Universitatea Babeş - Bolyai Cluj Napoca
Facultatea de Matematică şi Informatică

Teză de doctorat

CONTRIBUTIONS ON APPROXIMATION PROCESSES OF KING - TYPE

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Introduction

1. About Approximation Theory and approximation processes of King-type

Roughly speaking, Approximation Theory answers the following problem: how functions can be approximated with simpler and more easily calculated functions. Also it is concerned with quantitative characterization of the errors introduced by the approximation made. Approximation Theory is an area which has its primary roots in the mathematics of the 19^{th} century. A complicated function f usually is approximated by an easier function of the form $\theta(\cdot; a_0, ..., a_n)$, where $a_0, ..., a_n$ are parameters to be determined so as to characterize the best approximation of f. Depending on the sense in which the approximation is realized, there are different types of approaches such as interpolatory approximation, least-square approximation, min-max approximation. We say θ is a linear approximation of f if θ depends linearly on the parameters a_i, that is, if $\theta(\cdot; a_0, ..., a_n) = \sum_{k=0}^{n} a_k \varphi_k$, where φ_k are given functions. Choosing $\varphi_i(x) = x^i$, the approximation function θ becomes a polynomial.

To be more explicit and rigorous we present the notion of *approximation scheme*. Let (X, d) be a metric space. An approximation method requires a set of approximating functions, say $\mathcal{F}$, which is a subset of X. Specifically, the method is just a mapping from X to $\mathcal{F}$. In other words, given any $f \in X$, the method picks the element e_f, say, from $\mathcal{F}$, which is regarded as an approximation to f. To find how good is the chosen method, e_f should be compared with the best approximation to $f \in X$ from $\mathcal{F}$; this is an element $e^* \in \mathcal{F}$ such that

$$d(f, e^*) = \inf\{d(f, e) \ : \ e \in \mathcal{F}\} \ := \mathrm{dist}(f, \mathcal{F}).$$

Conditions for the existence, uniqueness and for the characterization of the best approximation in the case when $\mathcal{F}$ is a Hilbert space can be found, e.g., in the book of Coman, Chiorean, and Cătinaş [10, *Section 1.4*].

So, an important issue is to determine what type of approximating functions we use. One of the directions in Approximation Theory is given by positive linear approximation processes. It is a relatively new trend that came to light in the fifties due to the research of T. Popoviciu, H. Bohman and P.P. Korovkin. Their famous theorem for characterizing sequences of positive linear operators that approximate the identity operator, is based on easily checked, simple criteria.

The following three aspects are most vital in this direction: the construction of these processes, the study of the degree of approximation, their ability to mimic qualitative properties of the approximated function such as monotonicity, convexity, shape preservation. Concerning the construction of the approximation processes, an important principle was given by F. Altomare and M. Campiti [3, p. 229] regarding positive approximation process. The principle says that if E and F are two function spaces defined on the same locally compact Hausdorff space such that $E \subset F$, then a positive approximation process on E with respect to F is a net $(L_i)_{i \in I}$ of linear positive operators from E into F such that for every f belonging to E, $(L_i)_{i \in I}$ converges to f, the convergence being understood with respect to a suitable topology on F.

Many linear positive operators are preserving just one function from the set $\{e_0, e_1, e_2\}$ where these functions are similar and have the same properties with e^* defined above. J.P. King [18] presented a non-trivial sequence of positive linear operators which preserve e_0 and e_2. For documentation in this area, we used, as primary source, the monograph of O. Agratini [1]. All the basic information is concentrated in Chapter 1 of this thesis.

Among the many approaches to the quantitative error estimation we can mention the result of R. Mamedov [19], or later G. Shisha and B. Mond [22], that have obtained a more general result.

This vein of research was proved to be extremely fertile, consequently many mathematicians have developed this two subjects: construction of linear and positive processes and their quantitative error estimates. This thesis aims at this direction of investigation. The work combines classical results, new results appeared in the last decade and personal research aspects.

2. The architecture of the thesis

Our goal is to construct different classes of linear positive operators of discrete or integral type in the King sense, to study their approximation processes. For the approximation properties we study their error approximation and their convergence.

The thesis is structured in four chapters.

Chapter 1 gives a collection of some significant notions in the area of linear and positive operators. Here we give definitions, examples, properties that the linear and positive operators have, the classical Korovkin theorems, the notion of moduli of smoothness and their properties, results on the rate of convergence of a sequence of linear operators. All the involved mathematical entities are fully described and specified. Also, we give examples of linear and positive operators different from the classical operators. In a distinct section, we detailed the positive operators discovered by J.P. King in 2003 and we present their approximation properties.

Chapter 2 treats classes of modified Bernstein operators in King sense. We present the genuine Bernstein operators, we deal with a class of King-type operators which preserve the test functions e_1 and e_2. After a stopover on genuine Stancu operators and two classes of Stancu-type operators, we study two particular classes of Schurer-Stancu-type operators.

Chapter 3 begins by presenting the genuine Baskakov operators and continues with the presentation of a general class of Baskakov-type modified operators. Also, three particular classes of Baskakov-type operators modified in King sense are presented.

Chapter 4 is devoted to the study of integral operators. The genuine Durrmeyer operator is presented. Then it follows the construction of the particular classes of Durrmeyer-type operators modified in King sense.

In the construction of this work we tried first to include personal results.

Although the temptation was great, we did not want to present numerous existing results in the field. It would be transformed into a broad synthesis, which is not the primary purpose of a PhD thesis. Following the line, we inserted only the results we actually used in the published papers.

The presented results come from single or joint papers of the author of this thesis and the following coauthors: Ovidiu T. Pop, Petru I. Braica, Anamaria Indrea, Laurian I. Pişcoran. So, we have published eight articles in the following journals:

- *Applied Mathematics and Information Sciences*, 2013 ISI Impact Factor: 1.232,
- *Miskolc Mathematical Notes*, 2013 ISI Impact Factor: 0.357,
- *Creative Mathematics and Informatics*,
- *Annals of the University of Craiova*,
- *Acta Universitatis Apulensis* (three articles),
- *Acta Mathematica Universitatis Comenianae*.

We also mention that the author of this thesis has participated at the following conferences:

1) International Conference on Applied Mathematics, (ICAM 9), ninth edition, held in Baia Mare, September 25-28, 2013. The title of the talk was "On an operator of Stancu-type with fixed points e_1 and e_2".

2) International Conference on Numerical Analysis and Approximation Theory (NAAT 2014), third edition, held in Cluj Napoca, September 17-20, 2014. The title of the talk was "About a class of linear and positive Stancu-type operators".

3. Our original results

Our results are disseminated in Chapter 2, Chapter 3 and Chapter 4 as follows:

• Section 2.2: Lemma 2.2.1, Lemma 2.2.2, Theorem 2.2.1, Lemma 2.2.3, Theorem 2.2.2, Theorem 2.2.3

• Section 2.4: Lemma 2.4.1, Lemma 2.4.2, Lemma 2.4.3, Lemma 2.4.4, Theorem 2.4.1

• Section 2.5: Theorem 2.5.1, Theorem 2.5.2

• Section 2.6: Theorem 2.6.1, Theorem 2.6.2, Theorem 2.6.3, Lemma 2.6.1, Lemma 2.6.2, Theorem 2.6.4, Theorem 2.6.5, Theorem 2.6.6

• Section 3.2: Theorem 3.2.1, Theorem 3.2.2

• Section 3.3: Theorem 3.3.1, Theorem 3.3.2, Lemma 3.3.1, Lemma 3.3.2, Theorem 3.3.3, Theorem 3.3.4, Theorem 3.3.5, Theorem 3.3.6

• Section 4.2: Lemma 4.2.1, Theorem 4.2.1, Theorem 4.2.2, Lemma 4.2.2, Theorem 4.2.3, Theorem 4.2.4, Theorem 4.2.5

In short, in a synthesis approach, the new results obtained in this PhD thesis focus on the following aspects:
a) are constructed positive linear operators depending on several parameters, both discrete and continuous type, which generalizes some classical operators;
b) are investigated properties of new operators, insisting on the error of approximation and their asymptotic behavior; also they have the capacity to reproduce Korovkin test functions, usually the couples (e_0, e_1), (e_0, e_2), (e_1, e_2), in the spotlight being the operators of King-type.

Keywords: linear positive operator, approximation process, moduli of

smoothness, rate of convergence, Voronovskaja-type formula, Bernstein operator, Baskakov operator

Chapter 1

Preliminaries

1.1 Linear positive operators

Here are presented notions about operators, their properties, notion of approximation process, Korovkin's theorems, Korovkin's sets.

1.2 Moduli of smoothness

Moduli of smoothness are defined and there are presented some of their properties.

1.3 Quantitative error estimations

In this section we present some classical quantitative error estimates for the pointwise and uniform approximation by linear positive operators.

1.4 Examples of linear positive operators

We give some examples of linear and positive operators different from the classic ones.

1.5 Operators of King type

We define the operator introduced by J. P. King in 2003, we present some of its properties specifying notions of the iterates of this operator and result regarding the convergence of the iterates,

Chapter 2

Modified Bernstein-type operators in King sense

2.1 Bernstein operators

We present the genuine Bernstein operator, we give the properties and two Voronovskaja-type formulas for this operator and we remind about the convergence of Bernstein operator.

2.2 A new type of Bernstein-King operators

This section contains the results published in *Applied Mathematics and Information Sciences*, No. 6 (1) (2012), 191-197.

In [18], J.P. King defined linear positive operators which generalize the classical Bernstein operators. These operators have been presented in Chapter 1, see Section 5 and now, we define a new type of Bernstein-King operators. As we have seen in Chapter 1, Section 5, the King-type operators reproduce the test functions e_0 and e_2 and we have an approximation as smooth as the other operators. For these operators we discuss some of their properties and we give both an approximation theorem and a Voronovskaja-type formula. Also, in paper [8] can be found more applications for these operators.

Further on, we consider a fixed number $m_0 \in \mathbb{N}$, $m_0 > 2$. For the function $f : [0;1] \to R$, we define the sequence of operators $(B_m^* f)_{m \geq m_0}$ by

$$(B_m^* f)(x) = \frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\sum_{k=0}^{m}\binom{m}{k} \times (1-x)^{m-k}\left(x-\frac{1}{m}\right)^{k} f\left(\frac{k}{m}\right), \tag{2.2.1}$$

for any $m \geq m_0$ and any $x \in [(m_0-1)^{-1}, 1]$.

If $m \in N$, $m \geq m_0$, then the operator B_m^* is linear and positive.

Lemma 2.2.1. (Braica, P.I., Pop, O. T., **Indrea, D. Adrian**, [9]) *Let* $(B_m^*)_{m\in\mathbb{N}}$ *be defined by* (2.2.1). *The identities*

$$(B_m^* e_0)(x) = \frac{(m-1)x}{mx-1}, \tag{2.2.2}$$

$$(B_m^* e_1)(x) = x, \tag{2.2.3}$$

$$(B_m^* e_2)(x) = x^2, \tag{2.2.4}$$

$$(T_{m,0} B_m^*)(x) = \frac{(m-1)x}{mx-1}, \tag{2.2.5}$$

$$(T_{m,1} B_m^*)(x) = \frac{mx(x-1)}{mx-1} \tag{2.2.6}$$

and

$$(T_{m,2}B_m^*)(x) = \frac{m^2x^2(1-x)}{mx-1} \tag{2.2.7}$$

hold for any $m \in \mathbb{N}$, $m \geq m_0$ *and any* $x \in [(m_0-1)^{-1}, 1]$.

Lemma 2.2.2. (Braica, P.I., Pop, O. T., **Indrea, D. Adrian**, [9]) *Let* $(B_m^*)_{m\in\mathbb{N}}$ *be defined by* (2.2.1). *One has*

$$B_0(x) = \lim_{m\to\infty}(T_{m,0}B_m^*)(x) = 1, \tag{2.2.8}$$

$$B_2(x) = \lim_{m\to\infty}\frac{(T_{m,2}B_m^*)(x)}{m} = x(1-x) \tag{2.2.9}$$

and

$$(T_{m,0}B_m^*)(x) \leq m_0 - 1 = k_0, \tag{2.2.10}$$

$$\frac{(T_{m,2}B_m^*)(x)}{m} \leq \frac{m_0}{4} = k_2, \tag{2.2.11}$$

for any $x \in [(m_0-1)^{-1}, 1]$.

Theorem 2.2.1. (Braica, P.I., Pop, O. T., **Indrea, D. Adrian**, [9]) *Let* $(B_m^*)_{m\in\mathbb{N}}$ *be defined by* (2.2.1) *and* $f : [0,1] \to \mathbb{R}$ *be a continuous function on* $[0,1]$.

Then

$$\begin{aligned} &|(B_m^*f)(x) - f(x)| \\ &\leq |f(x)|\frac{1-x}{mx-1} + \frac{x}{mx-1}\left(m-1+\frac{1}{\delta}\sqrt{(m-1)x(1-x)}\right)\omega(f;\delta) \end{aligned} \tag{2.2.12}$$

and

$$|(B_m^*f)(x) - f(x)| \leq \frac{(m_0-2)M}{m-m_0+1} + \frac{2(m-1)}{m-m_0+1}\omega\left(f;\frac{1}{2\sqrt{m-1}}\right), \tag{2.2.13}$$

for any $\delta > 0, m \in \mathbb{N}, m \geq m_0$ *and* $x \in [(m_0 - 1)^{-1}, 1]$, *where*

$$M = \sup \left\{ |f(x)| : x \in \left[(m_0 - 1)^{-1}, 1 \right] \right\}.$$

Lemma 2.2.3. (Braica, P.I., Pop, O. T., **Indrea, D. Adrian**, [9]) *Let* $(B_m^*)_{m \in \mathbb{N}}$ *be defined by* (2.2.1) *and* $f : [0, 1] \to \mathbb{R}$ *be a continuous function on* $[0, 1]$.

There exists $m(0)$ *such that*

$$\left| (B_m^* f)(x) - \frac{(m-1)x}{mx - 1} f(x) \right| \leq \frac{5m_0 - 1}{4} \omega \left(f; \frac{1}{\sqrt{m}} \right) \tag{2.2.14}$$

and

$$|(B_m^* f)(x) - f(x)| \leq |f(x)| \frac{1 - x}{mx - 1} + \frac{5m_0 - 1}{4} \omega \left(f; \frac{1}{\sqrt{m}} \right) \tag{2.2.15}$$

for any $x \in [(m_0 - 1)^{-1}, 1]$, $m \in \mathbb{N}$, $m \geq m(0)$. *The quantity* $m(0)$ *was defined in Theorem* **??**.

Theorem 2.2.2. (Braica, P.I., Pop, O. T., **Indrea, D. Adrian**, [9]) *Let* $(B_m^*)_{m \in \mathbb{N}}$ *be defined by* (2.2.1) *and* $f : [0, 1] \to \mathbb{R}$ *be a continuous function on* $[0, 1]$. *Then*

$$\lim_{m \to \infty} (B_m^* f)(x) = f(x) \tag{2.2.16}$$

uniformly on $[(m_0 - 1)^{-1}, 1]$.

There exists $m(0)$ *such that*

$$|(B_m^* f)(x) - f(x)| \leq \frac{(m_0 - 2)M}{m - m_0 + 1} + \frac{5m_0 - 1}{4} \omega \left(f; \frac{1}{\sqrt{m}} \right) \tag{2.2.17}$$

for any $x \in [(m_0 - 1)^{-1}, 1]$ *and any* $m \in \mathbb{N}$, $m \geq m(0)$.

Theorem 2.2.3. (Braica, P.I., Pop, O. T., **Indrea, D. Adrian**, [9]) *Let* $(B_m^*)_{m\in\mathbb{N}}$ *be defined by* (2.2.1) *and* $f:[0,1]\to\mathbb{R}$ *be a continuous function on* $[0,1]$. *If* $x\in[(m_0-1)^{-1},1]$, f *is two times differentiable on* $[(m_0-1)^{-1},1]$ *and* $f^{(2)}$ *is continuous on* $[(m_0-1)^{-1},1]$, *then*

(2.2.18)
$$\lim_{m\to\infty} m\left((B_m^*f)(x)-\frac{(m-1)x}{mx-1}f(x)\right)=(x-1)f^{(1)}(x)+\frac{x(1-x)}{2}f^{(2)}(x)$$

and

(2.2.19)
$$\lim_{m\to\infty} m\left((B_m^*f)(x)-f(x)\right)=\frac{1-x}{x}f(x)+(x-1)f^{(1)}(x)+\frac{x(1-x)}{2}f^{(2)}(x).$$

2.3 Stancu operators

We present the operator introduced by D. D. Stancu [23] in 1969, we give the properties, a Korovkin-type theorem and a Voronovskaja-type formula for this operator. All these results are already known.

2.4 The study of a general Stancu-type approximation process

The main result of this section is based on the paper [14], appeared in *Acta Mathematica Universitatis Comenianae* 84 (2015), 1, 123-131.

Further on, we introduce a class of Stancu-type operators, with the properties that the test functions e_1 and e_2 are reproduced. Also, in our approach, an error approximation theorem and a Voronovskaja-type theorem are obtained.

Let $m_0 \in \mathbb{N}$ be given, $\mathbb{N}_1 = \{m \in \mathbb{N}_0 \mid m \geq m_0\}$ and $0 \leq \alpha \leq \beta$ are fixed real parameters. We consider the functions $a_m : J \longrightarrow \mathbb{R}$, $b_m : J \longrightarrow \mathbb{R}$ such that $a_m(x) \geq 0$, $b_m(x) \geq 0$ for any $x \in J$, $m \in \mathbb{N}_1$ and $I = [0,1]$. J will be defined later.

We define the operators of the following form

$$(H_m^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{m} \binom{m}{k} a_m^k(x) b_m^{m-k}(x) \cdot f\left(\frac{k+\alpha}{m+\beta}\right), \tag{2.4.1}$$

for any $m \in \mathbb{N}_1$, $x \in J$ and $f \in E([0,1])$, where $E([0,1])$ is a linear space of real valued functions defined on $[0,1]$.

In what follows, we impose some additional conditions to be fulfilled by our operators:

$$(H_m^{(\alpha,\beta)} e_0)(x) = 1 + u_m(x),\ m \in \mathbb{N}_1,\ x \in J \tag{2.4.2}$$

and

$$(H_m^{(\alpha,\beta)} e_1)(x) = x + v_m(x),\ m \in \mathbb{N}_1,\ x \in J, \tag{2.4.3}$$

Then, it follows

$$a_m(x) = (1 + u_m(x))^{\frac{1}{m}} \left(\frac{m+\beta}{m} \cdot \frac{x + v_m(x)}{1 + u_m(x)} - \frac{\alpha}{m}\right) \tag{2.4.4}$$

and

$$b_m(x) = (1 + u_m(x))^{\frac{1}{m}} \left(1 - \frac{m+\beta}{m} \cdot \frac{x + v_m(x)}{1 + u_m(x)} + \frac{\alpha}{m}\right), \tag{2.4.5}$$

for any $m \in \mathbb{N}_1$ and $x \in J$.

We have $I = [0,1]$, $E([0,1]) = C([0,1])$.

If $H_m^{(\alpha,\beta)} e_1 = e_1$, for any $m \in \mathbb{N}_1$, then $v_m(x) = 0$. We have

$$a_m(x) = (1 + u_m(x))^{\frac{1}{m}} \left(\frac{m+\beta}{m} \cdot \frac{x}{1+u_m(x)} - \frac{\alpha}{m} \right) \tag{2.4.6}$$

and

$$b_m(x) = (1 + u_m(x))^{\frac{1}{m}} \left(1 - \frac{m+\beta}{m} \cdot \frac{x}{1+u_m(x)} + \frac{\alpha}{m} \right), \tag{2.4.7}$$

for any $m \in \mathbb{N}_1$ and $x \in J$.

Lemma 2.4.1. (**Indrea, D. Adrian**, [14]) *Let $m \in \mathbb{N}_1$. The following relations are equivalent*

(i) $a_m(x) \geq 0$ and $b_m(x) \geq 0$,

(ii) $x \in J$ and $J = \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta} \right]$.

Next, from $H_m^{(\alpha,\beta)} e_2 = e_2$ and $v_m(x) = 0$, $m \in \mathbb{N}_1$, we have

(2.4.8)

$$(\alpha^2 + \alpha m) u_m^2(x) + \Big(m(m+\beta)^2 x^2 - (m+\beta)(m+2\alpha)x + 2(\alpha^2 + \alpha m) \Big) u_m(x) \\ + (m+\beta)^2 x^2 - (m+\beta)(m+2\alpha)x + (\alpha^2 + \alpha m) = 0.$$

The relation (2.4.8) is a second degree equation in $u_m(x)$ and we get the positive solution of this equation

$$u_m(x) = \frac{(m+\beta)(m+2\alpha)x - m(m+\beta)^2 x^2 - 2(\alpha^2 + \alpha m) + \sqrt{\Delta}}{2(\alpha^2 + \alpha m)} \tag{2.4.9}$$

Lemma 2.4.2. (**Indrea, D. Adrian**, [14]) *If $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta} \right]$ and $m \in \mathbb{N}_1$, then the operators $H_m^{(\alpha,\beta)}$ are linear and positive on $C([0,1])$.*

Lemma 2.4.3. (**Indrea, D. Adrian**, [14]) *If* $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right]$, *then*

$$\lim_{m \longrightarrow \infty} u_m(x) = 0 \tag{2.4.10}$$

and

$$\lim_{m \longrightarrow \infty} m u_m(x) = \frac{1-x}{x}. \tag{2.4.11}$$

Lemma 2.4.4. (**Indrea, D. Adrian**, [14]) *For* $m \in \mathbb{N}_1$ *and* $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right]$, *the following identities*

$$(T_{m,0} H_m^{(\alpha,\beta)})(x) = 1 + u_m(x), \tag{2.4.12}$$

$$(T_{m,1} H_m^{(\alpha,\beta)})(x) = m x u_m(x) \tag{2.4.13}$$

and

$$(T_{m,2} H_m^{(\alpha,\beta)})(x) = m^2 x^2 u_m(x) \tag{2.4.14}$$

hold.

Theorem 2.4.1. (**Indrea, D. Adrian**, [14]) *Let* $(H_m^{(\alpha,\beta)})_{m \in \mathbb{N}}$ *be defined by* (2.4.1) *and* $f : [0,1] \longrightarrow \mathbb{R}$ *be continuous function s times differentiable on* $[0,1]$, *having the s-order derivative continuous on* $[0,1]$.

For $s = 0$ *we have*

$$\lim_{m \longrightarrow \infty} H_m^{(\alpha,\beta)} f = f \tag{2.4.15}$$

uniformly on $J = \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right]$. *There exists* $m^* = max(m_0, m(0), m_1)$ *such that*

$$|(H_m^{(\alpha,\beta)} f)(x) - f(x)| \leq M \cdot \frac{m_0+\beta}{m\alpha} + \frac{13}{4}\omega\Big(f; \frac{1}{\sqrt{m}}\Big), \tag{2.4.16}$$

for any $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right], m \in \mathbb{N}, m \geq m^*$. *In the above* $M = \sup_{x \in J} |f(x)|$.

For $s = 2$, *we have*

$$\lim_{m \longrightarrow \infty} m\Big((H_m^{(\alpha,\beta)} f)(x) - f(x)\Big) = \frac{1-x}{x} f(x) + (1-x) f^{(1)}(x) + \frac{x(1-x)}{2} f^{(2)}(x). \tag{2.4.17}$$

2.5 A special class of Stancu-King-type operators

The section contains the results published in *Acta Universitatis Apulensis*, No. 31 (2012), 249-256.

At first, we introduce a particular class of Stancu-type operators, with the property that the test functions e_0 and e_1 are reproduced as in the classical case of Bernstein-type operator. Also, in our approach we give two theorems of error approximation and two Voronovskaja-type theorems.

Definition 2.5.1. *Let* $f \in C([0,1]$, $\beta \geq 0$ *and* $m_0 \in \mathbb{N}_0$ *be fixed, depending only on* β. *For any* $m \in \mathbb{N}, m \geq m_0$, *we define the operator*

$$(Q_m^\beta f)(x) = \frac{1}{m^m} \sum_{k=0}^{m} \binom{m}{k} ((m+\beta)x)^k (m-(m+\beta)x)^{m-k} f\left(\frac{k}{m+\beta}\right), \tag{2.5.1}$$

where $x \in \left[0, \frac{m_0}{m_0+\beta}\right]$.

Remark 2.5.1. *The operators* (Q_m^β), $m \geq m_0$, *are linear and positive. Also, for* $\beta = 0$ *in* (2.5.1), *we reobtain Bernstein's operators.*

Theorem 2.5.1. (**Indrea, D. Adrian**, [15]) *Let* $(Q_m^\beta)_{m\in\mathbb{N}}$ *be defined by* (2.5.1) *and* $f : [0,1] \longrightarrow \mathbb{R}$ *be a function* s *times differentiable on* $[0,1]$, *having the s order derivative continuous on $[0,1]$. Then, for $s=0$, we have*

$$\lim_{m\longrightarrow\infty} Q_m^\beta f = f \tag{2.5.2}$$

uniformly on $\left[0, \frac{m_0}{m_0+\beta}\right]$ and exists $m^* = \max(m_0, m(0), m(2))$ such that

$$|(Q_m^\beta f)(x) - f(x)| \leq \frac{9}{4}\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{2.5.3}$$

for any $x \in \left[0, \frac{m_0}{m_0+\beta}\right]$, any $m \in \mathbb{N}$, $m > m^*$.

Theorem 2.5.2. (**Indrea, D. Adrian**, [15]) *Let $(Q_m^\beta)_{m\in\mathbb{N}}$ be defined by (2.5.1) and $f : [0,1] \longrightarrow \mathbb{R}$ be a function s times differentiable on $[0,1]$, having s order derivative continuous on $[0,1]$.*
(i) For $s=2$, we have

$$\lim_{m\longrightarrow\infty} m((Q_m^\beta f)(x) - f(x)) = \frac{x(1-x)}{2} f^{(2)}(x) \tag{2.5.4}$$

uniformly on $\left[0, \frac{m_0}{m_0+\beta}\right]$ and exists $m_1 = \max(m^*, m(4))$ such that

$$m\left|(Q_m^\beta f)(x) - f(x)\right| \leq \frac{5}{8}M + \frac{39}{32}\omega\left(f^{(2)}, \frac{1}{\sqrt{m}}\right) \tag{2.5.5}$$

for any $x \in \left[0, \frac{m_0}{m_0+\beta}\right]$, $m \in \mathbb{N}, m > m^*$, where $M = \max\limits_{x\in[0,1]} |f^{(2)}(x)|$.
(ii) For $s=4$ we have

$$\begin{aligned}\lim_{m\longrightarrow\infty} m^2 \Bigg(&(Q_m^\beta f)(x) - f(x) - \frac{1}{2m}\left(-x + \frac{m}{m+\beta}x\right) f^{(2)}(x) \\ &- \frac{1}{6m^3} f^{(3)}(x)(T_{m,3}\ Q_m^\beta)(x)\Bigg) = \frac{3}{24}(x(1-x))^2 f^{(4)}(x),\end{aligned} \tag{2.5.6}$$

for any $x \in \left[0, \frac{m_0}{m_0+\beta}\right]$.

2.6 Two new classes of Schurer-Stancu-type operators

The main results of this section are based on [13, 17] appeared in *Acta Universitatis Apulensis*, 42 (2015), 1-8 and in *Creative Mathematics and Informatics*, 24 (2015), 1, 61-67, respectively.

Further on, a class of Schurer-Stancu-type operators is introduced. These operators reproduce the test functions e_0 and e_1. For these operators we give two theorems of error approximation and two Voronovskaja-type theorems.

Let α be a real number, $\alpha \geq 0$, $m_0 \in \mathbb{N}$ fixed.

We impose the condition that $m_0 \geq [\alpha]+1$. Consequently we have $\frac{\alpha}{m_0} < 1$.

Let $\mathbb{N}_1 = \{m \in \mathbb{N}_0 | m \geq m_0\}$.

If $m \geq m_0$ then $\left[\frac{\alpha}{m_0}; 1\right] \subset \left[\frac{\alpha}{m}; 1\right]$.

We consider the operators

$$(2.6.1) \quad \left(Q_m^{*\alpha} f\right)(x) = \frac{1}{m^m} \sum_{k=0}^{m} \binom{m}{k} (mx-\alpha)^k (m+\alpha-mx)^{m-k} f\left(\frac{k+\alpha}{m}\right)$$

for any $f \in C([0, 1+\alpha]), m \in \mathbb{N}_1$ and any $x \in \left[\frac{\alpha}{m_0}; 1\right]$.

Remark 2.6.1. (i) *The operators* $(Q_m^{*\alpha})_{m \geq m_0}$ *are linear and positive.*

(ii) *Choosing* $\alpha = 0$, *in* (2.6.1), *we obtain Bernstein's operators.*

Theorem 2.6.1. (**Indrea, D. Adrian**, Indrea, A., [13]) *Let* $Q_m^{*\alpha}$, $m \geq m_0$, *be defined by* (2.6.1). *If* $f \in C([0, 1+\alpha])$, *then we have*

$$(2.6.2) \qquad \lim_{m \longrightarrow \infty} Q_m^{*\alpha} f = f$$

uniformly on $J = \left[\frac{\alpha}{m_0}; 1\right]$. *There exists* $m^* = \max(m_0, m(0), m(2))$ *such that*

$$(2.6.3) \qquad |(Q_m^{*\alpha} f)(x) - f(x)| \leq \frac{13}{4} \omega\left(f; \frac{1}{\sqrt{m}}\right),$$

for any $x \in \left[\frac{\alpha}{m_0}; 1\right], m \in \mathbb{N}, m \geq m^*$.

Theorem 2.6.2. (**Indrea, D. Adrian**, Indrea, A., [13]) *Let* $Q_m^{*\alpha},\ m \geq m_0$, *be defined by* (2.6.1). *If* $f : [0, 1+\alpha] \longrightarrow \mathbb{R}$ *is two times differentiable on* $[0, 1+\alpha]$, *having the second order derivative continuous on* $[0, 1+\alpha]$, *then we have*

$$\lim_{m\longrightarrow\infty} m((Q_m^{*\alpha} f)(x) - f(x)) = \frac{x(1-x)}{2} f^{(2)}(x) \tag{2.6.4}$$

uniformly on $\left[\frac{\alpha}{m_0}; 1\right]$. *There exists* $m_1 = \max(m^*, m(4))$ *such that*

$$m|(Q_m^{*\alpha} f)(x) - f(x) - \frac{f^{(2)}(x)}{2m^2}(T_{m,2} Q_m^{*\alpha})(x)| \leq \frac{39}{32}\omega\left(f^{(2)}; \frac{1}{\sqrt{m}}\right), \tag{2.6.5}$$

for any $x \in \left[\frac{\alpha}{m_0}; 1\right], m \in \mathbb{N}, m \geq m_1$.

Theorem 2.6.3. (**Indrea, D. Adrian**, Indrea, A., [13]) *Let* $Q_m^{*\alpha},\ m \geq m_0$, *be defined by* (2.6.1). *If* $f : [0, 1+\alpha] \longrightarrow \mathbb{R}$ *is a four times differentiable on* $[0, 1+\alpha]$, *having the fourth order derivative continuous on* $[0, 1+\alpha]$, *then we have*

$$\begin{aligned} &\lim_{m\longrightarrow\infty} m^2 \left((Q_m^{*\alpha} f)(x) - f(x) - \frac{x(1-x)}{2m} f^{(2)}(x) - \frac{(2\alpha+1)x^2}{3m} f^{(3)}(x)\right) \\ &\qquad = \frac{\alpha(2x-1)}{2} f^{(2)}(x) + \frac{x(1-x)(1-2x)}{6} f^{(3)}(x) + \frac{x^2(1-x)^2}{8} f^{(4)}(x), \end{aligned} \tag{2.6.6}$$

for any $x \in \left[\frac{\alpha}{m_0}; 1\right]$.

Further on, we introduce a different class of Schurer-Stancu-type operators with the property that the test functions e_0 and e_1 are reproduced. Also, for these operators, we give an error approximation theorem and a

Voronovskaja-type theorem. The main aim is to study the convergence of the iterates for our new class of operators.

Let $p \in \mathbb{N}$ be fixed, $0 < \alpha + p < \beta$ and the functions $a_m, b_m : J \longrightarrow \mathbb{R}$ such that $a_m(x) \geq 0, b_m(x) \geq 0$ for any $x \in J, m \in \mathbb{N}$. Let $I = [0, 1+p]$. J will be defined later.

We define the operators of the following form

$$\left(S_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \binom{m+p}{k} a_m^k(x) b_m^{m+p-k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{2.6.7}$$

for any $m \in \mathbb{N}, x \in J$ and $f \in E([0, 1+p])$. Here $E([0, 1+p])$ is the linear space of all real valued functions defined on $[0, 1+p]$.

In what follows, we impose the additional condition to be fulfilled by our operators

$$\left(S_{m,p}^{(\alpha,\beta)} e_0\right)(x) = 1,\ m \in \mathbb{N},\ x \in J \tag{2.6.8}$$

and

$$\left(S_{m,p}^{(\alpha,\beta)} e_1\right)(x) = x,\ m \in \mathbb{N},\ x \in J. \tag{2.6.9}$$

It follows

$$a_m(x) = \frac{(m+\beta)x - \alpha}{m+p} \tag{2.6.10}$$

and

$$b_m(x) = \frac{m+p-(m+\beta)x+\alpha}{m+p}, \tag{2.6.11}$$

for any $m \in \mathbb{N}$ and $x \in J$.

We fix $m_0 \in \mathbb{N}$ and let $\mathbb{N}_1 = \{m \in \mathbb{N}_0 \mid m \geq m_0\}$.

Lemma 2.6.1. (**Indrea, D. Adrian**, Indrea, A., Braica, P.I.,[17]) *Let $m \in \mathbb{N}_1$, then the following relations are equivalent*

(i) $a_m(x) \geq 0$ *and* $b_m(x) \geq 0$*;*

(ii) $x \in J$ *and* $J = \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$.

Now, considering the relations (2.6.10) and (2.6.11), the operator (2.6.7) becomes

$$\begin{aligned}\left(S_{m,p}^{(\alpha,\beta)} f\right)(x) = \frac{1}{(m+p)^{m+p}} \sum_{k=0}^{m+p} \binom{m+p}{k} ((m+\beta)x - \alpha)^k \\ \times (m+p-(m+\beta)x+\alpha)^{m+p-k} \cdot f\left(\frac{k+\alpha}{m+\beta}\right)\end{aligned} \tag{2.6.12}$$

for any $m \in \mathbb{N}_1, x \in J$ and $f \in E([0, 1+p])$.

Remark 2.6.2. *If* $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$ *and* $m \in \mathbb{N}_1$*, then the operators* $S_{m,p}^{(\alpha,\beta)}$ *are linear and positive.*

Lemma 2.6.2. (**Indrea, D. Adrian**, Indrea, A., Braica, P.I., [17]) *Let* $S_{m,p}^{(\alpha,\beta)}$ *be defined by* (2.6.12)*. For* $m \in \mathbb{N}_1$ *and* $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$*, the following identities*

$$\left(T_{m,0} S_{m,p}^{(\alpha,\beta)}\right)(x) = 1, \tag{2.6.13}$$

$$\left(T_{m,1} S_{m,p}^{(\alpha,\beta)}\right)(x) = 0 \tag{2.6.14}$$

and

$$\left(T_{m,2} S_{m,p}^{(\alpha,\beta)}\right)(x) = m^2 \left(\left(S_{m,p}^{(\alpha,\beta)} e_2\right)(x) - x^2\right) \tag{2.6.15}$$

hold.

Theorem 2.6.4. (**Indrea, D. Adrian**, Indrea, A., Braica, P.I., [17]) *Let* $S_{m,p}^{(\alpha,\beta)}$ *be defined by* (2.6.12)*. If* $f : [0, 1 + p] \longrightarrow \mathbb{R}$ *is a continuous function on* $[0, 1]$*, then we have*

$$\lim_{m \longrightarrow \infty} S_{m,p}^{(\alpha,\beta)} f = f \tag{2.6.16}$$

uniformly on $J = \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$.

There exists $m^* = \max(m_0, m(0))$ *such that*

$$\left|\left(S_{m,p}^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq \frac{9 + 16\alpha^2 + 8\alpha}{4} \cdot \omega\left(f, \frac{1}{\sqrt{m}}\right), \tag{2.6.17}$$

for any $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right], m \in \mathbb{N}, m \geq m^*$.

Theorem 2.6.5. (**Indrea, D. Adrian**, Indrea, A., Braica, P.I., [17]) *Let* $S_{m,p}^{(\alpha,\beta)}$ *be defined by* (2.6.12)*. If* $f : [0, 1 + p] \longrightarrow \mathbb{R}$ *is a continuous function on* $[0, 1]$ *and is two times differentiable on* $[0, 1]$ *having the second-order derivative continuous on* $[0, 1]$*, then we have*

$$\lim_{m \longrightarrow \infty} m\left(\left(S_{m,p}^{(\alpha,\beta)} f\right)(x) - f(x)\right) = \frac{f^{(2)}(x)}{2} x(4\alpha + 1 - x). \tag{2.6.18}$$

Theorem 2.6.6. (**Indrea, D. Adrian**, Indrea, A., Braica, P.I., [17]) *Let* $I = [c, d]$*. Let* $S_{m,p}^{(\alpha,\beta)}, m \in \mathbb{N}_1$*, be defined by* (2.6.12) *such that* $\varphi_{m,0}(c) = \varphi_{m,m+p}(d) = 1$*. If* $f \in C([c, d])$*, then the sequence of iterates* $\left(\left(S_{m,p}^{(\alpha,\beta)}\right)^n\right)_{n\geq 1}$ *verifies*

$$\lim_{n \longrightarrow \infty} \left(\left(S_{m,p}^{(\alpha,\beta)}\right)^n f\right)(x) = f(c) + \frac{f(d) - f(c)}{d - c}(x - c), \tag{2.6.19}$$

uniformly on $[c, d]$*, where* $c = \frac{\alpha}{m+\beta}$*,* $d = \frac{m+p+\alpha}{m+\beta}$.

Chapter 3

Baskakov operators of King-type

3.1 The genuine Baskakov operators

In this section, we present the genuine Baskakov operator and their properties.

3.2 The study of a general Baskakov-type approximation process

The results included in this section have been published in *Miskolc Mathematical Notes*, 2 (2014), No. 2, 497-508.

In this section, we construct a general class of linear positive operators which generalize Baskakov's operators. For these operators, we study their convergence and we prove a Vornovskaja's type theorem.

Let $m_0 \in \mathbb{N}$ be given and $\mathbb{N}_1 = \{m \in \mathbb{N} \mid m \geq m_0\}$. We consider the functions $\alpha_m : J \longrightarrow \mathbb{R}$ and $\beta_m : J \longrightarrow \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_m(x) > 0,\\
&\beta_m(x) > 0,\\
&\beta_m(x) - \alpha_m(x) > 0,
\end{aligned}$$

for any $x \in J$ and any $m \in \mathbb{N}_1$, J will be indicated later, for each treated case, at the beginning.

We define the operators of the following form

$$(P_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \alpha_m^k(x)\beta_m^{-m-k}(x) f\left(\frac{k}{m}\right), \tag{3.2.1}$$

for any $m \in \mathbb{N}_1, x \in J$ and $f \in E([0,+\infty))$, where $E([0,\infty))$ is a linear space of real valued functions defined on $[0,\infty)$, for which the operators P_m are well defined.

We impose the condition

$$(P_m e_0)(x) = 1 + u_m(x), \tag{3.2.2}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$, where $u_m : J \longrightarrow \mathbb{R}, u_m(x) > -1$ and

$$(P_m e_1)(x) = x + v_m(x), \tag{3.2.3}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$, where $v_m : J \longrightarrow \mathbb{R}, v_m(x) > -x$.

It follows

$$\alpha_m(x) = \frac{x + v_m(x)}{1 + u_m(x)}(1 + u_m(x))^{-\frac{1}{m}} \tag{3.2.4}$$

and

$$\beta_m(x) = \left(1 + \frac{x + v_m(x)}{1 + u_m(x)}\right) (1 + u_m(x))^{-\frac{1}{m}}, \tag{3.2.5}$$

$m \in \mathbb{N}_1$, $x \in J$.

Taking (3.2.4) and (3.2.5) into account, the operator (3.2.1) becomes

$$\begin{aligned}(P_m f)(x) =&(1 + u_m(x)) \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x + v_m(x)}{1 + u_m(x)}\right)^k \\ &\times \left(1 + \frac{x + v_m(x)}{1 + u_m(x)}\right)^{-m-k} f\left(\frac{k}{m}\right),\end{aligned} \tag{3.2.6}$$

$m \in \mathbb{N}_1$, $x \in J$, $f \in E([0, +\infty))$.

We suppose that there exist the sequences $\Big(a_m(x)\Big)_{m\in\mathbb{N}_1}$, $\Big(b_m(x)\Big)_{m\in\mathbb{N}_1}$, such that

$$\lim_{m\longrightarrow\infty} a_m(x) = \lim_{m\longrightarrow\infty} b_m(x) = 0, \tag{3.2.7}$$

$$|u_m(x)| \le a_m(x), \tag{3.2.8}$$

$$|v_m(x)| \le b_m(x), \tag{3.2.9}$$

for any $m \in \mathbb{N}_1$ and any $x \in K = I \cap J$, K a compact interval.

We denote

$$\lim_{m\longrightarrow\infty} m(v_m(x) - x u_m(x)) = l(x) \tag{3.2.10}$$

and we suppose $l : J \longrightarrow \mathbb{R}$ is a bounded function on K.

Based on (3.2.7)-(3.2.9), if

$$\lim_{m\longrightarrow\infty} u_m(x) = \lim_{m\longrightarrow\infty} v_m(x) = 0, x \in K,$$

then we get

$$\lim_{m\longrightarrow\infty} m(v_m(x) - xu_m(x))^2 = 0.$$

This implies that there exists $m_1 \in \mathbb{N}$ such that

$$m(v_m(x) - xu_m(x))^2 \leq 1, m \in \mathbb{N}_1, m \geq m_1, x \in K. \tag{3.2.11}$$

Let us denote

$$M_1(K) = \sup_{\substack{m\in\mathbb{N}_1\\ x\in K}} a_m(x),\ M_2(K) = \sup_{\substack{m\in\mathbb{N}_1,\\ x\in K}} b_m(x)$$

and let $\mathbb{N}_2 = \{m \in \mathbb{N} \mid m \geq \max(m_0, m_1)\}$.

Theorem 3.2.1. (**Indrea, D. Adrian**, Pop, O. T., [16]) *Let the operators $P_m,\ m \geq m_0$, be defined by* (3.2.6)*. If $f \in C_2([0, +\infty))$, then*

$$\lim_{m\longrightarrow\infty} P_m f = f \tag{3.2.12}$$

uniformly on K.

There exists $m(0) \in \mathbb{N}, m(0)$ depending on K, such that the following inequalities

$$|(P_m f)(x) - (1 + u_m(x))f(x)| \leq (k_0 + k_2)\omega\left(f; \frac{1}{\sqrt{m}}\right), \tag{3.2.13}$$

$$|(P_m f)(x) - f(x)| \leq |u_m(x)| \cdot |f(x)| + (k_0 + k_2)\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{3.2.14}$$

and

$$|(P_m f)(x) - f(x)| \leq a_m(x)M + (k_0 + k_2)\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{3.2.15}$$

hold for any $m \in \mathbb{N}_2, m \geq m(0)$ and $x \in K$, where

$$M = \sup\{|f(x)| \mid x \in K\}.$$

Theorem 3.2.2. (**Indrea, D. Adrian**, Pop, O. T., [16]) *Let the operators* P_m, $m \geq m_0$, *be defined by* (3.2.6). *If* $f \in C_2([0, +\infty))$ *and* f *is two times differentiable on* K *and* $f^{(2)}$ *is continuous on* K, *then the following relation* (3.2.16)

$$\lim_{m \longrightarrow \infty} m\left((P_m f)(x) - (1 + u_m(x))f(x)\right) = l(x)f^{(1)}(x) + \frac{x(1+x)}{2} f^{(2)}(x)$$

holds, for any $m \in \mathbb{N}_1$, $x \in K$.

3.3 Classes of Baskakov-type operators

A. First class

We consider $K = [a, b]$, where $a > 0$. In this case $J = [0, +\infty)$ and $m_0 = 1$, $\mathbb{N}_1 = \mathbb{N}$. If the operators, $P_m, m \in \mathbb{N}$, preserve e_0 and e_1, we have

$$P_m e_0 = e_0 \text{ and } P_m e_1 = e_1,$$

for any $m \in \mathbb{N}$. Taking (3.2.2) and (3.2.3) into account, it results that

$$u_m(x) = v_m(x) = 0 \text{ and } l(x) = 0$$

for any $m \in \mathbb{N}$ and any $x \in [0, +\infty)$.

In this case, we get again the classical Baskakov operators. One has

$$a_m(x) = b_m(x) = 0,$$

for any $m \in \mathbb{N}, x \in [a, b]$, $k_0 = 1$ and $k_2 = b(1 + b) + 2$.

Theorem 3.3.1. (**Indrea, D. Adrian**, Pop, O. T., [16]) *Let the operators* P_m, $m \geq m_0$, *be defined by* (3.2.6). *If* $f \in C_2([0, +\infty))$, *one has*

$$\lim_{m \longrightarrow \infty} P_m f = f \tag{3.3.1}$$

uniformly on any compact interval $[a,b] \subset \mathbb{R}_+$. *There exists* $m(0) \in \mathbb{N}$, $m(0)$ *depending on* $K = [a,b]$ *such that*

$$|(P_m f)(x) - f(x)| \leq (3 + b + b^2)\omega\left(f; \frac{1}{\sqrt{m}}\right), \tag{3.3.2}$$

$m \in \mathbb{N}_2, m \geq m(0), x \in [a,b]$.

Theorem 3.3.2. (**Indrea, D. Adrian**, Pop, O. T., [16]) *Let the operators* P_m, $m \geq m_0$, *be defined by* (3.2.6). *If* $f \in C_2([0,+\infty))$, *f is two times differentiable on* $[a,b]$ *and* $f^{(2)}$ *is continuous on* $[a,b]$, *then*

$$\lim_{m \longrightarrow \infty} m((P_m f)(x) - f(x)) = \frac{x(1+x)}{2} f^{(2)}(x), \tag{3.3.3}$$

for any $x \in [a,b]$.

B. Second class

For the second type of operators we consider $J = [0,+\infty)$, $m_0 = 1$, $\mathbb{N}_1 = \mathbb{N}$. Because $P_m e_0 = e_0$ and $P_m e_2 = e_2$ for any $m \in \mathbb{N}$, taking (3.2.2) into account, it follows $u_m(x) = 0$ and

$$(m+1)(x + v_m(x))^2 + (x + v_m(x)) - mx^2 = 0, \tag{3.3.4}$$

for any $m \in \mathbb{N}$ and any $x \in [0,+\infty)$.

From (3.3.4) we get $v_m(x) = \frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)} - x$, then the operators (3.2.6) become

$$\begin{aligned}(P_m f)(x) = &\sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)}\right)^k \\ &\times \left(1 + \frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)}\right)^{-m-k} f\left(\frac{k}{m}\right),\end{aligned} \tag{3.3.5}$$

$m \in \mathbb{N},\ x \in [0,+\infty),\ f \in C_2([0,+\infty))$.

Lemma 3.3.1. (**Indrea, D. Adrian**, Pop, O. T., [16]) *The following relations hold*

$$v_m(x) \leq \frac{\sqrt{4m(m+1)a^2+1}-1}{2(m+1)} - a,\ m \in \mathbb{N},\ x \in K = [a,b], \tag{3.3.6}$$

and

$$\frac{\sqrt{4m(m+1)a^2+1}-1}{2(m+1)} - a \leq \sqrt{\frac{1}{2}a^2 + \frac{1}{16}} - a,\ m \in \mathbb{N}. \tag{3.3.7}$$

Lemma 3.3.2. (**Indrea, D. Adrian**, Pop, O. T., [16]) *The following relation*

$$\lim_{m \longrightarrow \infty} m v_m(x) = -\frac{1+x}{2} \tag{3.3.8}$$

holds, where $x \in K$.

Theorem 3.3.3. (**Indrea, D. Adrian**, Pop, O. T., [16]) *Let the operators P_m, $m \geq m_0$, be defined by* (3.3.5). *For any $f \in C_2([0,+\infty))$ it follows*

$$\lim_{m \longrightarrow \infty} P_m f = f \tag{3.3.9}$$

uniformly on compact $[a,b]$ and there exists $m(0) \in \mathbb{N}$, $m(0)$ depending on b, such that

(3.3.10)

$$|(P_m f)(x) - f(x)| \leq (3 + b(1+b))\, \omega\left(f; \frac{1}{\sqrt{m}}\right), m \in \mathbb{N}_2, m \geq m(0), x \in [a,b].$$

Theorem 3.3.4. (**Indrea, D. Adrian**, Pop, O. T., [16]) *Let the operators P_m, $m \geq m_0$, be defined by* (3.3.5). *If $f \in C_2([0,+\infty))$, $x \in [a,b]$, f is two times differentiable on $[a,b]$ and $f^{(2)}$ is continuous in $[a,b]$ then*

$$\lim_{m \longrightarrow \infty} m((P_m f)(x) - f(x)) = -\frac{1+x}{2} f^{(1)}(x) + \frac{x(1+x)}{2} f^{(2)}(x). \tag{3.3.11}$$

C. Third class

For the last case we consider $m_0 \in \mathbb{N}$, $m_0 \geq 2$ a fixed number and $J = [(m_0-1)^{-1}, +\infty)$. If $P_m e_1 = e_1$, then $v_m(x) = 0$, for any $x \in [(m_0-1)^{-1}, +\infty)$.

From $P_m e_1 = e_1$ and $P_m e_2 = e_2$, we have

$$\frac{m+1}{m} \cdot \frac{x^2}{1+u_m(x)} + \frac{x}{m} = x^2,\ m \in \mathbb{N}_1$$

and

$$u_m(x) = \frac{x+1}{mx-1},\ m \in \mathbb{N}_1\ ,\ x \in \Big[(m_0-1)^{-1}, +\infty\Big)\,. \tag{3.3.12}$$

The operators from (3.2.6) turn into

(3.3.13)
$$(P_m f)(x) = \frac{(m+1)x}{mx-1} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{mx-1}{m+1}\right)^k \left(1+\frac{x-1}{m+1}\right)^{-m-k} f\left(\frac{k}{m}\right)$$

for $m \in \mathbb{N}_1$, $x \in [(m_0-1)^{-1}, +\infty)$ and $f \in C_2([0,+\infty))$.

According to our relations (3.2.7), (3.2.9) and (3.2.10), we have

$$b_m\,(x) = 0,\ l(x) = -1-x,$$

for any $m \in \mathbb{N}_1, x \in [(m_0-1)^{-1}, +\infty)$. We also get

$$u_m(x) \leq \frac{m_0}{m-m_0+1} = a_m\,(x)$$

for any $x \in [(m_0-1)^{-1}, b]$ and $M_2\left([(m_0-1)^{-1}, b]\right) = 0$. Consequently,

$$k_0 = 1+m_0,\ k_2 = b(1+b)+2 \ \text{ and }\ M_1\left(\Big[(m_0-1)^{-1}, b\Big]\right) = m_0.$$

Theorem 3.3.5. (**Indrea, D. Adrian**, Pop, O. T., [16]) *Let $P_m,\ m \geq m_0$, be defined by* (3.3.13)*. For any $f \in C_2([0,+\infty))$ it follows*

$$\lim_{m\longrightarrow\infty} P_m f = f \tag{3.3.14}$$

uniformly on the compact $[(m_0-1)^{-1}, b]$. *Moreover, there exists* $m(0) \in \mathbb{N}$ *depending on* b, *such that*

$$\begin{aligned}|(P_m f)(x) - f(x)| \leq \frac{m_0}{m-m_0+1} M\left(\left[(m_0-1)^{-1}, b\right]\right) \\ + (3+m_0+b(1+b))\omega\left(f; \frac{1}{\sqrt{m}}\right)\end{aligned} \tag{3.3.15}$$

for any $m \in \mathbb{N}_2$, $m \geq m(0)$ *and* $x \in [(m_0-1)^{-1}, b]$. *In the above* $M([(m_0-1)^{-1}, b]) = \sup\{|f(x)| \mid x \in [(m_0-1)^{-1}, b]\}$.

Theorem 3.3.6. (**Indrea, D. Adrian**, Pop, O. T., [16]) *Let* P_m, $m \geq m_0$, *be defined by* (3.3.13). *If* $f \in C_2([0, +\infty))$, $x \in [(m_0-1)^{-1}, b]$, f *is two times differentiable on* $[(m_0-1)^{-1}, b]$ *and* $f^{(2)}$ *is continuous on* $[(m_0-1)^{-1}, b]$, *then*

$$\lim_{m \longrightarrow \infty} m((P_m f)(x) - f(x)) = \frac{1+x}{x} f(x) - (1+x) f^{(1)}(x) + \frac{x(1+x)}{2} f^{(2)}(x). \tag{3.3.16}$$

Proof. We have $\lim_{m \longrightarrow \infty} m u_m(x) = \dfrac{1+x}{x}$, $l(x) = -1-x$, for any $x \in [(m_0-1)^{-1}, b]$ and taking (3.2.16) into account, the relation (3.3.16) follows. □

Chapter 4

Durrmeyer operators of King-type

4.1 The genuine Durrmeyer operators

In this section, we present the genuine Durrmeyer operators and their properties.

4.2 Classes of Durrmeyer-King-type operators

The main results of this section are based on [20], published in *Annals of the University of Craiova*, Math. Comp. Sci. Ser. 39 (2012), No. 2, 288-298.

Three classes of linear and positive operators which preserve exactly two functions from the set $\{e_0, e_1, e_2\}$ are obtained. For each class of operators, results related to uniform convergence, error estimations in terms of the first modulus of continuity and Voronovskaja type theorems are established.

By using the idea of Durrmeyer operators construction, we consider operators of a general form defined by

$$(Q_m f)(x) = (m+1)\sum_{k=0}^{m}\binom{m}{k}(\alpha_m(x))^k(\beta_m(x))^{m-k}\int_0^1 p_{m,k}(t)f(t)dt, \tag{4.2.1}$$

where $x \in J$, $m \in \mathbb{N}_1$ and $f \in L_1([0,1])$.

Remark 4.2.1. *The operators $Q_m,\ m \in \mathbb{N}_1$ are linear and positive.*

In the following, we construct a sequence of Durrmeyer-type operators as defined in (4.2.1), which preserve the test functions e_0 and e_1.

Imposing the conditions $(Q_m e_0)(x) = e_0(x)$ and $(Q_m e_1)(x) = e_1(x)$, we obtain

$$\alpha_m(x) = \frac{(m+2)x-1}{m}, \tag{4.2.2}$$

$$\beta_m(x) = \frac{m+1-(m+2)x}{m}, \tag{4.2.3}$$

for any $x \in J$ and $m \in \mathbb{N}_1$.

Lemma 4.2.1. (Pop, O. T., **Indrea, D. Adrian**, Braica, P.I., [20]) *The following inclusion*

$$\left[\frac{1}{m_0+2}, \frac{m_0+1}{m_0+2}\right] \subset \left[\frac{1}{m+2}, \frac{m+1}{m+2}\right] \tag{4.2.4}$$

holds for any $m \in \mathbb{N}_1$.

Remark 4.2.2. *In what follows, we shall consider $J = \left[\frac{1}{m_0+2}, \frac{m_0+1}{m_0+2}\right]$. Thus, for α_m, β_m defined by (4.2.2) and (4.2.3) we have $\alpha_m(x) \geq 0$ and $\beta_m(x) \geq 0$, for any $x \in J$ and $m \in \mathbb{N}_1$.*

Taking into account the above remarks, we construct the sequence of Durrmeyer operators $(Q_{1,m})_{m\geq m_0}$. If $m \in \mathbb{N}_1$, we define the operator

$$(Q_{1,m}f)(x) = \frac{m+1}{m^m}\sum_{k=0}^{m}\binom{m}{k}((m+2)x-1)^k \times (m+1-(m+2)x)^{m-k}\int_0^1 p_{m,k}(t)f(t)dt, \tag{4.2.5}$$

for any $x \in \left[\frac{1}{m_0+2}, \frac{m_0+1}{m_0+2}\right]$, $f \in L_1([0,1])$.

Theorem 4.2.1. (Pop, O. T., **Indrea, D. Adrian**, Braica, P.I., [20]) *Let* $Q_{1,m}$, $m \geq m_0$, *be defined by* (4.2.5). *If* $f : [0,1] \longrightarrow \mathbb{R}$ *is a continuous function on* $[0,1]$, *then*

$$\lim_{m\longrightarrow\infty} Q_{1,m}f = f \tag{4.2.6}$$

uniformly on J. *Moreover, there exists* $m(0) \in \mathbb{N}$ *such that*

$$|(Q_{1,m}f)(x) - f(x)| \leq \frac{5}{2}\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{4.2.7}$$

for any $x \in J$ *and* $m \in \mathbb{N}_1, m \geq m(0)$.

Theorem 4.2.2. (Pop, O. T., **Indrea, D. Adrian**, Braica, P.I., [20]) *Let* $Q_{1,m}$, $m \geq m_0$, *be defined by* (4.2.5). *If* $f \in C([0,1]), x \in J, f$ *is two times differentiable on* $[0,1]$ *and* $f^{(2)}$ *is continuous on* $[0,1]$, *then*

$$\lim_{m\longrightarrow\infty} m((Q_{1,m}f)(x) - f(x)) = x(1-x)f^{(2)}(x), \tag{4.2.8}$$

for any $x \in J$.

B. Second class

Further on, we construct a sequence of Durrmeyer operators, which preserve the test functions e_0 and e_2.

We impose the conditions $(Q_m e_0)(x) = e_0(x)$ and $(Q_m e_2)(x) = e_2(x)$ and we obtain

$$\alpha_m(x) = \frac{-2m + \sqrt{\delta_m(x)}}{m(m-1)},$$
$$\beta_m(x) = \frac{m^2 + m - \sqrt{\delta_m(x)}}{m(m-1)}, \tag{4.2.9}$$

where

$$\delta_m(x) = m(2m + 2 + (m-1)(m+2)(m+3)x^2), \tag{4.2.10}$$

for $x \in J$ and $m \in \mathbb{N}_1$.

Lemma 4.2.2. (Pop, O. T., **Indrea, D. Adrian**, Braica, P.I., [20]) *Let $m \in \mathbb{N}_1$. Then $\beta_m(x) \geq 0$ for $x \geq 0$ if and only if*

$$0 \leq x \leq \sqrt{\frac{m+1}{m+3}}. \tag{4.2.11}$$

Remark 4.2.3. *Further on, we shall consider $J = \left[\sqrt{\frac{2}{(m_0+2)(m_0+3)}}, \sqrt{\frac{m_0+1}{m_0+3}}\right]$. Thus, for α_m, β_m defined by* (4.2.9) *we have $\alpha_m(x) \geq 0$ and $\beta_m(x) \geq 0$, for any $x \in J$ and $m \in \mathbb{N}_1$.*

If $m \in \mathbb{N}_1$ and $f \in L_1([0,1])$, we define the operator

$$(Q_{2,m}f)(x) = \frac{m+1}{(m(m-1))^m} \sum_{k=0}^{m} \binom{m}{k} \left(-2m + \sqrt{\delta_m(x)}\right)^k \times \left(m^2 + m - \sqrt{\delta_m(x)}\right)^{m-k} \cdot \int_0^1 p_{m,k}(t) f(t) dt \tag{4.2.12}$$

for any $x \in J$.

Theorem 4.2.3. (Pop, O. T., **Indrea, D. Adrian**, Braica, P.I., [20]) *Let $Q_{2,m}$, $m \geq m_0$, be defined by* (4.2.12).

(i) If $f : [0, 1] \longrightarrow \mathbb{R}$ is a continuous function on $[0, 1]$, then

$$\lim_{m \longrightarrow \infty} Q_{2,m} f = f \tag{4.2.13}$$

uniformly on J and there exists $m(0) \in \mathbb{N}$ such that

$$|(Q_{2,m} f)(x) - f(x)| \leq \frac{5}{2} \omega \left(f; \frac{1}{\sqrt{m}} \right) \tag{4.2.14}$$

for any $x \in J$ and $m \in \mathbb{N}_1, m \geq m(0)$.

(ii) If $f \in C([0, 1]), x \in J$, f is two times differentiable on $[0, 1]$ and $f^{(2)}$ is continuous on $[0, 1]$, then

$$\lim_{m \longrightarrow \infty} m((Q_{2,m} f)(x) - f(x)) = (x - 1) f^{(1)}(x) + x(1 - x) f^{(2)}(x). \tag{4.2.15}$$

C. Third class

Finally, we construct a sequence of Durrmeyer operators defined in (4.2.1), which preserve the test functions e_1 and e_2.

Imposing the conditions

$$(Q_m e_1)(x) = e_1(x)$$

and

$$(Q_m e_2)(x) = e_2(x).$$

Set $t_m(x) = \alpha_m(x) + \beta_m(x)$.

We obtain

$$\alpha_m(x) = \frac{(m + 2)}{m} \frac{x}{t_m^{m-1}(x)} - \frac{1}{m} t_m^m(x), \beta_m(x) = t_m(x) - \alpha_m(x) \tag{4.2.16}$$

and the following equation in $t_m^m(x)$

$$\begin{aligned} &(m + 1) t_m^{2m}(x) + (m(m + 3)(m + 2)x^2 - 2(m + 1)(m + 2)x) t_m^m(x) \\ &\quad + (1 - m)(m + 2)^2 x^2 = 0. \end{aligned} \tag{4.2.17}$$

For this equation, we have

$$\Delta_m = (m+2)^2 m x^2 (8(m+1) - 4(m+1)(m+3)x + m(m+3)^2 x^2) \geq 0, \tag{4.2.18}$$

for any $x \in J$ and $m \in \mathbb{N}_1$.

Set $\delta_m(x) = m\left(8(m+1) - 4(m+1)(m+3)x + m(m+3)^2x^2\right)$, for any $x \in J$ and $m \in \mathbb{N}_1$.

Because $m \geq 2$, one has

$$(1-m)(m+2)^2 x^2 \leq 0 \tag{4.2.19}$$

and then the equation (4.2.17) has exactly one positive solution

$$t_m^m(x) = \frac{(m+2)x\left(2(m+1) - m(m+3)x + \sqrt{\delta_m}\right)}{2(m+1)}. \tag{4.2.20}$$

Remark 4.2.4. *We set*

$$J = \left[\frac{2}{m_0+3}, \frac{m_0+2}{m_0+3}\right] \subset \left[\frac{2}{m+3}, \frac{m+2}{m+3}\right] \subset [0,1]. \tag{4.2.21}$$

If $m \in \mathbb{N}_1$ and $f \in L_1([0,1])$ we define the operator

$$(Q_{3,m}f)(x) = (m+1)\sum_{k=0}^{m} \binom{m}{k} (\alpha_m(x))^k (\beta_m(x))^{m-k} \int_0^1 p_{m,k}(t) f(t) dt, \tag{4.2.22}$$

where $\alpha_m(x)$ and $\beta_m(x)$ are given by the relations (4.2.16), for any $x \in J = \left[\frac{2}{m_0+3}, \frac{m_0+2}{m_0+3}\right]$.

Theorem 4.2.4. (Pop, O. T., **Indrea, D. Adrian**, Braica, P.I., [20]) *Let the operators $Q_{3,m}$, $m \in \mathbb{N}$, be defined by* (4.2.22). *If $f \in C([0,1])$, then*

$$\lim_{m \longrightarrow \infty} (Q_{3,m}f)(x) = f(x) \tag{4.2.23}$$

uniformly on J. Moreover, there exists $m(0) \in \mathbb{N}$ such that

$$|(Q_{3,m}f)(x) - f(x)| \leq \frac{7}{2}\omega\left(f; \frac{1}{\sqrt{m}}\right), \tag{4.2.24}$$

for any $x \in J$ and $m \in \mathbb{N}_1, m \geq m(0)$.

Theorem 4.2.5. (Pop, O. T., **Indrea, D. Adrian**, Braica, P.I., [20]) *Let the operators $Q_{3,m}, m \in \mathbb{N}$, be defined by* (4.2.22). *If $f \in C([0,1])$, $x \in J$, f is two times differentiable on $[0,1]$ and $f^{(2)}$ is continuous on $[0,1]$, then*

$$\lim_{m \longrightarrow \infty} m((Q_{3,m}f)(x) - f(x)) = \frac{2(1-x)}{x} f(x) + 2(x-1)f^{(1)}(x) + x(x-1)f^{(2)}(x). \tag{4.2.25}$$

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