FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
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On the geometry and applications of complex recurrent sequences

Doctoral Thesis Summary

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Contents

Introduction

Fibonacci numbers can be defined by the recurrence relation $F_{n+2} - F_{n+1} - F_n = 0$ starting from the initial values $F_0 = 1$ and $F_1 = 1$. Various types of closed forms expressions exist for the general sequence term (Binet formulae), which involve the two roots of the characteristic equation $x^2 - x - 1$. The larger root is referred to as the *golden ratio* $\frac{1+\sqrt{5}}{2} \approx 1.61803\,39887\cdots$, associated in arts with ideal proportions [115].

If the circle is divided into two arcs whose ratio is the golden number, then the shorter arc subtends an angle of about $\phi = 137.5°$, called the *golden angle*. This was used to simulate nature phyllotaxis [119], [95], in search algorithms to find minima of unimodal functions [51], or in optimal designs for concentrated power plants [96].

Following the work initiated by Horadam in the 1960's, general second order complex recurrences are called Horadam sequences. These can be expressed as

$$w_{n+2} = pw_{n+1} + qw_n, \quad w_0 = a, w_1 = b, \quad n \geq 0,$$

where in the most general context a, b, p, q are arbitrary complex coefficients.

Horadam first investigated basic properties of this recursion (both real and complex) [52, 53, 55], [54] and links to Tschebyscheff and other functions [56]. One may note that both types of Tschebyscheff polynomial—$T_n(x)$ (of the first kind) and $U_n(x)$ (of the second kind)—are solutions of the above recurrence when $p = 2x$ and $q = -1$, with (for $n \geq 0$) $T_n(x) = w_n(1, x; 2x, -1)$ and $U_n(x) = w_n(1, 2x; 2x, -1)$ [77]. Numerous results involving Horadam sequences followed.

During 1960-70's, Zeitlin investigated generating functions for products of recursive sequences, power identities, determinants and general identities for Horadam sequences [125–128]. During 1980-90's Horadam and Shannon established links with Catalan identities [61], elliptic functions and Lambert series [57], sequences of general order [58] and polynomial sequences [59, 60]. More identities were obtained by Lee [84], while Zhang explored integer Horadam sequences [129–131]. Extensions involving special recurrences [111], ideals [108] and partitions [109], have also been explored, together with basic results regarding third order recurrences in a general context [110], [124].

In 2000's, Kiliç et al. explored further identities for Horadam sequences [67], involving binomial sums [68], generating functions [69] and matrix methods [70]. Other results were presented in [47], [86] and recently, by Larcombe, Bagdasar and Fennesey [78, 79].

Links between Fibonacci and Horadam sequences were investigated by deKerf [36], Hilton [50] and Morgado [93], together with reciprocal sums of terms [2, 3], [44], [63], [76], [114], [116], polynomials [48], [62], [105], and algorithmic properties [104].

Upon completion of the survey paper [77], the authors sent a letter to Professor Horadam, to inform him about the recent developments in the field he initiated. In his response, Professor Horadam mentioned (for details presented see the Appendix)

> *"I am very flattered by the tone of the paper ... [which is] comprehensive and thorough with an insightful perspective on the history of the sequence."*

Linear recurrent sequences can be periodic [30, 106, 118]. First examples of periodic Horadam sequences were provided by Horadam in [53, (2.35), (2.36), p.166]. Clapperton, Larcombe and Fennessey provided further examples [31], [80], while necessary and sufficient conditions for periodicity were established by Bagdasar and Larcombe first for Horadam sequences [18], then for the general case [20].

This thesis represents the first comprehensive study aimed at unravelling the geometry and structure of Horadam sequences. We begin within the context of self-repeating orbits, followed by an atlas of non-periodic Horadam orbits. Then results are generalized to complex linear recurrent sequences of arbitrary order. The study so-far inspired the design of a pseudo-random number generator, while other applications are expected in areas like cryptography, search algorithms and geometric optimization.

Horadam sequences are expected to be useful for solving optimization problems in the complex plane. To this end, the unidimensional Fibonacci search method of Kiefer [66] can be generalized via novel unimodal-type functions, linked to those existing for the scalar [51], or vector (multi-objective) case [85].

This thesis is divided into five chapters.

Chapter 1 presents important notions and results. Section 1.1 presents key concepts regarding linear recurrent sequences, together with particular examples involving second order recurrences. Section 1.2 is dedicated to homographic recurrences in the complex plane [12], which motivate the study of periodicity conditions in the context of other complex recurrences. Section 1.3 presents key basic definitions regarding the geometry of the complex plane [33, 34], arithmetic functions [13, 14, 92], as well as certain density and linear independence results in number theory [10, 11, 43, 46].

Chapter 2 is dedicated to periodic orbits of complex Horadam sequences, depending on two arbitrary initial conditions and two complex recurrence coefficients. Section 2.1 presents basic concepts and binet-type formulae for the general sequence term for the cases when the roots of the quadratic characteristic equation (termed as *generators* for convenience) are equal (degenerate) or distinct (non-degenerate). In Section 2.2 are formulated necessary and sufficient conditions ensuring the periodicity of Horadam sequences in a general context. In Section 2.3 we discuss the geometry of self-repeating Horadam orbits. In Section 2.4 we establish enumerative and asymptotic properties for the number of self-repeating Horadam sequences of a given integer length $k \leq 1$, denoted by $H_P(k)$ The results in this chapter have been published by Bagdasar and Larcombe [18, 19], and by Bagdasar, Larcombe and Anjum [21, 22].

Chapter 3 is dedicated to non-periodic Horadam sequences and their applications. First, degenerate orbits are discussed in Section 3.1. In Section 3.2 showcases an atlas of Horadam patterns, obtained by Bagdasar [16]. This presents stable orbits (which are finite, or dense within 1D or 2D subsets of the complex plane). then quasi-convergent, convergent and divergent patterns in the complex plane. Section 3.3 analyzes a Horadam-based pseudo-random number generator designed by Bagdasar and Chen in [17].

In **Chapter 4** we extend the results presented in Chapters 2 and 3, for higher-order complex linear recurrent sequences (also called generalized Horadam sequences). In Section 4.1, we discuss the structure of the solution space for linear recurrent sequences. In Section 4.2, we present necessary and sufficient periodicity conditions for complex linear recurrent sequences of arbitrary order given by Bagdasar and Larcombe [20]. In Section 4.3 we establish geometric boundaries for regions containing generalized periodic Horadam orbits. We also determine the geometric structure and number of the periodic orbits of given length. Section 4.4 presents geometric properties of orbits produced by roots of unity. A mini-atlas of generalized complex Horadam patterns is presented, for third-order sequences in Section 4.5.

Chapter 5 presents certain integer sequences, relevant for the enumeration of periodic complex linear recurrent sequences. In Section 5.1 we find the number of ordered integer k-tuples having same lcm n. The results in this section have been published by Bagdasar in [15]. The arithmetic functions enumerating the (strictly) increasing tuples having the same lcm, are analyzed in Section 5.2. These are required for the enumeration of generalized periodic Horadam sequences, characterized in [17]. In Section 5.3 we discuss some newly added sequences to the OEIS database of integer sequences (A245019, A245020, A247513, A247516, A247517) by Bagdasar in 2014 [15], as well as various other contributions to already existing sequences.

List of published papers related to the thesis

Some results presented in this thesis have been published in the following papers:

- **Bagdasar, O.**, Larcombe, P. J., *On the characterization of periodic complex Horadam sequences*, Fib. Quart., **51.1** (2013), 28–37.
- **Bagdasar, O.**, Larcombe, P. J., *On the number of complex periodic complex Horadam sequences*, Fib. Quart., **51.4** (2013), 339–347.
- **Bagdasar, O.**, Larcombe, P. J., Anjum, A., *Particular Orbits of Periodic Horadam Sequences*, Octogon Math. Mag., **21.1**, (2013) 87–98.
- **Bagdasar, O.**, Chen., M., *A Horadam-based Pseudo-random Number Generator*, Proceedings of 16th UKSim, Cambridge (2014), 226–230.
- **Bagdasar, O.**, Larcombe, P. J., *On the characterization of periodic generalized Horadam sequences*, J. Differ. Equ. Appl. (**ISI**), **20.7** (2014), 1069–1090.
- **Bagdasar, O**., *On some functions involving the lcm and gcd of integer tuples*, Appl. Maths. Inform. and Mech., **6.2** (2014), 91–100.
- **Bagdasar, O.**, Popovici N., *Local maximum points of explicitly quasiconvex functions*, Optim. Lett. (**ISI**), **9** (2015), 769–777.
- **Bagdasar, O.**, Larcombe, P. J., Anjum, A., *On the structure of periodic complex Horadam sequences*, Carpathian J. Math. (**ISI**) (to appear).
- **Bagdasar, O.**, *An atlas of Horadam patterns* (submitted).
- Larcombe, P. J., **Bagdasar, O.**, Fennessey, E. J., *Horadam sequences: a survey*, Bull. Inst. Combin. Appl., **67** (2013), 49–72.
- Larcombe, P. J., **Bagdasar, O.**, Fennessey, E. J., *On A Result of Bunder Involving Horadam Sequences: A Proof and Generalization*, Fib. Quart., **51.2** (2013), 174–176.
- Larcombe, P. J., **Bagdasar, O.**, Fennessey, E. J., *On a result of bunder involving Horadam sequences: a new proof*, Fib. Quart., **52.2** (2014), 175–177.
- Vălcan, D., **Bagdasar, O.**, *Generalizations of some divisibility relations in* $\mathbb{N}$, Creative Math. & Inf., **18.1** (2009), 92–99.

New OEIS sequences: The results presented in [15] produced novel integer sequences, indexed in the OEIS as A247517, A245019, A245020, A247513, A247516 [97].

Some other articles are still in preparation.

Presentations related to the Thesis:

The results in this thesis have been disseminated within the following events:

- 6-10 July 2015: BCC25 (British Combinatorial Colloquium), Warwick University, UK; Talk: *On the enumeration of integer tuples having the same lcm*.
- 13-21 August 2014: ICM 2014 (International Congress of Mathematicians), Seoul, S. Korea; Talk: *On the enumeration of periodic generalized Horadam sequences*.
- 6-11 August 2014: ANTS XI (Algorithmic Number Theory Symposium), GyeongJu, S. Korea; Poster: *On certain computational and geometric properties of Horadam orbits*.
- 26-28 March 2014: UKSim2014, Cambridge, UK (Talk, Session Chair); Talk: A Horadam-based Pseudo-random Number Generator.
- 1-5 July 2013: BCC24 (British Combinatorial Colloquium), Royal Holloway, UK; Talk: *Enumeration of periodic Horadam sequences*.

The following invited or seminar talks on this subject were given:

- 6 June 2015: UBB Cluj, RO, Presentation for the PhD supervision committee: *"On the geometry and applications of complex recurrent sequences"*
- 2 April 2015: UBB Cluj, RO, Complex Analysis Research Seminar (30 min): *"On the geometric patterns produced by Horadam sequences in the complex plane"*
- 20 February 2015: TCS Research Group, Loughborough University, UK: *"Complex Horadam sequences: periodicity, enumeration and applications"*
- 16 February 2015: Guest Lecture, University of Derby: *"On the Visualisation of Number Sequences"*
- 22 August 2014: Hanbat National University, S. Korea: *"Mathematics of beauty: on recurrences and emotions"*
- 29 May 2014: UBB Cluj, RO, Faculty of Mathematics: *"On some properties and applications of Horadam sequences"*
- 29 October 2013: Open University Mathematics Seminar, UK: *"The geometric patterns of complex Horadam sequences"*

Keywords: complex linear recurrence, geometric patterns, Horadam sequence, Fibonacci numbers, periodicity, homographic recurrences, pseudo-random number generators, combinatorics, arithmetic functions, integer sequences, least common multiple.

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Were I to mention all those who prayed for me, I would need to double the page count of this work. It's better that I stopped here by saying a big THANK YOU to everyone!

CHAPTER 1

Theoretical background

In this chapter we present various key results and relevant examples regarding recurrent sequences, as well as concepts of complex plane geometry and number theory, useful for the formulation of results in this thesis.

1.1 Linear recurrent sequences (LRS)

In this section we present basic results in the theory of linear recurrent sequences.

Definition 1.1.1. *A linear recurrence sequence (LRS) represents an infinite sequence* $\boldsymbol{w} = (w_0, w_1, w_2, \ldots)$ *of numbers satisfying the recurrence relation*

$$w_n = c_1 w_{n-1} + c_2 w_{n-2} + \cdots + c_m w_{n-m}, \tag{1.1.1}$$

for $m \leq n \in \mathbb{N}$, *with* $c_1, \ldots, c_m$ $(c_m \neq 0)$ *fixed numbers. If numbers* $a_1, \ldots, a_m$ *satisfy* $w_{i-1} = a_i, i = 1, \ldots, m$, *the recurrence is said to have order* m, *and is uniquely defined.*

The *characteristic polynomial* associated with the LRS satisfying (1.1.1) is defined by

$$f(x) = x^m - c_1 x^{m-1} - \cdots - c_{m-1} x - c_m, \tag{1.1.2}$$

and dictates the recurrence properties. The general term of the recursion is given by

$$w_n = p_1(n) z_1^n + \cdots + p_m(n) z_1^m,$$

where $z_1, \ldots, z_m$ are roots of the characteristic polynomial (1.1.2). If these roots are distinct, the LRS is called *simple* and $p_1, \ldots, p_m$ are constants.

Numerous properties and results are presented in the monograph of Everest *et al.* [38]. Decision problems involving LRS with rational terms are discussed by Ouaknine in [99] and other of his papers.

Problem 1. Does $w_n = 0$ for some n ? (Skolem)
Problem 2. Is $w_n = 0$ for infinitely many n ?
Problem 3. Does $w_n \geq 0$ for all n ? (Positivity)
Problem 4. Does $w_n \geq 0$ for all but finitely many n ? (Ultimate Positivity)

Despite persistent efforts, Problem 1 is still open, while Berstel and Mignotte showed that Problem 2 is decidable [24]. Recently, Ouaknine reported progress on Problem 3 for simple LRS of order $m \leq 9$ [100], or arbitrary LRS of order $m \leq 5$ [101], respectively. Problem 4 has also received a positive answer for simple LRS [99].

Reduction of order for LRS

The order of a linear recurrence can be reduced while producing a non-linear recurrent sequence. For second-order recurrences, Andrica and Buzeţeanu showed that [8]:

Theorem 1.1.2. *If $w_n = c_1 w_{n-1} + c_2 w_{n-2}$ with $c_2 \neq 0$, $w_1 = a_1$ and $w_2 = a_2$, then*

$$w_n^2 - c_1 w_n w_{n-1} - c_2 w_{n-1}^2 = (-1)^n c_2^{n-2} \left(a_2^2 - c_1 a_1 a_2 - c_2 a_1^2\right).$$

For $c_1 = c_2 = 1$ and $a_1 = a_2 = 1$ one obtains the identity for Fibonacci numbers

$$F_n^2 - F_n F_{n-1} - F_{n-1}^2 = (-1)^{n-1}.$$

Applications to the theory of diophantine equations were also provided in that paper. The result for recurrences of arbitrary order was proved by the same authors in [9].

Second-order recurrent sequences

Fibonacci numbers and golden section

Definition 1.1.3. *The classical example of recurrence is represented by Fibonacci numbers, obeying the rule that each subsequent element is the sum of the previous two [1, Chapter 5]*

$$F_{n+2} - F_{n+1} - F_n = 0, \quad F_0 = 1, F_1 = 1.$$

The roots of the quadratic characteristic equation $x^2 - x - 1 = 0$ can be used to produce formulae for the general term (Binet formulae). An example is $F_n = (\varphi^n - \phi^n)/\sqrt{5}$, where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803\,39887\cdots$ is the golden ratio (sequence A001622 in OEIS).

In Mathematics they have applications in graph theory (Fibonacci numbers and graphs) [71], data structures (Fibonacci heap) [41], or search algorithms (Fibonacci search) [66]. Other results and further applications are given in [27–29], [73], [87–91], or [96].

Horadam sequences

Horadam sequences are a direct extension of the Fibonacci numbers in the complex plane. These are given by the recurrence

$$w_{n+2} = pw_{n+1} + qw_n, \quad w_0 = a, w_1 = b, \quad n \geq 0,$$

where in the most general context a, b, p, q are arbitrary complex coefficients.

The first systematic investigation into the Horadam literature produced over the past 50 years, was carried out in the survey paper of Larcombe, Bagdasar and Fennesey [77].

Horadam periodicity was first emphasized by Clapperton, Larcombe and Fennessey [31], and then by Larcombe and Fennesey [80]. The authors provided examples of periodic Horadam sequences and linked them to Catalan polynomials, such as

$$\{w_n(1,1;1,-1)\}_{n=0}^{\infty} = \{1,1,0,-1,-1,0,\dots\} = \{P_n(1)\}_0^{\infty}$$
$$\{w_n(1,\sqrt{2};\sqrt{2},-1)\}_{n=0}^{\infty} = \{1,\sqrt{2},1,0,-1,-\sqrt{2},-1,0,\dots\} = \{\sqrt{2}^n P_n(1/2)\}_0^{\infty},$$

where $P_n(x)$ is the $(n+1)$-th Catalan polynomial defined in [31].

1.2 Homographic recurrences: orbits and periodicity

In this section we present results concerning homographic recurrences. These include general terms for degenerate/non-degenerate sequences, as well as the convergence, divergence and periodicity conditions investigated by Andrica and Toader [12].

Definition 1.2.1. *A homographic sequence $\{z_n\}_{n=0}^{\infty}$ is defined by the recurrence*

$$z_{n+1} = \frac{a \cdot z_n + b}{c \cdot z_n + d} \quad z_0 \in \mathbb{C}, \tag{1.2.1}$$

where a, b, c, d are complex numbers and $c \neq 0 \neq ad - bc$.

The roots of the quadratics below can be used to express the general term of (1.2.1).

$$cq^2 + (d-a)q - b = 0, \quad p^2 - (a+d)p + ad - bc = 0. \tag{1.2.2}$$

The two quadratics share the common discriminant $D = (d-a)^2 + 4bc$, while the roots of (1.2.2) denoted by q_1, q_2 and p_1, p_2 satisfy the identities

$$q_{1,2} = \frac{a - d \pm \sqrt{D}}{2c}; \quad p_k = cq_k + d, \quad k = 1, 2. \tag{1.2.3}$$

Theorem 1.2.2 (Non-degenerate case, Theorem 2.1 [12])**.** *If $D \neq 0$ then z_n is given by*

$$z_n = \frac{q_1 \cdot (z_0 - q_2) \cdot (cq_1 + d)^n - q_2 \cdot (z_0 - q_1) \cdot (cq_2 + d)^n}{(z_0 - q_2) \cdot (cq_1 + d)^n - (z_0 - q_1) \cdot (cq_2 + d)^n}. \tag{1.2.4}$$

Equivalently, one can obtain formulae involving p_1, $p_2 c$, d, and z_0.

Theorem 1.2.3 (Degenerate case, Theorem 2.3 [12])**.** *If $D = 0$ then z_n is given by*

$$z_n = \frac{(a+d) \cdot z_0 + n\left[(a-d) \cdot z_0 + 2b\right]}{a + d + n(2cz_0 - a + d)}. \tag{1.2.5}$$

We can define the following notation $z = \frac{a+d+\sqrt{D}}{a+d-\sqrt{D}} = \frac{p_2}{p_1} = re^{2\pi i\theta}$, $\quad A = \frac{z_0 - q_2}{z_0 - q_1}$. The initial assumption $ad - bc \neq 0$ implies $a + d - \sqrt{D} \neq 0$. For $z \in \{0, 1\}$ it follows that $D = 0$, while for $D \neq 0$, formula (1.2.4) is equivalent to $z_n = \frac{Aq_1 - q_2 z^n}{A - z^n}$.

In what follows we reformulate [12, Theorem 3.1], in the single variable z.

Theorem 1.2.4. *Let $\{z_n\}_{n=0}^{\infty}$ be a homographic recurrence defined by* (1.2.1)*. One has:*

(a) If $z = 1$, then $\{z_n\}_{n=0}^{\infty}$ converges to $(a-d)/2c = (q_2 + q_1)/2$;

(b) If $|z| < 1$, then $\{z_n\}_{n=0}^{\infty}$ converges to q_1.

(c) If $|z| > 1$, then $\{z_n\}_{n=0}^{\infty}$ converges to q_2.

(d) If $|z| = 1$, then the following two distinct cases are possible.

(d1) For $\theta = p/q \in \mathbb{Q}$ irreducible fraction, $\{z_n\}_{n=0}^{\infty}$ is periodic within a curve.

(d2) For $\theta = p/q \in \mathbb{R} \setminus \mathbb{Q}$, $\{z_n\}_{n=0}^{\infty}$ is dense within the graph of $w = H(z) = \frac{q_2 z - q_1}{z - 1}$.

In Fig. 1.1 are plotted a periodic with 13 points (a), and a dense orbit (b).

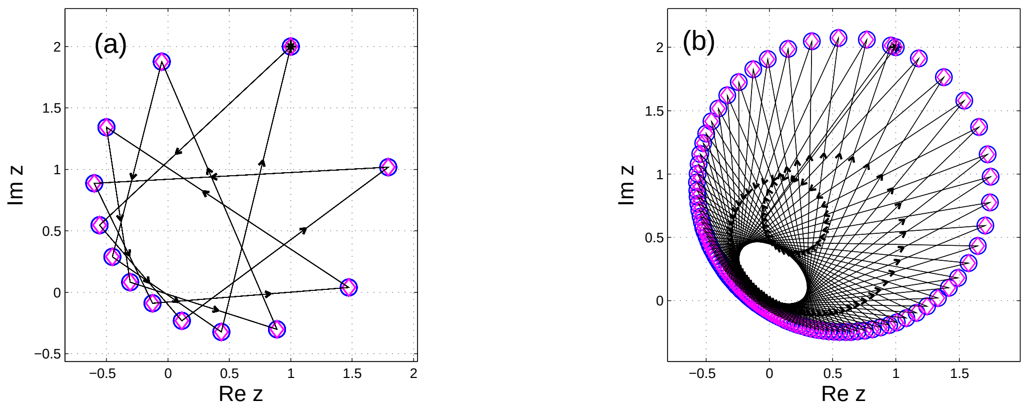

Figure 1.1: First 100 terms of $\{z_n\}_{n=0}^{\infty}$ computed by (1.2.1) (diamonds) and direct formula (circles) for $z_0 = 1 + 2i$ (star) and $z = re^{2\pi i x}$, where (a) $r = 1$, $x = 4/13$; (b) $r = 1$, $x = \sqrt{2}/4$. for $a = 3 + i$, $d = 1 - 2i$, $b = 2 - 2i$. Arrows indicate the increase of sequence index.

1.3 Key elements of complex geometry and number theory

Definition 1.3.1. *(Star polygons) For integers k and p the regular star polygon denoted by the Schläfli symbol $\{k/p\}$ can be considered as being constructed by connecting every pth point out of k points regularly spaced in a circular placement (see [33, Chapter 2], [34, Chapter 6]).*

Definition 1.3.2. *(Multipartite graph) For k a natural number, a k-partite graph W is a graph whose vertex set V is partitioned into k parts, with edges between vertices of different parts only: $G = (V_1, \dots, V_k, E)$ with $E \subset \{uv \mid u \in V_i, v \in V_j, i \neq j\}$. The vertices of $V_i, i = 1, \dots, k$ are called the ith level of G [83, p.4]. A 2-partite graph is called bipartite.*

LCM and GCD of integer tuples

Assume that $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \quad b = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ where $p_1 < p_2 < ... < p_k$ are primes and a_i, b_i are non-negative integers. The following identities hold:

$$\begin{aligned} \gcd(a,b) &= p_1^{\min(a_1,b_1)}\, p_2^{\min(a_2,b_2)} \cdots p_k^{\min(a_k,b_k)}, \\ \operatorname{lcm}(a,b) &= p_1^{\max(a_1,b_1)}\, p_2^{\max(a_2,b_2)} \cdots p_k^{\max(a_k,b_k)}, \\ a \cdot b = \operatorname{lcm}(a,b) \cdot \gcd(a,b) &= p_1^{a_1+b_1}\, p_2^{a_2+b_2} \cdots p_k^{a_k+b_k}. \end{aligned} \tag{1.3.1}$$

If $n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$, $|\{(a,b) : \operatorname{lcm}(a,b) = n\}| = (2n_1+1)(2n_2+1) \cdots (2n_r+1)$ [7].

The number of k-tuples of positive integers with the least common multiple n is

$$\operatorname{LCM}(n;k) = |\{(a_1, \dots, a_k) : \operatorname{lcm}(a_1, \dots, a_k) = n\}|.$$

A link between lcm and gcd of k-tuples was proved by Vălcan and Bagdasar [117].

Theorem 1.3.3. *Let $k \geq 2$ and $a_1, \dots, a_k$ be natural numbers. The following properties holds*

$$\operatorname{lcm}(a_1, a_2, \dots, a_k) = \frac{\prod\limits_{1 \leq i_1 < \cdots < i_u \leq k} \gcd(a_{i_1}, \dots, a_{i_u})}{\prod\limits_{1 \leq i_1 < \cdots < i_v \leq k} \gcd(a_{i_1}, \dots, a_{i_v})}, \tag{1.3.2}$$

$$\gcd(a_1, a_2, \dots, a_k) = \frac{\prod\limits_{1 \leq i_1 < \cdots < i_u \leq k} \operatorname{lcm}(a_{i_1}, \dots, a_{i_u})}{\prod\limits_{1 \leq i_1 < \cdots < i_v \leq k} \operatorname{lcm}(a_{i_1}, \dots, a_{i_v})}, \tag{1.3.3}$$

where u is odd and v is even.

Proof (sketch): If a prime p has multiplicities $m_1, \dots, m_k$ in $a_1, \dots, a_k$, (1.3.2) reduces to

$$\max(m_1, \dots, m_k) = \sum_{1 \leq i_1 < \ldots < i_u \leq n} \min(m_{i_1}, \dots, m_{i_u}) - \sum_{1 \leq i_1 < \ldots < i_v \leq n} \min(m_{i_1}, \dots, m_{i_v}),$$

where u is odd and v is even. To this end one just need to count the terms in the two sides. This argument can also be checked using an inclusion-exclusion principle. $\square$

Definition 1.3.4. *For an integer $n \in \mathbb{N}$, $\varphi(n)$ represents the number of integers $1 \le k \le n$ relatively prime with n [14, 46, 92]. If the factorisation of n is $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then*

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

The following identity valid is useful in Section 2.3.

Proposition 1.3.5. *For any positive integers $a, b \in \mathbb{N}$ one has*

$$\varphi(gcd(a,b)) \cdot \varphi(lcm(a,b)) = \varphi(a) \cdot \varphi(b). \tag{1.3.4}$$

Partitions and Stirling numbers

Let n and k be non-negative integers. The following results hold.

Proposition 1.3.6. *The number of k-tuples of positive integers with sum n is*

$$S_+^*(n,k) = \binom{n-1}{k-1}.$$

Proposition 1.3.7. *The number of k-tuples of non-negative integers with sum n is*

$$S^+(n,k) = \binom{n+k-1}{k-1}.$$

For more examples and detailed proofs one may consult [1], [4], [7], [39], or [42].

Definition 1.3.8. *The Stirling numbers of the second kind $S(n,k)$ (see [26], [72]), count the number of ways to partition a set of n labelled objects into k nonempty unlabelled subsets.*

Stirling numbers of second kind have the following properties [112]:

$$S(n+1,k) = kS(n,k) + S(n,k-1);$$
$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

Linear (in)dependence and density results

Here we present a number of useful linear independence and density results.

Definition 1.3.9. *The numbers $x_1, \ldots, x_k \in \mathbb{R}$, $k \ge 1$ are called linearly dependent over $\mathbb{Q}$ (or $\mathbb{Z}$) if there are coefficients $a_1, \ldots, a_k \in \mathbb{Q}$, such that*

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0, \quad \text{and} \quad (a_1, \ldots, a_k) \neq (0, \ldots, 0). \tag{1.3.5}$$

If the identity (1.3.5) *only takes place when $(a_1, \ldots, a_k) = (0, \ldots, 0)$, the numbers $x_1, \ldots, x_k$ are called linearly independent.*

Proposition 1.3.10. *For a prime p, the triplet $(1, \sqrt[3]{p}, \sqrt[3]{p^2})$ is linearly independent over $\mathbb{Z}$.*

For $k = 1$, the linear independence of 1 and x_1 implies that $x_1 \in \mathbb{R} \setminus \mathbb{Q}$, and one recovers the one-dimensional lemma of Kronecker [46, Theorem 339], [43].

Theorem 1.3.11. *If x is irrational, then $\{nx\}$ is dense in the interval $[0, 1]$.*

The following more general result is due to Weyl [121].

Theorem 1.3.12. *If x is irrational, then the points $\{nx\}$ are uniformly distributed in $[0, 1]$.*

Alternative formulations and generalizations of Kronecker's and Weyl's results were proposed by Andrica and Buzeţeanu in [10]. These include the following three results:

Theorem 1.3.13. *Let $P(X) = a_p X^p + \cdots + a_1 X + a_0 \in \mathbb{R}[X]$ be a polynomial such that at least one of the coefficients $a_p, \ldots, a_1$ is irrational. Then $\lim_{N\to\infty} \frac{1}{N} \sum_0^N e^{2\pi i P(n)} = 0$.*

Theorem 1.3.14. *If $x \in \mathbb{R} \setminus \mathbb{Q}$, $A = \{n + mx : m \in \mathbb{N}, \quad n \in \mathbb{Z}\}$ is dense everywhere in $\mathbb{R}$.*

Theorem 1.3.15. *Let $s > 0$, $a \geq 0$ and ≥ 0 be integers and x irrational. Then the set $A = \{n + mx : m \in \mathbb{N}, \quad n \in \mathbb{Z}, n = a(mod\ s), m = b(mod\ s)\}$ is dense everywhere in $\mathbb{R}$.*

In Chapter 3 we need the following multi-dimensional version of Kronecker's lemma.

Theorem 1.3.16. *([46, Theorem 442]) If $1, x_1, x_2 \ldots, x_k$ are linearly independent (over $\mathbb{N}$), $\alpha_1, \alpha_2, \ldots, \alpha_k$, and N and ε are positive, then there are integers $n > N$, $p_1, \ldots, p_k$ such that*

$$|nx_m - p_m - \alpha_m| < \varepsilon \quad (m = 1, \ldots, k)$$

Theorem 1.3.17. *([46, Theorem 443]) If $1, x_1, x_2 \ldots, x_k$ are linearly independent (over $\mathbb{N}$), then the set of points $\{nx_1\}, \{nx_2\}, \ldots, \{nx_k\}$, is dense in the unit cube.*

Proposition 1.3.18. *Let $x_1, x_2 \in \mathbb{R}$. If $(1, x_1, x_2)$ are linearly independent over $\mathbb{Q}$ (or $\mathbb{Z}$), then sequence $(\{nx_1\}, \{nx_2\})$ is dense within $[0, 1] \times [0, 1]$. Otherwise, $(1, x_1, x_2)$ are linearly dependent over $\mathbb{Q}$ (or $\mathbb{Z}$), hence one can find a_0, a_1, a_2 with the property $a_0 + a_1 x_1 + a_2 x_2 = 0$. The following cases are possible:*

1. $x_1, x_2 \in \mathbb{Q}$ ($a_0 = -a_1 x_1 - a_2 x_2$). In this case, the sequence $(\{nx_1\}, \{nx_2\})$ is periodic.

2. $x_1 = p/k \in \mathbb{Q}$ (irreducible), $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ ($a_1 = 0$, $a_0 = -a_2 x_2$). For these values the sequence $(\{nx_1\}, \{nx_2\})$ is dense within $\{0, \frac{1}{k}, \ldots \frac{k-1}{k}\} \times [0, 1]$.

3. $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$ ($x_2 = -\frac{a_1}{a_2} x_1 - \frac{a_0}{a_2} = b_1 x_1 + b_0$). Here the sequence $(\{nx_1\}, \{nx_2\})$ is dense within the graph of the function $f : [0, 1] \to \mathbb{R}^2$ defined by $f(x) = (x, b_1 x + b_0)$. Notable instances of this case are $b_1 = 0$ (i.e., $x_2/x_1 \in \mathbb{Q}$) or $b_1 = 1$ (i.e., $x_2 - x_1 \in \mathbb{Q}$).

CHAPTER 2

Periodic complex Horadam patterns

Complex Horadam sequences are a natural extension of Fibonacci numbers, involving four parameters (two initial values and two recursion coefficients), therefore successive sequence terms can be visualized in the complex plane. In this chapter we first provide formulae for the general sequence term. Then we discuss periodicity conditions used to classify and enumerate self-repeating Horadam patterns.

2.1 Horadam sequences

A sequence $\{w_n\}_{n=0}^{\infty} = \{w_n(a,b;p,q)\}_{n=0}^{\infty}$ defined by the recurrence

$$w_{n+2} = pw_{n+1} + qw_n, \quad w_0 = a, w_1 = b, \tag{2.1.1}$$

where the parameters a, b, p and q are complex numbers is called a Horadam sequence. For simplicity, the Horadam sequence $\{w_n(a,b;p,q)\}_{n=0}^{\infty}$ is written $\{w_n\}_{n=0}^{\infty}$ hereafter.

The second-order linear recurrent sequence (2.1.1) was named to honour the work of A.F. Horadam—who initiated the investigation of this general recursion in two seminal 1960's papers [53, 55]. When starting from $(a,b) = (0,1)$, for $(p,q) = (1,1)$ one recovers the Fibonacci, while for $(p,q) = (1,-1)$ the Lucas sequences, respectively.

The characteristic equation associated with the recurrence (2.1.1) is

$$P(x) = x^2 - px - q = 0, \tag{2.1.2}$$

whose roots termed *generators* are denoted by z_1 and z_2. Vieta's relations written for the polynomial P give

$$-p = z_1 + z_2, \quad q = z_1 z_2, \tag{2.1.3}$$

showing that the recurrence (2.1.1) defined for coefficients p, q may alternately be defined through the solutions z_1, z_2 of the characteristic polynomial, through (2.1.3).

General sequence term

We first present formulas for the general term w_n of the complex Horadam sequence (2.1.1), when the characteristic polynomial (2.1.2) has distinct, or equal roots.

Non-degenerate case: Distinct roots ($z_1 \neq z_2$)

The general term of the sequence $\{w_n\}_{n=0}^{\infty}$, for distinct roots z_1, z_2 of (2.1.2), is given by (see, for example, [1, Chapter 7], [38, Chapter 1] or [65])

$$w_n = Az_1^n + Bz_2^n = \frac{1}{z_2 - z_1}\left[(az_2 - b)z_1^n + (b - az_1)z_2^n\right], \tag{2.1.4}$$

with constants A and B obtained from $w_0 = a$ and $w_0 = b$ as

$$A = \frac{az_2 - b}{z_2 - z_1}, \quad B = \frac{b - az_1}{z_2 - z_1}, \tag{2.1.5}$$

Degenerate case: Equal roots ($z_1 = z_2$)

When the characteristic roots are equal ($z_1 = z_2 = z$, say) the general term of the associated Horadam sequence is given by

$$w_n = Az^n + Bnz^n = \left[a + \left(\frac{b}{z} - a\right)n\right]z^n \tag{2.1.6}$$

Particular Horadam orbits

The results in this section have been published in [21] and concern Horadam orbits produced by distinct generators. Some can be formulated for the equal generator case.

Orbits produced by conjugate generators (if $a, b \in \mathbb{R}$)

Theorem 2.1.1. *Let $\{w_n\}_{n=0}^{\infty}$ be a Horadam sequence of general term (2.1.4), whose generators satisfy $z_1 = \overline{z}_2$. The sequence $\{W_n\}_{n=0}^{\infty}$ represents a subset of the real line whenever $a, b \in \mathbb{R}$. In this case we obtain the classical Horadam sequence for real numbers.*

Concentric orbits produced by opposite entries

Theorem 2.1.2. *Let $k \in \mathbb{N}$ be even, z_1, z_2 opposite primitive k-th roots satisfying $z_2 = -z_1$ and $a, b \in \mathbb{C}$ arbitrary. The orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ defined in (2.1.1) is formed from two concentric regular k/2-gons, whose nodes represent a bipartite graph (see Fig. 2.1).*

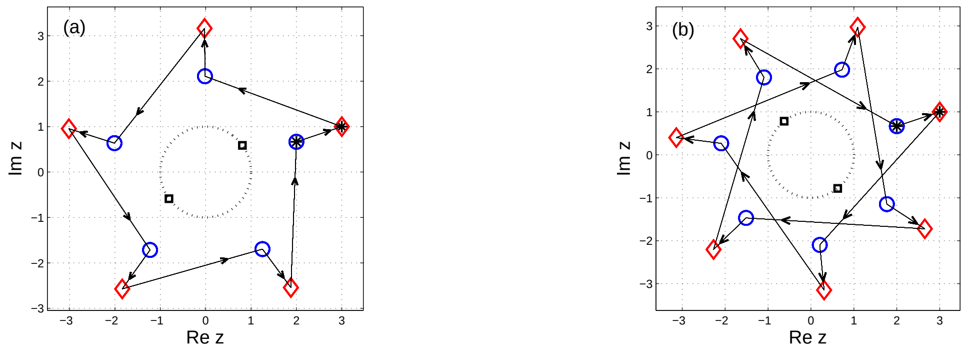

Figure 2.1: First $N = 100$ orbit terms of sequence $\{w_n\}_{n=0}^{N}$ obtained from (2.1.4), computed for pairs of opposite roots (a) $k = 10$, $z_1 = e^{2\pi i \frac{1}{10}}$, $z_1 = e^{2\pi i \frac{6}{10}}$; (b) $k = 14$, $z_1 = e^{2\pi i \frac{5}{14}}$, $z_1 = e^{2\pi i \frac{12}{14}}$. Arrows indicate the direction of the orbit from one term to the next. The dotted line is the unit circle. The initial conditions $w_0 = a = 2 + 2/3i$ and $w_1 = b = 3 + i$ are represented by stars.

Conjugate orbits produced by conjugate parameters

Theorem 2.1.3. *Let $\{w_n\}_{n=0}^{\infty}$ be the sequence defined in (2.1.4) for generators $z_1 \neq z_2$ and initial conditions a and b. The sequence $\{W_n\}_{n=0}^{\infty}$ generated by the conjugate generators $\overline{z_1}, \overline{z_2}$ and initial conditions $\overline{a}, \overline{b}$ satisfies*

$$W_n = \overline{w}_n, \quad n \in \mathbb{N}. \tag{2.1.7}$$

2.2 Periodicity of complex Horadam sequences

In one of his initial publications [53], Horadam himself made a passing remark about two periodic p, q sequence instances $\{w_n(a, b; \pm 1, 1)\}_{n=0}^{\infty}$. In this section necessary and sufficient conditions for the periodicity of the sequence $\{w_n\}_{n=0}^{\infty}$ are established when the characteristic solutions z_1, z_2 of (2.1.2) are distinct or identical. The results in this section have been published in [18].

Lemma 2.2.1. *The set $M = \{\{nx\} \mid n \in \mathbb{N}\}$ is dense in the interval $[0, 1]$ for every $x \in \mathbb{R} \backslash \mathbb{Q}$.*

The lemma below describes the behaviour of sequence $\{z^n\}_{n=0}^{\infty}$ for arbitrary $z \in \mathbb{C}$.

Lemma 2.2.2. *Let $z = re^{2\pi i x} \in \mathbb{C}$ be a complex number ($r > 0$). The orbit of $\{z^n\}_{n=0}^{\infty}$ is*

(i) a regular k-gon if $r = 1$, and $x = j/k \in \mathbb{Q}$ with $\gcd(j, k) = 1$;

(ii) a dense subset of the unit circle for $r = 1$ and $x \in \mathbb{R} \setminus \mathbb{Q}$;

(iii) an inward spiral for $r < 1$;

(iv) an outward spiral for $r > 1$.

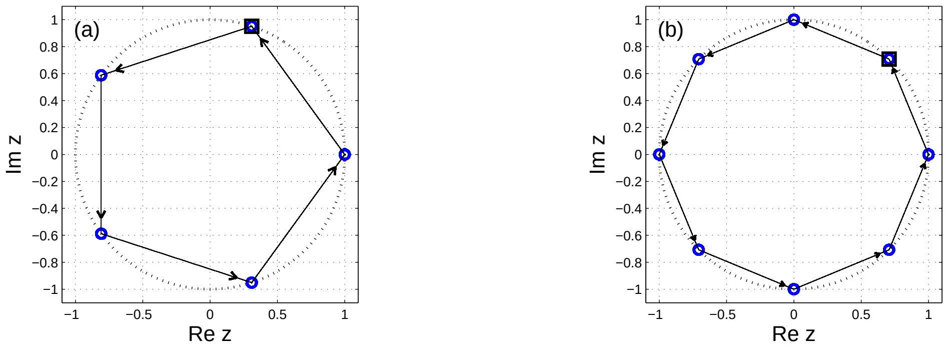

Figure 2.2: Orbit of $\{z^n\}_{n=0}^{\infty}$ obtained for $r = 1$ and (a) $x = 1/5$; (b) $x = 1/8$. Arrows indicate orbit's direction, dotted line the unit circle and generator $z = r\exp(2\pi ix)$ is shown as a square.

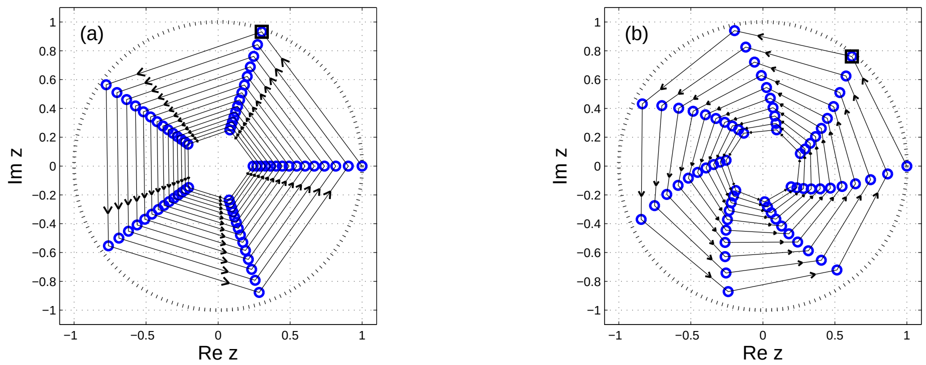

Figure 2.3: First 71 terms of $\{z^n\}_{n=0}^{\infty}$ obtained for $r = 0.98$ and (a) $x = 1/5$; (b) $x = \sqrt{2}/10$. Orbit's direction shown by arrows, unit circle by a dotted line and $z = r\exp(2\pi ix)$ by a square.

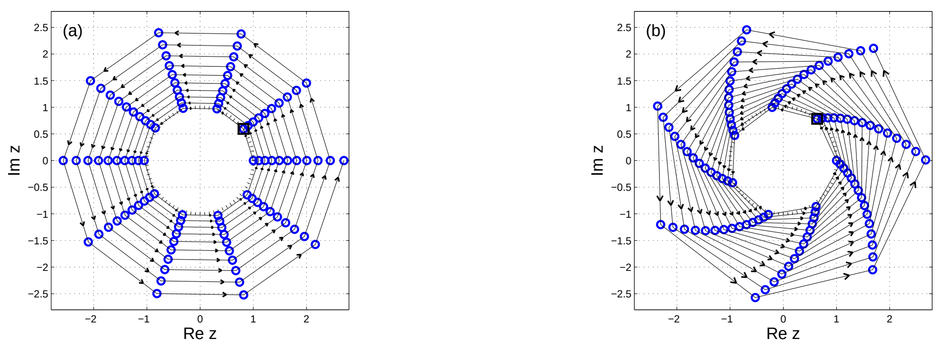

Figure 2.4: First 101 terms of $\{z^n\}_{n=0}^{\infty}$ obtained for $r = 1.01$ and (a) $x = 1/10$; (b) $x = \sqrt{2}/10$. Orbit's direction shown by arrows, unit circle by a dotted line and $z = r\exp(2\pi ix)$ by a square.

Periodicity conditions: Non-degenerate Case ($z_1 \neq z_2$)

Let $z_1 \neq z_2$ be distinct kth roots of unity ($k \geq 2$), and let the polynomial $P(x)$ be

$$P(x) = (x - z_1)(x - z_2), \quad x \in \mathbb{C}. \tag{2.2.1}$$

Theorem 2.2.3. *The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (2.2.1), and the arbitrary initial values $w_0 = a$, $w_1 = b$, is periodic.*

Theorem 2.2.4. *(Necessary condition for periodicity) Let $z_1 \neq z_2$ be the distinct roots of the characteristic polynomial (2.2.1). The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by z_1, z_2, and arbitrary initial values $w_0 = a$, $w_1 = b$, is periodic only if there exists $k \in \mathbb{N}$ s.t.*

$$A(z_1^k - 1)z_1 = 0, \quad B(z_2^k - 1)z_2 = 0, \tag{2.2.2}$$

where A and B are given by (2.1.5). Explicitly, these conditions allow for the following subcases:

(i) z_1 and z_2 are kth roots of unity (for some natural number $k \geq 2$) (non-degenerate);

(ii) z_1 or z_2 is a kth root of unity and the other is zero (regular polygon);

(iii) z_1 or z_2 is a kth root of unity and satisfies $b = az_1$ or $b = az_2$, resp. (regular polygon);

(iv) z_1 and z_2 are arbitrary, and $a = b = 0$ (degenerate orbit).

Periodicity conditions: Degenerate Case ($z_1 = z_2$)

Let z be a kth root of unity ($k \geq 2$), and let the polynomial $P(x)$ be

$$P(x) = (x - z)^2, \quad x \in \mathbb{C}. \tag{2.2.3}$$

Theorem 2.2.5. *The sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (2.2.3), and arbitrary initial values $w_0 = a$, $w_1 = b$, is periodic when $b = az$, being otherwise divergent.*

Proposition 2.2.6. *When generated by a repeated kth root of unity, the terms of the divergent subsequence $\{w_{Nk+j}\}_{N=0}^{\infty}$ are collinear for each value of $j \in \{0, \ldots, k-1\}$.*

Theorem 2.2.7. *(Necessary condition for periodicity) The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (2.2.3), and arbitrary initial values $w_0 = a$, $w_1 = b$, is periodic only if one of the following is true*

$$(z = 0) \text{ or } (z^k - 1 = 0,\ B = 0) \text{ or } (z^k - 1 \neq 0,\ A = B = 0). \tag{2.2.4}$$

Explicitly, these conditions give the subcases

(i) $z = 0$ (degenerate orbit)

(ii) z is a kth root of unity (for some natural number $k \geq 2$) and $b = az$ (regular polygon);

(iii) z is arbitrary and $a = b = 0$ (degenerate orbit).

2.3 The geometry of periodic Horadam orbits

A classification of the geometry patterns produced by generators $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$ is proposed. The results in this section are published in [19].

Regular star polygons

Theorem 2.3.1. *If $z_1 = e^{2\pi i p/k}$ is a primitive kth root $(k \geq 2)$ and $z_2 = 1$, the orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ is the regular star polygon $\{k/p\}$. The property is illustrated in Fig. 2.5.*

Proof. In this case, the general formula (2.1.4) gives $w_n = Az_1^n + B$. $\square$

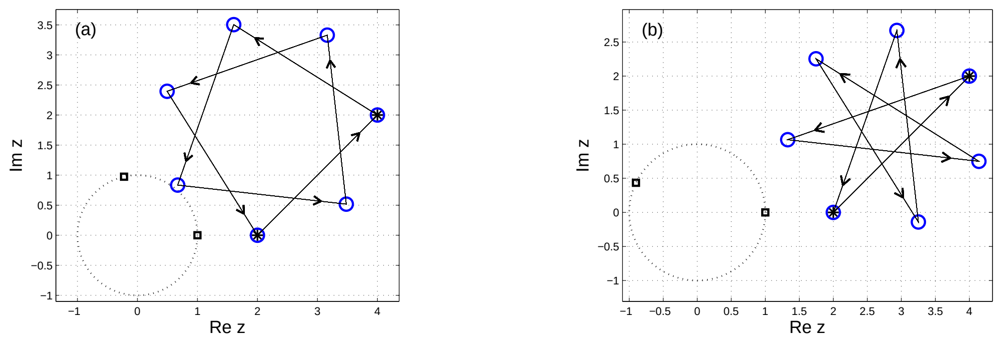

Figure 2.5: First $N = 100$ orbit terms of sequence $\{w_n\}_{n=0}^{N}$ obtained from (2.1.4), for $z_2 = 1$ and (a) $k = 7, z_1 = e^{2\pi i \frac{2}{7}}$; (b) $k = 7, z_1 = e^{2\pi i \frac{3}{7}}$; when $a = 2$ and $b = 4 + 2i$. Also shown are orbit's direction (arrows), starting points w_0, w_1 (star), $w_2, \ldots, w_N$ (circles), unit circle (dotted line).

Bipartite graphs

Theorem 2.3.2. *(k is odd) Let $k \geq 2$ be an odd number, z_1 a primitive kth root and $z_2 = -1$. The orbit of sequence $\{w_n\}_{n=0}^{\infty}$ is a 2k-gon, whose nodes can be divided into two regular k-gons representing a bipartite graph. The property is illustrated in Fig. 2.6 (a).*

Proof. In this case, the general formula (2.1.4) gives $w_n = Az_1^n + (-1)^n B$, and the orbit $W = \{w_0, w_1, \ldots, w_{2k-1}\}$ which can be partitioned into the disjoint ordered sets

$$\begin{aligned} W_0 &= \{A + B, Az_1^2 + B, \ldots, Az_1^{k-1} + B, Az_1^{k+1} + B, \ldots, Az_1^{2k-2} + B\}, \\ W_1 &= \{Az_1 - B, Az_1^3 - B, \ldots, Az_1^{k-2} - B, Az_1^k - B, \ldots, Az_1^{2k-1} - B\}, \end{aligned} \tag{2.3.1}$$

the vertices of two regular k-gons, visited alternatively by sequence $\{w_n\}_{n=0}^{\infty}$. $\square$

Theorem 2.3.3. *(k is even) Let $k \geq 2$ be an even number, z_1 a primitive kth root and $z_2 = -1$. The orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ is a k-gon, whose nodes can be divided into two regular $k/2$-gons representing a bipartite graph. The property is illustrated in Fig. 2.6 (b).*

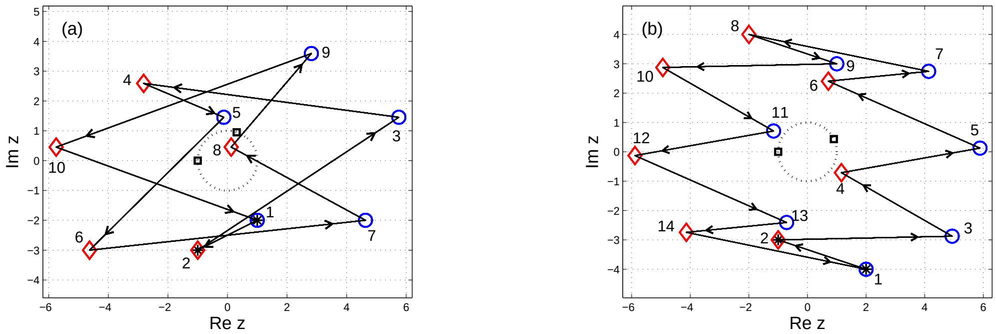

Figure 2.6: First $N = 100$ orbit terms of $\{w_n\}_{n=0}^{N}$ obtained from (2.1.4), for $z_2 = -1$ and (a) $k = 5$ (odd), $z_1 = e^{2\pi i \frac{1}{5}}$ and $a = 2 - 4i$, $b = -1 - 3i$; (b) $k = 14$ (even), $z_1 = e^{2\pi i \frac{1}{14}}$ and $a = 1 - 2i$, $b = -1 - 3i$. Arrows indicate the direction of the orbit visiting w_0, w_1 (star), $w_2, \ldots, w_N$.

Multipartite graphs

In the general case, multipartite periodic orbits of $\{w_n\}_{n=0}^{\infty}$ are obtained for the primitive roots of unity $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$.

Theorem 2.3.4. *Let $k_1, k_2, d \in \mathbb{N}$ s.t. $\gcd(k_1, k_2) = d$ and z_1, z_2 be k_1th,k_2th primitive roots, respectively. The orbit $\{w_n\}_{n=0}^{\infty}$ is a $k_1 k_2 / d$-gon, whose nodes can be divided into k_1 regular k_2/d-gons. By duality, the nodes of the orbit can also be divided into k_2 regular k_1/d-gons. The property is illustrated in Fig. 2.7 for $k_1 = 4$ and $k_2 = 5$, satisfying $(k_1, k_2) = 1$.*

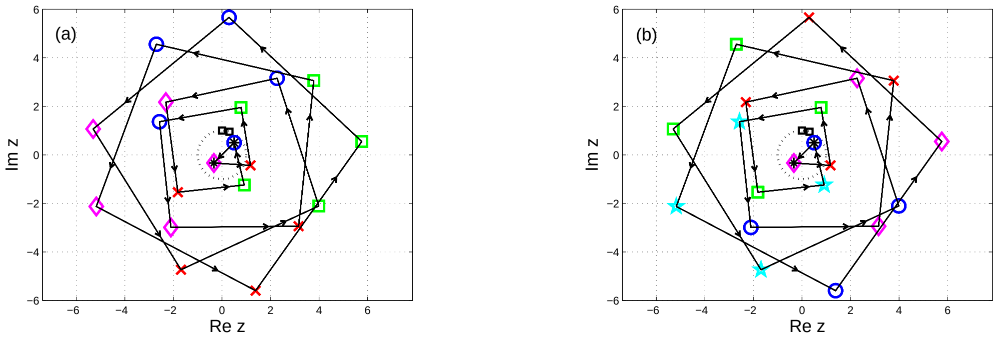

Figure 2.7: First $N = 100$ orbit terms of sequence $\{w_n\}_{n=0}^{N}$ obtained from (2.1.4), for $k_1 = 4$, $k_2 = 5$. We compute $w_0, \ldots, w_N$ for $z_1 = e^{2\pi i \frac{1}{5}}$, $z_2 = e^{2\pi i \frac{1}{4}}$ and initial conditions $a = (1 + i)/2$, $b = -(1 + i)/3$. The orbits are partitioned into (a) four regular pentagons; (b) five squares; Arrows indicate orbit's direction from one term to the next. The dotted line is the unit circle.

Corollary 2.3.5. *If $k_2 | k_1$ the orbit is a k_1-gon whose nodes can be divided into k_2 regular k_1/k_2-gons. The asymmetry in Fig. 2.8 illustrates this, as the only regular polygons that can be identified in the periodic orbit of $\{w_n\}_{n=0}^{\infty}$ are 1- and 2-gons.*

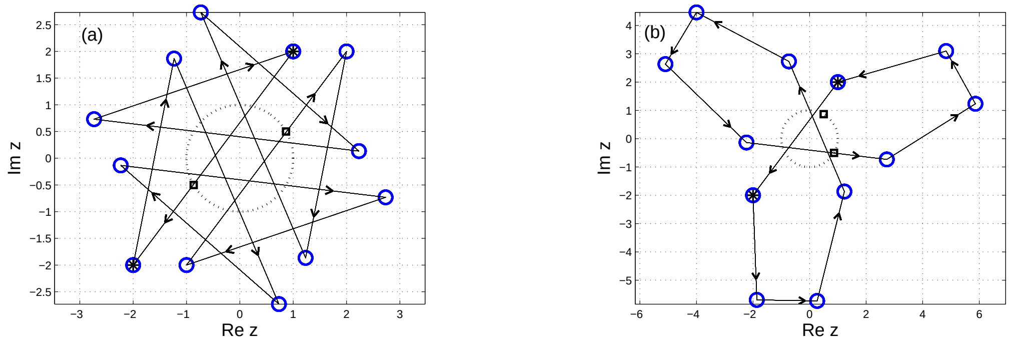

Figure 2.8: First $N = 100$ orbit terms of sequence $\{w_n\}_{n=0}^{N}$ obtained from (2.1.4), for initial conditions are $a = 1 + 2i$ and $b = -2 - 2i$. We compute $w_0, \dots, w_N$ for (a) $k_1 = k_2 = 12$, $z_1 = e^{2\pi i \frac{1}{12}}$, $z_2 = e^{2\pi i \frac{7}{12}}$; (b) $k_1 = 2k_2 = 12$, $z_1 = e^{2\pi i \frac{1}{6}}$, $z_2 = e^{2\pi i \frac{11}{12}}$. Arrows indicate the direction of the orbit from one term to the next. The dotted line is the unit circle.

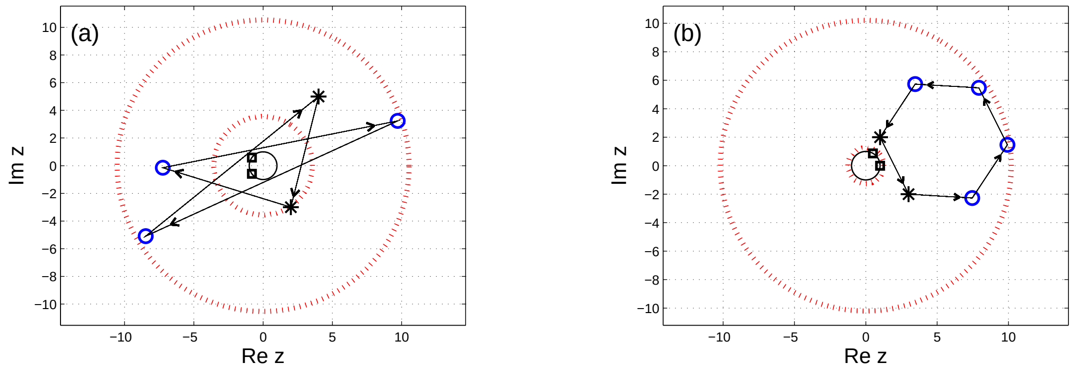

Figure 2.9: Orbit of a periodic Horadam sequence $\{w_n\}_{n=0}^{\infty}$ computed for (a) $z_1 = e^{2\pi i \frac{2}{5}}$, $z_2 = e^{2\pi i \frac{3}{5}}$, $a = 4 + 5i$, $b = 2 - 3i$; (b) $z_1 = e^{2\pi i \frac{1}{6}}$, $z_2 = e^{2\pi i \frac{5}{6}}$, $a = 1 + 2i$, $b = 3 - 2i$. Also plotted are the initial values a, b (stars), the generators z_1, z_2 (squares), the unit circle $S(0,1)$ (solid line) and boundaries of the annulus $U(0, ||A| - |B||, |A| + |B|)$ (dashed lines).

Geometric bounds of periodic orbits

Theorem 2.3.6. *When the Horadam sequence $\{w_n\}_{n=0}^{\infty}$ is periodic, the orbit is subject to the following geometric boundaries (see Fig. 2.9):*

(i) For $z_1 \neq z_2$, then one has (where A and B given by (2.1.5))

$$\{z \in \mathbb{C} : ||A| - |B|| \leq |w_n| \leq |A| + |B|\}, \quad \forall n \in \mathbb{N}.$$

(ii) For $z_1 = z_2 = z$ the orbit is regular k-gon within circle $S(0, |a|) = \{z \in \mathbb{C} : |z| = |a|\}$, for $a \neq 0$, or else the zero set $\{0\}$ for $a = 0$.

2.4 The enumeration of periodic Horadam patterns

In this section we investigate the number of distinct Horadam sequences which (for arbitrary initial conditions) have a fixed period, giving enumeration formulas in both degenerate and non-degenerate cases. The results have been published in [19].

2.4.1 The number of Horadam patterns of fixed length $H_P(k)$

Let $k \geq 2$ be a positive integer. The function for enumerating the Horadam sequences $\{w_n\}_{n=0}^{\infty}$ of period k is denoted by $H_P(k)$. This depends on the generators $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$, and the initial conditions a, b. There are two types of (degenerate and non-degenerate) periodic orbits to consider.

Degenerate orbits

A degenerate orbit is a regular polygon centered in 0 or point.

The number of distinct degenerate sequences having period k is given by

$$H_P(k) = |\{(p_1, k_1) : (p_1, k_1) = 1, \, k_1 = k\}| = \varphi(k),$$

where φ is Euler's totient function.

If no generator appears explicitly in the formulas (2.1.6) or (2.1.4) (this can be the case when $z_1 \neq z_2$, $A = 0$, $B = 0$ or $z_1 = z_2 = z$, $a = 0$, $b = 0$), the periodic sequence is constant and the number of generator configurations leading to periodicity $k \geq 2$ is therefore zero.

Non-Degenerate orbits

Here we cover periodic sequences producing non-degenerated orbits. In this case the generators are distinct roots of unity $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$, and the arbitrary initial conditions a, b are such that $AB \neq 0$ for A, B defined in (2.1.5).

The number of distinct sequences of period k can be enumerated from the quadruples

$$H_P(k) = |\{(p_1, k_1, p_2, k_2) : (p_1, k_1) = (p_2, k_2) = 1, \, [k_1, k_2] = k, \, k_1 \leq k_2\}| \,. \qquad (2.4.1)$$

Some formulas for this expression are identified, based on the properties of pairs (k_1, k_2) satisfying $[k_1, k_2] = k$, and corresponding generators $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$.

A first formula for $H_P(k)$

The first lemma counts the quadruples (p_1, k_1, p_2, k_2) in (2.4.1) for which $k_1 = k_2$.

Lemma 2.4.1. *If $k_1 = k_2$ and $[k_1, k_2] = k$ then $k_1 = k_2 = k$.*

The number of quadruples (p_1, k, p_2, k) fulfilling (2.4.1) produced in this case is

$$H'_P(k) = |\{(p_1, p_2) : (p_1, k) = (p_2, k) = 1, p_1 < p_2\}| = \frac{1}{2}\,\varphi(k)\,(\varphi(k) - 1)\,.$$

The second lemma counts the quadruples (p_1, k_1, p_2, k_2) when $k_1 \neq k_2$ and $[k_1, k_2] = k$.

Lemma 2.4.2. *If $[k_1, k_2] = k$ and $k_1 \neq k_2$, the number of quadruples (p_1, k_1, p_2, k_2) is*

$$H''_P(k) = |\{(p_1, k_1, p_2, k_2) : (p_1, k_1) = (p_2, k_2) = 1, [k_1, k_2] = k\}| = \varphi(k_1)\varphi(k_2).$$

Theorem 2.4.3. *The number of distinct Horadam sequences of period $k \geq 2$ is equal to*

$$H_P(k) = \sum_{[k_1,k_2]=k,\, k_1<k_2} \varphi(k_1)\varphi(k_2) + \frac{1}{2}\,\varphi(k)\,(\varphi(k) - 1)\,. \tag{2.4.2}$$

The number sequence $H_P(k)$ provided the first context for A102309 in OEIS [97].

$$1, 1, 3, 5, 10, 11, 21, 22, 33, 34, 55, 46, 78, 69, 92, 92, 136, 105, \ldots$$

Example 1: Prime numbers. When k is a prime number we have

$$H_P(k) = k(k-1)/2. \tag{2.4.3}$$

Example 2: Powers of a prime number. For $k = p^m$ with p prime, $m \geq 2$ we have

$$H_P(k) = \frac{\varphi(k)[2k - \varphi(k) - 1]}{2}. \tag{2.4.4}$$

Example 3: Products of two prime numbers. When p, q are prime and $k = pq$, we have

$$H_P(k) = (p-1)(q-1)(pq + p + q)/2.$$

For example, when $k = 6 = 2 \cdot 3$ there are 11 solutions produced by the pairs

$$\left(\frac{p_1}{k_1}, \frac{p_2}{k_2}\right) \in \left\{ \left(\frac{1}{1}, \frac{1}{6}\right), \left(\frac{1}{1}, \frac{5}{6}\right), \left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{2}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{6}\right), \right.$$
$$\left. \left(\frac{1}{2}, \frac{5}{6}\right), \left(\frac{1}{3}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{5}{6}\right), \left(\frac{2}{3}, \frac{1}{6}\right), \left(\frac{2}{3}, \frac{5}{6}\right), \left(\frac{1}{6}, \frac{5}{6}\right) \right\},$$

Some of the orbits realized for $k = 6$ are plotted in Fig. 2.10.

Example 4: More general numbers. The formula for $k = 12$ involves the divisor pairs

$$(k_1, k_2) \in \{(1, 12), (2, 12), (3, 4), (3, 12), (4, 6), (4, 12), (6, 12), (12, 12)\},$$

with multiplicities $\varphi(p)\varphi(q)$ for each pair (p, q) in the list satisfying $p < q$, and finally, $\varphi(12)(\varphi(12) - 1)/2$ for the pair $(12, 12)$. This gives the formula

$$H_P(12) = 4 + 4 + 4 + 8 + 4 + 8 + 8 + 4 \cdot 3/2 = 46.$$

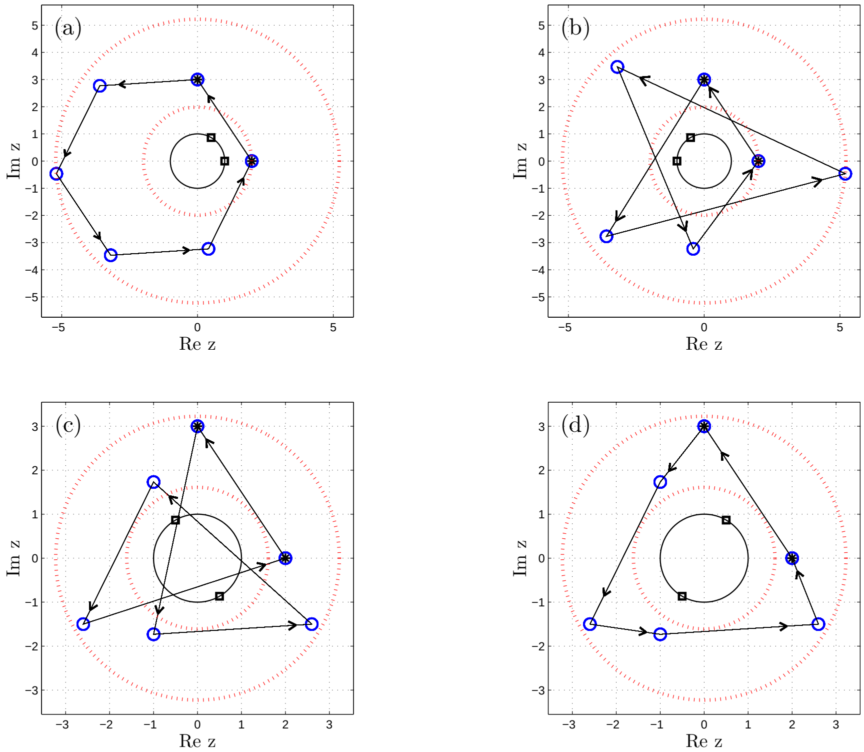

Figure 2.10: Sequence terms $\{w_n\}_{n=0}^{N}$ obtained from (2.1.4) for the pairs $(\frac{p_1}{k_1}, \frac{p_2}{k_2})$ (a) $(\frac{1}{1}, \frac{1}{6})$; (b) $(\frac{1}{2}, \frac{1}{3})$; (c) $(\frac{1}{3}, \frac{5}{6})$; (d) $(\frac{2}{3}, \frac{1}{6})$ when $a = 2$ and $b = 3i$ (stars). Arrows indicate the orbit's direction $w_0, w_1, \dots, w_6 = w_0$ (circles). Also plotted are generators z_1, z_2 (squares), unit circle (solid line) and boundaries of annulus $U(0, ||A| - |B||, |A| + |B|)$ (dotted line) with A, B from (2.1.5).

A second formula for $H_P(k)$

Lemma 2.4.4. *Let $d < k$ be two natural numbers s.t. $d|k$, whose prime decomposition is*

$$d = p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}, \quad k = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}, \quad (1 \leq d_i \leq m_i).$$

The number of pairs of natural numbers k_1, k_2 which satisfy $d = (k_1, k_2)$ and $k = [k_1, k_2]$ is

$$GL(d,k) = |\{(k_1,k_2) : d = (k_1,k_2) \text{ and } k = [k_1,k_2]\}| = 2^{\omega(k/d)-1}, \tag{2.4.5}$$

where $\omega(x)$ represents the number of distinct prime divisors for the integer x.

Theorem 2.4.5. *Formula $H_P(k)$ can be written more compactly as*

$$H_P(k) = \left[\sum_{d|k,\, d<k} \varphi(d) 2^{\omega(k/d)} + \varphi(k) - 1 \right] \frac{\varphi(k)}{2}. \tag{2.4.6}$$

Theorem 2.4.6 ($H_P(k)$ **for square-free numbers).** *When the period k is a square-free positive number $k = p_1 p_2 \dots p_m$ for $m \geq 2$ and $p_1, \dots, p_m$ prime numbers, we have*

$$H_P(k) = \Big[(p_1 + 1) \cdots (p_m + 1) - 1 \Big] \frac{(p_1 - 1) \cdots (p_m - 1)}{2}. \tag{2.4.7}$$

Remark 2.4.7. *The function $H_P(k)$ can also be generated from pairs (p_1, p_2) satisfying*

$$H_P(k) = |\{(p_1, p_2) : ((p_1, k), (p_2, k)) = 1, 1 \le p_1 < p_2 \le k\}| . \tag{2.4.8}$$

2.4.2 Computational complexity of evaluating $H_P(k)$

The number of pairs (k_1, k_2) s.t. $[k_1, k_2] = k$ required to evaluate $H_P(k)$ using (2.4.2) is

$$[(2m_1 + 1)(2m_2 + 1) \cdots (2m_n + 1) + 1]/2.$$

In formula (2.4.6) one needs to identify all the distinct divisors d of k, which are exactly

$$(m_1 + 1)(m_2 + 1) \cdots (m_n + 1).$$

Asymptotic bounds for $H_P(k)$

Lower and upper boundaries can be formulated for $H_P(k)$, as illustrated in Fig. 2.11.

Theorem 2.4.8. *For every $k \in \mathbb{N}$ one has the following bounds $\frac{\varphi(k)k}{2} \le H_P(k) \le \frac{(k-1)k}{2}$.*

Theorem 2.4.9. *If $k = p_1 p_2 \cdots p_m$ is square-free, the following lower bound holds*

$$H_P(k) = \Big[(p_1 + 1) \cdots (p_m + 1) - 1\Big] \frac{(p_1 - 1) \cdots (p_m - 1)}{2} \ge \frac{\varphi(k)[2k - \varphi(k) - 1]}{2}.$$

It is interesting that all these bounds are sharp whenever k is prime. An open question is whether this is indeed lower bound for $H_P(k)$ in general, as suggested in Fig. 2.11.

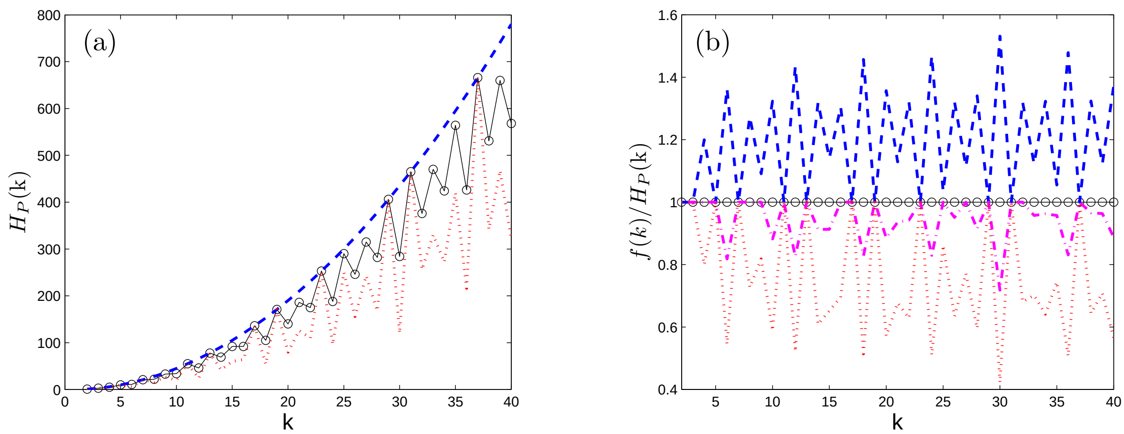

Figure 2.11: First 40 terms of the sequences (a) $H_P(k)$ (circles), $(k-1)k/2$ (dashed) and $\frac{\varphi(k)k}{2}$ (dotted); (b) $f(k)/H_P(k)$, where $f(k)$ is $H_P(k)$ (circles), $(k-1)k/2$ (dashed), $\frac{\varphi(k)k}{2}$ (dotted) and $\frac{\varphi(k)[2k-\varphi(k)-1]}{2}$ (dash-dotted).

CHAPTER 3

Non-periodic Horadam sequences and applications

In this chapter we investigate the orbits produced by non-periodic Horadam sequences. Section 3.1 presents degenerate orbits produced by equal generators, which are either singletons or simple spirals. An atlas of non-degenerate Horadam patterns is presented in Section 3.2. Certain dense Horadam patterns inspired the design a Horadam-based random-number generator, presented in Section 3.3.

3.1 Preliminary results and degenerate orbits

The notations $S = S(0;1)$, $U = U(0;1)$, $S(z_0, r)$ and $U(0; r_1, r_2)$ are used for the unit circle, unit disc, circle of centre z_0 and radius r, and annulus of radii r_1 and r_2.

General term of Horadam sequences

Details on these results can be found in [18] or in Section 2.1. We summarize them here for convenience. Recall that a Horadam sequence $\{w_n\}_{n=0}^{\infty} = \{w_n(a, b; p, q)\}_{n=0}^{\infty}$ is defined by the recurrence

$$w_{n+2} = pw_{n+1} + qw_n, \quad w_0 = a, w_1 = b,$$

where the parameters a, b, p, q are complex numbers. The characteristic equation is

$$x^2 - px - q = 0, \tag{3.1.1}$$

whose roots z_1 and z_2 are called generators.

For equal roots $z_1 = z_2$ of (3.1.1), the general term of Horadam's sequence $\{w_n\}_{n=0}^{\infty}$ is

$$w_n = \left[a + \left(\frac{b}{z} - a\right)n\right]z^n = \left[az + (b - az)n\right]z^{n-1}. \tag{3.1.2}$$

For distinct roots $z_1 \neq z_2$ of (3.1.1), the general term of sequence $\{w_n\}_{n=0}^{\infty}$ are

$$w_n = Az_1^n + Bz_2^n, \tag{3.1.3}$$

where the constants A and B are obtained from initial conditions $w_0 = a$ and $w_1 = b$ as

$$A = \frac{az_2 - b}{z_2 - z_1}, \quad B = \frac{b - az_1}{z_2 - z_1}. \tag{3.1.4}$$

When $AB = 0$, at least one of the generators z_1 and z_2 does not appear explicitly in the formula of w_n. For this reason, it will be assumed from now on that $AB \neq 0$.

Behaviour of sequence $\{z^n\}_{n=0}^{\infty}$

As linear combinations of $\{z_1^n\}_{n=0}^{\infty}$ and $\{z_2^n\}_{n=0}^{\infty}$ with coefficients A and B, the patterns of Horadam sequences largely depend on the behaviour of $\{z^n\}_{n=0}^{\infty}$, where $z \in \mathbb{C}$. This is described in the following result [18, Lemma 2.1]

Lemma 3.1.1. *Let $z = re^{2\pi ix}$ be a complex number ($r \geq 0, x \in \mathbb{R}$). The orbit of $\{z^n\}_{n=0}^{\infty}$ is*

(a) a regular k-gon if z is a primitive k-th root of unity;

(b) a dense subset of the unit circle if $r = 1$ and $x \in \mathbb{R} \setminus \mathbb{Q}$;

(c) an inward spiral for $r < 1$;

(d) an outward spiral for $r > 1$.

Whenever $x = j/k \in \mathbb{Q}$ is irreducible, the spirals in (c) and (d) are aligned along k rays.

Patterns produced by identical generators

We first examine the orbits produced by repeated roots of the quadratic (3.1.1).

Theorem 3.1.2. *Let $\{w_n\}_{n=0}^{\infty}$ be the sequence defined by (3.1.3) for initial conditions $w_0 = a$, $w_1 = b$ and let us assume that the polynomial (3.1.1) has a repeated root $z = z_1 = z_2 = re^{2\pi ix}$. The orbit of $\{w_n\}_{n=0}^{\infty}$ is reduced to a single point if $|a| + |b| = 0$. Otherwise, this represents:*

(a) the vertices of a k-gon when $b = az$ and z is a primitive k-th root of unity;

(b) a dense subset of S when $b = az$ and $|z| = 1$ and $x \in \mathbb{R} \setminus \mathbb{Q}$;

(c) a convergent spiral collapsing onto the origin for $|z| < 1$;

(d) a divergent spiral for $|z| > 1$ and $|a| + |b| > 0$, or for $|z| = 1$ and $b \neq az$.

3.2 An atlas of Horadam patterns

The aim of this section is to characterize the orbits of Horedam sequences obtained for arbitrary generators and initial conditions. The distinct generators are here denoted by

$$z_1 = r_1 e^{2\pi i x_1}, \quad z_2 = r_2 e^{2\pi i x_2}, \tag{3.2.1}$$

where r_1, r_2, x_1, x_2 are real numbers. We may assume that $0 \leq r_1 \leq r_2$.

Horadam patterns produced by formula (3.1.3) can be summarized below

1. Stable for $r_1 = r_2 = 1$;
2. Quasi-convergent for $0 \leq r_1 < r_2 = 1$;
3. Convergent for $0 \leq r_1 \leq r_2 < 1$;
4. Divergent for $r_2 \geq 1$.

3.2.1 Stable orbits: $r_1 = r_2 = 1$

The patterns recovered in this scenario are finite sets (periodic), or sets dense within certain 1D curves, or 2D annuli. Stable orbits are located inside the annulus

$$\{z \in \mathbb{C} : |\,|A| - |B|\,| \leq |z| \leq |A| + |B|\}. \tag{3.2.2}$$

Theorem 3.2.1. *Let $\{w_n\}_{n=0}^{\infty}$ be the sequence defined by (3.1.3). If the distinct roots (3.2.1) satisfy $r_1 = r_2 = 1$, the following orbit patterns of $\{w_n\}_{n=0}^{\infty}$ emerge for:*

(a) $x_1, x_2 \in \mathbb{Q}$: periodic orbit (finite set) (see Figs. 2.7, 2.8 or 2.10 in Chapter 2);

(b) $x_1 = p/k \in \mathbb{Q}$ (irreducible), $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ (or vice-versa): Orbit is a dense subset of the reunion of k distinct circles (see Fig. 3.1);

(c) $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$. In this case we have three distinct types of behaviour:

(c1) $x_2 - x_1 = q \in \mathbb{Q}$: the orbit is a finite set of concentric circles (see Fig. 3.2);

(c2) $x_2 = x_1 q$, $q \in \mathbb{Q}$: the orbit is a stable closed curve (flower) (see Fig. 3.3);

(c3) $1, x_1, x_2$ lin. indep. over $\mathbb{Q}$: dense orbit in $U(0, |\,|A| - |B|\,|, |A| + |B|)$ (see Fig. 3.4).

Proof. The dimension of the orbit's closure is zero for finite orbits, one for orbits dense within closed curves and two for orbits dense within an annulus.

(a) Stable periodic (finite) orbits

When $x_1, x_2 \in \mathbb{Q}$ the orbit of the sequence is finite.

(b) Stable orbits dense within unions of circles

When $x_1 = p/k$ is irreducible, z_1 is a k-th primitive root of unity, therefore $\{{z_1}^n\}_{n=0}^{\infty}$ only takes the distinct values $1, z_1, \ldots, z_1^{k-1}$ representing the vertices of a regular k-gon. This property is illustrated for $x_1 = 1/3$ in Fig. 3.1. The $k = 3$ circles are disjoint for $|A| > |B|$ (see Fig. 3.1(a)), and intersect for $|A| < |B|$ (see Fig. 3.1(b)), respectively.

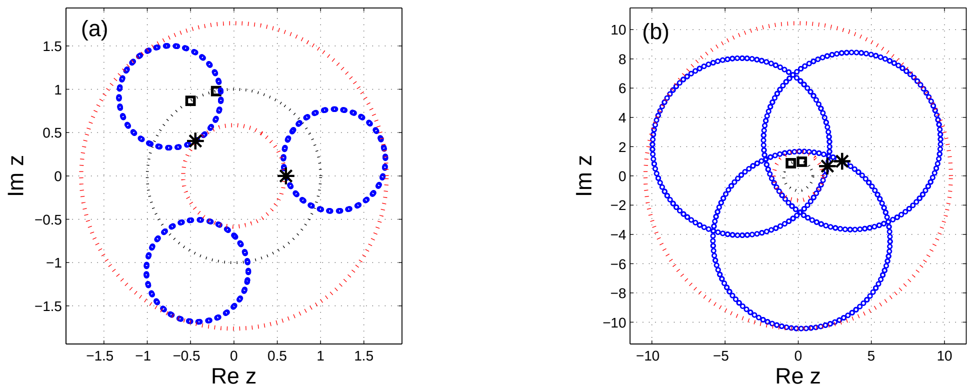

Figure 3.1: First 500 terms of $\{w_n\}_{n=0}^{\infty}$ from (3.1.3) for $r_1 = r_2 = 1$ and (a) $x_1 = 1/3$, $x_2 = \sqrt{2}/5$, when $a = 0.6$ and $b = 0.6\exp(2\pi i(\sqrt{2}/5 + 0.1))$; (b) $x_1 = \pi/15$, $x_2 = 1/3$, when $a = 2 + 2/3i$ and $b = 3 + i$. Also depicted are initial conditions w_0, w_1, (stars), generators z_1, z_2 (squares), unit circle (dotted line). Inner and outer circles show annulus $U(0, ||A| - |B||, |A| + |B|)$.

(c1) Orbits dense within concentric circles

This property is illustrated in Fig. 3.2 for the case when x_1 and x_2 are irrational and $x_2 - x_1 = 1/2$ in (a) and $x_2 - x_1 = 1/3$ in (b), respectively.

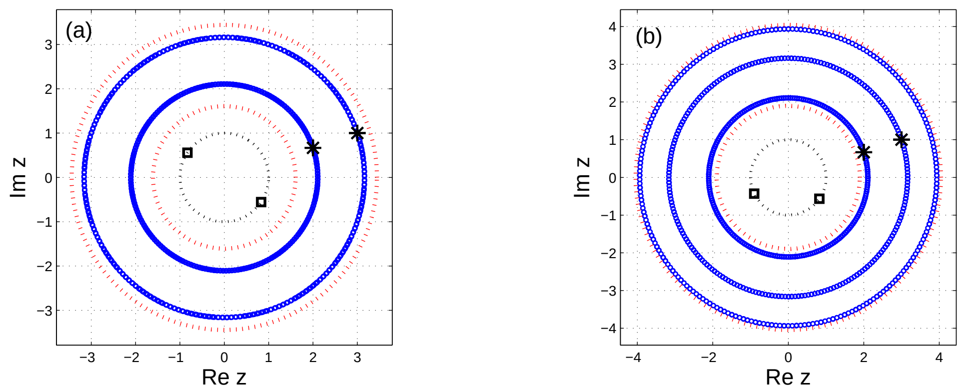

Figure 3.2: First 1000 terms of $\{w_n\}_{n=0}^{\infty}$ obtained from (3.1.3) for $r_1 = r_2 = 1$ and (a) $x_1 = e/3$, $x_2 = e/3 + 1/2$; (b) $x_1 = \pi/2$, $x_2 = \pi/2 + 1/3$, when $a = 2 + 2/3i$ and $b = 3 + i$.

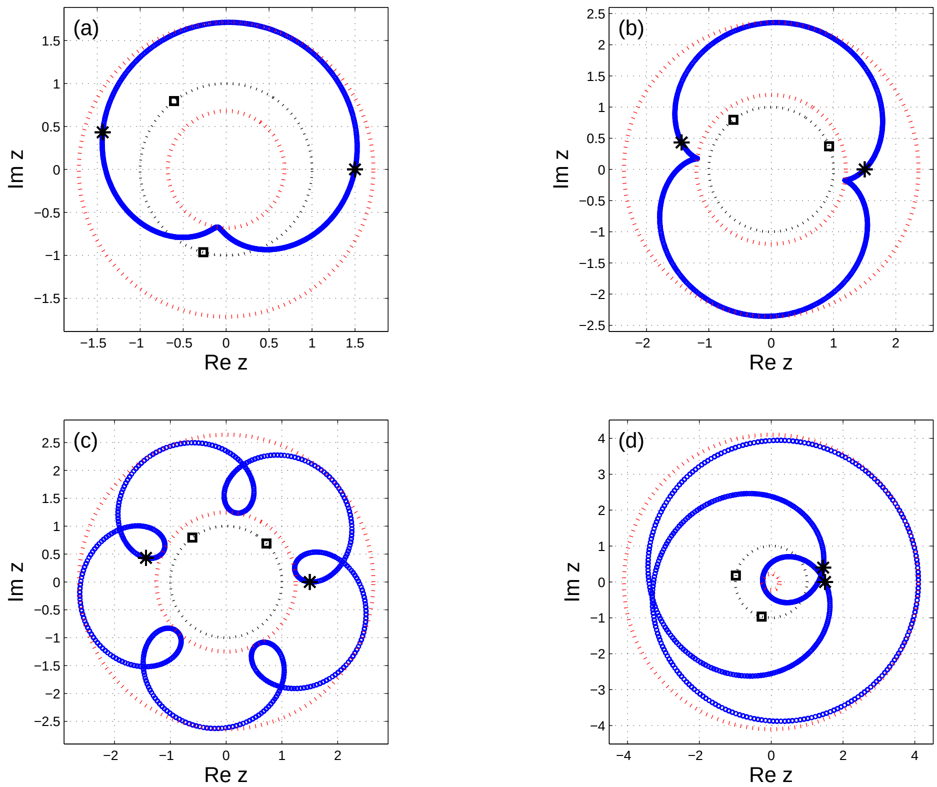

Figure 3.3: First 1000 sequence terms of $\{w_n\}_{n=0}^{\infty}$ obtained from (3.1.3) for $r_1 = r_2 = 1$, $x_2 = qx_1$, where (a) $q = 2$, $x_2 = \sqrt{2}/4$; (b) $q = 3$, $x_2 = \sqrt{2}/4$; (c) $q = 6$, $x_2 = \sqrt{2}/4$; and $a = 1.5$, $b = 1.5\exp(2\pi i(\sqrt{2}/4 + 1/10))$ and also for (d) $q = 2/3$, $x_2 = \sqrt{2}/3$, and $a = 1.5$, $b = 1.5\exp(2\pi i(2\sqrt{2}/3 + 1/10))$. Initial conditions a, b shown by stars, generators z_1, z_2 by squares, unit circle by dotted line. Inner and outer circles depict $U(0, ||A| - |B||, |A| + |B|)$.

(c2) Orbits dense within closed 1D curves

(c2) $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$, $x_2 = x_1 q$, $q \in \mathbb{Q}$. The general term w_n from (3.1.3) can be written as

$$w_n = Az_1^n + Bz_2^n = Az_1^n + B(z_1^q)^n.$$

From Lemma 3.1.1, the orbit of $\{z_1^n\}_{n=0}^{\infty}$ is dense within the unit circle, so the orbit of $\{w_n\}_{n=0}^{\infty}$ is dense within the graph of the complex function $f : S \to \mathbb{C}$ defined as

$$f(z) = Az + Bz^q.$$

This property is illustrated in Fig. 3.3. As shown in (a), $q = 6$ gives 5 self-intersections and winding number one, while for the case $q = 3/2$ illustrated in (d), winding number is three with two self-intersections. For an irreducible fraction $q = m/k$ one can show that the curve $g(x) = Ae^{2\pi xi} + Bz^{2\pi qxi}$, has m self-intersections and winding number k.

(c3) Stable orbits dense within 2D annuli

Whenever $1, x_1, x_2$ are linearly independent over $\mathbb{Q}$ (or $\mathbb{Z}$), the resulting Horadam orbit is dense within an annulus.

The property is illustrated in Fig. 3.4 (a) for $|A| \neq |B|$, obtained for $r_1 = r_2 = 1, x_1 = \frac{\sqrt{2}}{3}, x_2 = \frac{\sqrt{5}}{15}$ and $a = 2 + \frac{2}{3}i$, $b = 3 + i$. When $|A| = |B|$, the orbit is dense within the circle $U(0, 2|A|)$, as depicted in Fig 3.4(b) for the parameter values $r_1 = r_2 = 1$, $x_1 = \exp(1)/2, x_2 = \exp(2)/4$ and $a = 1 + 1/3i$, $b = 1.5a\exp(\pi(x_1 + x_2))$. These types of dense orbits are used to design a random number generator in Section 3.3.

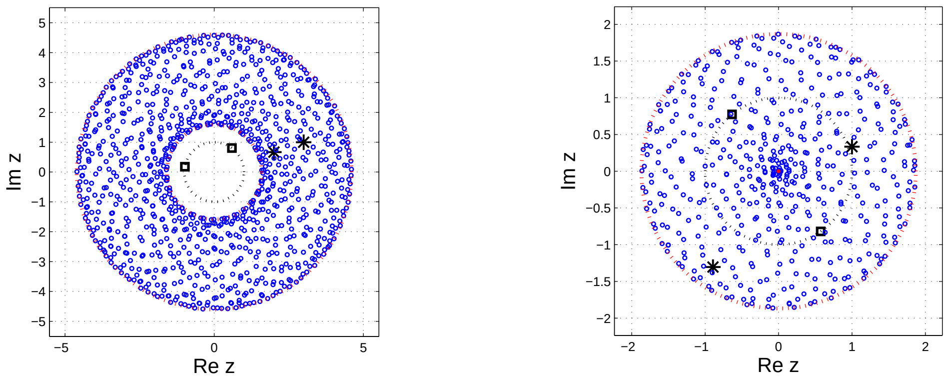

Figure 3.4: First N sequence terms of $\{w_n\}_{n=0}^{\infty}$ obtained from (2.1.4). (a) $N = 1000$, $|A| \neq |B|$; (b) $N = 500$, $|A| = |B|$. Also plotted, initial conditions a, b (stars), generators z_1, z_2 (squares), unit circle S (solid line) and circle $U(0, |A| + |B|)$ (dotted line).

3.2.2 Quasi-convergent orbits for $0 \leq r_1 < r_2 = 1$

Theorem 3.2.2. *Let $\{w_n\}_{n=0}^{\infty}$ be the sequence defined by (3.1.3) for initial conditions $w_0 = a$, $w_1 = b$ and let us assume that the polynomial (3.1.1) has distinct roots $z_1 \neq z_2$ such that $0 \leq r_1 < r_2 = 1$. The orbit of $\{w_n\}_{n=0}^{\infty}$ has the following profiles*

(a) For $x_2 \in \mathbb{Q}$: Orbits attracted onto discrete finite sets (vertices of regular polygons) along rays ($x_1 \in \mathbb{Q}$) or spirals ($x_1 \in \mathbb{R} \setminus \mathbb{Q}$) (see Fig. 3.5).

(b) For $x_2 \in \mathbb{R} \setminus \mathbb{Q}$: Orbits collapse onto a dense circle of radius $|B|$ (see Fig. 3.6).

(a) When $x_2 = p_2/k_2$ is an irreducible fraction, z_2 is a k_2-th primitive root of unity, therefore the sequence $\{{z_2}^n\}_{n=0}^{\infty}$ is periodic and has just k_2 distinct terms $1, z_2, \ldots, z_2^{k_2-1}$ representing the vertices of a regular k_2-gon. When $x_1 = p_1/k_1 \in \mathbb{Q}$ is irreducible, the subsequences $\{w_{nk_2+j}\}_{n=0}^{\infty}$ approach their limit along rays, as depicted in Fig. 3.5 (a). The number of rays is the same for each attractor.

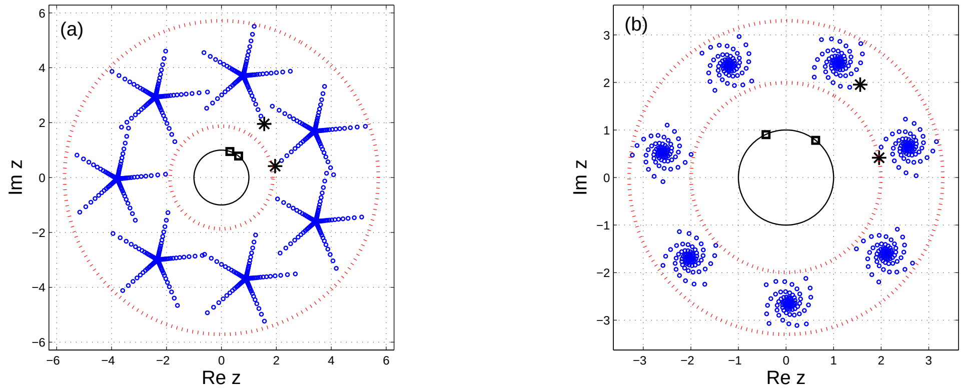

Figure 3.5: First 1000 terms of sequence $\{w_n\}_{n=0}^{\infty}$ obtained from (3.1.3) for $r_1 = 0.995, r_2 = 1$ and (a) $x_1 = 1/5, x_2 = 1/7$; (b) $x_1 = \sqrt{5}/7, x_2 = 1/7$, when $a = 2\exp(2\pi i/30)$ and $2.5\exp(2\pi i/7)$. Stars depict initial conditions a, b, squares the generators z_1, z_2 and the solid line the unit circle. Inner and outer circles represent the boundaries of the annulus $U(0, ||A| - |B||, |A| + |B|)$.

(b) When x_2 is irrational, $\{Bz_2^n\}_{n=0}^{\infty}$ is a dense subset of the circle of radius $|B|$ centred in the origin. As the sequence $\{Az_1{}^n\}_{n=0}^{\infty}$ converges to zero the orbit of sequence $\{w_n\}_{n=0}^{\infty}$ collapses onto the circle of radius $|B|$ centred in the origin, as illustrated in Fig. 3.6.

Note that sequence terms in Fig. 3.6 (a) are inside the circle $U(0, |A| + |B|)$, but not inside the annulus $U(0, ||A| - |B||, |A| + |B|)$, as seen in Fig. 3.6 (a). If $|A| > |B|$ we have $\lim_{n\to\infty} |w_n| = |B| < |A| - |B| = ||A| - |B||$, therefore such a condition is $|A| > 2|B|$, which requires $|az_2 - b| > 2|b - az_1|$. In the example shown in Fig. 3.6 (a) we have $|A| = 3.4669$ and $|B| = 1.5254$.

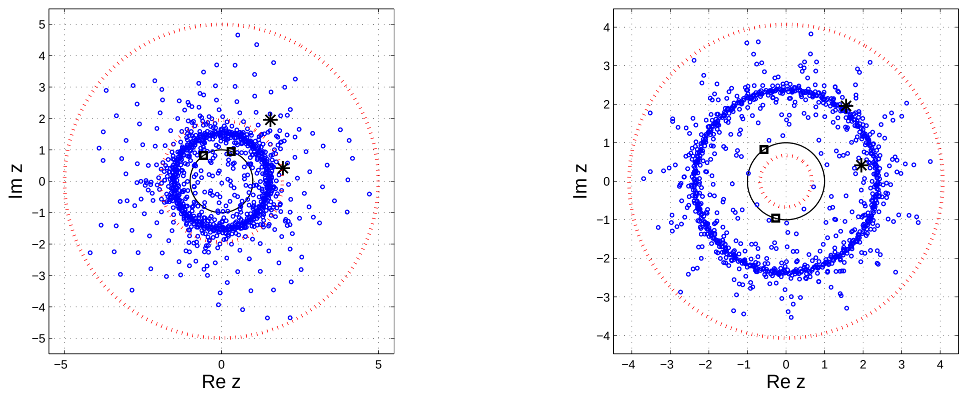

Figure 3.6: First 1000 sequence terms $\{w_n\}_{n=0}^{\infty}$ given by (3.1.3) for $r_1 = 0.995$, $r_2 = 1$ and (a) $x_1 = 1/5, x_2 = \sqrt{3}/5$; (b) $x_1 = \sqrt{2}/2, x_2 = \sqrt{3}/5$, when $a = 2\exp(2\pi i/30)$ and $2.5\exp(2\pi i/7)$. Stars represent initial conditions a, b, squares generators z_1, z_2, and the solid line the unit circle. Inner and outer circles represent the orbit boundaries $U(0, ||A| - |B||, |A| + |B|)$.

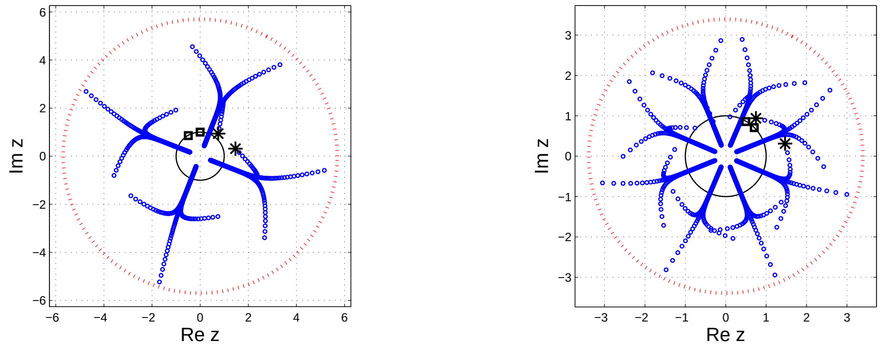

Figure 3.7: First 2000 terms of sequence $\{w_n\}_{n=0}^{\infty}$ obtained from (3.1.3) for $r_1 = 0.99$, $r_2 = 0.999$ and (a) $x_1 = 1/3$, $x_2 = 1/4$; (b) $x_1 = 1/6$, $x_2 = 1/8$; when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$. Also plotted, initial conditions w_0, w_1 (stars), generators z_1, z_2 (squares), unit circle S (solid line) and circle $U(0, |A| + |B|)$ (dotted line).

3.2.3 Convergent orbits for $0 \le r_1 \le r_2 < 1$

The sequence $\{w_n\}_{n=0}^{\infty}$ defined by (3.1.3) now converges to the origin. The cases when r_1 and r_2 are equal, or distinct have to be analyzed separately.

Theorem 3.2.3. *[Orbits for $r_1 < r_2$] Let $\{w_n\}_{n=0}^{\infty}$ be the sequence defined by (3.1.3) for initial conditions $w_0 = a$, $w_1 = b$ and let us assume that the polynomial (3.1.1) has distinct roots $z_1 \ne z_2$ such that $0 \le r_1 < r_2 < 1$. The following orbit profiles of $\{w_n\}_{n=0}^{\infty}$ emerge:*

(a) $x_1 = \frac{p_1}{k_1}, x_2 = \frac{p_2}{k_2} \in \mathbb{Q} : lcm(k_2, k_1)$ convergent branches merging onto k_2 rays (Fig. 3.7).

(b) $x_1 \in \mathbb{R} \setminus \mathbb{Q}$, $x_2 = p/k \in \mathbb{Q}$ (irreducible): k spirally perturbed arms. (see Fig. 3.8).

(c) $x_1 = p/k \in \mathbb{Q}$ (irreducible), $x_2 \in \mathbb{R} \setminus \mathbb{Q}$: Orbit converges concentrically to the origin, as a set of k petals or branches (see Fig. 3.9).

(d) $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$. In this case the orbit is an erratic or ordered spiral (see Fig. 3.10).

Theorem 3.2.4. *(Orbits for $r_1 = r_2 = r$) Let $\{w_n\}_{n=0}^{\infty}$ be the sequence defined by (3.1.3) for initial conditions $w_0 = a$, $w_1 = b$ and let us assume that the polynomial (3.1.1) has distinct roots $z_1 \ne z_2$ such that $0 < r_1 = r_2 = r < 1$. The following orbit profiles of $\{w_n\}_{n=0}^{\infty}$ emerge:*

(a) $x_1 = p_1/k_1, x_2 = p_2/k_2 \in \mathbb{Q} : \mathrm{lcm}(k_1, k_2)$ rays converging to origin (see Fig. 3.11 (a)).

(b) $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ (or vice-versa): k_2 perturbed spirals (see Fig. 3.11 (b),(c),(d)).

(c) $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$. In this case we identify the patterns

(c1). $x_2 - x_1 = q \in \mathbb{Q}$: multiple spiral (see Fig. 3.12 (a)).

(c2). $x_1 = x_2 q$, $q \in \mathbb{Q}$: multi-chamber contours (see Fig. 3.12 (b) and (c)).

(c3). $1, x_1, x_2$ linearly independent over $\mathbb{Q}$: erratic convergent spiral (see Fig. 3.12 (d)).

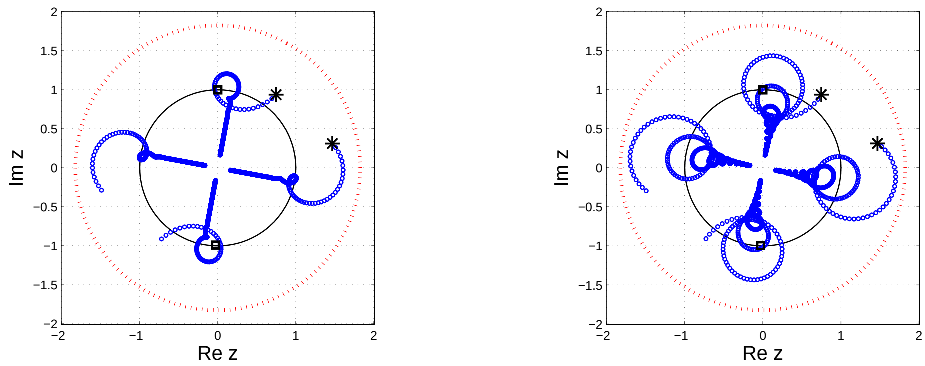

Figure 3.8: First 2000 terms of sequence $\{w_n\}_{n=0}^{\infty}$ obtained from (3.1.3) for parameters $x_1 = \sqrt{5}/3$, $x_2 = 1/4$ and $r_2 = 0.999$ for (a) $r_1 = 0.99$, (b) $r_1 = 0.997$, when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$. Also plotted, initial conditions w_0, w_1 (stars), generators z_1, z_2 (squares), unit circle S (solid line) and circle $U(0, |A| + |B|)$ (dotted line).

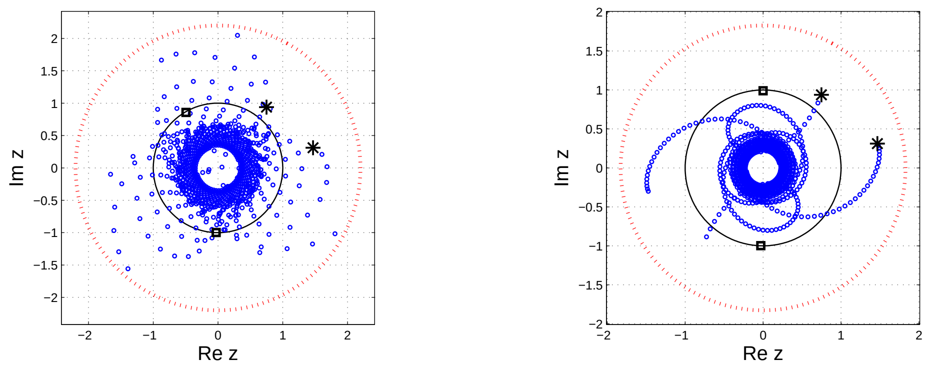

Figure 3.9: First 2000 terms of $\{w_n\}_{n=0}^{\infty}$ obtained for $r_1 = 0.99$, $r_2 = 0.999$ and (a) $x_1 = 1/3$, $x_2 = \sqrt{5}/3$; (b) $x_1 = 1/4$, $x_2 = \sqrt{5}/3$; when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$.

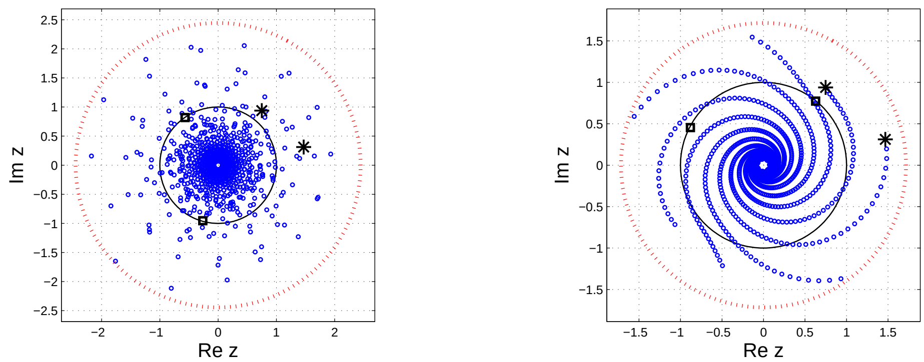

Figure 3.10: First 2000 terms of $\{w_n\}_{n=0}^{\infty}$ for $r_1 = 0.99$, $r_2 = 0.997$ and (a) $x_1 = \sqrt{2}/2$, $x_2 = \sqrt{5}/3$; (b) $x_1 = 3\sqrt{2}/10$, $x_2 = \sqrt{2}/10$, when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$.

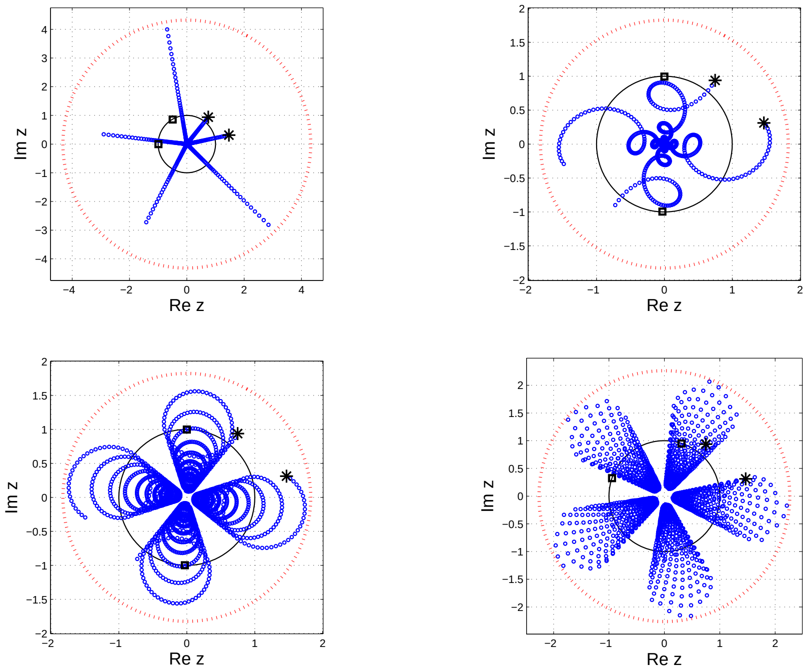

Figure 3.11: First 2000 terms of $\{w_n\}_{n=0}^{\infty}$ given by (3.1.3) for (a) $x_1 = 1/3$, $x_2 = 1/2$, $r = 0.99$; (b) $x_1 = \sqrt{5}/3$, $x_2 = 1/4$, $r = 0.995$; (c) $x_1 = \sqrt{5}/3$, $x_2 = 1/4$, $r = 0.999$; (d) $x_1 = \sqrt{5}/5$, $x_2 = 1/5$, $r = 0.999$ when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$ (stars).

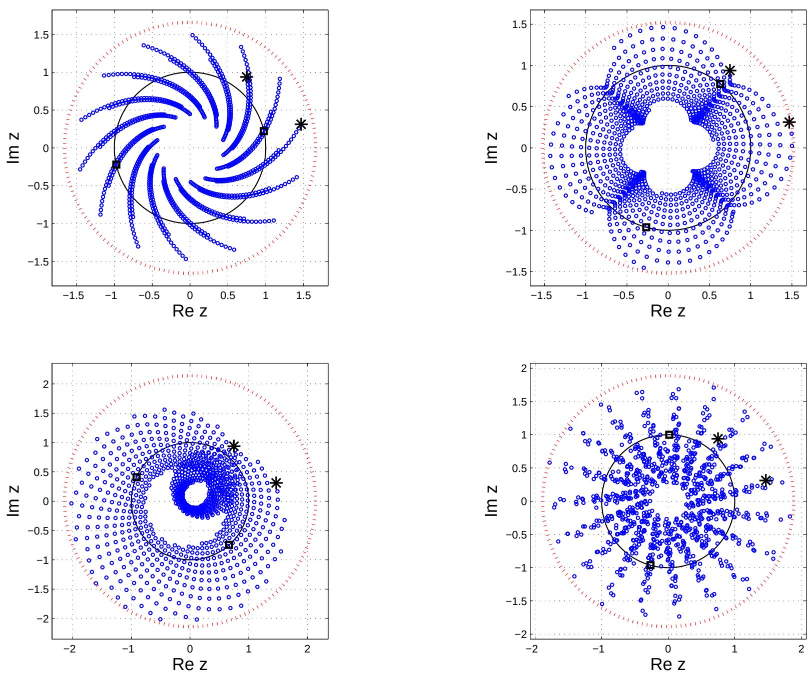

Figure 3.12: Sequence $\{w_n\}_{n=0}^{\infty}$ obtained for $r = 0.999$ and (a) $x_1 = 5\sqrt{2}/2 + 1/2$, $x_2 = 5\sqrt{2}/2$, (b) $x_1 = 5\sqrt{2}/10$, $x_2 = \sqrt{2}/10$; (c) $x_1 = \sqrt{3}/2$, $x_2 = \sqrt{3}/4$; (d) $x_1 = \sqrt{2}/2$, $x_2 = \sqrt{3}/7$.

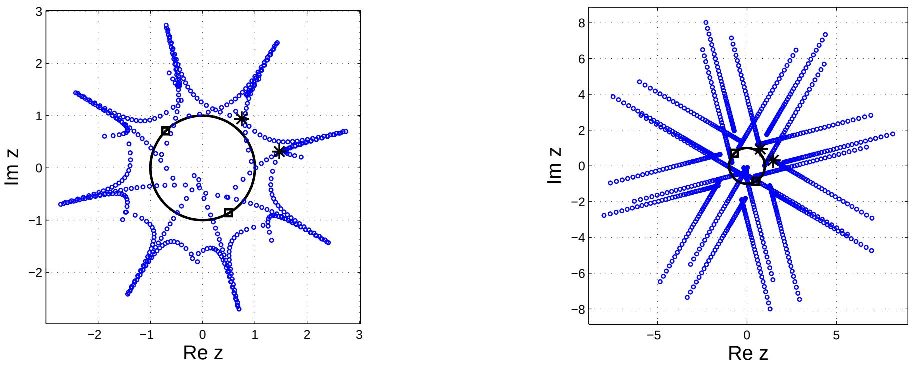

Figure 3.13: First N terms of $\{w_n\}_{n=0}^{\infty}$ for $r_2 = 1.002$, $x_1 = 5/6$, $x_2 = 3/8$ and (a) $N = 500$ and $r_2 = 0.999$; (b) $N = 1000$ and $r_2 = 1$, when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$.

3.2.4 Divergent orbits for $1 < r_2$

The sequence $\{w_n\}_{n=0}^{\infty}$ defined by (3.1.3) diverges to infinity.

Theorem 3.2.5. *[Orbits for $1 < r_2$, $r_1 < r_2$] Let $\{w_n\}_{n=0}^{\infty}$ be the sequence defined by (3.1.3) for initial conditions $w_0 = a$, $w_1 = b$ and assume that the polynomial (3.1.1) has distinct roots $z_1 \neq z_2$ such that $0 < r_1 < r_2$ and $1 < r_2$. The following orbit profiles of $\{w_n\}_{n=0}^{\infty}$ emerge:*

(a) $x_1 = p_1/k_1, x_2 = p_2/k_2 \in \mathbb{Q} : lcm(k_2, k_1)$ branches merging onto k_2 rays which diverge away from the origin to infinity (see Fig. 3.13).

(b) $x_1 \in \mathbb{R} \setminus \mathbb{Q}$, $x_2 = p/k \in \mathbb{Q}$ (irreducible): k divergent spiral arms. (see Fig. 3.14).

(c) $x_1 = p/k \in \mathbb{Q}$ (irreducible), $x_2 \in \mathbb{R} \setminus \mathbb{Q}$: Orbit diverges concentrically to infinity, as a set of k petals or branches (see Fig. 3.15).

(d) $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$. In this case the orbit is an erratic or ordered spiral (see Fig. 3.16).

Theorem 3.2.6. *(Orbits for $r_1 = r_2 = r$) Let $\{w_n\}_{n=0}^{\infty}$ be the sequence defined by (3.1.3) for initial conditions $w_0 = a$, $w_1 = b$ and let us assume that the polynomial (3.1.1) has distinct roots $z_1 \neq z_2$ such that $1 < r_1 = r_2 = r$. For different generators we have the following profiles for the orbit of $\{w_n\}_{n=0}^{\infty}$:*

(a) $x_1 = p_1/k_1, x_2 = p_2/k_2 \in \mathbb{Q} : \mathrm{lcm}(k_1, k_2)$ rays diverging to infinity (see Fig. 3.17 (a)).

(b) $x_1 \in \mathbb{Q} = p_1/k_1$ and $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ (or vice-versa): k_1 divergent spirals (see Fig. 3.17 (b)).

(c) $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$. In this case we identify the patterns

(c1) $x_2 - x_1 = q \in \mathbb{Q}$: multiple divergent spirals (see Fig. 3.17 (c)).

(c2) $x_1 = x_2 q$, $q \in \mathbb{Q}$: multi-chamber divergent contours (see Fig. 3.17 (d)).

(c3) $1, x_1, x_2$ linearly independent over $\mathbb{Q}$: erratic convergent spirals (see Fig. 3.18).

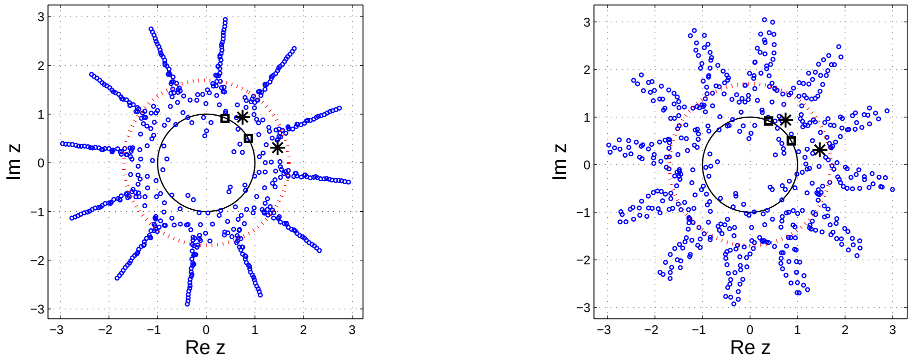

Figure 3.14: First 500 terms of the sequence $\{w_n\}_{n=0}^{\infty}$ obtained for $r_2 = 1.002$, $x_1 = \sqrt{5}/12$, $x_2 = 1/12$ and (a) $r_1 = 0.99$; (b) $r_1 = 0.997$, when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$.

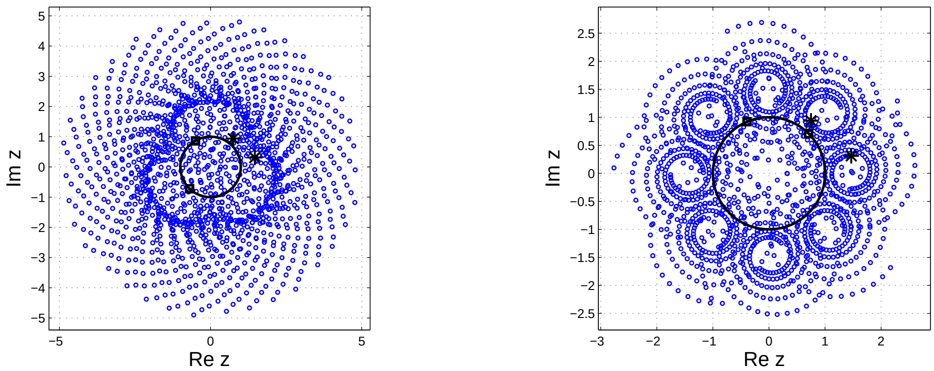

Figure 3.15: First 1500 terms of $\{w_n\}_{n=0}^{\infty}$ obtained for $r_2 = 1.001$, and (a) $x_1 = 1/3$, $x_2 = \pi/5$, $r_1 = 0.997$; (b) $x_1 = 1/8$, $x_2 = \pi/10$, $r_1 = 1$, when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$.

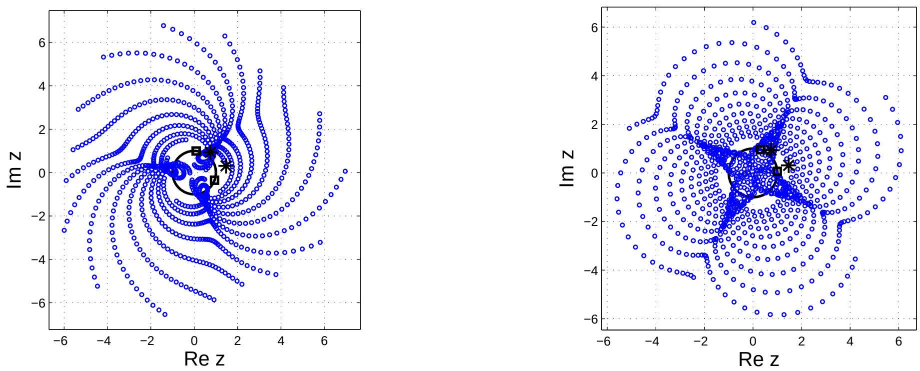

Figure 3.16: First 1000 terms of $\{w_n\}_{n=0}^{\infty}$ obtained for $r_2 = 1$, $r_2 = 1.002$, and (a) $x_1 = 4\sqrt{2}/6$, $x_2 = \sqrt{2}/6$; (b) $x_1 = 5\sqrt{2}/7$, $x_2 = \sqrt{2}/7$ when $a = 1.5\exp(2\pi i/30)$ and $b = 1.2\exp(2\pi i/7)$.

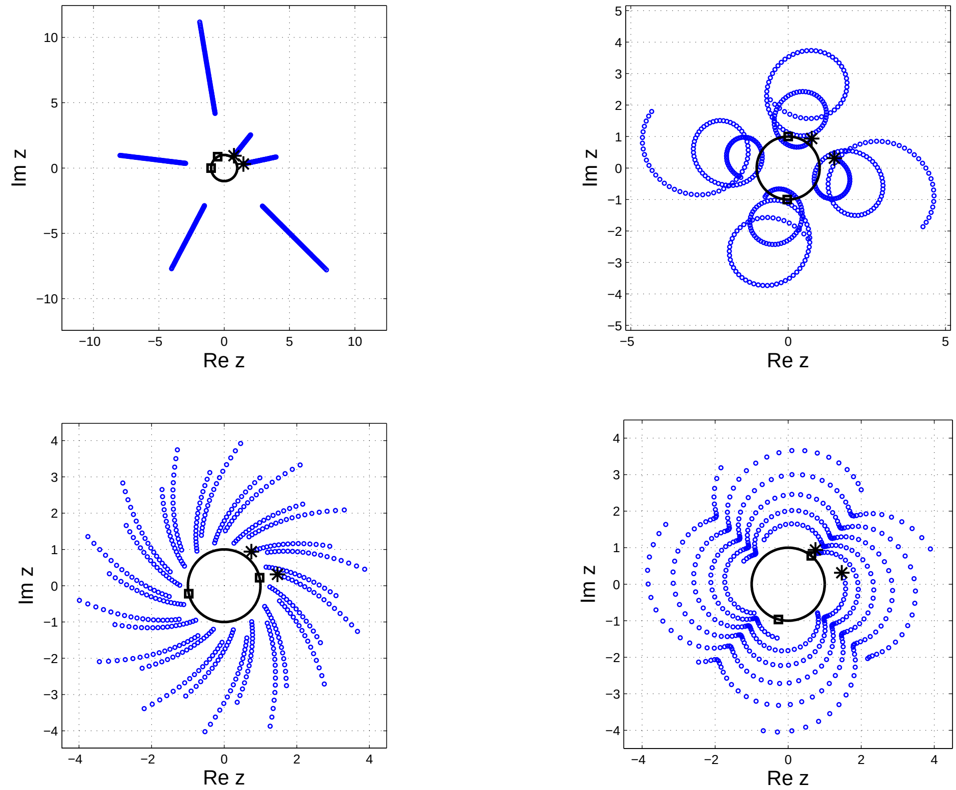

Figure 3.17: First 500 terms of the sequence $\{w_n\}_{n=0}^{\infty}$ obtained from (3.1.3) for $r = 1.002$ and (a) $x_1 = 1/3$, $x_2 = 1/2$; (b) $x_1 = \sqrt{5}/3$, $x_2 = 1/4$; (c) $x_1 = 5\sqrt{2}/2 + 1/2$, $x_2 = 5\sqrt{2}/2$; (d) $x_1 = \sqrt{2}/2$, $x_2 = \sqrt{2}/10$, when $a = 1.5\exp(2\pi i/30)$ and $1.2\exp(2\pi i/7)$.

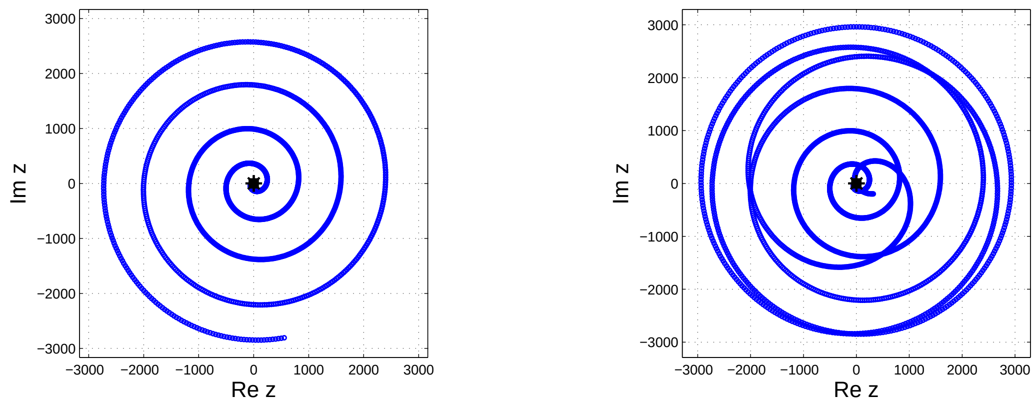

Figure 3.18: First N terms of the sequence $\{w_n\}_{n=0}^{\infty}$ obtained from (3.1.3) for (a) $x_1 = \exp(1)/1000$, $x_2 = \pi/1000$, $r = 1.001$, $N = 1500$; (b) $x_1 = \exp(1)/1000$, $x_2 = \pi/1000$, $r = 1.001$, $N = 2400$; when $a = 1.5\exp(2\pi i/30)$ and $1.2\exp(2\pi i/7)$. Also plotted, initial conditions w_0, w_1 (stars), generators z_1, z_2 (squares) and unit circle S (solid line).

3.3 A Horadam-based pseudo-random number generator

In this section certain complex Horadam sequences are used to design a pseudo-random number generator. This is evaluated using Monte Carlo π estimations against other generators like Multiplicative Lagged Fibonacci and the 'twister' Mersenne.

Pseudo-random generators and Horadam sequences

Pseudo-random number generators are a core component of numerical algorithms based on simulation and statistical sampling. Numerous implementations are based on recursive methods such as *Linear Congruences* and *Lagged Fibonacci Sequences* [98]. Random number generators require periodicity, uniformity and correlation [49].

3.3.1 The complex argument of 2D dense Horadam orbits

Certain bounded orbits are dense within a circle or an annulus centered in the origin. Specifically, if $r_1 = r_2 = 1$ and the generators $z_1 = e^{2\pi i x_1} \neq z_2 = e^{2\pi i x_2}$ are such that $1, x_1, x_2$ are linearly independent over $\mathbb{Q}$, then the orbit of the Horadam sequence $\{w_n\}_{n=0}^{\infty}$ is dense in the annulus $U(0, ||A| - |B||, |A| + |B|)$ These dense orbits are here examined and used to design a random number generator.

Argument of Horadam sequence terms

If $A = R_1 e^{i\phi_1}, B = R_2 e^{i\phi_2}, z_1 = e^{2\pi i x_1}, z_2 = e^{2\pi i x_2}$, the term w_n in polar form is

$$re^{i\theta} = Az_1^n + Bz_2^n = R_1 e^{i(\phi_1 + 2\pi n x_1)} + R_2 e^{i(\phi_2 + 2\pi n x_2)}.$$

Denoting $\theta_1 = \phi_1 + nx_2, \theta_2 = \phi_2 + nx_2$, one can write

$$re^{i\theta} = R_1 e^{i\theta_1} + R_2 e^{i\theta_2}, \quad \theta_1, \theta_2 \in \mathbb{R}, \quad R_1, R_2 > 0. \tag{3.3.1}$$

For $R = R_1 = R_2$, the formula for θ gives $\theta = \frac{1}{2}(\theta_1 + \theta_2)$.

Periodicity of arguments

From the formula of θ, the sequence of arguments for w_n is

$$\theta_n = \frac{\phi_1 + \phi_2}{2} + 2\pi n(x_1 + x_2).$$

For irrational and linearly independent $1, x_1, x_2$, $x_1 + x_2$ is also irrational, so the sequence of arguments θ_n is aperiodic. This property is also valid for the sequence of normalised arguments $(\theta_n + \pi)/(2\pi)$.

Uniform distribution of arguments

When the values of x_1 and x_2 are irrational, the arguments $\theta_1 = \phi_1 + 2\pi n x_2$, and $\theta_2 = \phi_2 + 2\pi n x_2 \in [-\pi, \pi]$ are uniformly distributed modulo 2π, hence θ is uniformly distributed on $[-\pi, \pi]$. The normalized argument $(\theta + \pi)/(2\pi)$ is then uniformly distributed in $[0, 1]$, as seen in Fig. 3.19 (a).

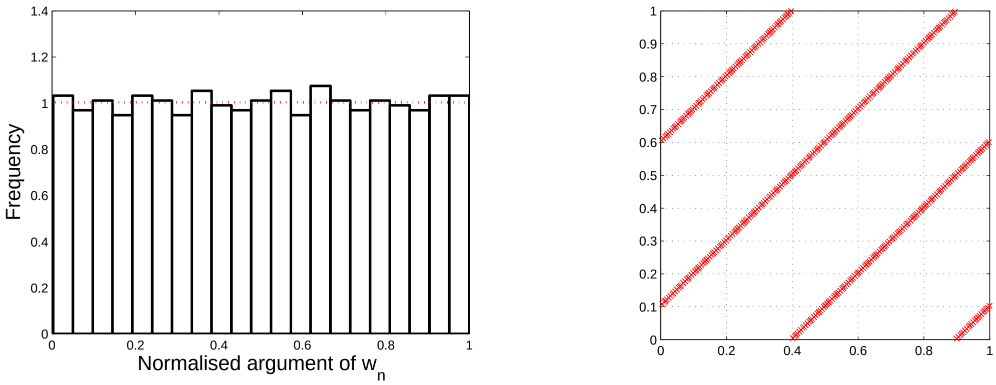

Figure 3.19: (a) Histogram of $\frac{\arg(w_n)+\pi}{2\pi}$ vs uniform density on $[0, 1]$. (b) Normalized angle correlations: $(\mathrm{Arg}(w_n), \mathrm{Arg}(w_{n+1}))$.

Autocorrelation of arguments

A test for the quality of pseudo-random number generators is the autocorrelation [49]. For a good quality generator, the 2D diagrams of normalised arguments (θ_n, θ_{n+1}) should uniformly cover the unit square. The plot depicted in Fig. 3.19 (b) suggests that consecutive arguments are very correlated.

3.3.2 Monte Carlo simulations

A Monte Carlo simulation approximating the value of π could involve randomly selecting points $(x_n, y_n)_{n=1}^{N}$ in the unit square and determining the ratio $\rho = m/N$, where m is number of points that satisfy $x_n^2 + y_n^2 \leq 1$. In our simulation two Horadam sequences $\{w_n^1\}$ and $\{w_n^2\}$ computed from formula (2.1.4) are used.

The parameters are $x_1 = \frac{e}{2}, x_2 = \frac{e^2}{4}$ for $\{w_n^1\}$, and $x_1 = \frac{e}{10}, x_2 = \frac{\pi}{10}$ for $\{w_n^2\}$, with initial conditions $a = 1 + \frac{1}{3}i, b = 1.5a\exp(\pi(x_1 + x_2))$. The 2D coordinates plotted in Fig. 3.20 represent normalized arguments of Horadam sequence terms, given by the formula

$$(x_n, y_n) = \left(\frac{\mathrm{Arg}(w_n^1) + \pi}{2\pi}, \frac{\mathrm{Arg}(w_n^2) + \pi}{2\pi} \right). \tag{3.3.2}$$

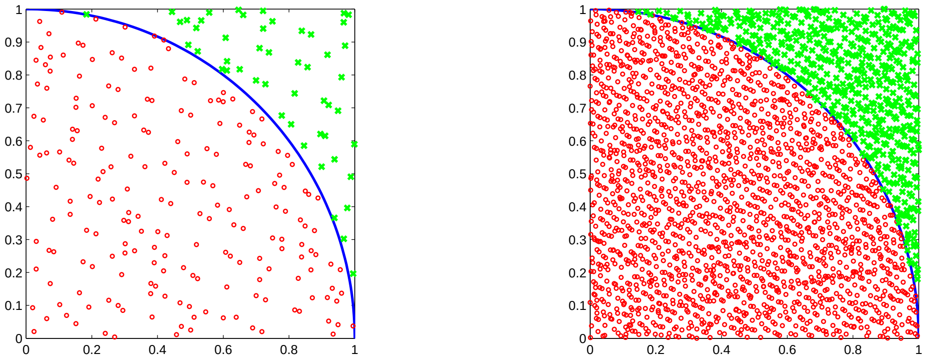

Figure 3.20: First (a) 1000; (b) 10000 points having coordinates (x_n, y_n) given by (3.3.2). Also represented, points inside (circles) and outside (crosses) the unit circle (solid line).

In the simulation shown in Fig. 3.20(a), the sample size is $N = 1000$ and 792 points satisfy $x_n^2 + y_n^2 \leq 1$. Using this data, one obtains $\rho = \frac{792}{1000} = 0.792$ and $\pi \sim 4\rho = 3.1680$. The value significantly improves with the increase in the number of sequence terms, to 3.1420 for $N = 10^4$ (depicted in Fig. 3.20 (b)) and to 3.141888 for $N = 10^6$.

Table 3.1: 10^N is the sample size used in each simulation.

10^N	H1	H2	F1	F2	MT1	MT2
1	0.0584	0.0584	-0.3297	-0.3297	-0.7258	0.8584
2	0.2584	-0.0216	0.0985	-0.0615	-0.0215	0.0985
3	0.0784	-0.0016	-0.0136	0.0304	-0.0456	0.0264
4	0.0104	0.0004	0.0092	-0.0200	0.0036	0.0096
5	0.0012	-0.0006	-0.0016	-0.0018	0.0004	-0.0034
6	0.0003	0.0000	-0.0001	-0.0010	-0.0026	-0.0015
7	0.0000	0.0000	0.0003	-0.0006	-0.0002	0.0004

A more detailed illustration of this convergence is shown in Table 3.1. There H1 and H2 are obtained from the pairs w_n^1 and w_n^2 of Horadam sequences, while H2 from sequences w_n^1 and w_n^3 given below in the form (x_1, x_2, a, b): $w_n^1 := \left(\frac{e}{2}, \frac{e^2}{4}, 1 + \frac{1}{3}i, 1.5ae^{\pi(x_1+x_2)}\right)$, $w_n^2 := \left(\frac{e}{10}, \frac{\sqrt{5}}{15}, 1 + \frac{2}{3}i, 1.5ae^{\pi(x_1+x_2)}\right)$, $w_n^3 := \left(\frac{\sqrt{2}}{3}, \frac{e}{15}, 1 + \frac{2}{3}i, 1.5ae^{\pi(x_1+x_2)}\right)$.

The 2D coordinates producing the results in the table are then given by the formulae

$$\text{H1}: \quad (x_n, y_n) = \left(\frac{\text{Arg}(w_n^1) + \pi}{2\pi}, \frac{\text{Arg}(w_n^2) + \pi}{2\pi}\right),$$

$$\text{H2}: \quad (x_n, y_n) = \left(\frac{\text{Arg}(w_n^1) + \pi}{2\pi}, \frac{\text{Arg}(w_n^3) + \pi}{2\pi}\right).$$

CHAPTER 4

Geometric patterns of complex linear recurrent sequences

This chapter investigates the geometric properties of complex linear recurrent sequences of arbitrary order, generalizing the results for Horadam sequences in Chapters 2 and 3.

The periodicity of linear recurrent sequences has been investigated mostly within the context of finite fields and general rings (see [38, Chapter 3] and [106, 118]). Bagdasar and Larcombe formulated necessary and sufficient periodicity conditions for complex linear recurrent sequences [17]. Here we enumerate the periodic sequence of fixed length and examine the geometric structure of their orbits. We then consider orbits produced by roots of unity and present a mini-atlas of non-periodic patterns.

4.1 Preliminary results

Let $m \geq 2$ be a natural number, $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{c} = (c_1, \ldots, c_m)$ be vectors of complex numbers and let $\{w_n(\mathbf{a}; \mathbf{c})\}_{n=0}^{\infty}$ be the sequence defined by the recurrence

$$w_n = c_1 w_{n-1} + c_2 w_{n-2} + \cdots + c_m w_{n-m}, \quad m \leq n \in \mathbb{N}, \tag{4.1.1}$$

satisfying the initial conditions $w_{i-1} = a_i, i = 1, \ldots, m$.

In this chapter are established necessary and sufficient conditions for the periodicity of generalized complex Horadam sequences. Results are derived using the formulas for the general term of arbitrary linear recurrences, formulated in terms of the initial conditions $a_1, \ldots, a_m$ and the generators $z_1, \ldots, z_m$, representing the non-zero roots (distinct or equal) of the characteristic equation of (4.1.1)

$$\lambda^n = c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_{m-1} \lambda^{n-m+1} + c_m \lambda^{n-m}, \quad n \in \mathbb{N}. \tag{4.1.2}$$

The structure of the solution space for general linear recurrences under arbitrary conditions is also discussed, based on the work of Ivanov [64]. Other means to obtain the closed-form formulae for the general term exist, as for example those given in Andrica and Toader [11], or Cobzaş [32, Chapter 6, Theorem 2.8].

The structure of the solution space for linear recurrences

A solution of the recurrence (4.1.1) is any function $w : \mathbb{N} \to \mathbb{C}$ satisfying the condition

$$w(n) = c_1 w(n-1) + c_2 w(n-2) + \cdots + c_m w(n-m), \quad n \geq m. \tag{4.1.3}$$

Note that for a non-zero value of λ, the characteristic equation (4.1.2) is equivalent to

$$\lambda^m = c_1 \lambda^{m-1} + c_2 \lambda^{m-2} + \cdots + c_{m-1} \lambda + c_m.$$

It may be assumed without loss of generality that the order of the recurrence can not be reduced, therefore $c_m \neq 0$. For finding a base of the vector space V, one may first check that the functions $w(n) = \lambda^n$ ($\lambda \neq 0$) are a solution of (4.1.3), whenever λ is a zero of the *characteristic polynomial*

$$f(x) = x^m - c_1 x^{m-1} - c_2 x^{m-2} - \cdots - c_{m-1} x - c_m. \tag{4.1.4}$$

As a complex polynomial, $f(x)$ has exactly m roots. Examples of bases for V for the cases when the roots of (4.1.4) are all distinct, all equal, or distinct with arbitrary multiplicities are presented below.

Theorem 4.1.1. *([64, Theorem 1]) If the characteristic polynomial $f(x)$ defined in (4.1.4) has m distinct roots $z_1, \ldots, z_m$, then the m sequences*

$$f_1(n) = z_1^n, f_2(n) = z_2^n, \ldots, f_m(n) = z_m^n,$$

form a basis of the vector space V containing the solutions of the recurrence (4.1.3).

Theorem 4.1.2. *([64, Corollary 1]) If the characteristic polynomial $f(x)$ defined in (4.1.4) has m roots equal to z, then the m sequences*

$$f_1(n) = z^n, f_2(n) = nz^n, \ldots, f_m(n) = n^{m-1} z^n,$$

form a basis of the vector space V containing the solutions of the recurrence (4.1.3).

Theorem 4.1.3. *([64, Theorem 2]) If a characteristic polynomial of a linear recurrence has m distinct roots $z_1, \ldots, z_m$ of multiplicities $d_1, \ldots, d_m$ ($d_1 + \cdots + d_m = d$), then the d sequences*

$$f_{ij}(n) = n^{j-1} z_i^n, \quad 1 \leq i \leq m, \quad 1 \leq j \leq d_i, \tag{4.1.5}$$

form a basis of the vector space V containing the solutions of the recurrence (4.1.3).

4.2 Periodicity conditions

For convenience, the sequence $\{w_n(\mathbf{a};\mathbf{c})\}_{n=0}^{\infty}$ defined in (4.1.1) for the complex m-tuples $a_1,\ldots,a_m$ (initial conditions) and $c_1,\ldots,c_m$ (coefficients) is denoted by $\{w_n\}_{n=0}^{\infty}$. It is assumed that recurrence order cannot be reduced, therefore $c_m \neq 0$.

4.2.1 Distinct roots

Let $z_1,\ldots,z_m$ be distinct kth roots of unity ($m \leq k$) and define $P(x)$ by

$$P(x) = (x-z_1)(x-z_2)\cdots(x-z_m), \quad x \in \mathbb{C}. \tag{4.2.1}$$

Also, consider the arbitrary initial conditions

$$w_{i-1} = a_i \in \mathbb{C}, \quad i = 1,\ldots,m. \tag{4.2.2}$$

Theorem 4.2.1. *The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (4.2.1) and arbitrary initial conditions* (4.2.2) *is periodic.*

Proof. Sequence $\{w_n\}_{n=0}^{\infty}$ of characteristic polynomial (4.2.1) satisfies the recurrence

$$w_n = c_1 w_{n-1} + c_2 w_{n-2} + \cdots + c_m w_{n-m}, \quad m \leq n \in \mathbb{N}, \tag{4.2.3}$$

with the coefficients $c_1,\ldots,c_m$ given by $c_i = (-1)^{i-1} S_i(z_1,\ldots,z_m)$, where $S_i(z_1,\ldots,z_m)$ represents the symmetric sum of products having i (unordered) factors chosen from $z_1,\ldots,z_m$. From Theorem 4.1.1, the sequences $f_1(n) = z_1^n, f_2(n) = z_2^n,\ldots,f_m(n) = z_m^n$, form a basis in the vector space V of solutions of the recurrence (4.2.3), therefore the n-th term of the sequence can be written as the linear combination

$$w_n = A_1 z_1^n + A_2 z_2^n + \cdots + A_m z_m^n. \tag{4.2.4}$$

The coefficients $A_1,\ldots,A_m$ can be obtained from (4.2.4) and the initial conditions (4.2.2) by solving the system of linear equations

$$\begin{cases} a_1 & = A_1 + A_2 + \cdots + A_m \\ a_2 & = A_1 z_1 + A_2 z_2 + \cdots + A z_m \\ \cdots & \\ a_m & = A_1 z_1^{m-1} + A_2 z_2^{m-1} + \cdots + A z_m^{m-1}. \end{cases}$$

As $z_1,\ldots,z_m$ are kth roots of unity, $z_i^n = z_i^{n+k}$, $i = 1,\ldots,m$, therefore $w_n = w_{n+k}$, $n \in \mathbb{N}$. This shows that the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic and its period divides k.

The initial condition (4.2.2) can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_m \\ \vdots & \vdots & & \vdots \\ z_1^{m-1} & z_2^{m-1} & \cdots & z_m^{m-1} \end{pmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = V_{m,m}(\mathbf{z}) \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}, \tag{4.2.5}$$

where $V_{m,m}(\mathbf{z})$ denotes the square Vandermonde matrix. For distinct $z_1, \ldots, z_m$, one obtains $\det\Big(V_{m,m}(\mathbf{z})\Big) = \prod_{1 \le i < j \le m}(z_j - z_i) \neq 0.$. Using the notation $\mathbf{a} = (a_1, \ldots, a_m)$, Cramer's rule [94] ensures that the unique solution of (4.2.5) is given by the expression

$$A_i = \det\Big(V^i_{m,m}(\mathbf{z}, \mathbf{a})\Big) / \det\Big(V_{m,m}(\mathbf{z})\Big), \quad i = 1, \ldots, m, \tag{4.2.6}$$

where $V^i_{m,m}$ is the matrix obtained by replacing the i-th column of $V_{m,m}(\mathbf{z})$ with $\mathbf{a}^T$.

The first N terms of the sequence $\{w_n\}_{n=0}^{\infty}$ can be computed from the formula

$$V_{N,m}(\mathbf{z}) * (A_1, A_2, \ldots, A_m)^T = (w_0, w_1, \ldots, w_{N-1})^T \tag{4.2.7}$$

Fig. 4.1 illustrates the periodic orbits of sequence $\{w_n\}_{n=0}^{\infty}$ obtained from the recurrence (4.2.3) (diamonds), or direct formula (4.2.7) (circles) when selecting (a) $m = 3$ and (b) $m = 5$ distinct roots respectively, from the 7th roots of unity.

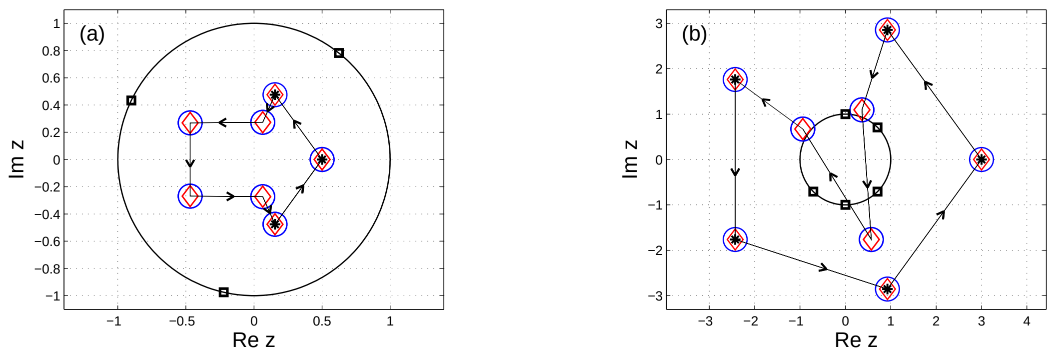

Figure 4.1: First 15 terms of $\{w_n\}_{n=0}^{\infty}$ computed from the recurrence (4.2.3) (diamonds) and direct formula (4.2.7) (circles), for (a) $m = 3$, $z_j = e^{\frac{2\pi i}{7}(2j-1)}$, and $a_j = .5e^{\frac{2\pi i}{5}(j+3)}$, $j = 1, 2, 3$; (b) $m = 5$, $z_j = e^{\frac{2\pi i}{7}j}$ for $j = 1, 2, 5, 6, 7$ and $a_j = 3e^{\frac{2\pi i}{5}(j+1)}$, $j = 1, \ldots, 5$. Also illustrated, initial conditions (stars), generators (squares), unit circle S (solid line) and orbit direction (arrows).

Theorem 4.2.2. *(Necessary condition for periodicity) Let us assume that the roots $z_1, \ldots, z_m$ of (4.2.1) are all distinct. The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (4.2.1) and initial conditions $a_1, \ldots, a_m$ is periodic only if there exist $k \in \mathbb{N}$ positive s. t.*

$$A_i(z_i^k - 1) = 0, \quad i = 1, \ldots, m, \tag{4.2.8}$$

where $A_1, \ldots, A_m$ are computed from formula (4.2.6).

4.2.2 Equal roots

Let z be a kth root of unity, $m \in \mathbb{N}$ and consider the polynomial $P(x)$ defined by

$$P(x) = (x - z)^m, \quad x \in \mathbb{C}. \tag{4.2.9}$$

Theorem 4.2.3. *The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (4.2.9) and initial conditions (4.2.2) is periodic when $A_2 = A_3 = \cdots = A_m = 0$, where $A_1, \ldots, A_m$ are the coefficients of $\{w_n\}_{n=0}^{\infty}$ in the basis defined in Theorem 4.1.2. The sequence $\{w_n\}_{n=0}^{\infty}$ is divergent otherwise.*

Proof. Similarly to Theorem 4.2.1, the sequence $\{w_n\}_{n=0}^{\infty}$ satisfies the linear recurrence (4.2.3), for the coefficients $c_1, \ldots, c_m$ given by $c_i = (-1)^{i-1}\binom{m}{i}z^i$, $i = 1, \ldots, m$. From Theorem 4.1.2, the sequences $f_1(n) = z^n, f_2(n) = nz_2^n, \ldots, f_m(n) = n^{m-1}z^n$, form a basis in the vector space V of solutions of the recurrence generated by the characteristic polynomial (4.2.9), hence the n-th term of the sequence can be written as

$$w_n = A_1 z^n + nA_2 z^n + \cdots + n^{m-1} A_m z^n. \tag{4.2.10}$$

For $A_i = 0$, $i = 2, \ldots, m$ we have $w_n = A_1 z^n$, so in this case the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic. Whenever there exists $i \geq 2$ s.t. $A_i \neq 0$, the behaviour of w_n is dictated by the divergent coefficient $n^i A_i$ of z^n, therefore the sequence $\{w_n\}_{n=0}^{\infty}$ diverges. This property is illustrated in Fig. 4.2. The sequence can either be (a) periodic or (b) divergent.

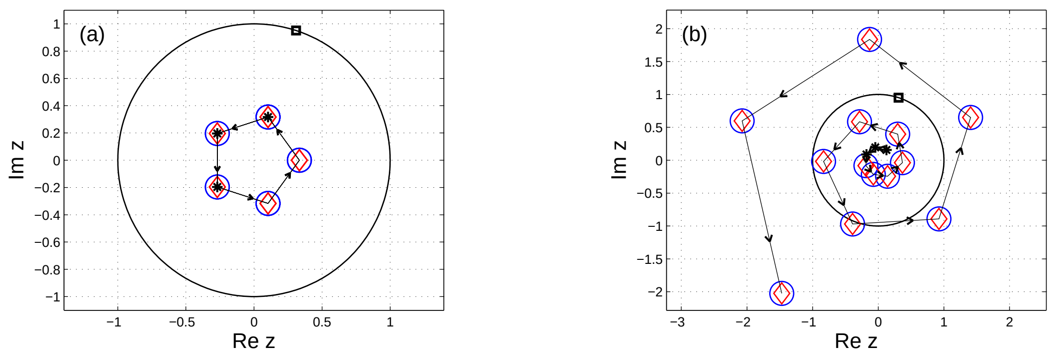

Figure 4.2: First 16 terms of sequence $\{w_n\}_{n=0}^{\infty}$ computed from the recurrence (4.2.3) (diamonds) and direct formula (circles) for $m = 3$, $z = e^{\frac{2\pi i}{5}}$ and initial conditions (a) $a_j = e^{\frac{2\pi i}{5}j}/3$, $j = 1, 2, 3$; (b) $a_j = e^{\frac{2\pi i}{7}j}/3$, $j = 1, 2, 3$. Also plotted are initial conditions (stars), generators (squares) and unit circle S (solid line). Orbit's diection is indicated by arrows.

Theorem 4.2.4. *(Necessary condition for periodicity) The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (4.2.9) and initial conditions (4.2.2) is only periodic when*

$$A_1(z^k - 1) = 0, A_2 = A_3 = \cdots = A_m = 0, \tag{4.2.11}$$

where $A_1, \ldots, A_m$ are computed from the initial conditions.

4.2.3 Distinct roots with arbitrary multiplicities

Let $2 \leq m \leq k$ and $d_1, \ldots, d_m$ be natural numbers, $z_1, \ldots, z_m$ distinct kth roots of unity and let the polynomial $P(x)$ be defined as

$$P(x) = (x - z_1)^{d_1}(x - z_2)^{d_2} \cdots (x - z_m)^{d_m}, \quad x \in \mathbb{C}. \tag{4.2.12}$$

Theorem 4.2.5. *The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (4.2.12) of degree $d = d_1 + \cdots + d_m$ and initial conditions $w_{i-1} = a_i$, $i = 1, \ldots, d$ is periodic when*

$$A_{ij} = 0, \quad 1 \leq i \leq m, \quad 2 \leq j \leq d_i, \tag{4.2.13}$$

where A_{ij} ($1 \leq i \leq m, 1 \leq j \leq d_i$) represent the coefficients of $\{w_n\}_{n=0}^{\infty}$ in the basis (4.1.5) defined in Theorem 4.1.3. The sequence $\{w_n\}_{n=0}^{\infty}$ is divergent otherwise.

As illustrated in Fig. 4.3 (a), the sequence may be periodic even when $d_i \geq 2$, $i = 1, \ldots, m$. When any of the coefficients A_{ij}, $1 \leq i \leq m$, $2 \leq j \leq d_i$ does not vanish, the sequence $\{w_n\}_{n=0}^{\infty}$ diverges as depicted in Fig. 4.3 (b). A detailed explanation of this behaviour is presented in Theorem 4.2.6.

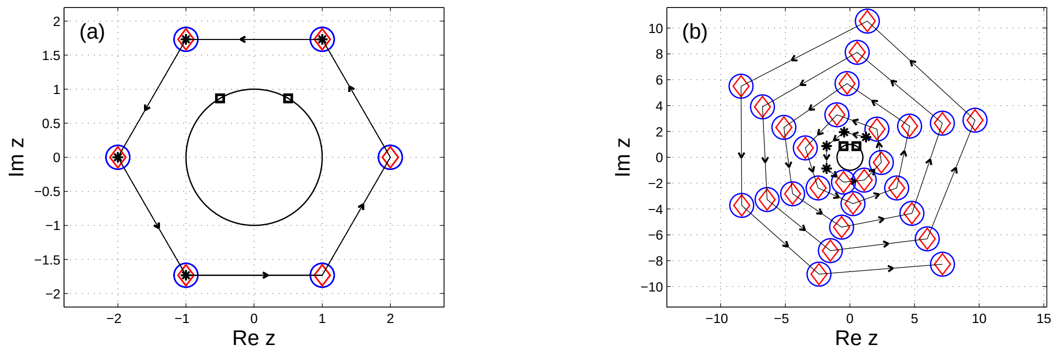

Figure 4.3: First 31 terms of sequence $\{w_n\}_{n=0}^{\infty}$ computed from the recurrence (diamonds) and direct formula (circles) for $z_1 = e^{\frac{2\pi i}{6}}, z_2 = e^{\frac{4\pi i}{6}}, d_1 = d_2 = 2$ and initial conditions (a) $a_j = 2e^{\frac{2\pi i}{6}j}$, $j = 1, \ldots, 4$; (b) $a_j = 2e^{\frac{2\pi i}{7}j}$, $j = 1, \ldots, 4$. Initial conditions (stars), generators (squares) and unit circle S (solid line). Arrows indicate the increase of sequence index.

Theorem 4.2.6. *(Necessary condition for periodicity) Let $2 \leq m \leq k$ and $d_1, \ldots, d_m$ be natural numbers. The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (4.2.12) of degree $d = d_1 + \cdots + d_m$ and initial conditions $w_{i-1} = a_i$, $i = 1, \ldots, d$ is periodic only if*

$$\begin{aligned} A_{i1}(z_i^k - 1) &= 0, \quad i = 1, \ldots, m, \\ A_{ij} &= 0, \quad j = 2, \ldots, d_i, \end{aligned} \tag{4.2.14}$$

where the coefficients A_{ij} are computed from initial conditions, for $1 \leq i \leq m, 1 \leq j \leq d_i$.

4.3 The geometry and enumeration of periodic patterns

In this section we examine the geometry and number of periodic orbits.

4.3.1 Geometric bounds of periodic orbits

Theorem 4.3.1. *Let $2 \leq m \leq k$ and $d_1, \ldots, d_m \in \mathbb{N}$. Let $\{w_n(\boldsymbol{a}; \boldsymbol{c})\}_{n=0}^{\infty}$ be a generalized Horadam sequence. Assume that the roots $z_1, \ldots, z_m$ of the polynomial $P(x)$ defined in (4.2.12) are distinct. If $\{w_n(\boldsymbol{a}; \boldsymbol{c})\}_{n=0}^{\infty}$ is periodic, all sequence terms are located inside the disk of radius $|A_{11}| + |A_{21}| + \cdots + |A_{m1}|$, with A_{j1}, $j = 1, \ldots, m$ computed from initial conditions.*

4.3.2 The geometric structure of periodic orbits

Theorem 4.3.2. *Let $m \geq 2$ and $k_1, k_2, \ldots, k_m$ be natural numbers and consider the distinct primitive roots of unity $z_j = e^{2\pi i p_j / k_j}$, $j = 1, \ldots, m$. Denote by K the least common multiple of numbers $k_1, \ldots, k_m$, i.e., $K = [k_1, k_2, \ldots, k_m]$. The orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ is then a K-gon, whose nodes can be divided into K/k_j regular k_j-gons representing a multipartite graph.*

4.3.3 The enumeration of periodic orbits

Let $m \geq 2$ and consider the distinct roots of unity $z_j = e^{2\pi i p_j / k_j}$ with $j = 1, \ldots, m$. We assume that the coefficients $A_1, \ldots, A_m$ in formula (4.2.4) are all non-zero.

Definition 4.3.3. *The number of distinct sequences of order m having period k for fixed $a_1, \ldots, a_m$ is denoted by $H_P^m(k)$. Function $H_P^m(k)$ is the number of tuples $(P, K) = (p_1, k_1, p_2, k_2, \ldots, p_m, k_m)$ ($1 \leq p_j \leq k_j, j = 1, \ldots, m$) satisfying the conditions*

$$H_P^m(k) = \left| \{ (p_1, k_1) = \cdots = (p_m, k_m) = 1,\ [k_1, \ldots, k_m] = k,\ k_1 \leq \cdots \leq k_m \} \right|. \quad (4.3.1)$$

Lemma 4.3.4. *If $k_1 = k_2 = \cdots = k_m$ and $[k_1, k_2, \ldots, k_m] = k$ then $k_1 = k_2 = \cdots = k_m = k$.*

Lemma 4.3.5. *If $[k_1, k_2, \ldots, k_m] = k$ and $k_1, \ldots k_m$ are distinct, the number of $2m$-tuples (P, K) satisfying (4.3.1) is $\varphi(k_1)\varphi(k_2) \cdots \varphi(k_m)$.*

Lemma 4.3.6. *If the numbers $k_1, k_2, \ldots, k_m$ satisfying $[k_1, k_2, \ldots, k_m] = k$ can be partitioned into $s > 0$ sets having d_j elements equal to a value K_s for $j = 1, \ldots, s$, then the $2m$-tuples (P, K) satisfying (4.3.1) corresponding to this m-tuple is*

$$\binom{\varphi(K_1)}{d_1} \binom{\varphi(K_2)}{d_2} \cdots \binom{\varphi(K_s)}{d_s}.$$

Theorem 4.3.7. *Let us consider the integers $k, m \geq 2$. The number of distinct generalized Horadam sequences order m and fixed period k is equal to*

$$H_P(m;k) = \sum_{s=1}^{m} \Bigg\{ \sum_{\substack{k_1 < k_2 < \cdots < k_s \\ d_1 + \cdots + d_s = k \\ 1 \leq d_i \leq k \\ [k_1, \ldots, k_s] = k}} \binom{\varphi(k_1)}{d_1} \cdots \binom{\varphi(k_s)}{d_s} \Bigg\}.$$

An example for $m = 3$

To find $H_P(3;k)$, one needs to count all configurations $(p_1, k_1) = (p_2, k_2) = (p_3, k_3) = 1$ with $[k_1, k_2, k_3] = k$. Then $H_P(3;k) = H_1 + H_2 + H_3 + H_4$ where

(1) $k_1 = k_2 = k_3$: $H_1 = \frac{\varphi(k)(\varphi(k)-1)(\varphi(k)-2)}{3!} = \binom{\varphi(k)}{3}$;

(2) $k_1 = k_2 < k_3$: $H_2 = \frac{\varphi(k_1)(\varphi(k_1)-1)}{2!}\varphi(k_3) = \binom{\varphi(k_1)}{2}\varphi(k_3)$;

(3) $k_1 < k_2 = k_3$: $H_3 = \varphi(k_1)\frac{\varphi(k_3)(\varphi(k_3)-1)}{2!} = \varphi(k_1)\binom{\varphi(k_3)}{2}$;

(4) $k_1 < k_2 < k_3$: $H_4 = \varphi(k_1)\varphi(k_2)\varphi(k_3)$.

As a number sequence $H_P(3;k)$ in $k \geq 1$ gives

k	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$H_P(3;k)$	0	0	1	4	9	19	35	52	83	110	165	$\cdots$

not currently indexed in OEIS.

4.4 Orbits generated by roots of unity

Theorem 4.4.1. *Let $2 \leq m \leq k$ and $d_1, \ldots, d_m$ be natural numbers, $z_1, \ldots, z_m$ distinct kth roots of unity and the sequence $\{w_n\}_{n=0}^{\infty}$ be generated by the characteristic polynomial (4.2.12). Defining the number*

$$d^* = \max\{j : A_{ij} \neq 0, i \in \{1, \ldots, m\}\}, \tag{4.4.1}$$

for the coefficients A_{ij} ($1 \leq i \leq m$, $1 \leq j \leq d_i$) given in Theorem 4.2.5, one obtains the following properties referring to the sequence $\{w_n\}_{n=0}^{\infty}$ and the subsequences $\{w_{Nk+j}\}_{N=0}^{\infty}$:

(a) For $d^* \leq 1$ the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic.

(b) For $d^* \geq 2$ the sequence $\{w_n\}_{n=0}^{\infty}$ is divergent.

(c) For $d^* \leq 2$ the terms of $\{w_{Nk+j}\}_{N=0}^{\infty}$ are collinear (including the periodic case $d^* = 1$).

(d) For $d^* \geq 3$ the terms of $\{w_{Nk+j}\}_{N=0}^{\infty}$ converge asymptotically towards straight lines.

4.5 A mini-atlas of non-periodic of complex LRS patterns

In this section we examine certain types non-periodic LRS patterns. The results extend those formulated in Chapter 3 for Horadam sequences. We shall again focus on non-degenerate orbits produced by distinct generators, where the formula of the general term is given by $w_n = A_1 z_1^n + A_2 z_2^n + \cdots + A_m z_m^n$, and the arbitrary initial conditions $a_1, a_2, \ldots, a_m$ are such that the coefficients $A_1, \ldots, A_m$ are all non-zero.

The $m \geq 2$ distinct generators are here denoted by

$$z_1 = r_1 e^{2\pi i x_1},\ z_2 = r_2 e^{2\pi i x_2}, \cdots,\ z_m = r_m e^{2\pi i x_m}, \tag{4.5.1}$$

where $r_1, \ldots, r_m, x_1, \ldots, x_m \in \mathbb{R}$. We may safely assume $0 \leq r_1 \leq r_2 \leq \cdots \leq r_m$. The generalized Horadam patterns produced by formula (4.2.10) for generators (4.5.1) are

1. Stable for $r_1 = r_2 = \cdots = r_m = 1$;
2. Quasi-convergent for $0 \leq r_1 \leq r_2 \leq \cdots \leq r_m = 1$;
3. Convergent for $0 \leq r_1 \leq r_2 \leq \cdots \leq r_m < 1$;
4. Divergent for $r_m \geq 1$.

The geometric patterns obtained in each case are presented below.

4.5.1 Stable orbits: $r_1 = r_2 = \cdots = r_m = 1$

Here the orbits corresponding to generators located on the unit circle are investigated. For convenience we restrict to the case $m = 3$, where for initial conditions $w_0 = a$, $w_1 = b$ and $w_2 = c$ ($a, b, c \in \mathbb{C}$), the general term formula for simple roots is given by

$$w_n = A_1 z_1^n + A_2 z_2^n + A_3 z_3^n, \tag{4.5.2}$$

where the constants A_1, A_2, A_3 can be recovered from the initial conditions.

The patterns recovered in this scenario are finite sets (periodic), or sets dense within certain 1D curves, or unions of 2D annuli and disks.

As illustrated in Section 4.3, sequence orbits are in this case located inside the disk of radius $|A_1| + |A_2| + |A_3|$, which is represented in the diagrams.

We here present a number of illustrative situations. Properties are linked to wether terms of the set $\{1, x_1, x_2, x_3\}$ are linearly dependent (or independent) over $\mathbb{Q}$, i.e., there exist $p_0, p_1, p_2, p_3 \in \mathbb{Q}$ (not all zero) s.t. $p_0 + p_1 x_1 + p_2 x_2 + p_3 x_3 = 0$.

(a) Stable periodic (finite) orbits

When $x_1, x_2, x_3 \in \mathbb{Q}$ the sequence orbit is finite (see Fig. 4.4). Indeed, when $x_1 = p_1/k_1$, $x_2 = p_2/k_2$ and $x_3 = p_3/k_3$ are irreducible, one has $z_1^{k_1} = z_2^{k_2} = z_3^{k_3} = 1$. For certain A_1, A_2, A_3 values, sequence terms $\{w_n\}_{n=0}^{\infty}$ repeat with periodicity $\mathrm{lcm}(k_1, k_2, k_3)$.

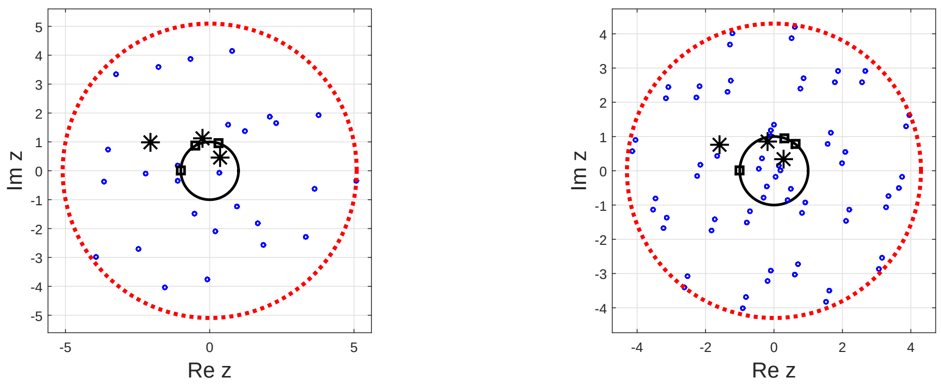

Figure 4.4: Orbit of $\{w_n\}_{n=0}^{\infty}$ obtained by (4.5.2) for $r_1 = r_2 = r_3 = 1$. $(a1)$ $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{5}$ (30 points); $(a2)$ $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{7}$, $x_3 = \frac{1}{5}$ (70 points). Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, |A_1| + |A_2| + |A_3|)$ (dotted line).

(b) Orbits dense in unions of circles (1D)

Horadam orbits may also be dense within unions of circles, as shown in Fig. 4.5.

(b1) $x_1, x_2 \in \mathbb{Q}$ and $x_3 \in \mathbb{R} \setminus \mathbb{Q}$: orbit is dense in a union of $[k_1, k_2]$ circles;

(b2) $x_1 \in \mathbb{R} \setminus \mathbb{Q}$ but $x_2 - x_1, x_3 - x_1 \in \mathbb{Q}$: orbit is dense in a union of concentric circles.

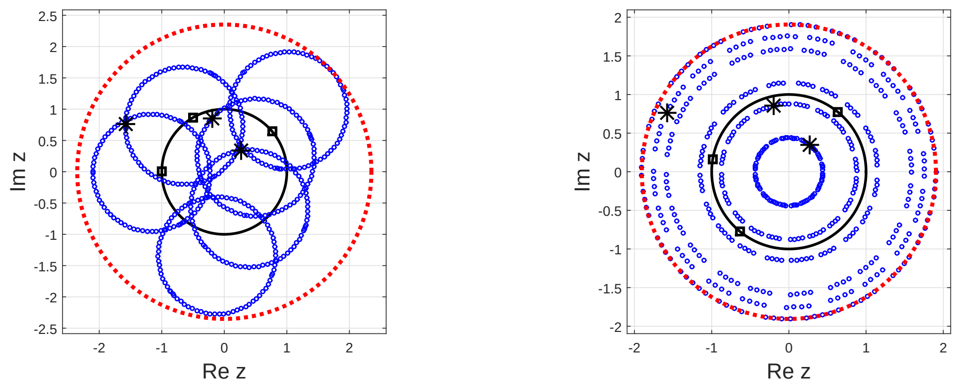

Figure 4.5: Orbit of $\{w_n\}_{n=0}^{\infty}$, given by (4.5.2) for $r_1 = r_2 = r_3 = 1$. $(b1)$ $x_1 = 1/3$, $x_2 = \sqrt{5}/20$, $x_3 = 1/2$; $(b2)$ $x_1 = \pi$, $x_2 = \pi + 1/3$, $x_3 = \pi + 1/3$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, |A_1| + |A_2| + |A_3|)$ (dotted line).

(c) Stable orbits dense within 1D complex curves

The orbits may also be dense within unions of rotated curves as in Fig. 4.6, or within more complex 1D curves. The closure of the orbit is

(c1) $x_1 \in \mathbb{Q}$, $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ and $x_3/x_2 \in \mathbb{Q}$: k_1 rotated copies of a complex curve.

(c2) $x_1 \in \mathbb{R} \setminus \mathbb{Q}$ but $x_2/x_1, x_3/x_1 \in \mathbb{Q}$: a complex curve of type $f(z) = az + bz^p + cz^q$.

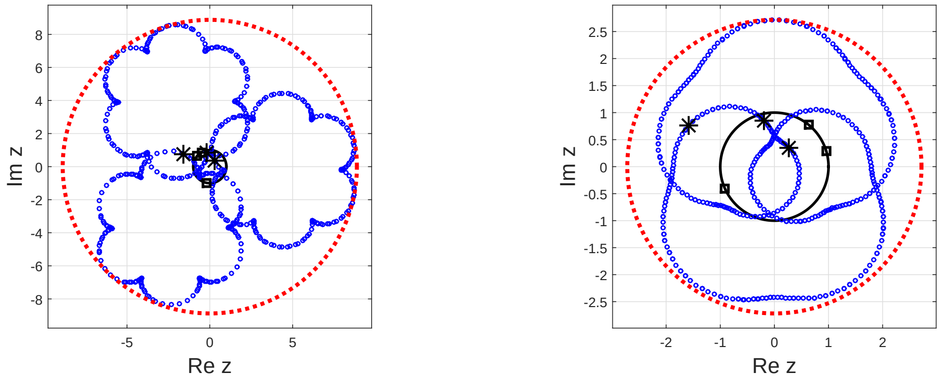

Figure 4.6: Orbits of $\{w_n\}_{n=0}^{\infty}$ dense within 1D curves, given by (4.5.2) for $r_1 = r_2 = r_3 = 1$. $(c1)$ $x_1 = \frac{1}{3}$, $x_2 = e$, $x_3 = \frac{e}{7}$; $(c2)$ $x_1 = \pi$, $x_2 = \frac{1\pi}{3}$, $x_3 = 4\pi$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0, 1)$ (solid line), $U(0, |A_1| + |A_2| + |A_3|)$ (dotted line).

(d) Stable orbits dense within a 2D disk

If $1, x_1, x_2, x_3$ are linearly independent over $\mathbb{Q}$, the orbit is dense within the disk of radius $|A_1| + |A_2| + |A_3|$. This usually happens when we combine square roots, e or π. This case is illustrated in Fig. 4.7.

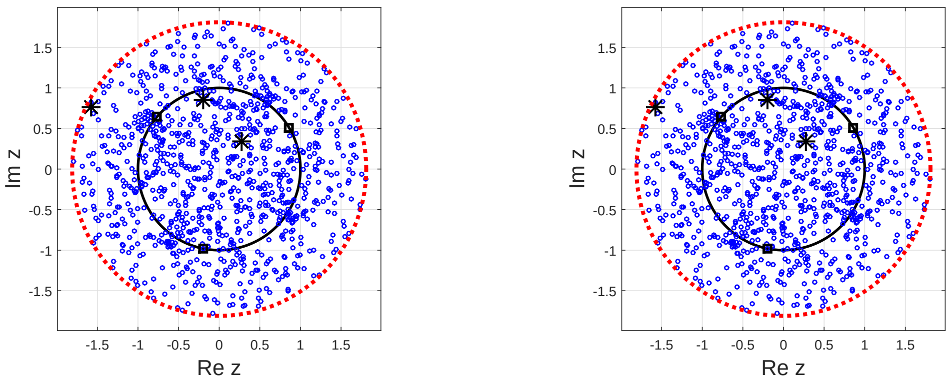

Figure 4.7: Orbits of $\{w_n\}_{n=0}^{\infty}$ dense within 2D regions, given by (4.5.2) for $r_1 = r_2 = r_3 = 1$. $(d1)$ $x_1 = e$, $x_2 = e^2$, $x_3 = e^3$; $(d2)$ $x_1 = \pi$, $x_2 = \pi^2$, $x_3 = \pi^3$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0, 1)$ (solid line), $U(0, ||A_1| + |A_2| + |A_3|)$ (dotted line).

(e) Stable orbits dense within a 2D region: Unions of annuli

If $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$ with $1, x_1, x_2$ linearly independent over $\mathbb{Q}$ and $x_3 \in \mathbb{Q}$, the orbit is a collection of k_1 annuli rotated around the origin. The situation is depicted in Fig. 4.8 for 3, and 4 annuli respectively.

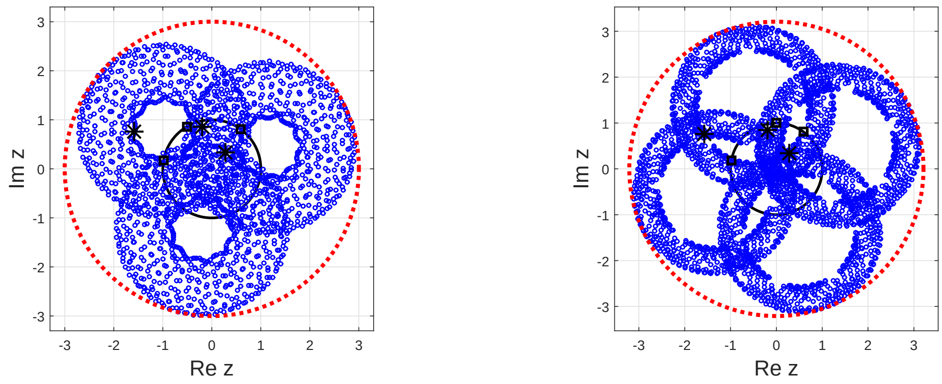

Figure 4.8: Orbits of $\{w_n\}_{n=0}^{\infty}$ dense within 2D regions, given by (4.5.2) for $r_1 = r_2 = r_3 = 1$. $(e1)$ $x_1 = \frac{\sqrt{2}}{3}$, $x_2 = \frac{\sqrt{5}}{15}$, $x_3 = \frac{1}{4}$; $(e2)$ $x_1 = \frac{\sqrt{2}}{3}$, $x_2 = \frac{\sqrt{5}}{15}$, $x_3 = \frac{1}{3}$. Also represented are the initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line) and $U(0, |A_1| + |A_2| + |A_3|)$ (dotted line).

(f) Stable orbits dense within a 2D region: "Buns"

If $x_1, x_2, x_3 \in \mathbb{R} \setminus \mathbb{Q}$ with $1, x_1$ and x_3 linearly dependent over $\mathbb{Q}$ (for example when $x_3/x_1 \in \mathbb{Q}$), the orbit is obtained by translating a disk along a closed 1D curve. This case is illustrated in Fig. 4.9 for 3, and 5 lobes respectively.

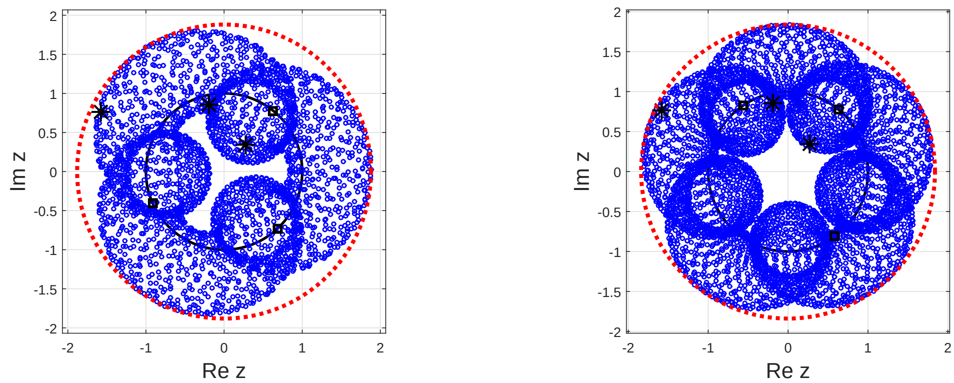

Figure 4.9: Orbits of $\{w_n\}_{n=0}^{\infty}$ dense within 2D regions, given by (4.5.2) for $r_1 = r_2 = r_3 = 1$. $(f1)$ $x_1 = \pi$, $x_2 = \pi^2$, $x_3 = 3\pi$; $(f2)$ $x_1 = \pi$, $x_2 = \pi^3/90$, $x_3 = 6\pi$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, |A_1| + |A_2| + |A_3|)$ (dotted line).

4.5.2 Quasi-convergent orbits: $0 \leq r_1 \leq r_2 \leq \cdots \leq r_m = 1$

Here one of the components vanishes. For simplicity we assume $0 < r_1 < r_2 = r_3 = 1$.

(a) Finite attractor set

When $x_2, x_3 \in \mathbb{Q}$ the orbit has $\mathrm{lcm}(k_2, k_3)$ attractor points. If $x_1 \in \mathbb{Q}$, there are k_1 rays towards each attractor, while for $x_1 \in \mathbb{R} \setminus \mathbb{Q}$ one obtains spirals (see Fig. 4.10).

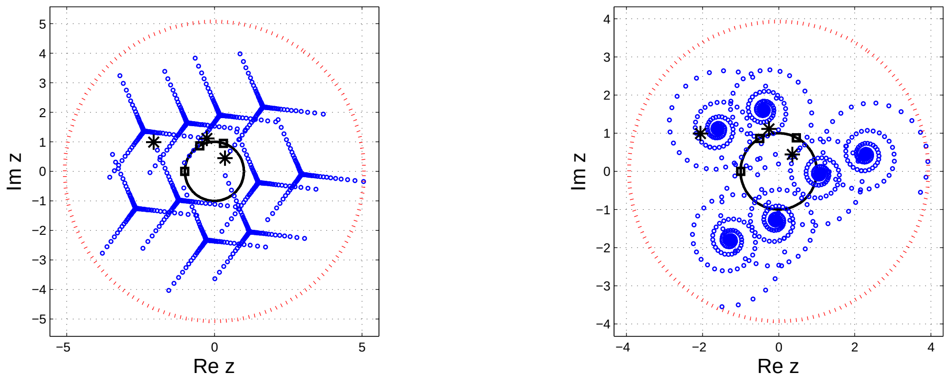

Figure 4.10: Orbit of $\{w_n\}_{n=0}^{\infty}$ obtained by (4.5.2) for $r_1 = .995$, $r_2 = r_3 = 1$. $(a1)$ $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{5}$; $(a2)$ $x_1 = \frac{\sqrt{3}}{10}$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{2}$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, ||A_1| + |A_2| + |A_3|)$ (dotted line).

(b) 1D attractor set - circles

When $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ the orbit's closure may consist of circles. If $x_2 - x_3 \in \mathbb{Q}$ one has concentric circles, while for $x_3 \in \mathbb{Q}$ one obtains k_3 rotated circles (see Fig. 4.11).

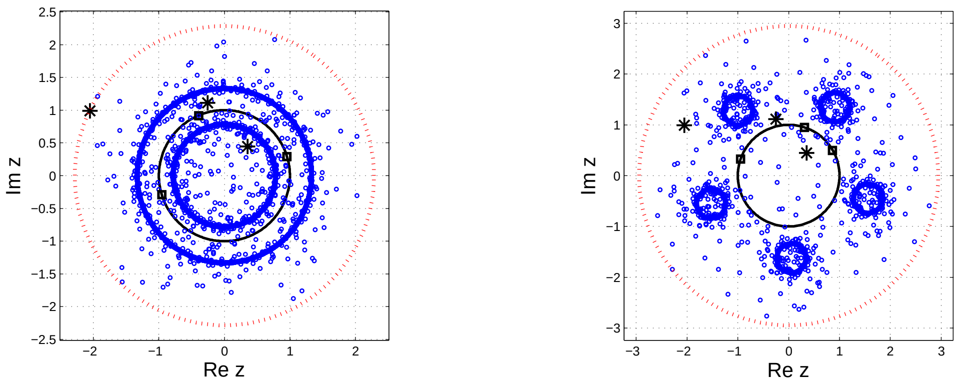

Figure 4.11: Orbit of $\{w_n\}_{n=0}^{\infty}$ obtained by (4.5.2) for $r_1 = .995$, $r_2 = r_3 = 1$. $(b1)$ $x_1 = \frac{\pi}{10}$, $x_2 = \frac{\pi}{3}$, $x_3 = \frac{\pi}{3} + \frac{1}{2}$; $(b2)$ $x_1 = \frac{1}{12}$, $x_2 = \frac{\sqrt{5}}{10}$, $x_3 = \frac{1}{5}$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, ||A_1| + |A_2| + |A_3|)$ (dotted line).

(c) 1D attractor set - curves

When $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ and $x_3/x_2 = q \in \mathbb{Q}$, the orbit is dense within a curve, representing the graph of the function $f : S \to \mathbb{C}$ defined by $f(z) = A_2 z + A_3 z^q$. The details of this statement are explained in Theorem 3.2.1 (c2). Two examples are shown in Fig. 4.12.

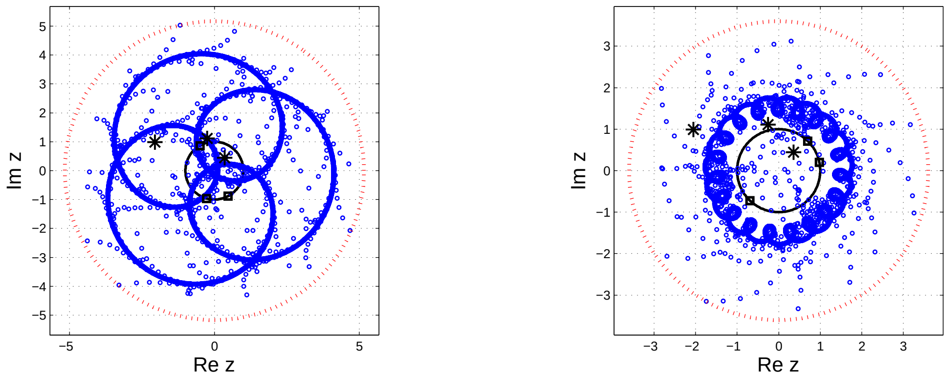

Figure 4.12: Orbit of $\{w_n\}_{n=0}^{\infty}$ obtained by (4.5.2) for $r_1 = .995$, $r_2 = r_3 = 1$. $(c1)$ $x_1 = \frac{1}{3}$, $x_2 = \frac{\sqrt{2}}{2}$, $x_3 = 4x_2$; $(c2)$ $x_1 = \frac{\pi}{25}$, $x_2 = \frac{x_1}{4}$, $x_3 = 20x_2$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, ||A_1| + |A_2| + |A_3|)$ (dotted line).

(d) 2D attractor set - annuli

When $1, x_2, x_3$ are linearly independent over $\mathbb{Q}$ the orbit's closure is dense within an annulus. The convergence property is illustrated in Fig. 4.13, when 2000 (d1), or 4000 (d2) sequence terms are evaluated, respectively.

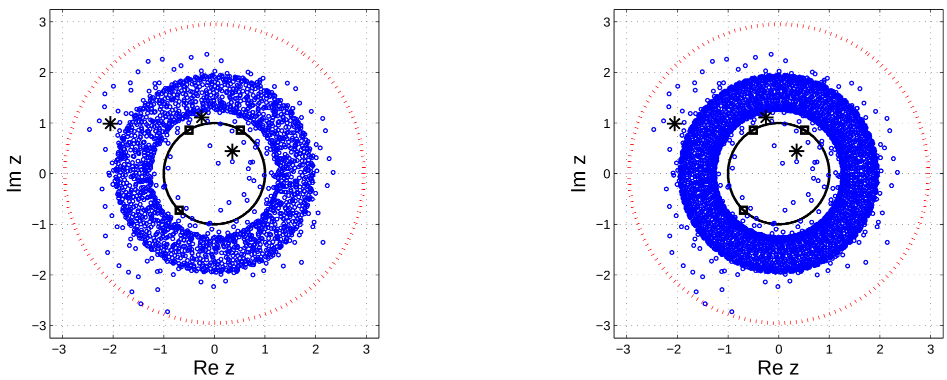

Figure 4.13: Orbit of $\{w_n\}_{n=0}^{\infty}$ obtained by (4.5.2) for $r_1 = .995$, $r_2 = r_3 = 1$ obtained for $x_1 = \frac{1}{3}$, $x_2 = \frac{\exp(.5)}{10}$, $x_3 = \frac{\pi}{5}$ and $(d1)$ 2000 terms; $(d1)$ 4000 terms;. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, ||A_1| + |A_2| + |A_3|)$ (dotted line).

4.5.3 Convergent orbits: $0 \leq r_1 \leq r_2 \leq \cdots \leq r_m < 1$

In this case the orbit has the origin as it's single attractor. As seen in Chapter 3, numerous patterns may emerge. We illustrate two of them in Fig. 4.14 below.

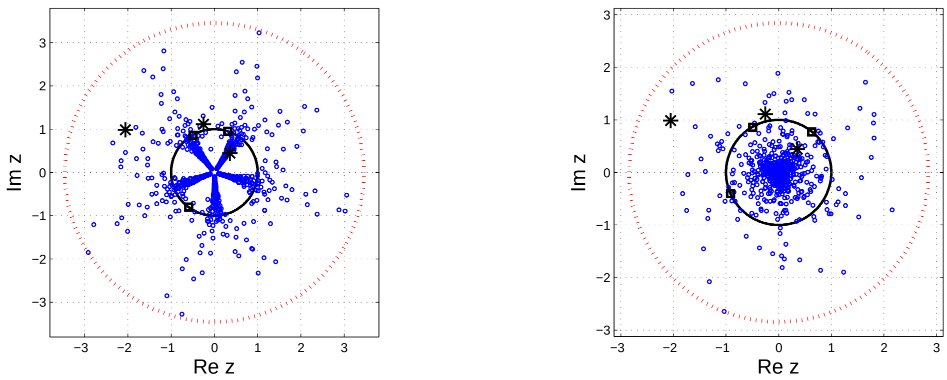

Figure 4.14: Orbit of $\{w_n\}_{n=0}^{\infty}$ obtained by (4.5.2) for $r_1 = .99$, $r_2 = .995$ and $r_3 = .997$. $(a1)$ $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{4}$; $(a2)$ $x_1 = \frac{1}{3}$, $x_2 = \pi$, $x_3 = 4\pi$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, ||A_1| + |A_2| + |A_3|)$ (dotted line).

4.5.4 Divergent orbits: $r_m > 1$

In this case the orbit diverges. As seen in Chapter 3, numerous patterns may emerge.

(a) Divergent rays and spirals

If $x_1, x_2, x_3 \in \mathbb{Q}$, orbit diverges along rays. If $x_3 \in \mathbb{R} \setminus \mathbb{Q}$ spirals emerge (see Fig. 4.15).

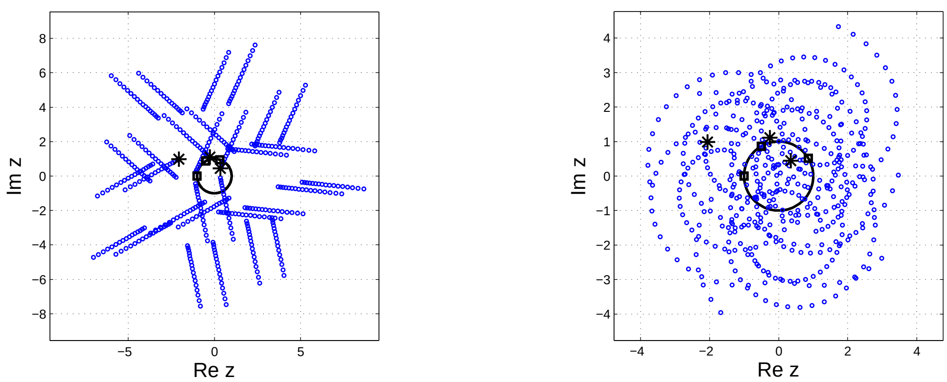

Figure 4.15: Orbit of $\{w_n\}_{n=0}^{\infty}$ obtained by (4.5.2) for $r_1 = r_2 = 1$ and $r_3 = 1.002$. $(a1)$ $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{5}$; $(a2)$ $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$, $x_3 = e^3$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, ||A_1| + |A_2| + |A_3|)$ (dotted line).

(b) General divergent patterns

Some other examples of divergent orbits are shown below in Fig. 4.16.

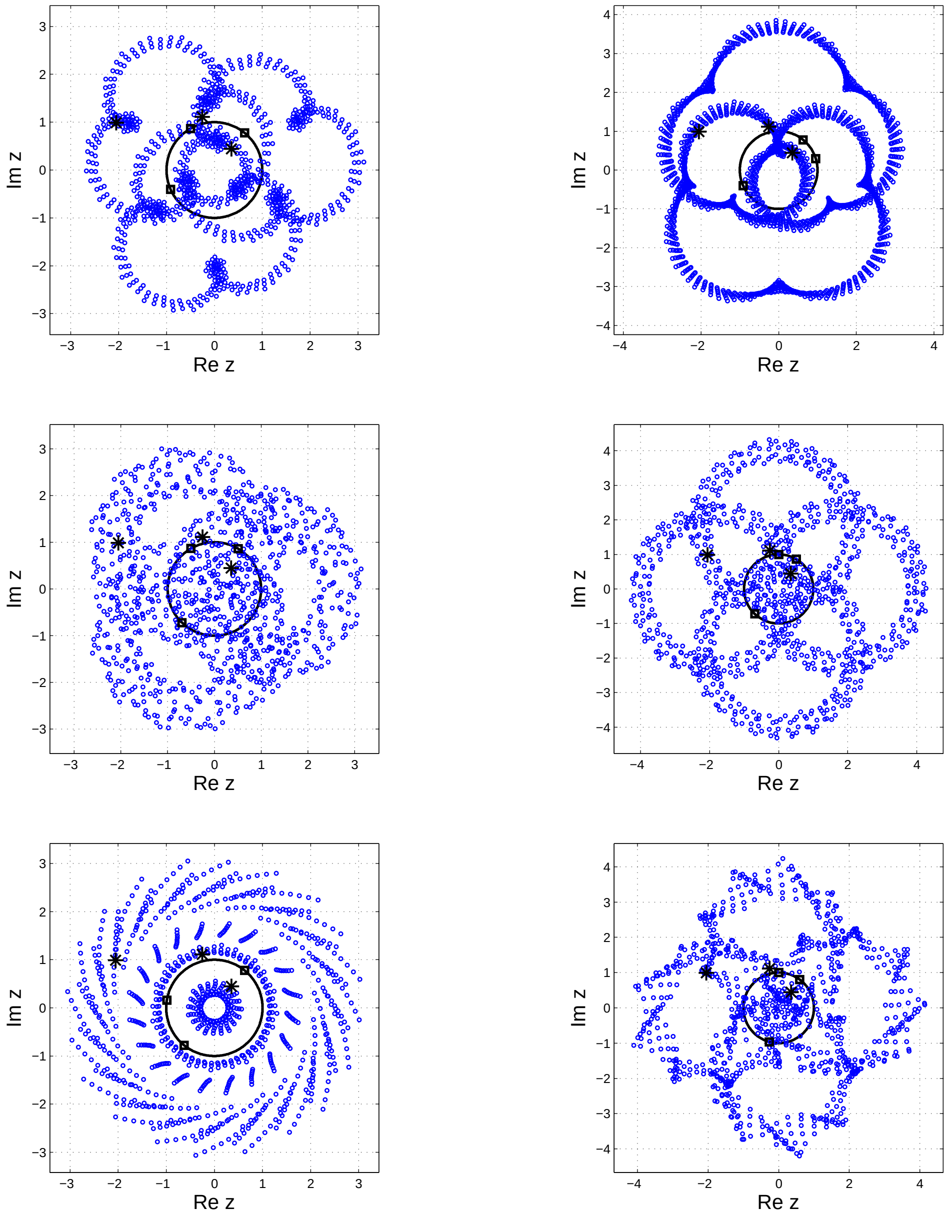

Figure 4.16: Divergent orbits of $\{w_n\}_{n=0}^{\infty}$ (various number of terms) obtained by (4.5.2) for $r_1 = r_2 = 1$ and $r_3 = 1.002$. Initial conditions w_0, w_1, w_2 (stars), generators z_1, z_2, z_3 (squares), $U(0,1)$ (solid line), $U(0, |A_1| + |A_2| + |A_3|)$ (dotted line).

CHAPTER 5

Enumerative sequences linked to periodic complex LRS

The enumeration of periodic Horadam sequences requires counting ordered or (strictly) increasing tuples having the same lcm. So far, arithmetic functions counting ordered tuples with the same lcm (and gcd), have been investigated by Bagdasar in [15]. The study led to novel additions to the OEIS database of integer sequences (A245019, A245020, A247513, A247516, A247517) in 2014, along with new meanings for existing sequences.

5.1 Ordered tuples with the same lcm and gcd

In this section we discuss arithmetic functions counting ordered tuples of positive integers with the same lowest common multiple and same greatest common divisor.

Tuples of integers with the same lcm

Definition 5.1.1. *The number of k-tuples of positive integers with lcm n is*

$$LCM(n;k) = \left|\{(a_1,\ldots,a_k) : \ lcm\,(a_1,\ldots,a_k) = n\}\right|. \tag{5.1.1}$$

Theorem 5.1.2. *Let k and n be naturals numbers. If n has the factorization $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, the number of ordered k-tuples whose* lcm *is n defined in* (5.1.1) *is given by the formula*

$$\mathrm{LCM}(n;k) = \prod_{i=1}^{r} \left[(n_i+1)^k - n_i^k \right]. \tag{5.1.2}$$

Remark 5.1.3. *For particular values of k, one recovers the following* OEIS *indexed sequences:* A048691 *for* LCM$(n;2)$ *(with numerous interpretations), and* A070919, A070920, A070921 *for* LCM$(n;3)$, LCM$(n;4)$, LCM$(n;5)$, *respectively.*

The following result illustrates the independence of $\mathrm{LCM}(n;k)$ on the prime factors.

Corollary 5.1.4. *Let k be a natural number and $n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$, $m = q_1^{n_1} q_2^{n_2} \dots q_r^{n_r}$, such that some, or all numbers $p_1, \dots, p_r, q_1, \dots, q_r$ are distinct. Then $\mathrm{LCM}(m;k) = \mathrm{LCM}(n;k)$.*

An arithmetic function $f(n)$ of the positive integer n is called

- *multiplicative* if $f(1) = 1$ and for any a and b coprime, then $f(ab) = f(a)f(b)$.
- *completely multiplicative* if $f(1) = 1$ and $f(ab) = f(a)f(b)$, even when a and b are not coprime.

Remark 5.1.5. *Let a, b be positive integers and $f : \mathbb{N} \to \mathbb{N}$ be a multiplicative arithmetic function. The following property holds*

$$f(\gcd(a,b)) \cdot f(\mathrm{lcm}(a,b)) = f(a) \cdot f(b). \tag{5.1.3}$$

Corollary 5.1.6. *Let m, n be integers satisfying $(m,n) = 1$. The following property holds*

$$\mathrm{LCM}(m \cdot n;k) = \mathrm{LCM}(m;k) \cdot \mathrm{LCM}(n;k).$$

Corollary 5.1.7. *Consider the natural numbers a and b. The following property holds:*

$$\mathrm{LCM}(\gcd(a,b);k) \cdot \mathrm{LCM}(\mathrm{lcm}(a,b);k) = \mathrm{LCM}(a;k) \cdot \mathrm{LCM}(b;k). \tag{5.1.4}$$

Theorem 5.1.8. *Let m, n and $k \geq 2$ be positive integers. The following inequality holds*

$$\mathrm{LCM}(m \cdot n;k) \leq \mathrm{LCM}(m;k) \cdot \mathrm{LCM}(n;k). \tag{5.1.5}$$

Lemma 5.1.9. *Let $k \geq 2$, p be a prime number and $\alpha, \beta \geq 1$ natural numbers. Then*

$$\mathrm{LCM}(p^{\alpha+\beta};k) \leq \mathrm{LCM}(p^{\alpha};k) \cdot \mathrm{LCM}(p^{\beta};k). \tag{5.1.6}$$

Remark 5.1.10. *The arithmetic function $\mathrm{LCM}(n;k)$ is not completely multiplicative. Indeed, for $p = 2$, $\alpha = \beta = 1$ one obtains the relation $3^k - 2^k < (2^k - 1^k)(2^k - 1^k)$, which is true for all values $k \geq 2$.*

Tuples of integers with the same lcm and gcd

The following lemma represents the motivation for the results in this section.

Lemma 5.1.11. *Let $d < n$ be positive integers, such that $d|n$. The number of ordered pairs (a,b) with the same greatest common divisor d and least common multiple n is*

$$|\{(a,b) : \gcd(a,b) = d, \mathrm{lcm}(a,b) = n\}| = 2^{\omega(n/d)}.$$

where $\omega(x)$ represents the number of distinct prime divisors for the integer x.

Definition 5.1.12. *The number of ordered k-tuples with the same gcd d and lcm n is*

$$GL(d,n;k) = |\{(a_1,\ldots,a_k) : gcd\,(a_1,\ldots,a_k) = d,\ lcm\,(a_1,\ldots,a_k) = n\}|\,, \tag{5.1.7}$$

Lemma 5.1.13. *Let k and $d|n$ be natural numbers. If $d = p_1^{d_1} p_2^{d_2} \ldots p_r^{d_r}$ and $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, the number of ordered k-tuples whose* gcd *is d, and* lcm *is n satisfies the property*

$$\mathrm{GL}(d,n;k) = \mathrm{GL}(1,n/d;k). \tag{5.1.8}$$

Lemma 5.1.14. *Let k and α be positive integers. The number of tuples $(\alpha_1,\ldots,\alpha_k)$ satisfying*

$$\mathrm{T}(\alpha;k) = |\{(\alpha_1,\ldots,\alpha_k) : \min\,(\alpha_1,\ldots,\alpha_k) = 0,\ \max\,(\alpha_1,\ldots,\alpha_k) = \alpha\}|\,, \tag{5.1.9}$$

is given by the formula

$$\mathrm{T}(\alpha;k) = (\alpha+1)^k - 2\alpha^k + (\alpha-1)^k. \tag{5.1.10}$$

Theorem 5.1.15. *Let k and n be naturals numbers. If n has the factorization $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, the number of ordered k-tuples whose* gcd *is* 1 *and* lcm *is n, is given by the formula*

$$\mathrm{L}(n;k) = \prod_{i=1}^{r} \left[(n_i+1)^k - 2n_i^k + (n_i-1)^k \right]. \tag{5.1.11}$$

Corollary 5.1.16. *Let k be a natural number and $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, $m = q_1^{n_1} q_2^{n_2} \ldots q_r^{n_r}$, such that all numbers $p_1,\ldots,p_r,q_1,\ldots,q_r$ are distinct. Then $\mathrm{L}(m;k) = \mathrm{L}(n;k)$.*

The following result states the multiplicity of the arithmetic function $\mathrm{L}(n;k)$ for $k \geq 2$.

Corollary 5.1.17. *Let m, n be comprime integers and $k \geq 2$. The following property holds*

$$\mathrm{L}(m \cdot n;k) = \mathrm{L}(m;k) \cdot \mathrm{L}(n;k).$$

Corollary 5.1.18. *Let a, b be natural numbers. The following property holds:*

$$\mathrm{L}\left(\mathrm{gcd}(a,b);k\right) \cdot \mathrm{L}\left(\mathrm{lcm}(a,b);k\right) = \mathrm{L}\left(a;k\right) \cdot \mathrm{L}\left(b;k\right).$$

Moreover, the arithmetic function $\mathrm{L}(n;k)$ is not completely multiplicative.

Remark 5.1.19. *Choosing $m = n = 2$ and $k \geq 2$, one has*

$$\mathrm{L}(m \cdot n;k) = 3^k - 2 \cdot 2^k + 1 < (2^k - 2) \cdot (2^k - 2) = \mathrm{L}(m;k) \cdot \mathrm{L}(n;k).$$

Some inequalities for $\mathrm{L}(n;k)$ can also be proved, for general values of m and n.

Theorem 5.1.20. *Let m, n and $k \geq 2$ be positive integers. The following inequality holds*

$$\mathrm{L}(m \cdot n;k) \leq \mathrm{L}(m;k) \cdot \mathrm{L}(n;k).$$

5.2 Monotonic integer tuples having same lcm

For increasing and strictly increasing k-tuples whose lcm is n we use the notations

$$\mathrm{LCM}^{\leq}(k,n) = |\{(a_1,\ldots,a_k) : [a_1,\ldots,a_k] = n,\ 1 \leq a_1 \leq \cdots \leq a_k \leq n\}|\,; \quad (5.2.1)$$

$$\mathrm{LCM}^{<}(k,n) = |\{(a_1,\ldots,a_k) : [a_1,\ldots,a_k] = n,\ 1 \leq a_1 < \cdots < a_k \leq n\}|\,. \quad (5.2.2)$$

In what follows we will establish the links between the above formulae. The main goal is to compute $\mathrm{LCM}^{\leq}$ and $\mathrm{LCM}^{<}$ as linear combinations involving LCM. For $k, n \geq 2$ the $k \times n$ matrices for LCM, $\mathrm{LCM}^{<}$ and $\mathrm{LCM}^{\leq}$, are denoted by $\mathcal{L}$, $\mathcal{L}^{<}$ and $\mathcal{L}^{\leq}$. Also, the $k \times k$ transition matrices will be denoted by $\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{R}$ as below.

$$\mathcal{L}^{<}(k,n) = \mathcal{M}(k,:) \cdot \mathcal{L}(:,n); \quad (5.2.3)$$

$$\mathcal{L}^{\leq}(k,n) = \mathcal{N}(k,:) \cdot \mathcal{L}(:,n); \quad (5.2.4)$$

$$\mathcal{L}^{\leq}(k,n) = \mathcal{P}(k,:) \cdot \mathcal{L}^{<}(:,n); \quad (5.2.5)$$

$$\mathcal{L}(k,n) = \mathcal{Q}(k,:) \cdot \mathcal{L}^{<}(:,n); \quad (5.2.6)$$

$$\mathcal{L}(k,n) = \mathcal{R}(k,:) \cdot \mathcal{L}^{\leq}(:,n). \quad (5.2.7)$$

We will show that $\mathcal{P}$ and $\mathcal{Q}$ are invertible and the following identities hold

$$\mathcal{N} = \mathcal{P}\mathcal{M}, \quad \mathcal{M} = \mathcal{Q}^{-1}, \quad \mathcal{R} = \mathcal{Q}\mathcal{P}^{-1}. \quad (5.2.8)$$

5.2.1 LCM$(k;n)$, LCM$^{<}(k;n)$ and LCM$^{\leq}(k;n)$ for $k = 3$

A triplet consisting of numbers $1 \leq a_1, a_2, a_3 \leq n$ with property $[a_1, a_2, a_3] = n$ may be

(1) (all equal): The only triplet with $a_1 = a_2 = a_3$ and $[a_1, a_2, a_3] = n$ is (n, n, n);

(2) (two equal): Any pair $a_1 < a_2$ creates the increasing triplets (a_1, a_1, a_2), (a_1, a_2, a_2), plus the ordered triples (a_1, a_2, a_1), (a_2, a_1, a_1), (a_2, a_1, a_2), (a_2, a_2, a_1);

(3) (all distinct): Any triplet $a_1 < a_2 < a_3$ creates six ordered triples (permutations).

The following relations can be obtained

$$\mathrm{LCM}^{\leq}(n;3) = 1 + 2 \cdot \mathrm{LCM}^{<}(2,n) + \mathrm{LCM}^{<}(3,n); \quad (5.2.9)$$

$$\mathrm{LCM}(3,n) = 1 + 6 \cdot \mathrm{LCM}^{<}(2,n) + 6 \cdot \mathrm{LCM}^{<}(3,n). \quad (5.2.10)$$

Rewriting (5.2.10) for $\mathrm{LCM}^{<}(3,n)$ and using the results for $m = 2$, we obtain

$$\mathrm{LCM}^{<}(3,n) = \frac{\mathrm{LCM}(3,n) - 3 \cdot \mathrm{LCM}(2,n) + 2}{6};$$

$$\mathrm{LCM}^{\leq}(3,n) = \frac{\mathrm{LCM}(3,n) + 3 \cdot \mathrm{LCM}(2,n) + 2}{6}.$$

5.2.2 Recurrence coefficients

Here we discuss the coefficients linking LCM, $\text{LCM}^{<}$ and $\text{LCM}^{\leq}$.

The link between $\text{LCM}^{\leq}$ and $\text{LCM}^{<}$

For a fixed n, the matrix $\mathcal{P}(k,j)$ $(1 \leq k, j \leq n)$ in (5.2.5) is given by

$$\mathcal{P}(k,j) = \begin{cases} \binom{k-1}{j-1}, & k \geq j \\ 0, & k < j. \end{cases}$$

The link between LCM and $\text{LCM}^{<}$

For a fixed n, the matrix $\mathcal{Q}(k,j)$ $(1 \leq k, j \leq n)$ in (5.2.7) is given by

$$\mathcal{Q}(k,j) = \begin{cases} j!S(k,j), & k \geq j \\ 0, & k < j. \end{cases}$$

where $S(k,j)$ stands for the Stirling numbers of the second kind seen in Chapter 1.

The transformation matrices $\mathcal{R}$, $\mathcal{M}$ and $\mathcal{N}$

By (5.2.8), matrices $\mathcal{P}$ and $\mathcal{Q}$ were sufficient for computing $\mathcal{R}$, $\mathcal{M}$ and $\mathcal{N}$. These will allow the computation of matrices for $\text{LCM}^{\leq}$ and $\text{LCM}^{<}$.

Theorem 5.2.1. *The following identities hold for $K \geq 2$ and $1 \leq j, k \leq K$:*

$$\mathcal{R}(k,j) = (-1)^{k+j}\mathcal{Q}$$
$$\mathcal{M}(k,j) = (-1)^{k+j}\mathcal{N}.$$

5.3 Novel additions to OEIS

New sequences obtained from $\mathrm{T}(k,n)$ and $\mathrm{L}(n,k)$

A245019: $\mathrm{T}(4,n) = 5^n - 2 \cdot 4^n + 3^n$.

A245020: $\mathrm{T}(5,n) = 6^n - 2 \cdot 5^n + 4^n$.

A247513: $\mathrm{L}(3,n) = 6^{\omega(n)} \prod_{i=1}^{r} n_i$.

A247516: $\mathrm{L}(4,n) = 2^{\omega(n)} \prod_{i=1}^{r} \left[6n_i^2 + 1\right]$.

A247517: $\mathrm{L}(5,n) = 10^{\omega(n)} \prod_{i=1}^{r} \left[2n_i^3 + n_i\right]$.

Contributions to other sequences

The results in [15] contributed with new meanings to the following OEIS sequences:

A000918: $\mathrm{T}(n, 1) = 2^n - 2$ (number of proper subsets for a set with n elements)

A028243: $\mathrm{T}(n, 2) = 3^n - 2^{n+1} + 1$ (Stirling numbers of the second kind).

A008588: $\mathrm{T}(3, n)$.

A038721: $\mathrm{T}(3, n + 1)$.

A005914: $\mathrm{T}(4, n) = 12n^2 + 2$ (counts the points on the surface of a hexagonal prism).

A068236: $\mathrm{T}(5, n) = 20n^3 + 10n$.

A101098: $\mathrm{T}(5, n + 1)$.

A048091: $\mathrm{LCM}(2, n)$

A070919: $\mathrm{LCM}(3, n)$

A070920: $\mathrm{LCM}(4, n)$

A070921: $\mathrm{LCM}(5, n)$

A102309: Our work provided the first enumerative context. Indeed, $a(n - 1)$ is the number of periodic complex Horadam orbits with period n, for $n > 2$ [19].

Contributions to OEIS from Section 5.2:

A063647: $\mathcal{L}^{<}(2, n)$. An interpretation for this sequence is the number of ways to write $1/n$ as sum of exactly two distinct unit fractions.

A086165: $\mathcal{L}^{<}(3, n)$

Axxxxxx: $\mathcal{L}^{<}(4, n)$ - not currently indexed.

A018892: $\mathcal{L}^{\leq}(2, n)$ number of ways to write $1/n$ as sum of exactly two unit fractions.

A086222: $\mathcal{L}^{\leq}(3, n)$

Axxxxxx: $\mathcal{L}^{\leq}(4, n)$ - not currently indexed.

A008778: $\mathcal{L}^{\leq}(n, 6)$ + fourth row of A022818.

A000292: $\mathcal{L}^{\leq}(n, 8)$ - tethraedral numbers $n \cdot (n + 1) \cdot (n + 2)/6$

Axxxxxx: $\mathcal{L}^{\leq}(n, 12)$ - not currently indexed.

Appendix

a bridge between this work and professor Horadam - excerpt from [77]

We here present some details regarding the correspondence between the authors of the Survey paper on Horadam sequences and Professor A. F. Horadam [77].

A Dedication to Professor A.F. Horadam

In the early part of April 2012 the author P.J.L. sent a draft version of this article to Professor A.G. Shannon, a longstanding and close friend of A.F. Horadam with whom he collaborated professionally. Professor Shannon, on a pre-planned visit to Armidale (New South Wales, Australia) shortly afterwards, took a copy of the paper with him from Sydney and read it to Professor Horadam who has not enjoyed good health for a number of years. Our sincere thanks go to Professor Shannon, who passed on the following comments from Professor Horadam:

> *"I am very flattered by the tone of the paper ... [which is] comprehensive and thorough with an insightful perspective on the history of the sequence."*

We are pleased to receive this personal endorsement from Professor Horadam, and we would like to dedicate this paper to him in recognition of both the work he conducted on the Horadam sequence and the motivation he has provided for others—ourselves included—to continue to study it.

P.J.L.
O.D.B.
E.J.F.

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