Fixed point theorems on spaces endowed with generalized metrics

Iulia Dorina Coroian

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Prof. Dr. Adrian-Olimpiu Petruşel

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President of the Committee: Prof. Dr. Radu Precup

Referees:

- Prof. Dr. Aurelian Cernea (University of Bucharest)
- Prof. Dr. Dorian Popa (Technical University Cluj-Napoca)
- Conf. Dr. Marcel-Adrian Şerban (Babeş-Bolyai University Cluj-Napoca)

Scientific Advisor: Prof. Dr. Adrian Petruşel

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Contents

Introduction

Fixed point theory is today one of the most dynamic research area with many theoretical and applicative results. Starting with the well know Banach-Caccioppoli Contraction Principle (in the '30), the field of the metric fixed point theory is, probably, the most important part of it. Fixed point theory is an interdisciplinary topic which can be applied in various disciplines of mathematics and mathematical sciences like game theory, mathematical economics, optimization theory, approximation theory and variational inequalities. Banach contraction principle has been studied and generalized in different spaces and for various generalized contraction conditions.

The development of this theory is related to the possibility to obtain different applications in various topics such as: Optimization Theory, Integral and Differential Equations and Inclusions, Fractal Theory, Mathematical Economics and many others.

In 1969, Nadler [47] initiated the development of the metric fixed point theory for multivalued mappings. Nadler used the concept of Haurdorff-Pompeiu metric to establish the multivalued contraction principle, containing the Banach contraction principle as a special case.

The distance between the subsets of $\mathbb{C}$ was introduced by the Romanian mathematician Dimitrie Pompeiu in 1905 in his Ph. D. Thesis [65]. Is worth that the concept of metric space appeared in the Ph. D. thesis of Maurice Frechet [26] published a little bit later in 1906. Felix Hausdorff studied the notion of set distance in the setting of metric spaces in his book from 1914 [30]. Hausdorff quoted Pompeiu as the author of the notion of distance between sets, but not in the main text of the book, only in the final notes, as was the procedure at that time to indicate the references. And thus the distance between sets came to be called the Haurdorff-Pompeiu distance.

The metric fixed point theory had a strong development including both the single-valued and the multivalued case. In fact, starting to the multivalued version of Banach-Caccioppoli's contraction principle, proved by S.B. Nadler jr. in 1969 (for multivalued operators with closed, bounded, nonempty values defined on a complete metric space and extend one year later by H. Covitz and S.B. Nadler jr. [20] for multivalued operators with nonempty and closed values), the metric fixed point theory for multivalued operators involves today more than 1000 papers in the literature.

The purpose of this thesis is to present several contributions to the metric fixed point theory for multivalued generalized contractions.

There are various extensions of the classical notion of multivalued α- contraction introduced by S.B. Nadler jr.. Most of them are working with the well known Haurdorff-Pompeiu metric H. In our study, we will also consider a Pompeiu type metric H_d^+, as well as some other related metrics (topologically equivalent to H) in order to obtain generalizations of the above mentioned principle.

Another research direction of this thesis is the coupled fixed point theory for multivalued operators in complete b-metric spaces.

Notice that b-metric spaces are a real generalization of the classical metric spaces. In 2006, Bhaskar and Laksmikantham [27] first studied the existence of coupled fixed points in partially ordered metric spaces. After that, Laksmikantham and *Ćirić* [39] have defined the mixed g monotone property for a function and generalized the results of Bhaskar and Laksmikantham. So far, many mathematicians have studied coupled fixed point results for mappings under various contractive type conditions in different metric spaces. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a b-metric space introduced and studied by Bakhtin [6] and Czerwik [22]. The abstract results presented in this thesis will be illustrated by several examples and some applications. After that a series of articles have been dedicated to the improvement of fixed point theory for singlevalued and multivalued operators in b-metric spaces.

The structure of this thesis is the following:

Chapter 1 presents the most important concepts and auxiliary results which will be necessary through this work.

In **Chapter 2** we will present the properties of the generalized Pompeiu type functional H_d^+ and of the multivalued operators satisfying a Lipschitz condition with respect to H_d^+. We give some examples which denotes the relations between multivalued k-contractions in the sense of Nadler and multivalued k-contractions wrt H_d^+. Author's results in this chapter are Theorem 2.1.3, Theorem 2.1.5, Theorem 2.1.6, Theorem 2.1.7, Theorem 2.1.18, Theorem 2.1.9, Lemma 2.1.5, Lemma 2.1.6, Lemma 2.1.7, Lemma 2.1.8, Lemma 2.1.11, Lemma 2.1.12 and Lemma 2.1.13.

Our contributions in this section is:

- **I. Coroian**, *On some generalizations of Nadler's contraction principle*, Stud. Univ. Babeş-Bolyai Math. 60(2015), no. 1, 123-133.
- **I. Coroian**, *Fixed point theorems for multivalued generalized contractions with respect to two topologically equivalent metrics*, Creative Math. Inform., 23 (2014), No. 1, 57 - 64.

Chapter 3 is dedicated to the study of the fixed point theory for multivalued operators satisfying different contractive conditions with respect to some generalized functionals of metric type. We will discuss here :

- Fixed point theorems for multivalued operators in the context of two topologically equivalent metric (Theorem 3.1.1)

- Fixed point theorems for locally multivalued contractions.

  Using the concept of metric transform we will show that any multivalued operator satisfying a contraction type condition is a locally multivalued contraction. (Theorem 3.2.1, Theorem 3.2.2, Theorem 3.2.3,Theorem 3.2.5, Definition 3.2.3 and Lemma 3.2.1).

- Fixed point and fixed set results for multivalued operators with respect to the Pompeiu type functional

  The purpose of this section is to present a fixed point theory for multivalued H_d^+- contractions from the following perspectives: existence/uniqueness of the fixed and strict fixed points, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point equation and other useful properties. (Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.3, Theorem 3.3.4, Theorem 3.3.5 and Lemma 3.3.5).

- Continuation results for contraction type multivalued operators with respect to the Pompeiu type functional

  In this section, we present a local result and a continuation result for a special kind of multivalued k-contractions wrt (H_d^+)(Theorem 3.4.1 and Theorem 3.4.2).

- Kikkawa-Suzuki fixed point theorems for contractive type multivalued operators with in respect to the Pompeiu type functional

  Kikkawa and Suzuki [35] generalized the well-known Banach contraction principle by introducing a new type of mappings, also Moţ and Petruşel [44] gave another generalization concerning the work of Kikkawa and Suzuki. We extend this results using generalized multivalued H_d^+- contractions of *Ćirić*. (Theorem 3.5.1, Theorem 3.5.2, Definition 3.5.1 and Definition 3.5.2)

The scientific papers which contains the original results of this section are:

I. Coroian, *Fixed points for multivalued contractions with respect to a Pompeiu type metric*, J. Nonlinear Sci. Appl. (SCIE, IF: 0.949) (accepted for publication).

I. Coroian, *Kikkawa-Suzuki fixed point theorems for contraction type multivalued operators with respect to the Pompeiu functional*, Annals of the Tiberiu Popoviciu Seminar (accepted for publication).

Finally, in **Chapter 4** we will consider the coupled fixed point problem (in the sens of Guo and Lakshmikkantan) for multivalued operators satisfying a contraction condition with in respect to the Pompeiu type functional. The approach is based on a fixed point theorem for a multi-valued operator in the setting of a b-metric space. We will discuss here:

1. Fixed point theorems

The purpose of this section is to give new fixed point results for multi-valued operators satisfying a contraction type condition with respect to the excess functional. We will consider here the context of a b-metric space. (Theorem 4.1.1 and Theorem 4.1.2)

2. Coupled fixed point theorems

   The section includes existence results, data dependence theorems, well-posedness, Ulam-Hyers stability, limit shadowing property of the coupled fixed point problem. (Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3, Theorem 4.2.4, Theorem 4.2.5, Theorem 4.2.6, Theorem 4.2.7 and Theorem 4.2.8)

3. Application

   Some applications to a system of integral inclusions and to a system of multivalued initial value problem for a first order differential inclusions are also given (Theorem 4.3.1, Theorem 4.3.2, Theorem 4.3.3, Theorem 4.3.4).

The scientific paper which contain the original results of this section is:
I. Coroian şi G. Petruşel, *Fixed point and coupled fixed point theorems for multivalued operators satisfying a symmetric contraction condition*, Communications in Non. Anal. (accepted for publication).

This thesis is based on more than 100 titles in the References list and on the following 5 papers of the author:

- **I. Coroian**, *On some generalizations of Nadler's contraction principle*, Stud. Univ. Babeş-Bolyai Math. 60(2015), no. 1, 123-133.

- **I. Coroian**, *Fixed point theorems for multivalued generalized contractions with respect to two topologically equivalent metrics*, Creative Math. Inform., 23 (2014), No. 1, 57 - 64.

- **I. Coroian**, *Fixed points for multivalued contractions with respect to a Pompeiu type metric*, J. Nonlinear Sci. Appl., I.F 0.949(accepted for publication).

- **I. Coroian**, *Kikkawa-Suzuki fixed point theorems for contraction type multivalued operators with respect to the Pompeiu functional*, Annals of the Tiberiu Popoviciu Seminar of Functional Eq., App. and Conv. (accepted for publication).

- **I. Coroian** şi G. Petruşel, *Fixed point and coupled fixed point theorems for multivalued operators satisfying a symmetric contraction condition*, Commun. Nonlinear Anal., 1 (2016), 42 - 62.

  A significant part of the original results proved in this thesis were also presented at the following scientific conferences:

  - The 9^{th} International Conference on Applied Mathematics (ICAM9), September, 25-28, 2013, North University of Baia Mare, Romania.

- The 10^{th} International Conference on Fixed Point Theory and its Applications, July 9 - 15, 2012, Babeş-Bolyai University, Cluj-Napoca, Romania.
- International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA), July $14^{th} - 17^{th}$, 2015, Babeş-Bolyai University of Cluj-Napoca, Romania.

Chapter 1

Preliminaries

In the first chapter we will present the main basic notions and auxiliary results of multivalued operators theory in metric spaces which are further considered in the next chapters of this work. Through this thesis we will use the classical notations and notions from Nonlinear Analysis. For this chapter we have used the following sources: Nadler Jr. S.B. [47], G. Moţ, G. Petruşel, A. Petruşel [44], A. Petruşel , I.A. Rus , M.A. Şerban [56], I.A. Rus [84], I.A. Rus , A. Petruşel, G. Petruşel [85], M. A. Khamsi, W.A. Kirk [33] and others.

1.1 Generalized functionals on metric spaces

The notion of metric space always plays a fundamental role, since continuity in analysis for real and complex functions, depends upon the notion of distance and the generalization of analysis totally depends upon to study the continuity. Metric space was introduced by French mathematician M. Frechet in 1906 and since then there corresponds several generalizations of metric space in the literature. The notion of distance between points of an abstract set leads naturally to the discussion of convergence of sequences and Cauchy sequences in the set.

Let (X, d) be a metric space and $\mathcal{P}$(X) be the set of all subsets of X. We denote $P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}$, $P_{cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is closed }\}$, $P_{b,cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is bounded and closed }\}$, $P_{cp}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is compact }\}$. By $B(x, r)$ and respectively $\tilde{B}(x, r)$ we will denote the open and respectively the closed ball centered at $x \in X$ with radius $r > 0$.

The following (generalized) functionals are used in the main sections of the paper.

1. The gap functional generated by d:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, D_d(A, B) = inf\{d(a, b) \mid a \in A, b \in B\}.$$

2. The diameter generalized functional:

$$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \delta(A, B) = \sup\{d(a, b) | a \in A, b \in B\}.$$

3. The excess generalized functional:

$$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \rho_d(A,B) = sup\{D_d(a,B)|a \in A\}.$$

4. The Hausdorff-Pompeiu generalized functional :

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, H_d(A,B) = \max\{\rho_d(A,B), \rho_d(B,A)\}.$$

5. The Pompeiu type generalized functional:

$$H_d^+ : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\},\ H_d^+(A,B) := \frac{1}{2}\{\rho_d(A,B) + \rho_d(B,A)\}.$$

Let (X,d) be a metric space. If $T : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for T if and only if $x \in T(x)$. We denote by F_T the fixed point set of T and by $(SF)_T$ the set al all strict fixed points of T, i.e., elements $x \in X$ such that $T(x) = \{x\}$.

For a multivalued operator $T : X \to P(Y)$ we will denote by

$$Graph(T) := \{(x,y) \in X \times Y : y \in T(x)\}$$

the graphic of T.

Some useful properties of the above functionals are the following.

Lemma 1.1.1. *[61] Let (X,d) be a metric space, $A,B \in P(X)$ and $\epsilon > 0$. If for each $a \in A$ there exists $b \in B$ such that $d(a,b) \leq \epsilon$ then $\rho(A,B) \leq \epsilon$.*

Lemma 1.1.2. *[61] Let (X,d) be a metric space, $A,B \in P(X)$ and $q > 1$. Then for each $a \in A$ there exists $b \in B$ such that $d(a,b) \leq q\rho(A,B)$.*

Lemma 1.1.3. *[61] Let (X,d) be a metric space, $A,B \in P(X)$ and $\epsilon > 0$. If for each $a \in A$ there exists $b \in B$ with $d(a,b) \leq \epsilon$ and if for $b \in B$ there exits $a \in A$ such that $d(a,b) \leq \epsilon$, then $H_d(A,B) \leq \epsilon$.*

Lemma 1.1.4. *[61] Let (X,d) be a metric space, $A,B \in P(X)$ and $\epsilon > 0$. Then for each $a \in A$ there exists $b \in B$ such that $d(a,b) \leq H_d(A,B) + \epsilon$.*

Lemma 1.1.5. *[61] Let (X,d) be a metric space, $A,B \in P(X)$ and $q > 1$. Then for each $a \in A$ there exists $b \in B$ such that $d(a,b) \leq qH_d(A,B)$.*

Lemma 1.1.6. *[61] Let (X,d) be a metric space, $A,B \in P(X)$ and $\epsilon > 0$. If $H_d(A,B) \leq \epsilon$, then for each $a \in A$ there exists $b \in B$ such that $d(a,b) \leq \epsilon$.*

1.2 Multivalued weakly Picard operators with respect to Hausdorff-Pompeiu metric

The concept of multivalued weakly Picard operator (briefly MWP operator) was introduced in close connection with the successive approximation method and the data dependence property for the fixed point set of multivalued operators on complete metric space, by I. A. Rus, A. Petruşel and A. Sîntămărian, see [86]. For the following notation see also A. Petruşel [60], I. A. Rus [88], I. A. Rus [89], I.A. Rus, A. Petruşel and G. Petruşel [85].

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Definition 1.2.1. *([55]) Let (X, d) be a metric space. Then, $T : X \to P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:*

1. $x_0 = x,\ x_1 = y$;
2. $x_{n+1} \in T(x_n)$, *for all* $n \in \mathbb{N}$;
3. *the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent and its limit is a fixed point of* T.

Definition 1.2.2. *([55]) Let (X, d) be a metric space and $T : X \to P(X)$ be a MWP operator. Then we define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x, y) = \{z \in F_T |$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.*

Definition 1.2.3. *([55]) Let (X, d) be a metric space and $T : X \to P(X)$ a MWP operator. Then T is said to be a c- multivalued weakly Picard operator (briefly $c-MWP$ operator) if and only if there exists a selection t^∞ of T^∞ such that*

$$d(x, t^\infty(x, y)) \le cd(x, y),\ for\ all\ (x, y) \in Graph(T).$$

We recall now the notion of multivalued Picard operator.

Definition 1.2.4. *([55]) Let (X, d) be a complete metric space and $T : X \to P(X)$. By definition, T is called a multivalued Picard operator (briefly MP operator) if and only if :*

1. $(SF)_T = F_T = \{x^*\}$;

2. $T^n(x) \stackrel{H_d}{\to} \{x^*\}$ *as* $n \to \infty$*, for each* $x \in X$*.*

Recall that, by definition, for $(A_n)_{n\in\mathbb{N}} \in P_{cl}(X)$, we will write $A_n \stackrel{H_d}{\to} A^*$ as $n \to \infty$ if and only if $H_d(A_n, A^*) \to 0$ as $n \to \infty$. Notice also that

$$A_n \stackrel{H_d}{\to} A^* \in P_{cl}(X) \text{ as } n \to \infty \text{ if and only if } A_n \stackrel{H_d^+}{\to} A^* \in P_{cl}(X) \text{ as } n \to \infty.$$

Further on, we present some examples of MWP operators. For other examples see also H. Covitz and S. B. Nadler [20], S. B. Nadler [47], A. Petruşel [60], I.A. Rus, A. Petruşel and A. Sîntămărian [86], I. A. Rus [84].

Example 1.2.1. *([85]) Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued* α*- contraction. Then* T *is a* $\dfrac{1}{1-\alpha}$*-MWP operator.*

Example 1.2.2. *([85]) Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued Reich-type operator there exist* $a, b, c \leq 0$ *with* $a + b + c < 1$ *such that* $H_d(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y)), \forall x, y \in X$*. Then* T *is a MWP operator.*

Example 1.2.3. *([85]) Let* (X, d) *be a complete metric space. A multivalued operator* $T : X \to P_{cl}(X)$ *is said to be a multivalued Rus-type graphic-contraction if Graph(T) is closed and the following condition is satisfied: there exist* $\alpha, \beta \in \mathbb{R}_+$ *with* $\alpha + \beta < 1$ *such that* $H_d(T(x), T(y)) \leq \alpha d(x, y) + \beta D(y, T(y)), \forall x \in X, \forall y \in T(x)$*. Then* T *is a* $\dfrac{1-\beta}{1-\alpha-\beta}$*- MWP operator.*

Chapter 2

Generalized metrics on the space $P_{b,cl}(X)$

The object of this chapter is to study different properties of the Pompeiu type functional H_d^+ and of the multivalued operators satisfying a Lipschitz condition with respect to H_d^+. The connections with some continuity notions for multivalued operators are also given. The second purpose of this chapter, is to present some general abstract results for the functionals introduced in *Chapter* 1 the metric space $P_{b,cl}(X)$ with respect to two equivalent metrics. The references which were used to develop this chapter are: H.K. Pathak, N. Shahzad [51], W.A. Kirk, N. Shahzad [52], A. Petruşel, I. A. Rus [55] and A. Petruşel, I.A. Rus, M.A. Serban [56].

2.1 Pompeiu type functional H_d^+ on $P_{b,cl}(X)$

Concerning the functional H_d^+, we will extend the properties given in H.K. Pathak, N. Shahzad [51] and W.A. Kirk, N. Shahzad [52]. We will present some semi-continuous properties of the multivalued operators satisfying a Lipschitz condition with respect to H_d^+. We will study also when the property (p^*) can be translated between equivalent metrics on a nonempty set X. In the second part of this section we will give some general abstract results for the metric space $P_{b,cl}(X)$ by defining some functionals. The references which were used to develop this section are: W.A. Kirk, N. Shahzad [37], K. Kuratowski [38] and G. Moţ, A. Petruşel, G. Petruşel [44].

Lemma 2.1.1. *([51]) H_d^+ is a metric on $P_{b,cl}(X)$.*

Lemma 2.1.2. *([51]) We have the following relations:*

$$\frac{1}{2}H_d(A,B) \leq H_d^+(A,B) \leq H_d(A,B),\ \textit{for all}\ A,B \in P_{b,cl}(X) \tag{2.1}$$

(i.e., H_d and H_d^+ are strongly equivalent metrics).

Proposition 2.1.1. *([51]) Let $(X, ||\cdot||)$ be a normed linear space. For any λ (real or complex), $A, B \in P_{b,cl}(X)$*

1. $H_d^+(\lambda A, \lambda B) = |\lambda| H_d^+(A, B)$.
2. $H_d^+(A + a, B + a) = H_d^+(A, B)$.

Theorem 2.1.1. *([51]) If $a, b \in X$ and $A, B \in P_{b,cl}(X)$, then the relations hold:*

1. $d(a, b) = H_d^+(\{a\}, \{b\})$.
2. $A \subset \overline{S}(B, r_1), B \subset \overline{S}(A, r_2) \Rightarrow H_d^+(A, B) \leq r$ *where* $r = \frac{r_1+r_2}{2}$.

Theorem 2.1.2. *([51]) If the metric space (X, d) is complete, then $(P_{b,cl}(X), H_d^+)$ and $(P_{b,cl}(X), H_d)$ are complete too.*

Definition 2.1.1. *([52]) Let (X,d) be a metric space. A multivalued mapping $T : X \to P_{b,cl}(x)$ is called k-contraction wrt H_d^+ if*

1. *there exists a fixed real number k, $0 < k < 1$ such that for every $x, y \in X$*

$$H_d^+(T(x), T(y)) \leq kd(x, y).$$

2. *for every x in X, y in $T(x)$ and $\varepsilon > 0$, there exists z in $T(y)$ such that*

$$d(y, z) \leq H_d^+(T(y), T(x)) + \varepsilon.$$

Lemma 2.1.3. *Let (X,d) be a metric space and $A, B \in P_{cl}(x)$. The the following assertions are equivalent:*

1. *Let $\epsilon >$ arbitrary. Then $\forall a \in A$ there exists $b \in B$ such that*

$$d(a, b) \leq H_d^+(A, B) + \epsilon.$$

2. *Let $q > 1$ arbitrary. Then $\forall a \in A$ there exists $b \in B$ such that*

$$d(a, b) \leq qH_d^+(A, B).$$

Definition 2.1.2. *[53] Let (X, d) be a Banach space and K a nonempty, convex, compact subset. Then $T : X \to P_{b,cl}(X)$ is called H_d^+-nonexpansive if:*

1. $H_d^+(T(x), T(y)) \leq ||x - y||$, *for every* $x, y \in K$
2. *for every $x \in X$, $y \in T(x)$ and for $\epsilon > 0$ there exists $z \in T(y)$ such that*

$$d(y, z) \leq H_d^+(T(x), T(y)) + \varepsilon.$$

The following concept was introduced by S.B. Nadler jr. as follows.

Definition 2.1.3. *([47]) Let (X,d) be a metric space. A mapping $T : X \to P_{cl}(X)$ is called a multivalued k-contraction wrt H_d if $k \in (0,1)$ and*

$$H_d(T(x),T(y)) \leq kd(x,y), \text{ for all } x,y \in X.$$

Notice that any multivalued k-contraction wrt H_d is a k-contraction wrt H_d^+, but the reverse implication does not hold.

We will now introduce a similar concept. For this purpose, we recall now the concept of (strong) comparison function.

Definition 2.1.4. *(see [55], [54]) A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$.*

As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0.

Definition 2.1.5. *([55], [54]) A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strong comparison function if it is a comparison function and $\sum\limits_{k=0}^{\infty} \varphi^k(t) < \infty$, for any $t > 0$.*

With respect to the Pompeiu type functional H_d^+, we define the following concept.

Definition 2.1.6. *([16])Let (X,d) be a metric space. Then, the multivalued operator $T : X \to P_{b,cl}(X)$ is called a φ-contraction wrt H_d^+ if:*

1. *$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function;*
2. *for all $x,y \in X$, we have that*

$$H_d^+(T(x),T(y)) \leq \varphi(d(x,y)).$$

In particular, if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\varphi(t) := kt$ (for some $k \in [0,1[$), then φ is a strong comparison function and the multivalued operator T is a k-contraction wrt H_d^+.

We recall now some useful concepts in the theory of multivalued operators.

Definition 2.1.7. *([55], [54]) Let (X,d) be a metric space and $T : X \to P_{b,cl}(X)$. Then, T is called upper semi-continuous (briefly u.s.c.) in $x \in X$ if, for all open subset U of X with $F(x) \subset U$, there exists $\eta > 0$ such that $T(B(x;\eta)) \subset U$. T is u.s.c. on X if it is u.s.c. in each $x \in X$.*

Definition 2.1.8. *([55], [54]) Let (X,d) be a metric space and $T : X \to P_{b,cl}(X)$. Then T is called lower semi-continuous (briefly l.s.c.) in $x \in X$ if, for all $(x_n)_{n\in\mathbb{N}^*} \subset X$ such that $\lim\limits_{n\to\infty} x_n = x$ and for all $y \in T(x)$, there exists a sequence $(y_n)_{n\in\mathbb{N}^*} \subset X$ such that $y_n \in T(x_n)$, for all $n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} y_n = y$. T is l.s.c. on X if it is l.s.c. in each $x \in X$.*

Definition 2.1.9. *([55], [54]) Let (X, d) be a metric space. $T : X \to P_{b,cl}(x)$ is called H_d-upper semi-continuous in $x_0 \in X$ (H_d-u.s.c) respectively $H-$ lower semi-continuous (H_d-l.s.c) if, for each sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that*

$$\lim_{n\to\infty} x_n = x_0,$$

we have

$$\lim_{n\to\infty} \rho(T(x_n), T(x_0)) = 0, \text{ respectively } \lim_{n\to\infty} \rho(T(x_0), T(x_n)) = 0.$$

It is well-known that if T is u.s.c. in $x \in X$, then T is H_d-u.s.c. in $x \in X$, while if T is H_d-l.s.c. in $x \in X$ implies that T is l.s.c. in $x \in X$.

Definition 2.1.10. *([55], [54]) Let (X, d) be a metric space and $T : X \to P(X)$. Then T is said to be with closed graph if, for each $x \in X$ and for all $(x_n)_{n\in\mathbb{N}^*} \subset X$ such that*

$$\lim_{n\to\infty} x_n = x$$

and for all $(y_n)_{n\in\mathbb{N}^*} \subset X$ with $y_n \in T(x_n)$, for all $n \in \mathbb{N}^*$ and

$$\lim_{n\to\infty} y_n = y,$$

we have $y \in T(x)$.

Definition 2.1.11. *Let (X,d) be a metric space. A multivalued mapping $T : X \to P_{cl}(X)$ is called k-Lipschitz wrt H_d if $k > 0$ and*

$$H_d(T(x), T(y)) \leq kd(x, y), \text{ for every } x, y \in X.$$

We define a similar concept with respect to the Pompeiu type functional H_d^+.

Let (X,d) be a metric space. A multivalued mapping $T : X \to P_{cl}(X)$ is called k-Lipschitz if $k > 0$ and

$$H_d^+(T(x), T(y)) \leq kd(x, y), \text{ for every } x, y \in X.$$

Some properties of a multivalued k-Lipschitz wrt H_d^+ operators are given now.

Theorem 2.1.3. *(I. Coroian [15]) Let (X, d) be a metric space and $T : X \to P_{b,cl}(X)$ be k-Lipschitz wrt H_d^+. Then:*

1. *T is has closed graph in $X \times X$;*
2. *T is $H_d - l.s.c$ on X;*
3. *T is $H_d - u.s.c$ on X;*
4. *T is l.s.c. on X;*

5. *If, additionally T has compact values, then T is l.s.c.*

Lemma 2.1.4. *(I. Coroian [15]) Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ such that*

$$H_d^+(T(x),T(y)) < d(x,y), \quad \text{for all } x,y \in X, \ x \neq y$$

Then T is u.s.c on X.

Theorem 2.1.4. *([47])(Nadler) Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued k-contraction wrt H_d. Then*

$$H_d(T(A),T(B)) \leq kH_d(A,B) \ \text{for all } A,B \in P_{cp}(X). \tag{2.2}$$

Lemma 2.1.5. *([55], [54]) Let (X,d) be a metric space and $A,B \in P_{cp}(X)$. Then for all $a \in A$ there exists $b \in B$ such that*

$$d(a,b) \leq H_d(A,B).$$

Theorem 2.1.5. *(I. Coroian [15]) Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ for which there exists $k > 0$ such that:*

$$H_d(T(x),T(y)) \leq kd(x,y), \ \text{for all } x,y \in X$$

Then

$$H_d^+(T(A),T(B)) \leq 2kH_d^+(A,B) \ \text{for all } A,B \in P_{cp}(X).$$

Let us recall the relations between u.s.c and $H_d - u.s.c$ of a multivalued operator. If (X,d) is a metric space, then $T : X \to P_{cp}(X)$ is $u.s.c$ on X if and only if T is $H_d - u.s.c.$

Theorem 2.1.6. *(I. Coroian [15]) Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued k-contraction wrt H_d^+. Then*

(a) T is H_d-l.s.c and u.s.c on X.

(b) for all $A \in P_{cp}(X) \Rightarrow T(A) \in P_{cp}(X)$

(c) there exists $k > 0$ such that

$$H_d^+(T(A),T(B)) \leq 2kH_d^+(A,B) \ \text{for all } A,B \in P_{cp}(X).$$

As a consequence of the previous result we obtain the following fixed set theorem for a multivalued contraction with respect to H_d^+.

Theorem 2.1.7. *(I. Coroian [15]) Let (X,d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued operator for which there exists $k \in [0, \frac{1}{2})$ such that*

$$H_d^+(T(x),T(y)) \leq kd(x,y), \ \text{for all } x,y \in X$$

Then, there exists a unique $A^* \in P_{cp}(X)$ such that $T(A^*) = A^*$.

In the second part of this section, we will study when the property (p^*) can be translated between equivalent metrics on a nonempty set X.

Definition 2.1.12. *Let (X,d) be a metric space and $H : P(X) \times P(X) \to \mathbf{R}_+$ be a functional. Then, the pair (d, H_d) has the property (p^*) if for $q > 1$ and $A, B \in P(X)$ the following assertion is true: for and any $a \in A$, there exists $b \in B$ such that:*

$$d(a,b) \leq qH_d(A,B).$$

Remark 2.1.1. *For example,the Hausdorff-Pompeiu functional H_d (generated by the metric d) has the property (p^*) (see Lemma 1.1.5). It is open question to give other functionals on $P(X) \times P(X)$ having this property.*

Lemma 2.1.6. *(I. Coroian [15]) Let X be a nonempty set, d_1, d_2 two Lipschitz equivalent metrics such that there exists $c_1, c_2 > 0$ with $c_1 \leq c_2$ i.e*

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y), \text{ for all } x, y \in X \tag{2.3}$$

If the pair (d_1, H_{d_1}) has the property (p^*), then the pair (d_2, H_{d_2}) has the property (p^*).

Lemma 2.1.7. *(I. Coroian [15]) Let X be a nonempty set, d_1, d_2 two metrics on X such that:*

$$\text{there exists } c > 0\text{: } d_2(x,y) \leq c d_1(x,y) \quad \text{for all } x, y \in X \tag{2.4}$$

and G_1, G_2 two metrics on $P_{b,cl}(X)$ such that:

$$\text{there exists } e > 0\text{: } eG_{d_1}(A,B) \leq G_{d_2}(A,B), \text{ for all } A, B \in P_{b,cl}(X) \tag{2.5}$$

with $e \leq c$. If the pair (d_1, G_1) has the property (p^*) then, the property (p^*) is also true for the pair (d_2, G_2).

Lemma 2.1.8. *(I. Coroian [15]) Let X be a nonempty set, d_1, d_2 two metrics on X such that:*

$$\text{there exists } c > 0\text{: } d_2(x,y) \leq c d_1(x,y) \text{ for all } x, y \in X \tag{2.6}$$

and G_1, G_2 two metrics on $P_{b,cl}(X)$ such that:

$$\text{there exists } e > 0\text{: } G_{d_2}(A,B) \leq eG_{d_2}(A,B), \text{ for all } A, B \in P_{b,cl}(X) \tag{2.7}$$

with $c \cdot e < 1$. If the pair (d_1, G_{d_2}) has the property (p^*) then, the property (p^*) is also true for the pair (d_2, G_{d_1}).

2.2 Others generalized metrics on $P_{b,cl}(X)$

In this section we will give some general abstract results for the metric space $P_{b,cl}(X)$.

Let (X,d) be a metric space, $U \subset P(X)$ and $\Psi : U \to \mathbb{R}_+$. We define some functionals on $U \times U$ as follows:

1. Let $x^* \in X$, $U \subset P_b(X)$

$$G_{\Psi_1}(A,B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

   where $\Psi_1(A) := \delta(A, x^*)$.

2. Let $U := P_b(X)$ and $A^* \in P_b(X)$

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

   Where $\Psi_2(A) = H_d(A, A^*)$.

Lemma 2.2.1. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_1}(A,B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

Where $\Psi_1(A) = \delta(A, A^*)$, $A^* \in P_{cp}(X)$. Then G_{Ψ_1} is a metric on $P_{cp}(X)$.

Lemma 2.2.2. *If (X,d) is a complete metric space, then $(P_{cp}(X), G_{\Psi_1})$ is complete metric space.*

Lemma 2.2.3. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_1}(A,B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

where $\Psi_1 : P_{cp}(X) \to \mathbb{R}_+, \Psi_1(A) = \delta(A, A^*)$ with $A^* \in P_{cp}(X)$. Then, the pair (d, G_{Ψ_1}) has the property (p^*).

Theorem 2.2.1. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued operator for which there exists $k \in (0,1)$ such that*

$$\delta(T(x), T(y) \leq k d(x,y).$$

For all $A, B \in P_{cp}(X)$ we consider

$$G_{\Psi_1}(A,B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B, \end{cases}$$

where $\Psi_1 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_1(A) = \delta(A, A^*)$ (with $A^* \in P_{cp}(X)$ is a given set satisfying $A^* = T(A^*)$). Then,

$$G_{\Psi_1}(T(A), T(B)) \leq k G_{\Psi_1}(A,B) \text{ for all } A, B \in P_{cp}(X).$$

Lemma 2.2.4. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A, A^*)$ with $A^* \in P_{cp}(X)$. Then G_{Ψ_2} is a metric on $P_{cp}(X)$.

Lemma 2.2.5. *If (X,d) is a complete metric space, then $(P_{cp}(X), G_{\Psi_2})$ is complete metric space.*

Theorem 2.2.2. *Let (X,d) be a metric space and $T : X \to P_{cp}(x)$ be a multivalued k-contraction with respect to H_d and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

Where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A, A^*)$ (with $A^* \in P_{cp}(X)$ is a given set satisfying $A^* = T(A^*)$). Then, there exists $k \in (0,1)$ such that

$$G_{\Psi_2}(T(A), T(B)) \leq kG_{\Psi_2}(A,B) \text{ for all } A, B \in P_{cp}(X).$$

Lemma 2.2.6. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B, \end{cases}$$

where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A, A^*)$ with $A^* \in P_{cp}(X)$. Then, the pair (d, G_{ψ_2}) has the property (p^*).

Lemma 2.2.7. *Let (X,d) be a metric space, $Z := X \times X$ and $\tilde{d} : Z \times Z \to \mathbb{R}_+$ given by*

$$\tilde{d}((x,y),(u,v)) = d(x,u) + d(y,v).$$

Then $H^+_{\tilde{d}}(A_1 \times B_1, A_2 \times B_2) = H^+_d(A_1, A_2) + H^+_d(B_1, B_2)$.

Remark 2.2.1. *Let (X,d) be a metric space, $Z := X \times X$ and $\bar{d} : Z \times Z \to \mathbb{R}_+$ given by*

$$\bar{d}((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}.$$

Then, in general,

$$H^+_{\bar{d}}(A_1 \times B_1, A_2 \times B_2) \nleq \max\{H^+_d(A_1, A_2), H^+_d(B_1, B_2)\}.$$

2.3 Multivalued operators with respect to Pompeiu type contraction functional

In this section we give some examples to illustrate that any multivalued k-contraction in sense of Nadler is k-contraction wrt H_d^+, but the reverse implication doesn't hold. An example of a multivalued H_d^+-nonexpansive operator is also given.

Definition 2.3.1. *([52]) Let (X,d) be a metric space. A multivalued mapping $T : X \to P_{b,cl}(x)$ is called k-contraction wrt H_d^+ if*

1. *there exists a fixed real number k, $0 < k < 1$ such that for every $x, y \in X$*

$$H_d^+(T(x), T(y)) \leq kd(x, y).$$

2. *for every x in X, y in $T(x)$ and $\varepsilon > 0$, there exists z in $T(y)$ such that*

$$d(y, z) \leq H_d^+(T(y), T(x)) + \varepsilon.$$

Definition 2.3.2. *[53] Let (X, d) be a Banach space and K a nonempty, convex, compact subset. Then $T : X \to P_{b,cl}(X)$ is called H_d^+-nonexpansive if:*

1. *$H_d^+(T(x), T(y)) \leq ||x - y||$, for every $x, y \in K$*
2. *for every $x \in X$, $y \in T(x)$ and for $\epsilon > 0$ there exists $z \in T(y)$ such that*

$$d(y, z) \leq H_d^+(T(x), T(y)) + \varepsilon.$$

Remark 2.3.1. *([52]) If T is a multivalued k-contraction in the sense of Nadler then T is a multivalued k-contraction wrt H_d^+ but not viceversa.*

Example 2.3.1. *(I. Coroian[16]) Let $X = \{0, \frac{1}{2}, 2\}$ and $d : X \times X \to \mathbb{R}$ be a standard metric. Let $T : X \to P_{b,cl}(X)$ be such that*

$$T(x) = \begin{cases} \{0, \frac{1}{2}\}, & \text{for } x = 0 \\ \{0\}, & \text{for } x = \frac{1}{2} \\ \{0, 2\}, & \text{for } x = 2 \end{cases}$$

Then T is a k-contraction wrt H_d^+ (with $k \in \left[\frac{5}{6}, 1\right)$) but is not an k- contraction in the sense of Nadler, since

$$H_d(T(0), T(2)) = H_d(\{0, \frac{1}{2}\}, \{0, 2\}) = 2 \leq kd(0, 2) = 2k \Rightarrow k \geq 1,$$

which is a contradiction with our assumption that $k < 1$.

Example 2.3.2. *(I. Coroian[16]) Let l_2 denote the Hilbert space of all square summable sequence of real numbers. Let $a = (1, \frac{1}{2}, ..., -\frac{1}{n})$ and for each $n = 1, 2, ...$, let e_n be the vector in l_2 with zeros in all its coordinates except the n^{th} coordinate which is equal to one.*

Let $A = \{a, e_1, e_2, ...\}$ and let $T : A \to P_{b,cl}(A)$ be such that

$$T(x) = \begin{cases} \{e_{i+1}, e_{i+2}, ...\}, & \textit{for } x = 0 \\ A, & \textit{for } x = \frac{1}{2} \end{cases}$$

It is easy to check that the first condition of the multivalued H_d^+- nonexpansive is satisfied for any $x, y \in A$.

Further, we see that if $\epsilon > 0$ then for every $x \in A$ and any $y \in T(x)$, there exists $z \in T(u)$ such that $d(y, z) \leq H_d^+(T(x), T(y)) + \epsilon$. Indeed,

(ia) if $\epsilon > 0$, $x = a$ and $y \in T(a) = a, e_1, e_2, ...$, say $y = a$, there exists $z \in T(a) = a, e_1, e_2, ...$, say $z = a$, such that

$$0 = d(y, z) \leq 0 + \epsilon = H_d^+(T(a), T(a)) + \epsilon$$

(ib) if $\epsilon > 0$, $x = a$ and $y \in T(a) = \{a, e_1, e_2, ...\}$, say $y = e_i$, there exists $z \in T(e_i) = \{e_{i+1}, e_{i+2}, ...\}$, say $z = e_{i+1}$, such that

$$\sqrt{2} = d(y, z) \leq (||a||^2 + 1)^{\frac{1}{2}} + \epsilon = H_d^+(T(a), T(e_i)) + \epsilon$$

(ic) if $\epsilon > 0$, $x = e_i$ and $y \in T(e_i) = \{e_{i+1}, e_{i+2}, ...\}$, say $y = e_{i+1}$,there exists $z \in T(e_{i+1}) = \{e_{i+2}, e_{i+3}, ...\}$, say $z = e_{i+2}$, such that

$$\sqrt{2} = d(y, z) \leq \sqrt{2} + \epsilon = H_d^+(T(e_i), T(e_{i+1})) + \epsilon$$

Hence the condition (ii) from Definition 4.1.6 is also satisfied and T is a multivalued H_d^+-nonexpansive.

Chapter 3

Fixed point theorems for multivalued operators satisfying generalized contractive conditions

In this chapter, we will present some fixed point results for multivalued almost contractions with respects to two topologically equivalent Hausdorff type metrics and for multivalued operators satisfying different contractive conditions with respect to some generalized functionals of metric type from the following perspectives: existence/uniqueness of the fixed and strict fixed points, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point equation, well-posedness of the fixed point problem, limit shadowing property for a multivalued operator, set-to-set operatorial equations and fractal operator theory. The purpose of this chapter is to study some properties of the fixed point set of H^+-contraction with constant α from the MWP operator theory point of view. An application to the continuous dependence of the solution set for a Cauchy problem associated to a differential inclusion, with respect to the initial condition, is also given.

The results complement and extend some recent results proved by M. Kikkawa, T. Suzuki ([35]), T. Lazăr, D. O'Regan, A. Petruşel ([40]) şi T. Suzuki ([92]), M. A. Khamsi, W.A. Kirk ([33]), W.A. Kirk, N. Shahzad([34]) and I.A. Rus ([84]), E. Lami-Dozo [41], J.T. Markin [46] and

3.1 Fixed point theorems for multivalued operators in the context of two topologically equivalent metrics

In this section, we will present some fixed point results for multivalued almost contractions with respects to two topologically equivalent Hausdorff type metrics. The references which were used to develop this chapter are: [55] A. Petruşel, I. A. Rus [55] and A. Petruşel, I.A. Rus, M.A. Serban [56].

Theorem 3.1.1. *(I. Coroian[15]) Let (X, d) be a complete metric space and let $H^\star$ be any metric on $P_{b,cl}(X)$ which is topologically equivalent to the Hausdorff metric H_d.*
Suppose that $T : X \to P(X)$ satisfies :

1. *For each $x \in X$ the set $T(x)$ is bounded (with respect to d) and closed (with respect to the metric topology generated by d).*
2. *there exist $a, b, c \in \mathbf{R}$, $a + b + c < 1$ such that*
$$H^\star(T(x), T(y)) \leq ad(x, y) + bD_d(x, T(x)) + cD_d(y, T(x)), \forall x, y \in X$$
3. *if $(x, y) \in Graph(T))$ then :*
$$D_d(y, T(y)) \leq H^\star(T(y), T(x))$$

Then T has a fixed point, i.e, there exists $x \in X$ such that $x \in T(x)$.

3.2 Fixed point theorems for locally multivalued contractions

Using the concept of metric transform we will show that any multivalued operator satisfying a contraction type condition is a locally multivalued contraction. The references which were used to develop this section are: M. A. Khamsi, W.A. Kirk ([33]), W.A. Kirk, N. Shahzad([34]) şi I.A. Rus ([84]) and V. L. Lazăr [42].

Definition 3.2.1. *([33]) Let (X, d) be a metric space. A strictly increasing concave function $\phi : [0, \infty) \to \mathbb{R}$ for which $\phi(0) = 0$ is called a metric transform.*

Definition 3.2.2. *([34]) ([2])Let (X, d) be a metric space. A mapping $T : X \to P_{b,cl}(X)$ is said to be an (ε, k)-uniform local multivalued contraction (where $\varepsilon > 0$ and $k \in (0, 1)$) if for $x, y \in X$ $d(x, y) < \varepsilon \Rightarrow H_d(T(x), T(y)) \leq kd(x, y)$.*

As an extension of the above concept, we introduce the following notion.

Definition 3.2.3. *(I. Coroian[16]) Let (X, d) be a metric space. A mapping $T : X \to P_{b,cl}(X)$ is said to be an $(\varepsilon, \alpha, \beta, \gamma)$-uniform local multivalued contraction (where $\varepsilon > 0$ and $\alpha + \beta + \gamma \in (0, 1)$) if for $x, y \in X$ $d(x, y) < \varepsilon \Rightarrow H_d(T(x), T(y)) \leq \alpha d(x, y) + \beta D_d(x, T(x)) + \gamma D_d(y, T(y))$.*

Lemma 3.2.1. *(I. Coroian[16]) Let (X,d) be a metric space and $T : X \to P_{b,cl}(X)$. And let $\phi : [0, \infty) \to \mathbb{R}$ be a strictly increasing function which $\phi(0) = 0$, such that the following conditions hold :*

1. *There exist a,b,c,k* $\in \mathbb{R}$, *a+b+k* $<$ *1 such that*

$$\phi(H_d(T(x),T(y))) \leq ad(x,y) + bD_d(x,T(x)) + cD_d(y,T(y)) + kD_d(x,T(y)),$$

$$\forall(x,y) \in Graph(T)$$

2. *There exists* $e \in (0,1)$ *such that for* $t > 0$ *sufficiently small,*

$$(a+b+k)t \leq \psi(et)$$

where $\psi : [0,\infty) \to \mathbb{R}, \psi(t) = \phi(t) - (c+k)t$ *is strictly increasing.*

Then for $\varepsilon > 0$ *sufficiently small we have*

$$H_d(T(x),T(y)) \leq ed(x,y),\ \forall(x,y) \in Graph(T) \text{ with } d(x,y) < \epsilon.$$

The next result is a Nadler type fixed point theorem on the graphic.

Theorem 3.2.1. *(I. Coroian[16]) Let* (X,d) *be a complete metric space and* $T : X \to P_{b,cl}(X)$ *with the closed graphic. Suppose that exist* $e < 1$ *such that* $H_d(T(x),T(y)) \leq ed(x,y), \forall(x,y) \in Graph(T)$ *then* $Fix(T) \neq \emptyset$ *.*

We recall that a metric space (X,d) is said to be ε-chainable (where $\varepsilon > 0$ is fixed) if and only if given $a, b \in X$ there is an ε-chain from a to b (that is, a finit set of points $x_0, x_1, \ldots, x_n \in X$ such that $x_0 = a, x_n = b$ and $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \ldots, n$).

The following result is a slight extension of a theorem due to Nadler [47].

Theorem 3.2.2. *(I. Coroian[16]) Let* (X,ε) *be a complete* ε*-chainable metric space. If* $T : X \to P_{b,cl}(X)$ *has closed graphic and there exists* $e < 1$ *such that* $H(T(x),T(y)) \leq ed(x,y)$ *for all* $(x,y) \in Graph(T)$ *with* $d(x,y) < \epsilon$*, then* $Fix(T) \neq \emptyset$*.*

By combining the above result with $Theorem$ 2.2 we obtain the following result :

Theorem 3.2.3. *(I. Coroian[16]) Let (X,d) be a complete and connected metric space and* $T : X \to P_{b,cl}(X)$*. Let* $\phi : [0,\infty) \to \mathbb{R}$ *be a strictly increasing function which* $\phi(0) = 0$ *such that the following conditions hold :*

1. *There exist a,b,c,k*$\in \mathbb{R}$, *a+b+k* $<$ *1 such that*

$$\phi(H_d(T(x),T(y))) \leq ad(x,y) + bD_d(x,T(x)) + cD_d(y,T(y)) + kD_d(x,T(y)),$$

$$\forall(x,y) \in Graph(T)$$

2. *There exists* $e \in (0,1)$ *such that for* $t > 0$ *sufficiently small, we have*

$$(a+b+k)t \leq \psi(et)$$

where $\psi : [0,\infty) \to \mathbb{R}, \psi(t) = \phi(t) - (c+k)t$ *is strictly increasing*

Then $Fix(T) \neq \emptyset$.

We recall now Nadler's fixed point theorem for multivalued contractions.

Theorem 3.2.4. *([47]) Let* (X,d) *be a complete metric space. If* $T : X \to P_{b,cl}(X)$ *is a multivalued contraction mapping, then* T *has a fixed point.*

As a consequences of the previous results we have :

Theorem 3.2.5. *(I. Coroian[16]) Let* (X,ε) *be a complete* ε*-chainable metric space. If* $T : X \to P_{b,cl}(X)$ *is an* $(\varepsilon,\alpha,\beta,\gamma)$*-uniformly locally contractive multivalued mapping, then* T *has a fixed point.*

3.3 Fixed point and fixed set results for multivalued operators with respect to the Pompeiu functional

The purpose of this section is to present a fixed point theory for multivalued H_d^+-contractions from the following perspectives: existence/uniqueness of the fixed and strict fixed points, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point equation, well-posedness of the fixed point problem, limit shadowing property for a multivalued operator, set-to-set operatorial equations and fractal operator theory. The references which were used to develop this section are: J.-P. Aubin, H. Frankowska [4], J.-P. Aubin and A. Cellina [5], V. Berinde [7], C. Castaing [12], A. Filippov [23], I. A. Rus [81], I.A. Rus [79] and I.A. Rus [78].

We will start by presenting some auxiliary results.

Lemma 3.3.1. *(see, for example, [54]) Let* (X,d) *be a metric space and* $A, B \in P_{cl}(X)$. *Suppose that there exists* $\eta > 0$ *such that for each* $a \in A$ *there exists* $b \in B$ *such that* $d(a,b) \leq \eta$ *and for each* $b \in B$ *there exists* $a \in A$ *such that* $d(a,b) \leq \eta$. *Then* $H(A,B) \leq \eta$.

Lemma 3.3.2. *([54]) Let* (X,d) *be a metric space,* $A, B \in P(X)$ *and* $q > 1$. *Then, for every* $a \in A$ *there exists* $b \in B$ *such that* $d(a,b) \leq qH(A,B)$.

Lemma 3.3.3. *([55]) Let* (X,d) *be a metric space and* $A, B \in P_{cp}(X)$. *Then for every* $a \in A$ *there exists* $b \in B$ *such that* $d(a,b) \leq H(A,B)$.

Lemma 3.3.4. *([54]) Let (X,d) be a metric space. If $A,B\in P(X)$ and $\epsilon>0$ then for every $a\in A$ there exists $b\in B$ such that*

$$d(a,b)\leq H(A,B)+\epsilon.$$

Lemma 3.3.5. *(I. Coroian[17]) Let (X,d) be a metric space, $A,B\in P_{cl}(X)$ and $\varepsilon>0$. If $H_d^+(A,B)<\varepsilon$ then:*

1. *for all $a\in A$ there exists $b\in B$ such that $d(a,b)<\varepsilon$*

   or

2. *for all $b\in B$ there exists $a\in A$ such that $d(a,b)<\varepsilon$.*

Lemma 3.3.6. *(Cauchy)(see [87]) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of non-negative real numbers, such that*

$$\sum_{k=0}^{+\infty} a_k<+\infty \text{ and } \lim_{n\to+\infty} b_n=0.$$

Then,

$$\lim_{n\to\infty}\sum_{k=0}^{n} a_{n-k}b_k=0.$$

Theorem 3.3.1. *(I. Coroian[17]) Let (X,d) be a complete metric space and $T:X\to P_{cl}(X)$ be a multivalued H_d^+-contraction with constant α. Then we have:*

(i) $F_T\neq\emptyset$;

(ii) T is a $\dfrac{1}{1-\alpha}$ -MWP operator;

(iii) Let $S:X\to P_{cl}(X)$ be a H_d^+-contraction with constant α and $\eta>0$ such that $H_d^+(S(x),T(x))\leq\eta$, for each $x\in X$. Then $H_d^+(F_S,F_T)\leq\dfrac{2\cdot\eta}{1-\alpha}$;

(iv) Let $T_n:X\to P_{cl}(X), n\in\mathbb{N}$ be a sequence of multivalued H_d^+-contraction with constant α such that $T_n(x)\overset{H_d^+}{\to}T(x)$ as $n\to\infty$, uniformly with respect to $x\in X$. Then, $F_{T_n}\overset{H_d^+}{\to}F_T$ as $n\to\infty$;

If, additionally $T(x)\in P_{cp}(X)$ for each $x\in X$, then we also have:

(v) (Ulam-Hyers stability of the inclusion $x\in T(x)$) Let $\epsilon>0$ and $x\in X$ be such that $D(x,T(x))\leq\epsilon$. Then there exists $x^*\in F_T$ such that $d(x,x^*)\leq\dfrac{\epsilon}{1-\alpha}$;

(vi) The fractal operator $\hat{T}:P_{cp}(X)\to P_{cp}(X)$, $\hat{T}(Y):=\bigcup\limits_{x\in Y}T(x)$ is a 2α-contraction;

(vii) If, additionally, $\alpha\in[0,\frac{1}{2}[$, then $F_{\hat{T}}=\{A_T^*\}$ and $T^n(x)\overset{H_d^+}{\to}A_T^*$ as $n\to\infty$, for each $x\in X$. Moreover, $F_T\subset A_T^*$, F_T is compact and

$$A_T^*=\bigcup_{n\in\mathbb{N}^*}T^n(x)\ ,\ \text{for each } x\in F_T.$$

Some new conclusions with respect to the fixed point and the strict fixed point sets are given in our next result.

Theorem 3.3.2. *(I. Coroian[17]) Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued (H_d^+, α)-contraction with $(SF)_T \neq \emptyset$. Then, the following assertions hold:*

(i) $(SF)_T = \{x^*\}$;

If additionally, $\alpha \in [0, \frac{1}{2}[$, then:

(ii) $F_T = (SF)_T = (SF_{T^n}) = \{x^*\}$ for $n \in \mathbb{N}^*$;

(iii) $T^n(x) \overset{H_d^+}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$;

(iv) Let $S : X \to P_{cl}(X)$ be a multivalued operator such that $F_S \neq \emptyset$ and suppose there exists $\eta > 0$ such that:

$$H_d^+(S(x), T(x)) \leq \eta, \text{ for each } x \in X.$$

Then

$$H_d^+(F_S, F_T) \leq \frac{2\eta}{1-2\alpha};$$

(vi) (Well-posedness of the fixed point problem wrt to H_d^+) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that

$$H_d^+(x_n, T(x_n)) \to 0 \text{ as } n \to \infty,$$

then $x_n \overset{d}{\to} x^*$ as $n \to \infty$;

(vii) (Limit shadowing property of the multivalued operator) If $(y_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(y_{n+1}, T(y_n)) \to 0$ as $n \to \infty$, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T, such that $d(x_n, y_n) \to 0$ as $n \to \infty$.

Remark 3.3.1. *(I. Coroian[17]) Similar results can be given for the case of multivalued φ-contraction wrt H_d^+. The results of this type can be viewed as generalizations of some theorems given in [42].*

In [45], Markin proved a stability theorem on the set of solutions to (3.1) using the L^2 norm, while Lim [43] proved a stability result in terms of the Hausdorff-Pompeiu functional. We will prove now a similar theorem using the *sup* norm and the Pompeiu functional generated by it. For more results see J.T. Markin [45], H. Hermes [31], C.J. Himmelberg and F.S. Van Vleck,[32].

We now give an application of the above results to the continuous dependence of the solution set for a Cauchy problem associated to a differential inclusion, with respect to the initial condition. The existence of a solution to the initial value problem :

$$\begin{cases} \dot{x}(t) \in T(t, x(t)) \\ x(0) = b \end{cases} \tag{3.1}$$

was proved by Filippov [23] and Castaing [12] under certain conditions on T.

We recall first the concept of solution.

Definition 3.3.1. *([43]) Let $D = [0, a] \times \mathbb{R}^n$ and $T : D \to P(\mathbb{R}^n)$ be a continuous operator. Then, a mapping $x : [0, a] \to \mathbb{R}^n$ is said to be a solution of the differential inclusion (3.1) if x is an absolutely continuous mapping and $x'(t) \in T(t, x(t))$, a.e on $[0, a]$.*

Let B be an origin-centered closed ball in R^n and $P_{cl,cv}(B)$ endowed with the H_d^+ metric generated by the Euclidean norm $||\cdot||$ of R^n. Let $C[0, a]$ be the set of the continuous maps of $[0, a]$ into R^n with the *sup* norm $||\cdot||_C$.

Assume that T is a continuous map of $[0, a] \times B$ into $P_{cl,cv}(B)$ satisfying, for some $k > 0$, the condition

$$H_d^+(T(t, u), T(t, v)) \leq k||u - v||_C, \text{ for all } t \in [0, a] \text{ and } u, v \in B.$$

For $b \in B$, we will denote $S(b)$ the set of solutions of (3.1) on $[0, a]$. $S(b)$ is nonempty and compact, by [3] and [2].

Theorem 3.3.3. *(I. Coroian[17]) If the following conditions hold :*

1. *$T : [0, a] \times B \to P_{cl,cv}(B)$ is continuous;*
2. *There exists $k > 0$ such that*

$$H_d^+(T(t, u), T(t, v)) \leq k||u - v||_C, for\ all\ t \in [0, a]\ and\ for\ all\ u, v \in B \subseteq \mathbb{R}^n;$$

3. *$2ka < 1$.*

Then $S(b)$ is continuous from B into the family of nonempty compact subsets of $C[0, a]$ equipped with the H_d^+ metric.

More generally, we have the following result.

Theorem 3.3.4. *(I. Coroian[17]) For each $n = 0, 1, 2, ...$, let T_n be a continuous map of $[0, a] \times B$ into $C(B)$ satisfying, for some $k > 0$, the condition*

$$H_d^+(T_n(t, u), T_n(t, v)) \leq k||u - v||_C, for\ all\ t \in [0, a]\ and\ for\ all\ u, v \in B$$

Assume that $T_n \to T_0$ uniformly on $[0, a] \times B$. For each $b \in B$ and $n = 0, 1, 2, ...$ Let $S_n(b)$ be the set of solutions of

$$\begin{cases} \dot{x}(t) \in T_n(t, x(t)) \\ x(0) = b \end{cases} \tag{3.2}$$

If $2ka < 1$ and $b_n \to b_0$ in B, then $S_n(b_n) \to S_0(b_0)$.

Next, an application to Ulam-Hyers stability of the inclusion $x \in T(x)$ ($Theorem$ 3.1(v)) is given. The notion of Ulam-Hyers stability for a differential inclusion is defined as follows.

Definition 3.3.2. *([43]) Let*

$$x' \in T(t, x(t)),\ t \in [0, a] \tag{3.3}$$

and $T : [0, a] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *be a continuous operator. We say that* (3.12) *is Ulam-Hyers stable if for any* $\epsilon > 0$ *and an* $y \in C[0, a]$ *an* ϵ*- solution of (3.3) (which means that*

$$D\Big(y(t), y(0) + \int_0^t T(s, y(s))ds\Big) \leq \epsilon,\ t \in [0, a])$$

there exists a solution x^* *of (3.3) and* $c > 0$ *such that* $||x^* - y|| \leq c \cdot \epsilon$.

Definition 3.3.3. *([43]) Let* $F : [0, a] \to P_cl(\mathbb{R}^n)$ *be a measurable multivalued operator. If* $L^1([0, a], \mathbb{R}^n)$ *denotes the set of all measurable and integrable mappings from* $[0, a]$ *to* $\mathbb{R}^n$*, then* S_F *will denote the set of all integrable selections of* F*, i.e :*

$$S_F := \{f \in L^1([0, a], \mathbb{R}^n) | f(t) \in F(t) a.e.t \in [0, a]\}.$$

Remark 3.3.2. *([43]) In particular, if* $x : [0, a] \to \mathbb{R}^n$ *and* $T : [0, a] \times \mathbb{R}^n \to P_{cl}(\mathbb{R}^n)$*, then the set of all integrable selections of* T *will be denoted by*

$$S_{T(\cdot,x(\cdot))} := \{f \in L^1([0, a], \mathbb{R}^n) | f(t) \in T(t, x(t)) a.e.t \in [0, a]\}.$$

Theorem 3.3.5. *(I. Coroian[17]) Let us consider the inclusion (3.3). We assume:*

(a) $T : [0, a] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *is a continuous, measurable and integrably bounded multivalued operator.*

(b)There exists $L > 0$ *such that*

$$H_d^+(T(t, u_1), T(t, u_2)) \leq L||u_1 - u_2||,\ for\ each\ (t, u_1), (t, u_2) \in [0, a] \times \mathbb{R}^n.$$

Then the differential inclusion (3.3) with initial condition $x(0) = x^0$ *has at least one solution. Moreover the differential inclusion (3.3) is Ulam-Hyers stable.*

3.4 Continuation results for contraction type multivalued operators with respect to the Pompeiu functional

In this section we present some fixed point and continuation results for multivalued of H_d^+-contractions in a metric space. The first first fixed point result for multivalued contraction was obtained by Nadler [?], while the first continuation result is due to Frigon and Granas [25] , see also M. Frigon [24].

In this section, we present a local result and a continuation result for a special kind of multivalued (H_d^+, α)-contractions. Following *Kirk* and *Shahzad* ([34]) , we will replace the second condition of the Definition 2.3.1:

$(*)$ for every $x \in X$, $y \in T(x)$ and for every $\epsilon > 0$ there exists z in $T(y)$ such that

$$d(y,z) \leq H_d^+(T(x),T(y)) + \varepsilon$$

with the following one:

$(**)$ for every $x \in X$ and every $y \in T(x)$ we have that

$$D(y,T(y)) \leq H_d^+(T(x),T(y)).$$

Notice that $(**)$ implies $(*)$. Moreover if we consider the following condition

$(***)$ for every $\epsilon > 0$, for every $x \in X$, $y \in T(x)$ there exists z in $T(y)$ such that

$$d(y,z) \leq H_d^+(T(x),T(y)) + \varepsilon,$$

then it is easy to see that $(**)$ is equivalent with $(***)$. In this last case, we also notice that for each $x \in X$ and every $y \in T(x)$ we have $\rho(T(x),T(y) \leq H_d^+(T(x),T(y))$. As a consequence, $\rho(T(x),T(y) \leq \rho(T(y),T(x))$ and so

$$H_d^+(T(x),T(y)) \leq \rho(T(y),T(x)), \text{ for each } x \in X \text{ and } y \in T(x).$$

Homotopy results for multivalued operators of contractive types are well-know in the literature, see [**?**], [**?**], [**?**]. The approach is based in all cases on a local fixed point theorem. The first result of this section is the following local fixed point theorem.

Theorem 3.4.1. *(I. Coroian[17]) Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \tilde{B}(x_0,r) \to P_{cl}(X)$ is a multivalued operator. We suppose that :*

(i) T is a multivalued (H_d^+,α) - contraction, i.e., $\alpha \in]0,1[$ and

$$H_d^+(T(x),T(y)) \leq \alpha d(x,y), \text{ for every } x,y \in X;$$

(ii) for every $x \in X$ and every $y \in T(x)$ we have that $D(y,T(y)) \leq H_d^+(T(x),T(y))$;

(iii) $D(x_0,T(x_0)) < (1-\alpha)r$.

Then, there exists $x^* \in \tilde{B}(x_0,r)$ such that $x^* \in T(x^*)$.

Theorem 3.4.2. *(I. Coroian[17]) Let (X,d) be a complete metric space. Let U be an open subset of (X,d). Let $G : U \times [0,1] \to P(X)$ be a multivalued operator such that the following conditions are satisfied :*

1. *$x \neq G(x,t)$ for each $x \in \partial B$ and each $t \in [0,1]$;*
2. *there exists $\alpha \in [0,1[$ such that for each $t \in [0,1]$ and each $x,y \in U$ we have :*

$$H_d^+(G(x,t),G(y,t)) \leq \alpha d(x,y)$$

3. *there exists a continuous increasing functions $\phi : [0,1] \to \mathbb{R}$ such that*

$$H_d^+(G(x,t),G(x,s)) < |\phi(t) - \phi(s)|, \text{ for all } t,s \in [0,1], \ t \neq s \text{ and each } x \in U;$$

4. *$G : U \times [0,1] \to P((X,d))$ is closed.*

Then $G(\cdot,0)$ has a fixed point if and only if $G(\cdot,1)$ has a fixed point.

3.5 Kikkawa-Suzuki fixed point theorems for contractive type multivalued operators with in respect to the Pompeiu functional

Suzuki ([93]) generalized the well-known Banach contraction principle by introducing a new type of mappings. Then Kikkawa and Suzuki([35])gave another generalization which generalized the results of Suzuki and Nadler. Also Moţ and Petruşel ([44]) gave another generalization concerning the work of Kikkawa and Suzuki. We want to extend this results using generalized multivalued H_d^+- contractions of Ciric. The results complement and extend some recent results proved by M. Kikkawa, T. Suzuki ([35]), T. Lazăr, D. O'Regan, A. Petruşel ([40]) şi T. Suzuki ([92]). Following Reich ([69]), we introduce the following concept.

Definition 3.5.1. *(I. Coroian[18]) Let (X, d) be a metric space. Then, $T : X \to P_{cl}(X)$ is called an $(a, b, c)-$ KSR multivalued operator if $a, b, c \in \mathbb{R}_+$ with $a + b + c \in (0, 1)$ and the following two assertions hold:*

(i) $\frac{1-2b-2c}{1+2a} D(x, T(x)) \leq d(x, y)$ *implies that*

$$H_d^+(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y)), \; for \; x, y \in X$$

(ii) for every $x \in X$, $y \in T(x)$ and for every $q > 1$ there exists $z \in T(y)$ such that $d(y, z) \leq qH_d^+(T(x), T(y))$.

Theorem 3.5.1. *(I. Coroian[18]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a (a, b, c)- KSR multivalued operator. Then $Fix(T) \neq \emptyset$.*

Definition 3.5.2. *(I. Coroian[18]) Let (X, d) be a metric space. Then, $T : X \to P_{cl}(X)$ is called an α KSR multivalued operator if $\alpha \leq \frac{1}{2}$ and the following two assertions hold:*

(i) $(1 - 2\alpha)D(x, T(x)) \leq d(x, y)$ *implies that*

$$H_d^+(T(x), T(y)) \leq \alpha \max\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}(D(x, T(y)) + D(y, T(x))\}, \; for \; x, y \in X$$

(ii) for every $x \in X$, $y \in T(x)$ and for every $q > 1$ there exists $z \in T(y)$ such that $d(y, z) \leq qH_d^+(T(x), T(y))$.

Theorem 3.5.2. *(I. Coroian[18]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a α- KSR multivalued operator. Then $Fix(T) \neq \emptyset$.*

Chapter 4

Coupled fixed point theorems for contractive type multivalued operators with respect to the Pompeiu functional

In this chapter, we will consider the coupled fixed point problem for a multi-valued operator satisfying a contraction condition with respect to the excess functional in b-metric spaces. The approach is based on a fixed point theorem for a multi-valued operator in the setting of a b-metric space. On one hand, we will consider the problem of the existence of the solutions and, on the other hand, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing property of the coupled fixed point problem are discussed. Some applications to a system of integral inclusions and to a system of multi-valued initial value problem for a first order differential inclusions are also given.

The references which were used to develop this chapter are: T. Gnana Bhaskar and V. Lakshmikantham [27], A. Petruşel, D. Guo, Y.J. Cho, J. Zhu [28], D. Guo, V. Lakshmikantham [29], A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao [57], G. Petruşel, B. Samet şi J.-C. Yao [58], A. Petruşel, C. Urs, O. Mleşniţe [62], A.I. Perov [64], R. Precup [66], C. Urs [94], V.I. Opoitsev [48], V.I. Opoitsev, T.A. Khurodze [49], D. O'Regan, N. Shahzad, R.P. Agarwal [50], R.S. Varga [95] and P.P. Zabrejko [96].

4.1 Fixed point theorems

Nadler's contraction principle is an extension to the multivalued case of the classical Banach's contraction principle. There are many applications of these results, mainly in the theory of operator inclusions. The purpose of this paper is to give new fixed point results for multi-valued operators satisfying a contraction type condition with respect to the excess functional. We will consider here the context of a b-metric space. Our results are new even for the case of metric spaces and they extend some theorems given in [27], [57], [58], A. Petruşel, I.A. Rus [59], A. Petruşel, G. Petruşel, C. Urs [63] and other papers in the literature.

On the other hand, several theorems were given for the so called coupled fixed point problem. In the multivalued setting, this problem is as follows:

Let (X, d) be a metric space an $P(X)$ be the family of all nonempty subsets of X. If $G : X \times X \to P(X)$ is a multivalued operator, the, by definition, a coupled fixed point problem for G means to find a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* \in G(x^*, y^*) \\ y^* \in G(y^*, x^*) \end{cases} . \tag{4.1}$$

We introduce the definition of a b-metric space.

Definition 4.1.1. *(Bakhtin([6]), Czerwik([21])) Let X be a set and let s $\geq$1 be a given real number. a functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric with constant s if all the axioms of the metric take place with the following modification of the triangle inequality axiom :*

$$d(x, z) \leq s[d(x, y) + d(y, z)], \; for\ all\ x, y, z \in X.$$

Under the above hypotheses the pair (X, d) is called a b-metric space with constant s.

Definition 4.1.2. *([91]) Let X be a nonempty set. Then the triple $(X, \leq, d)$ is called an ordered b-metric space if :*

i. " $\leq$ " be a partial order on X;

ii. d is a b-metric on X with constant $s \leq 1$;

iii. $(x_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence on X, $x_n \to x^*$ where $x^* \in X$, then $x_n \leq x^*$, for all $n \in \mathbb{N}$;

iv. $(y_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence on X, $y_n \to y^*$ where $y^* \in X$, then $y_n \geq y^*$, for all $n \in \mathbb{N}$.

Definition 4.1.3. *([91]) Let $(X, \leq)$ be a partially ordered set and endow the product space $X \times X$ with the following partial ordered:*

for $(x, y), (u, v) \in X \times X$, $(x, y) \leq_p (u, v) \Leftrightarrow x \leq, y \geq v$

Definition 4.1.4. *([90]) Let (X, d) be a b-metric space and $G : X \times X \to P(X)$ be a multivalued operator. Then, by definition, a couple fixed point for G is a pair $(x^*, y^*) \in X \times X$ satisfying*

$$(P1) \begin{cases} x^* \in G(x^*, y^*) \\ y^* \in G(y^*, x^*) \end{cases} \tag{4.2}$$

We will denote by $CFix(G) = \{(x, y) \in X \times X | x \in G(x, y) \text{ and } y \in G(y, x)\}$ the coupled fixed point set of G.

Definition 4.1.5. *([90]) Let $(X, \leq)$ be a partially ordered set and $T : X \times X \to P(X)$. We say that T has the mixed monotone property if $T(\cdot, y)$ is monotone increasing for any $y \in X$ and $T(y, \cdot)$ is monotone decreasing for any $x \in X$.*

Lemma 4.1.1. *([55]) Let (X,d) be a metric space, $A, B \in P(X)$ and $q > 1$. Then, for every $a \in A$ there exists $b \in B$ such that $\rho(A, B) \leq qd(a, b)$.*

In this part we will present some fixed point theorems in b-metric spaces for a multivalued operator.

Definition 4.1.6. *[52] Let (X,d) be a metric space. A multivalued mapping $T : X \to P_{b,cl}(X)$ is called H^+-contraction with constant α if:*

1. *there exists a fixed real number k, $0 < k < 1$ such that for every $x, y \in X$*

$$H_d^+(T(x), T(y)) \leq kd(x, y),$$

2. *for every $x \in X$, $y \in T(x)$ we have*

$$D_d(y, T(y)) \leq H_d^+(T(x), T(y)).$$

Definition 4.1.7. *([91]) Let (X, d) be a partially ordered set and $A, B \in P(X)$. We will denote $A \leq_{st} B \Leftrightarrow \forall a \in A, \forall b \in B$ we have $a \leq b$.*

Remark 4.1.1. *(I. Coroian, G. Petruşel [19]) Notice that if $A, B, C \in P(X)$, then $A \leq_s B$ and $B \leq_s C$ implies $A \leq_s C$.*

The first result of this section is a fixed point theorem in an ordered b-metric space.

Theorem 4.1.1. *(I. Coroian, G. Petruşel [19]) Let $(X, \leq, d)$ be an order b-metric space and $d : X \times X \to \mathbb{R}_+$ a complete b-metric with constant $s \geq 1$. Let $T : X \to P_{cl}(X)$ be a multivalued operator strong increasing with respect to " $\leq$ ". Suppose that*

i) there exists $k \in \left(0, \frac{1}{s}\right)$ and an element $x_0 \in X$ such that $\rho_d(T(x), T(y)) \leq kd(x, y)$, for all $x, y \in X$ with $x \leq y$,

ii) $x_0 \leq_w T(x_0)$.

Then :

a. $Fix(T) \neq \emptyset$ and there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximation of T starting from $x_0 \in X$ which converges to a fixed point of T.

b. In particular, if d is continuous b-metric on X then we have:

$$d(x_n, x^*) \leq sk^n \frac{1}{1 - sk} d(x_0, x_1), \forall n \in \mathbb{N}^* \text{ and } x_1 \in T(x_0).$$

The second result is a global fixed point theorem in a b-metric space.

Theorem 4.1.2. *(I. Coroian, G. Petruşel[19]) Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $T : X \to P_{cl}(X)$ a multivalued operator which satisfies the following condition:*

$$\text{there exists } k \in \left(0, \frac{1}{s}\right) \text{ such that } \rho(T(x), T(y)) \leq kd(x,y), \text{ for all } x, y \in X$$

Then:

a) Fix(T) $\neq \emptyset$ and there exits a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximation of T starting from any $(x_0, x_1) \in Graph(T)$ which converges to a fixed point x^* of T;

b) If additionally $SFix(T) \neq \emptyset$, then $Fix(T) = SFix(T) = \{x^*\}$;

c) In particular, if d is a continuous b-metric then we have:

$$d(x_n, x^*) \leq \frac{sk^n}{1-sk} d(x_0, x_1), \forall n \in \mathbb{N}^*$$

4.2 Coupled fixed point theorems

In this chapter we will give new coupled fixed point theorems for multi-valued operators satisfying a contraction type condition with respect to the excess functional. We will consider here the context of a b-metric space. Our results are new even for the case of metric spaces and they extend some theorems given in V. Berinde [9], M. Boriceanu-Bota, A. Petruşel, I.A. Rus [10], M. Boriceanu-Bota, A. Petruşel, G. Petruşel, B. Samet [11],[27], [57], [58]and other papers in the literature. We will start this section by recalling some useful notion and results.

We will start this section by recalling some useful notion and results.

Definition 4.2.1. *([57]) Let $(X, \preceq)$ a partially ordered set and $G : X \times X \to P(X)$. We say that G has the strict mixed monotone property with respect to the partial order "$\preceq$", if the following implications holds:*

1. $x_0 \preceq x_1 \Rightarrow G(x_0, y) \leq_{st} G(x_1, y)$, $\forall y \in X$.

2. $y_0 \succeq y_1 \Rightarrow G(x, y_0) \leq_{st} G(x, y_1)$, $\forall x \in X$.

Remark 4.2.1. *(I. Coroian, G. Petruşel [19]) Let (X,d) be a b-metric space with constant $s \geq 1$ and $Z := X \times X$. Then the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by*

$$\tilde{d}((x,y),(u,v)) \leq d(x,u) + d(y,v), \text{ for all } (x,y),(u,v) \in Z$$

is a b-metric on Z with the same constant $s \geq 1$ and if (X,d) is a complete b-metric space, then $(Z, \tilde{d})$ is a complete b-metric space, too.
Moreover, for $x, y \in X$, $A, B, U, V \in P(X)$ we have:

$$D_{\tilde{d}}((x,y), A \times B) = D_d(x, A) + D_d(y, B),$$

$$\rho_{\tilde{d}}(U \times V, A \times B) = \rho_d(U, A) + \rho_d(V, B)$$

$$H_{\tilde{d}}(U \times V, A \times B) \leq H_d(U, A) + H_d(V, B)$$

and

$$H^+_{\tilde{d}}(U \times V, A \times B) = H^+_d(U, A) + H^+_d(V, B).$$

Additionally, by the properties of the gap functional D_d, if $(x, y) \in X \times X$ and $A, B \in P_{cl}(X)$, then

$$D_{\tilde{d}}((x, y), A \times B) = 0 \text{ if and only if } (x, y) \in A \times B.$$

The first main result of this section is the following theorem.

Theorem 4.2.1. *(I. Coroian, G. Petruşel [19]) Let $(X, \preceq, d)$ be an ordered b-metric space with constant $s \geq 1$ such that the b-metric d is complete. Let $G : X \times X \to P(X)$ be a multi-valued operator having the strict mixed monotone property with respect to "$\preceq$" and G has closed graph.*

Assume that:

(i) there exists $k \in \left(0, \frac{1}{s}\right)$ such that

$$\rho_d(G(x, y), G(u, v)) + \rho_d(G(y, x), G(v, u)) \leq k[d(x, u) + d(y, v)], \ \forall x \preceq u \text{ and } y \succeq v;$$

(ii) there exist $(x_0, y_0) \in X \times X$ and $(x_1, y_1) \in G(x_0, y_0) \times G(y_0, x_0)$ such that $x_0 \preceq x_1$ and $y_0 \succeq y_1$.

Then, the following conclusions hold:

(a) there exist $x^*, y^* \in X$ and there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X with

$$\begin{cases} x_{n+1} \in G(x_n, y_n) \\ y_{n+1} \in G(y_n, x_n) \end{cases}$$

for all $n \in \mathbb{N}$ such that $x_n \to x^*$, $y_n \to y^*$ as $n \to \infty$ and

$$\begin{cases} x^* \in G(x^*, y^*) \\ y^* \in G(y^*, x^*). \end{cases}$$

(b) If, in addition, the b-metric d is continuous, then we have the following estimation holds

$$d(x_n, x^*) + d(y_n, y^*) \leq \frac{sk^n}{1 - sk}[d(x_0, x_1) + d(y_0, y_1)], \text{ for all } n \in \mathbb{N}.$$

Remark 4.2.2. *(I. Coroian, G. Petruşel [19]) Notice that the contraction condition (i) from Theorem 4.2.1 can be re-written, using the functional H^+, as follows:*

$$H^+_d(T(x, y), T(u, v)) \leq 2k(d(x, u)+d(y, v)), \ \forall (x, y), (u, v) \in X\times X \text{ with } (x, y) \leq_p (u, v).$$

Remark 4.2.3. *(I. Coroian, G. Petruşel [19]) If instead of the hypothesis that the triple $(X, \preceq, d)$ is a complete ordered b-metric space, we only assume that X is a nonempty set, $\preceq$ is a partially ordering on X and d is a complete b-metric with constant $s \geq 1$, then the conclusions of the above theorem hold if, additionally, the graph of G is a closed set.*

The following result is about the uniqueness of the coupled fixed point.

Theorem 4.2.2. *(I. Coroian, G. Petruşel [19])*
In addition to the hypotheses of Theorem 4.2.1 we suppose that:
(i) there exists $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} G(x^*, y^*) = \{x^*\} \\ G(y^*, x^*) = \{y^*\} \end{cases}$$

(ii) for any solution $(\overline{x}, \overline{y})$ of the coupled fixed point problem (P1) we have $\overline{x} \leq x^*$ and $\overline{y} \geq y^*$ or $(\overline{x}, \overline{y}) \leq_p (x^*, y^*)$.
Then, we obtain that the coupled fixed point problem (P1) has an unique solution.

Theorem 4.2.3. *(I. Coroian, G. Petruşel [19]) We suppose that all the hypotheses of Theorem 4.2.2 take place and $x^* \leq y^*$ or $y^* \leq x^*$ where (x^*, y^*) is the unique coupled fixed point of G. Then $x^* = y^*$ i.e $G(x^*, x^*) = \{x^*\}$.*

If the contraction condition withe respect to the excess functional is assumed for every elements $x, y \in X$, then no monotonicity assumptions are needed for G. Hence, we get the following result.

Theorem 4.2.4. *(I. Coroian, G. Petruşel [19])Let (X, d) be a complete b-metric space with constant $s \geq 1$. Let $G : X \times X \to P_{cl}(X)$ be a multi-valued operator for which there exists $k \in (0, 1)$ such that*

$$\rho_d(G(x, y), G(u, v)) + \rho_d(G(y, x), G(v, u)) \leq k[d(x, u) + d(y, v)] \ , \forall (x, y), (u, v) \in X \times X.$$

Then, the following conclusions hold:
(a) $CFix(G) \neq \emptyset$ and for any initial point $(x_0, y_0) \in X \times X$, the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} x_{n+1} \in G(x_n, y_n) \\ y_{n+1} \in G(y_n, x_n) \end{cases}$$

converge to x^* and respectively to y^* as $n \to \infty$ where

$$\begin{cases} x^* \in G(x^*, y^*) \\ y^* \in G(y^*, x^*) \end{cases}$$

(b) In particular, if the b-metric d is continuous, then we have the following estimation:

$$d(x_n, x^*) + d(y_n, y^*) \leq \frac{sk^n}{1 - sk}[d(x_0, x_1) + d(y_0, y_1)].$$

(c) If additionally, there exists a pair $(u^*, v^*) \in X \times X$ such that

$$\begin{cases} G(u^*, v^*) = \{u^*\} \\ G(v^*, u^*) = \{v^*\} \end{cases} ,$$

then $CFix(G) = \{(x^*, y^*)\}$.

In the next part of this section, we will present some proprieties of the coupled fixed point problem (P1).

The first one is the data dependence problem for the coupled fixed point problem (P1).

Theorem 4.2.5. *(I. Coroian, G. Petruşel [19])Let (X, d) be a complete b-metric space with constant $s \geq 1$ and let $G : X \times X \to P_{cl}(X)$, $S : X \times X \to P(X)$ be two multi-valued operators. We suppose that:*

(i) there exists $k \in \left(0, \frac{1}{s}\right)$ such that

$$\rho_d(G(x,y), G(u,v)) + \rho_d(G(y,x), G(v,u)) \leq k[d(x,u) + d(y,v)],\ \forall (x,y),(u,v) \in X \times; X$$

(ii) there exists $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} G(u^*, v^*) = \{u^*\} \\ G(v^*, u^*) = \{v^*\} \end{cases};$$

(iii) there exists $(u^*, v^*) \in X \times X$ such that

$$\begin{cases} u^* \in S(u^*, v^*) \\ v^* \in S(v^*, u^*) \end{cases};$$

(iv) there exists $\eta > 0$ such that $\rho_d(G(x,y), S(x,y)) \leq \eta$, for all $(x,y) \in X \times X$. Then

$$\rho_{\tilde{d}}(CFix(S), CFix(G)) \leq \frac{2s\eta}{1 - sk},$$

where $\tilde{d}((x,y),(u,v)) = d(x,u) + d(y,v)$, $\forall (x,y),(u,v) \in X \times X$.

The second problem we intend to study is the well-posedness of the coupled fixed point problem.

Definition 4.2.2. *We consider the coupled fixed point problem $(P1)$. By definition, $(P1)$ is well-posed for G with respect to D_d if:*

(i) $CFix(G) = \{w^*\}$, where $w^* = (u^*, v^*) \in X \times X$;

(ii) if there exists a sequence $w_n = (u_n, v_n) \in X \times X$ with

$$\begin{cases} D_d(u_n, G(u_n, v_n)) \to 0 \\ D_d(v_n, G(v_n, u_n)) \to 0 \end{cases},$$

then $d(u_n, u^*) + d(v_n, v^*) \to 0$ as $n \to \infty$.

Theorem 4.2.6. *(I. Coroian, G. Petruşel [19]) We suppose that all the hypotheses of Theorem 4.2.4 take place. Then the coupled fixed point problem $(P1)$ is well-posed for G with respect to D_d.*

Now we will present the Ulam-Hyers property of the coupled fixed point problem.

Definition 4.2.3. *Let (X, d) be a b-metric space with constant $s \geq 1$ and let $G : X \times X \to P(X)$ be a multi-valued operator. Let $\tilde{d}$ any b-metric on $X \times X$ generated by d. Let us consider the system of inclusions*

$$\begin{cases} x \in G(x, y) \\ y \in G(y, x) \end{cases} \tag{4.3}$$

and the inequality

$$D_d(x, G(x, y)) + D_d(y, G(x, y)) \leq \epsilon, \tag{4.4}$$

where $\epsilon > 0$ and $(x, y) \in X \times X$.

By definition, the system of inclusions (4.3) is called Ulam-Hyers stable if and only if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\epsilon > 0$ and for each solution $(u^*, v^*) \in X \times X$ of the inequality (4.4), there exists a solution $(x^*, y^*) \in X \times X$ of (4.3) such that

$$d(u^*, v^*) + d(v^*, y^*) \leq \psi(\epsilon) \Leftrightarrow \tilde{d}((x^*, y^*), (u^*, v^*)) \leq \psi(\epsilon).$$

Theorem 4.2.7. *(I. Coroian, G. Petruşel [19]) Let $G : X \times X \to P_{cl}(X)$ be a multi-valued operator which verifies the hypotheses of Theorem 4.2.4. Then the system of inclusions (4.3) is Ulam-Hyers stable.*

For the next result, we need the following theorem (known as Cauchy's Lemma).

Lemma 4.2.1. *(Cauchy's Lemma) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of non-negative real numbers, such that $\sum\limits_{p=0}^{\infty} a_p < +\infty$ and $\lim\limits_{n\to\infty} b_n = 0$. Then, $\lim\limits_{n\to\infty} \left(\sum\limits_{p=0}^{n} a_{n-p} b_p \right) = 0$.*

Using the above result and the global existence result for the coupled fixed point problem (see Theorem 4.2.4), we can prove the limit shadowing property of the coupled fixed point problem (P1).

Definition 4.2.4. *Let (X, d) be a b-metric space with constant $s \geq 1$ and $G : X \times X \to P(X)$ be a multi-valued operator. Let $\tilde{d}$ be any b-metric generated by d. By definition, the coupled fixed problem $(P1)$ has the limit shadowing property if, for any sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $X \times X$ for which*

$$D\tilde{d}((x_{n+1}, y_{n+1}), G(x_n, y_n) \times G(y_n, x_n)) \to 0 \text{ as } n \to \infty,$$

there exists a sequence $(u_n, v_n)_{n\in\mathbb{N}}$ in $X \times X$ such that

$$\tilde{d}((x_n, y_n), (u_n, v_n)) \to 0, \text{ as } n \to \infty.$$

Theorem 4.2.8. *(I. Coroian, G. Petruşel [19]) Let (X, d) be a b-metric space with constant $s \geq 1$ and $G : X \times X \to P_{cl}(X)$ be a multi-valued operator which verifies all the hypotheses of Theorem 4.2.4. Then, the coupled fixed point problem $(P1)$ has the limit shadowing property.*

4.3 Applications

In this chapter we will present some applications to integral and differential inclusion of the previous results. The references which were used to develop this section are: R. Precup, A. Viorel [67], R.P. Agarwal [1] and T.P. Petru, A. Petruşel, J.-C. Yao [68]. Let us consider the following system of integral inclusions:

Let us consider first the following system of integral inclusions

$$\begin{cases} x(t) \in \int\limits_0^t K(s, x(s), y(s))ds + g(t) \\ y(t) \in \int\limits_0^t K(s, y(s), x(s))ds + g(t) \end{cases} for\ t \in [0, T], \tag{4.5}$$

where $g : [0, T] \to \mathbb{R}$ and $K : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ are operators satisfying some appropriate conditions.

A solution of the above system is a pair $(x, y) \in C([0, T], \mathbb{R}^n) \times C([0, T], \mathbb{R}^n)$ satisfying the above relations for all $t \in [0, T]$.

We consider $X := C([0, T], \mathbb{R}^n)$ endowed with the partial order relation

$$x \leq_C y \Leftrightarrow x(t) \leq y(t), \text{ for all } t \in [0, T],$$

where " $\leq$ " is the component wise ordering relation on $\mathbb{R}^n$. We will also denote by $|\cdot|$ a norm in $\mathbb{R}^n$.

We also consider (for $p \in \mathbb{N}, p \geq 2$ an even number) the following functional:

$$d(x, y) := \max_{t \in [0,T]} [(x(t) - y(t))^p e^{-\tau t}].$$

Notice that d is a b-metric, for any $\tau > 0$ arbitrary chosen. Indeed, is easy to check that the first conditions are satisfied for all $x, y \in X$. We show that the triangle's inequality is verified too. Using Hölder's inequality we obtain that:

$$(x(t) - y(t))^p \leq 2^{\frac{p}{q}} [(x(t) - z(t))^p + (z(t) - y(t))^p] \cdot e^{-\tau t} \cdot e^{\tau t} \leq$$

$$\leq 2^{\frac{p}{q}} \max_{t \in [0,T]} ((x(t) - z(t))^p e^{\tau t} + (z(t) - y(t))^p e^{\tau t}) \cdot e^{\tau t} =$$
$$= 2^{\frac{p}{q}} (d(x, z) + d(y, z)) \cdot e^{\tau t}$$

with $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

After a multiplication with $e^{-\tau t}$ and taking then maximum over $t \in [0, T]$ we get that

$$d(x, y) \leq 2^{p-1}(d(x, z) + d(z, y)).$$

Notice also that the triple $(X, \leq, d)$ is an ordered metric space.

We can prove now have the following existence result.

Theorem 4.3.1. *(I. Coroian, G. Petruşel [19]) Consider the integral system (4.5). We suppose that:*

(i) $g : [0,T] \to \mathbb{R}$ is a continuous function and $K : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable in the first variable and jointly $H_{|\cdot|}$- continuous in the last two variables;

(ii) K is integrably bounded, i.e., there exists a mapping $r \in L^1[0,T]$ such that for each $(s,u,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and for any $w \in K(s,u,v)$, we have $|w| \le r(t)$, a.e. $t \in [0,T]$;

(iii) $K(s,\cdot,\cdot)$ has the strict mixed monotone property with respect to the last two variables, for all $s \in [0,T]$;

(iv) there exist $\alpha, \beta \in L^1([0,T], \mathbb{R}_+)$ such that, for each $s \in [0,T]$, we have

$$\rho_{|\cdot|}(K(s,x,y), K(s,u,v)) \le \alpha(s)|x-u| + \beta(s)|y-v|, \ \forall x,y,u,v \in \mathbb{R}^n,$$

with $(x \le u,\ y \ge v)$ or $(x \le u,\ y \le v)$;

(v) there exist $x_0, y_0 \in C[0,T]$ and two measurable selections $f_{x_0,y_0} : [0,T] \to \mathbb{R}^n$ of $K(\cdot, x_0(\cdot), y_0(\cdot))$ and $f_{y_0,x_0} : [0,T] \to \mathbb{R}^n$ of $K(\cdot, y_0(\cdot), x_0(\cdot))$, such that

$$\begin{cases} x_0(t) \le g(t) + \int\limits_0^t K(s, x_0(s), y_0(s)) f_{x_0,y_0}(s)ds \\ y_0(t) \ge g(t) + \int\limits_0^t K(s, y_0(s), x_0(s)) f_{y_0,x_0}(s)ds \end{cases} \tag{4.6}$$

or

$$\begin{cases} x_0(t) \ge g(t) + \int\limits_0^t K(s, x_0(s), y_0(s)) f_{x_0,y_0}(s)ds \\ y_0(t) \le g(t) + \int\limits_0^t K(s, x_0(s), y_0(s)) f_{y_0,x_0}(s)ds \end{cases} \tag{4.7}$$

for all $t \in [0,T]$.

Then, there exists at least one solution (x^*, y^*) of the system (4.5).

Now we can give an application of the previous result using the system of differential inclusions with initial values. Lets consider the following system :

$$\begin{cases} x'(t) \in F(t, x(t), y(t)) \\ y'(t) \in F(t, y(t), x(t)) \\ x(0) = x^0 \\ y(0) = x^0 \end{cases} . \tag{4.8}$$

where $x^0 \in \mathbb{R}^n$ and $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is an operator satisfying some appropriate conditions.

We can observe that the system (4.8) is equivalent with the following system of integral inclusions:

$$\begin{cases} x(t) \in \int\limits_0^t F(s, x(s), y(s))ds + x^0 \\ y(t) \in \int\limits_0^t F(s, y(s), x(s))ds + x^0 \end{cases} for\ t \in [0, T], \tag{4.9}$$

By applying Theorem 4.3.1 to the above system we immediately obtain the following existence result.

Theorem 4.3.2. *(I. Coroian, G. Petruşel, [19]) Consider the integral system (4.8). We suppose that:*

(i) $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is upper semicontiunous and Lebesgue measurable in the first variable and jointly $H_{|\cdot|}$- continuous in the last two variables;

(ii) F is integrably bounded, i.e., there exists a mapping $r \in L^1[0, T]$ such that for each $(s, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ and for any $w \in F(s, u, v)$, we have $|w| \leq r(t)$, a.e. $t \in [0, T]$;

(iii) $F(s, \cdot, \cdot)$ has the strict mixed monotone property with respect to the last two variables, for all $s \in [0, T]$;

(iv) there exist $\alpha, \beta \in L^1([0, T], \mathbb{R}_+)$ such that, for each $s \in [0, T]$, we have

$$\rho_{|\cdot|}(F(s, x, y), F(s, u, v)) \leq \alpha(s)|x - u| + \beta(s)|y - v|,\ \forall x, y, u, v \in \mathbb{R}^n,$$

with $(x \leq u,\ y \geq v)$ or $(x \leq u,\ y \leq v)$;

(v) there exist $x_0, y_0 \in C[0, T]$ and two measurable selections $f_{x_0,y_0} : [0, T] \to \mathbb{R}^n$ of $F(\cdot, x_0(\cdot), y_0(\cdot))$ and $f_{y_0,x_0} : [0, T] \to \mathbb{R}^n$ of $F(\cdot, y_0(\cdot), x_0(\cdot))$, such that

$$\begin{cases} x_0(t) \leq x^0 + \int\limits_0^t F(s, x_0(s), y_0(s)) f_{x_0,y_0}(s)ds \\ y_0(t) \geq x^0 + \int\limits_0^t F(s, y_0(s), x_0(s)) f_{y_0,x_0}(s)ds \end{cases} \tag{4.10}$$

or

$$\begin{cases} x_0(t) \geq x^0 + \int\limits_0^t F(s, x_0(s), y_0(s)) f_{x_0,y_0}(s)ds \\ y_0(t) \leq x^0 + \int\limits_0^t F(s, x_0(s), y_0(s)) f_{y_0,x_0}(s)ds \end{cases} \tag{4.11}$$

for all $t \in [0, T]$.

Then, there exists at least one solution (x^*, y^*) of the system (4.9).

An extension of the coupled fixed point problem is given now. We will discuss the existence of the solution for the operator system:

$$\begin{cases} x \in T_1(x, y) \\ y \in T_2(x, y) \end{cases} \tag{4.12}$$

Let $Z \neq \emptyset$ and $\tilde{d} : Z \times Z \to \mathbb{R}^m_+$ be a vector valued metric on Z given by

$$\tilde{d}(z,w) = \begin{pmatrix} d_1(z,w) \\ \cdots \\ d_m(z,w) \end{pmatrix}.$$

We denote by $\rho_{\tilde{d}} : P_{cp} \times P_{cp} \to \mathbb{R}^m_+$ the excess functional generated by $\tilde{d}$ as follows:

$$\rho_{\tilde{d}}(A,B) = \begin{pmatrix} \rho_{d_1}(A,B) \\ \cdots \\ \rho_{d_m}(A,B) \end{pmatrix}.$$

Theorem 4.3.3. *Let $(Z, \tilde{d})$ be a complete metric space. Let $G : Z \to P_{cl}(Z)$ a mapping which satisfies*

$$\rho_{\tilde{d}}(G(z_1), G_(z_2)) \leq A\tilde{d}(z_,z_2), \forall z_1, z_2 \in Z$$

where $A \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ is a matrix convergent to zero. Then $Fix(G) \neq \emptyset$.

For our next result we will work with the following vector valued metric on $Z = X \times X$ and the excess functional:

1. $\tilde{d} : Z \times Z \to \mathbf{R}^2_+$, $\tilde{d}((x,y),(u,v)) = \begin{pmatrix} d_1(x,u) \\ d_2(y,v) \end{pmatrix}$.

2. $\rho_{\tilde{d}} : P_{cp}(Z) \times P_{cp}(Z) \to \mathbb{R}^2_+$, $\rho_{\tilde{d}}(A,B) = \begin{pmatrix} \rho_{d_1}(A,B) \\ \rho_{d_2}(A,B) \end{pmatrix}$.

Theorem 4.3.4. *Let (X, d_1) and (Y, d_2) two complete metrics spaces. Let $T_1 : X\times X \to P_{cl}(X)$ and $T_1 : Y \times Y \to P_{cl}(Y)$ two multivalued operators which satisfies:*

$$\rho_{d_1}(T_1(x,y), T_1(u,v)) \leq k_1 d_1(x,u) + k_2 d_2(y,v) \tag{4.13}$$

$$\rho_{d_2}(T_2(x,y), T_2(u,v)) \leq k_3 d_1(x,u) + k_4 d_2(y,v), \tag{4.14}$$

where, we consider $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in \mathbf{M}_{2\times 2}$ a matrix convergent to zero. Then the operator system (4.20) has at least one solution.

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