Babeş-Bolyai University
Faculty of Mathematics and Computer Science

Contributions in the theory of Loewner chains

Ph.D. Thesis - Summary

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Contents

Introduction

The theory of univalent functions is an important subject in the geometric function theory of one complex variable. In view of the Riemann mapping theorem, it is natural to focus on the study of the class S of normalized univalent functions on the unit disc $\mathbb{U}$. Various results in the theory of univalent functions, with emphasis on the study of the class S, may be found in the monographs of the following authors, who made significant contributions to this subject: Duren [33], Goluzin [38], Graham, Kohr [50], Hayman [61], Pommerenke [79], Rosenblum, Rovnyak [88], and others.

Since the class S is compact, one may consider various extremal problems on S. Probably the most famous one is the Bieberbach conjecture [14], which asserts that the absolute value of the kth Taylor coefficient of every function in S is dominated by k, for all $k \in \mathbb{N}$, $k \geq 2$. A fundamental property of the class S, which is useful in the study of associated extremal problems, is that the single-slit functions (i.e. the univalent functions which map $\mathbb{U}$ onto the complement of an arc) in S form a dense subfamily (see e.g. [33], [79]). Motivated by this, Loewner [70] obtained in 1923 the parametric representation of the single-slit functions, by means of certain differential equations. Later on, Kufarev [69] and Pommerenke [78] developed Loewner's parametric method, by considering a continuously growing family of simply connected domains in $\mathbb{C}$ to which they associated a Loewner chain $f : \mathbb{U} \times [0, \infty) \to \mathbb{C}$, which is determined by the Loewner differential equation:

$$\frac{\partial f}{\partial t}(z,t) = \frac{\partial f}{\partial z}(z,t)zp(z,t), \text{ a.e. } t \geq 0, \text{ for all } z \in \mathbb{U},$$

where $p : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ is such that $p(\cdot, t)$ is holomorphic with $p(0,t) = 1$ and $\Re p(z,t) > 0$, $z \in \mathbb{U}$, for all $t \geq 0$, and $p(z, \cdot)$ is measurable on $[0, \infty)$, for all $z \in \mathbb{U}$. Pommerenke [79] proved that every function in S embeds as the first element of a Loewner chain and has parametric representation on $\mathbb{U}$: $f = \lim_{t\to\infty} e^t v(\cdot, t)$, where $v(z, \cdot)$ is the unique locally absolutely continuous solution on $[0, \infty)$ of the Loewner differential equation:

$$\frac{dv}{dt} = -vp(v,t), \text{ a.e. } t \geq 0,$$

with the initial condition $v(z, 0) = z$, for all $z \in \mathbb{U}$, where p satisfies the same conditions as above. Later on, we shall see that this result is not true in higher dimensions for the full class of normalized univalent mappings on the unit ball in $\mathbb{C}^n$.

A strong contribution to the theory is given by the recent generalizations due to Bracci, Contreras, Díaz-Madrigal [16] and Contreras, Díaz-Madrigal, Gumenyuk [23], regarding the general notion of L^d-Loewner chain.

The theory of Loewner chains in one complex variable has important applications regarding extremal problems (see e.g. [33]), analytic characterizations of geometric properties (see e.g. [50]), univalence criteria (see Becker, Pommerenke [13], Pascu [72, 73]), quasiconformal extensions (see Becker [12]), etc. One of the most important applications is due to de Branges [19], who used Loewner's parametric method to prove the Bieberbach conjecture. The study of extremal problems associated with parametric representations on $\mathbb{U}$ has motivated the development of a optimal control–theoretic approach to the Loewner differential equation, considered by Goodman [40], Prokhorov [84], Roth [89, 91], and others. An important contribution to this study is the Ph.D. thesis of Roth [90].

The theory of univalent functions has important applications in different branches of mathematics (see e.g. Pascu [71], for an interesting application related to PDEs and stochastic processes). Many applications of the Loewner theory in various subjects (e.g. statistical mechanics, fluid mechanics) may be found in [1].

The family $S(\mathbb{B}^n)$ of normalized univalent mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ was introduced and studied by Cartan [20]. In contrast with the one dimensional case, he proved that $S(\mathbb{B}^n)$ is not locally uniformly bounded, hence $S(\mathbb{B}^n)$ is not compact and there are no growth/distortion theorems or coefficient estimates for the full family $S(\mathbb{B}^n)$, for $n \geq 2$ (see [20]; see also [50]). Cartan [20] conjectured that we may have some upper estimates for the Jacobian determinant of the univalent mappings in $S(\mathbb{B}^n)$, but this was disproved by Duren and Rudin [35]. Another basic difference is provided by fact that the Riemann mapping theorem is not true in higher dimensions. For example, Poincaré [77] proved that the Euclidean unit ball and the unit polydisc in $\mathbb{C}^n$ are not biholomorphic, for $n \geq 2$. In view of this, it is not clear which is the analogue of the single-slit function in the case of several complex variables. However, in Chapter 3 of the thesis, we have an n-dimensional version of the density result regarding single-slit functions in the class S. On the other hand, the Andersén-Lempert theory [2], which does not hold in dimension one, has applications in the study of Loewner chains in higher dimensions as we cam see in Chapter 2 of the thesis.

The first generalizations of the theory of Loewner chains to higher dimensions are due to Pfaltzgraff [75], who considered the connection between a Loewner chain $f : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ and the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \text{ a.e. } t \geq 0, \text{ for all } z \in \mathbb{U},$$

where $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is such that $h(\cdot,t)$ is a normalized holomorphic mapping with $\Re \langle h(z,t), z\rangle \geq 0$, $z \in \mathbb{B}^n$, for all $t \geq 0$, and $h(z,\cdot)$ is measurable on $[0,\infty)$, for all $z \in \mathbb{B}^n$. Poreda [80] was the first to use the Loewner differential equation to study the family of univalent mappings with parametric representation on the unit polydisc in $\mathbb{C}^n$. Graham, Hamada and Kohr [41] developed the theory of Loewner chains on $\mathbb{B}^n$ and studied the family $S^0(\mathbb{B}^n)$ of univalent mappings with parametric representation on $\mathbb{B}^n$, introduced by Kohr [68], which can be seen as a natural generalization of the class S (cf. [50]). The authors in [41] proved that $S^0(\mathbb{B}^n)$ is a locally uniformly bounded family in $H(\mathbb{B}^n)$, and pointed out important differences between the case of one complex variable and the case of several complex variables, in view of the work of Poreda [80]. Graham, Kohr and Kohr [51] proved that every Loewner chain satisfies a Loewner differential equation (see also Curt, Kohr [28, 29]) and that $S^0(\mathbb{B}^n)$ is a compact (proper) subset of $S(\mathbb{B}^n)$, when $n \geq 2$. We also mention here the example of a bounded support point of the compact family $S^0(\mathbb{B}^2)$ due to Bracci [15], which is in contrast with the one dimensional case, since every support point of the class S is unbounded (see e.g. [33]). The theory was further developed by Poreda [82], Graham, Kohr, Pfaltzgraff [52], Curt [27], Graham, Hamada, Kohr, Kohr [44], Duren, Graham, Hamada, Kohr [34], Hamada [57], Arosio [4], Voda [97], and others.

As in the case of one complex variable, the parametric method has applications in the study of extremal problems, like growth theorems or coefficient estimates (see [41], [50]). Graham, Hamada, Kohr, Kohr [46] obtained generalizations to higher dimensions of a certain control-theoretic approach and studied extreme/support points associated to compact families of univalent mappings with parametric representation on $\mathbb{B}^n$ (see also [53], [93]). Various applications of the Loewner theory in several complex variables refer also to analytic characterizations of geometric properties (see [50]), quasiconformal extensions (see Pfaltzgraff [76]; see also [27]), univalence criteria (see Curt and Pascu [30]; see also [27]), univalence criteria for mappings that are not necessarily holomorphic and related quasiconformal extensions (see Cristea [25, 26]). Finally, we mention that a modern approach to the Loewner theory was given by Bracci, Contreras, Díaz-Madrigal [17] and

Arosio, Bracci, Hamada, Kohr [8], regarding L^d-Loewner chains in higher dimensions.

In the thesis, first, we focus on the development of the theory of Loewner chains in one and higher dimensions and we present various applications, with emphasis on extremal problems. Then, in view of this, we present some original results. The thesis is divided into four chapters, which we summarize in the following.

In **Chapter 1**, we deal, first, with the theory of Loewner chains on the unit disc $\mathbb{U}$. We present some definitions and general results which developed Loewner's parametric method. In particular, we refer to the work of Kufarev [69] and Pommerenke [78]. The theory is closely related to the Carathéodory class $\mathcal{P}$ of holomorphic functions with positive real part on $\mathbb{U}$, and the class S of normalized univalent functions on $\mathbb{U}$. We present these families and associated growth theorems and coefficient estimates. Next, we point out the relation between the Loewner chains and the Loewner differential equations. In connection with this, we consider the notion of parametric representation on $\mathbb{U}$. We note that every normalized univalent function on $\mathbb{U}$ has parametric representation. We also refer to the particular parametric representations of the single-slit functions, studied by Loewner [70], and to that of the bounded normalized univalent functions (see e.g. [84]). In each subsection, we discuss certain associated extremal problems. For example, we mention the results of Kirwan [66] and Pell [74], concerning extremal properties of the Loewner chains.

Next, we present the development of the theory of Loewner chains on the Euclidean unit ball $\mathbb{B}^n$. We refer to the initial results of Pfaltzgraff [75], which relate the Loewner chains with the Loewner differential equations, and to the results of Poreda [80], regarding the parametric representation on the polydisc. Further, we consider the family $S^0(\mathbb{B}^n)$ of mappings that have parametric representation on $\mathbb{B}^n$, introduced by Kohr [68]. We present results of Graham, Hamada and Kohr [41], concerning the compactness of the Carathéodory family $\mathcal{M}$, which is a generalization to higher dimensions of the class $\mathcal{P}$. Certain consequences in the study of the family $S^0(\mathbb{B}^n)$, such as growth theorems, coefficient estimates and extremal properties due to the previous mentioned authors are also presented. On the other hand, we point out the characterization of the Loewner chains, using the Loewner differential equations, due to Graham, Kohr and Pfaltzgraff [52]. Further, we discuss a related notion, namely, that of A-parametric representation on $\mathbb{B}^n$ and the connection with the A-normalized univalent subordination chains, due to Graham, Hamada, Kohr, Kohr [44] and Duren, Graham, Hamada, Kohr [34], where A is a linear operator in $L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$ ($k_+(A)$ is the Lyapunov index of A and $m(A) = \min_{\|z\|=1} \Re\langle A(z), z\rangle$). In this case, we present the compact family $S^0_A(\mathbb{B}^n)$ of mappings with A-parametric representations on $\mathbb{B}^n$, which generalizes the family $S^0(\mathbb{B}^n)$ (see [44]). Finally, we consider the general study of L^d-Loewner chains and their associated evolution families and Herglotz vector fields, due to Bracci, Contreras, Díaz-Madrigal [17] and Arosio, Bracci, Hamada, Kohr [8]. In connection with this study, we present the recent results due to Arosio, Bracci and Wold [10].

This chapter will serve both as a background and a motivation for the forthcoming chapters.

In **Chapter 2**, we are concerned with the variational method developed by Bracci, Graham, Hamada and Kohr [18], which provides a way to construct Loewner chains, by means of variations of certain Loewner chains. We obtain applications of this method, regarding some families of normalized univalent mappings on the unit ball, which are given in terms of Loewner chains with range $\mathbb{C}^n$ (i.e. the growing family of the images of $\mathbb{B}^n$, given by the Loewner chain, eventually cover the whole space $\mathbb{C}^n$). For the first application given in Theorem 2.2.1, we prove a topological property of the family $S^0(\mathbb{B}^n)$ of mappings with parametric representation on the unit ball $\mathbb{B}^n$, which for $n \geq 2$ immediately implies a main result given in Theorem 2.2.2. This result was initially conjectured by Schleißinger [94], namely, the density of the automorphisms of $\mathbb{C}^n$ that have parametric representation on $\mathbb{B}^n$. Next, we apply the variational method in order to obtain a partial answer (Theorem 2.3.1) to a question of Arosio, Bracci and Wold [9]. Namely, we prove that every normalized univalent mapping on $\mathbb{B}^n$ whose image is Runge and which is C^1 up to the boundary embeds into a Loewner chain with range $\mathbb{C}^n$.

This chapter is based on the original results obtained by the author of the thesis in [62].

A first study of some extremal problems associated to parametric representations in several complex variables has been given by Graham, Kohr and Pfaltzgraff [53]. A generalization of the control–theoretic approach to the Loewner differential equation, in the case of the unit ball $\mathbb{B}^n$, has been given by Graham, Hamada, Kohr and Kohr [46, 48] (see also [18], [49]). Also, Roth has recently obtained a Pontryagin maximum principle for the Loewner differential equation in $\mathbb{C}^n$ (see [92]).

In **Chapter 3**, we consider the notion of reachable family associated with the Loewner differential equation and some related results due to Graham, Hamada, Kohr and Kohr [48]. Then, we prove some results conjectured by the previously mentioned authors. We note that an important source of control-theoretic results and ideas, regarding the compactness and the density of certain finite-time reachable families, is the work in one complex variable due to Roth [89, 90]. Our main result (Theorem 3.2.7) in the first section of this chapter is the proof of [48, Conjecture 4.16], which yields that the family of mappings which have an A–parametric representation on $\mathbb{B}^n$ obtained by solving the Loewner differential equation associated to Herglotz vector fields which take values in $\operatorname{ex}\mathcal{N}_A$ (i.e. the set of extreme points of the Carathéodory family $\mathcal{N}_A$) is dense in $S^0_A(\mathbb{B}^n)$ (i.e. the family of all mappings with A–parametric representation on $\mathbb{B}^n$), where A is a linear operator in $L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$. We point out that this result generalizes in a certain sense the density result regarding single-slit functions in the class S, in view of Loewner's result [70] and an observation due to Roth [89]. Our main result (Theorem 3.3.5) in the second section of this chapter is the proof of [48, Conjecture 4.19], which yields that the normalized time-T-reachable family $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)$ of the Loewner differential equation, which is generated by the Carathéodory mappings with values in a compact and convex subfamily Ω of the Carathéodory family $\mathcal{N}_A$, is compact, and the corresponding normalized reachable family $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \operatorname{ex}\Omega)$ is dense in it, where $T \in [0,\infty]$, $A \in L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$, and $\operatorname{ex}\Omega$ denotes the corresponding subset of Ω consisting of extreme points. Since $S^0_A(\mathbb{B}^n)$ is equal to the normalized infinite-time reachable family $\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \mathcal{N}_A)$ and $\mathcal{N}_A$ is a compact and convex family in $H(\mathbb{B}^n)$, we remark that this result generalizes the first one (Theorem 3.2.7). However, our approach to the second result given in Theorem 3.3.5 is quite different from the first one.

This chapter contains original results, including the above mentioned results, obtained by the author of the thesis in [63] and [64].

The univalent subordination chains with normalization given by a time-dependent linear operator and the connection with the Loewner differential equation on the unit ball in $\mathbb{C}^n$ have been first considered by Graham, Hamada, Kohr and Kohr [45]. They also introduced the notion of generalized parametric representation and generalized spirallikeness on $\mathbb{B}^n$ with respect to a time-dependent operator and obtained characterizations in terms of univalent subordination chains on $\mathbb{B}^n$. Further related results, regarding the study of the Loewner differential equation, have been obtained by Graham, Hamada, Kohr [42], Voda [96] and Arosio [6].

In **Chapter 4**, in the first section, we consider the family $\widetilde{S}^t_A(\mathbb{B}^n)$ of normalized univalent mappings on $\mathbb{B}^n$ that have generalized parametric representation with respect to time-dependent operators $A \in \widetilde{\mathcal{A}}$, where $t \geq 0$ and $\widetilde{\mathcal{A}}$ is a family of measurable mappings from $[0,\infty)$ into $L(\mathbb{C}^n)$ which satisfy certain natural assumptions. In particular, we have (Theorems 4.1.12 and 4.1.17) that the mappings in $\widetilde{S}^t_A(\mathbb{B}^n)$ embed in certain Loewner chains at time t, and the family $\widetilde{S}^t_A(\mathbb{B}^n)$ is compact. On the other hand, certain examples (Examples 4.1.13 and 4.1.18) yield that the family $\widetilde{S}^t_A(\mathbb{B}^n)$ for time-dependent operators $A \in \widetilde{\mathcal{A}}$ is basically different from that in the case of constant time-dependent linear operators.

In the second section of this chapter, we consider extreme points and support points associated with the compact family $\widetilde{S}^t_A(\mathbb{B}^n)$, where $A \in \widetilde{\mathcal{A}}$ and $t \geq 0$. We are concerned with the generalization of the results due to Kirwan [66] and Pell [74], in the case of time-dependent operators, which are given in Theorems 4.2.2 and 4.2.4, in view of the recent results due to Graham, Hamada, Kohr and Kohr [48], [49] (see also [21], [43], [46], [53], [93]). Also, we present an example (Theorem 4.2.9) of a bounded support point for the family $\widetilde{S}_A^t(\mathbb{B}^2)$, where $A \in \widetilde{\mathcal{A}}$ is a certain time-dependent operator, in view of Bracci's example [15] (see also [18], [49]). We also consider the notion of a reachable family with respect to time-dependent linear operators $A \in \widetilde{\mathcal{A}}$, and the corresponding characterizations of extreme/support points associated with these families of bounded univalent mappings on $\mathbb{B}^n$, by generalizing some results due to Graham, Hamada, Kohr and Kohr [48, 49]. Useful examples and applications (e.g. Example 4.2.6) point out differences between the case of non-constant time-dependent operators and the case of constant time-dependent linear operators.

The first two sections of this chapter contain original results based on the joint work of M. Iancu with H. Hamada and G. Kohr [58].

In the last section of this chapter, we discuss certain convergence results for the family $\widetilde{S}_A^t(\mathbb{B}^n)$ with respect to the Hausdorff metric ρ on $H(\mathbb{B}^n)$, where $A \in \widetilde{\mathcal{A}}$ and $t \geq 0$. The case of the reachable families is also considered. These results (Theorems 4.3.1 and 4.3.2) may be seen as dominated convergence type theorems for time-dependent operators $A \in \widetilde{\mathcal{A}}$. As applications, we have related convergence results (Theorems 4.3.3 and 4.3.4) for the compact families $S_{\mathbf{A}}^0(\mathbb{B}^n)$ of univalent mappings with **A**-parametric representation on $\mathbb{B}^n$ and $\widehat{S}_{\mathbf{A}}(\mathbb{B}^n)$ of spirallike mappings with respect to **A** on $\mathbb{B}^n$, where $\mathbf{A} \in L(\mathbb{C}^n)$ is such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$. Another application (Theorem 4.3.7) provides some sufficient conditions related to $A \in \widetilde{\mathcal{A}}$ which yield the equality $\widetilde{S}_A^t(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, for all $t \geq 0$.

The last section of this chapter is based on the joint work of M. Iancu with H. Hamada and G. Kohr [59].

The original results in the thesis have been obtained in the following papers:

1. **Iancu, M.**: *Some applications of variation of Loewner chains in several complex variables*, J. Math. Anal. Appl., **421** (2015), 1469–1478.

2. **Iancu, M.**: *A density result for parametric representations in several complex variables*, Comput. Methods Funct. Theory, **15** (2015), 247–262.

3. **Iancu, M.**: *On reachable families of the Loewner differential equation in several complex variables*, Complex Anal. Oper. Theory, to appear, DOI: 10.1007/s11785-015-0461-z.

4. Hamada, H., **Iancu, M.**, Kohr, G.: *Extremal problems for mappings with generalized parametric representation in* $\mathbb{C}^n$, submitted.

5. Hamada, H., **Iancu, M.**, Kohr, G.: *Convergence results for families of univalent mappings on the unit ball in* $\mathbb{C}^n$, submitted.

We mention the main original results in the thesis:
Chapter 2: Theorems 2.2.1, 2.2.2, 2.3.1.
Chapter 3: Theorems 3.2.7, 3.3.4, 3.3.5.
Chapter 4: Theorems 4.1.12, 4.1.17, 4.2.2, 4.2.4, 4.2.9, 4.2.16, 4.3.1, 4.3.2, 4.3.3, 4.3.4, 4.3.7.

Keywords
Carathéodory family, Carathéodory mapping, control, embedding problem, extreme point, Herglotz vector field, Hausdorff metric, Loewner chain, Loewner differential equation, parametric representation, reachable family, support point, transition mapping, univalent mapping, variational method.

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Chapter 1

The theory of Loewner chains in $\mathbb{C}$ and $\mathbb{C}^n$

In this chapter, we focus on the development of the theory of Loewner chains in the context of geometric function theory in one and higher dimensions. We also present several important applications, with emphasis on extremal problems. This chapter will serve both as a background and a motivation for the forthcoming chapters.

In the first section, we deal with the theory of Loewner chains in the case of one complex variable. We present some basic definitions and results which developed Loewner's method. For example, we refer to the work of Kufarev [69] and Pommerenke [78]. The theory is closely related to the Carathéodory class and the class S. We present these families and associated growth theorems and coefficient estimates. Next, we point out the relation between the Loewner chains and the Loewner differential equations. In connection with this, we consider the notion of parametric representation on the unit disc. We see that every normalized univalent function on the unit disc has parametric representation. We also refer to the particular parametric representations of the single-slit functions, studied by Loewner [70], and to that of the bounded normalized univalent functions. In each subsection, we discuss certain associated extremal problems. For example, we mention the results of Kirwan [66] and Pell [74], concerning extremal properties of the Loewner chains.

In the second section, we present the development of the theory of Loewner chains in several complex variables. We refer to the initial results of Pfaltzgraff [75], which relates the Loewner chains with the Loewner differential equations, and of Poreda [80], regarding the parametric representation on the polydisc. In view of this, we present the results of Graham, Hamada and Kohr [41], concerning the compactness of the Carathéodory family and its consequences in the study of the family of mappings that have parametric representation on the unit ball, introduced by Kohr [68]. Related growth theorems, coefficient estimates and extremal properties are also presented. On the other hand, we point out the characterization of the Loewner chains, using the Loewner differential equations, due to Graham, Kohr and Pfaltzgraff [52]. Further, we discuss a related notion, namely, that of A-parametric representation on the unit ball and the connection with the A-normalized univalent subordination chains, due to Graham, Hamada, Kohr, Kohr [44] and Duren, Graham, Hamada, Kohr [34], where A is a linear operator that satisfies certain properties which will be mentioned in a forthcoming section. Finally, we consider the study of L^d-Loewner chains and their associated evolution families and Herglotz vector fields, due to Bracci, Contreras, Díaz-Madrigal [17] and Arosio, Bracci, Hamada, Kohr [8].

1.1 The theory of Loewner chains in $\mathbb{C}$

In this section, we present basic notions and results from the theory of Loewner chains in the complex plane. We also consider related families of holomorphic functions on the unit disc and we point out associated extremal problems, like growth theorems and coefficient estimates.

First, we refer to the Carathéodory class of holomorphic functions with positive real part on the unit disc, which plays an important role in various analytic characterizations of geometric properties of univalent functions on the unit disc. We recall the Herglotz representation formula and its consequences: the growth theorem, the compactness of the class, the sharp coefficient estimates and the characterization of the extreme points and support points. Next, we regard the class S of normalized univalent functions on the unit disc and we present the growth theorem, the compactness of the class and the sharp coefficient estimates. We also point out an important property of the corresponding extreme points and support points.

Next, we focus on the fundamental results in the theory of Loewner chains in the complex plane. We present the definitions of the Loewner chains and the related notions. We point out the characterizations given by the Loewner differential equations. For the most of this part, we rely on the exposition given by Pommerenke [79]. A particular interest is provided by some extremal properties of the Loewner chains due to Kirwan [66] and Pell [74]. Further, we consider the notion of parametric representation on the unit disc. We shall see that every normalized univalent function on the unit disc has parametric representation and can be embedded in a Loewner chain. Next, we take into account the special case of single-slit functions, first studied by Loewner [70], and we pay attention to some associated extremal properties observed by Roth [89]. Also, we consider the particular parametric representation of the bounded normalized univalent functions on the unit disc.

It is worth mentioning here the important monographs in the theory of univalent functions, which refer to the theory of Loewner chains, of Duren [33], Hayman [61], Goluzin [38], Graham, Kohr [50], Pommerenke [79], Rosenblum, Rovnyak[88]. Besides these, the main references for this section are [3], [24], [56], [84], [90].

Now, we establish some notations which will be useful in the next sections. Let $\mathbb{C}$ be the complex plane. The unit disc $\{z \in \mathbb{C} \big| \, \|z\| < 1\}$ is denoted by $\mathbb{U}$ and the unit circle is denoted by $\partial\mathbb{U}$. We denote by $H(\mathbb{U})$ the family of holomorphic functions defined on $\mathbb{U}$ with values in $\mathbb{C}$. We consider the topology of local uniform convergence on $H(\mathbb{U})$. Then, it is well known that $H(\mathbb{U})$ becomes a Fréchet space. If $f \in H(\mathbb{U})$, we say that f is normalized if $f(0) = 0$ and $f'(0) = 1$.

1.1.1 The Carathéodory class

The definition of the *Carathéodory class* is (see e.g. [33]):

$$\mathcal{P} := \{p \in H(\mathbb{U}) \big| \, p(0) = 1 \text{ and } \Re p(z) > 0, \text{ for all } z \in \mathbb{U}\}.$$

Analytic characterizations of certain geometric properties are closely related to the study of this class (see e.g. [33], [50], [56], [79]). Later on, we shall see the strong connection between this class and the Loewner theory.

A simple way to provide examples of functions in $\mathcal{P}$ is to consider the following characterization in terms of *Schwarz functions* (see [33] and [79]; e.g. [50]).

Remark 1.1.1. A function $p : \mathbb{U} \to \mathbb{C}$ is in $\mathcal{P}$ if and only if there exists a Schwarz function v (i.e. $v \in H(\mathbb{U})$ and $|v(z)| \leq |z|$, $z \in \mathbb{U}$) such that $p(z) = \dfrac{1 + v(z)}{1 - v(z)}$, $z \in \mathbb{U}$.

We point out an important example of a function in $\mathcal{P}$ (see e.g. [33] and [79]; e.g. [50]).

Example 1.1.2. Let $\lambda \in \partial\mathbb{U}$. Then the function $p : \mathbb{U} \to \mathbb{C}$ given by $p(z) = \dfrac{1 + \lambda z}{1 - \lambda z}$, $z \in \mathbb{U}$, is in $\mathcal{P}$.

A very useful characterization of the functions in $\mathcal{P}$ is given by the *Herglotz* (*integral*) *representation formula* (see e.g. [33], [56], [79]).

Theorem 1.1.3. *A function $p : \mathbb{U} \to \mathbb{C}$ is in $\mathcal{P}$ if and only if there exists a non-decreasing function $\mu : [0, 2\pi] \to \mathbb{R}$ such that $\mu(2\pi) - \mu(0) = 1$ and*

$$p(z) = \int_0^{2\pi} \frac{1 + e^{-it}z}{1 - e^{-it}z} d\mu(t), \quad z \in \mathbb{U}. \tag{1.1.1}$$

An immediate consequence of formula (1.1.1) is the following growth theorem (see [33] and [79]; e.g. [50]).

Theorem 1.1.4. *Let $p \in \mathcal{P}$. Then the following sharp inequalities hold:*

$$\frac{1 - |z|}{1 + |z|} \leq |p(z)| \leq \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{U}.$$

In view of the previous theorem, one can deduce the following consequence (see [79]; e.g. [50]).

Theorem 1.1.5. *$\mathcal{P}$ is a compact subset of $H(\mathbb{U})$.*

Next, we will briefly discuss some extremal problems related to the class $\mathcal{P}$. To do this, we first recall the notions of extreme point and support point for a compact subset of a topological vector space (see e.g. [32]).

Definition 1.1.6. Let X be a topological vector space and $\Omega \subseteq X$ be a nonempty compact set.

(i) A point $f \in \Omega$ is called an *extreme point* of Ω if: $f = \lambda g + (1 - \lambda)h$, for some $\lambda \in (0, 1)$, $g, h \in \Omega$, implies that $f = g = h$.

(ii) A point $f \in \Omega$ is called a *support point* of Ω if there exists a continuous linear functional $L : X \to \mathbb{C}$ such that $\Re L$ is nonconstant on E and $\Re L(f) = \max_{g \in \Omega} \Re L(g)$.

We denote by $\operatorname{ex}\Omega$ the set of extreme points of Ω, by $\operatorname{supp}\Omega$ the set of support points of Ω and by $\operatorname{co}\Omega$ the convex hull of Ω. We mention that $\operatorname{ex}\Omega \neq \emptyset$ and, if Ω has at least two points, then also $\operatorname{supp}\Omega \neq \emptyset$ (see [56]). Throughout this section, we consider $X = H(\mathbb{U})$.

The integral formula (1.1.1) provides a complete characterization of the extreme points and the support points of the Carathéodory class $\mathcal{P}$ (see [56]).

Theorem 1.1.7.

(i) $\operatorname{ex}\mathcal{P} = \{z \mapsto \frac{1 + \lambda z}{1 - \lambda z} | \lambda \in \partial\mathbb{U}\}$.

(ii) $\operatorname{supp}\mathcal{P} = \operatorname{co}\operatorname{ex}\mathcal{P}$.

An important example of an extremal problem is that of finding sharp estimates of the coefficients of certain families of holomorphic functions. Using formula (1.1.1), one can obtain sharp bounds for the coefficients of the functions in $\mathcal{P}$ (see e.g. [33], [50], [79]).

Theorem 1.1.8. *Let $p \in \mathcal{P}$ be such that $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$, $z \in \mathbb{U}$. Then $|p_k| \leq 2$, for all $k \geq 1$. These estimates are sharp.*

For other interesting results regarding extremal problems associated to various compact families of holomophic functions on $\mathbb{U}$, related to $\mathcal{P}$, we refer to the monograph of Hallenbeck and MacGregor [56].

1.1.2 Normalized univalent functions on the unit disc

We denote by S the well known class of normalized univalent functions (holomorphic and injective) on $\mathbb{U}$ (see e.g. [33], [50], [79]):

$$S = \{f \in H(\mathbb{U}) \big| \, f \text{ is univalent and } f(0) = 0, f'(0) = 1\}.$$

An important subfamily of S is the family of normalized starlike functions on the unit disc. We present the definition (see e.g. [33], [50], [79]).

Definition 1.1.9. A function $f : \mathbb{U} \to \mathbb{C}$ is called starlike if f is a univalent function with $f(0) = 0$ and $f(\mathbb{U})$ is a starlike domain with respect to the origin.

We denote by S^* the family of normalized univalent functions on $\mathbb{U}$ that are starlike.

We present an example of well known functions in S (see [33], [50], [56], [79]), which are solutions to certain relevant extremal problems, as we shall see in the following.

Example 1.1.10. The Koebe function $k : \mathbb{U} \to \mathbb{C}$, given by $k(z) = \dfrac{z}{(1-z)^2}$, $z \in \mathbb{U}$, and it's rotation $k_\theta : \mathbb{U} \to \mathbb{C}$, given by $k_\theta(z) = e^{-i\theta}k(e^{i\theta}z) = \dfrac{z}{(1-e^{-i\theta}z)^2}$, $z \in \mathbb{U}$, where $\theta \in \mathbb{R}$, are normalized starlike functions, and thus they belong to the class S. Moreover, they are both unbounded extreme and support points for the class S.

Next, we present the growth theorem for functions in S (see [33], [79]; see also e.g. [50]).

Theorem 1.1.11. *Let $f \in S$. Then the following sharp inequalities hold:*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \text{ for all } z \in \mathbb{U}.$$

In particular, $\mathbb{U}_{1/4} \subseteq f(\mathbb{U})$.

In view of the previous theorem, one can deduce the compactness of the class S (see [33], [79]; see also e.g. [50]).

Theorem 1.1.12. *S is a compact subset of $H(\mathbb{U})$.*

Since S is compact, one can consider various extremal problems for S. Bieberbach [14] considered in 1916 the following very important extremal problem, known as the *Bieberbach conjecture*, which was proved by de Branges [19] in 1985.

Theorem 1.1.13. *Let $f \in S$ be such that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $z \in \mathbb{U}$, and let $n \in \mathbb{N}$, $n \geq 2$. Then $|a_n| \leq n$ and equality holds if and only if f is a rotation of the Koebe function.*

The Bieberbach conjecture has a long history of many attempts to prove it, which lead to the development of various ideas and techniques to approach extremal problems. De Branges [19] proved the conjecture by making use of Loewner's parametric method [70] (cf. [24], [61], [88]).

In the following, we point out an important property of the functions in $\operatorname{ex} S$ and $\operatorname{supp} S$ (see [33], [79]).

Remark 1.1.14. If $f \in \operatorname{ex} S \cup \operatorname{supp} S$, then $\mathbb{C} \setminus f(\mathbb{U})$ is a single Jordan arc with increasing modulus. In particular, if $f \in \operatorname{ex} S \cup \operatorname{supp} S$, then $f(\mathbb{U})$ is an unbounded domain.

1.1.3 The Loewner chains and the Loewner differential equation in $\mathbb{C}$

Throughout this subsection, we deal with the definition of the Loewner chains on the unit disc and the characterization by the Loewner differential equation. The following basic results represent the foundation for the generalization of the theory of Loewner chains to several complex variables. The main bibliographic source for this subsection is the exposition of Pommerenke [79, Chapter 6]. We also rely on the monograph of Graham and Kohr [50]. Also, the following bibliographic sources [3],[12], [24], [40], [69], [78], [88] have been useful in this subsection.

Definition 1.1.15. A function $f : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ is called a univalent subordination chain if $f(\cdot, t)$ is univalent on $\mathbb{U}$ with $f(0, t) = 0$, for all $t \geq 0$, and $f(\mathbb{U}, s) \subseteq f(\mathbb{U}, t)$, for all $0 \leq s \leq t$. If, in addition, $f'(0, t) = e^t$, for all $t \geq 0$, where $f'(0, t) = \frac{\partial f}{\partial z}(0, t)$, then f is called a Loewner chain.

From the previous definition we have that, if f is a univalent subordination chain, then for every $0 \leq s \leq t$, there exists a unique univalent Schwarz function $v(\cdot, s, t)$ such that $f(z, s) = f(v(z, s, t), t)$, for all $z \in \mathbb{U}$. In this case, v is called the *transition function* associated to f. The transition function v satisfies the *semigroup property*: $v(z, s, u) = v(v(z, s, t), t, u)$, for all $z \in \mathbb{U}$, $0 \leq s \leq t \leq u$. Moreover, if f is a Loewner chain, then $v'(0, s, t) = e^{s-t}$, for all $0 \leq s \leq t$, where $v'(0, s, t) = \frac{\partial v}{\partial z}(0, s, t)$.

We mention that there is a geometric interpretation of the notion of univalent subordination chain given in terms of the Carathéodory kernel convergence (see [24] and [79]; see also e.g. [50]).

Next, we remark that the study of univalent subordination chains can be reduced to the study of Loewner chains, by a renormalization (see [24], [50], [79]). Such renormalization is not valid for the full family of univalent subordination chains in higher dimensions, as we will see in a forthcoming section.

Remark 1.1.16. If f is univalent subordination chain, then $f_* : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ given by $f_*(z, t^*) = f(e^{-i\theta(t)}z, t)$, $z \in \mathbb{U}$, $t \geq 0$, where $t^* = \ln |f'(0, t)|$ and $\theta(t) = \arg f'(0, t)$, is a Loewner chain.

A simple example of a a Loewner chain can be provided by starting with a starlike function on the unit disc (see [79]).

Example 1.1.17. Let $f \in S$. Then $f \in S^*$ if and only if $F : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ given by $F(z, t) = e^t f(z)$, $z \in \mathbb{U}$, $t \geq 0$, is a Loewner chain.

Now, we present the Loewner differential equation and we point out the connection with the Loewner chains (see [78] and [79]; see also e.g. [50]).

Theorem 1.1.18. *Let* $p : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ *be such that:*

(i) $p(\cdot, t) \in \mathcal{P}$, *for all* $t \geq 0$,

(ii) $p(z, \cdot)$ *is measurable on* $[0, \infty)$, *for all* $z \in \mathbb{U}$.

Then, for every $z \in \mathbb{U}$ *and* $s \geq 0$, *the differential equation*

$$\frac{dv}{dt} = -vp(v, t), \quad a.e. \ t \geq s, \tag{1.1.2}$$

has a unique locally absolutely continuous solution $v(z, s, \cdot)$ *with the initial condition* $v(z, s, s) = z$. *Moreover, for every* $s \geq 0$, $v(z, s, \cdot)$ *is Lipschitz continuous on* $[s, \infty)$, *locally uniformly with respect to* $z \in \mathbb{U}$, *and* $v(\cdot, s, t)$ *is a univalent Schwarz function with* $v'(0, s, t) = e^{s-t}$, *for all* $t \geq s$. *Furthermore, for every* $s \geq 0$, *the following limit exists locally uniformly with respect to* $z \in \mathbb{U}$:

$$f(z, s) := \lim_{t\to\infty} e^t v(z, s, t) \tag{1.1.3}$$

and $f : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ *given by* (1.1.3) *is a Loewner chain such that* v *is the transition function associated to* f *and satisfies, for almost every* $t \geq 0$:

$$\frac{\partial f}{\partial t}(z, t) = zf'(z, t)p(z, t), \quad z \in \mathbb{U}. \tag{1.1.4}$$

The differential equation (1.1.2) is called the *Loewner (ordinary) differential equation*, while the differential equation (1.1.4) is called the *Loewner differential equation* (or *Loewner-Kufarev differential equation*). We mention that the related *Polubarinova-Galin equation* has important applications in fluid mechanics, regarding the study of Hele-Shaw flows (see [37], [55]). On the other hand, we mention the applications regarding quasiconformal extensions, initially obtained by Becker [12] (see also [13]).

Next, we present the characterization of the Loewner chains as certain solutions of the Loewner differential equation (see [78] and [79]; see also e.g. [50]).

Theorem 1.1.19. *Let $f : \mathbb{U} \times [0, \infty) \to \mathbb{C}$. Then f is a Loewner chain if and only if the following two conditions are satisfied:*

(i) there exist $r \in (0, 1)$ and $M > 0$ such that $f(\cdot, t)$ is holomorphic on $\mathbb{U}_r = \{z \in \mathbb{C} \,|\, |z| < r\}$ with $f(0, t) = 0$, $f'(0, t) = e^t$, for all $t \geq 0$, $f(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in \mathbb{U}_r$, and $|f(z, t)| \leq Me^t$, for all $z \in \mathbb{U}_r$, $t \geq 0$;

(ii) there exists $p : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ that satisfies conditions (i) and (ii) from Theorem 1.1.18 such that, for almost every $t \geq 0$:

$$\frac{\partial f}{\partial t}(z, t) = zf'(z, t)p(z, t,), \quad z \in \mathbb{U}_r.$$

In the following, we recall how can be recovered the Loewner chain from the transition function. Also, we point out the relation with the Loewner differential equations (see [78] and [79]; see also e.g. [50]).

Theorem 1.1.20. *Let f be a Loewner chain and v be the transition function associated to f. Then $f(z, \cdot)$ is locally Lipschitz continuous on $[0, \infty)$, locally uniformly with respect to $z \in \mathbb{U}$, and there exists $p : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ that satisfies the conditions (i) and (ii) from Theorem 1.1.18 such that f satisfies the Loewner differential equation (1.1.4) associated to p. Moreover, for every $s \geq 0$ and $z \in \mathbb{U}$, $v(z, s, \cdot)$ is the unique locally absolutely continuous solution on $[s, \infty)$ of the Loewner differential equation (1.1.2) associated to p with the initial condition $v(z, s, s) = z$. Furthermore, $\lim_{t\to\infty} e^t v(z, s, t) = f(z, s)$, locally uniformly with respect to $z \in \mathbb{U}$, for all $s \geq 0$.*

Remark 1.1.21. The function p in Theorem 1.1.20 is essentially unique in the following sense: if q is another function that satisfies the conditions (*i*) and (*ii*) from Theorem 1.1.18 such that f satisfies the Loewner differential equation (1.1.4) associated to q, then $p(\cdot, t) = q(\cdot, t)$, for a.e. $t \geq 0$ (see [16], [50]).

Every Loewner differential equation has a unique normalized solution that is a Loewner chain, as we shall note in the next theorem (see [12], [78]).

Theorem 1.1.22. *Let $p : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ be a function that satisfies the conditions (i) and (ii) from Theorem 1.1.18. Then there exists a unique Loewner chain f that satisfies the Loewner differential equation (1.1.4) associated to p.*

In the following we consider some extremal properties of the Loewner chains, which are very useful in the study of extremal problems for the class S (see [67]). Namely, we consider the results due to Kirwan [66] and Pell [74].

Theorem 1.1.23. *Let f be a Loewner chain. Then the following statements hold:*

(i) If $f(\cdot, 0) \in \operatorname{ex} S$, then $e^{-t}f(\cdot, t) \in \operatorname{ex} S$, for all $t \geq 0$.

(ii) If $f(\cdot, 0) \in \operatorname{supp} S$, then $e^{-t}f(\cdot, t) \in \operatorname{supp} S$, for all $t \geq 0$.

At the end of this subsection, we mention that a unified general study of the theory of Loewner chains and the theory of semigroups of holomorphic functions on the unit disc has been given in [16] and [23]. Also, an interesting treatment of basic properties of Loewner chains and the Loewner differential equation on the unit disc may be found in [88].

1.1.4 Parametric representations on the unit disc

First, we recall the definition of a univalent function that has parametric representation on the unit disc (see [79]; see also e.g. [50]).

Definition 1.1.24. A function $f : \mathbb{U} \to \mathbb{C}$ has parametric representation if there exists $p : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ such that

(i) $p(\cdot, t) \in \mathcal{P}$, for all $t \geq 0$,

(ii) $p(z, \cdot)$ is measurable on $[0, \infty)$, for all $z \in \mathbb{U}$, and

$$f = \lim_{t\to\infty} e^t v(\cdot, t), \text{ locally uniformly on } \mathbb{U},$$

where, for every $z \in \mathbb{U}$, $v(z, \cdot)$ is the unique locally absolutely continuous solution on $[0, \infty)$ of the Loewner differential equation associated to p:

$$\frac{dv}{dt} = -vp(v, t), \text{ a.e. } t \geq 0,$$

with the initial condition $v(z, 0) = z$.

Next, we point out that every normalized univalent function on the unit disc has parametric representation (see [79]). In the case of several complex variables this is not true for the full family of normalized univalent mappings on the unit ball in $\mathbb{C}^n$, as we shall see later.

Theorem 1.1.25. *Let $f : \mathbb{U} \to \mathbb{C}$. Then $f \in S$ if and only if f has parametric representation.*

Taking into account the previous subsection, we present the characterization of the class S in terms of Loewner chains, namely, that every normalized univalent function on the unit disc can be embedded as the first element of a Loewner chain (see [79]).

Theorem 1.1.26. *Let $f : \mathbb{U} \to \mathbb{C}$. Then $f \in S$ if and only if there exists a Loewner chain F such that $f = F(\cdot, 0)$.*

1.1.5 Parametric representations of single-slit functions

In this subsection we present Loewner's parametric result [70] from 1923, which represents the cornerstone of the theory of Loewner chains.

First, we recall the definition of a single-slit function (see [33], [79]).

Definition 1.1.27. A function $f \in S$ is called a single-slit function if $\mathbb{C} \setminus f(\mathbb{U})$ is a single Jordan arc.

A very useful property of the single-slit functions is that they are dense in the class S (see [33], [79]). This played an important role in the motivation of Loewner's work [70].

Theorem 1.1.28. *Let $f \in S$. Then there exists a sequence of single-slit functions $(f_n)_{n\in\mathbb{N}}$ in S such that $f_n \to f$, as $n \to \infty$, locally uniformly on $\mathbb{U}$.*

Next, we present an important result due to Loewner [70]. A detailed proof of this result may be found in [24], [33], [61].

Theorem 1.1.29. *Let $f \in S$ be a single-slit function. Then there exists a continuous function $\kappa : [0, \infty) \to \partial\mathbb{U}$ such that $f = \lim_{t\to\infty} e^t v(\cdot, t)$, locally uniformly on $\mathbb{U}$, where, for every $z \in \mathbb{U}$, $v(z, \cdot)$ is the unique solution of the differential equation:*

$$\frac{dv}{dt} = -v\frac{1 + \kappa(t)v}{1 - \kappa(t)v}, \quad t \geq 0, \tag{1.1.5}$$

with the initial condition $v(z, 0) = z$.

Regarding the converse of Theorem 1.1.29, we have the following remark (see e.g. [33], [90]).

Remark 1.1.30. If a function $f \in S$ has a parametric representation given by (1.1.5), then f is not necessary a single-slit function.

In Chapter 3, we shall prove a generalization of Theorem 1.1.28, in the case of several complex variables, taking into account Theorem 1.1.29 and using control theory. To this end, we mention the following interesting observation due to Roth [89], which, in view of the previous theorem and Theorem 1.1.7 (i), relates the Loewner differential equation (1.1.5) with the set of extreme points of the class $\mathcal{P}$.

Remark 1.1.31. For every single-slit function $f \in S$, there exists $p : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ such that

(i) $p(\cdot, t) \in \operatorname{ex} \mathcal{P}$, for all $t \geq 0$,

(ii) $p(z, \cdot)$ is continuous on $[0, \infty)$, for all $z \in \mathbb{U}$,
and

$$f = \lim_{t \to \infty} e^t v(\cdot, t), \text{ locally uniformly on } \mathbb{U},$$

where, for every $z \in \mathbb{U}$, $v(z, \cdot)$ is the unique solution on $[0, \infty)$ of the differential equation:

$$\frac{dv}{dt} = -vp(v, t), \quad t \geq 0,$$

with the initial condition $v(z, 0) = z$.

Consequently, taking into account Theorem 1.1.25, every function that has parametric representation can be approximated locally uniformly on $\mathbb{U}$ with functions that have the above particular parametric representation which is generated by functions from $\operatorname{ex} \mathcal{P}$ (see [89] and [90]).

1.1.6 Parametric representations of bounded univalent functions on the unit disc

The class of bounded normalized univalent functions for a given $M \in [1, \infty)$ is an important subclass of the class S (see [84]). We begin this subsection with the definition of this class.

Definition 1.1.32. We denote by

$$S(M) := \{f \in S \big| \, |f(z)| < M, \text{ for all } z \in \mathbb{U}\}$$

the class of normalized bounded univalent functions on $\mathbb{U}$ for $M \in [1, \infty)$.

In view of the Schwarz Lema, we observe that for $M = 1$ we have $f \in S(1)$ if and only if $f(z) \equiv z$ (see e.g. [84]).

Some examples of bounded normalized univalent functions can be provided by starting with a starlike function (see [84]).

Example 1.1.33. Let $f \in S^*$ and $M \in [1, \infty)$. Then the function $f^M : \mathbb{U} \to \mathbb{C}$ given by

$$Mf\left(\frac{1}{M} f^M(z)\right) = f(z), \, z \in \mathbb{U},$$

is in $S(M)$.

In particular, we have the following important example of a bounded normalized univalent function (see [84]).

Example 1.1.34. The Pick function $p_\theta^M : \mathbb{U} \to \mathbb{C}$, given by

$$Mk_\theta\left(\frac{1}{M}p_\theta^M(z)\right) = k_\theta(z),\ z \in \mathbb{U},$$

is in $S(M)$, where $M \in [1,\infty)$, $\theta \in \mathbb{R}$, and k_θ is the corresponding rotation of the Koebe function.

Since $S(M)$ is a normal family, one can easy deduce the compactness of the class $S(M)$, for $M \in [1,\infty)$ (see [84]).

Theorem 1.1.35. *Let $M \in [1,\infty)$. Then $S(M)$ is a compact subset of $H(\mathbb{U})$.*

In the case of bounded normalized univalent functions, we have the following particular parametric representation (see [24], [40], [79]).

Theorem 1.1.36. *Let $M \in [1,\infty)$ and $f : \mathbb{U} \to \mathbb{C}$. Then $f \in S(M)$ if and only if there exists $p : \mathbb{U} \times [0, \log M] \to \mathbb{C}$ such that $p(\cdot,t) \in \mathcal{P}$, for all $t \in [0,\log M]$, $p(z,\cdot)$ is measurable on $[0, \log M]$, for all $z \in \mathbb{U}$, and $f = Mv(\cdot, \log M)$, where, for every $z \in \mathbb{U}$, $v(z,\cdot)$ is the unique absolutely continuous solution on $[0,\log M]$ of the differential equation:*

$$\frac{dv}{dt} = -vp(v,t),\ \ a.e.\ \ t \in [0,M],$$

with the initial condition $v(z,0) = z$.

The following result provides another characterization of the class $S(M)$ in terms of Loewner chains (see [24], [40], [79]).

Theorem 1.1.37. *Let $f : \mathbb{U} \to \mathbb{C}$ and $M \in [1,\infty)$. Then $f \in S(M)$ if and only if there exists a Loewner chain F such that $f = F(\cdot,0)$ and $F(z,t) = e^t z$, $z \in \mathbb{U}$, $t \geq \log M$.*

Using the parametric representation of the bounded univalent functions, one can solve some extremal problems. As an example, we present the following growth theorem (see [84]).

Theorem 1.1.38. *Let $M \in [1,\infty)$ and $f \in S(M)$. Then the following sharp inequalities hold:*

$$p_\pi^M(|z|) \leq |f(z)| \leq p_0^M(|z|),\ \ \textit{for all } z \in \mathbb{U}.$$

The particular interest in the study of extremal problems on this class (see e.g. [84]) has motivated the development of a control-theoretic method (see e.g. [84], [90], [91]). Later on, we shall focus on the control-theoretic approach to the study of the Loewner differential equation.

1.2 The theory of Loewner chains in $\mathbb{C}^n$

In this section, we briefly survey the development of the theory of Loewner chains in several complex variables. The first results in this direction, regarding the Loewner differential equations and the univalent subordination chains on the Euclidean unit ball, are due to Pfaltzgraff [75]. Important contributions, concerning parametric representations on the unit polydisc in $\mathbb{C}^n$, are due to Poreda [80, 81], who also obtained some generalizations for the unit ball in a complex Banach space [82]. We also mention that Poreda [80] obtained the growth theorem and second coefficient bounds for the family of mappings with parametric representation on the unit polydisc and pointed out both similarities and differences between dimension one and higher dimensions, regarding this family.

The results of Graham, Hamada, Kohr [41] had a notable impact on the development of the theory on the unit ball, in higher dimensions. For example, we refer here to the generalization of the class S to the family $S^0(\mathbb{B}^n)$ of mappings with parametric representation on the unit ball $\mathbb{B}^n$, the compactness of the Carathéodory family on $\mathbb{B}^n$, associated growth theorems, coefficient estimates, etc. They also pointed out relevant differences between the one variable case and the several variables case. Further results regarding the study of the solutions of the Loewner differential equation, in connection with the Loewner chains, are due to Curt, Kohr [28] and Graham, Kohr, Pfaltzgraff [52]. We remark that applications of the theory of Loewner chains to univalence and quasiconformal extension results in $\mathbb{C}^n$ may be found in the monograph of Curt [27].

Throughout this section, we also focus on the generalizations that refer to the A-parametric representations on $\mathbb{B}^n$ due to Graham, Hamada, Kohr, Kohr [44] and Duren, Graham, Hamada, Kohr [34], which are in connection with the work of Gurganus [54] and Poreda [83], where A is a linear operator in $L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$ ($k_+(A)$ is the Lyapunov index of A and $m(A) = \min_{\|z\|=1} \Re\langle A(z), z\rangle$). In this case, we consider the compact family $S^0_A(\mathbb{B}^n)$ of mappings with A-parametric representation on $\mathbb{B}^n$, which generalizes the family $S^0(\mathbb{B}^n)$ (see [44]). Also, we present an example due to Graham, Hamada, Kohr and Kohr [49] of mapping in $S^0_A(\mathbb{B}^2)$, which is not in $S^0(\mathbb{B}^n)$, where $A \in L(\mathbb{C}^n)$ is such that $k_+(A) < 2m(A)$. At the end, we consider the general approach to the theory of Loewner chains due to Bracci, Contreras, Díaz-Madrigral [17] and Arosio, Bracci, Hamada, Kohr [8], regarding the study of L^d-Loewner chains and the associated notions of evolution families and Herglotz vector fields. In view of this, we present some important results due to Arosio, Bracci and Wold [10], regarding certain properties of the Loewner range.

In each subsection, we pay attention also to some extremal problems associated to Loewner chains. For example, we refer to the results of Graham, Hamada, Kohr, Kohr [46, 48, 49] and Schleißinger [93]. A basic difference between dimension one and higher dimensions is provided by an example of a bounded support point of $S^0(\mathbb{B}^2)$ due to Bracci [15] (cf. Remark 1.1.14).

An important bibliographic source for this subject is the monograph of Graham and Kohr [50]. Besides the previous mentioned works, the main references for this section are [3], [5], [27], [48], [49], [54], [60], [83], [94], [95], [96].

Now, we establish some notations for this chapter and the next ones. Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n \big| \, \|z\| < r\}$ is denoted by $\mathbb{B}^n_r$ and the unit ball $\mathbb{B}^n_1$ is denoted by $\mathbb{B}^n$. In the case $n = 1$, the unit disc $\mathbb{B}^1$ is denoted by $\mathbb{U}$.

Let $L(\mathbb{C}^n)$ denote the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm. Also, let I_n be the identity operator in $L(\mathbb{C}^n)$. If $A \in L(\mathbb{C}^n)$, we denote by A^* the adjoint of the operator A.

Let $D \subset \mathbb{C}^n$ be a domain. Then we denote by $H(D)$ the family of holomorphic mappings from D into $\mathbb{C}^n$ and we consider the compact-open topology on it. In this case, $H(D)$ becomes a Fréchet space. For every $\Omega \subseteq H(D)$, $\overline{\Omega}$ denotes the closure of Ω with respect to this topology. For every $f \in H(D)$ and $K \subset D$, we denote $\|f\|_K = \sup_{z\in K} \|f(z)\|$.

If $f \in H(\mathbb{B}^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$. Let $S(\mathbb{B}^n)$ be the family of normalized univalent mappings on $\mathbb{B}^n$. If $n = 1$, then the family $S(\mathbb{U})$ is denoted by S. A mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ is called locally univalent if, for every $z \in \mathbb{B}^n$, there is an open neighborhood V of z such that f is univalent on V (or, equivalently, for every $z \in \mathbb{B}^n$, $Df(z)$ is invertible, where $Df(z)$ is the Fréchet differential of f at z). Also, we use the notations introduced in Definition 1.1.6 for $X = H(\mathbb{B}^n)$.

1.2.1 The Carathéodory family in $\mathbb{C}^n$

The following family of holomorphic mappings on $\mathbb{B}^n$ is closely related to the Caratheéodory class $\mathcal{P}$ (see [50], [54], [75], [95]). This family plays an essential role in the generalization of the Loewner theory to several complex variables, as we shall see in the next subsection.

Definition 1.2.1.

$$\mathcal{M} = \{h \in H(\mathbb{B}^n) \big| \, h(0) = 0, Dh(0) = I_n \text{ and } \Re\langle h(z), z\rangle \geq 0,\ z \in \mathbb{B}^n\}.$$

In the case $n = 1$, every function $h \in \mathcal{M}$ can be expressed as $h(z) = zp(z)$, $z \in \mathbb{U}$, where $p \in \mathcal{P}$. In view of this, we point an example of a mapping in the family $\mathcal{M}$, for $n \geq 1$ (see [50], [95]).

Example 1.2.2. Let $p_1, \cdots, p_n \in \mathcal{P}$. Then the mapping $h : \mathbb{B}^n \to \mathbb{C}^n$ given by

$$h(z) = \big(z_1 p_1(z_1), \ldots, z_n p_n(z_n)\big),\ z = (z_1, \ldots, z_n) \in \mathbb{B}^n,$$

is in $\mathcal{M}$.

For $n = 2$, we present the following simple but important example of a mapping in $\mathcal{M}$ (see [50], [95]).

Example 1.2.3. Let $n = 2$ and $h : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$h(z) = \big(z_1 - a z_2^2, z_2\big), \quad z = (z_1, z_2) \in \mathbb{B}^2,$$

where $a \in \mathbb{C}$. Then $h \in \mathcal{M}$ if and only if $|a| \leq \frac{3\sqrt{3}}{2}$.

For the family $\mathcal{M}$, we have the following growth theorem. The lower bound is due to Pfaltzgraff [75] (cf. [54]) and the upper bound is due to Graham, Hamada and Kohr [41] (cf. [50]). In case of the unit polydisc, a sharp upper bound was obtained by Poreda [80].

Theorem 1.2.4. *Let $h \in \mathcal{M}$. Then*

$$\|z\| \frac{1 - \|z\|}{1 + \|z\|} \leq \|h(z)\| \leq \frac{4\|z\|}{(1 - \|z\|)^2}, \quad z \in \mathbb{B}^n.$$

As a consequence of Theorem 1.2.4, Graham, Hamada and Kohr [41] obtained the compactness of the family $\mathcal{M}$.

Theorem 1.2.5. *$\mathcal{M}$ is a compact subset of $H(\mathbb{B}^n)$.*

In view of the Theorem 1.2.5, one can consider various extremal problems for the family $\mathcal{M}$. Graham, Hamada and Kohr [41] obtained the following coefficient estimates, which are useful in the study of other extremal problems (e.g. coefficient estimates for starlike mappings; see [39], [50]). Poreda [80] obtained a corresponding sharp coefficient estimate in the case of the polydisc.

Theorem 1.2.6. *Let $h \in \mathcal{M}$. Then the following sharp estimates hold*

$$\frac{1}{m!} \big| \langle D^m h(0)(w^m), w \rangle \big| \leq 2, \quad w \in \mathbb{C}^n, \|w\| = 1,\ m \geq 2.$$

Also, the following estimates hold

$$\left\| \frac{1}{m!} D^m h(0)(w^m) \right\| \leq 2k_m, \quad w \in \mathbb{C}^n, \|w\| = 1,\ m \geq 2,$$

where $k_m = m^{m/(m-1)}$.

A complete characterization of the set of extreme points and the set of support points of the Carathéodory family $\mathcal{M}$ seems to be much more complicated in the case of several complex variables (cf. Theorem 1.1.7), as some examples due to Voda [96] suggest. The following example due to Voda [96] points out a mapping that is a support point of $\mathcal{M}$, but it is not an extreme point of $\mathcal{M}$ for $n = 2$ (cf. Theorem 1.1.7 (i)).

Example 1.2.7. Let $h : \mathbb{B}^2 \to \mathbb{C}^2$ be given by $h(z_1, z_2) = \big(z_1 \frac{1 + z_1}{1 - z_1}, z_2 \frac{1 + z_2}{1 - z_2}\big)$, $(z_1, z_2) \in \mathbb{B}^2$. Then $h \in \operatorname{supp} \mathcal{M} \setminus \operatorname{ex} \mathcal{M}$.

The family $\mathcal{M}$ is very useful in the study of geometric properties of the univalent mappings on $\mathbb{B}^n$ (see [50], [95]). In the following, we consider the definition of starlike mappings on $\mathbb{B}^n$. We mention that there is an analytic characterization for starlike mappings in terms of mappings in $\mathcal{M}$ (see e.g. [50], [95]).

Definition 1.2.8. A mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ is called starlike if f is univalent on $\mathbb{B}^n$ with $f(0) = 0$ and $f(\mathbb{B}^n)$ is a starlike domain with respect to the origin.

We denote by $S^*(\mathbb{B}^n)$ the family of normalized univalent mappings on $\mathbb{B}^n$ that are starlike.

For $n = 2$, in connection with Example 1.2.3, we have the following example of a starlike mapping on $\mathbb{B}^2$ (see [95]; see also [87]).

Example 1.2.9. Let $n = 2$ and $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$f(z) = \big(z_1 + az_2^2, z_2\big),\ z = (z_1, z_2) \in \mathbb{B}^2,$$

where $a \in \mathbb{C}$. Then $f \in S^*(\mathbb{B}^2)$ if and only if $|a| \leq \dfrac{3\sqrt{3}}{2}$.

1.2.2 Loewner chains and the Loewner differential equation in $\mathbb{C}^n$

In this subsection, we present the definition of the Loewner chains on $\mathbb{B}^n$ and related results regarding the Loewner differential equation in $\mathbb{C}^n$ (see [3], [27], [41], [50], [75], [82]).

Definition 1.2.10. A mapping $f : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ is called a univalent subordination chain if $f(\cdot, t)$ is univalent on $\mathbb{B}^n$ with $f(0, t) = 0$, for all $t \geq 0$, and $f(\mathbb{B}^n, s) \subseteq f(\mathbb{B}^n, t)$, for all $0 \leq s \leq t$. If, in addition, $Df(0, t) = e^t I_n$, for all $t \geq 0$ (where D stands for the differential with respect to the complex variables), then f is called a Loewner chain. Also, if f is a Loewner chain such that $\{e^{-t} f(\cdot, t)\}_{t\geq 0}$ is a normal family, then f is called a normal Loewner chain.

From the previous definition we have that, if f is a univalent subordination chain, then, for every $0 \leq s \leq t$, there exists a unique univalent Schwarz mapping $v(\cdot, s, t)$ (i.e. $\|v(z, s, t)\| \leq \|z\|$, $z \in \mathbb{B}^n$) such that $f(z, s) = f(v(z, s, t), t)$, for all $z \in \mathbb{B}^n$. In this case, v is called the *transition mapping* associated to f. The transition mapping v satisfies the *semigroup property*: $v(z, s, u) = v(v(z, s, t), t, u)$, for all $z \in \mathbb{B}^n$, $0 \leq s \leq t \leq u$. Moreover, if f is a Loewner chain, then $Dv(0, s, t) = e^{s-t} I_n$, for all $0 \leq s \leq t$.

For a geometric interpretation of the univalent subordination chains in $\mathbb{C}^n$, one can use the notion of Carathéodory kernel convergence (see [34], [52], [96]; see [38], in the case $n = 1$).

For applications regarding quasiconformal extensions in higher dimensions, we refer to the initial results due to Pfaltzgraff [76] (see also [22], [27], and [50, Chapter 8]).

As in the case of one complex variable, one can provide a simple example of a normal Loewner chain starting with a starlike mapping on $\mathbb{B}^n$ (see e.g. [50]).

Example 1.2.11. Let $f \in S(\mathbb{B}^n)$. Then $f \in S^*(\mathbb{B}^n)$ if and only if the mapping $F : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ given by $F(z, t) = e^t f(z)$, $z \in \mathbb{B}^n$, $t \geq 0$, is a normal Loewner chain.

Now, we consider the Loewner differential equation in $\mathbb{C}^n$. The first results, regarding the existence, uniqueness and some properties of the solution on the unit ball, are due to Pfaltzgraff [75]. We summarize this results in the following theorem, taking into account the improvement due to Graham, Hamada and Kohr [41] (see also [50], [51]). We mention that the related case of the unit polydisc in $\mathbb{C}^n$ was first considered by Poreda [80].

Theorem 1.2.12. *Let $h : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ be such that:*

(i) $h(\cdot, t) \in \mathcal{M}$, for all $t \geq 0$,

(ii) $h(z,\cdot)$ *is measurable on* $[0,\infty)$*, for all* $z \in \mathbb{B}^n$*.*

Then, for every $z \in \mathbb{B}^n$ *and* $s \geq 0$*, the differential equation*

$$\frac{dv}{dt} = -h(v,t), \;\; a.e. \;\; t \geq s, \tag{1.2.1}$$

has a unique locally absolutely continuous solution $v(z,s,\cdot)$ *with the initial condition* $v(z,s,s) = z$*. Moreover, for every* $s \geq 0$*,* $v(z,s,\cdot)$ *is Lipschitz continuous on* $[s,\infty)$*, locally uniformly with respect to* $z \in \mathbb{B}^n$*,* $v(\cdot,s,t)$ *is a univalent Schwarz function with* $Dv(0,s,t) = e^{s-t}$*, for all* $t \geq s$*, and the following inequalities hold:*

$$\frac{e^{t-s}\|v(z,s,t)\|}{(1-\|v(z,s,t)\|)^2} \leq \frac{\|z\|}{(1-\|z\|)^2},$$

$$\frac{e^{t-s}\|v(z,s,t)\|}{(1+\|v(z,s,t)\|)^2} \geq \frac{\|z\|}{(1+\|z\|)^2},$$

for all $z \in \mathbb{B}^n$ *and* $t \geq s \geq 0$*.*

The mapping $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ that satisfies the conditions (i) and (ii) from Theorem 1.2.12 is called a *Herglotz vector field* or a *generating vector field* (see [17], [34]). The differential equation (1.2.1) is called the *Loewner* (*ordinary*) *differential equation* associated to h.

The next theorem shows how to generate a normal Loewner chain, whose transition mapping is given by the solution of the Loewner differential equation. Poreda [82] obtained this result under some extra assumptions that have been proved to be not necessary by Graham, Hamada and Kohr [41] (see also [50], [51]).

Theorem 1.2.13. *Let* $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ *be a Herglotz vector field and* $v(z,s,\cdot)$ *be the unique locally absolutely continuous solution of the Loewner differential equation* (1.2.1) *associated to* h *on* $[s,\infty)$ *with the initial condition* $v(z,s,s) = z$*, for all* $s \geq 0$*,* $z \in \mathbb{B}^n$*. Then, for every* $s \geq 0$*, the following limit exists locally uniformly on* $\mathbb{B}^n$*:*

$$f(z,s) := \lim_{t\to\infty} e^t v(z,s,t) \tag{1.2.2}$$

and $f : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ *given by* (1.2.2) *is a normal Loewner chain whose transition mapping is* v*. Moreover,* $f(z,\cdot)$ *is locally Lipschitz continuous on* $[0,\infty)$ *locally uniformly with respect to* $z \in \mathbb{B}^n$ *and* f *satisfies for almost every* $t \geq 0$*:*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad z \in \mathbb{B}^n. \tag{1.2.3}$$

The differential equation (1.2.3) is called the (*generalized*) *Loewner differential equation* associated to h.

Graham and Kohr [50] proved that if f is a Loewner chain, then $f(z,\cdot)$ is locally Lipschitz on $[0,\infty)$, locally uniformly with respect to $z \in \mathbb{B}^n$. Taking into account also the works of Curt, Kohr [28, 29], Graham, Hamada, Kohr [41] (see also [51]), we have the following theorem which states that every Loewner chain satisfies a Loewner differential equation and provides a characterization for normal Loewner chains. Applications of the Loewner differential equation in higher dimensions may be also found in [27].

Theorem 1.2.14. *Let* $f : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ *be a Loewner chain. Then* $f(z,\cdot)$ *is locally Lipschitz continuous on* $[0,\infty)$ *locally uniformly with respect to* $z \in \mathbb{B}^n$ *and there exists* $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ *that satisfies the conditions* (i) *and* (ii) *from Theorem* 1.2.12 *such that, for almost every* $t \geq 0$*, we have:*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \text{ for all } z \in \mathbb{B}^n.$$

Moreover, if f *is a normal Loewner chain, then* $f(z,s) = \lim_{t\to\infty} e^t v(z,s,t)$*, locally uniformly with respect to* $z \in \mathbb{B}^n$*, for all* $s \geq 0$*, where* $v(z,s,\cdot)$ *is the unique locally absolutely continuous solution* *of the Loewner differential equation* (1.2.1) *associated to* h *on* $[s,\infty)$ *with the initial condition* $v(z,s,s)=z$.

Remark 1.2.15. The Herglotz vector field h from Theorem 1.2.14 is essentially unique in the following sense: if k is another Herglotz vector field such that f satisfies the Loewner differential equation (1.2.3) associated to k, then $h(\cdot,t)=k(\cdot,t)$, for a.e. $t\geq 0$ (see [8], [50]).

Theorems 1.2.12 and 1.2.14 imply the following growth theorem for normal Loewner chains (see [29], [50], [51]).

Theorem 1.2.16. *Let* $f:\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ *be a normal Loewner chain. Then*

$$\frac{\|z\|}{(1+\|z\|)^2}\leq e^{-t}\|f(z,t)\|\leq\frac{\|z\|}{(1-\|z\|)^2},\quad z\in\mathbb{B}^n,\ t\geq 0.$$

As a consequence of Theorem 1.2.16, we have the following remark (see [52]).

Remark 1.2.17. If $f:\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ is a normal Loewner chain, then $\bigcup_{t\geq 0}f(\mathbb{B}^n,t)=\mathbb{C}^n$.

Now, we shall see how to generate a Loewner chain, by solving a Loewner differential equation. Furthermore, we shall also point out the characterization of all Loewner chains in terms of normal Loewner chains and entire univalent mappings. To this end, we first consider some definitions (see [52], [34], [8], [18]).

Definition 1.2.18. A mapping $f:\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ is called a standard solution to the Loewner differential equation (1.2.3) associated to a Herglotz vector field h if $f(0,t)=0$, for all $t\geq 0$, $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z\in\mathbb{B}^n$ and f satisfies the differential equation (1.2.3) associated to h.

A mapping $f:\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ is called a normalized univalent solution to the Loewner differential equation (1.2.3) associated to a Herglotz vector field h if f is a standard solution to the Loewner differential equation (1.2.3) associated to h and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a family in $S(\mathbb{B}^n)$. If, in addition, $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family, then f is called a canonical solution to the Loewner differential equation (1.2.3) associated to h.

We have the following important theorem due to Pfaltzgraff [75] and Graham, Hamada, Kohr [41] (see also [51], [52]), which, in view of Theorems 1.2.13 and 1.2.14, yields that the notions of normal Loewner chain and the canonical solution are equivalent.

Theorem 1.2.19. *Let* $h:\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ *be a Herglotz vector field. A mapping* $f:\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ *is a canonical solution to the Loewner differential equation* (1.2.3) *associated to* h *if and only if* f *is the normal Loewner chain given by* (1.2.2). *In particular, there exists a unique canonical solution to the Loewner differential equation* (1.2.3) *associated to* h.

For standard solutions and normalized univalent solutions, we have the following characterization due to Graham, Kohr and Pfaltzgraff [52] (cf. [34]). In the last subsection, we refer to an abstract approach to this characterization, in a very general context, due to Arosio, Bracci, Hamada and Kohr [8].

Theorem 1.2.20. *Let* h *be a Herglotz vector field and* f *be the canonical solution of the Loewner differential equation* (1.2.3) *associated to* h. *Then* $g:\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ *is a standard solution of the Loewner differential equation* (1.2.3) *associated to* h *if and only if there exists a holomorphic mapping* $\Phi:\mathbb{C}^n\to\mathbb{C}^n$ *with* $\Phi(0)=0$ *such that*

$$g(z,t)=\Phi(f(z,t)),\quad z\in\mathbb{B}^n,\ t\geq 0.$$

In particular, $g:\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ *is a normalized univalent solution of the Loewner differential equation* (1.2.3) *associated to* h *if and only if there exists a univalent mapping* $\Phi:\mathbb{C}^n\to\mathbb{C}^n$ *with* $\Phi(0)=0$ *and* $D\Phi(0)=I_n$ *such that*

$$g(z,t)=\Phi(f(z,t)),\quad z\in\mathbb{B}^n,\ t\geq 0.$$

In view of Theorem 1.2.20, we have the following characterization of Loewner chains due to Graham, Kohr and Pfaltzgraff [52] (see also [8], [34]; cf. Theorem 1.1.22).

Theorem 1.2.21. *A mapping $g : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ is a Loewner chain if and only if g is a normalized univalent solution of the Loewner differential equation* (1.2.3) *associated to some Herglotz vector field. Moreover, g is a Loewner chain if and only if there exists a unique normal Loewner chain and a unique univalent mapping $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ with $\Phi(0) = 0$ and $D\Phi(0) = I_n$ such that*

$$g(z,t) = \Phi(f(z,t)), \quad z \in \mathbb{B}^n,\ t \geq 0.$$

In particular, if f is a normal Loewner chain and $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ is a univalent mapping with $\Phi(0) = 0$ and $D\Phi(0) = I_n$, then: $\Phi \circ f$ is normal Loewner chain if and only if $\Phi = I_n$.

In view of Theorem 1.2.21, one can easily provide examples of Loewner chains which are not normal, as the following example shows (cf. [9], [41], [50]).

Example 1.2.22. Let $n \geq 2$ and let $\phi : \mathbb{C}^n \to \mathbb{C}^n$ be an automorphism of $\mathbb{C}^n$ be such that $\Phi(0) = 0$, $D\Phi(0) = I_n$ and $\Phi \not\equiv I_n$. Also, let $F : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ be given by $F(z, t) = \Phi(e^t z)$, $z \in \mathbb{B}^n$, $t \geq 0$. Then, by Theorem 1.2.21, F is a Loewner chain which is not a normal Loewner chain in dimension $n \geq 2$.

1.2.3 Parametric representations on the unit ball

The family of mappings that have parametric representation on the unit polydisc in $\mathbb{C}^n$ was first considered by Poreda [80, 81] and then by Kohr [68] on the Euclidean unit ball. In fact, Kohr [68] (see also [41]) considered the more general family of mappings that have g-parametric representation on $\mathbb{B}^n$. Graham, Hamada and Kohr [41] considered the case of an arbitrary norm.

Definition 1.2.23. A mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ has parametric representation if there exists a Herglotz vector field $h : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ such that:

(i) $h(\cdot, t) \in \mathcal{M}$, for all $t \geq 0$,

(ii) $h(z, \cdot)$ is measurable on $[0, \infty)$, for all $z \in \mathbb{B}^n$,
and

$$f = \lim_{t \to \infty} e^t v(\cdot, t), \text{ locally uniformly on } \mathbb{B}^n,$$

where, for every $z \in \mathbb{B}^n$, $v(z, \cdot)$ is the unique locally absolutely continuous solution on $[0, \infty)$ of the Loewner differential equation associated to h:

$$\frac{dv}{dt} = -h(v, t), \text{ a.e. } t \geq 0,$$

with the initial condition $v(z, 0) = z$.

We denote by $S^0(\mathbb{B}^n)$ the family of mappings that have parametric representation on $\mathbb{B}^n$.

Although the family $S^0(\mathbb{B}^n)$ is a proper subset of $S(\mathbb{B}^n)$ (see [41], [50]; cf. [80]; see also Example 1.2.33), the family $S^0(\mathbb{B}^n)$ represents a natural generalization to higher dimensions of the class S of normalized univalent functions on the unit disc. In the following we shall see some similarities between the class S and the family $S^0(\mathbb{B}^n)$. We mention that, by Theorem 1.1.25, we have that $S^0(\mathbb{U}) = S$.

In view of the previous subsection, we have the following characterization of the family $S^0(\mathbb{B}^n)$, in terms of normal Loewner chains, due to Graham, Kohr and Kohr [51] (see also [41], [50]). Poreda [81] obtained this result for the case of the polydisc, under some restrictive conditions.

Theorem 1.2.24. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$. Then $f \in S^0(\mathbb{B}^n)$ if and only if there exists a normal Loewner chain F such that $F(\cdot, 0) = f$.*

Taking into account Example 1.2.11, we immediately have that every normalized starlike mapping is in $S^0(\mathbb{B}^n)$ (see [50]; cf. [80]).

Remark 1.2.25. $S^*(\mathbb{B}^n) \subset S^0(\mathbb{B}^n)$.

A consequence of Theorems 1.2.16 and 1.2.24 is the following growth theorem for mappings in $S^0(\mathbb{B}^n)$ due to Graham, Hamada and Kohr [41] (see also [68]; see Poreda [80], for the case of the polydisc). Barnard, FitzGerald and Gong [11] obtained this growth theorem for the family $S^*(\mathbb{B}^n)$, by using the analytical characterization of starlikeness on $\mathbb{B}^n$ (cf. [22]).

Theorem 1.2.26. *Let $f \in S^0(\mathbb{B}^n)$. Then the following sharp inequalities hold:*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in \mathbb{B}^n.$$

In particular, $\mathbb{B}^n_{1/4} \subseteq f(\mathbb{B}^n)$.

Graham, Kohr and Kohr [51] proved the compactness of the family $S^0(\mathbb{B}^n)$.

Theorem 1.2.27. *$S^0(\mathbb{B}^n)$ is a compact subset of $H(\mathbb{B}^n)$.*

In view of the previous theorem, one can consider various extremal problems for the family $S^0(\mathbb{B}^n)$. The problem of finding sharp coefficient estimates for the mappings that have parametric representation on $\mathbb{B}^n$ seems to be more complicated for $n \geq 2$ (see [41], [50]). The following coefficient estimates are due to Graham, Hamada and Kohr [41] (see also [50], [68]). The corresponding result of (1.2.4) for the case of the polydisc was obtained by Poreda [80]. We mention that Graham, Hamada and Kohr [41] formulated the n-dimensional version of the Bieberbach conjecture (see also [50]; cf. [39]).

Theorem 1.2.28. *Let $f \in S^0(\mathbb{B}^n)$. Then the following sharp inequality holds:*

$$\left| \frac{1}{2} \left\langle D^2 f(0)(w^2), w \right\rangle \right| \leq 2, \quad w \in \mathbb{C}^n,\ \|w\| = 1. \tag{1.2.4}$$

Also, the following inequality holds:

$$\left\| \frac{1}{2} D^2 f(0)(w^2) \right\| \leq 8, \quad w \in \mathbb{C}^n,\ \|w\| = 1.$$

Taking into account the results of Kirwan [66] and Pell [74] (see Theorem 1.1.23), Graham, Kohr and Pfaltzgraff [53] considered extremal properties associated to compact families of univalent mappings on $\mathbb{B}^n$, $n \geq 2$, generated by normal Loewner chains. Graham, Hamada, Kohr and Kohr [46] generalized Theorem 1.1.23 (*i*) (cf. [21]). Schleißinger [93] proved the corresponding generalization of Theorem 1.1.23 (*ii*), by using deep ideas based on properties of Runge pairs in $\mathbb{C}^n$ (see also [46] for a partial result; cf. [21]).

Theorem 1.2.29. *Let f be a normal Loewner chain.*

(i) If $f(\cdot, 0) \in \operatorname{ex} S^0(\mathbb{B}^n)$, then $e^{-t} f(\cdot, t) \in \operatorname{ex} S^0(\mathbb{B}^n)$, for all $t \geq 0$.

(ii) If $f(\cdot, 0) \in \operatorname{supp} S^0(\mathbb{B}^n)$, then $e^{-t} f(\cdot, t) \in \operatorname{supp} S^0(\mathbb{B}^n)$, for all $t \geq 0$.

Remark 1.2.30. The condition that f is a normal Loewner chain in the previous theorem is essential, in the view of the following fact: if f is a normal Loewner chain, then $e^{-t} f(\cdot, t) \in S^0(\mathbb{B}^n)$, for all $t \geq 0$ (see [46]).

In connection with Example 1.2.7, we present an example of a starlike mapping that is a support point of $S^0(\mathbb{B}^n)$, but which is not an extreme point of $S^0(\mathbb{B}^n)$ for $n = 2$, due to Graham, Hamada, Kohr, Kohr [48] and Voda [96].

Example 1.2.31. Let $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by $f(z_1, z_2) = \big(\frac{z_1}{(1-z_1)^2}, \frac{z_2}{(1-z_2)^2}\big)$, $(z_1, z_2) \in \mathbb{B}^2$. Then $f \in S^*(\mathbb{B}^2)$ and $f \in \operatorname{supp} S^0(\mathbb{B}^2) \setminus \operatorname{ex} S^0(\mathbb{B}^2)$.

In view of Example 1.1.10, we remark that, in spite of the fact that the above mapping is not an extreme point of $S^0(\mathbb{B}^2)$, every component of the mapping is an extreme point of the class S (see [48]).

As in dimension one, one can use optimal control theory to study extremal problems associated to the family of mappings that have parametric representation on $\mathbb{B}^n$ (see [46]). A relevant result in this direction is the Pontryagin maximum principle for the Loewner differential equation in higher dimensions due to Roth [92]. We present a consequence of this result, due to Roth [92], which gives a necessary condition for a normal Loewner chain such that the first element of the chain is a support point of $S^0(\mathbb{B}^n)$. This improves a related result due to Bracci, Graham, Hamada, Kohr [18].

Theorem 1.2.32. *Let f be a normal Loewner chain such that $f(\cdot, 0) \in \operatorname{supp} S^0(\mathbb{B}^n)$. Then*

$$\inf_{z\in\mathbb{B}^n\setminus\{0\}} \Re \left\langle \big[Df(z,t)\big]^{-1} \frac{\partial f}{\partial t}(z,t), \frac{z}{\|z\|^2} \right\rangle = 0, \ \ a.e. \ t \geq 0.$$

We close this subsection with an example of a mapping that provides important differences between the case of one complex variable and that of higher dimensions (see [15], [41], [50]). In particular, we present an example of a bounded support point of $S^0(\mathbb{B}^2)$ due to Bracci [15].

Example 1.2.33. We consider again Example 1.2.9. Let $n = 2$ and $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$f(z) = \big(z_1 + az_2^2, z_2\big), \quad z = (z_1, z_2) \in \mathbb{B}^2,$$

where $a \in \mathbb{C}$. Then:

(i) For every $a \in \mathbb{C}$, we have that $f \in S(\mathbb{B}^2)$. Since the absolute value of a can be arbitrarily chosen, we have that $S(\mathbb{B}^2)$ is not a normal family (see [41], [50]). Hence, in view of Theorem 1.2.27, $S^0(\mathbb{B}^2)$ is a proper subset of $S(\mathbb{B}^2)$.

(ii) Graham, Hamada and Kohr [41] pointed out another difference between dimension one (cf. Theorem 1.1.13) and higher dimensions: if $|a| = \frac{3\sqrt{3}}{2}$, then $f \in S^0(\mathbb{B}^2)$ and

$$\left\| \frac{1}{2} D^2 f(0)(w^2) \right\| = |a| = \frac{3\sqrt{3}}{2} > 2, \quad w \in \mathbb{C}^2, \, \|w\| = 1.$$

Moreover, this is also in contrast with the coefficient estimate in the case of the unit polydisc due to Poreda [80].

(iii) Bracci [15] proved that: $f \in S^0(\mathbb{B}^2)$ if and only if $|a| \leq \frac{3\sqrt{3}}{2}$. Hence, taking into account Example 1.2.9, $f \in S^0(\mathbb{B}^2)$ if and only if $f \in S^*(\mathbb{B}^2)$. Moreover, Bracci [15] proved that: if $a = \dfrac{3\sqrt{3}}{2}$, then f is a bounded support point of $S^0(\mathbb{B}^2)$, with respect to the continuous linear functional $L : H(\mathbb{B}^2) \to \mathbb{C}$ given by $L(g) = \dfrac{1}{2} \dfrac{\partial^2 g_1}{\partial z_2^2}(0,0)$, $g = (g_1, g_2) \in H(\mathbb{B}^2)$.

1.2.4 *A*-parametric representations on the unit ball

In this subsection, we shall present some definitions and results due to Graham, Hamada, Kohr, Kohr [44] and Duren, Graham, Hamada, Kohr [34], which generalize the results from the previous subsections. Also, we refer to some results regarding some extremal properties (see [48], [49]). On the other hand, we mention also the contributions of Arosio [4, 5] and Voda [96, 97].

We shall use the following notations related to an operator $A \in L(\mathbb{C}^n)$ (see e.g. [36]):

$$\begin{aligned} m(A) &= \min\{\Re\langle A(z), z\rangle \big| \, \|z\| = 1\}, \\ k(A) &= \max\{\Re\langle A(z), z\rangle \big| \, \|z\| = 1\}, \\ |V(A)| &= \max\{|\langle A(z), z\rangle| \big| \, \|z\| = 1\}, \\ k_+(A) &= \max\{\Re\lambda \big| \, \lambda \in \sigma(A)\}, \end{aligned}$$

where $\sigma(A)$ is the spectrum of A. Note that $|V(A)|$ is the numerical radius of the operator A and $k_+(A)$ is the upper exponential index (Lyapunov index) of A. Then $m(A) \le k_+(A) \le |V(A)| \le \|A\|$ (see e.g. [48]) and it is known that $\|A\| \le 2|V(A)|$ and $k_+(A) = \lim_{t\to\infty} \frac{\log \|e^{tA}\|}{t}$ (see e.g. [31], [36]).

First, we consider some generalization of the Carathéodory class $\mathcal{P}$, in the case of several complex variables, related to the Carathéodory family $\mathcal{M}$ (see [54], [95]).

Definition 1.2.34. We denote:

$$\mathcal{N} = \{h \in H(\mathbb{B}^n) \big| \, h(0) = 0, \, \Re\langle h(z), z\rangle \ge 0, \, z \in \mathbb{B}^n\}$$

If $A \in L(\mathbb{C}^n)$ is such that $m(A) \ge 0$, then we denote:

$$\mathcal{N}_A = \{h \in \mathcal{N} \big| \, Dh(0) = A\}.$$

We observe that the Carathéodory family $\mathcal{M}$ coincides with $\mathcal{N}_{I_n}$.

We have the following compactness result due to Graham, Hamada, Kohr and Kohr [44].

Theorem 1.2.35. *Let $A \in L(\mathbb{C}^n)$ be such that $m(A) \ge 0$. Then $\mathcal{N}_A$ is a compact subset of $H(\mathbb{B}^n)$.*

The family $\mathcal{N}_A$ is useful in the characterization of the normalized spirallike mappings on $\mathbb{B}^n$ with respect to A, where $A \in L(\mathbb{C}^n)$ with $m(A) > 0$ (see [50], [54], [95]). In the following, we present the definition of spirallike mappings on $\mathbb{B}^n$ (see [95]).

Definition 1.2.36. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. A mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ is called spirallike with respect to A if f is a univalent mapping on $\mathbb{B}^n$ with $f(0) = 0$ and $f(\mathbb{B}^n)$ is a spirallike domain with respect to A, i.e. $e^{-tA} f(\mathbb{B}^n) \subseteq f(\mathbb{B}^n)$, for all $t \ge 0$.

We denote by $\widehat{S}_A(\mathbb{B}^n)$ the family of normalized univalent mappings on $\mathbb{B}^n$ that are spirallike with respect to A.

Graham, Hamada and Kohr [41] gave the following example of a spirallike mapping on the unit ball in $\mathbb{C}^2$ which does not have parametric representation (see also [44], [50]).

Example 1.2.37. We consider Example 1.2.33. Let $n = 2$ and $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$f(z) = \big(z_1 + a z_2^2, z_2\big), \quad z = (z_1, z_2) \in \mathbb{B}^2,$$

where $a \in \mathbb{C}$. f is spirallike with respect to $A \in L(\mathbb{C}^2)$ given by $A(z_1, z_2) = (2z_1, z_2)$, $(z_1, z_2) \in \mathbb{C}^2$, but if the absolute value of a is sufficiently large, then f does not have parametric representation on $\mathbb{B}^2$.

This example gives one of the motivations to extend the study of parametric representations to the study of A-parametric representations due to Graham, Hamada, Kohr and Kohr [44]. In the following, we consider the definition of a mapping that has A-parametric representation on $\mathbb{B}^n$.

Definition 1.2.38. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$. A mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ has A-parametric representation if there exists $h : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ such that:

(i) $h(\cdot, t) \in \mathcal{N}_A$, for all $t \ge 0$,

(ii) $h(z,\cdot)$ is measurable on $[0,\infty)$, for all $z \in \mathbb{B}^n$,
and

$$f = \lim_{t\to\infty} e^{tA} v(\cdot,t), \text{ locally uniformly on } \mathbb{B}^n, \tag{1.2.5}$$

where, for every $z \in \mathbb{B}^n$, $v(z,\cdot)$ is the unique locally absolutely continuous solution on $[0,\infty)$ of the Loewner differential equation associated to h:

$$\frac{dv}{dt} = -h(v,t), \text{ a.e. } t \geq 0, \tag{1.2.6}$$

with the initial condition $v(z,0) = z$.

We denote by $S_A^0(\mathbb{B}^n)$ the family of mappings that have A-parametric representation on $\mathbb{B}^n$.

We note that, in general, the limit (1.2.5) may not exist, as the following example shows (see [97, Example 3.7]; cf. [4]).

Example 1.2.39. Let $n = 2$ and let $h : \mathbb{B}^2 \times [0,\infty) \to \mathbb{C}^2$ be given by

$$h(z,t) = (\lambda z_1 + a(t)z_2^2, z_2), \quad z = (z_1,z_2) \in \mathbb{B}^2,\ t \geq 0, \tag{1.2.7}$$

where $\lambda \in \mathbb{C}$ with $\Re\lambda \geq 2$ and $a : [0,\infty) \to \mathbb{C}$ is measurable with $|a(t)| \leq 1$, $t \geq 0$. Then h satisfies the conditions (i) and (ii) from Definition 1.2.38 with respect to $A = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, and, for every $z = (z_1,z_2) \in \mathbb{B}^2$, $v(z,\cdot)$ given by

$$v(z,t) = \Big(e^{-\lambda t}\Big(z_1 - \Big(\int_0^t a(s)\,ds\Big)z_2^2\Big), e^{-t}z_2\Big), \quad t \geq 0, \tag{1.2.8}$$

is the unique locally absolutely continuous solution on $[0,\infty)$ of the Loewner differential equation (1.2.6) associated to h.

Now, fix $\lambda = 2$ and $a(t) = \cos(t)$, $t \geq 0$. Then we observe that $e^{tA}v(z,t) = \big(z_1 - \sin(t)z_2^2, z_2\big)$, $z = (z_1,z_2) \in \mathbb{B}^2$, $t \geq 0$, and thus the limit (1.2.5) does not exist, in this case.

Graham, Hamada, Kohr and Kohr [44] found a sufficient condition for $A \in L(\mathbb{C}^n)$, namely $k_+(A) < 2m(A)$, such that the limit (1.2.5) always exists and then they obtained certain characterizations of the family $S_A^0(\mathbb{B}^n)$. In the following, we shall present some of these results. To this end, first we consider the definition of A-normalized univalent subordination chains, where $A \in L(\mathbb{C}^n)$ (see [34], [44], [48]).

Definition 1.2.40. Let $A \in L(\mathbb{C}^n)$. A univalent subordination chain f is called A-normalized if $Df(0,t) = e^{tA}$, for all $t \geq 0$.

A simple example of an A-normalized univalent subordination chain, where $A \in L(\mathbb{C}^n)$ with $m(A) > 0$, can be provided by using a spirallike mapping with respect to A (see [44]).

Example 1.2.41. Let $A \in L(\mathbb{C}^n)$ be such that $m(A) > 0$ and let $f \in S(\mathbb{B}^n)$. Then $f \in \widehat{S}_A(\mathbb{B}^n)$ if and only if the mapping $F : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ given by $F(z,t) = e^{tA}f(z)$, $z \in \mathbb{B}^n$, $t \geq 0$, is an A-normalized univalent subordination chain.

Graham, Hamada, Kohr and Kohr [44] characterized the family $S_A^0(\mathbb{B}^n)$ for $A \in L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$, in terms of A-normalized univalent subordination chains.

Theorem 1.2.42. *Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. If $f : \mathbb{B}^n \to \mathbb{C}^n$, then $f \in S_A^0(\mathbb{B}^n)$ if and only if there exists an A-normalized univalent subordination chain F such that $\{e^{-tA}F(\cdot,t)\}_{t\geq 0}$ is a normal family and $F(\cdot,0) = f$.*

Also, authors in [44] proved the compactness of the family $S_A^0(\mathbb{B}^n)$, for $A \in L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$.

Theorem 1.2.43. *Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. Then $S_A^0(\mathbb{B}^n)$ is a compact subset of $H(\mathbb{B}^n)$.*

In view of Theorem 1.2.43, we note that Voda provided an example [97, Example 3.7] (see also [4]) which shows that, without the condition $k_+(A) < 2m(A)$ for $A \in L(\mathbb{C}^n)$, the family $S_A^0(\mathbb{B}^n)$ may not be compact.

We mention that most of the results in Subsection 1.2.2 remain valid for A-normalized univalent subordination chains for $A \in L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$ (see [34], [44]). However, we point out an example due to Graham, Hamada, Kohr and Kohr [49] of mapping that has A-parametric representation for some $A \in L(\mathbb{C}^2)$ with $k_+(A) < 2m(A)$, but which does not have parametric representation on $\mathbb{B}^2$. Moreover, this example shows the existence of a bounded support point for $S_A^0(\mathbb{B}^2)$ (cf. [15]; see Example 1.2.33).

Example 1.2.44. Let $\lambda \in (1, 2)$ and let $A = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. Also, let $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$f(z) = \left(z_1 + \frac{a_0}{\lambda - 2} z_2^2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2,$$

where $a_0 = \max\left\{a > 0 \,\middle|\, \lambda x^2 + y^2 - axy^2 \geq 0, x, y \geq 0, x^2 + y^2 \leq 1\right\}$. Then $k_+(A) < 2m(A)$ and $f \in S_A^0(\mathbb{B}^2) \setminus S^0(\mathbb{B}^2)$. Moreover, $f \in \operatorname{supp} S_A^0(\mathbb{B}^2)$.

Since the family $S_A^0(\mathbb{B}^n)$ is compact for $A \in L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$, one may consider the study of associated extremal problems. For a study based on control theory, we refer to [48, 49]. In Chapter 3, we shall focus on this study. Graham, Hamada, Kohr and Kohr [48] obtained the following generalization of Theorem 1.2.29.

Theorem 1.2.45. *Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$ and let f be an A-normalized univalent subordination chain such that $\{e^{-tA} f(\cdot, t)\}_{t \geq 0}$ is a normal family.*

(i) If $f(\cdot, 0) \in \operatorname{ex} S_A^0(\mathbb{B}^n)$, then $e^{-tA} f(\cdot, t) \in \operatorname{ex} S_A^0(\mathbb{B}^n)$, for all $t \geq 0$.

(ii) If $f(\cdot, 0) \in \operatorname{supp} S_A^0(\mathbb{B}^n)$, then $e^{-tA} f(\cdot, t) \in \operatorname{supp} S_A^0(\mathbb{B}^n)$, for all $t \geq 0$.

Remark 1.2.46. Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. In connection with Remark 1.2.30, we note that: if f is an A-normalized univalent subordination chain such that $\{e^{-tA} f(\cdot, t)\}_{t \geq 0}$ is a normal family, then $e^{-tA} f(\cdot, t) \in S_A^0(\mathbb{B}^n)$, for all $t \geq 0$ (see [48]).

Some of the results presented in this chapter have been extended to the case of reflexive complex Banach spaces by Graham, Hamada, Kohr and Kohr [47]. For a general study regarding the A-normalized univalent subordination chains for $A \in L(\mathbb{C}^n)$ with $m(A) > 0$, we mention the work of Poreda [82] and the more recent works of Arosio [4, 5] and Voda [96, 97] (see also [57]). The general case of time-dependent normalization for univalent subordination chains in $\mathbb{C}^n$ was considered first by Graham, Hamada, Kohr and Kohr [45] (see also [42]), then by Arosio [6] and Voda [96]. We shall focus on this case in the last chapter. On the other hand, we mention the topological method of Cristea [25, 26], concerning the theory of Loewner chains for mappings that are not necessarily holomorphic.

1.2.5 L^d-Loewner chains

In this subsection, we consider the recent geometric approach to the theory of Loewner chains in higher dimensions due to Bracci, Contreras, Diaz-Madrigal [17], Arosio, Bracci, Hamada, Kohr [8], Arosio, Bracci, Wold [10] (see also [4], [5], [7], [60]), which does not rely on any particular normalization. Moreover, this study has been developed on complete hyperbolic complex manifolds, but, for the sake of simplicity and the purpose of this exposition, we consider, throughout this subsection, only the case of the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. To be more precise, we focus on the recent work of Arosio, Bracci and Wold [10], in which the authors collected several results in this direction and pointed out some new and interesting properties that turned out to be very useful even in the case of some particular normalizations of the univalent subordination chains.

The main notions that we consider in this subsection were first introduced and studied in the case of one complex variable by Bracci, Contreras, Díaz-Madrigal [16] (*L^d-evolution family* and *L^d-Herglotz vector field*) and by Contreras, Díaz-Madrigal, Gumenyuk [23] (*L^d-Loewner chain*) and then generalized to higher dimensions by Bracci, Contreras, Díaz-Madrigal [17] and by Arosio, Bracci, Hamada, Kohr [8].

In the following, for every $d \in [1, \infty]$ and $T > 0$, we denote by $L^d([0, T])$ the corresponding Lebesgue space of real-valued functions.

We begin with the definition of L^d-evolution families (or evolution families of order d) on $\mathbb{B}^n$ (see [17]; cf. [7], [8], [10]).

Definition 1.2.47. Let $d \in [1, \infty]$. A family $(\varphi_{s,t})_{0 \le s \le t}$ of holomorphic self-mappings of $\mathbb{B}^n$ is an L^d-evolution family on $\mathbb{B}^n$ if:

(i) $\varphi_{s,s} = \mathrm{id}_{\mathbb{B}^n}$, for all $s \ge 0$,

(ii) $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$, for all $0 \le s \le u \le t$,

(iii) for every compact set $K \subset \mathbb{B}^n$ and $T > 0$, there exists $c_{K,T} \in L^d([0, T])$ such that

$$\left\|\varphi_{s,t}(z) - \varphi_{s,u}(z)\right\| \le \int_u^t c_{K,T}(\tau)\, d\tau, , \quad \text{for all } z \in K,\ 0 \le s \le u \le t \le T.$$

We present a simple example of an L^d-evolution family on $\mathbb{B}^n$, by using a *star-shaped mapping* on $\mathbb{B}^n$ (see [8]; cf. [16]). A mapping $F : \mathbb{B}^n \to \mathbb{C}^n$ is called star-shaped if F is univalent and $e^{-t}F(z) \in F(\mathbb{B}^n)$, for all $z \in \mathbb{B}^n$ and $t \ge 0$ (see [8]; cf. [36]). In this case, $0 \in \overline{F(\mathbb{B}^n)}$.

Example 1.2.48. Let $d \in [1, \infty]$ and let $\lambda : [0, \infty) \to [0, \infty)$ be a measurable function such that $\lambda\big|_{[0,T]} \in L^d([0, T])$, for all $T > 0$. Also, let $F : \mathbb{B}^n \to \mathbb{C}^n$ be a star-shaped mapping and let $\varphi_{s,t}(z) = F^{-1}\Big(e^{-\int_s^t \lambda(\tau)d\tau} F(z)\Big)$, for $0 \le s \le t$ and $z \in \mathbb{B}^n$. Then $(\varphi_{s,t})_{0 \le s \le t}$ is an L^d-evolution family on $\mathbb{B}^n$.

We point out that every mapping in an L^d-evolution family on $\mathbb{B}^n$ is univalent (see [17]; see also [8], [10]).

Remark 1.2.49. Let $d \in [1, \infty]$ and let $(\varphi_{s,t})_{0 \le s \le t}$ be an L^d-evolution family on $\mathbb{B}^n$. Then $\varphi_{s,t}$ is a univalent mapping, for all $0 \le s \le t$.

Next, we consider the definition of the L^d-Loewner chains (or Loewner chains of order d) on $\mathbb{B}^n$ (see [8]; cf. [10], [60]).

Definition 1.2.50. Let $d \in [1, \infty]$. A family $(f_t)_{t \ge 0}$ of univalent mappings from $\mathbb{B}^n$ to $\mathbb{C}^n$ is called an L^d-Loewner chain on $\mathbb{B}^n$ if:

(i) $f_s(\mathbb{B}^n) \subseteq f_t(\mathbb{B}^n)$, for all $0 \le s \le t$,

(ii) for every compact set $K \subset \mathbb{B}^n$ and $T > 0$, there exists $c_{K,T} \in L^d([0, T])$ such that

$$\left\|f_s(z) - f_t(z)\right\| \le \int_s^t c_{K,T}(\tau)\, d\tau, , \quad \text{for all } z \in K,\ 0 \le s \le t \le T.$$

Also, we use the notation: $R(f_t) := \bigcup_{t \ge 0} f_t(\mathbb{B}^n)$. This set is called the Loewner range of the L^d-Loewner chain $(f_t)_{t \ge 0}$.

In connection with Example 1.2.48, we present an example of an L^d-Loewner chain on $\mathbb{B}^n$ (see [8]).

Example 1.2.51. Let $d \in [1,\infty]$ and let $\lambda : [0,\infty) \to [0,\infty)$ be a measurable function such that $\lambda\big|_{[0,T]} \in L^d([0,T])$, for all $T > 0$. Also, let $F : \mathbb{B}^n \to \mathbb{C}^n$ be a star-shaped mapping and let $f_t(z) = e^{\int_0^t \lambda(\tau)d\tau} F(z)$, for $t \geq 0$ and $z \in \mathbb{B}^n$. Then $(f_t)_{t\geq 0}$ is an L^d-Loewner chain on $\mathbb{B}^n$.

A connection between L^d-evolution families and L^d-Loewner chains is given by the following theorem (see [10]; cf. [17], [60]).

Theorem 1.2.52. *Let $d \in [1,\infty]$. If $(f_t)_{t\geq 0}$ is an L^d-Loewner chain on $\mathbb{B}^n$ and $(\varphi_{s,t})_{0\leq s\leq t}$ is the family given by $\varphi_{s,t} = f_t^{-1} \circ f_s$, $0 \leq s \leq t$, then $(\varphi_{s,t})_{0\leq s\leq t}$ is an L^d-evolution family on $\mathbb{B}^n$. Conversely, if $(\varphi_{s,t})_{0\leq s\leq t}$ is an L^d-evolution family on $\mathbb{B}^n$ and $(f_t)_{t\geq 0}$ is a family of univalent mappings from $\mathbb{B}^n$ to $\mathbb{C}^n$ such that $\varphi_{s,t} = f_t^{-1} \circ f_s$, $0 \leq s \leq t$, then $(f_t)_{t\geq 0}$ is an L^d-Loewner chain.*

We say that an L^d-Loewner chain $(f_t)_{t\geq 0}$ and an L^d-evolution family $(\varphi_{s,t})_{0\leq s\leq t}$ are associated if $f_s = f_t \circ \varphi_{s,t}$, $0 \leq s \leq t$ (see [8], [60]).

Now, we present the definition of L^d-Herglotz vector fields (or Herglotz vector fields of order d) on $\mathbb{B}^n$ (see [17]; cf. [7], [8], [10]).

Definition 1.2.53. Let $d \in [1,\infty]$. A mapping $G : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is called an L^d-Herglotz vector field on $\mathbb{B}^n$ if:

(*i*) $G(z,\cdot)$ is measurable, for all $z \in \mathbb{B}^n$,

(*ii*) $G(\cdot,t)$ is holomorphic on $\mathbb{B}^n$, for a.e. $t \geq 0$,

(*iii*) for every compact set $K \subset \mathbb{B}^n$ and $T > 0$, there exists $C_{K,T} \in L^d([0,T])$ such that

$$\|G(z,t)\| \leq C_{K,T}(t), \quad \text{for all } z \in K, \text{ for a.e. } t \in [0,T],$$

(*iv*) for almost every fixed $t \geq 0$, $G(\cdot,t)$ is an infinitesimal generator.

A mapping $F \in H(\mathbb{B}^n)$ is an infinitesimal generator if the Cauchy problem

$$\begin{cases} \dfrac{dx}{d\tau}(\tau) = F(x(\tau)), & \text{a.e. } \tau \in [0,\infty) \\ x(0) = x_0 \end{cases}$$

has a locally absolutely continuous solution $x : [0,\infty) \to \mathbb{B}^n$, for all $x_0 \in \mathbb{B}^n$ (see e.g. [86]).

The connection between L^d-evolution families and L^d-Herglotz vector fields is given by the following theorem (see [17]; cf. [7], [10], [60]).

Theorem 1.2.54. *Let $d \in [1,\infty]$. Then, for every L^d-Herglotz vector field G on $\mathbb{B}^n$, there exists a unique L^d-evolution family $(\varphi_{s,t})_{0\leq s\leq t}$ on $\mathbb{B}^n$ such that, for all $s \geq 0$, there exists a null set $E_s \subset [s,\infty)$ such that*

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z),t), \quad z \in \mathbb{B}^n,\ t \in [s,\infty) \setminus E_s. \tag{1.2.9}$$

Conversely, for every L^d-evolution family $(\varphi_{s,t})_{0\leq s\leq t}$ on $\mathbb{B}^n$, there exists an L^d-Herglotz vector field G on $\mathbb{B}^n$ such that (1.2.9) *is satisfied. Moreover, if H is another L^d-Herglotz vector field G on $\mathbb{B}^n$ such that* (1.2.9) *associated to H is satisfied, then $G(\cdot,t) = H(\cdot,t)$, for a.e. $t \geq 0$.*

The equation (1.2.9) is called the (*generalized*) *Loewner ODE* (see [8], [10]). Theorem 1.2.52 shows that there is an one-to-one correspondence between L^d-evolution families and L^d-Herglotz vector fields, via the generalized Loewner ODE. We say that an L^d-evolution family $(\varphi_{s,t})_{0\leq s\leq t}$ and an L^d-Herglotz vector field G are associated if $(\varphi_{s,t})_{0\leq s\leq t}$ is obtained by solving the Loewner ODE (1.2.9) associated to G (see [8], [60]).

The connection between L^d-Loewner chains and L^d-Herglotz vector fields is given by the following theorem (see [8]; cf. [10], [60]).

Theorem 1.2.55. *Let $d \in [1,\infty]$. If G is an L^d-Herglotz vector field on $\mathbb{B}^n$ and $(f_t)_{t\geq 0}$ is a family of univalent mappings in $H(\mathbb{B}^n)$ such that $t \mapsto f_t(z)$ is locally absolutely continuous, locally uniformly with respect to $z \in \mathbb{B}^n$, and there is a null set $E \subseteq [0,\infty)$ such that*

$$\frac{\partial f_t}{\partial t}(z) = -Df_t(z)G(f_t(z),t), \quad z \in \mathbb{B}^n,\ t \in [0,\infty) \setminus E, \tag{1.2.10}$$

then $(f_t)_{t\geq 0}$ is an L^d-Loewner chain on $\mathbb{B}^n$. Moreover, if $(\varphi_{s,t})_{0\leq s\leq t}$ is the L^d-evolution family on $\mathbb{B}^n$ associated to G, then $(f_t)_{t\geq 0}$ and $(\varphi_{s,t})_{0\leq s\leq t}$ are associated.

Conversely, if $(f_t)_{t\geq 0}$ is an L^d-Loewner chain on $\mathbb{B}^n$, then there exists an L^d-Herglotz vector field G such that (1.2.10) *holds. Moreover, if $(\varphi_{s,t})_{0\leq s\leq t}$ is the L^d-evolution family on $\mathbb{B}^n$ associated to $(f_t)_{t\geq 0}$, then $(\varphi_{s,t})_{0\leq s\leq t}$ and G are associated.*

The differential equation (1.2.10) is called the (*generalized*) *Loewner PDE* (see [8], [10]).

Before we present the following result, we recall the definition of Runge pairs and Runge domains in $\mathbb{C}^n$ (see e.g. [85]).

Definition 1.2.56. Let $D_1 \subseteq D_2 \subseteq \mathbb{C}^n$ be two domains. Then (D_1, D_2) is called a Runge pair if $\mathcal{O}(D_2)$ is dense in $\mathcal{O}(D_1)$, with respect to the local uniform convergence, where $\mathcal{O}(D_j)$ is the family of holomorphic functions on D_j into $\mathbb{C}$, for $j = 1, 2$. A domain $D \subseteq \mathbb{C}^n$ is called Runge if $(D, \mathbb{C}^n)$ is a Runge pair.

In the following, we present the main result of Arosio, Bracci and Wold [10] (cf. [8]), which yields that every generalized Loewner PDE has a solution that is an L^d-Loewner chain (cf. Theorem 1.2.20 and [4], [57], [97]). Moreover, they pointed out important properties regarding the Loewner range.

Theorem 1.2.57. (*see* [*10*]) *Let $d \in [1,\infty]$ and G be an L^d-Herglotz vector field on $\mathbb{B}^n$. Then there exists an L^d-Loewner chain $(f_t)_{t\geq 0}$ on $\mathbb{B}^n$ that satisfies the Loewner PDE* (1.2.10) *associated to G. Moreover, $R(f_t)$ is a Runge domain in $\mathbb{C}^n$ and for every family $(g_t)_{t\geq 0}$ in $H(\mathbb{B}^n)$ such that $t \mapsto g_t(z)$ is locally absolutely continuous, locally uniformly with respect to $z \in \mathbb{B}^n$, and that satisfies the Loewner PDE* (1.2.10) *associated to G, there exists a holomorphic mapping $\Phi : R(f_t) \to \mathbb{C}^n$ such that $g_t = \Phi \circ f_t$, $t \geq 0$. Furthermore, $(g_t)_{t\geq 0}$ is an L^d-Loewner chain on $\mathbb{B}^n$ that satisfies the Loewner PDE* (1.2.10) *associated to G if and only if there exists a univalent mapping $\Phi : R(f_t) \to \mathbb{C}^n$ such that $g_t = \Phi \circ f_t$, $t \geq 0$.*

We mention that $R(f_t)$ is also a Stein domain, in Theorem 1.2.57, in view of [10]. For the definition and various characterizations of the Stein domains, one may consult [85].

In view of Theorems 1.2.52, 1.2.55 and 1.2.57, we have the following remark (see [8], [10]).

Remark 1.2.58. Let $d \in [1,\infty]$. Then, for every L^d-evolution family $(\varphi_{s,t})_{0\leq s\leq t}$ on $\mathbb{B}^n$, there exists an L^d-Loewner chain $(f_t)_{t\geq 0}$ on $\mathbb{B}^n$ associated to $(\varphi_{s,t})_{0\leq s\leq t}$. Moreover, if $(g_t)_{t\geq 0}$ is an other L^d-Loewner chain on $\mathbb{B}^n$ associated to $(\varphi_{s,t})_{0\leq s\leq t}$, then there exists a univalent mapping $\Phi : R(f_t) \to \mathbb{C}^n$ such that $g_t = \Phi \circ f_t$, $t \geq 0$.

In connection with Theorem 1.2.57, we have the following result due to Arosio, Bracci and Wold [10] (cf. [94]).

Theorem 1.2.59. *Let $d \in [1,\infty]$.*

(i) If $(\varphi_{s,t})_{0\leq s\leq t}$ is an L^d-evolution family on $\mathbb{B}^n$, then $\varphi_{s,t}(\mathbb{B}^n)$ is a Runge domain, for all $0 \leq s \leq t$.

(ii) If $(f_t)_{t\geq 0}$ is an L^d-Loewner chain on $\mathbb{B}^n$, then $\big(f_s(\mathbb{B}^n), f_t(\mathbb{B}^n)\big)$ is a Runge pair, for all $0 \leq s \leq t$, and $\big(f_s(\mathbb{B}^n), R(f_t)\big)$ is a Runge pair, for all $s \geq 0$.

We close this subsection with the following remark (see [8], [10], [60]). We mention that, in view of this remark, Theorem 1.2.59 has been very useful in the study of some extremal properties associated to univalent subordination chains (see Theorems 1.2.29 and 1.2.45; see also [93], [48], [49]). Other applications will be considered in the forthcoming chapters.

Remark 1.2.60. Taking into account the definitions presented in Subsection 1.2.2, every Loewner chain in the usual sense is an L^d-Loewner chain on $\mathbb{B}^n$, every transition mapping is an L^d-evolution family on $\mathbb{B}^n$ and every Herglotz vector field in the usual sense is an L^d-Herglotz vector field on $\mathbb{B}^n$, for all $d \in [1, \infty]$. Hence, all results in this subsection are valid for the notions introduced in Subsection 1.2.2. Moreover, the same is true for the corresponding notions introduced in Subsection 1.2.4.

Chapter 2

Some applications of variation of Loewner chains in $\mathbb{C}^n$

In this chapter, we consider the variational method developed by Bracci, Graham, Hamada and Kohr [18], which provides a way to construct Loewner chains, by means of variations of certain Loewner chains. Then, we obtain applications of this method, regarding some families of normalized univalent mappings on the unit ball. We present a priori these families in terms of Loewner chains with range $\mathbb{C}^n$. For the first application, we prove a topological property of the family $S^0(\mathbb{B}^n)$ of mappings with parametric representation on the unit ball $\mathbb{B}^n$. This result immediately implies, for $n \geq 2$, a main result suggested by Schleißinger [94], namely, the density of the automorphisms of $\mathbb{C}^n$ that have parametric representation on $\mathbb{B}^n$. Next, we give a partial answer to a question of Arosio, Bracci and Wold [9], by proving that every normalized univalent mapping on $\mathbb{B}^n$ whose image is Runge and which is C^1 up to the boundary embeds into a Loewner chain with range $\mathbb{C}^n$.

We mention that this chapter contains original results obtained by the author in [62], which are given in Sections 2.2 and 2.3.

2.1 Families of univalent mappings and variation of Loewner chains in $\mathbb{C}^n$

In this section we prepare the ground for our forthcoming results. We present some families of normalized univalent mappings on $\mathbb{B}^n$ (see [9], [94]), and then the variational method for Loewner chains in $\mathbb{C}^n$ due to Bracci, Graham, Hamada, Kohr [18].

2.1.1 Families of normalized univalent mappings on the unit ball

In this subsection, we consider some families of univalent mappings on $\mathbb{B}^n$ that embed into Loewner chains with range $\mathbb{C}^n$ (see [9], [94]), which are involved in our forthcoming results.

In view of Subsection 1.2.5, we establish some notations for this chapter. We say that a family $(f_t)_{t\geq 0}$ of mappings in $H(\mathbb{B}^n)$ is a Loewner chain if there is a Loewner chain $f : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ (see Definition 1.2.10) such that $f(\cdot, t) = f_t$, for all $t \geq 0$. Moreover, if, in addition, f is a normal Loewner chain, then we say that $(f_t)_{t\geq 0}$ is a normal Loewner chain. Also, we recall that the range of a Loewner chain $(f_t)_{t\geq 0}$ is denoted by $R(f_t)$ and is given by $R(f_t) = \bigcup_{t\geq 0} f_t(\mathbb{B}^n)$.

We say that a mapping $f \in S(\mathbb{B}^n)$ embeds into a Loewner chain $(f_t)_{t\geq 0}$ if $f_0 = f$.

By Theorem 1.2.24, we have (see [41, 50]):

$$S^0(\mathbb{B}^n) = \left\{ f \in S(\mathbb{B}^n) \,\middle|\, f \text{ embeds into a normal Loewner chain } (f_t)_{t\geq 0} \right\}.$$

In this chapter, we consider the following families of normalized univalent mappings on $\mathbb{B}^n$ (see [9]; cf. [94]):

$$S^1(\mathbb{B}^n) := \big\{ f \in S(\mathbb{B}^n) \big| \, f \text{ embeds into a Loewner chain } (f_t)_{t\geq 0} \text{ with } R(f_t) = \mathbb{C}^n \big\}$$

and

$$S_R(\mathbb{B}^n) := \big\{ f \in S(\mathbb{B}^n) \big| \, f(\mathbb{B}^n) \text{ is Runge} \big\}.$$

For the definition and basic properties of the Runge domains, one can consult [85, Chapter VI, Section 1.4] (cf. Definition 1.2.56).

Further, we also consider the following families (see [9]; cf. [94]):

$$Aut_0(\mathbb{C}^n) := \big\{ \Phi : \mathbb{C}^n \to \mathbb{C}^n \big| \, \Phi \text{ is an automorphism of } \mathbb{C}^n \text{ with } \Phi(0) = 0, D\Phi(0) = I_n \big\},$$

$$\mathcal{A}(\mathbb{B}^n) := \big\{ \varphi : \mathbb{B}^n \to \mathbb{C}^n \big| \, \varphi = \Phi\big|_{\mathbb{B}^n}, \text{ where } \Phi \in Aut_0(\mathbb{C}^n) \big\}$$

and

$$\mathcal{A}^0(\mathbb{B}^n) := \mathcal{A}(\mathbb{B}^n) \cap S^0(\mathbb{B}^n).$$

We mention that examples of mappings in $\mathcal{A}^0(\mathbb{B}^n)$ can be found in [50, Problems 6.2.1–2] (see also Example 1.2.33).

If $n = 1$, then we have $\mathcal{A}^0(\mathbb{U}) = \mathcal{A}(\mathbb{U}) = \{\mathrm{id}_{\mathbb{U}}\}$ and $S^0(\mathbb{U}) = S^1(\mathbb{U}) = S(\mathbb{U}) = S_R(\mathbb{U})$ (see Subsection 1.1.4 and [79, Section 6.1]). The last equality is a consequence of the classical result that a domain in $\mathbb{C}$ is Runge if and only if it is simply connected (see e.g. [85, Chapter VI, Section 1.4])

If $n \geq 2$, then, in view of Remark 1.2.17, we have that $S^0(\mathbb{B}^n) \subsetneq S^1(\mathbb{B}^n)$ (this inclusion is strict, by [50, Example 8.3.12]).

By Theorem 1.2.21, we can deduce that (see [52]):

$$S^1(\mathbb{B}^n) = \big\{ f \in S(\mathbb{B}^n) \big| \, f = \Phi \circ g, \text{ where } g \in S^0(\mathbb{B}^n) \text{ and } \Phi \in Aut_0(\mathbb{C}^n) \big\}. \tag{2.1.1}$$

As mentioned in the introduction of [9], by [10, Section 4] for $n \geq 2$ we have (cf. [94] and Theorem 1.2.59):

$$S^1(\mathbb{B}^n) \subset S_R(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n).$$

The last inclusion is strict by [9, Example 2.2].

Taking into account the Andersén-Lempert theorem (see [2, Theorem 2.1]), for $n \geq 2$ we have (see [9, Proposition 3.2, Corollary 3.3] and [94, Theorem 2.5.6]):

$$S_R(\mathbb{B}^n) = \overline{\mathcal{A}(\mathbb{B}^n)}.$$

Now, we can observe that for $n \geq 2$ we have (see [9], [94]):

$$\mathcal{A}(\mathbb{B}^n) \subsetneq S^1(\mathbb{B}^n) \subset S_R(\mathbb{B}^n) = \overline{\mathcal{A}(\mathbb{B}^n)} \subsetneq S(\mathbb{B}^n). \tag{2.1.2}$$

We mention that for $n \geq 2$ Schleißinger proved in [93, Corollary 2.4] (see also [94, Corollarry 2.6.10]) that

$$S^0(\mathbb{B}^n) \subsetneq \overline{\mathcal{A}(\mathbb{B}^n)}. \tag{2.1.3}$$

2.1.2 Variation of Loewner chains in $\mathbb{C}^n$

In this subsection, we present the variational method developed by Bracci, Graham, Hamada and Kohr [18]. The authors used this method to study extreme and support points of the family $S^0(\mathbb{B}^n)$. We mention that a related variational method was developed by Roth [92] to obtain a Pontryagin maximum principle for the Loewner differential equation and to study nonlinear extremal problems associated to $S^0(\mathbb{B}^n)$. In the forthcoming sections, we give some new applications (see [62]) of the variational method from [18], but in the context presented in the previous section.

In the following, we recall some definitions, notations and results from [18].

Definition 2.1.1. (see [18, Proposition 2.9 and Definition 2.15])

A Loewner chain $(f_t)_{t\geq 0}$ is geräumig in $[0,T)$, for some $T>0$, if there exists $a,b>0$ and $c\in(0,1]$ such that

(1) for all $t\in[0,T)$ and all $z\in\mathbb{B}^n$, $\mu(Df_t(z)):=\min_{\|v\|=1}\|Df_t(z)v\|\geq a$,

(2) for a.e. $t\in[0,T)$ and all $z\in\mathbb{B}^n$, $\left\|\frac{\partial f_t}{\partial t}(z)\right\|\leq b$,

(3) for a.e. $t\in[0,T)$ and all $z\in\mathbb{B}^n$, $\Re\left\langle\big(Df_t(z)\big)^{-1}\frac{\partial f_t}{\partial t}(z),z\right\rangle\geq c\|z\|^2$.

We mention that, by [18, Lemma 2.14], we have $\mu(A)=\frac{1}{\|A^{-1}\|}$, for all invertible linear operators $A:\mathbb{C}^n\to\mathbb{C}^n$.

The proofs of our main results in the next sections heavily rely on the following result of Bracci, Graham, Hamada and Kohr [18, Theorem 3.1].

Theorem 2.1.2. *Assume that $(f_t)_{t\geq 0}$ is a Loewner chain, respectively a normal Loewner chain. If $(f_t)_{t\geq 0}$ is geräumig in $[0,T)$, for some $T>0$, then there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0]$, setting*

$$\alpha(t):=\begin{cases}\varepsilon\Big(1-\frac{t}{T}\Big), & t\in[0,T)\\ 0, & t\in[T,\infty)\end{cases},$$

the family $\big(f_t+\alpha(t)h\big)_{t\geq 0}$ is a Loewner chain, respectively a normal Loewner chain, for every $h:\mathbb{B}^n\to\mathbb{C}^n$ holomorphic with $h(0)=0$, $Dh(0)=0$, $\sup_{z\in\mathbb{B}^n}\|h(z)\|\leq 1$ and $\sup_{z\in\mathbb{B}^n}\|Dh(z)\|\leq 1$.

Throughout this chapter, we consider the following family of normalized holomorphic mappings on $\mathbb{B}^n$:

$$H_0(\mathbb{B}^n):=\big\{f\in H(\mathbb{B}^n)\big|\, f(0)=0 \text{ and } Df(0)=I_n\big\}.$$

For any $g\in H_0(\mathbb{B}^n)$ and $r\in(0,1)$ we denote by g_r the mapping which satisfies: $g_r(z)=\frac{1}{r}g(rz)$, for all $z\in\mathbb{B}^n$. If the mapping has some index, e.g. g_α, then we denote by $g_{r,\alpha}$ the corresponding re-scaled mapping.

Following [18, Proposition 4.5], we have a lemma which plays an important role in the proofs of the main results of this chapter (see [62]).

Lemma 2.1.3. *Let $n\in\mathbb{N}^*$, $g\in S^0(\mathbb{B}^n)$ and $(g_t)_{t\geq 0}$ be a normal Loewner chain into which g embeds. Then*

i) for every $r\in(0,1)$, $(g_{r,t})_{t\geq 0}$ is a normal Loewner chain which is geräumig in $[0,T)$, for any $T>0$; in particular, $g_r\in S^0(\mathbb{B}^n)$;

ii) for every $r\in(0,1)$ and for every univalent mapping $\phi:\mathbb{C}^n\to\mathbb{C}^n$ with $\phi(0)=0$ and $D\phi(0)=I_n$, $\big(\phi\circ g_{r,t}\big)_{t\geq 0}$ is a Loewner chain which is geräumig in $[0,T)$, for any $T>0$; in particular, $\phi\circ g_r\in S^1(\mathbb{B}^n)$, for $\phi\in Aut_0(\mathbb{C}^n)$.

2.2 A density result

In view of (2.1.3), we consider the following question due to Schleißinger [94, Question 2.6.11]: is it true that $\overline{\mathcal{A}^0(\mathbb{B}^n)}=S^0(\mathbb{B}^n)$ for $n\geq 2$?

Our first result is that (roughly speaking) $S^0(\mathbb{B}^n)$ is "absorbing" in $H_0(\mathbb{B}^n)$ in the following sense: if a sequence in $H_0(\mathbb{B}^n)$ converges to a mapping in $S^0(\mathbb{B}^n)$, locally uniformly on $\mathbb{B}^n$, then there exists a subsequence which re-scaled in a natural prescribed way is in $S^0(\mathbb{B}^n)$ and still converges to the same mapping (cf. [50, Theorem 6.1.18]). As a consequence of this result we get a simple proof of the fact that $\overline{\mathcal{A}^0(\mathbb{B}^n)}=S^0(\mathbb{B}^n)$ for $n\geq 2$. The same "absorbing" property holds also for $S^1(\mathbb{B}^n)$. This result was obtained in [62].

Theorem 2.2.1. *Let $n\in\mathbb{N}^*$, $g\in S^0(\mathbb{B}^n)$ and $(g_j)_{j\in\mathbb{N}}$ be a sequence in $H_0(\mathbb{B}^n)$ which converges, locally uniformly on $\mathbb{B}^n$, to g. Then for every sequence $(r_k)_{k\in\mathbb{N}}$ in $(0,1)$ convergent to 1 there exists a subsequence of indexes $(j_k)_{k\in\mathbb{N}}$ such that $g_{r_k,j_k}\in S^0(\mathbb{B}^n)$, for all $k\in\mathbb{N}$, and $\big(g_{r_k,j_k}\big)_{k\in\mathbb{N}}$ converges to g, locally uniformly on $\mathbb{B}^n$. The same property holds for $S^1(\mathbb{B}^n)$.*

Schleißinger proved that $S^0(\mathbb{B}^n) \subset \overline{\mathcal{A}(\mathbb{B}^n)}$, for $n \geq 2$ (see [93, Corollary 2.4]; cf. [94, Corollary 2.6.10]), then suggested that we may have $S^0(\mathbb{B}^n) = \overline{\mathcal{A}^0(\mathbb{B}^n)}$, for $n \geq 2$. In the following, we prove that, indeed, this is true (see [62]).

Theorem 2.2.2. *If $n \in \mathbb{N}$ is such that $n \geq 2$, then $S^0(\mathbb{B}^n) = \overline{\mathcal{A}^0(\mathbb{B}^n)}$.*

2.3 On an embedding problem

Arosio, Bracci and Wold [9] have been interested in the following question: is $S^1(\mathbb{B}^n) = S_R(\mathbb{B}^n)$? Our main result in this section gives a partial answer to this question, namely, we prove that every mapping in $S_R(\mathbb{B}^n)$ which is of class C^1 up to the boundary is in $S^1(\mathbb{B}^n)$ (see [62]).

We denote:

$$C^1(\overline{\mathbb{B}^n}) := \left\{ f \in C^1(\mathbb{B}^n) \middle| f \text{ and } df \text{ extend continuously to } \overline{\mathbb{B}^n} \right\}.$$

The following main result was obtained in [62]. This result improves a recent result due to Arosio, Bracci and Wold [9] (see Remark 2.3.2).

Theorem 2.3.1. *$S_R(\mathbb{B}^n) \cap C^1(\overline{\mathbb{B}^n}) \subset S^1(\mathbb{B}^n)$, for all $n \in \mathbb{N}^*$.*

Remark 2.3.2. (see [62]) If $f \in S_R(\mathbb{B}^n)$ and $f(\mathbb{B}^n)$ is a bounded strongly pseudoconvex domain with C^∞ boundary, then, in view of Fefferman's Mapping Theorem, $f \in C^\infty(\overline{\mathbb{B}^n})$, hence, by Theorem 2.3.1, we have that $f \in S^1(\mathbb{B}^n)$. So Theorem 2.3.1 generalizes, in some sense, the result contained in [9, Theorem 1.2].

Chapter 3

On reachable families of the Loewner differential equation in $\mathbb{C}^n$

The study of extremal problems on compact families of univalent functions with parametric representation on the unit disc has motivated the development of a control–theoretic approach, considered by Goodman [40], Prokhorov [84], Roth [89, 90], and others. A first study of some extremal problems associated to parametric representations in several complex variables has been given by Graham, Kohr and Pfaltzgraff [53] and a generalization of the control–theoretic approach has been given by Graham, Hamada, Kohr and Kohr [46, 48] (see also [18]). We mention that Roth obtained a Pontryagin maximum principle for the Loewner differential equation in [92] (cf. [91], for dimension one).

In this chapter, we consider the notion of reachable family of the Loewner differential equation and related results due to Graham, Hamada, Kohr and Kohr [48]. Then, we prove some results conjectured by the previously mentioned authors. We note that an important source of control-theoretic results and ideas in this direction is the work in one complex variable of Roth [89, 90]. Our first main result is the proof of [48, Conjecture 4.16], which yields that the family of mappings which have an A–parametric representation on $\mathbb{B}^n$ obtained by solving the Loewner differential equation associated to Herglotz vector fields which take values in $\operatorname{ex}\mathcal{N}_A$ (i.e. the set of extreme points of the Carathéodory family $\mathcal{N}_A$) is dense in $S^0_A(\mathbb{B}^n)$ (i.e. the family of all mappings with A–parametric representation on $\mathbb{B}^n$), where $A \in L(\mathbb{C}^n)$ is such that $k_+(A) < 2m(A)$. We point out that this result generalizes Loewner's result [70] (cf. Remark 1.1.31). Our second main result is the proof of [48, Conjecture 4.19], which yields that the normalized time-T-reachable family $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)$ of the Loewner differential equation, which is generated by the Carathéodory mappings with values in a compact and convex subfamily Ω of the Carathéodory family $\mathcal{N}_A$, is compact and the corresponding normalized reachable family $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \operatorname{ex}\Omega)$ is dense in it, where $T \in [0, \infty]$ and $A \in L(\mathbb{C}^n)$ is such that $k_+(A) < 2m(A)$. Since $S^0_A(\mathbb{B}^n)$ is equal to the normalized infinite-time reachable family $\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \mathcal{N}_A)$ and $\mathcal{N}_A$ is a compact and convex family in $H(\mathbb{B}^n)$, we remark that this result generalizes the first one. However, our approach to the second result is different from the first one.

We mention that this chapter contains original results obtained in [63] and [64], which are given in Sections 3.2 and 3.3.

3.1 Reachable families of the Loewner differential equation in $\mathbb{C}^n$

In this section, we consider some control-theoretic definitions and results regarding the Loewner differential equation and the A-parametric representations on $\mathbb{B}^n$, due to Graham, Hamada, Kohr and Kohr [48], which will be useful in the forthcoming sections. We also refer to the characterization in terms of A-normalized univalent subordination chains and to some extremal properties (see [18], [46], [48], [49], [93]).

We begin with the definition of the Carathéodory mappings (see [48, Section 4]; cf. [46], [90]).

Definition 3.1.1. Let $I \subseteq [0, \infty)$ be an interval and $\Omega \subseteq H(\mathbb{B}^n)$. A mapping $h : \mathbb{B}^n \times I \to \mathbb{C}^n$ is called a Carathéodory mapping on I with values in Ω if:

(i) $h(\cdot, t) \in \Omega$, for all $t \in I$,

(ii) $h(z, \cdot)$ is measurable on I, for all $z \in \mathbb{B}^n$.

We denote by $\mathcal{C}(I, \Omega)$ the family of Carathéodory mappings on I with values in Ω. We say that the Carathéodory mappings on I with values in Ω represent the *controls* of the *control system* $\mathcal{C}(I, \Omega)$ and Ω represents the *input family.*

Next, we consider the solution of the Loewner differential equation associated to a Carathéodory mapping (see [48, Section 4]; cf. [46], [90]).

Definition 3.1.2. Let $T \in [0, \infty]$ and $A \in L(\mathbb{C}^n)$ with $m(A) \geq 0$. Let I be either the interval $[0, T]$, if $T \in [0, \infty)$, or the interval $[0, \infty)$, if $T = \infty$. For every $h \in \mathcal{C}(I, \mathcal{N}_A)$, we denote by $v(\cdot, \cdot; h) : \mathbb{B}^n \times I \to \mathbb{B}^n$ the unique locally absolutely continuous solution on I of the initial value problem

$$\text{(3.1.1)} \qquad \begin{cases} \dfrac{\partial v}{\partial t}(z, t; h) = -h(v(z, t; h), t), & \text{for a.e. } t \in I, \\ v(z, 0; h) = z, \end{cases}$$

for all $z \in \mathbb{B}^n$.

Note that $v(\cdot, t; h)$ is a univalent Schwarz mapping with $Dv(0, t; h) = e^{-tA}$, for all $t \in I$, and $v(z, \cdot; h)$ is Lipschitz continuous on I, locally uniformly with respect to $z \in \mathbb{B}^n$ (see [44]).

Following [48], we consider the notation

$$\mathscr{A} = \big\{ A \in L(\mathbb{C}^n) \big| \, k_+(A) < 2m(A) \big\}.$$

We note that, if $A \in \mathscr{A}$ and $h \in \mathcal{C}([0, \infty), \mathcal{N}_A)$, then the limit $f := \lim_{t \to \infty} e^{tA} v(\cdot, t; h)$ exists locally uniformly on $\mathbb{B}^n$ and $f \in S_A^0(\mathbb{B}^n)$ (see [44]; see also Subsection 1.2.4).

Now, we present the notion of reachable family of the Loewner differential equation in $\mathbb{C}^n$, with respect to an operator $A \in \mathscr{A}$ (see [48, Section 4]; cf. [46], [90]).

Definition 3.1.3. For every $A \in \mathscr{A}$, $\Omega \subseteq \mathcal{N}_A$ and $T \in [0, \infty)$ we denote by

$$\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega) := \Big\{ e^{TA} v(\cdot, T; h) \Big| \, h \in \mathcal{C}([0, T], \Omega) \Big\}$$

the *normalized time-T-reachable family* of the control system $\mathcal{C}([0, T], \Omega)$ and by

$$\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \Omega) := \Big\{ \lim_{t \to \infty} e^{tA} v(\cdot, t; h) \Big| \, h \in \mathcal{C}([0, \infty), \Omega) \Big\}$$

the *normalized infinite-time-reachable family* of the control system $\mathcal{C}([0, \infty), \Omega)$.

We easily observe that, for every $A \in \mathscr{A}$, $\Omega \subseteq \mathcal{N}_A$ and $T \in [0,\infty]$, we have $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\Omega) \subseteq \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\mathcal{N}_A)$ and $\widetilde{\mathcal{R}}_0(id_{\mathbb{B}^n},\Omega) = \{\mathrm{id}_{\mathbb{B}^n}\}$.

A simple example of a mapping in a normalized finite time reachable family can be provided by considering a spirallike mapping on $\mathbb{B}^n$ (see [48, Section 4]).

Example 3.1.4. Let $A \in \mathscr{A}$ and $T \in [0,\infty)$. Also, let $F \in \widehat{S}_A(\mathbb{B}^n)$. Then the mapping $F_A^T : \mathbb{B}^n \to \mathbb{C}^n$ given by $F_A^T(z) = e^{TA}F^{-1}\big(e^{-TA}F(z)\big)$, $z \in \mathbb{B}^n$, is in $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\mathcal{N}_A)$.

In the following, we see the characterization of the normalized reachable families $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\mathcal{N}_A)$ in terms of A-normalized univalent subordination chains, where $A \in \mathscr{A}$ and $T \in [0,\infty)$, due to Graham, Hamada, Kohr and Kohr [48, Section 4] (cf. [46], [90]).

Theorem 3.1.5. *Let $A \in \mathscr{A}$ and $T \in [0,\infty)$. Also, let $f : \mathbb{B}^n \to \mathbb{C}^n$. Then $f \in \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\mathcal{N}_A)$ if and only if there exits an A-normalized univalent subordination chain $F : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ such that $F(\cdot,0) = f$, $F(\cdot,T) = e^{TA}\mathrm{id}_{\mathbb{B}^n}$ and $\{e^{tA}F(\cdot,t)\}_{t\geq 0}$ is a normal family in $H(\mathbb{B}^n)$.*

In connection with Definition 1.1.32, for every $A \in \mathscr{A}$ and $M \in [1,\infty)$, we denote (see [48]; cf. Definition 1.1.32 and [46], [84], [90]):

$$S_A^0(M,\mathbb{B}^n) := \{f \in S_A^0(\mathbb{B}^n) \big| \, \|f(z)\| < M,\, z \in \mathbb{B}^n\}.$$

If $A = I_n$, then we denote $S^0(M,\mathbb{B}^n) := S_{I_n}^0(M,\mathbb{B}^n)$, $M \in [1,\infty)$. If $n = 1$, then we have $S^0(M,\mathbb{U}) = S(M)$, $M \in [1,\infty)$ (see Definition 1.1.32).

Next, we observe that every mapping in a normalized reachable family has A-parametric representation, when $A \in \mathscr{A}$ (see [48]). Moreover, we point out some extremal properties and an important difference between the one dimension case and the higher dimensions case (see [18], [46], [48], [49], [93]; cf. [84], [90]).

Remark 3.1.6.

(i) Let $A \in \mathscr{A}$. Then $\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n},\mathcal{N}_A) = S_A^0(\mathbb{B}^n)$ and $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\mathcal{N}_A) \subset S_A^0(\|e^{TA}\|,\mathbb{B}^n)$, for all $T \in [0,\infty)$ (see [48]). Moreover, $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\mathcal{N}_A) \cap \Big(\mathrm{ex}\, S_A^0(\mathbb{B}^n) \cup \mathrm{supp}\, S_A^0(\mathbb{B}^n)\Big) = \emptyset$, for all $T \in [0,\infty)$ (see [46], [48], [49], [93]).

(ii) For $n = 1$, we have $\widetilde{\mathcal{R}}_{\log M}(id_{\mathbb{U}},\mathcal{M}) = S(M)$, for all $M \in [1,\infty)$ (see Theorems 1.1.36 and 1.1.37). For $n \geq 2$, we have $\widetilde{\mathcal{R}}_{\log M}(id_{\mathbb{B}^n},\mathcal{M}) \subsetneq S^0(M,\mathbb{B}^n)$, for all $M \in [1,\infty)$ (see [18]; cf. Example 1.2.33).

The following growth theorem for normalized finite time reachable families, in terms of the Pick functions given in Example 1.1.34, is due to Graham, Hamada, Kohr and Kohr [48] (cf. Theorem 1.1.38).

Theorem 3.1.7. *Let $A \in \mathscr{A}$, $T \in [0,\infty)$ and $a = e^{k(A)T}$, $b = e^{m(A)T}$. Also, let $f \in \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\mathcal{N}_A)$. Then*

$$\frac{b}{a}p_\pi^a(\|z\|) \leq \|f(z)\| \leq \frac{a}{b}p_0^b(\|z\|), \quad z \in \mathbb{B}^n.$$

Also, the authors in [48] proved the compactness of the following finite time normalized reachable families (cf. Theorem 1.1.35).

Theorem 3.1.8. *Let $A \in \mathscr{A}$ and $T \in [0,\infty)$. Then $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},\mathcal{N}_A)$ is a compact set in $H(\mathbb{B}^n)$.*

Since these finite time normalized reachable families are compact, one can consider various associated extremal problems. For such results, we refer to the work in [48, 49] (see also [18], for the case $A = I_n$). In the last chapter, we obtain some generalizations of these results to the case of time-dependent linear operators in $\mathbb{C}^n$.

3.2 A density result for A-parametric representations in $\mathbb{C}^n$

Our goal in this section is to prove a generalization of Loewner's result [70], in the setting of several complex variables, which was conjectured by Graham, Hamada, Kohr and Kohr [48], in view of some of the work of Roth [89, 90].

To be more precise, we obtain a control theoretical proof of the n-dimensional generalization of the well known approximation property for the class S, namely every function in S can be approximated locally uniformly on $\mathbb{U}$ by a sequence of single-slit mappings (see also Remark 1.1.31; cf. [89]). Because of the absence of an adequate and analogue of the notion of single–slit function in higher dimensions, the control–theoretic approach considered in [48] and [90] will play a central role in our proof.

First, we refer to the following density result for normalized finite time reachable families, due to Graham, Hamada, Kohr and Kohr [48, Theorem 4.14].

Theorem 3.2.1. *Let $A \in \mathscr{A}$ and $T \in [0, \infty)$. Then*

$$\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \mathcal{N}_A) = \overline{\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \operatorname{ex}\mathcal{N}_A)}.$$

Let $A \in \mathscr{A}$. In view of Remark 3.1.6 (i), we prove the infinite time version of the previous theorem, conjectured in [48, Conjecture 4.16]:

$$S_A^0(\mathbb{B}^n) = \overline{\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \operatorname{ex}\mathcal{N}_A)}. \tag{3.2.1}$$

We mention that this section is based on original results obtained in [63].

3.2.1 Preliminary results

In this subsection, we present some partial results related to (3.2.1).

First, we consider the following notation (see [63]). Let $T \in [0, \infty]$ and $A \in \mathscr{A}$. Also, let I be either the interval $[0, T]$, if $T \in [0, \infty)$, or the interval $[0, \infty)$, if $T = \infty$. For every $\mathcal{F} \subseteq \mathcal{C}(I, \mathcal{N}_A)$, we denote:

$$\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \mathcal{F}) := \{e^{TA}v(\cdot, T; h) \,|\, h \in \mathcal{F}\},$$

where, if $T = \infty$, $f \in \widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \mathcal{F})$ if and only if $f = \lim_{t\to\infty} e^{tA}v(\cdot, t; h)$, for some $h \in \mathcal{F}$.

We observe that $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \mathcal{F}) \subseteq \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \mathcal{N}_A)$.

The next theorem is related to (3.2.1) and has an important role in the next subsection (see [63]).

Theorem 3.2.2. *Let $A \in \mathscr{A}$ and let*

$$\mathcal{F} = \Big\{h \in \mathcal{C}([0, \infty), \mathcal{N}_A) \Big|\ \text{there exist } T \geq 0 \text{ and } h_0 \in \operatorname{co}(\operatorname{ex}\mathcal{N}_A) \text{ such that}$$
$$h(\cdot, t) \in \operatorname{ex}\mathcal{N}_A,\ \text{for all } 0 \leq t \leq T,\ \text{and } h(\cdot, t) = h_0,\ \text{for all } t > T\Big\}.$$

Then

$$S_A^0(\mathbb{B}^n) = \overline{\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \mathcal{F})}.$$

In particular,

$$S_A^0(\mathbb{B}^n) = \overline{\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \operatorname{co}(\operatorname{ex}\mathcal{N}_A))}.$$

In view of the previous result, we have a remark from [63].

Remark 3.2.3. Let $A \in \mathscr{A}$. In view of the arguments used for previous result, one could simply consider

$$\mathcal{F} = \Big\{h \in \mathcal{C}\big([0, \infty), \big(\operatorname{ex}\mathcal{N}_A\big) \cup \{A\}\big) \Big|\ \text{there exists } T \geq 0 \text{ such that}$$
$$h(\cdot, t) \in \operatorname{ex}\mathcal{N}_A,\ \text{for } 0 \leq t \leq T,\ \text{and } h(\cdot, t) = A,\ \text{for } t > T\Big\}$$

and get $S_A^0(\mathbb{B}^n) = \overline{\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \mathcal{F})}$. In particular, we have $S_A^0(\mathbb{B}^n) = \overline{\widetilde{\mathcal{R}}_\infty\big(id_{\mathbb{B}^n}, \big(\operatorname{ex}\mathcal{N}_A\big) \cup \{A\}\big)}$.

3.2.2 A density result

In this section, we present the main ideas of our proof for (3.2.1), based on [65, Chapter 3, Section 2.1]. The next results are contained in [63].

Let $A \in \mathscr{A}$. For every $t \geq 0$ and $\Omega \subseteq \mathcal{N}_A$, we denote:

$$\Phi_t(\Omega) := \big\{h \in \mathcal{C}([0,\infty),\mathcal{N}_A)\big|h(\cdot,s) \in \mathrm{ex}\,\mathcal{N}_A,\ \text{for all } 0 \leq s \leq t, h(\cdot,s) \in \Omega,$$

$$\text{for all } s > t,\ \text{and } h\big|_{(t,\infty)} \text{ is periodic and piece-wise constant on intervals}\big\}$$

and

$$\Phi^0_t(\Omega) := \big\{h \in \mathcal{C}([0,\infty),\mathcal{N}_A)\big|h(\cdot,s) \in \mathrm{ex}\,\mathcal{N}_A,\ \text{for all } 0 \leq s \leq t, h(\cdot,s) \in \Omega,$$

$$\text{for all } s > t,\ \text{and } h\big|_{(t,\infty)} \text{ is constant }\big\}.$$

In view of the previous subsection and the above notation, we have a remark. Our proof of (3.2.1) is based on it.

Remark 3.2.4. (see [63]) Let $A \in \mathscr{A}$. In view of Theorem 3.2.2, we have

$$S^0_A(\mathbb{B}^n) = \overline{\bigcup_{t\geq 0} \widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \Phi^0_t(\mathrm{co}\,(\mathrm{ex}\,\mathcal{N}_A)))}.$$

If X is a complex (or real) linear space, $\mathcal{F} \subseteq X$ and $k \in \mathbb{N}$, then we denote

$$\mathrm{co}_k(\mathcal{F}) := \big\{\lambda_1 f_1 + \ldots + \lambda_k f_k \big|\, \lambda_1, \ldots, \lambda_k \geq 0 \text{ with } \lambda_1 + \ldots + \lambda_k = 1 \text{ and } f_1, \ldots, f_k \in \mathcal{F}\big\}.$$

The following lemma is pictured in [65, Fig. 3.9, p. 82] (see [63]).

Lemma 3.2.5. *Let X be a complex (or real) linear space. For every $\mathcal{F} \subseteq X$ and $k \in \mathbb{N}$, we have:*

$$\mathrm{co}_{k+1}(\mathcal{F}) \subseteq \mathrm{co}_2(\mathrm{co}_k(\mathcal{F})).$$

The next result was obtained by the author in [63]. Combining this result with the previous lemma, we obtain a proof for (3.2.1), in view of Remark 3.2.4.

Proposition 3.2.6. *Let $A \in \mathscr{A}$. Then*

$$\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \Phi_T(\mathrm{co}_2(\Omega))) \subseteq \overline{\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \Phi_T(\Omega))},$$

for all $T > 0$ and $\Omega \subseteq \mathcal{N}_A$.

Now, we present the main result of this section. This result yields that [48, Conjecture 4.16] is true (see [63]).

Theorem 3.2.7. *Let $A \in \mathscr{A}$. Then*

$$S^0_A(\mathbb{B}^n) = \overline{\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \mathrm{ex}\,\mathcal{N}_A)}.$$

3.3 Compactness and density of certain reachable families

The authors in [48] studied the normalized reachable families $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)$, which are generated by solving the Loewner differential equation associated to the Carathéodory mappings with values in a subfamily Ω of the Carathéodory family $\mathcal{N}_A$, when $T \in [0,\infty)$ and $A \in L(\mathbb{C}^n)$ is such that $k_+(A) < 2m(A)$. They obtained some compactness and density results, as generalizations of related results due to Roth [90], and conjectured that if Ω is compact and convex, then $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)$ is compact and $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, ex\,\Omega)$ is dense in $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)$, for all $T \in [0,\infty]$. In this section, we confirm this, by embedding the Carathéodory mappings in a suitable Bochner space. We remark that the main result in the previous section is a particular case of this result, since $S_A^0(\mathbb{B}^n)$ is equal to the normalized infinite-time reachable family $\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \mathcal{N}_A)$ (see Remark 3.1.6 (i)) and $\mathcal{N}_A$ is compact and convex family in $H(\mathbb{B}^n)$ (see Theorem 1.2.35). However, our proof for this more general result is quite different from the one given before. Our arguments are based on general results from measure theory and functional analysis (see e.g. [32]).

We mention that this section is based on original results obtained in [64].

To be more precise, we recall some results. Graham, Hamada, Kohr and Kohr proved the following (see [48, Corrolary 4.7, Lemma 4.12, Lemma 4.13, Theorem 4.14]):

$$\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \mathcal{N}_A) \text{ is compact, for all } T \in [0,\infty] \text{ and } A \in \mathscr{A},$$

and

$$\overline{\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)} = \overline{\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n} \operatorname{ex} \Omega)},$$

for all $T \in [0,\infty)$, $A \in \mathscr{A}$ and all compact and convex families $\Omega \subseteq \mathcal{N}_A$. In particular, they deduced that

$$\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \mathcal{N}_A) = \overline{\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n} \operatorname{ex} \mathcal{N}_A)}, \text{ for all } T \in [0,\infty) \text{ and } A \in \mathscr{A}.$$

In Section 3.2, we have proved [48, Conjecture 4.16] (see [64]):

$$\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n}, \mathcal{N}_A) = \overline{\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n} \operatorname{ex} \mathcal{N}_A)}, \text{ for all } A \in \mathscr{A}.$$

In this section, we are concerned with [48, Conjecture 4.19] (see also [64]):

Conjecture 3.3.1. $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)$ is compact and $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega) = \overline{\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n} \operatorname{ex} \Omega)}$, for all $T \in [0,\infty]$, $A \in \mathscr{A}$ and all compact and convex families $\Omega \subseteq \mathcal{N}_A$.

To prove the conjecture, we need to work with slightly different *controls* (see [64]; cf. [90, (I.28), p. 47]). Using the new notations, we provide the outline of our proof.

Let $T \in [0,\infty]$ and $A \in \mathscr{A}$ be arbitrary. Let I be either the interval $[0,T]$, if $T \in [0,\infty)$, or the interval $[0,\infty)$, if $T = \infty$. For every $h \in \mathcal{C}(I, \mathcal{N}_A)$, we denote by $\widetilde{h} : \mathbb{B}^n \times I \to \mathbb{C}^n$ that $\widetilde{h}(z,t) := e^{tA}\big(h(e^{-tA}z, t) - Ae^{-tA}z\big)$, for all $z \in \mathbb{B}^n$ and $t \in I$, and by $\widetilde{v}(\cdot,\cdot;\widetilde{h}) : \mathbb{B}^n \times I \to \mathbb{C}^n$ that $\widetilde{v}(z,t;\widetilde{h}) := e^{tA}v(z,t;h)$, for all $z \in \mathbb{B}^n$ and $t \in I$, and we observe that $\widetilde{v}(z,\cdot;\widetilde{h})$ is the unique locally absolutely continuous solution on I of the initial value problem

$$\begin{cases} \dfrac{\partial \widetilde{v}}{\partial t}(z,t;\widetilde{h}) = -\widetilde{h}(\widetilde{v}(z,t;\widetilde{h}),t), & \text{for a.e. } t \in I, \\ \widetilde{v}(z,0;\widetilde{h}) = z, \end{cases} \tag{3.3.1}$$

for all $z \in \mathbb{B}^n$.

For every $\Omega \subseteq \mathcal{N}_A$, we denote $\widetilde{\mathcal{C}}(I,\Omega) := \big\{\widetilde{h} \big| h \in \mathcal{C}(I,\Omega)\big\}$. For every $\widetilde{\mathcal{F}} \subseteq \widetilde{\mathcal{C}}(I,\mathcal{N}_A)$, we denote $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \widetilde{\mathcal{F}}) := \big\{\widetilde{v}(\cdot,T;\widetilde{h}) \big| \widetilde{h} \in \widetilde{\mathcal{F}}\big\}$, where, if $T = \infty$, $\widetilde{v}(\cdot,\infty;\widetilde{h}) := \lim_{t\to\infty} \widetilde{v}(\cdot,t;\widetilde{h})$, $\widetilde{h} \in \widetilde{\mathcal{F}}$.

Let $\Omega \subseteq \mathcal{N}_A$ be arbitrary. We immediately observe that $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega) = \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \widetilde{\mathcal{C}}(I,\Omega))$ (see Definition 3.1.3). The outline of the next subsections is as follows. In Subsection 3.3.1, we choose a Bochner space X such that $\widetilde{\mathcal{C}}(I,\Omega)$ is a weakly relatively compact set in X and

$$cl_X^w \widetilde{\mathcal{C}}(I,\Omega) = \widetilde{\mathcal{C}}(I, \overline{\operatorname{co}\Omega}),$$

where cl_X^w denotes the closure with respect to the weak topology of X. In Subsection 3.3.2, we prove that

$$\overline{\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \widetilde{\mathcal{C}}(I,\Omega))} = \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, cl_X^w \widetilde{\mathcal{C}}(I,\Omega)).$$

Consequently, we obtain a proof of the above conjecture.

We note that all the key-ideas in this section come from [90, Chapter I], especially the proof of [90, Theorem I.50], [48, Section 4] and [63] (see [64]).

3.3.1 On Carathéodory mappings

In this subsection, we consider some useful results regarding the Carathéodory mappings (see [64]).

First, we prepare the ground with some notations and definitions. For any (X, Σ, μ) measurable space with σ-finite measure μ, $(Y, \|\cdot\|_Y)$ Banach space and $p \in (1, \infty)$ we use the notation

$$L^p(X, Y) := \Big\{ f : X \to Y \Big| f \text{ is strongly measurable and } \int_X \|f(s)\|_Y^p \, d\mu(s) < \infty \Big\}.$$

It is well known that, identifying the functions which are equal almost everywhere, this is a Banach space, known as a Bochner space, with respect to the following norm:

$$\|f\|_{L^p(X,Y)} := \Big(\int_X \|f(s)\|_Y^p \, d\mu(s) \Big)^{\frac{1}{p}},$$

for all $f \in L^p(X, Y)$ and $p \in (1, \infty)$.

Let $q \in (1, \infty)$ and $r \in (0, 1)$. We denote

$$\mathcal{A}^q(\mathbb{B}_r^n) := H(\mathbb{B}_r^n) \cap L^q(\mathbb{B}_r^n, \mathbb{C}^n)$$

and we mention that $\mathcal{A}^q(\mathbb{B}_r^n)$ is a reflexive Banach subspace of $L^q(\mathbb{B}_r^n, \mathbb{C}^n)$, with respect to the induced norm $\|\cdot\|_{L^q(\mathbb{B}_r^n, \mathbb{C}^n)}$. $\mathcal{A}^q(\mathbb{B}_r^n)$ is the cartesian nth power of a Bergman space (cf. [85]).

In the following, under the assumption that $A \in \mathscr{A}$, we embed $\widetilde{\mathcal{C}}(I, \Omega)$ in $L^p\big(I, \mathcal{A}^q(\mathbb{B}_r^n)\big)$ and we present some properties (see [64]).

Proposition 3.3.2. *Let $I \subseteq [0, \infty)$ be an interval, $r \in (0, 1)$, $p, q \in (1, \infty)$, $A \in \mathscr{A}$ and $\Omega \subseteq \mathcal{N}_A$. Then*

i) there exists $M > 0$ such that

$$\big\|\widetilde{h}(\cdot, s)\big\|_{\mathbb{B}_r^n} \le M \big\| e^{(A - 2m(A) I_n) s} \big\|,$$

for a.e. $s \in I$ and all $\widetilde{h} \in \widetilde{\mathcal{C}}(I, \mathcal{N}_A)$;

ii) $\widetilde{\mathcal{C}}(I, \Omega)$ is a bounded subset of $L^p(I, \mathcal{A}^q(\mathbb{B}_r^n))$;

iii) if Ω is closed, then $\widetilde{\mathcal{C}}(I, \Omega)$ is closed in $L^p(I, \mathcal{A}^q(\mathbb{B}_r^n))$;

iv) if Ω is convex, then $\widetilde{\mathcal{C}}(I, \Omega)$ is convex;

v) the weak closure of $\widetilde{\mathcal{C}}(I, \Omega)$ in $L^p(I, \mathcal{A}^q(\mathbb{B}_r^n))$ is a subset of $\widetilde{\mathcal{C}}(I, \overline{\operatorname{co}\Omega})$;

vi) $\widetilde{\mathcal{C}}(I, \Omega)$ is relatively weakly compact (i.e. its weak closure is weakly compact) in $L^p(I, \mathcal{A}^q(\mathbb{B}_r^n))$;

vii) the weak closure of $\widetilde{\mathcal{C}}(I, \Omega)$ in $L^p(I, \mathcal{A}^q(\mathbb{B}_r^n))$ is a metrizable space with respect to the weak topology.

Taking into account [90, Lemma I.33], we characterize $cl^w_{L_r^{p,q}} \widetilde{\mathcal{C}}(I, \Omega)$, for all intervals $I \subseteq [0, \infty)$, $p, q \in (1, \infty)$, $r \in (0, 1)$, $A \in \mathscr{A}$ and $\Omega \subseteq \mathcal{N}_A$, where $cl^w_{L_r^{p,q}}$ denotes the closure with respect to the weak topology of $L^p(I, \mathcal{A}^q(\mathbb{B}_r^n))$. The next result was obtained in [64].

Proposition 3.3.3. *Let $I \subseteq [0, \infty)$ be an interval, $r \in (0, 1)$, $p, q \in (1, \infty)$, $A \in \mathscr{A}$ and $\Omega \subseteq \mathcal{N}_A$. Then $cl^w_{L_r^{p,q}} \widetilde{\mathcal{C}}(I, \Omega) = \widetilde{\mathcal{C}}(I, \overline{\operatorname{co}\Omega})$.*

3.3.2 A compactness and density result

In this section, we present first some results which yield that Conjecture 3.3.1 is true. These results have been recently obtained in [64].

Our following theorem, obtained in [64], is in connection with [90, Theorem I.38, Corollary I.39].

Theorem 3.3.4. *For every $r \in (0,1)$, $p,q \in (1,\infty)$, $T \in [0,\infty]$, $A \in \mathscr{A}$ and $\Omega \subseteq \mathcal{N}_A$ we have*

$$\overline{\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \widetilde{\mathcal{C}}(I,\Omega))} = \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, cl^w_{L^{p,q}_r}\widetilde{\mathcal{C}}(I,\Omega)) = \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \widetilde{\mathcal{C}}(I, \overline{co\Omega})),$$

where I is either $[0,T]$, if $T \in [0,\infty)$, or $[0,\infty)$, if $T=\infty$.

In view of the previous theorem and the Krein-Milman Theorem, we are able to confirm that [48, Conjecture 4.19] is true (see [64]). For partial results see [48, Lemma 4.13, Theorem 4.14] and [63] (for $n=1$, cf. [90, Theorem I.42, Corollary I.43]).

Theorem 3.3.5. *Let $A \in \mathscr{A}$ and $\Omega \subseteq \mathcal{N}_A$ be a compact and convex family.*
Then $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)$ is compact and $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega) = \overline{\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \mathrm{ex}\,\Omega)}$, for all $T \in [0,\infty]$.

Next, we recall the definition of the Hausdorff metric on $H(\mathbb{B}^n)$ (cf. [90]), in order to present the following result.

Definition 3.3.6. Let δ denote the well known metric on $H(\mathbb{B}^n)$ such that $(H(\mathbb{B}^n),\delta)$ is a Fréchet space with respect to the compact-open topology. For all nonempty subsets V and W of $H(\mathbb{B}^n)$, let

$$\delta(V,W) = \sup_{f\in V} \inf_{g\in W} \delta(f,g).$$

Also, let ρ be the Hausdorff metric on $H(\mathbb{B}^n)$ given by

$$\rho(V,W) = \max\{\delta(V,W), \delta(W,V)\},$$

for all nonempty compact subsets V and W of $H(\mathbb{B}^n)$.

Taking into account [90, Theorem I.45] and [48, Proposition 4.20], we present the following result from [64]. We note that this result makes sense in the view of Theorem 3.3.5.

Proposition 3.3.7. *Let $A \in \mathscr{A}$ and $\Omega \subseteq \mathcal{N}_A$ be a compact and convex family. Then $T \mapsto \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, \Omega)$ is a continuous mapping from $[0,\infty]$ to the space of compact subsets of $H(\mathbb{B}^n)$ endowed with the Hausdorff metric ρ.*

Chapter 4

Generalized parametric representations and related problems in $\mathbb{C}^n$

The univalent subordination chains with normalization given by a time-dependent linear operator and the connection with the Loewner differential equation on the unit ball in $\mathbb{C}^n$ have been first considered by Graham, Hamada, Kohr and Kohr [45]. They also introduced the notion of generalized parametric representation and generalized spirallikeness on $\mathbb{B}^n$ with respect to a time-dependent operator and obtained characterizations in terms of univalent subordination chains on $\mathbb{B}^n$. Further related results, regarding the study of the Loewner differential equation, have been obtained by Graham, Hamada, Kohr [42], Voda [96] and Arosio [6].

In this chapter, we are concerned with the family $\widetilde{S}^t_A(\mathbb{B}^n)$ $(t \geq 0)$ of normalized univalent mappings on $\mathbb{B}^n$ that have generalized parametric representation with respect to time-dependent operators $A \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is a family of measurable mappings from $[0,\infty)$ into $L(\mathbb{C}^n)$ which satisfy certain natural assumptions. In particular, we have that the mappings in $\widetilde{S}^t_A(\mathbb{B}^n)$ embed in normal Loewner chains with respect to A at time t, and the family $\widetilde{S}^t_A(\mathbb{B}^n)$ is compact. On the other hand, certain examples yield that the family $\widetilde{S}^t_A(\mathbb{B}^n)$ for time-dependent operators $A \in \widetilde{\mathcal{A}}$ is basically different from that in the case of constant time-dependent linear operators (cf. Subsection 1.2.4).

Next, we consider extreme points and support points associated with the compact family $\widetilde{S}^t_A(\mathbb{B}^n)$, where $A \in \widetilde{\mathcal{A}}$ and $t \geq 0$. We are concerned with the generalization of the results due to Kirwan [66] and Pell [74] (see Theorem 1.1.23; cf. Theorems 1.2.29 and 1.2.45) in the case of time-dependent operators, in view of the recent results obtained in [48], [49] (see also [21], [43], [46], [53], [93]). Also, we present an example of a bounded support point for the family $\widetilde{S}^t_A(\mathbb{B}^2)$, where $A \in \widetilde{\mathcal{A}}$ is a certain time-dependent operator, in view of Bracci's example [15] (see also [18], [49]). We also consider the notion of a reachable family with respect to time-dependent linear operators $A \in \widetilde{\mathcal{A}}$, and the corresponding characterizations of extreme/support points associated with these families of bounded univalent mappings on $\mathbb{B}^n$ (cf. [48], [49]). Useful examples and applications point out differences between the case of non-constant time-dependent operators and the case of constant time-dependent linear operators.

Finally, we discuss certain convergence results for the family $\widetilde{S}^t_A(\mathbb{B}^n)$ with respect to the Hausdorff metric ρ on $H(\mathbb{B}^n)$, where $A \in \widetilde{\mathcal{A}}$ and $t \geq 0$. The case of the reachable families is also considered. These results may be seen as dominated convergence type theorems for time-dependent operators $A \in \widetilde{\mathcal{A}}$. As applications, we have related convergence results for the families $S^0_{\mathbf{A}}(\mathbb{B}^n)$ and $\widehat{S}_{\mathbf{A}}(\mathbb{B}^n)$, where $\mathbf{A} \in L(\mathbb{C}^n)$ is such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$ (cf. Subsection 1.2.4). Another application provides some sufficient conditions related to $A \in \widetilde{\mathcal{A}}$ which yield the equality $\widetilde{S}^t_A(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, for all $t \geq 0$.

We mention that this chapter is based on the joint work of the author of the thesis with H. Hamada and G. Kohr [58, 59].

4.1 Generalized parametric representations on the unit ball

In this section, we consider the notion of generalized parametric representation on $\mathbb{B}^n$ and that of normal Loewner chain with respect to time-dependent operators $A \in \widetilde{\mathcal{A}}$, where the family $\widetilde{\mathcal{A}}$ of certain measurable mappings from $[0, \infty)$ into $L(\mathbb{C}^n)$ is introduced and studied a priori. We take into account the previous work of Graham, Hamada, Kohr, Kohr [45] (see also the works in [42] and Voda [96]). Next, we discuss general properties of the family $\widetilde{S}_A^t(\mathbb{B}^n)$ of normalized univalent mappings on $\mathbb{B}^n$ that have generalized parametric representation with respect to $A \in \widetilde{\mathcal{A}}$, where $t \geq 0$. For example, we are concerned with the characterization in terms of normal Loewner chains with respect to $A \in \widetilde{\mathcal{A}}$, certain growth theorems and the compactness of the family. If $A \in \widetilde{\mathcal{A}}$ is constant, then $A(t) = \mathbf{A}$, $t \geq 0$, for some $\mathbf{A} \in L(\mathbb{C}^n)$ with $k_+(\mathbf{A}) < 2m(\mathbf{A})$, and $\widetilde{S}_A^t(\mathbb{B}^n) = S_{\mathbf{A}}^0(\mathbb{B}^n)$, $t \geq 0$ (cf. Subsection 1.2.4). Moreover, if $n = 1$ and $a \in \widetilde{\mathcal{A}}$, then $\widetilde{S}_a^t(\mathbb{U}) = S$, $t \geq 0$. However, certain examples reveal some important differences between the case of non-constant time-dependent operators and the case of constant time-dependent operator in $\widetilde{\mathcal{A}}$, for dimension $n \geq 2$. For example, there exists $A \in \widetilde{\mathcal{A}}$, for $n = 2$, such that $\widetilde{S}_A^t(\mathbb{B}^n) \neq \widetilde{S}_A^s(\mathbb{B}^n)$, for some $t > s \geq 0$.

We mention that this section contains original results due to Hamada, Iancu, Kohr [58].

4.1.1 Definitions, examples and general results

In this subsection, we prepare the ground the forthcoming results.

Let $A : [0, \infty) \to L(\mathbb{C}^n)$ be a measurable mapping that is locally integrable on $[0, \infty)$. For every $s \geq 0$, we denote by $V(s, \cdot) : [s, \infty) \to L(\mathbb{C}^n)$ the unique locally absolutely continuous solution of the initial value problem (see [31], [32]; cf. [96])

$$\frac{\partial V}{\partial t}(s,t) = -A(t)V(s,t), \text{ a.e. } t \in [s, \infty), \ V(s,s) = I_n. \tag{4.1.1}$$

Also, let $V(t) = V(0, t)$, for all $t \geq 0$. Then it is not difficult to see that $V(s, t) = V(t)V(s)^{-1}$, for $0 \leq s \leq t < \infty$ (see [31], [32]; cf. [96]).

Remark 4.1.1. Let $A : [0, \infty) \to L(\mathbb{C}^n)$ be a measurable mapping which is locally integrable on $[0, \infty)$ and let $s \geq 0$. If $A(t)$ and $\int_s^t A(\tau)d\tau$ commute for all $t \geq s$, then

$$V(s,t) = e^{-\int_s^t A(\tau)d\tau}, \quad \forall\, t \in [s, \infty),$$

in view of [32, Exercise VII.2.22] (cf. [45, Remark 1.6]).

The following estimates related to a measurable and locally integrable mapping $A : [0, \infty) \to L(\mathbb{C}^n)$ are very useful (see [96, Proposition 1.2.1, Remark 1.2.2]; cf. [45, Remark 1.6 (v)]).

Lemma 4.1.2. *Let $A : [0, \infty) \to L(\mathbb{C}^n)$ be a measurable mapping that is locally integrable, and let $V(s, t)$ be the unique solution on $[s, \infty)$ of the initial value problem* (4.1.1) *related to A. Then*

$$e^{\int_s^t m(A(\tau))d\tau} \leq \left\|V(s,t)^{-1}\right\| \leq e^{\int_s^t k(A(\tau))d\tau} \tag{4.1.2}$$

and

$$e^{-\int_s^t k(A(\tau))d\tau} \leq \left\|V(s,t)\right\| \leq e^{-\int_s^t m(A(\tau))d\tau}, \tag{4.1.3}$$

for all $t \geq s \geq 0$.

Next, we present the definition of a mapping with generalized parametric representation with respect to a time-dependent linear operator. This notion was first considered by Graham, Hamada, Kohr and Kohr [45], and then generalized by Hamada, Iancu and Kohr [58] (cf. [96, Proposition 1.5.1]).

Definition 4.1.3. Let $A : [0,\infty) \to L(\mathbb{C}^n)$ be a measurable mapping, which is locally integrable, such that $m(A(t)) \geq 0$ for a.e. $t \geq 0$, and let $T \geq 0$. Also, let $V(s,t)$ be the unique solution on $[s,\infty)$ of the initial value problem (4.1.1) related to A. We say that a mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ has generalized parametric representation with respect to A on $[T,\infty)$ if there exists a mapping $h = h(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ which satisfies the following conditions:

(i) $h(z,\cdot)$ is measurable on $[0,\infty)$, for all $z \in \mathbb{B}^n$,

(ii) $h(\cdot,t) \in \mathcal{N}$, for all $t \geq 0$,

(iii) $Dh(0,t) = A(t)$, for all $t \geq 0$,
such that

$$f(z) = \lim_{t\to\infty} V(T,t)^{-1} v(z,T,t)$$

locally uniformly on $\mathbb{B}^n$, where $v(z,T,\cdot) : [T,\infty) \to \mathbb{C}^n$ is the unique locally absolutely continuous solution of the initial value problem

$$\frac{\partial v}{\partial t}(z,T,t) = -h(v(z,T,t),t), \text{ a.e. } t \in [T,\infty),\ v(z,T,T) = z, \tag{4.1.4}$$

for all $z \in \mathbb{B}^n$.

Let $\widetilde{S}^T_A(\mathbb{B}^n)$ be the family of mappings with generalized parametric representation with respect to A on $[T,\infty)$.

Clearly, $\widetilde{S}^T_A(\mathbb{B}^n) \neq \emptyset$, since $\mathrm{id}_{\mathbb{B}^n} \in \widetilde{S}^T_A(\mathbb{B}^n)$, for all $T \geq 0$ and for all measurable and locally integrable mapping $A : [0,\infty) \to L(\mathbb{C}^n)$ with $m(A(t)) \geq 0$, for a.e. $t \geq 0$.

We remark that the above definition generalizes the corresponding notions presented in Chapter 1 (see [58]).

Remark 4.1.4. Let $\mathbf{A} \in L(\mathbb{C}^n)$ be such that $m(\mathbf{A}) > 0$ and let $A : [0,\infty) \to L(\mathbb{C}^n)$ be such that $A(t) = \mathbf{A}$, for all $t \geq 0$. Then $\widetilde{S}^0_A(\mathbb{B}^n)$ reduces to the family $S^0_{\mathbf{A}}(\mathbb{B}^n)$ of mappings with $\mathbf{A}$-parametric representation on $\mathbb{B}^n$ (see [44]). In particular, if $A \equiv I_n$, then $\widetilde{S}^0_{I_n}(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, where $S^0(\mathbb{B}^n)$ is the family of mappings with the usual parametric representation on $\mathbb{B}^n$ (see [50] and Subsection 1.2.3).

Various properties and a geometric construction related to the notion of generalized parametric representation on $\mathbb{B}^n$ may be found in [42] and [45] (cf. [44]).

In view of Definition 4.1.3, we consider the following definition from [58] (cf. [17], [34]).

Definition 4.1.5. Let $A : [0,\infty) \to L(\mathbb{C}^n)$ be a measurable mapping, which is locally integrable on $[0,\infty)$, such that $m(A(t)) \geq 0$, for a.e. $t \geq 0$. A mapping $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ which satisfies the conditions *(i)*–*(iii)* of Definition 4.1.3 is called a Herglotz vector field (or a generating vector field) with respect to A (cf. [17] and [34]).

In the following, we consider the notion of univalent subordination chain in connection with the notion of generalized parametric representation (see [45], [58], [96]; cf. [50, Chpater 8]).

Definition 4.1.6. A univalent subordination chain $f : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is called a normal Loewner chain with respect to A, if $Df(0,t) = V(t)^{-1}$, for $t \geq 0$, and $\{V(t)f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$, where $A : [0,\infty) \to L(\mathbb{C}^n)$ is a measurable and locally integrable mapping and $V(t) = V(0,t)$ is the unique solution on $[0,\infty)$ of the initial value problem (4.1.1) related to A.

The following family of time-dependent operators, introduced by Hamada, Iancu, Kohr [58], plays an important role in this chapter.

Definition 4.1.7. Let $\widetilde{\mathcal{A}}$ be the family of all measurable mappings $A : [0,\infty) \to L(\mathbb{C}^n)$ which satisfy the following conditions:
(i) $m(A(\tau)) \geq 0$, for a.e. $\tau \geq 0$;
(ii) $\operatorname{ess\,sup}_{s\geq 0}\|A(s)\| < \infty$;
(iii) $\sup_{s\geq 0}\int_s^\infty \|V(s,t)^{-1}\|e^{-2\int_s^t m(A(\tau))d\tau}dt < \infty$, where $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (4.1.1) related to A.

The following remark provides a way to construct time-dependent operators in $\widetilde{\mathcal{A}}$ (see [58]).

Remark 4.1.8. Let $T > 0$, $\mathbf{A} \in L(\mathbb{C}^n)$ and let $A : [0,\infty) \to L(\mathbb{C}^n)$ be such that $m(A(t)) \geq 0$, for a.e. $t \in [0,T]$, $\operatorname{ess\,sup}_{t\in[0,T]}\|A(t)\| < \infty$ and $A(t) = \mathbf{A}$, for a.e. $t > T$. Then $A \in \widetilde{\mathcal{A}}$ if and only if $k_+(\mathbf{A}) < 2m(\mathbf{A})$, by Lemma 4.1.2, [34, Remark 2.8] and [45, Remark 2.2]. In particular, $I_n \in \widetilde{\mathcal{A}}$.

Taking into account Remark 4.1.1, we may construct an operator $A \in \widetilde{\mathcal{A}}$ for which there exists some $t > 0$ such that $V(t) \neq e^{-\int_0^t A(\tau)d\tau}$, where $V(t) = V(0,t)$ and $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (4.1.1) – see [58].

Example 4.1.9. Let $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$. Also, let $A : [0,\infty) \to L(\mathbb{C}^2)$ be given by

$$A(t) = \begin{cases} A_1, & \text{for } t \in [0,1) \\ A_2, & \text{for } t \in [1,\infty). \end{cases}$$

Then $A \in \widetilde{\mathcal{A}}$ and $V(2) \neq e^{-\int_0^2 A(\tau)d\tau}$.

We close this subsection with the notion of spirallikeness with respect to a time-dependent linear operator (see [58]; cf. [45, Definition 3.1]). This notion is a generalization of the usual notion of spirallikeness (see [95]; see also [36], [86] and Subsection 1.2.4).

Definition 4.1.10. Let $A : [0,\infty) \to L(\mathbb{C}^n)$ be a measurable and locally integrable mapping such that $m(A(t)) \geq 0$, for a.e. $t \geq 0$. A mapping $f \in S(\mathbb{B}^n)$ is said to be generalized spirallike with respect to A if $V(s,t)f(z) \in f(\mathbb{B}^n)$, for all $z \in \mathbb{B}^n$ and $0 \leq s \leq t < \infty$, where $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (4.1.1) related to A.

Next, we present the following characterization of generalized spirallikeness on $\mathbb{B}^n$ with respect to time-dependent operators (see [58]; cf. [45, Theorems 3.3 and 3.5]).

Proposition 4.1.11. *Let $A \in \widetilde{\mathcal{A}}$ and let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized locally univalent mapping. Then the following statements are equivalent:*
(i) *f is a generalized spirallike mapping with respect to A.*
(ii) *$\Re\langle Df(z)^{-1}A(t)f(z), z\rangle \geq 0$, for a.e. $t \geq 0$ and for all $z \in \mathbb{B}^n$.*
(iii) *$F : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ given by $F(z,t) = V(t)^{-1}f(z)$, $z \in \mathbb{B}^n$, $t \geq 0$, is a normal Loewner chain with respect to A, where $V(t) = V(0,t)$ is the unique solution on $[0,\infty)$ of the initial value problem* (4.1.1) *related to A.*

4.1.2 On generalized parametric representations on the unit ball

In this subsection, we present various results related to the notion of generalized parametric representation on $\mathbb{B}^n$.

We present the following characterization of mappings in $\widetilde{S}_A^T(\mathbb{B}^n)$, obtained in [58], in terms of normal Loewner chains with respect to time-dependent operators $A \in \widetilde{\mathcal{A}}$ (cf. [45, Corollary 2.7], [96]). In the case $A(t) \equiv \mathbf{A}$, this result was obtained in [44] (see also [41] and [80], [81], for $A = I_n$; cf. Theorem 1.2.42 and Remark 1.2.46).

Note that $\widetilde{S}_A^T(\mathbb{B}^n) \subseteq S(\mathbb{B}^n)$, for all $T \geq 0$, in view of Theorem 4.1.12.

Theorem 4.1.12. *Let $T \geq 0$, $A \in \widetilde{\mathcal{A}}$, and let $g \in H(\mathbb{B}^n)$ be a normalized mapping. Then $g \in \widetilde{S}^T_A(\mathbb{B}^n)$ if and only if there exists a normal Loewner chain $f = f(z,t)$ with respect to A such that $g = V(T)f(\cdot, T)$, where $V(t) = V(0,t)$ and $V(s,t)$ is the unique locally absolutely continuous solution on $[s,\infty)$ of the initial value problem* (4.1.1) *related to A.*

Next, we consider an example of a time-dependent operator $A \in \widetilde{\mathcal{A}}$ such that $\widetilde{S}^t_A(\mathbb{B}^2) \neq \widetilde{S}^s_A(\mathbb{B}^2)$, for some $s, t \in [0,\infty)$, $s \neq t$ (cf. Remark 1.2.46). This example, due to Hamada, Iancu, Kohr [58], gives one of the motivations for considering the family $\widetilde{S}^s_A(\mathbb{B}^n)$ rather than $\widetilde{S}^0_A(\mathbb{B}^n)$, for $n \geq 2$.

Example 4.1.13. Let $T > 0$, $\varepsilon \in (0,1)$, and let $\mathbf{A} \in L(\mathbb{C}^2)$ be given by

$$\mathbf{A}(z) = (2z_1, (1+\varepsilon)z_2), \text{ for all } z = (z_1, z_2) \in \mathbb{C}^2. \tag{4.1.5}$$

Let $A \in \widetilde{\mathcal{A}}$ be given by

$$A(t) = \begin{cases} \mathbf{A}, & \text{for } t \in [0,T) \\ I_2, & \text{for } t \in [T,\infty). \end{cases} \tag{4.1.6}$$

Then $\widetilde{S}^t_A(\mathbb{B}^2) = S^0(\mathbb{B}^2)$, for all $t \in [T,\infty)$. However, $\widetilde{S}^0_A(\mathbb{B}^2) \neq S^0(\mathbb{B}^2)$, for sufficiently small $\varepsilon \in (0,1)$ and sufficiently large $T > 0$.

Taking into account Example 4.1.13, it is natural to ask the following question (see [58]):

Question 4.1.14. *Let $A \in \widetilde{\mathcal{A}}$ and $T \geq 0$. Does there exist $\mathbf{A} \in L(\mathbb{C}^n)$ such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$ and $\widetilde{S}^T_A(\mathbb{B}^n) = S^0_{\mathbf{A}}(\mathbb{B}^n)$, for $n \geq 2$?*

The following result that is related to the Question 4.1.14 (see [79, Chapter 6], in the case $n = 1$; cf. Remark 1.2.46), was obtained in [58].

Proposition 4.1.15. *Let $a : [0,\infty) \to \mathbb{R}$ be a measurable function such that*

$$\operatorname{ess\,inf}_{t\geq 0} a(t) > 0 \ \ \textit{and} \ \ \operatorname{ess\,sup}_{t \geq 0} a(t) < \infty. \tag{4.1.7}$$

Also, let $\mathbf{A} \in L(\mathbb{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$ and let $A : [0,\infty) \to L(\mathbb{C}^n)$ be given by $A(t) = a(t)\mathbf{A}$, for a.e. $t \geq 0$. Then $A \in \widetilde{\mathcal{A}}$ and $\widetilde{S}^T_A(\mathbb{B}^n) = S^0_{\mathbf{A}}(\mathbb{B}^n)$, for $T \geq 0$.

From Proposition 4.1.15, we obtain the following particular case related to the families S and $\widetilde{S}^t_a(\mathbb{U})$, for $t \geq 0$ (see [58]; cf. [79, Theorem 6.1], Remark 1.1.16).

Corollary 4.1.16. *Let $n = 1$ and $a \in \widetilde{\mathcal{A}}$. Then $\widetilde{S}^t_a(\mathbb{U}) = S$, for all $t \geq 0$.*

In view of [51, Lemma 2.8 and Theorem 2.9], we have the following compactness result related to the family $\widetilde{S}^T_A(\mathbb{B}^n)$ (see [58]; cf. [44, Theorem 2.15] and [51, Theorem 2.9]).

Theorem 4.1.17. *Let $T \geq 0$ and $A \in \widetilde{\mathcal{A}}$. Then $\widetilde{S}^T_A(\mathbb{B}^n)$ is a compact set.*

We have seen in Example 4.1.13 that there exists a time-dependent operator $A \in \widetilde{\mathcal{A}}$ such that $\widetilde{S}^0_A(\mathbb{B}^2) \not\subseteq \widetilde{S}^T_A(\mathbb{B}^2)$, for some $T > 0$. Next, we consider an example, due to Hamada, Iancu, Kohr [58], of a time-dependent operator $A \in \widetilde{\mathcal{A}}$ such that $\widetilde{S}^T_A(\mathbb{B}^2) \not\subseteq \widetilde{S}^0_A(\mathbb{B}^2)$, for some $T > 0$.

Example 4.1.18. Let $T > 0$, $\varepsilon \in (0,1)$ and $a > 0$. Also, let $\mathbf{A} \in L(\mathbb{C}^2)$ by given by (4.1.5), and let $f : \mathbb{B}^2 \to \mathbb{C}^2$ be defined by

$$f(z) = (z_1 + az_2^2, z_2), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Also, let $\mathbf{E} \in L(\mathbb{C}^2)$ be such that $\mathbf{E} + \mathbf{E}^* = 2\lambda I_2$, for some $\lambda > 0$. Let $A : [0, \infty) \to L(\mathbb{C}^2)$ be given by

$$A(t) = \begin{cases} \mathbf{E}, & \text{for } t \in [0, T) \\ \mathbf{A}, & \text{for } t \in [T, \infty). \end{cases}$$

Then there exist $T > 0$, $\varepsilon \in (0, 1)$ and $a > 0$ such that $A \in \widetilde{\mathcal{A}}$ and $f \in \widetilde{S}_A^t(\mathbb{B}^2) \setminus \widetilde{S}_A^0(\mathbb{B}^2)$, for all $t \geq T$.

Next, we consider the dependence of $\widetilde{S}_A^T(\mathbb{B}^n)$ on $T \geq 0$, where $A \in \widetilde{\mathcal{A}}$ (see [58]; cf. [48] and [90]). In view of Theorem 4.1.17, the following result is related to [48, Proposition 4.20] and was obtained in [58] (cf. Proposition 3.3.7; cf. [90, Theorem I.45], for $n = 1$). For the definition of the Hausdorff metric ρ on $H(\mathbb{B}^n)$, see Definition 3.3.6.

Proposition 4.1.19. *Let $A \in \widetilde{\mathcal{A}}$. Then $T \mapsto \widetilde{S}_A^T(\mathbb{B}^n)$ is a continuous mapping on $[0, \infty)$ to the metric space of nonempty compact subsets of $H(\mathbb{B}^n)$ with respect to ρ.*

4.2 Extremal problems for mappings with generalized parametric representation in $\mathbb{C}^n$

In this section, we consider extremal problems associated with the compact family $\widetilde{S}_A^s(\mathbb{B}^n)$, where $A \in \widetilde{\mathcal{A}}$ and $s \geq 0$. We are interested in generalizations of the results due to Kirwan [66] and Pell [74], regarding extremal properties of the Loewner chains in one complex variable (see Theorem 1.1.23). A first generalization of this result to several complex variables is due to Graham, Kohr and Pfaltzgraff [53]. Further results for the family $S^0(\mathbb{B}^n)$ are due to Graham, Hamada, Kohr, Kohr [46] and Schleissinger [93]. We mention also the recent results due to Graham, Hamada, Kohr, Kohr [48, 49] for the family $S_{\mathbf{A}}^0(\mathbb{B}^n)$, where $\mathbf{A} \in L(\mathbb{C}^n)$ with $k_+(\mathbf{A}) < 2m(\mathbf{A})$ (see also the results due to Chirilă, Hamada, Kohr [21] and Graham, Hamada, Kohr [43]). In this section, we are concerned with the generalization in the case of time-dependent operators $A \in \widetilde{\mathcal{A}}$. We have that if $f(z, t) = V(t)^{-1}z + \cdots$ is a normal Loewner chain such that $V(s)f(\cdot, s) \in \operatorname{ex} \widetilde{S}_A^s(\mathbb{B}^n)$ (resp. $V(s)f(\cdot, s) \in \operatorname{supp} \widetilde{S}_A^s(\mathbb{B}^n)$), then $V(t)f(\cdot, t) \in \operatorname{ex} \widetilde{S}_A^t(\mathbb{B}^n)$, for all $t \geq s$ (resp. $V(t)f(\cdot, t) \in \operatorname{supp} \widetilde{S}_A^t(\mathbb{B}^n)$, for all $t \geq s$), where $V(t)$ is the unique solution on $[0, \infty)$ of the initial value problem (4.1.1). Moreover, we present an example that points out the importance of the time-dependent normalization in this result. Also, we present an example of a bounded support point for the family $\widetilde{S}_A^t(\mathbb{B}^2)$, where $A \in \widetilde{\mathcal{A}}$ is a certain time-dependent operator, taking into account the examples due to Bracci [15] and due to Graham, Hamada, Kohr, Kohr [49]. Similar results for reachable families with respect to time-dependent linear operators $A \in \widetilde{\mathcal{A}}$ are also considered.

We mention that this section is based on original results due to Hamada, Iancu, Kohr [58].

4.2.1 Extreme points, support points, and mappings in $\widetilde{S}_A^t(\mathbb{B}^n)$

In this subsection, we are concerned with extreme points and support points associated with the compact family $\widetilde{S}_A^s(\mathbb{B}^n)$, where $A \in \widetilde{\mathcal{A}}$ and $s \geq 0$ (see [58]).

In view of the proof of [48, Theorem 3.1] (cf. the proof of [46, Theorem 2.1]), we have the following useful lemma (see [58]).

Lemma 4.2.1. *Let $T \geq 0$ and let $A \in \widetilde{\mathcal{A}}$. Also, let f be a normal Loewner chain with respect to A. Let v be the transition mapping associated with f. Then the following statements hold:*

(i) If g is a normal Loewner chain with respect to A, then the mapping $G : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ given by

$$G(z, t) = \begin{cases} g(v(z, t, T), T), & 0 \leq t \leq T \\ g(z, t), & t > T \end{cases} \tag{4.2.1}$$

is a normal Loewner chain with respect to A.

(ii) If $h \in \widetilde{S}_A^T(\mathbb{B}^n)$, then $V(t,T)^{-1}h(v(\cdot,t,T)) \in \widetilde{S}_A^t(\mathbb{B}^n)$, for all $t \in [0,T]$. In particular, $V(t,T)^{-1}v(\cdot,t,T) \in \widetilde{S}_A^t(\mathbb{B}^n)$, for all $t \in [0,T]$, where $V(t) = V(0,t)$ and $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (4.1.1) *related to A.*

The following result, due to Hamada, Iancu, Kohr [58], is a generalization of [46, Theorem 2.1] and [48, Theorem 3.1] (see Theorems 1.2.29 and 1.2.45; cf. [21]) to the case of time-dependent operators (see [66] and [74], in the case $n = 1$).

Theorem 4.2.2. *Let $A \in \widetilde{\mathcal{A}}$ and $s \geq 0$. Let f be a normal Loewner chain with respect to A. If $V(s)f(\cdot,s) \in \operatorname{ex}\widetilde{S}_A^s(\mathbb{B}^n)$, then $V(t)f(\cdot,t) \in \operatorname{ex}\widetilde{S}_A^t(\mathbb{B}^n)$, for all $t \geq s$, where $V(t) = V(0,t)$ and $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem* (4.1.1) *related to A.*

In view of [48, Propositions 3.2 and 3.4], we have the following result from [58] (see also Remark 3.1.6 (i); cf. [21], [43], [46], [93], in the case $A(t) = I_n$, for all $t \geq 0$).

Proposition 4.2.3. *Let $s \geq 0$ and let $A \in \widetilde{\mathcal{A}}$ be such that $\operatorname{ess\,inf}_{t\geq 0} m(A(t)) > 0$. Let f be a normal Loewner chain with respect to A and let v be the transition mapping associated with f. Then $V(s,t)^{-1}v(\cdot,s,t) \in \widetilde{S}_A^s(\mathbb{B}^n) \setminus (\operatorname{ex}\widetilde{S}_A^s(\mathbb{B}^n) \cup \operatorname{supp}\widetilde{S}_A^s(\mathbb{B}^n))$, for all $t \geq s$, where $V(t) = V(0,t)$ and $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem* (4.1.1) *related to A. In particular, $\mathrm{id}_{\mathbb{B}^n} \in \widetilde{S}_A^s(\mathbb{B}^n) \setminus (\operatorname{ex}\widetilde{S}_A^s(\mathbb{B}^n) \cup \operatorname{supp}\widetilde{S}_A^s(\mathbb{B}^n))$.*

Now, we consider the following result due to Hamada, Iancu, Kohr [58], which is a generalization of [93, Theorem 1.1] and [48, Theorem 3.5] (cf. [21]) to the case of time-dependent operators (cf. Theorems 1.2.29 and 1.2.45; cf. [66] and [74], in the case $n = 1$).

Theorem 4.2.4. *Let $s \geq 0$ and $A \in \widetilde{\mathcal{A}}$ be such that $\operatorname{ess\,inf}_{t\geq 0} m(A(t)) > 0$. Let f be a normal Loewner chain with respect to A. If $V(s)f(\cdot,s) \in \operatorname{supp}\widetilde{S}_A^s(\mathbb{B}^n)$, then $V(t)f(\cdot,t) \in \operatorname{supp}\widetilde{S}_A^t(\mathbb{B}^n)$, for all $t \geq s$, where $V(t) = V(0,t)$ and $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem* (4.1.1) *related to A.*

In view of Theorems 4.2.2 and 4.2.4, we have the following example (see [58]):

Example 4.2.5. Let $g \in \operatorname{supp}S^0(\mathbb{B}^n)$ (resp. $g \in \operatorname{ex}S^0(\mathbb{B}^n)$) and let $f(z,t) = e^t z + \cdots$ be a Loewner chain such that $\left\{e^{-t}f(\cdot,t)\right\}_{t\geq 0}$ is a normal family and $g = f(\cdot,0)$. Also, let $T > 0$ and let $F : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ be defined by

$$F(z,t) = \begin{cases} e^T g(e^{t-T}z), & z \in \mathbb{B}^n,\ t \in [0,T) \\ e^T f(z,t-T), & z \in \mathbb{B}^n,\ t \in [T,\infty). \end{cases}$$

If $A : [0,\infty) \to \mathbb{C}^n$ is given by $A(t) = I_n$, for all $t \geq 0$, then F is a normal Loewner chain with respect to A such that $V(s)F(\cdot,s) \notin \operatorname{supp}\widetilde{S}_A^s(\mathbb{B}^n)$ (resp. $V(s)F(\cdot,s) \notin \operatorname{ex}\widetilde{S}_A^s(\mathbb{B}^n)$), for all $s \in [0,T)$, and $V(t)F(\cdot,t) \in \operatorname{supp}\widetilde{S}_A^t(\mathbb{B}^n)$ (resp. $V(t)F(\cdot,t) \in \operatorname{ex}\widetilde{S}_A^t(\mathbb{B}^n)$), for all $t \in [T,\infty)$, where $V(t) = V(0,t)$ is the unique solution on $[0,\infty)$ of the initial value problem (4.1.1) related to A.

Next, we point out the role of the normalization of the normal Loewner chain in Theorem 4.2.4, taking into account [93, Theorem 1.1]. We present an example, due to Hamada, Iancu, Kohr [58], of a normal Loewner chain F with respect to an $A \in \widetilde{\mathcal{A}}$ (with $\operatorname{ess\,inf}_{t\geq 0} m(A(t)) > 0$) such that the following conditions hold for $n \geq 2$:

- $V(t)F(\cdot,t) \in S^0(\mathbb{B}^n)$, for all $t \geq 0$
- $F(\cdot,0) \in \operatorname{supp}S^0(\mathbb{B}^n)$, but $V(t)F(\cdot,t) \notin \operatorname{supp}S^0(\mathbb{B}^n)$, for all $t > 0$,

where $V(t) = V(0,t)$ is the unique solution on $[0,\infty)$ of the initial value problem (4.1.1) related to A (cf. Remark 1.2.30).

This example (see [58]) is obtained in view of Example 4.1.13, [15], [18, Section 5] (see also [49, Remark 7.4]). For simplicity, we consider only the case $n = 2$ in Example 4.2.6 below, but the construction may be done in any dimension $n \geq 2$.

Example 4.2.6. Let $\varepsilon > 0$ and let $\mathbf{A} \in L(\mathbb{C}^2)$ be given by (4.1.5). Also, let $A \in \widetilde{\mathcal{A}}$ be given by (4.1.6) and $V(t) = V(0,t)$ be the unique solution on $[0,\infty)$ of the initial value problem (4.1.1) related to A. Let $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by $f(z) = (z_1 + 3\sqrt{3}z_2^2, z_2)$, for all $z = (z_1, z_2) \in \mathbb{B}^2$, and let $\varepsilon > 0$ be sufficiently small such that f is spirallike with respect to $\mathbf{A}$.

Let $T = \frac{\ln 2}{2\varepsilon}$ and let $F : \mathbb{B}^2 \times [0,\infty) \to \mathbb{C}^2$ be given by

$$F(z,t) = \begin{cases} e^{T\mathbf{A}} f^{-1}\big(e^{(t-T)\mathbf{A}} f(z)\big), & z \in \mathbb{B}^2,\ 0 \leq t < T \\ e^{t-T} e^{T\mathbf{A}} z, & z \in \mathbb{B}^2,\ t \geq T. \end{cases}$$

Then F is a normal Loewner chain with respect to A such that $V(t)F(\cdot,t) \in S^0(\mathbb{B}^2)$, for all $t \geq 0$, $F(\cdot,0) \in \operatorname{supp} S^0(\mathbb{B}^2)$, and $V(t)F(\cdot,t) \notin \operatorname{supp} S^0(\mathbb{B}^2)$, for all $t > 0$.

We note that, by Theorem 4.1.12, Theorem 4.2.4 and Proposition 4.2.3, we have (cf. [18] and [49])

$$V(t)F(\cdot,t) \in \widetilde{S}_A^t(\mathbb{B}^2) \setminus \big(\operatorname{supp} \widetilde{S}_A^t(\mathbb{B}^2) \cup \operatorname{ex} \widetilde{S}_A^t(\mathbb{B}^2)\big), \text{ for all } t \geq 0.$$

In view of [48, Theorem 3.8 and Example 3.10], we present a non-trivial example of a time-dependent operator $A \in \widetilde{\mathcal{A}}$ and a support point for $\widetilde{S}_A^s(\mathbb{B}^2)$, for all $s \geq 0$ (see [58]).

Example 4.2.7. Let $\alpha, \beta : [0,\infty) \to \mathbb{C}$ be essentially bounded measurable functions such that $\operatorname{ess\,inf}_{t\geq 0}\big(\Re\beta(t) - \Re\alpha(t)\big) \geq 0$ and $\operatorname{ess\,inf}_{t\geq 0}\big(2\Re\alpha(t) - \Re\beta(t)\big) > 0$. Let $A : [0,\infty) \to L(\mathbb{C}^2)$ be given by

$$A(t) = \begin{pmatrix} \alpha(t) & 0 \\ 0 & \beta(t) \end{pmatrix}, \quad t \geq 0. \tag{4.2.2}$$

Also, let $f_1 : \mathbb{U} \to \mathbb{C}$ be the Koebe function given by $f_1(\zeta) = \frac{\zeta}{(1-\zeta)^2}$, $\zeta \in \mathbb{U}$, and let $f_2 \in S$ be an arbitrary function. Let $f \in S(\mathbb{B}^2)$ be given by $f(z) = (f_1(z_1), f_2(z_2))$, for all $z = (z_1, z_2) \in \mathbb{B}^2$. Then $A \in \widetilde{\mathcal{A}}$ and $f \in \operatorname{supp} \widetilde{S}_A^s(\mathbb{B}^2)$, for all $s \geq 0$.

Remark 4.2.8. (see [58]) Let

$$\alpha(t) = \begin{cases} 1+\varepsilon, & t \in [0,T) \\ 1, & t \in [T,\infty) \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} 2, & t \in [0,T) \\ 1, & t \in [T,\infty) \end{cases},$$

where $\varepsilon \in (0,1)$ is sufficiently small and $T > 0$ is sufficiently large such that $\widetilde{S}_A^0(\mathbb{B}^2) \neq \widetilde{S}_A^t(\mathbb{B}^2)$, for all $t \geq T$ (see Example 4.1.13), where $A \in \widetilde{\mathcal{A}}$ is given by (4.2.2). Also, let $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by $f(z) = \left(\dfrac{z_1}{(1-z_1)^2}, z_2\right)$, for all $z = (z_1, z_2) \in \mathbb{B}^2$. Thus, by Example 4.2.7, f is an example of a support point for different families of mappings with generalized parametric representation with respect to A.

Moreover, f is generalized spirallike with respect to A. Let $F : \mathbb{B}^2 \times [0,\infty) \to \mathbb{C}^2$ be given by $F(z,t) = V(t)^{-1} f(z)$, for all $z \in \mathbb{B}^2$, where $V(t) = V(0,t)$ is the unique solution on $[0,\infty)$ of the initial value problem (4.1.1) related to A. Then F is a simple example of a normal Loewner chain with respect to A such that $V(t)F(\cdot,t) \in \operatorname{supp} \widetilde{S}_A^t(\mathbb{B}^2)$, for all $t \geq 0$ (cf. Theorem 4.2.4).

4.2.2 Bounded support points for the family $\widetilde{S}_A^t(\mathbb{B}^2)$

In this subsection, we present an example, due to Hamada, Iancu, Kohr [58], of a bounded support point for a family of mappings with generalized parametric representation with respect to a certain time-dependent operator, using [15] and [49, Section 7] (cf. Examples 1.2.33 and 1.2.44).

In view of [49, Proposition 7.2], we define $a_0 : [1,2) \to [\frac{3\sqrt{3}}{2}, \frac{4\sqrt{3}}{2})$ by

$$a_0(\lambda) = \max\Big\{a > 0 : \lambda x^2 + y^2 - axy^2 \geq 0, x, y \geq 0, x^2 + y^2 \leq 1\Big\}, \quad \lambda \in [1,2). \tag{4.2.3}$$

We have the following result from [58], which provides an example of a bounded support point for the compact family $\widetilde{S}_A^s(\mathbb{B}^2)$, where $A : [0,\infty) \to L(\mathbb{C}^2)$ is given in Theorem 4.2.9 below. This result is a generalization of [15, Theorem 1.2] and [49, Theorem 7.6] to the case of time-dependent linear operators.

Theorem 4.2.9. *Let $T > 0$, $\lambda_1, \lambda_2 \in [1,2)$, $\lambda : [0,\infty) \to [1,2)$ be given by*

$$\lambda(t) = \begin{cases} \lambda_1, & t \in [0,T) \\ \lambda_2, & t \in [T,\infty) \end{cases} \tag{4.2.4}$$

and $A : [0,\infty) \to L(\mathbb{C}^2)$, given by $A(t) = \begin{pmatrix} \lambda(t) & 0 \\ 0 & 1 \end{pmatrix}$, for all $t \geq 0$. Then $A \in \widetilde{\mathcal{A}}$, and for every $s \in [0,T)$ the mapping $\varphi_s : \mathbb{B}^2 \to \mathbb{C}^2$ given by

$$\varphi_s(z) = \left(z_1 + \left(\frac{a_0(\lambda_1)}{2-\lambda_1}(1 - e^{(s-T)(2-\lambda_1)}) + \frac{a_0(\lambda_2)}{2-\lambda_2}e^{(s-T)(2-\lambda_1)}\right) z_2^2, z_2\right), \tag{4.2.5}$$

for all $z = (z_1, z_2) \in \mathbb{B}^2$, is a bounded support point for $\widetilde{S}_A^s(\mathbb{B}^2)$.

4.2.3 Extremal problems associated with reachable families

In this subsection, we study extreme points and support points associated with reachable families generated by time-dependent linear operators $A \in \widetilde{\mathcal{A}}$. The following results have been obtained in [58],as generalizations of recent results obtained in [46] and [48].

In view of Section 3.1, we adapt some control-theoretic notions (see [58]).

Definition 4.2.10. Let I be an interval and $A \in \widetilde{\mathcal{A}}$. A mapping $h : \mathbb{B}^n \times I \to \mathbb{C}^n$ is called a Carathéodory mapping on I with respect to A if the following conditions hold:

(i) $h(\cdot, t) \in \mathcal{N}_{A(t)}$, for all $t \in I$,

(ii) $h(z, \cdot)$ is measurable on I, for all $z \in \mathbb{B}^n$.

Let $\mathcal{C}(I, (\mathcal{N}_{A(t)})_{t\in I})$ denote the family of Carathéodory mappings on I with respect to A. We say that the Carathéodory mappings on I with respect to A represent the *controls* of the *control system* $\mathcal{C}(I, (\mathcal{N}_{A(t)})_{t\in I})$, and $(\mathcal{N}_{A(t)})_{t\in I}$ represents the *input family*.

Definition 4.2.11. Let I be either the interval $[T_0, T_1]$, where $T_1 > T_0 \geq 0$, or the interval $[T_0, \infty)$, where $T_0 \geq 0$, and $A \in \widetilde{\mathcal{A}}$. For every $h \in \mathcal{C}(I, (\mathcal{N}_{A(t)})_{t\in I})$ we denote by $v(z, T_0, \cdot; h)$ the unique locally absolutely continuous solution on I of the initial value problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(z, T_0, t; h) = -h(v(z, T_0, t; h), t), & \text{for a.e. } t \in I, \\ v(z, T_0, T_0; h) = z, \end{cases}$$

for all $z \in \mathbb{B}^n$.

Note that $v(\cdot, T_0, t; h)$ is a univalent Schwarz mapping with $Dv(0, T_0, t; h) = V(T_0, t)$, for all $t \in I$ (see [58]; cf. [45] and [96]), where $V(T_0, \cdot)$ is the unique solution on $[T_0, \infty)$ of the initial value problem (4.1.1) related to A.

Now, we consider the notion of the reachable family with respect to time-dependent linear operators, introduced in [58] (cf. [48]).

Definition 4.2.12. Let $T_0 \geq 0$ and $A \in \widetilde{\mathcal{A}}$. For every $T > T_0$ we denote the *normalized time-T-reachable family* of the *control system* $\mathcal{C}\big([T_0,T],(\mathcal{N}_{A(t)})_{t\in[T_0,T]}\big)$ by

$$\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},(\mathcal{N}_{A(t)})_{t\in[T_0,T]})$$
$$= \Big\{V(T_0,T)^{-1}v(\cdot,T_0,T;h) : h \in \mathcal{C}\big([T_0,T],(\mathcal{N}_{A(t)})_{t\in[T_0,T]}\big)\Big\}.$$

We also denote the *normalized infinite-time-reachable family* of the *control system* $\mathcal{C}\big([T_0,\infty),(\mathcal{N}_{A(t)})_{t\geq T_0}\big)$ by

$$\widetilde{\mathcal{R}}_\infty(id_{\mathbb{B}^n},(\mathcal{N}_{A(t)})_{t\geq T_0})$$
$$= \Big\{\lim_{t\to\infty} V(T_0,t)^{-1}v(\cdot,T_0,t;h) : h \in \mathcal{C}\big([T_0,\infty),(\mathcal{N}_{A(t)})_{t\geq T_0}\big)\Big\}.$$

In view of [48, Theorem 4.5] (cf. [46, Theorem 3.7]), we have the following characterization, obtained in [58], of the normalized reachable families in terms of normal Loewner chains with respect to a time-dependent operator (cf. Theorem 3.1.5).

Lemma 4.2.13. *Let $T > T_0 \geq 0$ and $A \in \widetilde{\mathcal{A}}$. Also, let $f \in H(\mathbb{B}^n)$.*

Then $f \in \widetilde{\mathcal{R}}_T\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T]}\big)$ if and only if there exists a normal Loewner chain F with respect to A such that $V(T_0)F(\cdot,T_0) = f$ and $V(T)F(\cdot,T) = \mathrm{id}_{\mathbb{B}^n}$, where $V(t) = V(0,t)$, for all $t \geq 0$, and $V(s,\cdot)$ is the unique solution on $[s,\infty)$ of the initial value problem (4.1.1) *related to A, for all $s \geq 0$.*

In particular, $\widetilde{\mathcal{R}}_\infty\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\geq T_0}\big) = \widetilde{S}_A^{T_0}(\mathbb{B}^n)$ and $\widetilde{\mathcal{R}}_T\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T]}\big) \subset \widetilde{S}_A^{T_0}(\mathbb{B}^n)$.

In view of [48, Corollary 4.7], we have the compactness of the following normalized finite-time-reachable families (see [58]; cf. Theorem 3.1.8 and [90, Theorem I.42]).

Proposition 4.2.14. *Let $T_0 \geq 0$ and $A \in \widetilde{\mathcal{A}}$. Then $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n},(\mathcal{N}_{A(t)})_{t\in[T_0,T]})$ is compact, for all $T > T_0$.*

Now, we consider the generalization of [49, Proposition 6.7] (cf. [43, Corollary 7]) to the case of time-dependent operators in $\mathbb{C}^n$, which was obtained in [58] (cf. Proposition 4.2.3).

Proposition 4.2.15. *Let $T_1 > T > T_0 \geq 0$ and let $A \in \widetilde{\mathcal{A}}$ be such that* $\operatorname{ess\,inf}_{t\geq 0} m(A(t)) > 0$. *Let $f \in \widetilde{\mathcal{R}}_T\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T]}\big)$. Then $f \in \widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T_1]}\big)$, but*

$$f \notin \operatorname{ex}\widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T_1]}\big) \cup \operatorname{supp}\widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T_1]}\big).$$

The following result due to Hamada, Iancu, Kohr [58], is related to [48, Theorem 4.8] and [49, Theorem 6.8] (see [90], in the case $n = 1$; cf. Theorems 4.2.2 and 4.2.4).

Theorem 4.2.16. *Let $T_1 > T_0 \geq 0$, $A \in \widetilde{\mathcal{A}}$, and $f \in \widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T_1]}\big)$. Let F be a normal Loewner chain with respect to A such that $V(T_0)F(\cdot,T_0) = f$ and $V(T_1)F(\cdot,T_1) = \mathrm{id}_{\mathbb{B}^n}$, where $V(t) = V(0,t)$, for all $t \geq 0$, and $V(s,\cdot)$ is the unique solution on $[s,\infty)$ of the initial value problem* (4.1.1) *related to A, for all $s \geq 0$.*

If $f \in \operatorname{ex}\widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T_1]}\big)$, then

$$V(T)F(\cdot,T) \in \operatorname{ex}\widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T,T_1]}\big),\ \textit{for all } T \in [T_0,T_1). \tag{4.2.6}$$

Moreover, if $\operatorname{ess\,inf}_{t\geq 0} m(A(t)) > 0$ *and $f \in \operatorname{supp}\widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T_0,T_1]}\big)$, then*

$$V(T)F(\cdot,T) \in \operatorname{supp}\widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n},\big(\mathcal{N}_{A(t)}\big)_{t\in[T,T_1]}\big),\ \textit{for all } T \in [T_0,T_1). \tag{4.2.7}$$

Remark 4.2.17. (see [58]) Let $T_0 = 0$ and let $A \in \widetilde{\mathcal{A}}$ be a constant time-dependent operator. Then Theorem 4.2.16 implies [49, Theorem 6.8], in view of the following fact:

$$\widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^n}, \big(\mathcal{N}_{A(t)}\big)_{t\in[T,T_1]}\big) = \widetilde{\mathcal{R}}_{T_1-T}\big(\mathrm{id}_{\mathbb{B}^n}, \big(\mathcal{N}_{A(t)}\big)_{t\in[0,T_1-T]}\big),$$

for all $T_1 > T > 0$.

Taking into account [18, Theorem 5.9] (see also [49, Theorem 7.11]), we present the following example, due to Hamada, Iancu, Kohr [58], of a bounded support point for a normalized reachable family with respect to a time-dependent operator in $\mathbb{C}^2$.

Proposition 4.2.18. *Let $T > 0$, $\lambda_1, \lambda_2 \in [1, 2)$ and $A : [0,\infty) \to L(\mathbb{C}^2)$ be given by $A(t) = \begin{pmatrix} \lambda(t) & 0 \\ 0 & 1 \end{pmatrix}$, for all $t \geq 0$, where $\lambda : [0,\infty) \to [1, 2)$ is given by (4.2.4). Then for all $T_1 > T > T_0 \geq 0$, the mapping $\phi_{T_0,T_1} : \mathbb{B}^2 \to \mathbb{C}^2$ given by*

$$\begin{aligned}\phi_{T_0,T_1}(z) = \bigg(z_1 + \Big(\frac{a_0(\lambda_1)}{2-\lambda_1}(1 - e^{(T_0-T)(2-\lambda_1)}) \\ + \frac{a_0(\lambda_2)}{2-\lambda_2}e^{(T_0-T)(2-\lambda_1)}\big(1 - e^{(T-T_1)(2-\lambda_2)}\big)\Big)z_2^2, z_2\bigg),\end{aligned} \tag{4.2.8}$$

for $z = (z_1, z_2) \in \mathbb{B}^2$, is a bounded support point of $\widetilde{\mathcal{R}}_{T_1}(\mathrm{id}_{\mathbb{B}^2}, (\mathcal{N}_{A(t)})_{t\in[T_0,T_1]})$, where $a_0 : [1, 2) \to [\frac{3\sqrt{3}}{2}, \frac{4\sqrt{3}}{2})$ is given by (4.2.3).

The result contained in [49, Theorem 6.8] does not necessarily hold in the case of nonconstant time-dependent operators (cf. Remark 4.2.17), as we shall see in the following example from [58].

Example 4.2.19. Let us consider the same notations as in Proposition 4.2.18 and assume that $\lambda_1 < \lambda_2$. Then there exists a transition mapping v associated to a normal Loewner chain with respect to A such that $G : \mathbb{B}^2 \times [0,\infty) \to \mathbb{C}^2$ given by

$$G(\cdot, t) = \begin{cases} V(T_1)^{-1}v(\cdot, t, T_1), & t \in [0, T_1) \\ V(t)^{-1}\mathrm{id}_{\mathbb{B}^2}, & t \in [T_1,\infty), \end{cases}$$

is a normal Loewner chain with respect to A with $V(T_1)G(\cdot, T_1) = \mathrm{id}_{\mathbb{B}^2}$,

$$G(\cdot, 0) \in \operatorname{supp}\widetilde{\mathcal{R}}_{T_1}\big(\mathrm{id}_{\mathbb{B}^2}, \big(\mathcal{N}_{A(\tau)}\big)_{\tau\in[0,T_1]}\big),$$

and there exists $t \in (0, T_1)$ such that

$$V(t)G(\cdot, t) \notin \operatorname{supp}\widetilde{\mathcal{R}}_{T_1-t}\big(\mathrm{id}_{\mathbb{B}^2}, \big(\mathcal{N}_{A(\tau)}\big)_{\tau\in[0,T_1-t]}\big),$$

where $V(t) = V(0, t)$, for $t \geq 0$, and $V(0, \cdot)$ is the unique solution on $[0,\infty)$ of the initial value problem (4.1.1) related to A.

4.3 Convergence results for compact families of univalent mappings

In this section, we consider a certain convergence result for the compact family $\widetilde{S}^t_A(\mathbb{B}^n)$ with respect to the Hausdorff metric ρ on $H(\mathbb{B}^n)$, where $A \in \widetilde{\mathcal{A}}$ and $t \geq 0$. To be more precise, if a sequence of time-dependent operators $(A_k)_{k\in\mathbb{N}}$ in $\widetilde{\mathcal{A}}$ is dominated in a certain sense and converges point-wise to a time-dependent operator $A \in \widetilde{\mathcal{A}}$ with $\operatorname{ess\,inf}_{t\geq T} m(A(t)) > 0$, then $\rho(\widetilde{S}^T_{A_k}(\mathbb{B}^n), \widetilde{S}^T_A(\mathbb{B}^n)) \to 0$, as $k \to \infty$, where $T \geq 0$ (cf. Proposition 4.1.19). A similar result holds for the normalized time-T-reachable families $\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, (\mathcal{N}_{A(t)})_{t\in[T_0,T]})$ ($T > T_0 \geq 0$) with respect to time-dependent operators $A \in \widetilde{\mathcal{A}}$, introduced in Subsection 4.2.3 (cf. [48], [90]). These results may be seen as dominated convergence type theorems for time-dependent operators $A \in \widetilde{\mathcal{A}}$. In particular, we have related convergence results for the compact families $S^0_{\mathbf{A}}(\mathbb{B}^n)$ and $\widehat{S}_{\mathbf{A}}(\mathbb{B}^n)$ (cf. Subsection 1.2.4), in the case that $\mathbf{A} \in L(\mathbb{C}^n)$ is a linear operator with $k_+(\mathbf{A}) < 2m(\mathbf{A})$. We also obtain a convergence result for the Carathéodory family $\mathcal{N}_{\mathbf{A}}$, where $\mathbf{A} \in L(\mathbb{C}^n)$ with $m(\mathbf{A}) > 0$. Finally, we consider an application of the above mentioned dominated convergence type theorem that gives some sufficient conditions related to $A \in \widetilde{\mathcal{A}}$ such that $\widetilde{S}^t_A(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, for all $t \geq 0$.

We mention that this section is based on original results due to Hamada, Iancu, Kohr [59].

4.3.1 Convergence results for $\widetilde{S}^t_A(\mathbb{B}^n)$ and reachable families

In this subsection, we consider the dependence of $\widetilde{S}^T_A(\mathbb{B}^n)$ on $A \in \widetilde{\mathcal{A}}$, where $T \geq 0$ (see [59]; cf. [58, Proposition 3.15]). Note that the following results may be seen as *dominated convergence* type theorems. In the next subsection we shall apply Theorem 4.3.2 to obtain certain results which involve time-dependent operators that are step functions (cf. Proposition 4.3.5 and Theorem 4.3.7).

We present the first main result of this section, which is due to Hamada, Iancu, Kohr [59]. This is a convergence result for the normalized finite time families introduced in Subsection 4.2.3.

Theorem 4.3.1. *Let I be the interval $[T_0, T]$, where $T > T_0 \geq 0$, and $A \in \widetilde{\mathcal{A}}$ be such that $\operatorname{ess\,inf}_{t\in I} m(A(t)) > 0$. Also, let $M > 0$ and let $(A_k)_{k\in\mathbb{N}}$ be a sequence in $\widetilde{\mathcal{A}}$ such that $\|A_k(t)\| \leq M$, for a.e. $t \in I$ and for all $k \in \mathbb{N}$. If*

$$A_k(t) \to A(t), \text{ as } k \to \infty, \text{ for a.e. } t \in I,$$

then

$$\rho(\widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, (\mathcal{N}_{A_k(t)})_{t\in I}), \widetilde{\mathcal{R}}_T(id_{\mathbb{B}^n}, (\mathcal{N}_{A(t)})_{t\in I})) \to 0, \text{ as } k \to \infty.$$

We present the next main result of this section, due to Hamada, Iancu, Kohr [59]. This is a dominated convergence type theorem for families of mappings with generalized parametric representation on $\mathbb{B}^n$ with respect to time-dependent operators $A \in \widetilde{\mathcal{A}}$.

Theorem 4.3.2. *Let $T \geq 0$ and $A \in \widetilde{\mathcal{A}}$ be such that $\operatorname{ess\,inf}_{t\geq T} m(A(t)) > 0$. Also, let $M > 0$, $\alpha \in L^1([T,\infty), \mathbb{R})$ and $(A_k)_{k\in\mathbb{N}}$ be a sequence in $\widetilde{\mathcal{A}}$ such that, for a.e. $t \geq T$ and for every $k \in \mathbb{N}$, we have: $\|A_k(t)\| \leq M$ and $\|V_k(T,t)^{-1}\| e^{-2\int_T^t m(A_k(\tau))d\tau} \leq \alpha(t)$, where $V_k(T,\cdot)$ is the unique solution on $[T,\infty)$ of the initial value problem* (4.1.1) *related to A_k. If*

$$A_k(t) \to A(t), \text{ as } k \to \infty, \text{ for a.e. } t \geq T,$$

then

$$\rho(\widetilde{S}^T_{A_k}(\mathbb{B}^n), \widetilde{S}^T_A(\mathbb{B}^n)) \to 0, \text{ as } k \to \infty.$$

For constant time-dependent operators (cf. Remark 4.1.8), we have the following result, due to Hamada, Iancu, Kohr [59].

Theorem 4.3.3. *Let $\mathbf{A} \in L(\mathbb{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$, and let $(\mathbf{A}_l)_{l\in\mathbb{N}}$ be a sequence in $L(\mathbb{C}^n)$ such that $\mathbf{A}_l \to \mathbf{A}$, as $l \to \infty$. Then there is $l_0 \in \mathbb{N}$ such that $S^0_{\mathbf{A}_l}(\mathbb{B}^n)$ is compact for $l \geq l_0$, and $\rho(S^0_{\mathbf{A}_l}(\mathbb{B}^n), S^0_{\mathbf{A}}(\mathbb{B}^n)) \to 0$, as $l \to \infty$.*

We close this subsection with the following convergence result for the family $\widehat{S}_{\mathbf{A}}(\mathbb{B}^n)$ of spirallike mappings with respect to $\mathbf{A} \in L(\mathbb{C}^n)$, where $k_+(\mathbf{A}) < 2m(\mathbf{A})$, due to Hamada, Iancu, Kohr [59].

Theorem 4.3.4. *Let $\mathbf{A} \in L(\mathbb{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$, and let $(\mathbf{A}_l)_{l\in\mathbb{N}}$ be a sequence in $L(\mathbb{C}^n)$ such that $\mathbf{A}_l \to \mathbf{A}$, as $l \to \infty$. Then there is $l_0 \in \mathbb{N}$ such that $\widehat{S}_{\mathbf{A}_l}(\mathbb{B}^n)$ is compact for $l \geq l_0$, and $\rho(\widehat{S}_{\mathbf{A}_l}(\mathbb{B}^n), \widehat{S}_{\mathbf{A}}(\mathbb{B}^n)) \to 0$, as $l \to \infty$.*

4.3.2 Analytical characterizations of mappings in $\widetilde{S}_A^t(\mathbb{B}^n)$

In this subsection we present some sufficient conditions related to $A \in \widetilde{\mathcal{A}}$, which guarantee the equality $\widetilde{S}_A^t(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, for $t \geq 0$. The first result is a generalization of [44, Theorem 3.12] and was obtained in [59].

Proposition 4.3.5. *Let $k \in \mathbb{N}$, $\alpha_1, \dots, \alpha_k > 0$, and let $E_1, \dots, E_k \in L(\mathbb{C}^n)$ be such that $E_i + E_i^* = 2\alpha_i I_n$, for all $i \in \{1, \dots, k\}$. Also, let $0 = T_0 < T_1 < \dots < T_{k-1} < T_k = \infty$ and let $A : [0, \infty) \to L(\mathbb{C}^n)$ be given by*

$$A(t) = \begin{cases} E_1, & \text{for } t \in [T_0, T_1) \\ \vdots & \\ E_k, & \text{for } t \in [T_{k-1}, T_k). \end{cases}$$

Then $\widetilde{S}_A^T(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, for all $T \geq 0$.

In view of Propositions 4.1.15 and 4.3.5, we have the following example from [59].

Example 4.3.6. Let $E = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $T > 0$. Let $A \in \widetilde{\mathcal{A}}$ be given by

$$A(t) = \begin{cases} E, & \text{for } t \in [0, T) \\ I_2, & \text{for } t \in [T, \infty). \end{cases}$$

Then $\widetilde{S}_A^s(\mathbb{B}^2) = S^0(\mathbb{B}^2)$, for all $s \geq 0$, but there do not exist $\mathbf{A} \in L(\mathbb{C}^2)$ with $k_+(\mathbf{A}) < 2m(\mathbf{A})$ and a measurable function $a : [0, \infty) \to \mathbb{R}$ such that (4.1.7) holds and $A(t) = a(t)\mathbf{A}$ for a.e. $t \geq 0$.

Taking into account Theorem 4.3.2 and Proposition 4.3.5, we have the following result, due to Hamada, Iancu, Kohr [59]. This result improves Proposition 4.3.5 (cf. Proposition 4.1.15 for $\mathbf{A} = I_n$, and [44, Theorem 3.12]).

Theorem 4.3.7. *Let $A : [0, \infty) \to L(\mathbb{C}^n)$ be a measurable mapping such that $\operatorname{ess\,inf}_{t \geq 0} m(A(t)) > 0$, $\operatorname{ess\,sup}_{t \geq 0} \|A(t)\| < \infty$, and for a.e. $t \geq 0$ there is $\alpha(t) > 0$ such that $A(t) + A(t)^* = 2\alpha(t) I_n$. Then $A \in \widetilde{\mathcal{A}}$ and $\widetilde{S}_A^T(\mathbb{B}^n) = S^0(\mathbb{B}^n)$, for all $T \geq 0$.*

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