \leq (A^U)_r^-$ and $(A^L)_r^+ \leq (A^U)_r^+$, for every $r \in [0, 1]$.

Definition 6. If $A^{TL} = (a^L, b^L, c^L, d^L)$ and $A^{TU} = (a^U, b^U, c^U, d^U)$ are two trapezoidal fuzzy numbers, then $\tilde{A}^T = [A^{TL}, A^{TU}]$ is an interval of trapezoidal fuzzy numbers if and only if $a^L \leq a^U, b^L \leq b^U, c^L \leq c^U$ and $d^L \leq d^U$.

In the case of intervals of trapezoidal fuzzy numbers, the sum and the scalar multiplication are:

$$\begin{aligned}
 \tilde{A}_1^T + \tilde{A}_2^T &= [(a_1^L, b_1^L, c_1^L, d_1^L), (a_1^U, b_1^U, c_1^U, d_1^U)] + [(a_2^L, b_2^L, c_2^L, d_2^L), (a_2^U, b_2^U, c_2^U, d_2^U)] = \\
 &= [(a_1^L + a_2^L, b_1^L + b_2^L, c_1^L + c_2^L, d_1^L + d_2^L), (a_1^U + a_2^U, b_1^U + b_2^U, c_1^U + c_2^U, d_1^U + d_2^U)], \\
 \lambda \cdot \tilde{A}^T &= \lambda \cdot [(a^L, b^L, c^L, d^L), (a^U, b^U, c^U, d^U)] \\
 &= \begin{cases} [(\lambda a^L, \lambda b^L, \lambda c^L, \lambda d^L), (\lambda a^U, \lambda b^U, \lambda c^U, \lambda d^U)], & \text{if } \lambda \geq 0 \\ [(\lambda d^L, \lambda c^L, \lambda b^L, \lambda a^L), (\lambda d^U, \lambda c^U, \lambda b^U, \lambda a^U)], & \text{if } \lambda < 0, \end{cases}
 \end{aligned}$$

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2.4.2 Intuitionistic fuzzy numbers

Definition 7. (see [36]) A trapezoidal intuitionistic fuzzy number $\widetilde{A}^T = \langle (\underline{a}, \underline{b}, \underline{c}, \underline{d}), (\overline{a}, \overline{b}, \overline{c}, \overline{d}) \rangle$ is an intuitionistic fuzzy set on \mathbb{R} with the membership function $\mu_{\widetilde{A}^T}$ and the non-membership function $\nu_{\widetilde{A}^T}$ defined as:

$$\mu_{\widetilde{A}^T}(x) = \begin{cases} \frac{x-\underline{a}}{\underline{b}-\underline{a}}, & \text{if } x \in [\underline{a}, \underline{b}] \\ 1, & \text{if } x \in [\underline{b}, \underline{c}] \\ \frac{\underline{d}-x}{\underline{d}-\underline{c}}, & \text{if } x \in (\underline{c}, \underline{d}] \\ 0, & \text{otherwise} \end{cases}$$

$$\nu_{\widetilde{A}^T}(x) = \begin{cases} \frac{\overline{b}-x}{\overline{b}-\overline{a}}, & \text{if } x \in [\overline{a}, \overline{b}] \\ 0, & \text{if } x \in [\overline{b}, \overline{c}] \\ \frac{x-\overline{c}}{\overline{d}-\overline{c}}, & \text{if } x \in (\overline{c}, \overline{d}] \\ 1, & \text{otherwise,} \end{cases}$$

where $\underline{a} \leq \underline{b} \leq \underline{c} \leq \underline{d}$, $\overline{a} \leq \overline{b} \leq \overline{c} \leq \overline{d}$, $\overline{a} \leq \underline{a}$, $\overline{b} \leq \underline{b}$, $\underline{c} \leq \overline{c}$ and $\underline{d} \leq \overline{d}$.

The imposed conditions on \underline{a} , \underline{b} , \underline{c} , \underline{d} , \overline{a} , \overline{b} , \overline{c} , \overline{d} ensure that \widetilde{A}^T is an intuitionistic fuzzy set.

Definition 8. (see [33]) A trapezoidal intuitionistic fuzzy number $\widetilde{A}^T = \langle (\underline{a}, \underline{b}, \underline{c}, \underline{d}), (\overline{a}, \overline{b}, \overline{c}, \overline{d}) \rangle$ is non-negative if and only if $\overline{a} \geq 0$.

The sum and the scalar multiplication on the set of the intuitionistic fuzzy numbers (see [2] or [4]) become in the particular case of trapezoidal intuitionistic fuzzy numbers as follows:

$$\begin{aligned} \widetilde{A}_1^T + \widetilde{A}_2^T &= \langle (\underline{a}_1, \underline{b}_1, \underline{c}_1, \underline{d}_1), (\overline{a}_1, \overline{b}_1, \overline{c}_1, \overline{d}_1) \rangle + \langle (\underline{a}_2, \underline{b}_2, \underline{c}_2, \underline{d}_2), (\overline{a}_2, \overline{b}_2, \overline{c}_2, \overline{d}_2) \rangle \\ &= \langle (\underline{a}_1 + \underline{a}_2, \underline{b}_1 + \underline{b}_2, \underline{c}_1 + \underline{c}_2, \underline{d}_1 + \underline{d}_2), (\overline{a}_1 + \overline{a}_2, \overline{b}_1 + \overline{b}_2, \overline{c}_1 + \overline{c}_2, \overline{d}_1 + \overline{d}_2) \rangle, \end{aligned} \quad (2.8)$$

$$\begin{aligned} \lambda \cdot \widetilde{A}^T &= \lambda \cdot \langle (\underline{a}, \underline{b}, \underline{c}, \underline{d}), (\overline{a}, \overline{b}, \overline{c}, \overline{d}) \rangle \\ &= \begin{cases} \langle (\lambda \underline{a}, \lambda \underline{b}, \lambda \underline{c}, \lambda \underline{d}), (\lambda \overline{a}, \lambda \overline{b}, \lambda \overline{c}, \lambda \overline{d}) \rangle, & \text{if } \lambda \in \mathbb{R}, \lambda \geq 0, \\ \langle (\lambda \underline{d}, \lambda \underline{c}, \lambda \underline{b}, \lambda \underline{a}), (\lambda \overline{d}, \lambda \overline{c}, \lambda \overline{b}, \lambda \overline{a}) \rangle, & \text{if } \lambda \in \mathbb{R}, \lambda < 0. \end{cases} \end{aligned} \quad (2.9)$$

In addition, if the trapezoidal intuitionistic fuzzy numbers are non-negative, it can be defined the multiplication and the rise to positive power, as follows (see [60]):

$$\begin{aligned} \widetilde{A}_1^T \otimes \widetilde{A}_2^T &= \langle (\underline{a}_1, \underline{b}_1, \underline{c}_1, \underline{d}_1), (\overline{a}_1, \overline{b}_1, \overline{c}_1, \overline{d}_1) \rangle \otimes \langle (\underline{a}_2, \underline{b}_2, \underline{c}_2, \underline{d}_2), (\overline{a}_2, \overline{b}_2, \overline{c}_2, \overline{d}_2) \rangle \\ &= \langle (\underline{a}_1 \underline{a}_2, \underline{b}_1 \underline{b}_2, \underline{c}_1 \underline{c}_2, \underline{d}_1 \underline{d}_2), (\overline{a}_1 \overline{a}_2, \overline{b}_1 \overline{b}_2, \overline{c}_1 \overline{c}_2, \overline{d}_1 \overline{d}_2) \rangle, \end{aligned} \quad (2.10)$$

$$\begin{aligned}\widetilde{A}^{\lambda} &= \langle (\underline{a}, \underline{b}, \underline{c}, \underline{d}), (\bar{a}, \bar{b}, \bar{c}, \bar{d}) \rangle^{\lambda} \\ &= \langle (\underline{a}^{\lambda}, \underline{b}^{\lambda}, \underline{c}^{\lambda}, \underline{d}^{\lambda}), (\bar{a}^{\lambda}, \bar{b}^{\lambda}, \bar{c}^{\lambda}, \bar{d}^{\lambda}) \rangle, \lambda \geq 0.\end{aligned}\quad (2.11)$$

It is obvious that the neutral element for the sum is $\langle (0, 0, 0, 0), (0, 0, 0, 0) \rangle$ and for the product is $\langle (1, 1, 1, 1), (1, 1, 1, 1) \rangle$.

We recall in the following two aggregation operators for trapezoidal intuitionistic fuzzy numbers. Suppose that $\widetilde{A}_i, i = \{1, \dots, n\}$ is a set of non-negative trapezoidal intuitionistic fuzzy numbers and $\widetilde{\omega}_i$ a non-negative trapezoidal intuitionistic fuzzy number as the fuzzy weight of the criterion \widetilde{A}_i , for every $i = \{1, \dots, n\}$, then it can be defined the arithmetical aggregation operator $WAA_{\widetilde{\omega}} : TIFN^n(\mathbb{R}) \rightarrow TIFN(\mathbb{R})$ (see [61]) as follows:

$$WAA_{\widetilde{\omega}}(\widetilde{A}_1, \dots, \widetilde{A}_n) = (1/n) \cdot (\widetilde{\omega}_1 \otimes \widetilde{A}_1 + \dots + \widetilde{\omega}_n \otimes \widetilde{A}_n), \quad (2.12)$$

using (2.8), (2.9) and (2.10). If $\omega_i, i = \{1, \dots, n\}$ are positive crisp numbers, then it can be defined the geometrical aggregation operator $WGA_{\omega} : TIFN^n(\mathbb{R}) \rightarrow TIFN(\mathbb{R})$ (see [54]) as follows:

$$WGA_{\omega}(\widetilde{A}_1, \dots, \widetilde{A}_n) = \widetilde{A}_1^{\omega_1} \otimes \dots \otimes \widetilde{A}_n^{\omega_n}, \quad (2.13)$$

using (2.10) and (2.11).

2.4.3 Interval-valued trapezoidal and triangular fuzzy numbers

Definition 9. (see, e.g., [55]) An interval-valued trapezoidal fuzzy number \widetilde{A}^T is an interval-valued fuzzy set on \mathbb{R} , defined by $\widetilde{A}^T = [A^{TL}, A^{TU}]$, where $A^{TL} = (a_1^L, a_2^L, a_3^L, a_4^L)$ and $A^{TU} = (a_1^U, a_2^U, a_3^U, a_4^U)$ are trapezoidal fuzzy numbers such that $A^{TL} \subseteq A^{TU}$.

If $a_2^L = a_3^L$ and $a_2^U = a_3^U$ in the above definition, then we obtain an interval-valued triangular fuzzy number denoted by $\widetilde{A}^{\Delta} = [(a_1^L, a_2^L, a_3^L), (a_1^U, a_2^U, a_3^U)]$.

2.4.4 Relations between intuitionistic fuzzy numbers and interval-valued fuzzy numbers

At the end of this section we recall some relationships between trapezoidal (triangular) intuitionistic fuzzy numbers and interval-valued trapezoidal (triangular) fuzzy numbers, as well as immediate properties of these relationships.

2.5 Numerical characteristics of fuzzy numbers

In [5] and [6] it was proved that the expected value is a simple but effective ranking method of fuzzy numbers. More exactly, for $A, B \in FN(\mathbb{R})$ we introduce the following relations:

$$A \prec_{EV} B \text{ if and only if } EV(A) < EV(B), \quad (2.14)$$

$$A \sim_{EV} B \text{ if and only if } EV(A) = EV(B), \quad (2.15)$$

$$A \preceq_{EV} B \text{ if and only if } EV(A) \leq EV(B). \quad (2.16)$$

In the case of a trapezoidal fuzzy number $A^T = (a, b, c, d)$, with $a \leq b \leq c \leq d$, having the r -level sets $(A^T)_r = [(A^T)_r^-, (A^T)_r^+]$, for $r \in [0, 1]$, with

$$(A^T)_r^- = a + (b - a)r, \quad (A^T)_r^+ = d + (c - d)r,$$

the defuzzification measures have the following expressions:

$$\begin{aligned} Amb(A^T) &= \frac{-a - 2b + 2c + d}{6}, \\ Val(A^T) &= \frac{a + 2b + 2c + d}{6}, \\ card A^T &= \frac{-a - b + c + d}{2}, \\ core(A^T) &= [b, c], \\ supp(A^T) &= [a, d], \\ EI(A^T) &= \left[\frac{a + b}{2}, \frac{c + d}{2} \right], \\ EV(A^T) &= \frac{a + b + c + d}{4}. \end{aligned} \quad (2.17)$$

The expected value of a triangular fuzzy number $A^\Delta = (a, \alpha, \beta)$ is:

$$EV(A^\Delta) = a + \frac{\beta - \alpha}{4}. \quad (2.18)$$

The expected value of an interval of fuzzy number is:

$$EV(\tilde{A}) = EV([A^L, A^U]) = \frac{1}{2}(EV(A^L) + EV(A^U)).$$

Among many ranking methods on trapezoidal intuitionistic fuzzy numbers from the literature (see, e.g., [33], [36], [38], [60], [57]), we consider in the following four of them.

2.5 Numerical characteristics of fuzzy numbers

We denote by $\widetilde{\widetilde{A^T}} = \langle (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \rangle$ the trapezoidal intuitionistic fuzzy number.

Firstly, we consider a ranking method on trapezoidal intuitionistic fuzzy numbers based on the index $M_\mu^{\beta,k}$ for membership function and index $M_\nu^{\beta,k}$ for non-membership function (see [33]). In the particular case when $\beta = 1/3$ and $k = 0$, these indexes are:

$$\begin{aligned} M_\mu^{1/3,0}(\widetilde{\widetilde{A^T}}) &= \frac{1}{6}(a_1 + 2b_1 + 2c_1 + d_1), \\ M_\nu^{1/3,0}(\widetilde{\widetilde{A^T}}) &= \frac{1}{6}(a_2 + 2b_2 + 2c_2 + d_2). \end{aligned} \quad (2.19)$$

Further, for simplification, we denote $M_\mu(\widetilde{\widetilde{A^T}}) = M_\mu^{1/3,0}(\widetilde{\widetilde{A^T}})$ and respectively $M_\nu(\widetilde{\widetilde{A^T}}) = M_\nu^{1/3,0}(\widetilde{\widetilde{A^T}})$.

Definition 10. (see [33]) Let $\widetilde{\widetilde{A^T}}$ and $\widetilde{\widetilde{B^T}}$ be two trapezoidal intuitionistic fuzzy numbers. Then

$$\begin{aligned} \widetilde{\widetilde{A^T}} \prec_M \widetilde{\widetilde{B^T}} &\Leftrightarrow M_\mu(\widetilde{\widetilde{A^T}}) < M_\mu(\widetilde{\widetilde{B^T}}) \text{ or} \\ &(M_\mu(\widetilde{\widetilde{A^T}}) = M_\mu(\widetilde{\widetilde{B^T}}) \text{ and } -M_\nu(\widetilde{\widetilde{A^T}}) < -M_\nu(\widetilde{\widetilde{B^T}})). \end{aligned}$$

The second ranking method on trapezoidal intuitionistic fuzzy numbers is based on the value-index V_λ and ambiguity-index A_λ (see [35]). The value of the membership function is given by $V_\mu(\widetilde{\widetilde{A^T}}) = \frac{1}{6}(a_1 + 2b_1 + 2c_1 + d_1)$ and the value of the non-membership function is given by $V_\nu(\widetilde{\widetilde{A^T}}) = \frac{1}{6}(a_2 + 2b_2 + 2c_2 + d_2)$. Analogously, the ambiguity of the membership function is given by $A_\mu(\widetilde{\widetilde{A^T}}) = \frac{1}{6}(-a_1 - 2b_1 + 2c_1 + d_1)$ and the ambiguity of the non-membership function is given by $A_\nu(\widetilde{\widetilde{A^T}}) = \frac{1}{6}(-a_2 - 2b_2 + 2c_2 + d_2)$. Then the value-index and the ambiguity-index of $\widetilde{\widetilde{A^T}}$ are given by

$$V_\lambda(\widetilde{\widetilde{A^T}}) = \lambda V_\mu(\widetilde{\widetilde{A^T}}) + (1 - \lambda)V_\nu(\widetilde{\widetilde{A^T}}), \quad A_\lambda(\widetilde{\widetilde{A^T}}) = \lambda A_\mu(\widetilde{\widetilde{A^T}}) + (1 - \lambda)A_\nu(\widetilde{\widetilde{A^T}}), \quad (2.20)$$

where $\lambda \in [0, 1]$ is a weight which represents the decision-maker's preference information, namely $\lambda \in [0, 0.5)$ shows that the decision-maker prefers certainty, $\lambda \in (0.5, 1]$ shows that the decision-maker prefers uncertainty and $\lambda = 0.5$ shows that the decision-maker is indifferent between certainty and uncertainty.

Definition 11. (see [35]) Let $\widetilde{\widetilde{A^T}}$ and $\widetilde{\widetilde{B^T}}$ be two trapezoidal intuitionistic fuzzy numbers. Then

$$\widetilde{\widetilde{A^T}} \prec_{VA} \widetilde{\widetilde{B^T}} \Leftrightarrow V_\lambda(\widetilde{\widetilde{A^T}}) < V_\lambda(\widetilde{\widetilde{B^T}}) \text{ or } (V_\lambda(\widetilde{\widetilde{A^T}}) = V_\lambda(\widetilde{\widetilde{B^T}}) \text{ and } A_\lambda(\widetilde{\widetilde{A^T}}) > A_\lambda(\widetilde{\widetilde{B^T}})).$$

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For a third ranking method, introduced in [61], we recall the following definition of the score S and of the accuracy E of $\widetilde{\widetilde{A^T}}$:

$$\begin{aligned} S(\widetilde{\widetilde{A^T}}) &= (a_1 - a_2 + b_1 - b_2 + c_1 - c_2 + d_1 - d_2)/4, \\ E(\widetilde{\widetilde{A^T}}) &= (a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + d_1 + d_2)/4. \end{aligned} \quad (2.21)$$

If $a_i, b_i, c_i, d_i \in [0, 1]$, for $i \in \{1, 2\}$, then $S(\widetilde{\widetilde{A^T}}) \in [-1, 1]$ and $E(\widetilde{\widetilde{A^T}}) \in [0, 2]$.

Definition 12. (see [61]) Let $\widetilde{\widetilde{A^T}}$ and $\widetilde{\widetilde{B^T}}$ be two trapezoidal intuitionistic fuzzy numbers. Then

$$\widetilde{\widetilde{A^T}} \prec_{SE} \widetilde{\widetilde{B^T}} \Leftrightarrow S(\widetilde{\widetilde{A^T}}) < S(\widetilde{\widetilde{B^T}}) \text{ or } (S(\widetilde{\widetilde{A^T}}) = S(\widetilde{\widetilde{B^T}}) \text{ and } E(\widetilde{\widetilde{A^T}}) < E(\widetilde{\widetilde{B^T}})).$$

Last ranking method, but not the least important, because it is simple and has suitable properties, is based on the expected value EV (see, e.g., [7]):

$$EV(\widetilde{\widetilde{A^T}}) = (a_1 + b_1 + c_1 + d_1 + a_2 + b_2 + c_2 + d_2)/8. \quad (2.22)$$

Definition 13. (see [7]) Let $\widetilde{\widetilde{A^T}}$ and $\widetilde{\widetilde{B^T}}$ be two trapezoidal intuitionistic fuzzy numbers. Then

$$\widetilde{\widetilde{A^T}} \prec_{EV} \widetilde{\widetilde{B^T}} \Leftrightarrow EV(\widetilde{\widetilde{A^T}}) < EV(\widetilde{\widetilde{B^T}}).$$

2.6 The Zadeh's extension principle

The Zadeh's extension principle (see [58], [59] and the recent book [13]) allows to extend real functions and particularly, the basic operations for crisp numbers, to fuzzy numbers.

Definition 14. (see, e.g., [28], p. 41) Let X_1, \dots, X_n, Z be non-empty sets and the function $F : X \rightarrow Z$, where X is the product space $X = X_1 \times X_2 \times \dots \times X_n$. Furthermore, we consider the fuzzy sets A_1, \dots, A_n such that $A_i : X_i \rightarrow [0, 1]$, $i \in \{1, \dots, n\}$. Taking use of the function F we can define the fuzzy set $F(A_1, A_2, \dots, A_n) : Z \rightarrow [0, 1]$ by:

$$F(A_1, \dots, A_n)(z) = \begin{cases} \sup_{(x_1, \dots, x_n) \in F^{-1}(z)} \min\{A_1(x_1), \dots, A_n(x_n)\}, & \text{if } z \in F(X), \\ 0, & \text{otherwise.} \end{cases} \quad (2.23)$$

Theorem 15. ([46], Proposition 5.1) Let us consider the continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and the fuzzy numbers A_1, \dots, A_n . Then $F(A_1, \dots, A_n)$ obtained by the extension principle (2.23) is a fuzzy number given by

$$(F(A_1, \dots, A_n))_r = f((A_1)_r, \dots, (A_n)_r), r \in [0, 1].$$

The Zadeh's extension principle based on a triangular norm T extends an arithmetical operation $*$ with crisp numbers to an arithmetical operation \otimes with fuzzy numbers such as (see [31], [59]):

$$(A \otimes B)(z) = \sup_{x*y=z} T(A(x), B(y)). \quad (2.24)$$

The T_M -based operations are the most used, but the T_W -based operations have some advantages, namely, the calculation is much simplified, the fuzziness of the results is small, and the sum and multiplication preserves the form of the fuzzy numbers and therefore, in particular, of the triangular fuzzy numbers (see [29], [30], [32], [41]).

2.7 Linguistic variables

If we consider a 5-level Likert scale, the linguistic variables used to assess the performances of the alternatives may be the set $\{\text{very poor}, \text{poor}, \text{medium}, \text{good}, \text{very good}\}$ whose representation by triangular fuzzy numbers can be one of the Figure 2.1. In this way, the data can be used in mathematical methods (see [21]).

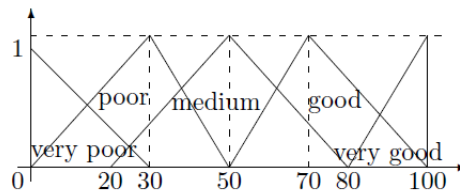


Figure 2.1: Linguistic variables represented by triangular fuzzy numbers

Chapter 3

Decision theory preliminaries

3.1 Multicriteria analysis

Multicriteria analysis is (according to [48]) a structured approach used in order to determine a common preference, choosing from several options, each must meet a number of objectives.

Classical steps of multicriteria analysis (see [40] or [48]) are:

1. decision problem context definition: the aim, the decision makers etc;
2. decision alternative definition $A = \{A_1, \dots, A_m\}$;
3. criteria definition $C = \{C_1, \dots, C_n\}$;
4. creating the decision matrix (having values for consequences $R = \{r_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$);
5. determination of the weights of the criteria $P = \{p_1, \dots, p_n\}$ which give the importance of the criteria in decision making;
6. applying a suitable MCDM method in order to determine the final score of each alternative (e.g. the weighted average of the performances and the weights);
7. examination and interpretation of the results (computing a hierarchy of alternatives);
8. sensitivity analysis of the method, by changing the consequences and/or the weights and evaluating the deviations scores.

3.2 Determination of the weights of the criteria

3.2.1 Direct method

Let us consider n criteria C_1, \dots, C_n of a service and k decision makers D_1, \dots, D_k . We denote by W_{tj} the weight of the criterion C_j , $j \in \{1, \dots, n\}$ in the opinion of the decision maker D_t , $t \in \{1, \dots, k\}$, obtained by answers to questions as "How important is ...?". The weight of a criterion C_j can be given by a direct method, aggregating the values W_{tj} . As an example, using the triangular fuzzy numbers as in 2.1 and the arithmetical operations generated by the drastic triangular norm T_W , we get the Algorithm 1 used to obtain a hierarchy of the criteria according to fuzzy weights determined by direct method.

Algorithm 1 The hierarchy of the criteria according to fuzzy weights determined by direct method.

for $j = 1 \rightarrow n$ **do**

compute (see (2.2), (2.4))

$$\widetilde{W}_j^* = \frac{1}{k} \cdot (\widetilde{W}_{1j}^* + \dots + \widetilde{W}_{kj}^*) = \left(\frac{1}{k} \sum_{t=1}^k w_{tj}^*, \max_{t \in \{1, \dots, k\}} \alpha_{tj}^*, \max_{t \in \{1, \dots, k\}} \beta_{tj}^* \right)$$

compute (see (2.18))

$$W_j = EV(\widetilde{W}_j^*) = \frac{1}{k} \sum_{t=1}^k w_{tj}^* + \frac{1}{4} \max_{t \in \{1, \dots, k\}} \beta_{tj}^* - \frac{1}{4} \max_{t \in \{1, \dots, k\}} \alpha_{tj}^*$$

end for

order the vector \widetilde{W}^* according to W in the following way: if $W_{j_1} \geq W_{j_2}$ then $\widetilde{W}_{j_1}^* \geq \widetilde{W}_{j_2}^*$, otherwise $\widetilde{W}_{j_1}^* < \widetilde{W}_{j_2}^*$ (see (2.14)-(2.16))

3.2.2 Indirect correlation method

An indirect method for the calculation of the weights of the criteria in the crisp case is based on correlation coefficient between the performances related to each criterion and the overall customer satisfaction (see, e.g., [19], [44], [43]). Using the notation from the previous section, the weight of the criterion C_j , $j \in \{1, \dots, n\}$, denoted by W_j , is

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given by the correlation coefficient between variables (X_{1j}, \dots, X_{kj}) and (X_1, \dots, X_k) , therefore

$$W_j = \frac{\sum_{t=1}^k (X_{tj} - X_j^M) (X_t - X^M)}{\sqrt{\sum_{t=1}^k (X_{tj} - X_j^M)^2 \sum_{t=1}^k (X_t - X^M)^2}}, \quad (3.1)$$

where $X_j^M = \frac{1}{k} \sum_{t=1}^k X_{tj}$ and $X^M = \frac{1}{k} \sum_{t=1}^k X_t$.

It is well-known that the correlation coefficient takes values from -1 to 1, therefore $W_j \in [-1, 1]$. Even if sometimes are preferred other methods that avoid negative values, regression analysis and correlation coefficient are considered most appropriate methods used to measure the derived weights from a survey. The negativity of the weights of the criteria is not a problem when the results have subsequent employments, between these the best example being the IPA.

3.3 Importance-performance analysis

IPA is a simple and effective marketing technique which can help in identifying improvement priorities and quality-based marketing strategies. IPA consists in placing the points with performance as the first coordinate and weight as the second coordinate on a two-dimensional plot called an IPA grid. The two axes, the horizontal one, which shows the performance related to criterion in the customer opinion and the vertical one, which shows the weight of the criterion, determine four quadrants and, implicitly, a classification of criteria as in Table 3.1.

Table 3.1: Quadrants in traditional IPA

Quadrant	Performance	Weight
1: <i>Keep up the good work</i> (GW)	high	high
2: <i>Concentrate here</i> (CH)	low	high
3: <i>Low priority</i> (LP)	low	low
4: <i>Possible overkill</i> (PO)	high	low

The placement of the axes is subject of a continuous debate. On the other hand, most classes from the real world are fuzzy rather than crisp. The tools which identify

natural structure of data - like fuzzy clustering - seem to be more suitable than other artificial approaches in IPA.

3.3.1 Fuzzy partition using fuzzy c -means algorithm

Proposition 16. *Let A_1, \dots, A_s , $s > 2$, be fuzzy sets on X . The family $P = \{A_1, \dots, A_s\}$ is a fuzzy partition of X if and only if $\sum_{i=1}^s A_i(x) = 1$, for every $x \in X$.*

Let $X = \{\mathbf{x}^1, \dots, \mathbf{x}^n\} \subset \mathbb{R}^p$ be a set of vectors, where n is the number of objects and p is the number of characteristics, $\mathbf{x}^j = (x_1^j, \dots, x_p^j)$, and $L = \{L^1, \dots, L^s\}$ be a s -tuple of prototypes, $L^i = (L_1^i, \dots, L_p^i)$, each of them characterizing one of the s clusters of the data set. A partition of X into s fuzzy clusters is performed by minimizing the objective function (see [18], [24], [25], [47])

$$J(P, L) = \sum_{i=1}^s \sum_{j=1}^n \left(A_i(\mathbf{x}^j) \right)^2 d^2(\mathbf{x}^j, L^i), \quad (3.2)$$

where $P = \{A_1, \dots, A_s\}$ is the fuzzy partition of X , $A_i(\mathbf{x}^j) \in [0, 1]$ represents the membership degree of a point \mathbf{x}^j to cluster A_i and d is a distance on \mathbb{R}^p , usually the Euclidean distance, that is

$$d^2(\mathbf{x}^j, L^i) = \sum_{k=1}^p (x_k^j - L_k^i)^2. \quad (3.3)$$

For a given set of prototypes L , the minimum of the function $J(\cdot, L)$ is obtained for (see [25])

$$A_i(\mathbf{x}^j) = \frac{1}{\sum_{k=1}^s \frac{d^2(\mathbf{x}^j, L^i)}{d^2(\mathbf{x}^j, L^k)}}, i \in \{1, \dots, s\}, j \in \{1, \dots, n\}. \quad (3.4)$$

For a given partition P , the minimum of the function $J(P, \cdot)$ is obtained for (see [25])

$$L^i = \frac{\sum_{j=1}^n (A_i(\mathbf{x}^j))^2 \mathbf{x}^j}{\sum_{j=1}^n (A_i(\mathbf{x}^j))^2}, i \in \{1, \dots, s\}. \quad (3.5)$$

Therefore, the optimal fuzzy partition of X is determined by using the iterative method described before, where J is successively minimized with respect to P and L . The procedure is called fuzzy c -means algorithm (see [14] and [25], p. 293-295).

Chapter 4

Indirect methods for determining the fuzzy weighting of the criteria

4.1 Method for determining the derived fuzzy weighting of the criteria based on correlation coefficient as well as on the arithmetic generated by triangular norm T_M

The correlation coefficient of fuzzy numbers as a fuzzy number was introduced in [39]. We benefit from this contribution and we propose an indirect method for computing the fuzzy weights of the criteria. More exactly, the calculus is based on the arithmetic generated by the triangular norm T_M through Zadeh's extension principle. The numerical results are obtained by working on different r -level sets, taking into account that an analytical solution is difficult to be given. Even so, the amount of calculation is very big, so that the proposed practical method is preferable when the number of customers and/or criteria is large. The practical method has another important advantage, namely that it furnish us trapezoidal fuzzy numbers which can be easily compared, interpreted and handled in subsequent processing of data.

4.1 Method for determining the derived fuzzy weighting of the criteria based on correlation coefficient as well as on the arithmetic generated by triangular norm T_M

4.1.1 General method description

We introduce

$$\widetilde{w}_j = \frac{\sum_{t=1}^k (\widetilde{X}_{tj} - \widetilde{X}_j^M) \cdot (\widetilde{X}_t - \widetilde{X}^M)}{\sqrt{\sum_{t=1}^k (\widetilde{X}_{tj} - \widetilde{X}_j^M)^2 \cdot \sum_{t=1}^k (\widetilde{X}_t - \widetilde{X}^M)^2}} \quad (4.1)$$

as the fuzzy weight of the criterion $C_j, j \in \{1, \dots, n\}$, where \widetilde{X}_{tj} denotes the performance related to criterion $C_j, j \in \{1, \dots, n\}$ in the opinion of the decision maker $D_t, t \in \{1, \dots, k\}$, expressed as a fuzzy number, \widetilde{X}_t denotes the overall satisfaction in the opinion of the decision maker $D_t, t \in \{1, \dots, k\}$, expressed as a fuzzy number, and, in addition, we denote

$$\widetilde{X}_j^M = \frac{1}{k} \cdot \sum_{t=1}^k \widetilde{X}_{tj} \text{ and } \widetilde{X}^M = \frac{1}{k} \cdot \sum_{t=1}^k \widetilde{X}_t.$$

Formula (4.1) is rather formal, the effective calculus of the fuzzy number \widetilde{w}_j is based on the Zadeh's extension principle, more exactly we obtain the r -level sets $(\widetilde{w}_j)_r^-$ and $(\widetilde{w}_j)_r^+$, for every $r \in [0, 1]$, by solving the following pair of crisp mathematical programs:

$$(\widetilde{w}_j)_r^- = \min f_j(x_{1j}, \dots, x_{kj}, x_1, \dots, x_k) \quad (4.2)$$

such that

$$(\widetilde{X}_{tj})_r^- \leq x_{tj} \leq (\widetilde{X}_{tj})_r^+, \forall t \in \{1, \dots, k\}, \quad (4.3)$$

$$(\widetilde{X}_t)_r^- \leq x_t \leq (\widetilde{X}_t)_r^+, \forall t \in \{1, \dots, k\} \quad (4.4)$$

and

$$(\widetilde{w}_j)_r^+ = \max f_j(x_{1j}, \dots, x_{kj}, x_1, \dots, x_k) \quad (4.5)$$

such that

$$(\widetilde{X}_{tj})_r^- \leq x_{tj} \leq (\widetilde{X}_{tj})_r^+, \forall t \in \{1, \dots, k\}, \quad (4.6)$$

$$(\widetilde{X}_t)_r^- \leq x_t \leq (\widetilde{X}_t)_r^+, \forall t \in \{1, \dots, k\}, \quad (4.7)$$

where

$$f_j(x_{1j}, \dots, x_{kj}, x_1, \dots, x_k) = \frac{\sum_{t=1}^k \left(x_{tj} - \frac{1}{k} \sum_{t=1}^k x_{tj}\right) \left(x_t - \frac{1}{k} \sum_{t=1}^k x_t\right)}{\sqrt{\sum_{t=1}^k \left(x_{tj} - \frac{1}{k} \sum_{t=1}^k x_{tj}\right)^2 \sum_{t=1}^k \left(x_t - \frac{1}{k} \sum_{t=1}^k x_t\right)^2}},$$

4. INDIRECT METHODS FOR DETERMINING THE FUZZY WEIGHTING OF THE CRITERIA

for every $j \in \{1, \dots, n\}$. The fuzzy number \widetilde{w}_j can be reconstituted by the Negoitǎ-Ralescu characterization theorem ([45]).

It is difficult to give analytical solutions of the systems given by Eqs. (4.2) - (4.4) and Eqs. (4.5)-(4.7), even if the constrained variable method and reduced gradient method could be useful. Nevertheless, numerical solutions can be easily obtained by finding a finite set of r -level, $r \in \{r_0 = 0 < r_1 < \dots < r_{s-1} < r_s = 1\}$, for every $\widetilde{w}_j, j \in \{1, \dots, n\}$. The idea was launched in [39] related with the calculation of the fuzzy correlation coefficient. Of course, we obtain a good solution by choosing small differences $r_{h+1} - r_h$ for any $h \in \{0, \dots, s-1\}$ and a large s , but we pay a better quality by an increasing volume of computation.

The results obtained by fuzzy methods can be easily interpreted after defuzzification. A ranking method based on the expected value and adjusted to our structure of data consists in the following. Let A be a fuzzy number for which we know the r -level sets $r_h = \frac{h}{s}, h \in \{0, \dots, s\}, s \geq 2$. Then:

$$EV^-(A) = \frac{A_0^- + A_0^+ + A_1^- + A_1^+}{4s} + \frac{1}{2s} \sum_{h=1}^{s-1} A_{\frac{h}{s}}^- + \frac{1}{2s} \sum_{h=1}^{s-1} A_{\frac{h}{s}}^+. \quad (4.8)$$

4.1.2 Practical method description

A good choice for the approximation of a fuzzy number A , having the r -level sets $A_r = [A_r^-, A_r^+], r \in [0, 1]$ is the trapezoidal fuzzy number $(A_0^-, A_1^-, A_1^+, A_0^+)$. In this case it is enough to solve the following four problems to find the fuzzy weight $\widetilde{w}_j = (p_j, q_j, r_j, s_j), j \in \{1, \dots, n\}$:

$$p_j = \min f_j(x_{1j}, \dots, x_{kj}, x_1, \dots, x_k)$$

such that

$$\left(\widetilde{X}_{tj}\right)_0^- \leq x_{tj} \leq \left(\widetilde{X}_{tj}\right)_0^+, \forall t \in \{1, \dots, k\},$$

$$\left(\widetilde{X}_t\right)_0^- \leq x_t \leq \left(\widetilde{X}_t\right)_0^+, \forall t \in \{1, \dots, k\},$$

$$q_j = \min f_j(x_{1j}, \dots, x_{kj}, x_1, \dots, x_k)$$

such that

$$\left(\widetilde{X}_{tj}\right)_1^- \leq x_{tj} \leq \left(\widetilde{X}_{tj}\right)_1^+, \forall t \in \{1, \dots, k\},$$

4.1 Method for determining the derived fuzzy weighting of the criteria based on correlation coefficient as well as on the arithmetic generated by triangular norm T_M

$$\left(\widetilde{X}_t\right)_1^- \leq x_t \leq \left(\widetilde{X}_t\right)_1^+, \forall t \in \{1, \dots, k\},$$

$$r_j = \max f_j(x_{1j}, \dots, x_{kj}, x_1, \dots, x_k)$$

such that

$$\left(\widetilde{X}_{tj}\right)_1^- \leq x_{tj} \leq \left(\widetilde{X}_{tj}\right)_1^+, \forall t \in \{1, \dots, k\},$$

$$\left(\widetilde{X}_t\right)_1^- \leq x_t \leq \left(\widetilde{X}_t\right)_1^+, \forall t \in \{1, \dots, k\}$$

and

$$s_j = \max f_j(x_{1j}, \dots, x_{kj}, x_1, \dots, x_k)$$

such that

$$\left(\widetilde{X}_{tj}\right)_0^- \leq x_{tj} \leq \left(\widetilde{X}_{tj}\right)_0^+, \forall t \in \{1, \dots, k\},$$

$$\left(\widetilde{X}_t\right)_0^- \leq x_t \leq \left(\widetilde{X}_t\right)_0^+, \forall t \in \{1, \dots, k\},$$

where

$$f_j(x_{1j}, \dots, x_{kj}, x_1, \dots, x_k) = \frac{\sum_{t=1}^k \left(x_{tj} - \frac{1}{k} \sum_{t=1}^k x_{tj}\right) \left(x_t - \frac{1}{k} \sum_{t=1}^k x_t\right)}{\sqrt{\sum_{t=1}^k \left(x_{tj} - \frac{1}{k} \sum_{t=1}^k x_{tj}\right)^2 \sum_{t=1}^k \left(x_t - \frac{1}{k} \sum_{t=1}^k x_t\right)^2}},$$

for every $j \in \{1, \dots, n\}$.

Regarding the ranking method, the expected value of the naive trapezoidal approximation $(A_0^-, A_1^-, A_1^+, A_0^+)$ of A is (see (2.17)):

$$EV^-(A) = \frac{A_0^- + A_0^+ + A_1^- + A_1^+}{4}. \quad (4.9)$$

4.1.3 The algorithms

4.1.4 Numerical examples and a case study on the quality of hotel services

4.1.5 Conclusions

The obtained results regarding to the weights of the criteria can be used in subsequent studies as classical or fuzzy MCDM methods or IPA analysis. On the other hand, we emphasize here that the choice of appropriate sets of criteria are a very important step in any analysis related to different parts of the service industry (see, e.g., [34]).

The obtained results presented in Section 4.1 were published in the paper [8].

4. INDIRECT METHODS FOR DETERMINING THE FUZZY WEIGHTING OF THE CRITERIA

4.2 Method for determining the derived fuzzy weighting of the criteria based on correlation coefficient as well as on the arithmetic generated by triangular norm T_W

In this section it is also proposed an indirect method for calculation of the fuzzy weights of the criteria based on the correlation coefficient, the triangular fuzzy numbers, but on the fuzzy arithmetic generated by the triangular norm T_W .

4.2.1 Method description

It is well-known that the shape of fuzzy numbers is preserved by the generated addition and multiplication from T_W , the calculus is simple and, moreover, the ambiguity of the output data is preserved in reasonable limits. It is considered that all operations in (4.1) are obtained by the extension principle using T_W norm (see (2.24)). If the input data are triangular fuzzy numbers, that is $\widetilde{X}_{ij} \in \Delta FN(\mathbb{R})$ and $\widetilde{X}_i \in \Delta FN(\mathbb{R})$ for every $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, then for the calculation of the fuzzy weights of the criteria in (4.1) we can use the operations introduced by (2.2)-(2.7). By defuzzification we get, in the same time, a hierarchy of the criteria too.

4.2.2 The algorithm

4.2.3 Numerical examples and a case study on the quality of hotel services

4.2.3.1 The global case and the dependence on subjective choices of the fuzzy numbers. Symmetric case versus drastic case.

4.2.3.2 Segmented case and the dependence on various characteristics

4.2.4 Comparison with other methods of calculations

4.2.4.1 Proposed method versus direct method

4.2.4.2 Proposed method versus indirect method in [8]

4.2.5 Conclusions

Based on the presented case study regarding the quality of the hotel services in Oradea, Romania, we can conclude that in the calculation of the weights of the criteria

4.2 Method for determining the derived fuzzy weighting of the criteria based on correlation coefficient as well as on the arithmetic generated by triangular norm T_W

based on the correlation method, already classical arithmetical operations obtained by the standard Zadeh extension principle could be replaced by the T_W -based arithmetical operations generated by (2.24) (see (2.2)-(2.7)). The main advantages are related with the possibility of an analytical calculation and a less complicated calculation from the point of view of computational resources, both important for subsequent developments. Also, the case study illustrates both the similarities between the results obtained by the two indirect proposed methods and the differences between them and the direct method for calculation of the fuzzy weights of the criteria.

The obtained results presented in Section 4.2 were published in the papers [9, 12].

Chapter 5

Fuzzy multiple criteria decision making methods

5.1 Fuzzy MCDM method based on intervals of trapezoidal fuzzy numbers

In this section we intend to generalize the method from [3] by using the intervals of trapezoidal fuzzy numbers, to be applied into situations in which people surveyed want to choose two answers or an intermediate answer from the given response options.

5.1.1 Method description

A standard multicriteria decision making problem assumes a committee of k decision makers D_1, \dots, D_k , which is responsible for evaluating m alternatives A_1, \dots, A_m under n criteria C_1, \dots, C_n . We consider that C_1, \dots, C_h are subjective criteria, C_{h+1}, \dots, C_p are objective criteria of benefit kind and C_{p+1}, \dots, C_n are objective criteria of cost kind. In addition, as a generalization of the fuzzy multicriteria decision making method proposed in [3] we consider that the evaluations are given by intervals of trapezoidal fuzzy numbers.

If $\widetilde{r}_{ijt} = [(e_{ijt}^L, f_{ijt}^L, g_{ijt}^L, h_{ijt}^L), (e_{ijt}^U, f_{ijt}^U, g_{ijt}^U, h_{ijt}^U)]$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, h\}$, $t \in \{1, \dots, k\}$ is the performance of alternative A_i versus subjective criterion C_j in the

5.1 Fuzzy MCDM method based on intervals of trapezoidal fuzzy numbers

opinion of the decision maker D_t , then for $i \in \{1, \dots, m\}$, $j \in \{1, \dots, h\}$,

$$\begin{aligned} \widetilde{r}_{ij} &= [(e_{ij}^L, f_{ij}^L, g_{ij}^L, h_{ij}^L), (e_{ij}^U, f_{ij}^U, g_{ij}^U, h_{ij}^U)] = \\ &= \left[\left(\sum_{t=1}^k \frac{e_{ij}^L}{k}, \sum_{t=1}^k \frac{f_{ij}^L}{k}, \sum_{t=1}^k \frac{g_{ij}^L}{k}, \sum_{t=1}^k \frac{h_{ij}^L}{k} \right), \left(\sum_{t=1}^k \frac{e_{ij}^U}{k}, \sum_{t=1}^k \frac{f_{ij}^U}{k}, \sum_{t=1}^k \frac{g_{ij}^U}{k}, \sum_{t=1}^k \frac{h_{ij}^U}{k} \right) \right] \end{aligned} \quad (5.1)$$

is the averaged rating of A_i under C_j . On the other hand, if $\widetilde{x}_{ij} = [(a_{ij}^L, b_{ij}^L, c_{ij}^L, d_{ij}^L), (a_{ij}^U, b_{ij}^U, c_{ij}^U, d_{ij}^U)]$, $i \in \{1, \dots, m\}$, $j \in \{h+1, \dots, n\}$ is the performance of alternative A_i versus objective criterion C_j , then the normalized values of performances with respect to benefit criteria are $\widetilde{r}_{ij} = [(e_{ij}^L, f_{ij}^L, g_{ij}^L, h_{ij}^L), (e_{ij}^U, f_{ij}^U, g_{ij}^U, h_{ij}^U)]$, $i \in \{1, \dots, m\}$, $j \in \{h+1, \dots, p\}$, where

$$e_{ij}^L = \frac{a_{ij}^L - a_j^{L*}}{m_j^{L*}}, \quad f_{ij}^L = \frac{b_{ij}^L - a_j^{L*}}{m_j^{L*}}, \quad g_{ij}^L = \frac{c_{ij}^L - a_j^{L*}}{m_j^{L*}}, \quad h_{ij}^L = \frac{d_{ij}^L - a_j^{L*}}{m_j^{L*}}, \quad (5.2)$$

$$e_{ij}^U = \frac{a_{ij}^U - a_j^{U*}}{m_j^{U*}}, \quad f_{ij}^U = \frac{b_{ij}^U - a_j^{U*}}{m_j^{U*}}, \quad g_{ij}^U = \frac{c_{ij}^U - a_j^{U*}}{m_j^{U*}}, \quad h_{ij}^U = \frac{d_{ij}^U - a_j^{U*}}{m_j^{U*}}, \quad (5.3)$$

and the normalized values of performances with respect to cost criteria are $\widetilde{r}_{ij} = [(e_{ij}^L, f_{ij}^L, g_{ij}^L, h_{ij}^L), (e_{ij}^U, f_{ij}^U, g_{ij}^U, h_{ij}^U)]$, $i \in \{1, \dots, m\}$, $j \in \{p+1, \dots, n\}$, where

$$e_{ij}^L = \frac{d_j^{L*} - d_{ij}^L}{m_j^{L*}}, \quad f_{ij}^L = \frac{d_j^{L*} - c_{ij}^L}{m_j^{L*}}, \quad g_{ij}^L = \frac{d_j^{L*} - b_{ij}^L}{m_j^{L*}}, \quad h_{ij}^L = \frac{d_j^{L*} - a_{ij}^L}{m_j^{L*}}, \quad (5.4)$$

$$e_{ij}^U = \frac{d_j^{U*} - d_{ij}^U}{m_j^{U*}}, \quad f_{ij}^U = \frac{d_j^{U*} - c_{ij}^U}{m_j^{U*}}, \quad g_{ij}^U = \frac{d_j^{U*} - b_{ij}^U}{m_j^{U*}}, \quad h_{ij}^U = \frac{d_j^{U*} - a_{ij}^U}{m_j^{U*}}, \quad (5.5)$$

and for $j \in \{h+1, \dots, n\}$

$$\begin{aligned} a_j^{L*} &= \min_{i \in \{1, \dots, m\}} a_{ij}^L, & d_j^{L*} &= \max_{i \in \{1, \dots, m\}} d_{ij}^L, & m_j^{L*} &= d_j^{L*} - a_j^{L*}, \\ a_j^{U*} &= \min_{i \in \{1, \dots, m\}} a_{ij}^U, & d_j^{U*} &= \max_{i \in \{1, \dots, m\}} d_{ij}^U, & m_j^{U*} &= d_j^{U*} - a_j^{U*}. \end{aligned}$$

If $\widetilde{w}_t = [(o_{jt}^L, p_{jt}^L, q_{jt}^L, s_{jt}^L), (o_{jt}^U, p_{jt}^U, q_{jt}^U, s_{jt}^U)]$, $j \in \{1, \dots, n\}$, $t \in \{1, \dots, k\}$ is the weight of the criterion C_j in opinion of the decision maker D_t , then the averaged weight of the criterion C_j is \widetilde{w}_j , $j \in \{1, \dots, n\}$, given by

$$\begin{aligned} \widetilde{w}_j &= [(o_j^L, p_j^L, q_j^L, s_j^L), (o_j^U, p_j^U, q_j^U, s_j^U)] = \\ &= \left[\left(\sum_{t=1}^k \frac{o_{jt}^L}{k}, \sum_{t=1}^k \frac{p_{jt}^L}{k}, \sum_{t=1}^k \frac{q_{jt}^L}{k}, \sum_{t=1}^k \frac{s_{jt}^L}{k} \right), \left(\sum_{t=1}^k \frac{o_{jt}^U}{k}, \sum_{t=1}^k \frac{p_{jt}^U}{k}, \sum_{t=1}^k \frac{q_{jt}^U}{k}, \sum_{t=1}^k \frac{s_{jt}^U}{k} \right) \right]. \end{aligned} \quad (5.6)$$

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The final evaluation value \widetilde{G}_i of alternatives A_i is the aggregation of the weighted ratings by interval of fuzzy numbers, under formula:

$$\widetilde{G}_i = \frac{1}{n}((\widetilde{r}_{i1} \otimes \widetilde{w}_1) + \dots + (\widetilde{r}_{in} \otimes \widetilde{w}_n)), i \in \{1, \dots, m\}.$$

We obtain:

$$EV(\widetilde{G}_i) = \frac{1}{n}(EV(\widetilde{r}_{i1} \otimes \widetilde{w}_1) + \dots + EV(\widetilde{r}_{in} \otimes \widetilde{w}_n)), \quad (5.7)$$

where the calculus of each terms in the sum can be performed for every $j \in \{1, \dots, n\}$, such as:

$$\begin{aligned} EV(\widetilde{r}_{ij} \otimes \widetilde{w}_j) &= \\ &= \frac{1}{24}(2e_{ij}^L o_j^L + f_{ij}^L o_j^L + e_{ij}^L p_j^L + 2f_{ij}^L p_j^L) + \frac{1}{24}(2g_{ij}^L q_j^L + h_{ij}^L q_j^L + g_{ij}^L s_j^L + 2h_{ij}^L s_j^L) + \\ &+ \frac{1}{24}(2e_{ij}^U o_j^U + f_{ij}^U o_j^U + e_{ij}^U p_j^U + 2f_{ij}^U p_j^U) + \frac{1}{24}(2g_{ij}^U q_j^U + h_{ij}^U q_j^U + g_{ij}^U s_j^U + 2h_{ij}^U s_j^U). \end{aligned}$$

Finally, knowing the crisp values of the final scores of alternatives, after defuzzification using the expected value, we get a hierarchy of alternatives, using the order relation on the real numbers.

5.1.2 The algorithm

5.1.3 Numerical example

5.1.4 Comparison with other methods

5.1.5 Conclusions

The comparison made in Section 5.1.4, from the fact that were obtained the same hierarchies, it does nothing but confirm the validity of the proposed method. We can assume with conviction that the hierarchy in a generalized case that contains many different inputs than in the individual case, will be different from that of the individual case.

The obtained results presented in Section 5.1 were published in the paper [51].

5.2 Fuzzy MCDM method based on intuitionistic fuzzy numbers

Due to the simple form and easy computation, the trapezoidal intuitionistic fuzzy numbers can be successfully used in the intuitionistic fuzzy MCDM methods. The proposed method is based on two aggregation operators and four ranking methods for trapezoidal intuitionistic fuzzy numbers.

5.2.1 Method description

We consider that the performance of an alternative A_i under the criterion C_j in the opinion of the decision maker D_t is given by a non-negative trapezoidal intuitionistic fuzzy number $\widetilde{r}_{ijt} = \langle (a_{1ijt}, b_{1ijt}, c_{1ijt}, d_{1ijt}), (a_{2ijt}, b_{2ijt}, c_{2ijt}, d_{2ijt}) \rangle$ and the weight of the criterion C_j in the opinion of the decision maker D_t it is also given by a non-negative trapezoidal intuitionistic fuzzy number $\widetilde{w}_{jt} = \langle (e_{1jt}, f_{1jt}, g_{1jt}, h_{1jt}), (e_{2jt}, f_{2jt}, g_{2jt}, h_{2jt}) \rangle$.

The first step of the proposed method consists in calculating the average rating \widetilde{r}_{ij} of performances of alternative A_i under criteria C_j , $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$, in order to obtain the decision matrix, as follows:

$$\widetilde{r}_{ij} = (1/k) \cdot (\widetilde{r}_{ij1} + \dots + \widetilde{r}_{ijk}), \quad (5.8)$$

more exactly, using the operations (2.8) and (2.9) from Section 2.4,

$$\begin{aligned} \widetilde{r}_{ij} = \langle & \left(\frac{1}{k} \cdot \sum_{t=1}^k a_{1ijt}, \frac{1}{k} \cdot \sum_{t=1}^k b_{1ijt}, \frac{1}{k} \cdot \sum_{t=1}^k c_{1ijt}, \frac{1}{k} \cdot \sum_{t=1}^k d_{1ijt} \right), \\ & \left(\frac{1}{k} \cdot \sum_{t=1}^k a_{2ijt}, \frac{1}{k} \cdot \sum_{t=1}^k b_{2ijt}, \frac{1}{k} \cdot \sum_{t=1}^k c_{2ijt}, \frac{1}{k} \cdot \sum_{t=1}^k d_{2ijt} \right) \rangle. \end{aligned} \quad (5.9)$$

Next step is the calculation of the average weight \widetilde{w}_j of the criterion C_j , $j \in \{1, \dots, n\}$, as follows:

$$\widetilde{w}_j = (1/k) \cdot (\widetilde{w}_{j1} + \dots + \widetilde{w}_{jk}), \quad (5.10)$$

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and also using the operations (2.8) and (2.9) from Section 2.4,

$$\begin{aligned} \widetilde{w}_j = & \langle (\frac{1}{k} \cdot \sum_{t=1}^k e_{1jt}, \frac{1}{k} \cdot \sum_{t=1}^k f_{1jt}, \frac{1}{k} \cdot \sum_{t=1}^k g_{1jt}, \frac{1}{k} \cdot \sum_{t=1}^k h_{1jt}), \\ & (\frac{1}{k} \cdot \sum_{t=1}^k e_{2jt}, \frac{1}{k} \cdot \sum_{t=1}^k f_{2jt}, \frac{1}{k} \cdot \sum_{t=1}^k g_{2jt}, \frac{1}{k} \cdot \sum_{t=1}^k h_{2jt}) \rangle. \end{aligned} \quad (5.11)$$

For the next step we have to normalize the values of average performances with respect to criteria and the values of averaged weights of criteria. This is only necessary if the maximum value of the performances and/or respectively the maximum value of the weights are greater than 1. We normalize as follows: if $\widetilde{r}_{ij} = \langle (a_{1ij}, b_{1ij}, c_{1ij}, d_{1ij}), (a_{2ij}, b_{2ij}, c_{2ij}, d_{2ij}) \rangle$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ and we find that $\alpha = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} d_{2ij} > 1$, then

$$\widetilde{r}_{ij} = (1/\alpha) \cdot \widetilde{r}_{ij}, \quad (5.12)$$

using (2.9) from Section 2.4. For simplicity, we used the same notation \widetilde{r}_{ij} for the normalized values in decision matrix too. In the same way, if $\widetilde{w}_j = \langle (e_{1j}, f_{1j}, g_{1j}, h_{1j}), (e_{2j}, f_{2j}, g_{2j}, h_{2j}) \rangle$, $j \in \{1, \dots, n\}$ and we find that $\beta = \max_{1 \leq j \leq n} h_{2j} > 1$, then

$$\widetilde{w}_j = (1/\beta) \cdot \widetilde{w}_j, \quad (5.13)$$

using also the operation (2.9) from Section 2.4. We also used the same notation \widetilde{w}_j for the normalized values of the weights of the criteria. Next step is to evaluate the alternatives A_i , $i \in \{1, \dots, m\}$ by the aggregation of the performances with weights using the $WAA_{\widetilde{\otimes}}$ operator, developed as:

$$\widetilde{G}_i = (1/n) \cdot (\widetilde{r}_{i1} \otimes \widetilde{w}_1 + \dots + \widetilde{r}_{in} \otimes \widetilde{w}_n), \quad (5.14)$$

and using the operations (2.8), (2.9) and (2.10) from Section 2.4,

$$\begin{aligned} \widetilde{G}_i = & \langle (\frac{1}{n} \sum_{j=1}^n (a_{1ij} \cdot e_{1j}), \frac{1}{n} \sum_{j=1}^n (b_{1ij} \cdot f_{1j}), \frac{1}{n} \sum_{j=1}^n (c_{1ij} \cdot g_{1j}), \frac{1}{n} \sum_{j=1}^n (d_{1ij} \cdot h_{1j})), \\ & (\frac{1}{n} \sum_{j=1}^n (a_{2ij} \cdot e_{2j}), \frac{1}{n} \sum_{j=1}^n (b_{2ij} \cdot f_{2j}), \frac{1}{n} \sum_{j=1}^n (c_{2ij} \cdot g_{2j}), \frac{1}{n} \sum_{j=1}^n (d_{2ij} \cdot h_{2j})) \rangle. \end{aligned} \quad (5.15)$$

5.2 Fuzzy MCDM method based on intuitionistic fuzzy numbers

Then, if we use the WGA_ω operator, then, for the beginning, the weights must be defuzzified using the expected value (see (2.22)), namely

$$w_j = EV(\widetilde{w}_j) = (e_{1j} + f_{1j} + g_{1j} + h_{1j} + e_{2j} + f_{2j} + g_{2j} + h_{2j})/8, \quad (5.16)$$

for $j = \{1, \dots, n\}$, then

$$\widetilde{H}_i = \widetilde{r}_{i1}^{w_1} \otimes \dots \otimes \widetilde{r}_{in}^{w_n}, \text{ pentru } i \in \{1, \dots, m\}, \quad (5.17)$$

and using the operations (2.10) and (2.11) from Section 2.4,

$$\begin{aligned} \widetilde{H}_i = \langle & (\prod_{j=1}^n a_{1ij}^{w_j}, \prod_{j=1}^n b_{1ij}^{w_j}, \prod_{j=1}^n c_{1ij}^{w_j}, \prod_{j=1}^n d_{1ij}^{w_j}), \\ & (\prod_{j=1}^n a_{2ij}^{w_j}, \prod_{j=1}^n b_{2ij}^{w_j}, \prod_{j=1}^n c_{2ij}^{w_j}, \prod_{j=1}^n d_{2ij}^{w_j}) \rangle. \end{aligned} \quad (5.18)$$

In order to obtain the ranking of alternatives, we used, one at a time, all four criteria from Definitions 10, 11, 12 and 13, separately for \widetilde{G}_i and \widetilde{H}_i , thus obtaining eight hierarchies.

5.2.2 The algorithm

5.2.3 Numerical examples

5.2.4 Comparison with other methods

5.2.5 Conclusions

A MCDM method based on more aggregating operators and/or ranking methods is particularly effective because, when we obtain the same hierarchies, the solution is a secure one and when we obtain different hierarchies, we can review the evaluations, then the solution of the problem will certainly lead to finding the best alternative. In addition, the proposed method that uses trapezoidal intuitionistic fuzzy numbers can be easily translated into a method that uses interval-valued trapezoidal fuzzy numbers, based on bijections proven in Section 2.4.

The obtained results presented in Section 5.2 were published in the paper [52] and the results in Section 2.4 in the paper [7].

Chapter 6

Fuzzy criteria partitioning methods in IPA

6.1 Fuzzy partitioning method using four categories

In this section we propose a method of determination of a fuzzy partition of a set of criteria, using the same four categories as in traditional IPA, namely:

$$\begin{aligned} P &= \{A_1, A_2, A_3, A_4\} = \\ &= \{ \text{"Keep up the good work"}, \text{"Concentrate here"}, \\ &\quad \text{"Low priority"}, \text{"Possible overkill"} \} = \{GW, CH, LP, PO\}. \end{aligned}$$

6.1.1 Method description

Let $\{a^1, \dots, a^n\}$ be a set of criteria and we denote by x^j , $j \in \{1, \dots, n\}$, the pair (*performance*, *weight*) corresponding to the criterion a^j , that is $x^j = (p^j, w^j)$. A fuzzy partition of the set $X = \{x^1, \dots, x^n\} \subset \mathbb{R}^2$ determines a degree of membership of each criteria to a set $A_i, i \in \{1, 2, 3, 4\}$. The method is based on the fuzzy c -means algorithm, using the atoms from P and the most suitable initial prototypes L^1, L^2, L^3, L^4 of the

6.1 Fuzzy partitioning method using four categories

fuzzy sets $A_1 = GW$, $A_2 = CH$, $A_3 = LP$ and respectively $A_4 = PO$, namely:

$$(L^1)_0 = ((L_1^1)_0, (L_2^1)_0) = \left(\max_{1 \leq j \leq n} p^j, \max_{1 \leq j \leq n} w^j \right), \quad (6.1)$$

$$(L^2)_0 = ((L_1^2)_0, (L_2^2)_0) = \left(\min_{1 \leq j \leq n} p^j, \max_{1 \leq j \leq n} w^j \right), \quad (6.2)$$

$$(L^3)_0 = ((L_1^3)_0, (L_2^3)_0) = \left(\min_{1 \leq j \leq n} p^j, \min_{1 \leq j \leq n} w^j \right), \quad (6.3)$$

$$(L^4)_0 = ((L_1^4)_0, (L_2^4)_0) = \left(\max_{1 \leq j \leq n} p^j, \min_{1 \leq j \leq n} w^j \right). \quad (6.4)$$

We chose the Euclidean distance on \mathbb{R}^2 and fixed the permissible error as $\varepsilon = 10^{-5}$. We present below the needed calculation formulas. For calculating the distances we have

$$(d_{ij})_l = d^2(x^j, (L^i)_l) = (p^j - (L_1^i)_l)^2 + (w^j - (L_2^i)_l)^2, \quad (6.5)$$

for calculating the partition matrix we have

$$q_{ij}^l = \frac{1}{\frac{(d_{ij})_l}{(d_{1j})_l} + \frac{(d_{ij})_l}{(d_{2j})_l} + \frac{(d_{ij})_l}{(d_{3j})_l} + \frac{(d_{ij})_l}{(d_{4j})_l}}, \quad (6.6)$$

and for calculating the prototypes we have

$$(L^i)_l = ((L_1^i)_l, (L_2^i)_l) = \left(\frac{\sum_{j=1}^n (q_{ij}^{l-1})^2 p^j}{\sum_{j=1}^n (q_{ij}^{l-1})^2}, \frac{\sum_{j=1}^n (q_{ij}^{l-1})^2 w^j}{\sum_{j=1}^n (q_{ij}^{l-1})^2} \right), \quad (6.7)$$

for every $i \in \{1, \dots, 4\}$ and $j \in \{1, \dots, n\}$.

Sometimes, for an easy interpretation and subsequent developments, especially in practice, it is mandatory to obtain a crisp partition $\bar{P} = \{\bar{A}_1, \dots, \bar{A}_s\}$ of X , derived from the fuzzy partition P , even if this means a loss of information. The most natural definition of \bar{P} is (see [23], p. 333)

$$x^j \in \bar{A}_i \text{ if and only if } A_i(x^j) = q_{ij} = \max_{1 \leq p \leq s} q_{pj} = \max_{1 \leq p \leq s} A_p(x^j). \quad (6.8)$$

Subsequent decisions can be taken on the basis of this crisp partition, but the practitioners must be very careful with respect to criteria which have large membership degrees to two or more atoms of the fuzzy partition.

6. FUZZY CRITERIA PARTITIONING METHODS IN IPA

6.1.2 The algorithm

6.1.3 Case studies and comparisons

6.1.4 Conclusions

The proposed method was applied on data from some studies in the recent literature and the obtained results were compared with results from the traditional IPA. They were given both examples in which the results obtained by fuzzy clustering were not very different from those obtained by traditional IPA and examples in which the results showed great differences. The comparison it was possible only after the defuzzification of the fuzzy partition, even if this means a loss of information. At our level of knowledge do not exist other methods related with IPA having fuzzy output data such that, at the moment, a such comparison is not possible. On the other hand, a mathematical proof on the quality of the method in comparison with other methods, crisp or fuzzy, will be very difficult, rather impossible, to be given.

The proposed method, besides the fact that determine a more natural partition of the criteria, because the fuzzy c -means algorithm uses the data structure, still has an advantage, namely that it improves the traditional IPA in at least two aspects: the points that are very close or even on the border between partitions are treated better and it is avoided the subjective drawing of axes.

The obtained results presented in Section 6.1 were published in the paper [11].

6.2 Fuzzy partitioning method using s categories

6.2.1 Method description

For the proposed method, we consider a traditional IPA model with fixed categories $\{A_1, \dots, A_s\}$, $s \geq 2$ and, for determining a fuzzy partition of the criteria, or, in other words, the membership degree of each criterion to each category A_i , $i \in \{1, \dots, s\}$, we apply an adapted fuzzy c -means algorithm. The crisp partition is obtain by a simple procedure of defuzzification. The method is exemplified to the case of nine categories of criteria. The solution of partitioning of a set of criteria certainly depends from the input data, such that the fuzzy clustering is a technique that can furnish indeed useful methods in relationship with IPA.

Further we describe the proposed method.

6.2 Fuzzy partitioning method using s categories

Let $X = \{a^1, \dots, a^n\}$ be the set of the criteria and we denote by x^j , $j \in \{1, \dots, n\}$, the pair (*performance, weight*) corresponding to the criterion a^j , that is $x^j = (p^j, w^j)$.

We present below the needed calculation formulas. For calculating the distances we have

$$(d_{ij})_l = d^2(x^j, (L^i)_l) = (p^j - (L^i)_l)^2 + (w^j - (L^i)_l)^2, \quad (6.9)$$

for calculating the partition matrix we have

$$q_{ij}^l = \frac{1}{\frac{(d_{ij})_l}{(d_{1j})_l} + \dots + \frac{(d_{ij})_l}{(d_{sj})_l}}, \quad (6.10)$$

and for calculating the prototypes we have

$$(L^i)_l = ((L^i)_l, (L^i)_l) = \left(\frac{\sum_{j=1}^n (q_{ij}^{l-1})^2 p^j}{\sum_{j=1}^n (q_{ij}^{l-1})^2}, \frac{\sum_{j=1}^n (q_{ij}^{l-1})^2 w^j}{\sum_{j=1}^n (q_{ij}^{l-1})^2} \right), \quad (6.11)$$

for every $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, n\}$.

The crisp partition $\bar{P} = \{\bar{A}_1, \dots, \bar{A}_s\}$ of X can be obtain from the fuzzy partition $P = \{A_1, \dots, A_s\}$ of X following (6.8). The crisp partition allows a simple interpretation and subsequent developments related with the problem under study.

In [1] it is assigned a ranking to both the weight scores and performance scores of each criteria, namely "High", "Medium" or "Low" for the weight scores and respectively "A", "B" or "C" for performance scores. This created nine possible categories of criteria, as follows:

<i>High</i>	A_7 : <i>Competitive vulnerability</i>	A_8 : <i>High B</i>	A_1 : <i>Competitive strength</i>
Weight	A_6 : <i>Medium C</i>	A_9 : <i>Gray zone</i>	A_2 : <i>Medium A</i>
<i>Low</i>	A_5 : <i>Relative indifference</i>	A_4 : <i>Low B</i>	A_3 : <i>Irrelevant superiority</i>
	<i>Low (C)</i>	Performance	<i>High (A)</i>

Our aim is to determine for the set X of criteria, the fuzzy partition $P = \{A_1, \dots, A_9\}$, corresponding to the above scheme, that is $A_1 =$ "Competitive strength", $A_2 =$ "Medium A", $A_3 =$ "Irrelevant superiority", $A_4 =$ "Low B", $A_5 =$ "Relative indifference", $A_6 =$ "Medium C", $A_7 =$ "Competitive vulnerability", $A_8 =$ "High B" and $A_9 =$ "Gray zone". Let $X = \{a^1, \dots, a^n\} \subset \mathbb{R}^2$ and $x^j = (p^j, w^j)$ be the set of points (*performance, weight*) corresponding to criteria. The following natural choice of the initial prototypes of the

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atoms A_i , $i \in \{1, \dots, 9\}$ can be considered as

$$\begin{aligned} (L^1)_0 &= \left(\max_{1 \leq j \leq n} p^j, \max_{1 \leq j \leq n} w^j \right), & (L^2)_0 &= \left(\max_{1 \leq j \leq n} p^j, \sum_{1 \leq j \leq n} w^j/n \right), \\ (L^3)_0 &= \left(\max_{1 \leq j \leq n} p^j, \min_{1 \leq j \leq n} w^j \right), & (L^4)_0 &= \left(\sum_{1 \leq j \leq n} p^j/n, \min_{1 \leq j \leq n} w^j \right), \\ (L^5)_0 &= \left(\min_{1 \leq j \leq n} p^j, \min_{1 \leq j \leq n} w^j \right), & (L^6)_0 &= \left(\min_{1 \leq j \leq n} p^j, \sum_{1 \leq j \leq n} w^j/n \right), \\ (L^7)_0 &= \left(\min_{1 \leq j \leq n} p^j, \max_{1 \leq j \leq n} w^j \right), & (L^8)_0 &= \left(\sum_{1 \leq j \leq n} p^j/n, \max_{1 \leq j \leq n} w^j \right), \\ (L^9)_0 &= \left(\sum_{1 \leq j \leq n} p^j/n, \sum_{1 \leq j \leq n} w^j/n \right). \end{aligned}$$

6.2.2 The algorithm

6.2.3 Case studies

6.2.4 Conclusions

In the second section of this chapter we proposed a generalization of the method from the first section of this chapter, namely it has been proposed a method for categorization of a set of criteria using s fuzzy partitions. The generalized achieved method was exemplified in the case of nine possible categories, whose prototypes were obtained in a natural way. We used four case studies taken from the recent literature.

The proposed method has the same advantages as the proposed method in Section 6.1, namely that it cause a natural partition on the set of criteria, avoid the rigorous classification of the criteria located near the axes and avoid the subjective drawing of axes.

The obtained results presented in Section 6.2 were included in the submitted paper [10].

Chapter 7

Conclusions

This thesis contains original contributions in the field of the fuzzy MCDM methods, namely six fuzzy methods published in the papers [7, 8, 9, 11, 12, 51, 52], two fuzzy methods for indirect calculations of the fuzzy weights of the criteria, two fuzzy MCDM methods in order to obtain the hierarchy of the alternatives and another two fuzzy partitioning methods for a classification of criteria.

7.1 Fulfilling the research objectives

The first specific objective O_1 has been assessed by two proposed indirect methods for calculating the fuzzy weights of the criteria based on correlation coefficient between alternatives performances and the overall level of satisfaction, both of them in the opinion of the decision makers. The proposed methods use the correlation coefficient between two fuzzy numbers, the result being a fuzzy number too. The effective calculus is based on Zadeh's extension principle. As far as the first method described in Section 4.1.1 is concerned, the numerical results are obtained considering the r -level sets because it is difficult to give an analytical solution when using the T_M -based arithmetic. Even though, the amount of calculation is very big, so that the proposed practical method described in Section 4.1.2 is to be preferred when the number of the decision makers and/or criteria is big. The proposed practical method thus obtained has one more important advantage, that is it furnishes trapezoidal fuzzy numbers, which can be easily compared, interpreted and used in further calculus. In order to improve the proposed method in Section 4.1, we proposed in Section 4.2 an indirect method to

7. CONCLUSIONS

obtain fuzzy weights of the criteria, but using the T_W -based arithmetic. The method provide a hierarchy of the given criteria, considering the obtained weights, beginning with the most important criterion and ending with the least important. The proposed methods represent an important stage within a fuzzy IPA analysis development, as well as in the fuzzy MCDM methods. They also bring something new because the existing contributions in the literature (see [17], [20]) contain a premature defuzzification of the data, which leads to denaturation of the results.

The second specific objective O_2 was realized, first of all, by proposed fuzzy MCDM method from Section 5.1, which models the situation when in a survey you can answer to certain questions with an intermediate response or even with two responses. Its applicability was illustrated by an example in Section 5.1.3. As far as this objective is concern, we proposed another fuzzy MCDM method in Section 5.2 based on trapezoidal intuitionistic fuzzy numbers, which models well the uncertainties. The proposed method improves other existing methods in the literature (see, e.g., [38], [53]) because it stocks more information using operations with trapezoidal intuitionistic fuzzy numbers all along the method and only at the end the results are defuzzified for easy interpretation of the results.

For the achieving of the third specific objective O_3 , we proposed a fuzzy partitioning method. The fuzzy clustering is known as an excellent tool to identify the structure in data, therefore, we adapt the fuzzy c -means algorithm. As it is illustrated in the scientific literature, the problem of categorization of criteria in IPA is very important. We do not have the possibility to prove by mathematical means that the proposed method is better than other methods, therefore we cannot affirm that our method completely solve this problem. Nevertheless, the developed method has some significant advantages. Besides the results are more suitable for deriving managerial decisions than in traditional IPA, we overcome the important problem of categorization of the criteria which are in the nearness of axes in the traditional IPA.

7.2 Future research directions

The results obtained by the proposed methods in Chapter 4 can be exploited in other directions. One of them consists, for example, in choosing a representative hotel for which to study the customer behavior approaching to the general level of satisfaction.

Also, future researches will be dedicated to the application of the elaborated method to different categories of customers, according with certain criteria, or to the identification of the representative hotel in an area or resort. As it was already remarked even in the crisp case (see, e.g., [26]), it is very difficult to give a final answer to the question of the best method of calculation of the weights of the criteria. The choosing of a method is context dependent and it is a matter of preference. Nevertheless, in the case of a derived fuzzy weighting based on the correlation coefficient, the method proposed in Sections 4.1 and 4.2 has some already pointed out advantages.

In Section 2.4 were demonstrated bijections between the set of trapezoidal (triangular) intuitionistic fuzzy numbers and the set of interval-valued trapezoidal (triangular) fuzzy numbers. These bijections lead immediately to extending and generalizing the existing methods, as shown in Section 5.2.5. The proposed method in Section 5.2 can be easily applied for interval-valued trapezoidal fuzzy numbers too, by a simple adaptation. As future research directions too, it can be seen that the proposed method in Section 5.2 can be easily extended to other types of intuitionistic fuzzy numbers. Also, because of the fact that trapezoidal intuitionistic fuzzy numbers are a generalization of trapezoidal fuzzy numbers, other existing fuzzy MCDM methods that use fuzzy numbers can be extended to intuitionistic fuzzy numbers. Last but not least, we intend to search for other effective aggregation operators and/or ranking methods for the trapezoidal intuitionistic fuzzy numbers which will be integrated in our method.

The problem of partitioning a set of criteria depends certainly on the input data, such that fuzzy grouping is a technique that can provide really useful methods regarding IPA analysis. We do not consider that our proposed method in Section 6.1 completely solves the very important problem of categorization of criteria, but it proposes a new possible approach in this topic and has clear advantages, pointed out in the thesis. The new approach can be continued at least in the directions of research discussed in the sequel. Even if the traditional IPA assumes the categorization of criteria by partitioning the (*performance*, *weight*) plane in four sets, there have been attempts to divide the plane in two (see [49], [50]), three (see [42]) or even nine (see [1]) areas. Based on the idea developed in Section 6.1, it would be very interesting to apply the same adapted fuzzy *c*-means algorithm in these cases. On the other hand, we can deduce that it is not very clear whether a good IPA means a partitioning of the *performance*, *weight*) plane in two, three, four, nine or another number of sets. We believe that the

7. CONCLUSIONS

number depends rather from the input data. Because the fuzzy clustering identify the structure of data in a natural way (see, e.g., [14] and [25]), by applying an adequate algorithm we can simultaneously obtain the optimal number of categories of criteria and the categorization of criteria. Such a result would be valuable for determining the optimal number of criteria from the point of view of their significance related to the problem under study, an important subject of debate in the recent literature (see [34]). A more laborious project looks on the development of a fuzzy IPA, that is an analysis starting from fuzzy data and having as output data fuzzy sets too. On the other hand, the IPA was expanded from two dimensional grid (*performance, weight*) to three dimensions (*performance, weight, competitors' performance*) (see [15]) or four dimensions (*performance, weight, competitors' performance, criterion determinacy*) (see [56]) such that the methods and algorithms of fuzzy clustering, discussed in the present thesis or not, could be useful in the developing of these directions of research.

Appendices

Appendix 1

Application to Chapter 4, Section 4.1. Indirect calculus of fuzzy weights based on correlation coefficient as well as on the arithmetic generated by triangular norm T_M .

Appendix 2

Application to Chapter 4, Section 4.2. Indirect calculus of fuzzy weights based on correlation coefficient as well as on the arithmetic generated by triangular norm T_W .

Appendix 3

Application to Chapter 5, Section 5.1. Fuzzy multicriteria decision making method based on intervals of trapezoidal fuzzy numbers.

Appendix 4

Application to Chapter 5, Section 5.2. Fuzzy multicriteria decision making method based on trapezoidal intuitionistic fuzzy numbers.

Appendix 5

Application to Chapter 6, Section 6.1. Fuzzy criteria partitioning using four categories and an adapted form of the fuzzy c -means algorithm.

Appendix 6

Application to Chapter 6, Section 6.2. Fuzzy criteria partitioning using nine categories and an adapted form of the fuzzy c -means algorithm.

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